Composite Quadrature and Romberg quadrature

Monday, November 4, 2024 9:26 AM

Class 30: November 4, 2024

Recall: Last time, we developed the idea of **composite quadrature** for Newton-Cotes rules. The idea is to partition the interval [a,b] into **N = mn** sub-intervals with endpoints at equispaced nodes a=x0 < x1 < … < xN=b, with **xj = a + h j** and **h = (b-a)/N**. We then think of this partition as consisting of m "panels" with n+1 nodes each, and use the interpolation-based Newton-Cotes quadrature for polynomials of degree <=n (n=1 -> Trapezoidal, n=2 -> Simpsons) to approximate the integral *on each panel*.

Using this scheme, we built two composite rules:

1. **Composite Trapezoidal**, with weight vector **w = h[1/2, 1, 1, 1, …, 1, 1/2]** . We derived an error estimate

   E_h^T[f] = -(f''(eta)(b-a)/12) h^2

   To reach absolute error target eps=10^{-p}, we need N = O(eps^{-1/2})=O(10^{p/2}). In other words, increasing N by a factor of 10 gets us about two digits.
2. **Composite Simpsons**, with weight vector **w = (h/3) [1, 4, 2, 4, …,2, 4, 1]** . We derived an error estimate

   E_h^S[f] = -(f^{iv}(eta)(b-a)/180) h^4

   This means to reach an absolute error target eps=10^{-p}, we need N = O(eps^{-1/4})=O(10^{p/4}). In other words, increasing N by a factor of 10 gets us about four digits.

We finished our session talking about additional interesting results:

- Trapezoidal (and Simpson's to a bit lesser degree) do incredibly well with smooth, periodic functions: the error goes to zero faster than any power of h (power of 1/N).
- We can use a formula (Euler-Maclaurin) to write the error for Trapezoidal as an infinite expansion in powers of h. Today, we will see one way this can be used to increase the order of the quadrature rule.

IDEA: Error for C. Trapezoidal as

$$I - T_h[f] = E_h^T[f] = K_2 h^2 + K_4 h^4 + \dots$$

EULER-MACLAURIN

$$K_2 = \frac{B_2}{2!}\left(f'(b) - f'(a)\right)$$

$$K_4 = \frac{B_4}{4!}\left(f'''(b) - f'''(a)\right)$$

$$\vdots$$

$$K_{2k} = \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right)$$

"Richardson Extrapolation"

Quantity of Int: I , Estimate $N_1(h)$

Assumption:

- $N_1(h) \to I$ as $h \to 0$.
- $I - N_1(h) = K_1 h + K_2 h^2 + K_3 h^3 + \dots,$

IDEA:

(I) $\quad I - N_1(h) = K_1 h + K_2 h^2 + K_3 h^3 + \dots$

$$\text{(I)} \quad I - N_1(h) = K_1 h + K_2 h^2 + K_3 h^3 + \dots$$

$$\text{(II)} \quad I - N_1(h/2) = \frac{K_1}{2} h + \frac{K_2}{4} h^2 + \frac{K_3}{8} h^3 + \dots$$

$$2\text{(II)} - \text{(I)}: \quad I - \underbrace{\left(2N_1\left(\tfrac{h}{2}\right) - N_1(h)\right)}_{} = \left(\tfrac{1}{2} - 1\right) K_2 h^2 + \left(\tfrac{1}{4} - 1\right) K_3 h^3 + \dots$$

$$\boxed{N_2(h) = 2N_1\left(\tfrac{h}{2}\right) - N_1(h)}$$

$$\text{I} \quad I - N_2(h) = C_2 h^2 + C_3 h^3 + \dots$$

$$\text{II} \quad I - N_2\left(\tfrac{h}{2}\right) = \frac{C_2}{4} h^2 + \frac{C_3}{8} h^3 + \dots$$

$$4\text{II} - \text{I}: \quad 3I - \left(4N_2\left(\tfrac{h}{2}\right) - N_2(h)\right) = \left(\tfrac{1}{2} - 1\right) C_3 h^3 + \dots$$

$$I - \underbrace{\left(\frac{4N_2(h/2) - N_2(h)}{3}\right)}_{N_3(h)} = D_3 h^3 + D_4 h^4 + \dots$$

In general:

$$\boxed{N_j(h) = \frac{2^{j-1} N_{j-1}\left(\tfrac{h}{2}\right) - N_{j-1}(h)}{2^{j-1} - 1}}$$

$$\boxed{\text{this is } O(h^j)}$$

$$\begin{array}{ccccc}
N_1(h) & & N_1(h/2) & & N_1(h/4) \\
\downarrow & \swarrow & \downarrow & \swarrow & \\
N_2(h) & & N_2(h/2) & & \\
\downarrow & \swarrow & & & \\
N_3(h) & & & &
\end{array}$$

Newton tableau!

Romberg:

$$I - T_h[f] = K_2 h^2 + K_4 h^4 + \dots$$

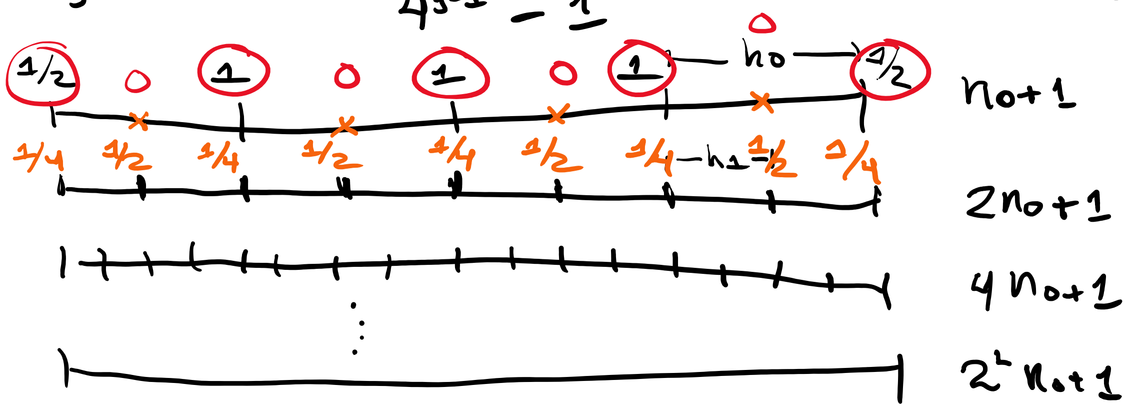
$$I - T_{\frac{h}{2}}[f] = \frac{K_2}{4} h^2 + \frac{K_4}{16} h^4 + \dots$$

$$I - T_{\frac{h}{2}}[f] = \frac{K_2}{4} h^2 + \frac{K_4}{16} h^4 + \dots$$

$$4I - I: \quad 3I - \left(4T_{\frac{h}{2}}[f] - T_h[f]\right) = \left(\frac{1}{4} - 1\right) K_4 h^4 + \dots$$

$$I - \underbrace{\left(\frac{4T_{h/2}[f] - T_h[f]}{3}\right)}_{R_2[h]} = C_4 h^4 + C_6 h^6 + \dots$$

$$R_j[h] = \frac{4^{j-1} R_{j-1}[h/2] - R_{j-1}[h]}{4^{j-1} - 1} \rightarrow O(h^{2j})$$



$$\begin{bmatrix} R_1(h) & R_1(h/2) & R_1(h/4) & \dots & R_1(h/2^i) \\ R_2(h) & R_2(h/2) & \dots & \dots & R_2(h/2^{i-1}) \\ \vdots & & \ddots & & \\ R_i(h) & & & & \end{bmatrix}$$

R_2:

NODES $\rightarrow h/2 \quad x_j = a + j(h/2) \quad j = 0, \dots, 2n_0$

WEIGHTS: $\frac{h}{6}[1\ 4\ 2\ 4\ 2\ 4 \cdots 2\ 4\ 1]$

R_3: (Boole's rule, NC $p = 4$)

$$\frac{2h}{45}[7 \quad 32 \quad 12 \quad 32 \quad 14 \quad 32 \quad 12 \quad 32 \quad 14 \dots$$

$$\frac{2h}{45}\left[7 \quad 32 \quad 12 \quad 32 \quad 14 \quad 32 \quad 12 \quad 32 \quad 14 \ldots\right.$$

R_4: $$\frac{h}{2835}\left[217 \quad 1024 \quad 352 \quad 1024 \quad 436 \quad 1024 \quad 352 \quad 1024 \quad 434 \ldots\right.$$

(NOT NC!)