L2 approximation and rational approximation

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Class 26: October 25, 2024

Recall: Last class we introduced the problem of continuous function approximation, talked about a few commonly used function norms (L2, weighted L2, Linf, Lp) and then spent some time going over the L2 approximation problem. We concluded the following:

1. We can once again write our objective function as a quadratic, which is very similar to that for Discrete Least Squares:
   q(a) = a^T G a - 2 b^T y + c

   Where G(i,j) = <phi_i, phi_j> and b(i) = <phi_i, f >.
2. As long as the basis functions phi_j(x) used to build our approximation p(x) are linearly independent, this is a strictly con vex (concave up) quadratic, and G is SPD. It has a unique global minimum at the unique solution of G*a = b.
3. Same as for interpolation, if we want to use polynomials to approximate, the monomial basis (powers of x) gives us a really b adly conditioned G. We can once again build a better basis: an orthogonal basis of polynomials makes G diagonal, and means we have a formula for each coefficient a_j:
   a_j = <phi_j, f> / <phi_j, phi_j>

We started a discussion on how to build orthogonal basis of polynomials using Gram-Schmidt. Today, we continue the construction of the Legendre polynomials ([-1,1], w(x)=1) and talk about an important result to generate them using a 3-term recursion formula. We will then cover a different family of orthogonal polynomials (Tchebyshev).

GRAM-SCHMIDT:

$$\boxed{P_0(x) = 1}$$

$$P_1(x) = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle}\cdot 1$$

$$= x - \left(\frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} 1\,dx}\right) 1 = x$$

$$P_2(x) = x^2 - \underbrace{\frac{\langle x^2, x\rangle}{\langle x, x\rangle}}_{\frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx}} x - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle} 1$$

$$\left[\frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} 1\,dx}\right] \to (2/3)/2$$

$$\boxed{P_2(x) = x^2 - 1/3}$$

$$P_3(x) = x^3 - \frac{\langle x^3, x^2 - 1/3\rangle}{\langle x^2 - 1/3, x^2 - 1/3\rangle}(x^2 - 1/3) - \frac{\langle x^3, x\rangle}{\langle x, x\rangle} x$$

$$- \frac{\langle x^3, 1\rangle}{\ldots} 1$$

$$- \frac{\langle x^3, 1\rangle}{\langle 1, 1\rangle} 1$$

$$= \boxed{x^3 - \frac{3}{5}x}$$

- There is a better way to generate $P_n(x)$!

"3-term recurrence"

$$\begin{cases} P_{n+1}(x) = \frac{(2n+1)}{(n+1)} x P_n(x) - \frac{n}{(n+1)} P_{n-1}(x) \\ P_0(x) = 1, \quad P_{-1}(x) = 0. \end{cases}$$

G-S generated:

"$P_{n+1}(x) = x P_n(x) - \left(\frac{n^2}{4n^2-1}\right) P_{n-1}(x)$"

General: $[a,b]$, $w(x) \geq 0$ (int).

$\to$ $\langle f, g\rangle_w = \int_a^b f(x) g(x) w(x) dx$.

Family orth. poly $\{\Phi_j\}_{j=0}^{\infty}$

(•) $\Phi_j \in \mathcal{P}_j$ (•) $\{\Phi_0, \Phi_1, \dots, \Phi_j\}$ is an orthogonal basis of $\mathcal{P}_j$.

(•) $\langle \Phi_j, q\rangle_w = 0$ for all $q \in \mathcal{P}_{j-1}$.

$$\Phi_{n+1}(x) = (x - \beta_n) \Phi_n(x) - \gamma_n \Phi_{n-1}(x)$$

$\beta_n = \frac{\langle x\phi_n, \phi_n\rangle_w}{\ }$, $\gamma_n = \frac{\langle \phi_n, \phi_n\rangle_w}{\ }$

$$\beta_n = \frac{\langle x\phi_n, \phi_n\rangle_w}{\langle \phi_n, \phi_n\rangle_w}, \quad \gamma_n = \frac{\langle \phi_n, \phi_n\rangle_w}{\langle \phi_{n-1}, \phi_{n-1}\rangle_w}.$$

Induction:

- Base case $(n=0)$: $\Phi_1(x) = (x - \beta_0) = x - \frac{\langle x, 1\rangle_w}{\langle 1, 1\rangle_w}$ ✓
- Case n true $\Rightarrow$ Case $n+1$ is true.

$$\langle \Phi_{n+1}, \Phi_j\rangle_w = \langle x\phi_n, \phi_j\rangle_w - \beta_n\langle \phi_n, \phi_j\rangle_w - \gamma_n\langle \phi_{n-1}, \phi_j\rangle_w$$

$\boxed{j=n}$: $$= \langle x\phi_n, \phi_n\rangle - \frac{\langle x\phi_n, \phi_n\rangle}{\langle \phi_n, \phi_n\rangle}\cdot\langle \phi_n, \phi_n\rangle = 0.$$

$\boxed{j=n-1}$: $$= \langle x\phi_n, \phi_{n-1}\rangle - \frac{\langle \phi_n, \phi_n\rangle}{\cancel{\langle \phi_{n-1}, \phi_{n-1}\rangle}}\cdot\cancel{\langle \phi_{n-1}, \phi_{n-1}\rangle}$$
$$= \langle \phi_n, x\phi_{n-1}\rangle - \langle \phi_n, \phi_n\rangle$$
$$= \langle \phi_n, \underbrace{x\phi_{n-1} - \phi_n}_{\beta_{n-1}\phi_{n-1} + \gamma_{n-1}\phi_{n-2}}\rangle = 0$$

$\boxed{j<n-1}$ $$= \langle x\phi_n, \phi_j\rangle = \langle \phi_n, \underbrace{x\phi_j}_{\in \mathcal{P}_{n-1}}\rangle = 0$$

Tchebyshev poly: $[-1,1]$, $w(x) = \frac{1}{\sqrt{1-x^2}}$

$T_n(x) = \cos n\theta \qquad x = \cos\theta, \ \theta \in [0,\pi]$

$n=0 \to T_0(x) \equiv 1, \quad n=1: \ T_1(x) = x.$

$$T_{n+1}(x) = \cos(n+1)\theta = \cos n\theta\cos\theta - \sin n\theta\sin\theta$$
$$T_{n-1}(x) = \cos(n-1)\theta = \cos n\theta\cos\theta + \sin n\theta\sin\theta$$

$T_{n-1}(x) = \cos(n-1)\theta = \cos n\theta \cos\theta + \sin n\theta \sin\theta$

$T_{n+1}(x) + T_{n-1}(x) = 2x\, T_n(x)$

$$\boxed{T_{n+1}(x) = 2x\,T_n(x) - T_{n-1}(x)}$$

$$\langle T_n, T_m \rangle_\omega = \int_{-1}^{1} T_n(x) T_m(x) \frac{dx}{\sqrt{1-x^2}} = \int_{\cancel{\pi}\,0}^{\cancel{0}\,\pi} \cos n\theta \cos m\theta \frac{\cancel{+}\cancel{\sin\theta}\, d\theta}{\cancel{\sin\theta}}$$

$x = \cos\theta$

$dx = -\sin\theta\, d\theta$

$$= \begin{cases} 0 & ; n \neq m \\ \pi & ; n = m = 0 \\ \pi/2 & ; n = m > 0 \end{cases}$$

Coeff's Tchebyshev:

$$a_j = \frac{\langle T_j(x), f \rangle}{\langle T_j(x), T_j(x) \rangle} = \begin{cases} a_0 = \frac{1}{\pi}\int_0^\pi f(\cos\theta)\, d\theta \\ a_j = \frac{2}{\pi}\int_0^\pi f(\cos\theta)\cos j\theta\, d\theta \end{cases}$$

$\boxed{\text{Fast Cosine Transf.}}$

(o) Chebyshev → fast, near minimax

↦ "Cheb fun" – (N. Trefethen)

(*) Chebyshev nodes → zeroes of $T_{n+1}(x)$

$$|\psi(x)| = \left| \prod_{j=0}^{n} (x - x_j) \right| = \left| \frac{1}{2^n} T_{n+1}(x) \right|$$

$$\leq \underline{\underline{1/2^n}}$$

Approx w/ other functions – rational, trigonometric.

[illegible]

trigonometric.