Bisection method (Part 2) / Fixed Point Iteration

Wednesday, September 11, 2024 1:14 PM

Class 07: September 11, 2024

Recall: Last time, we introduced the **rootfinding problem** as a general framework to solve one nonlinear equation in one variable. We then came up with the **bisection method**, which is guaranteed to work to find a root of a function f(x) on an interval [a,b] assuming f is continuous and f(a)f(b)<0.

We derived a pseudocode for the method and talked about termination criteria. In this lecture, we continue our discussion and perform error / convergence analysis on the bisection method.

WARM UP: Termination criteria & Error analysis

WHILE ($n \leq n_{max}$ AND ___?___)

(o) ERROR ESTIMATE $\geq$ TOL $\longrightarrow$ REL ERROR?

$$|r - x_n| \leq \boxed{\frac{(b_n - a_n)}{2}} = \frac{(b_{n-1} - a_{n-1})}{4} = \dots = \frac{(b-a)}{2^{n+1}}$$

(o) CONVERGENCE $\longrightarrow$ $|x_n - x_{n-1}| \geq$ TOL

(o) $|f(x_n)| \geq$ TOL $\longrightarrow$ depends on f'

$$|r - x_n| = |e_n| \leq \frac{(b-a)}{2^{n+1}} < TOL$$

$$2^{n+1} > \frac{(b-a)}{TOL}$$

$$n+1 > \log_2\left(\frac{b-a}{TOL}\right)$$

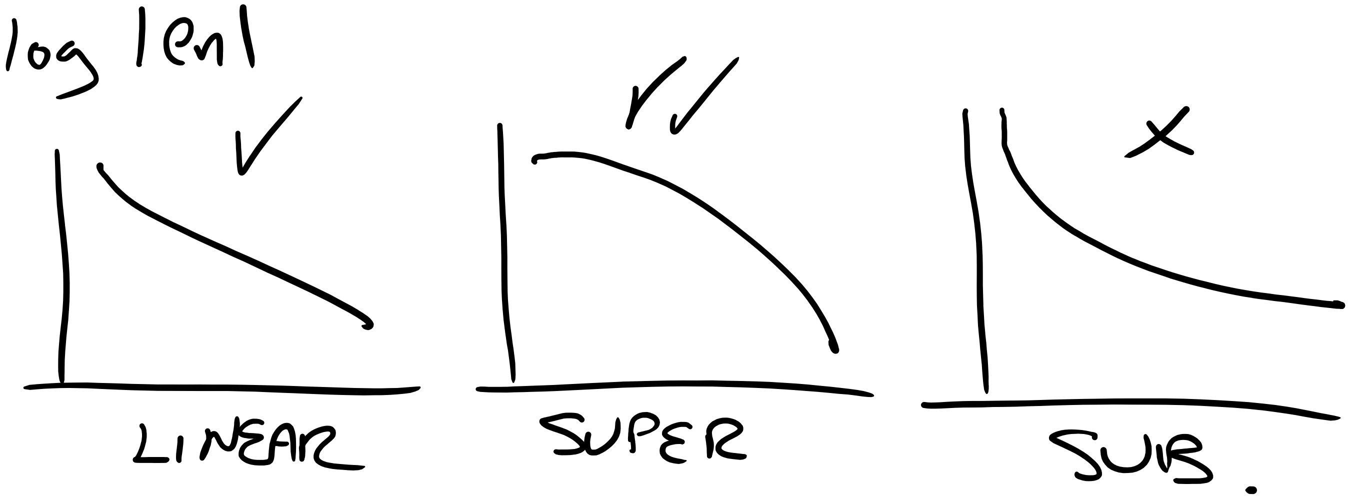
$$n > \left\lceil \log_2\left(\frac{b-a}{TOL}\right) - 1 \right\rceil$$

$$e_{n+1} = r - x_{n+1} \approx \underline{\underline{0.5}}\, e_n = 0.5\,(r - x_n)$$

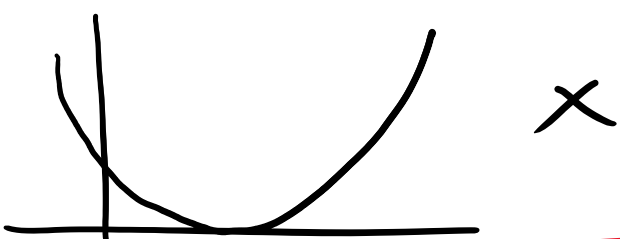
"RATE"

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} = C \longrightarrow C \in (0,1) \quad \text{LINEAR}$$

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} = C \quad \begin{cases} \to C \in (0,1) & \text{LINEAR} \\ \to C = 0 & \text{SUPERLINEAR} \\ \to C = 1 & \text{SUBLINEAR} \end{cases}$$



FIXED POINT METHODS:



- Given a function $g(x)$, r is a FIXED POINT of g IF $\underline{g(r) = r}$.
    - **STABLE**: If I start a bit away from it, I get drawn towards it.
    - **UNSTABLE** → I get repelled.

Take x_0, $x_1 = g(x_0)$, $x_2 = g(x_1) = g(g(x_0))$,

$\dots$ $\underline{\underline{x_n = g(x_{n-1})}}$ "FIXED POINT ITERATION"

I want to solve find r s.t. $f(r) = 0$.

say $f(x) = x + \cos x - 3$.

Find g such that

$$\boxed{f(r) = 0 \rightleftarrows g(r) = r}$$

<u>Example:</u> $g(r) = r + f(r)$ ✓

$g_c(r) = r + c f(r)$ ✓
$(c \neq 0)$

$g_c(r) = r + c(r) f(r)$ ✓
$(c(x) \neq 0)$