Composite numerical quadrature

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Recall: Last time, we introduced the concept of ***numerical quadrature*** as a methodology to approximate definite integrals numerically. In essence, given a function f defined in [a,b], we approximate its definite integral I[f] with a weighted sum defined by n+1 quadrature nodes x0,x1,…,xn an d n+1 quadrature weights w0,w1,…,wn. That is:

I[f] ~ Q[f] = w0 f(x0) + w1 f(x1) + … + wn f(xn)

The choice of nodes and weights defines a particular quadrature rule / method.

We then built the ***Interpolation-based Newton-Cotes*** quadrature methods. On an interval [a,b], we take n+1 equispaced points, and interpolate f(x) with a polynomial of degree <= n. The quadrature is then given by the integral of the polynomial interpolant pn(x). So,

- Nodes xj = a + h * j, with h = (b-a)/n .
- Weights are given by the Integral of Lj(x) from a to b.

We computed what the weights are for Trapezoidal: h[1/2, 1/2] and for Simpson: h[1/3, 4/3, 1/3] and then went over the Error Analysis for each of these two:

- **Trapezoidal** is exact for polynomials of degree <=1, and the error I[f] - Q[f] = **- (f''(eta)/12) h^3**
- **Simpson's** rule is exact for polynomials of degree <=3 (we gain a degree!) and the error is I[f] - Q[f] = **-(f^(iv) (eta) / 90) h^5**

Today, we will go over doing a **composite rule**: we partition our interval [a,b] into m sub-intervals and then approximate the integral as the sum of the quadrature rules applied to one (or a few) sub-interval(s).

what about Newton-Cotes $n>2$?

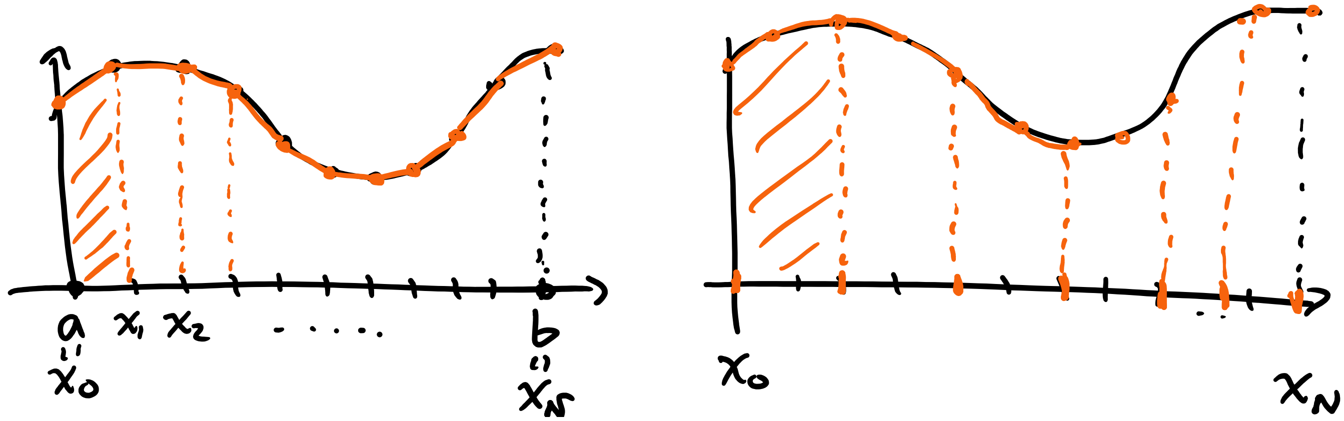
- $n+1$ is even (Trapezoidal) $\to$ Error $\sim C_n f^{(n+1)}(\eta)\, h^{n+2}$
- $n+1$ is odd (Simpson) $\to$ Error $\sim C_n f^{(n+2)}(\eta)\, h^{n+3}$

$\to$ as n grows, $\to$ Runge (equispaced are not good)

$\to$ $n+1 \geq 5$ $\to$ some weights are negative.

$\Rightarrow$ Unstable.

COMPOSITE QUAD.



$a = x_0 \quad x_1 \quad x_2 \quad \cdots\cdots \quad b = x_N \qquad x_0 \qquad\qquad x_N$

deg n rule, m panels $\Rightarrow$ $N+1$ pts $N = mn$.

$$\int_a^b f(x)\,dx = \sum_{j=0}^{m-1} \int_{x_{jn}}^{x_{(j+1)n}} f(x)\,dx \approx \sum_{j=0}^{m-1} Q_{x_{jn}}^{x_{(j+1)n}}[f]$$

COMPOSITE TRAPZ:

$h = (b-a)/N$.

$$\sum_{j=0}^{m-1} \frac{h}{2}\left(f(x_j) + f(x_{j+1})\right)$$

$$= \frac{h}{2}\left(f(a) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + \ldots + f(x_{N-1}) + f(b)\right]$$

$$= \frac{h}{2}\left(f(a) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{N-1}) + f(b)\right]$$

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x = linspace(a, b, N+1);  ->
w = h/2 [1 2 2 2 ... 2 1];  ->
return sum(w*f(x));
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ERROR ANALYSIS

$$E_h^T[f] = \int_a^b f(x)\,dx - T_h[f]$$

$$= \sum_{j=0}^{m-1}\left(\int_{x_j}^{x_{j+1}} f(x)\,dx - \frac{h}{2}\left(f(x_j) + f(x_{j+1})\right)\right)$$

Assuming $f \in C^2[a,b]$,

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$$= \sum_{j=0}^{m} \left(-\frac{f''(\eta_j)}{12} \cdot h^3 \right)$$

$$= \underline{\frac{-h^3}{12} \sum_{j=0}^{m} f''(\eta_j)} = -\frac{h^3}{12}\, m \left(\underbrace{\frac{1}{m} \sum_{j=0}^{m} f''(\eta_j)}_{\text{value between } \min f'',\ \max f''.} \right)$$

by IVT, $\exists\, \eta \in [a,b]$ s.t. $f''(\eta) = \frac{1}{m} \sum f''(\eta_j)$.

$$E_h^T = -\frac{f''(\eta)}{12} \cdot m \cdot h^3 \qquad m = N = \frac{(b-a)}{h}$$

$$\boxed{E_h^T = -\frac{f''(\eta)(b-a)}{12} h^2} \qquad O(h^2) = O(N^{-2})$$

SIMPSONS

$$E_h^S[f] = \int_a^b f(x)\,dx - S_h[f]$$

$$= \sum_{j=0}^{m} \left(\int_{x_{2j}}^{x_{2j+2}} f(x)\,dx - \frac{h}{3}\left(f(x_{2j}) + 4f(x_{2j+1}) + f(x_{2j+2})\right) \right)$$

Assuming $f \in C^{iv}([a,b])$

$$= \sum_{j=0}^{m} \left(-\frac{f^{iv}(\eta_j)}{90} \cdot h^5 \right)$$

$$= -\frac{h^5}{90} \sum_{j=0}^{m} f^{(iv)}(\eta_j)$$

$$= -h^5 m \left(\frac{1}{m} \sum^{m} f^{(iv)}(\eta_j) \right)$$

$$= -\frac{h^5 m}{90}\left(\frac{1}{m}\sum_{j=0}^{m} f^{(iv)}(x_j)\right)$$

IVT

$N = 2m$

$$= -\frac{h^5 m}{90}\cdot f^{iv}(\eta) = -\frac{h^5(b-a)}{90\cdot 2h}\cdot f^{iv}(\eta)$$

$$= \underline{-\frac{f^{iv}(\eta)(b-a)}{180}\cdot h^4} = O(h^4)$$

$$O(N^{-4})$$

$f(x) = \sin(2x)$ $\quad [a,b] = [0, 2\pi]$.

What is h (what is N) to guarantee $|E_h^T[f]| < \varepsilon$.

$$|E_h^T[f]| = \frac{|f''(\eta)|(2\pi)}{12} h^2$$

$f''(x) = -4\sin(2x) \longrightarrow |f''(\eta)| \leq 4$.

$$|E_h^T[f]| \leq \frac{8\pi h^2}{12} < \varepsilon$$

$\left(N = \frac{2\pi}{h},\ h = \frac{2\pi}{N}\right)$ $\quad \frac{8\pi}{12}\cdot\left(\frac{2\pi}{N}\right)^2 < \varepsilon$

$$\left(\frac{N}{2\pi}\right)^2 > \frac{8\pi}{12\varepsilon}$$

$$\boxed{N > 2\pi\sqrt{\frac{8\pi}{12\varepsilon}}} \quad O(\varepsilon^{-1/2})$$
