The Spanning-Linear Independence Duality

Given vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k \in \mathbb{R}^n$ and suppose $\mathsf{A} = [\boldsymbol{a}_1\ \boldsymbol{a}_2\ \ldots \boldsymbol{a}_k]$. We have two "dual" sets of equivalent statements:

a. $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k\}$ spans $\mathbb{R}^n$.

b. For every $\boldsymbol{b} \in \mathbb{R}^n$, the matrix equation $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution.

c. For every $\boldsymbol{b} \in \mathbb{R}^n$, the system with augmented matrix $[\mathsf{A}\ \boldsymbol{b}] = [\boldsymbol{a}_1\ \boldsymbol{a}_2\ \ldots\ \boldsymbol{a}_k\ \boldsymbol{b}]$ has a solution.

d. A has a pivot position in every row.

e. The linear function $T\colon \mathbb{R}^k \to \mathbb{R}^n$ defined by $T(\boldsymbol{x}) = \mathsf{A}\boldsymbol{x}$ is onto.

a. $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_k\}$ is linearly independent.

b. The matrix equation $\mathsf{A}\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution.

c. The system with augmented matrix $[\mathsf{A}\ \boldsymbol{0}] = [\boldsymbol{a}_1\ \boldsymbol{a}_2 \ldots \boldsymbol{a}_k\ \boldsymbol{0}]$ has only the trivial solution.

d. A has a pivot position in every column.

e. The linear function $T\colon \mathbb{R}^k \to \mathbb{R}^n$ defined by $T(\boldsymbol{x}) = \mathsf{A}\boldsymbol{x}$ is one-to-one.