Spline Interpolation (I) and Hermite wrap-up

Wednesday, October 16, 2024 11:54 AM

Class 22: October 16, 2024

Recall: Last time, we proposed an extension to the polynomial interpolation problem on n+1 points (nodes) x0,x1,...,xn. In this extension, we provide function data yi = f(xi) and derivative data zi = f'(xi) at these nodes, resulting in 2n+2 data points. The Hermite interpolant is the unique polynomial of degree <= 2n+1 that satisfies all of these conditions (so, h(xi)=yi and h'(xi)=zi for i=0,1,...,n).

We briefly touched on the following:

1. We can build a Vandermonde matrix for this problem, known as the Confluent-Vandermonde. Its determinant is non-zero if and only if the xi's are distinct. Its condition number is terrible (grows exponentially with n).
2. We can build a Hermite-Lagrange basis {H_j(x), K_j(x)}, consisting of polynomials of degree <= 2n+1 that satisfy one condition = 1, and the rest = 0. The interpolant is then the sum of yj * H_j(x) + zj * K_j(x), and the formulas for both depend on M_j(x) = L_j(x)^2
3. We can build a Hermite-Newton basis, and come up with a process to compute a Hermite-Newton tableau and then use Horner to evaluate h(x). This requires us to "repeat" each xi twice, and define f[xi , xi] = f'(xi).
4. Finally, we mentioned that the same tricks for Barycentric Lagrange can be played to find a 2nd barycentric formula for Hermite-Lagrange.

Today, we will talk about error estimates for Hermite, wrapping up this subject. We will then move on to talk about an extremely useful interpolation method: the use of "splines" (a kind of piecewise polynomial). Splines are ubiquitous in curve fitting, approximation, computer graphics (e.g. Pixar) and scientific computing.

ERROR EST FOR HERMITE:

FOR INT ON n+1 points given (x_i, y_i) data.
Assumed $y_i = f(x_i)$ (Exact), $f \in C^{n+1}([a,b])$
$[a,b]$ contains $x_0, x_1, \ldots, x_n$.

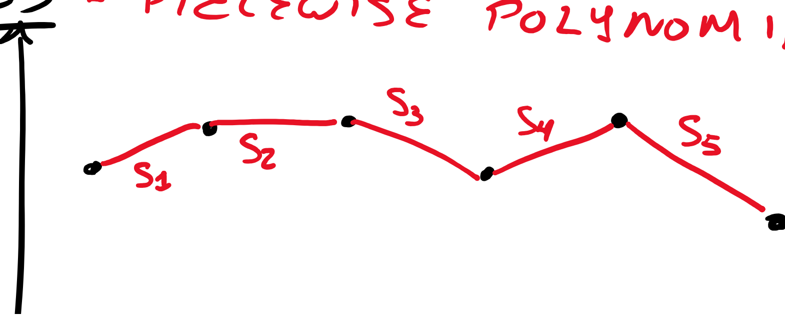
$$f(x) - p(x) = \frac{f^{(n+1)}(\eta_x)}{(n+1)!} \overbrace{(x-x_0)(x-x_1)\cdots(x-x_n)}^{\psi(x)}$$

HERMITE:

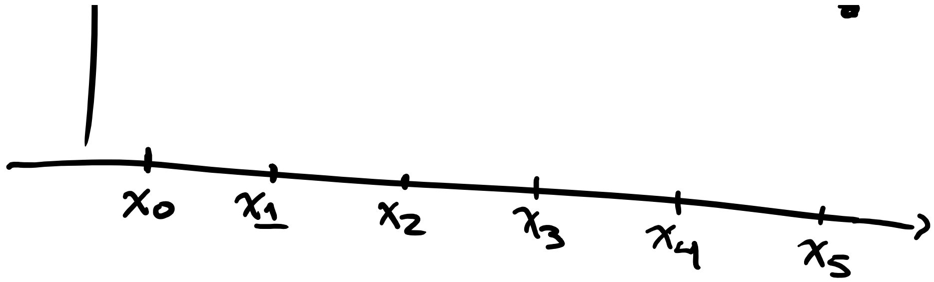
Assume: $y_i = f(x_i)$, $z_i = f'(x_i)$, $f \in C^{2n+2}([a,b])$

$$f(x) - h(x) = \frac{f^{(2n+2)}(\eta_x)}{(2n+2)!} \underbrace{(x-x_0)^2(x-x_1)^2\cdots(x-x_n)^2}_{\psi(x)^2}$$

SPLINES - PIECEWISE POLYNOMIAL



$s(x)$ piecewise linear on [each int.]



each int. $I_i = [x_{i-1}, x_i]$, $i = 1, \dots, n$

$x_0 < x_1 < x_2 < \dots < x_5 < \dots$

$S_i(x) = s(x)\big|_{I_i}$

$S_i \in \mathcal{P}_1$

$s(x)$ is continuous.

$S_1(x_1) = S_2(x_1)$

$S_2(x_2) = S_3(x_2)$

$\vdots$

$\{S_i(x_i) = S_{i+1}(x_i)\}_{i=1}^{n-1}$

FORMULA FOR $S_i(x)$:

Given data (x_i, y_i) $i = 0, \dots, n$

S_i is defined by (x_{i-1}, y_{i-1}) and (x_i, y_i)

$$S_i(x) = y_{i-1}\left(\frac{x - x_i}{x_{i-1} - x_i}\right) + y_i\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)$$

$h_i = x_i - x_{i-1}$.

$$\boxed{S_i(x) = \frac{1}{h_i}\left(y_{i-1}(x_i - x) + y_i(x - x_{i-1})\right)}$$

Given (x_i, y_i) and x_{eval} (m pts)

$\longrightarrow s(x_{EVAL})$.

ERROR ANALYSIS?

Error Analysis:

$$f(x) - s(x) = \ ?$$

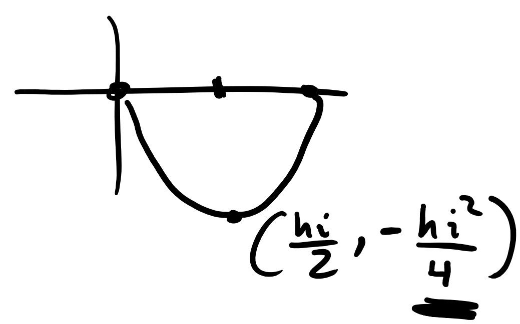
$$|f(x) - s(x)| \le \ ?$$

Assume $f \in C^2([a,b])$, $x \in I_i = [x_{i-1}, x_i]$ with width h_i.

$$\boxed{f(x) - s_i(x) = \frac{f''(\eta_x)}{2!}(x - x_{i-1})(x - x_i)}$$

$$|f(x) - s_i(x)| \le \underbrace{\frac{\overset{\|f''\|_\infty}{\max_{x \in [a,b]} |f''(x)|}}{2}} \ \underbrace{\max_{x \in I_i} |(x - x_{i-1})(x - x_i)|}$$

$$\max_{x \in [0, h_i]} |\underbrace{x(x - h_i)}|$$



$$s_i(x) = x(x - h_i)$$
$$= x^2 - h_i x$$
$$s_i'(x) = 2x - h_i \rightarrow x = \frac{h_i}{2}$$

$$|f(x) - s_i(x)| \le \frac{\|f''\|_\infty}{2} \cdot \left(\frac{h_i^2}{4}\right) \rightarrow O(h^2)$$

Partition is equispaced $h_i = h$. $n \to \infty$, $h \to 0$

CUBIC SPLINES

CUBIC SPLINES

$$S_i(x) = s(x)\big|_{I_i} \in \mathcal{P}_3 .$$

- Continuity.
- s', s'' → Continuity.
- Space of PW CUBIC → problem is well-defined.