Function Approximation and Discrete Least Squares

Monday, October 21, 2024 9:05 AM

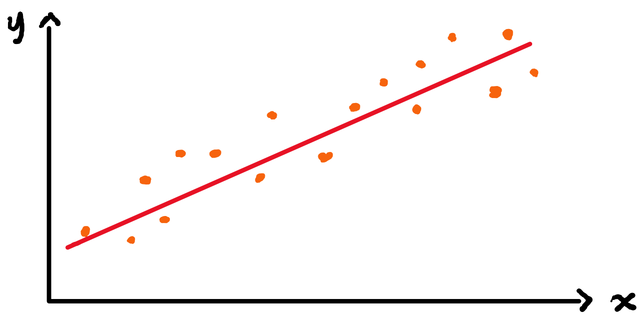
Class 24: October 21, 2024

Recall: Last class, we discussed how to construct and how to use cubic splines to interpolate data (x_i,y_i), for i=0,…,n . We went over how one can construct the case for normal splines using formulas for each cubic piece s_i(x) and its derivatives, resulting in an (n-1) by (n-1) system of linear equations (with a very nice, tri-diagonal and SPD matrix).

Today, we move on from discussing the interpolation problem (which assumed our data was of the form y_i = f(x_i), for a nice enough f) to problems in **function approximation**. We separate those into two kinds of problems: discrete and continuous:

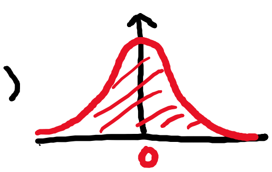
1. **Discrete:** Given n data points (x_i, y_i), it is often more realistic to assume that y_i = f(x_i) + eps_i , where eps_i is a noise term. We do not want to interpolate this data! (we would be fitting noise along with our signal). Instead, we want to come up with a model p(x) = a_0 phi_0(x) + a_1 phi_1(x) + … + a_{m-1} phi_{m-1}(x) such that some norm of the error vector err_i = y_i - p(x_i) is as small as possible (minimized).
2. **Continuous:** Given a function f(x), we want to know the answer to questions like: "what is the closest quadratic polynomial to this function in the interval [0,1]"? We can then work with our approximation instead of the original function.

NOISE ε → Measurement / Computation
→ Modeling error

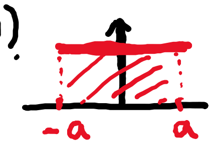


$y_i = f(x_i) + \varepsilon_i$
→ vector $\vec{y} \in \mathbb{R}^n$

$\varepsilon \sim N(0, \sigma)$



$\varepsilon \sim U([-a, a])$



What do I mean by p(x) being close to the data?

$\boxed{m<n}$ (We don't want to overfit).

Interpolation:

$$\underset{n\times m}{M}\vec{a} = \vec{y}$$

Approx: This has no solution!

$$q(\vec{a}) = \| M\vec{a} - y \|_2^2$$

What is $\vec{a}$ such that it minimizes $q(\vec{a})$?

$$p(x) = a_0 + a_1 x$$

$$q(a_0, a_1) = \sum_{i=0}^{n-1} (a_0 + a_1 x_i - y_i)^2$$

Disc. Least Squares

Model

$$p(x) = \sum_{j=0}^{m-1} a_j x^j$$

$$\vec{p} = \underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots \\ 1 & x_1 & x_1^2 & \\ \vdots & \vdots & \vdots & \\ 1 & x_{n-1} & x_{n-1} & \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{bmatrix}}_{\vec{a}}$$

$$q(a_0, a_1) = \sum_{i=0} (a_0 + a_1 x_i - y_i)^2$$ Disc. Least Squares

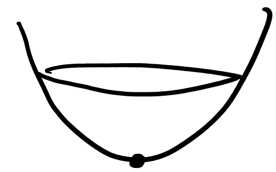
Other norms:

$$\min \{ \max \{ |p(x_i) - y_i| \} \} \to \| \vec{p} - \vec{y} \|_\infty$$

$$\min \left\{ \sum_{i=0}^{n-1} |p(x_i) - y_i| \right\} \to \| p - y \|_1$$

$\boxed{\text{DLS}}$

$$q(a_0, a_1) = \sum_{i=0}^{n-1} (a_0 + a_1 x_i - y_i)^2$$



$$\left\{ \begin{aligned} \frac{\partial q}{\partial a_0} &= \sum_{i=0}^{n-1} \cancel{2} (a_0 + a_1 x_i - y_i) = 0 \\ \frac{\partial q}{\partial a_1} &= \sum_{i=0}^{n-1} \cancel{2} (a_0 + a_1 x_i - y_i) x_i = 0 \end{aligned} \right\}$$

$$\left(\sum_{i=0}^{n-1} 1 \right) a_0 + \left(\sum_{i=0}^{n-1} x_i \right) a_1 = \sum_{i=0}^{n-1} y_i$$

$$\left(\sum_{i=0}^{n-1} x_i \right) a_0 + \left(\sum_{i=0}^{n-1} x_i^2 \right) a_1 = \sum_{i=0}^{n-1} x_i y_i$$

$$\begin{bmatrix} n & \sum_{i=0}^{n-1} x_i \\ \sum_{i=0}^{n-1} x_i & \sum_{i=0}^{n-1} x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

NORMAL EQUATIONS

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polynomial case:

$$\vec{p} = \underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 \\ 1 & x_1 & x_1^2 \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 \end{bmatrix}}_{n \times m} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}}_{m \times 1} = M \vec{a}$$

$$q(\vec{a}) = \| M a - y \|_2^2$$

$$q(\vec{a}) = \| Ma - y \|_2^2$$
$$= (Ma - y)^T (Ma - y)$$
$$= a^T \underbrace{M^T M}\, a - 2\, \vec{a}^T M^T y + y^T y$$

$$\nabla q(\vec{a}) = 2 M^T M \vec{a} - 2 M^T y = \vec{0}$$

$$\boxed{\underset{m\times n}{M^T}\ \underset{n\times m}{M}\ \underset{m\times 1}{\vec{a}} = \underset{m\times n}{M^T}\ \underset{n\times 1}{y}}$$

NORMAL EQS.

Linear case

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_{n-1} \end{bmatrix} \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \\ \vdots & \vdots \\ 1 & x_{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_{n-1} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

$$x^T M^T M x > 0 \quad \text{for } x \neq 0$$

$$\| Mx \|_2^2 > 0 \quad \text{for } x \neq 0$$

Cond $\rightarrow$ columns of M have to be LI.

functions are LI.

$M^T M \rightarrow$ Gramian Matrix G

$$\boxed{G(i,j) = \langle M(:,i),\ M(:,j) \rangle}$$

Polynomial fit using monomial basis,

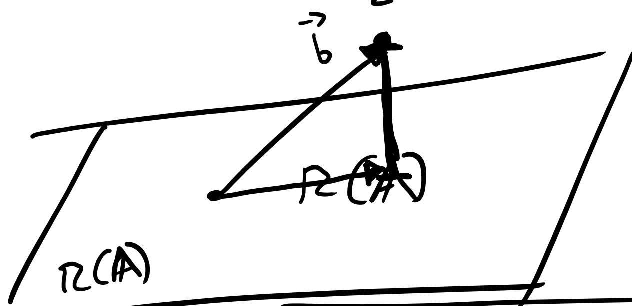
$\kappa(M^T M) \rightarrow$ horrible.

$M = QR$, $M = U\Sigma V^T$

(qr) (svd)

$\boxed{Ra = Q^T b}$

$\odot$ $\min \|Ax - b\|_2^2$



solve $\boxed{Ax = P_{R(A)}\, b}$