APPM 4600: Algorithms & Stability II

Wednesday, September 4, 2024 9:24 AM

Class 04: September 4, 2024

Recall: We defined an **algorithm** to be a sequence of instructions to be executed in order, and a **pseudocode** as a "code-like" description for an algorithm, including inputs, outputs, and distinct, clear steps. We then began a discussion on some of the aspects we care about when evaluating an algorithm to solve a given mathematical problem:

- We care, of course, that the algorithm is **mathematically correct**; that given reasonable assumptions on our inputs, the instructions carried out compute the desired outputs to a satisfactory target accuracy.
- We want our algorithm to be **efficient** in its use of resources (e.g. memory) and "fast".
- Finally, we began a discussion on an crucially important feature of algorithms, related to loss of precision. That is: we want our algorithms to be **numerically stable.** Intuitively, this means we do not want an algorithm to *unnecessarily lose precision* as it computes outputs from inputs.

WARM UP: EXAMPLES

① SUM OF N numbers $x_i \in \mathbb{R}$

INPUTS: $N, \{x_i\}_{i=1}^{N}$

OUTPUT S

(1) SET $S = 0$

(2) FOR $i = 1$ to N

(3) $\quad S = S + x_i;$

(4) RETURN $S;$

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② SOLVE QUADRATIC EQ ax^2+bx+c (assuming 2 distinct $\mathbb{R}$ roots)

INPUTS: a, b, c

OUTPUTS: r_1, r_2

(1) SET $r_1 = r_2 = 0$

(2) COMPUTE $d = b^2 - 4ac$

(3) IF $a == 0$ OR $d \leq 0$

(4) $\quad$ PRINT ("ROOTS ARE NOT REAL OR DISTINCT"); RETURN r_1, r_2;

ELSE

  IF $b == 0$

    $r_1 = \sqrt{-c/a}$, $r_2 = -r_1$;

  ELSEIF $b > 0$

    $r_1 = \dfrac{2c}{(-b-\sqrt{d})}$; $r_2 = \dfrac{-b-\sqrt{d}}{2a}$;

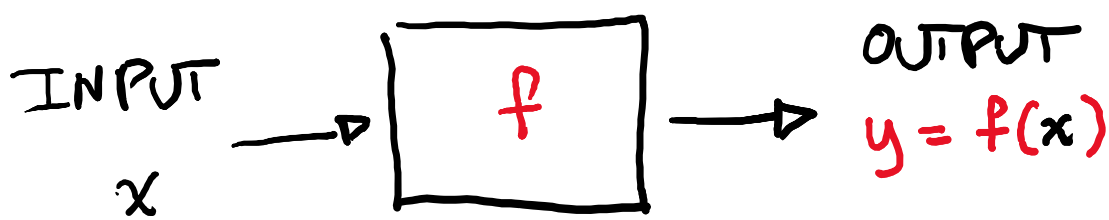
  ELSE

    $r_1 = \dfrac{-b+\sqrt{d}}{2a}$; $r_2 = \dfrac{2c}{-b+\sqrt{d}}$

RETURN r_1, r_2.

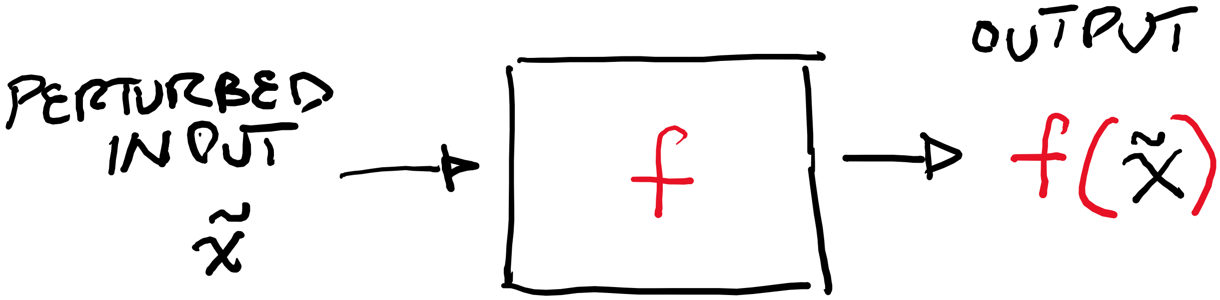
CONDITION NUMBER & STABILITY

MATH PROBLEM



What happens when we plug in $\tilde{x} = x + \Delta x$?

→ perturbation Δx — noise, measurement error, sensitivity analysis, ...



REL ERROR $\qquad\qquad\qquad\qquad$ REL ERROR

$$\frac{|x-\tilde{x}|}{|x|} \xrightarrow{K_f(x)} \frac{|f(x)-f(\tilde{x})|}{f(x)}$$

CONDITION NUMBER $K_f(x)$:

We want to say that for small Δx,

$$\frac{|f(x)-f(\tilde{x})|}{|f(x)|} \approx K_f(x)\,\frac{|x-\tilde{x}|}{|x|}$$

REL ERR OUTPUTS $\approx$ (COND NUMBER) (REL ERR INPUTS)

COND NUMBER $\approx 1 \longrightarrow$ MATH PROBLEM IS "WELL CONDITIONED"

COND NUMBER $\simeq 10^p$, $p>1$

$\Rightarrow$ "I lose $\approx p$ digits"

LARGE COND $\longrightarrow$ "PROBLEM IS ILL CONDITIONED"

DEFINITION & EXAMPLES

DEF: The condition number for function $f:\mathbb{R}\to\mathbb{R}$ is given by:

$$K_f(x) = \lim_{\Delta x\to 0} \frac{\frac{|f(x+\Delta x)-f(x)|}{|f(x)|}}{\frac{|\Delta x|}{|x|}}$$

$$= \lim_{\Delta x\to 0} \frac{|f(x+\Delta x)-f(x)|}{|\Delta x|}\cdot\frac{|x|}{|f(x)|}$$

$$\boxed{\text{If } f \text{ is diff at } x, \quad K_f(x) = |f'(x)|\cdot\frac{|x|}{|f(x)|}}$$

EXAMPLES

$$f_+(x) = x+a \longrightarrow K_{f_+}(x) = \frac{|x|}{|x+a|}$$

$$f_\times(x) = ax \longrightarrow K_{f_\times}(x) = |a|\frac{|x|}{|ax|}$$

$$= \underline{1}\ !$$

$$f_b(x) = \frac{-b+\sqrt{b^2+4}}{2b} \longrightarrow K(x) = \frac{|b|}{\sqrt{b^2+4}}$$

STABILITY $\rightarrow$ SIMILAR STATEMENT, BUT NOW IT IS ABOUT **ALGORITHMS**

A **STABLE** ALGORITHM

APPROX INPUT $\tilde{x}$ $\longrightarrow$ $\boxed{\text{ALGORITHM } \tilde{f}}$ OUTPUT $\tilde{f}(\tilde{x})$

$$\begin{pmatrix}\text{REL ERR}\\ \text{OUTPUTS}\end{pmatrix} \approx \left(\quad ? \quad \right)\begin{pmatrix}\text{REL ERR}\\ \text{INPUTS}\end{pmatrix}$$

STABLE: ? is $\approx K_f(x)$

UNSTABLE: ? $\gg K_f(x)$