Spline Interpolation / Beginning of Discrete Least Squares

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Recall: Last time, we wrapped up Hermite interpolation, and then we began talking about a way to interpolate using piecewise polynomials, called Splines. Given a partition of [a,b] into n subintervals I_i, with a=x0<x1<...<xn-1<xn=b and Ii = [x_{i-1},xi], we can define:

1. A space of piecewise linear splines s(x) such that the restriction to I_i, which we denote as s_i(x), is a linear polynomial and s(x) is continuous.

An element of this space is defined by s_i(x) = (1/hi) [y_{i-1} (x_i - x) + y_i (x - x_{i-1}) .

We then used the error estimates for linear polynomial interpolation to come up with an error bound |s(x) - f(x)| <= ||f''||_inf h^2 / 8.

Today, we will develop the same idea for piecewise cubic polynomials (Cubic Splines). We will then, time permitting, start our next topic on Discrete Least Squares and function data approximation.

CUBIC SPLINE

$S_i(x) \to$ cubic — 4 coefficients.

$\left\{\begin{array}{l} S(x) \text{ cont and } S(x_i) = y_i \\ S' \text{ is cont.} \\ S'' \text{ is cont.} \end{array}\right\}$ — $S_2, S_3, \dots, S_{n-1}$ (Interior) ✓

— $S_1, S_n \to$ 2 extra cond (1 each)

4 familes:

Natural: $S_1''(x_0) = 0, \; S_n''(x_n) = 0$.

Clamped: $S_1'(x_0) = 0, \; S_n'(x_n) = 0$ (give me values)

Periodic: $\to$ S, S', S'' periodic & continuous.

Not-a-knot: $S_1'''(x_1) = S_2'''(x_1), \; S_{n-1}'''(x_{n-1}) = S_n'''(x_{n-1})$

Natural: $(h_i = h)$

Look at $S''(x)$.

$$S_i''(x) = a_{i-1} \cdot \frac{(x_i - x)}{h} + a_i \frac{(x - x_{i-1})}{h}.$$

$a_i = S''(x_i)$, $\to$ continuous. ✓

antidiff (2x)

$$S_i(x) = \frac{1}{?}\left(a_{i-1}(x_i - x)^3 + a_i (x - x_{i-1})^3 \right.$$

$$S_i(x) = \frac{1}{h}\left(a_{i-1}\frac{(x_i - x)^3}{6} + a_i \frac{(x - x_{i-1})^3}{6} + b_i(x_i - x) + c_i(x - x_{i-1}) \right]$$

$$S_i(x_{i-1}) = y_{i-1} = \frac{1}{h}\left(a_{i-1}\cdot\frac{h^3}{6} + b_i h \right)$$

$$y_{i-1} = a_{i-1}\frac{h^2}{6} + b_i$$

$$\boxed{b_i = y_{i-1} - a_{i-1}\, h^2/6} \quad \checkmark \quad \text{Int Left}$$

$$S_i(x_i) = y_i \longrightarrow y_i = \frac{1}{h}\left(a_i \frac{h^3}{6} + c_i h \right)$$

$$= a_i\, h^2/6 + c_i$$

$$\boxed{c_i = y_i - a_i\, h^2/6} \quad \checkmark \quad \text{Int Right}$$

② $S_i'(x_i) = S_{i+1}'(x_i) \qquad i = 1, \dots, n-1$

$$S_i'(x) = \frac{1}{h}\left(-\underline{\underline{a_{i-1}}}\frac{(x - x_i)^2}{2} + \underline{\underline{a_i}}\frac{(x - x_{i-1})^2}{2} \underbrace{- b_i + c_i}_{(y_i - y_{i-1}) - (\underline{\underline{a_i}} - \underline{\underline{a_{i-1}}})\, h^2/6.} \right)$$

"$y_i = f(x_i)$"

$$\cancel{\tfrac{1}{12}a_0} + \frac{1}{3}a_1 + \frac{1}{12}a_2 = f[x_0, x_1, x_2]$$

$$\frac{1}{12}a_1 + \frac{1}{3}a_2 + \frac{1}{12}a_3 = f[x_1, x_2, x_3]$$

$$\frac{1}{12}a_2 + \frac{1}{3}a_3 + \frac{1}{12}a_4 = f[x_2, x_3, x_4]$$

$$\vdots$$

$\vdots$

$n-1$ eqs in $n+1$ var $\cancel{a_0}, a_1, \ldots, a_n$.

$S''(x_0) = 0 - a_0 = 0$

$S''(x_n) = 0 - \ a_n = 0$

$$\frac{1}{12}\begin{bmatrix} 4 & 1 & 0 & \cdots & & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & & \\ \vdots & \vdots & \vdots & & \ddots & \\ & & & & 1 & 4 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} f[x_0, x_1, x_2] \\ f[x_1, x_2, x_3] \\ \vdots \\ \\ \end{bmatrix}$$

Tridiagonal,

SPD, Diag. Dominant.

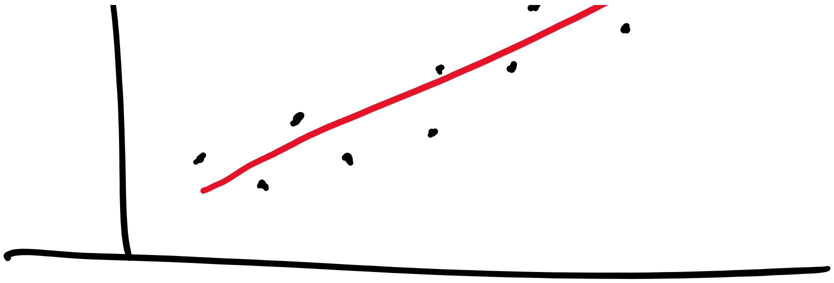
Gauss Elim $\rightarrow$ Thomas alg. $O(n)$

Periodic: (CIRCULANT)

$$\frac{1}{12}\begin{bmatrix} 4 & 1 & & & & 1 \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & 1 & 4 & 1 & \\ & & & & \ddots & \\ 1 & & & & 1 & 4 \end{bmatrix}$$

Approximation

$y_i = f(x_i) + \varepsilon_i$



$y_i = f(x_i) + \varepsilon_i$

C. SPLINE ERROR EST: $f \in C^{(4)}([a,b])$, $x \in [a,b]$

$$
\left\{
\begin{aligned}
|f(x) - s(x)| &\le C_0 \|f^{(4)}\|_\infty \cdot h^4 \longrightarrow O(h^4) \checkmark \\
|f'(x) - s'(x)| &\le C_1 \|f^{(4)}\|_\infty h^3 \\
|f''(x) - s''(x)| &\le C_2 \|f^{(4)}\|_\infty h^2 \\
|f'''(x) - s'''(x)| &\le C_3 \|f^{(4)}\|_\infty h
\end{aligned}
\right.
$$