# Continuous L2 approximation

Wednesday, October 23, 2024 9:46 AM

Class 25: October 23, 2024

Recall: Last time, we discussed a general way to frame a discrete approximation problem: Given n data points (xi,yi), can we find a model p(x) = a0 phi_0(x) + a1 phi1(x) + ... + am phim(x) such that it minimizes some norm ||p(x) - y|| ? We then discussed the case where the norm being minimized is the euclidean norm (2-norm). We called this **"Discrete Least Squares"**:

**Min q(a) = Min Sum_{i=0}^{n-1} ( p(x_i) - y_i)^2 = Min ||M a - y||_2^2**

We then realized that the function q(a) = || ||^2 being minimized is a concave up quadratic. We can write its formula using inner products and matrix-vector algebra as:

**q(a) = a^T G a - 2b^T a + c**

With G = (M^T)*M the SPS Grammian matrix, b = M^Ty and c = y^ty (constant). If M has LI columns (trivial nullspace), then G is SPD and invertible, and q has a unique minimizer at the one value of a such that:

**Grad q(a) = 2(G a - b) = 0**
**Ga = b**
**(M^T)M a = (M^T) y**

We called this linear system the "Normal Equations", and its solution a_{LS} is the vector of coefficients of our model of best fit (in the LS sense), p = M*a_{LS}.

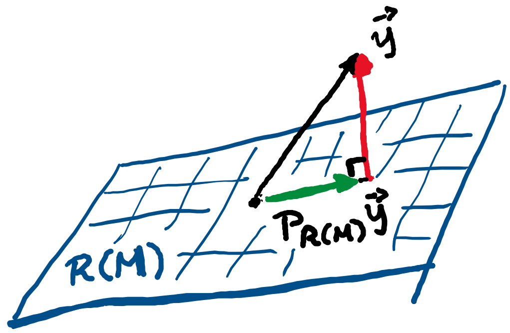
Finally, we mentioned that even if G is invertible, it will sometimes be very ill-conditioned (e.g. polynomial model using monomial basis). So, stabler methods to solve this problem involve some factorization of M instead of solving Ga=b.

Today, I will briefly explain one of these methods, show a demo, and then we will move on to another problem of function approximation: continuous L2 approximation.

Recall this interpretation of DLS:

$$\min \|M\vec{a} - \vec{y}\|^2$$

Solve $M\vec{a} = P_{R(M)}\vec{y}$




**Algorithm Idea:**

$$\underset{n\times m}{M} = \underset{n\times m}{Q}\ \underset{m\times m}{R}$$

$Q$: unitary, $\{q_1, q_2, \dots, q_m\}$ orth basis of $R(M)$

$R$: upper triang

$$Ma = P_{R(M)}y$$
$$\downarrow$$
$$QRa = QQ^T\vec{y}$$
$$\updownarrow$$

SOLVE: $\boxed{R\vec{a} = Q^T\vec{y}}$

{FOR MORE, FINAL PROJECTS!}

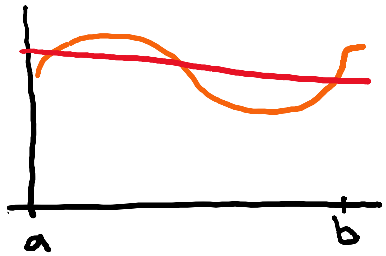
**Generalizations:**

- Weighted LS: $\min \sum_{i=0}^{n-1} w_i (p(x_i) - y_i)^2$
- Regularized LS: $\min \{ \|p - y\|^2 + \underbrace{\lambda \|\Gamma a\|}_{\text{penalty}} \}$
  - Ridge regression / Shrinkage
  - Tikhonov
  - LASSO

**CONTINUOUS FUNCTION APPROX:**


DISCRETE APPROX ✓

HOW? FORM $\vec{p} = p(x_i)$,
Find coeffs $a_0, \dots, a_m$ to
minimize $\| \vec{p} - \vec{y} \|$.




CONT. APPROX ?

$\min \| p - f \|$

What does $\| f \|$ mean?

$f \in C[a,b]$,

$$\| f \|_2 = \left( \int_a^b f(x)^2 \, dx \right)^{1/2} \longrightarrow L^2 \text{ norm.}$$

space $L^2([a,b])$

$$\langle f, g \rangle = \int_a^b f(x) g(x) \, dx$$

inner product sp.
(Hilbert).

- Find $\vec{a}$ such that $\| f - p \|^2$ is minimized.

$$\| f \|_{\omega,2} = \int_a^b f(x)^2 \, \omega(x) \, dx \quad (\omega(x) \geq 0, \text{ integrable.})$$

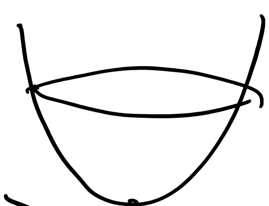
$$\langle f, g \rangle_\omega = \int_a^b f(x) g(x) \, \omega(x) \, dx \qquad \text{Hilbert sp.}$$

- $\| f \|_\infty = \sup_{x \in [a,b]} \{ |f(x)| \} \longrightarrow$ "minimax" $L^\infty$
- $\| f \|_p = \left( \int_a^b f(x)^p \, dx \right)^{1/p} \qquad L^p$

$L^2$:

$$q(\vec{a}) = \int_a^b \left( \sum_{j=0}^{m-1} a_j x^j - f(x) \right)^2 dx$$

$$= \left\langle \sum_{i=0}^{m-1} a_i x^i - f, \sum_{j=0}^{m-1} a_j x^j - f \right\rangle$$



$$q(\vec{a}) = a^T G a - 2 b^T a + c$$

DISC

| | DISC |
|---|---|
| $q(\vec{a}) = a^T G a - 2 b^T a + c$ | |
| $G(i,j) = \langle x^i, x^j \rangle = \int_a^b x^{i+j}\,dx$ | $M(:,i)^T M(:,j)$ |
| $b(i) = \langle x^i, f \rangle = \int_a^b x^i f(x)\,dx$ | $M(:,i)^T y$ |
| $c = \langle f, f \rangle = \int_a^b f(x)^2\,dx$ | $y^T y$ |

---

Powers of $x$ (poly approx) $\longrightarrow$ Hilbert mat.

$\{1, x, x^2, x^3, \ldots, x^{m-1}\} \rightarrow$ LI but poorly cond.

$$G(i,j) = \langle \phi_i, \phi_j \rangle$$

$\langle \phi_i, \phi_j \rangle = 0$ if $i \neq j$ ORTHOGONAL

and $\langle \phi_i, \phi_i \rangle = 1$ — ORTHONORMAL.

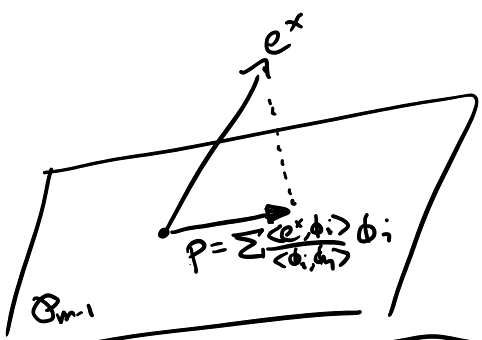
---

ORTHOGONAL BASIS $\{\phi_0, \phi_1, \ldots, \phi_{m-1}\}$

$$G a = b \qquad \begin{bmatrix} \langle \phi_1, \phi_1 \rangle & 0 & 0 & \cdots & 0 \\ 0 & \langle \phi_2, \phi_2 \rangle & 0 & \cdots & 0 \\ \vdots & 0 & \langle \phi_3, \phi_3 \rangle & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & \langle \phi_{m-1}, \phi_{m-1} \rangle \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_{m-1} \end{bmatrix} = \begin{bmatrix} \langle f, \phi_1 \rangle \\ \langle f, \phi_2 \rangle \\ \vdots \\ \vdots \\ \langle f, \phi_{m-1} \rangle \end{bmatrix}$$

$$\boxed{a_j = \frac{\langle f, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}}, \quad p(x) = \sum_{j=0}^{m-1} \frac{\langle f, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle} \phi_j(x)$$

$$a_j = \frac{\int_a^b f(x)\phi_j(x)\,dx}{\int_a^b \phi_j(x)^2\,dx}$$




---

ERROR $\|f - p\|^2 = \sum_{j=m}^{\infty} \left( \frac{\langle f, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle} \right)^2$

ERROR $\|f-p\|^{2} = \sum_{j=m} \left(\frac{\langle f, \phi_j\rangle}{\langle \phi_j, \phi_j\rangle}\right)$

---

Polynomial approx, $[-1,1]$, $w(x)\equiv 1$

→ Find orthogonal basis of polynomials
$\{P_0(x), P_1(x), P_2(x), \ldots\ldots\}$ → $\boxed{\text{LEGENDRE}}$

spans the same as
$\{1, x, x^2, x^3, \ldots\}$

---

GRAM-SCHMIDT:

$P_0(x) = 1$

$$P_1(x) = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle}\cdot 1$$

$$= x - \left(\frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} 1\,dx}\right)1 = x$$

$$P_2(x) = x^2 - \underbrace{\frac{\langle x^2, x\rangle}{\langle x, x\rangle}}_{\frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx}} x - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} 1$$

$$\left[\frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} 1\,dx}\right] \to (2/3)/2$$

$$\boxed{P_2(x) = x^2 - 1/3}$$

$$P_3(x) = x^3 - \frac{\langle x^3, x^2 - 1/3\rangle}{\langle x^2-1/3, x^2-1/3\rangle}(x^2 - 1/3) - \frac{\langle x^3, x\rangle}{\langle x, x\rangle}x - \frac{\langle x^3, 1\rangle}{\langle 1, 1\rangle}1.$$

---