Numerical quadrature: Newton-Cotes

Wednesday, October 30, 2024 11:01 AM

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Recall: Last time we covered two important cases of approximation with non-polynomials:

1. **Rational approximation:** We covered the general idea of rational approximation, why it might be a superior extension of polynomial approximation, what are some modern methods being researched (e.g. AAA, minimax) and an example of a classical method (Pade approximation).
2. **Trigonometric approximation**: We started by covering the L2 approximation problem, and the formulas obtained from the Fourier basis being orthogonal. We then discussed very briefly what the DFT and the FFT are, and how they solve an interpolation and an approximation problem for trigonometric polynomials.

In L2 approximation, we often had formulas for coefficients in terms of integrals. Also, we know from applications of calculus and differential equations that integration is an extremely important operation along with differentiation. And of course, we know it can't always be done analytically (we can't always find an antiderivative, not even with Mathematica or ChatGPT).

Today, we begin a review on algorithms to compute integrals numerically. We start with algorithms that are familiar to us from Calculus I and II, but we are now able to derive error estimates and understand where these algorithms (and extensions) come from.

Numerical Quadrature

$$I_a^b[f] = \int_a^b f(x)\,dx \approx Q_a^b[f] = \sum_{j=0}^{n} \omega_j f(x_j)$$

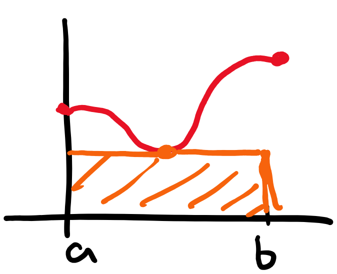
QUADRATURE: $\{x_j\}_{j=0}^{n}$ nodes, ω_j weights

Galerkin:

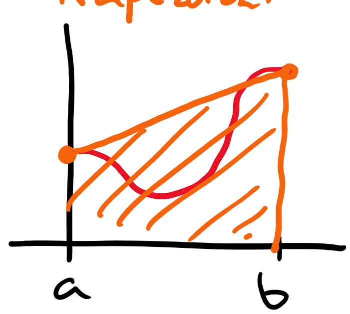
$$f(x) \approx \sum_{k=0}^{m} \alpha_k \phi_k(x), \quad \int_a^b f(x)\,dx \approx \sum_{k=0}^{m} \alpha_k \left[\int_a^b \phi_k(x)\,dx\right]$$

Interpolation-based Quadrature (Newton-Cotes)

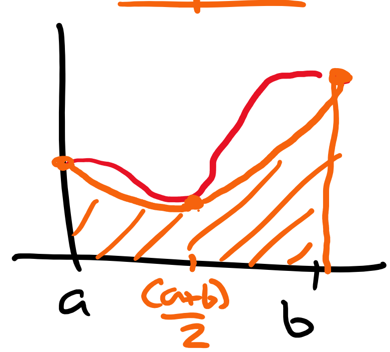
Riemann sum



Trapezoidal



Simpson



Select n+1 equisp nodes from a to b.

⊙ Select $n+1$ equisp nodes from a to b.

↳ Interpolate using $p \in \mathcal{P}_n$.

$$\rightarrow \int_a^b f(x)\,dx \approx \int_a^b p(x)\,dx$$

$$p(x) = \sum_{j=0}^{n} f(x_j) L_j(x)$$

$$\int_a^b p(x)\,dx = \sum_{j=0}^{n} f(x_j) \underbrace{\left(\int_a^b L_j(x)\,dx \right)}_{\omega_j}$$

Trapezoidal Rule:

Linear interp.

$$P_1(x) = f(a)\frac{(b-x)}{(b-a)} + f(b)\frac{(x-a)}{(b-a)}$$

Notation: $h = b-a$

$$P_1(x) = \frac{1}{h}\left(f(a)(b-x) + f(b)(x-a) \right)$$

$$\int_a^b P_1(x)\,dx = \frac{1}{h}\left(f(a)\underbrace{\int_a^b (b-x)\,dx}_{-\frac{(b-x)^2}{2}\Big|_a^b = \boxed{\frac{h^2}{2}}} + f(b)\underbrace{\int_a^b (x-a)\,dx}_{\frac{(x-a)^2}{2}\Big|_a^b} \right)$$

$$= \boxed{\frac{h}{2}\left(f(a) + f(b) \right)}$$

$$= (b-a)\left(\frac{f(a)+f(b)}{2} \right)$$

$$= (b-a)\left(\frac{f(a)+f(b)}{2}\right)$$

$$\left\{\begin{matrix} x_0 = a & x_1 = b \\ \omega_0 = h/2 & \omega_1 = h/2 \end{matrix}\right\}$$

Simpsons

$x_0 = a \quad x_1 = (a+b)/2 \quad x_2 = b \qquad h = (b-a)/2$

$$P_2(x) = f(a)\,L_0(x) + f\left(\frac{a+b}{2}\right)L_1(x) + f(b)\,L_2(x)$$

Weights: $\omega_0 = \omega_2 = h/3, \quad \omega_1 = 4h/3.$

Simpson's rule: $\frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

```
def quad_int (f, a, b, type)
    x = linspace (a, b, n+1)
    fvec = f(x);
    w = h[              ]
return sum (w*f)
```

$\boxed{\text{ERROR ANALYSIS}}$:

$$E_n[f] = f(x) - P_n(x) = \frac{f^{(n+1)}(\eta_x)}{(n+1)!}\underbrace{\prod_{j=0}^{n}(x-x_j)}_{\psi_n(x)}$$

$$\int_a^b E_n[f]\,dx = \int_a^b f(x)\,dx - \int_a^b P_n(x)\,dx$$

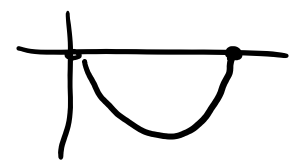
$=$

Trapezoidal:

$$f(x) - P_1(x) = \frac{f''(\eta_x)}{2} \overbrace{(x-a)(x-b)}^{\psi_1(x)}$$

$$\int_a^b \frac{f''(\eta_x)}{2}(x-a)(x-b)\,dx$$

$\psi_1(x) \le 0$



IMVT

$$\frac{f''(\eta)}{2}\int_a^b (x-a)(x-b)\,dx$$

$$= -\frac{f''(\eta)\cdot h^3}{12}$$

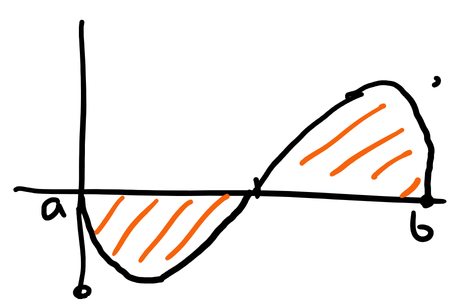
linear poly exactly.

error 1 int $O(h^3)$

$h^3 = h^{n+2}$

$\boxed{\text{SIMPSON'S}}$

$$f(x) - P_2(x) = \frac{f'''(\eta_x)}{6} \overbrace{(x-a)\left(x-\frac{a+b}{2}\right)(x-b)}^{\psi_2(x)}$$



- Show that Simpsons integrates cubics exactly.

$$c(x) = \left(x - \frac{a+b}{2}\right)^3$$

$$c(x) = \left(x - \frac{a+b}{2}\right)$$

$$\int_a^b c(x)\,dx = 0 \qquad S_a^b[c] = \frac{h}{3}\left[c(a) + c\left(\cancel{\frac{a+b}{2}}\right) + c(b)\right] = 0$$

$$p \in \mathcal{P}_3 \to p = \alpha c + q, \quad q \in \mathcal{P}_2.$$

$$f(x) = \underbrace{T_3(x)}_{\text{Taylor centered at } a} + \underbrace{\frac{f^{IV}(\eta_x)}{4!}(x-a)^4}_{\text{Taylor res}}$$

$$\int_a^b \frac{f^{IV}(\eta_x)}{4!}(x-a)^4\,dx - \text{Simpsons applied to it.}$$

$$\frac{f^{IV}(\eta)}{4!}\int_a^b (x-a)^4\,dx$$

$$\boxed{h = \frac{(b-a)}{2}}$$

ERROR $\to -\frac{f^{IV}(\eta)}{90} h^5$