Newton-Raphson method (II)

Wednesday, September 18, 2024 10:30 AM

Class 10: September 18, 2024

Recall: Last class we introduced perhaps the most famous rootfinding method: the **Newton-Raphson** method. Given a twice continuously differentiable function f, we assumed we have access to evaluate f(x) and f'(x) at a given guess x0 for our root. We find the equation of the tangent line (best linear approximation at x0) l(x) = f(x0) + f'(x0)(x-x0), and set it equal to zero. This gives us the Newton step:

xn = x_{n-1} - f(x_{n-1}) / f'(x_{n-1})

The main result on the Newton-Raphson method (and why it is so popular) is the following theorem, which we proved using the fact that Newton-Raphson is a Fixed Point Iteration for the function

g_{NR}(x) = x - f(x) / f'(x)

Theorem 1: (Convergence of Newton) If f in C^2([a,b]) and r a "single root" in [a,b] (such that f'(r) is not zero), then there exists a delta>0 such that if |x0 - r | < delta, then starting Newton at x0 converges *quadratically* to r.

This theorem is easily proven by showing that, given the assumptions, g'_{NR}(r) = 0.

What does this theorem tell us? In so many words: if we start Newton "close enough" to a single root, it converges quadratically. We saw examples of this quadratic convergence: once it gets going, you get to machine precision *very* fast; digits of precision double every step (e.g. 1,2,4,8,16).

However, there are some issues with Newton: for one, this gives us no guarantees if we start away from this "basin of quadratic convergence" (which we generally have no way to know) and if r is a multiple root, Newton only converges linearly. Today we will continue this discussion and give more examples.

WARM UP: NEWTON-RAPHSON

PSEUDOCODE:

INPUTS - $f(x)$, $f'(x)$, x_0, TOL, n_{max}

OUTPUTS - r

Initialize $n=0$

While ($n \le n_{max}$ AND $|x_n - x_{n-1}| \ge$ TOL)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)};$$

$$n = n+1$$

Return r

LAST TIME WE ASKED: WHAT IF $f(r)=0$, $f'(r)=0$?

- ROOT MULTIPLICITY: $f(r) = f'(r) = \dots = f^{(m-1)}(r) = 0$, $f^{(m)}(r) \neq 0$

$\hookrightarrow \exists\, h(x)$ s.t. $h(r) \neq 0$, $f(r) = (x-r)^m h(r)$.

$$g_{NR}(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\cancel{(x-r)^m}\,(x-r)\, h(x)}{\cancel{(x-r)^{m-1}}\,[m h(x) + (x-r) h'(x)]}$$

$$g'_{NR}(x) = 1 - \frac{(h(x)+(x-r)\quad)[m h(x) + \_\_\_] - ((x-r)h(x))[\qquad]'}{[m h(x) + (x-r) h'(x)]^2}$$

$$\lim_{x\to r} g'_{NR}(x) = 1 - \frac{\cancel{m}\, \cancel{h(r)^2}}{m^{\cancel{2}}\, \cancel{h(r)^2}} = \underline{1 - 1/m} \;\begin{cases} \to m=1 \to 0 \\ \to m>1 \in [1/2, 1) \end{cases}$$

$\lim_{x\to r} g_{NR}'(x) = 1 - \frac{\cancel{\ldots}}{m^{\cancel{2}}\, \cancel{h(r)^2}} = \underline{1 - 1/m} \to m>1 \quad \in [1/2, 1)$

FIXES:

(a) Given $f(x)$ w/ multiple root at r.

$\to f(x) = (x-r)^m h(r)$,

$f'(x) = (x-r)^{m-1}[\qquad]$

$$\boxed{f(x)/f'(x)} = \frac{h(r)(x-r)}{[\qquad]}$$

Apply Newton to f/f'. (requires $(f/f')'$)

(b) $g_{NR}(x) = x - f(x)/f'(x)$

$\tilde{g}_{NR}(x) = x - \alpha f(x)/f'(x)$ what is α s.t. $\tilde{g}_{NR}'(r) = 0$?

"Quasi Newton" $\to$ Secant method.

- Historically what is $f'(x)$?
- Currently — $f'(x)$ can be expensive, or hard to compute accurately.

$$\frac{f(x+h) - f(x)}{h} \approx f'(x)$$

$\to$ Quasi Newton:

- Want to evaluate $f'(x)$ 1 or 0 times.
- Want superlinear convergence close to r.

1ST IDEA: "Lazy Newton" (Chord Iteration)

$$x_n = x_{n-1} - \left(\frac{1}{f'(x_0)}\right) f(x_{n-1}).$$

- f' is only evaluated once ✓
- Convergence → Linear (when it converges).

Secant method

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

x_0, x_1 close to each other and close to r,

$$l_{sec}(x) = f(x_1) + \underbrace{\left[\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right]}_{\approx f'(x_1)} (x - x_1)$$

$$x_2 = x_1 - \frac{1}{\underbrace{m_{sec}}_{\left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right)}} \cdot (f(x_1))$$