Linear Algebra Review

Monday, November 11, 2024 9:37 AM

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Recall: Last class we wrapped up our discussion on Gaussian quadratures for a given interval [a,b] and weight function w(x). The main results for these quadratures are as follows:

- We construct or obtain the **3-term recursion** for the **family of orthogonal polynomials Q_j(x).**
- The **nodes {x_j}_{j=0}^n** for the n+1 point quadrature are the **unique n+1 zeroes of Q_{n+1}(x)**. There is a theorem that shows Q_{n+1} always has n+1 unique zeroes, and that all of them are in [a,b].
- These zeroes can be obtained via **rootfinding (good first guess using interlacing + Newton)** or the **Golub-Welch** algorithm.
- The **weights w_j** can be obtained via the integrals of Lagrange polynomials or via the linear system that results from asking the rule be exact for polynomials of degree <=n. However, the most efficient way to compute them is via formulas involving Q_n, Q_{n+1} and Q'_{n+1} (all which can be computed using the 3-term recursion and formulas obtained from it).
- **Weights are positive and small;** they add up to the integral of w(x) from a to b. This makes Gaussian quadratures very stable.
- We can either use one interval and let n go to infinity OR do a composite quadrature for a given n+1 point rule. In the first case, order of accuracy goes faster to 0 than any power of h = (b-a)/N. In the second case, we get an O(h^{2n+2}) composite quadrature (the best for an n+1 pt quadrature).

We then introduced the concept of Adaptive Quadrature, what it is and how the algorithm to subdivide an interval adaptively works. We finished our discussion describing a recursive and non-recursive algorithm to perform Adaptive Quadrature.

Today, we start our section on numerical methods for linear algebra by going over a review of important linear algebra concepts.

LOOKING FORWARD

1. Gauss Elimination – LU decomp
2. Gram Schmidt – QR decomp
3. Eigenvalues, eigenvectors < UDU^* / $U\Sigma V^*$

KEY CONCEPTS:

- Vector space / subspace – $(\mathcal{V}, +, *)$
  - → Scalar field $\mathbb{F} \to \mathbb{R} / \mathbb{C}$.
- Linear combination $\quad \alpha_j \in \mathbb{F}$

  $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \quad v_j \in \mathcal{V}$
- Linear span $(\{v_1, \dots, v_n\})$
  - → Set of all possible (finite) linear comb.
- Linear dependence / Linear independence

  $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$, α_j not all 0.

$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = 0$, a_j not all 0.

$\updownarrow$

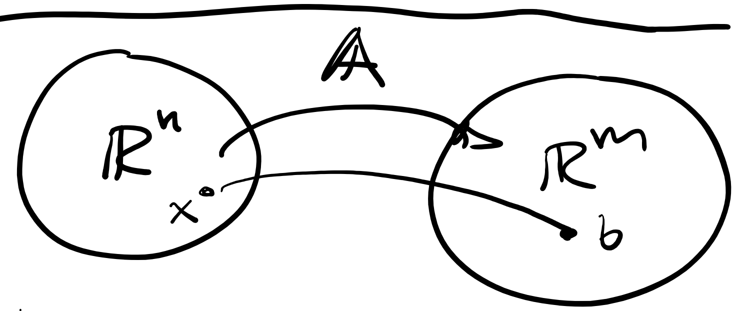
Can I write v_j as a LC of the rest?

- Basis & Dimension.

  $\hookrightarrow$ $B \subseteq V$ is a basis of V if span $(B) = V$ and B is LI set.

  $\rightarrow$ $|B| = \dim V$

$$\underset{m\times n}{A}\ \underset{m\times 1}{\vec{x}} = \vec{b}_{m\times 1}$$



$\boxed{1}$ $R(A)$, Range / Column space of A

$R(A) = \{ b \in \mathbb{R}^m \mid \text{There is } x \text{ s.t. } Ax = b \}$

$$Ax = x_1 v_1 + x_2 v_2 + \dots + x_n v_n \qquad A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

$\dim(R(A)) = r(A)$ "rank of A".

$\boxed{2}$ $N(A)$ Nullspace / Kernel

$$N(A) = \{ x \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$$

$$Ax = x_1 v_1 + x_2 v_2 + \dots + x_n v_n = 0$$

<u>Uniqueness</u>: If $Ax_p = b$, then $A(x_p + w) = b$ for all $w \in N(A)$.

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- $N(A) = \{\vec{0}\}$, then if $Ax = b$ has a sol, it is unique.
- $N(A) \neq \{\vec{0}\}$, then if $Ax = b$ has a sol, it is not unique, and is $\underline{\underline{x_p + N(A)}}$

$\dim(N(A)) = \nu(A)$ "nullity of A"

(o) Rank & Nullity thm: A $m \times n$,

$$r(A) + \nu(A) = n \quad (\text{\# of cols})$$

$Ax = b$, A $m \times n$ matrix. A.

<u>Case I</u>: Underdetermined $(m < n)$

$r(A) \leq 5 \rightarrow \nu(A) > 0 \; (\geq n - m)$

Best case: $r(A) = m$, $\nu(A) = n - m$.

<u>Case II</u>: Overdetermined $(m > n)$

$r(A) \leq n$, $\nu(A) = 0$ but $\underline{r(A) < m}$.

$\rightarrow$ there are b for which $Ax = b$ has no solution.

<u>Case III</u>: $(m = n)$ (Determined)

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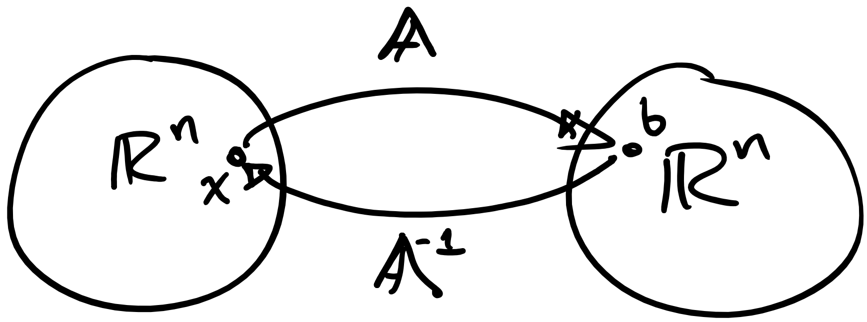
↳ I can have $\nu(A)=0 \iff r(A)=n$

Existence (for any b) and Uniqueness ?

$A \to$ invertible (non-singular)



$$\boxed{A A^{-1} = A^{-1} A = I}$$

Equivalent conditions:

- $Ax=b$ is uniquely solvable for every b.
- A is invertible
- $r(A)=n$ (cols of A are LI)
- $\nu(A)=0$
- A has no zero eigenvalues
- $\det(A) \neq 0$.

Best case (Invertible A)

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$\hookrightarrow$ Solve $Ax=b$
- Compute A^{-1} ✗.
- DIRECT METHODS
- ITERATIVE METHODS

DIRECT METHODS – Gauss Elim. (LU)

QR-based

UDU^* / SVD.

DO BEST WHEN
n is not too big
OR
I want to solve many $Ax=b_n$.

- Less dependent on cond #.

Iterative Methods (CG, GMRES, Multigrid, Fixed pt method)

Producing seq of $x_n \xrightarrow[n\to\infty]{} x$.

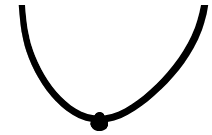
Benefit: Cost p/iter is low
(usually $A\cdot x$) Mat Vec

Cons: Cost = (# iters)(Cost p iter)

A not invertible:

$\hookrightarrow$ min $\|Ax-b\|^2$ ( )

$\hookrightarrow \min \|Ax-b\|^2$



$\hookrightarrow$ IF NOT

$\rightarrow$ **min norm**