Quasi Newton and Secant methods

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Class 10: September 20, 2024

Recall: Last time, we finished our analysis and examples demonstrating the performance of the Newton-Raphson method for rootfinding. *The main advantage Newton gives us is that if our initial guess is close enough to the root, it converges quadratically, which usually gets us to machine precision in 4-5 iterations.* Disadvantages of this method are:

1. If we do not start in the basin of quadratic convergence, it might be slow to get there or might even diverge.
2. For certain problems, evaluating f'(x) accurately might be very expensive. We will see that for systems of equations, this becomes much worse.

The last set of methods we are discussing in this unit is **Quasi Newton** methods; there are attempts at "imitating" Newton while removing the requirement to evaluate f'(x) every step. Last class, we introduced the ideas for two methods:

1. **Chord Iteration ("Lazy Newton"):** For this method, we take f'(x0) (or a good approximation of it given by the user) and we use it instead of f'(xn) (we never update f'(xn)). That gives us a step of the form:

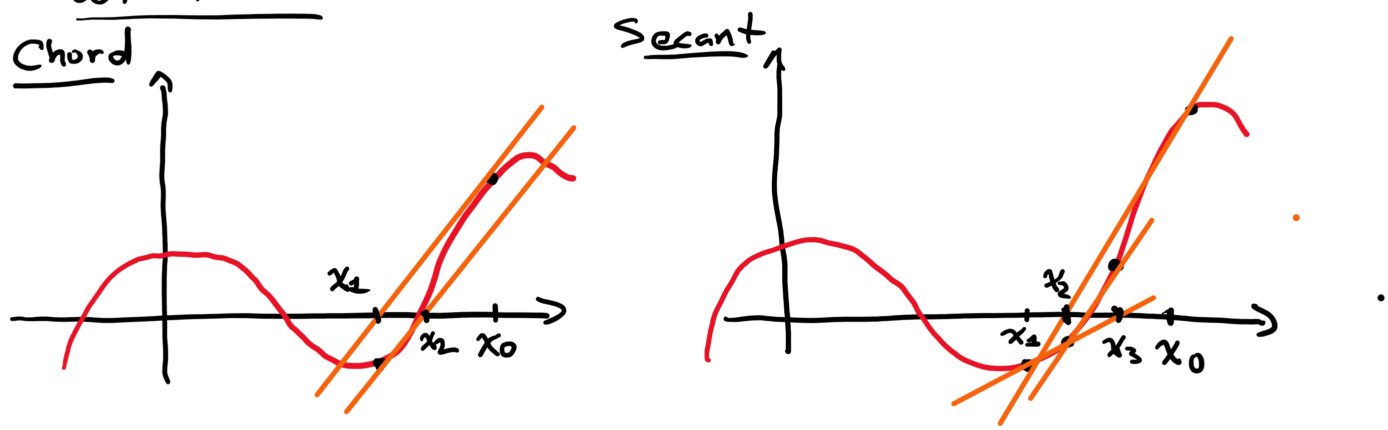
   xn = x_{n-1} - f(x_{n-1}) / f'(x0)

2. **Secant method:** We take two user provided initial guesses x0 and x1, and we use them to find the slope of the line secant to the points (x0,f(x0)) and (x1,f(x1)). We use the root of the secant line as our next iterate. This gives us a step of the form:

   m_{sec} = (f(x_{n-1}) - f(x_{n-2})) / (x_{n-2} - x_{n-1})
   xn = x_{n-1} - f(x_{n-1}) / m_{sec}

Out of these two, chord iteration fails to retain the superlinear convergence near the root that makes Newton attractive. However, secant succeeds! Today, we will go over some results around showing that secant converges superlinearly, with rate phi = 1.618... . We will then go over some examples applying the secant method.

WARM UP: CHORD & SECANT



THM (NEWTON): $f \in C^2([a,b])$ on int around root r. and root is simple ($f(r)=0$, $f'(r)\neq 0$), then there is $\delta>0$ s.t. $|x_0 - r| < \delta$ then Newton converges quadratically.

$$g_{NR}(x) = x - \frac{f(x)}{f'(x)}, \quad \text{Showed } g'_{NR}(r) = 0.$$

$$\boxed{e_{n+1} \approx M e_n^2} \qquad M = \frac{1}{2} g''_{NR}(r)$$

$$\boxed{e_{n+1} \approx M e_n^2} \qquad M = \frac{1}{2} g_{NR}''(r)$$

$$\boxed{g_{NR}''(r) = \frac{f''(r)}{f'(r)}}$$

THM (Secant): If (Newton assumptions are true) then $\exists \delta > 0$ s.t. $|x_0 - r| < \delta$ then secant converges superlinearly (order ϕ).

What do we know about secant.

$$\boxed{e_{n+1} \approx M e_n \cdot e_{n-1}} \qquad \left(M = \frac{f''(r)}{2f'(r)} \right)$$

Apply $\log_{10}$:

$$\underbrace{\log_{10}(e_{n+1})}_{L_{n+1}} \approx \underbrace{\log_{10}(e_n)}_{L_n} + \underbrace{\log_{10}(e_{n-1})}_{L_{n-1}} + \log_{10} M$$

$$L_{n+1} \quad L_n \quad + \quad L_{n-1}$$

NEWTON: $-1, -2, -4, -8, -16$

SECANT: $-1, -1, -2, -3, -5, -8, -13, -21$

$$e_{n+1} = M e_n e_{n-1} \qquad e_{n+1} = C e_n^{q}$$

$$e_n = C e_{n-1}^{q}$$

$$C e_n^{q} = M e_n e_{n-1} \qquad C^q = M \rightarrow C = M^{1/q}$$

$$C^q e_{n-1}^{q(q-1)} = M e_{n-1} \qquad \boxed{q^2 - q = 1} \quad (q > 0)$$

$$1 \pm \sqrt{5}$$

$$q = \underbrace{\frac{1+\sqrt{5}}{2}}_{\phi} \approx 1.618$$

SECANT — QUASINEWTON.

Method	Assumptions	Cost per Iteration	Convergence
Bisection			
FPI			
Newton			
Chord			
Bisection.			