Newton-Raphson method

Monday, September 16, 2024 10:21 AM

Class 09: September 16, 2024

Recall: Last time, we proved two very important theorems about both the **Fixed Point problem** and the performance of the **Fixed Point Iteration (FPI)**. Those are:

Theorem 1 (*Existence, Uniqueness and Convergence of FPI*): Let g(x) be continuous on [a,b] (a member of C[a,b]).
(I) If a <= g(x) <= b for all x in [a,b], then there exists at least one fixed point.
(II) If additionally, g'(x) exists for all x in (a,b) and |g'(x)| <= k < 1 for all x in (a,b), then there is a unique fixed point r in [a,b].
(III) If the assumptions of (I) and (II) are true, then for any x0 in (a,b), FPI converges at least linearly to r (with rate k or smaller).

Theorem 2 (*Local convergence and convergence order for FPI*): Let r be a fixed point of g(x), and let g be at least two times continuously differentiable on an interval around r (g in C^2[r-eps,r+eps]). Then,
(I) If g'(r) is not zero, then there exists a delta>0 such that, if |x0 - r| < delta, then FPI starting at x0 converges to r linearly with rate |g'(r)| .
(II) If g'(r) is zero and g''(r) is not zero, then there exists a delta>0 such that, if |x0 - r| < delta, then FPI starting at x0 converges to r quadratically.

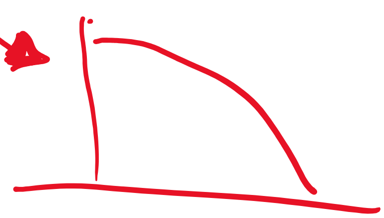
These two results give us powerful tools to analyze the convergence of Fixed Point Iterations, and to have a better idea of what to expect in terms of the performance of this family of methods. Theorem 2, in particular, tells us that unless we have derivative information on a whole interval around the FP r, we will only know that "an interval exists" around r such that FPI converges at a certain order / rate.

Finally, we went through examples for which FPI converges at various linear rates or with quadratic order, and some for which it diverges (displaying either cycles or going off to infinity).

In today's session, we will discuss the powerful (and rather famous) **Newton-Raphson method**. After deriving it from first principles, we will then realize it is a particular case of a FPI. We will analyze its performance closely and discuss examples of success and failure for this method.

WARM UP: FP EXAMPLES

1. $g_1(x) = 1 + 0.5\sin x$; $r \approx 1.4987$, $g_1'(x) = 0.5\cos x$, $g_1'(r) \approx 0.036$
2. $g_2(x) = 3 + 2\sin x$; $r \approx 3.1$ → $g_2'(x) = 2\cos x$, $g_2'(r) \approx -2$
3. $g_3(x) = \frac{1}{2}x + \frac{1}{x}$; $r = \sqrt{2}$



$g_3'(x) = \frac{1}{2} - \frac{1}{x^2}$

$g_3'(\sqrt{2}) = \frac{1}{2} - \frac{1}{\sqrt{2}^2} = \underline{0}$

BANACH FP THEOREM

> **REMARK:** More general condition (Contraction map) → $\forall x, y \in [a,b]$, $|g(x) - g(y)| \leq K|x-y|$ for $K \in (0,1)$.

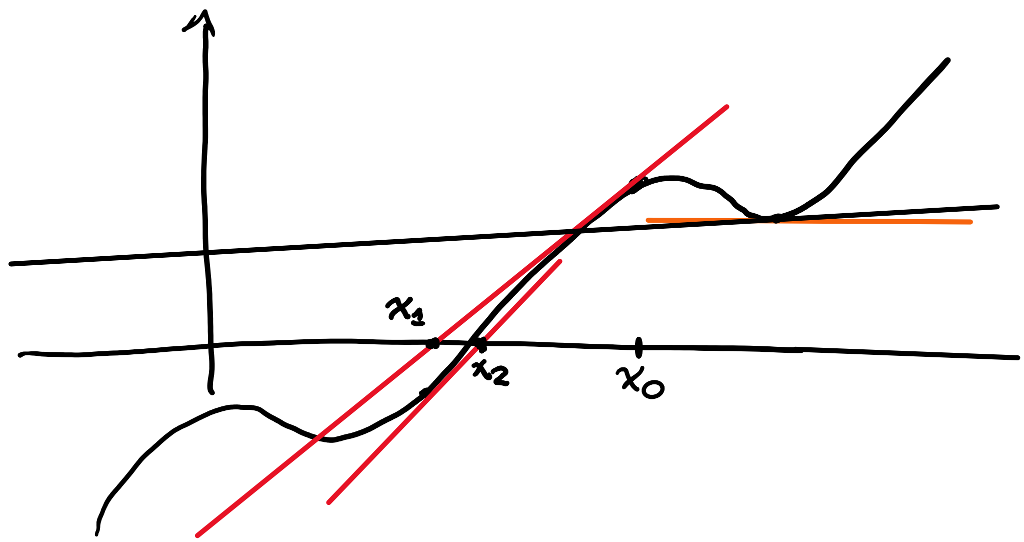
NEWTON-RAPHSON METHOD.

NEWTON-RAPHSON METHOD.

Rootfinding $f(r)=0$, access to f'.

Initial guess x_0. Data: $f(x_0)$, $f'(x_0)$.

$$\ell(x) = f(x_0) + f'(x_0)(x-x_0)$$



$\boxed{f'(x_0) \neq 0}$.

$f'(x_0) \approx 0$ NOT GOOD

$$0 = \ell(x_1) = f(x_0) + f'(x_0)(x_1 - x_0)$$

$$f'(x_0)(x_1-x_0) = -f(x_0)$$

$f'(x_0) \neq 0$

$$x_1 - x_0 = -f(x_0)/f'(x_0)$$

$$\boxed{x_1 = x_0 - f(x_0)/f'(x_0)}$$

$$\boxed{x_n = x_{n-1} - f(x_{n-1})/f'(x_{n-1})}$$

"NEWTON STEP"

$$g_{NR}(x) = x - \frac{f(x)}{f'(x)}$$

Family $x + c(x)f(x)$

NEWTON IS FPI FOR THAT g_{NR}?

NEWTON IS FPI FOR THAT g_{NR}!

<u>Theorem (Newton)</u>: Let f be twice cont. diff in $[a,b]$ and $r \in (a,b)$ a root of f, such that $f'(r) \neq 0$. Then, there exists $\delta > 0$ s.t. $|x_0 - r| < \delta$ then Newton starting at x_0 converges to r <u>quadratically</u>.

$$\lim_{n \to \infty} \frac{|x_{n+1} - r|}{|x_n - r|^2} = C .$$

$$g_{NR}(x) = x - f(x)/f'(x)$$

$$g'_{NR}(x) = 1 - \frac{f'(x)^2 - f(x) f''(x)}{f'(x)^2}$$

$$= \frac{f'(x)^2 - f'(x)^2 + f(x) f''(x)}{f'(x)^2} = \frac{f(x) f''(x)}{f'(x)^2}$$

$$g'_{NR}(r) = \frac{f(r) f''(r)}{f'(r)^2} = \underline{0}$$

What happens if all assumptions are true except that $f(r) = 0$, $f'(r) = 0$?

$$f(x) = x^2 (3 + \cos x)$$

$$r = 0 .$$

In this case, Newton converges but linearly!

linearly?

<u>Root multiplicity</u>: r is a root of multiplicity m if $f(r) = f'(r) = \dots = f^{(m-1)}(r) = 0$ and $f^{(m)}(r) \neq 0$.

$\hookrightarrow$ there is $h(x)$ s.t. $h(r) \neq 0$ and

$$\underline{f(x) = (x-r)^m h(x)}.$$