Fixed Point Iteration (II)

Friday, September 13, 2024 10:31 AM

Class 08: September 13, 2024

Recall: In our wrap up of the discussion on bisection method, we discussed potential termination criteria, and then went on to do analysis of how quickly the error e_n converges to zero (x_n converges to a root r). We looked at the reduction in error from taking one step: e_{n+1} / e_n . We said that if in the limit:

- e_{n+1} / e_n tends to a constant C in (0,1), we say our sequence converges **linearly with rate C**.
- e_{n+1} / e_n tends to 0, we say our sequence converges **superlinearly**.
- e_{n+1} / e_n tends to 1, we say our sequence converges **sublinearly**.

We then discussed 3 ways to distinguish between these cases and to better understand what that means: by looking at how quickly we gain digits in r, how quickly the log error goes to zero, and plotting log(e_{n+1}) against log(e_n) or log(e_n) against n.

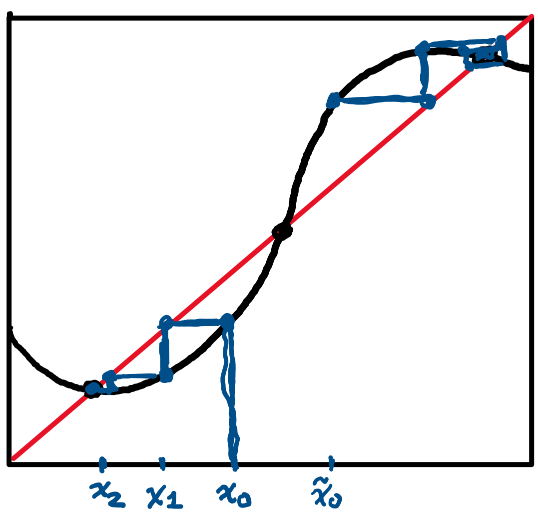
We then introduced the **fixed point problem**, and how given a rootfinding problem f(r) = 0, we can always find many g(x) such that f(r) = 0 if and only if g(r) = r. In particular, we gave two families of functions:

- Given a nonzero c, set g_c(x) = x + cf(x) and run FPI.
- Given c(x) nonzero (at least in a neighborhood around the root), set g2_c(x) = x + c(x)f(x) and run FPI.

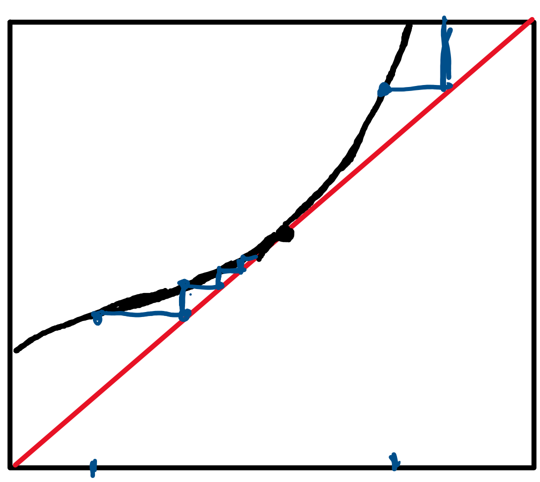
Please note these are not the only g(x) one could try, they are just families of g's that work in general.

We described the FPI, which is a very simple algorithm: apply g repeatedly until convergence or until you run out of iterations. We will continue our discussion of the Fixed Point Iteration algorithm and ask: when is there a fixed point at all? When is it unique? Under what conditions can I expect FPI to converge, and if it does, how fast does it converge?

WARM UP: RUNNING FPI



3 FIXED POINTS r_1, r_2, r_3.
What happens when we apply g repeatedly?



EXAMPLE 2: What happens to $x_0 < r$ vs $x_0 > r$?

What can we say about the FP problem?
What do we need to assume about g(x)?

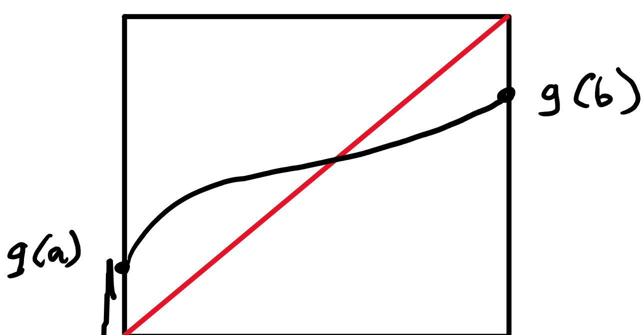
What do we need to assume about $g(x)$?

3 QUESTIONS
- EXISTENCE
- UNIQUENESS
- PERFORMANCE OF FPI

① Existence: "$g \in C([a,b])$",

$$a \le g(x) \le b \quad \text{for } x \in [a,b].$$

$\Rightarrow$ there exists at least one FP $r \in [a,b]$.



Define $f(x) = g(x) - x$

By IVT, $\exists\, r \in (a,b)$ s.t. $f(r) = g(r) - r = 0$

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(2) If in addition $g'(x)$ exists for $x \in (a,b)$ and $\exists\, K \in (0,1)$ s.t. $|g'(x)| \le K < 1$, then the FP r is unique!

Let $r_1 < r_2$ FPs of g. By MVT, there exists $\xi \in (r_1, r_2)$ s.t.

$$g'(\xi) = \frac{g(r_2) - g(r_1)}{r_2 - r_1} = \frac{r_2 - r_1}{r_2 - r_1} = 1$$

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(3) If the conds of 2 hold, then for all $x_0 \in [a,b)$, starting FPI at x_0 will [illegible]

$x_0 \in [a,b]$, starting FPI at x_0 will converge at least linearly to FP r.

$$|r - x_n| = |g(r) - g(x_{n-1})| \underset{MVT}{=} |g'(\xi_n)(r - x_{n-1})|$$

$$= \underbrace{|g'(\xi_n)|}|r - x_{n-1}|$$

$$|r - x_n| \le K|r - x_{n-1}|$$

$$|e_n| \le K|e_{n-1}|$$

$$|e_n| \le K|e_{n-1}| \le K^2|e_{n-2}| \le \ldots \le K^n|e_0| \xrightarrow[n\to\infty]{} 0.$$

Consider r FP of g, x_n is the nth iterate of FPI and $x_n \to r$.

$$\underline{g(x_n)} = g(r) + g'(r)\underbrace{(x_n - r)} + \frac{1}{2}g''(\xi)(x_n - r)^2$$

$$x_{n+1} = r + \ldots$$

$$e_{n+1} = g'(r)\,e_n + \frac{1}{2}g''(\xi)\,e_n^2$$

$$\lim_{n\to\infty}\frac{e_{n+1}}{e_n} = g'(r)$$

$\ne 0$ but $|g'(r)| < 1$ then linear

$g'(r) = 0$

$$g'(r) = 0 \to e_{n+1} = \frac{1}{2}g''(\xi)\,e_n^2$$

$$g'(r) = 0 \rightarrow \underline{e_{n+1} = \frac{1}{2} g''(\xi)\, e_n^2}$$

ORDER OF CONVERGENCE: $q > 0$ s.t.

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^q} = C \neq 0.$$

If $|g'(r)| < 1$ then $\boxed{\exists\, \delta > 0}$ s.t. $x_0 \in (r-\delta, r+\delta)$ then FPI converges at least linearly w/ rate $|g'(r)|$.