Rational and Trigonometric Approximation Class 27

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Recall: Last time, we wrapped up the topic of L2 continuous approximation, especially that focused on using orthogonal families of polynomials to do L2 and weighted L2 polynomial approximation. Given a weight w(x) and an interval [a,b], that defines a family of orthogonal polynomials {Q0(x), Q1(x), ..., Qk(x),...} where Qk is of degree k and Qk is orthogonal to all polynomials of degree < k . The coefficients of the approximation are then given by

ak = < f, Qk>_w / <Qk, Qk>_w

Last time, we discussed a key property of these families of orthogonal polinomials: the **3-term recursion formula**. We showed a general formula of the form

Q_{k+1} = (x - bk) Qk - gk Q_{k-1}

Where bk and gk are defined in terms of inner products of Qk and Q_{k-1}. We then covered two important families: the Legendre polynomials (corresponding to [-1,1] with w(x)=1) and the Tchebyshev polynomials (corresponding to [-1,1] and w(x) = 1/sqrt(1-x^2)). Finally, we mentioned some interesting properties of Tchebyshev polynomials (near-minimax approximation, zeroes being the Tchebyshev points, proving that Chebyshev points solve the Runge phenomenon).

Today, we will introduce two kinds of function approximation with things other than polynomials: **rational and trigonometric approximation**.

POLYNOMIAL APPROX ✓

L2 - Legendre, Tchebyshev

MINIMAX: EQUIOSCILLATION

Rational Functions

$$f(x) \approx \frac{p(x) \to \in \mathcal{P}_m \to m+1}{q(x) \to \in \mathcal{P}_n \to n+1}$$

$$\boxed{f(x)\, q(x) \approx p(x)}$$

- Padé approx
- Continued fractions
- Minimax (Rational Remez)
- AAA - (Trefethen et al)

Instead of $f(x) \to$ Taylor poly Π_{n+m}

Idea is

$$\Pi_{n+m}(x) \cdot q(x) = p(x) \quad \text{match coeffs}$$

Example: $f(x) = e^{-x}$, $m=3$, $n=2$

$$r(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2}$$

$$\Pi_5(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}$$

$$\Pi_5(x) \cdot (1 + b_1 x + b_2 x^2) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\left.\begin{array}{l} a_0 = 1 \\ a_1 = b_1 - 1 \\ a_2 = 1/2 - b_1 + b_2 \\ a_3 = -1/6 + b_1/2 - b_2 \end{array}\right\} \text{ Evaluate.}$$

$$\left\{\begin{array}{l} 0 = a_4 = \frac{1}{24} - b_1/6 + b_2/2 \\ 0 = a_5 = \frac{-1}{120} + b_1/24 - b_2/6 \end{array}\right\} \longrightarrow \text{Solve for } b_1, b_2$$

TRIGONOMETRIX APPROX:

f defined $[-\pi, \pi]$, periodic, smooth.

$$f \approx a_0 + \sum_{k=0}^{m} a_k \cos(kx) + b_k \sin(kx)$$

$-m$

$k=0$

$$\{1, \cos(kx), \sin(kx)\}_{k=0}^{m} \longrightarrow 2m+1$$

model – "trigonometric polynomial"

$\mathcal{T}_m$ deg $\le m$.

$$\|f\|_2^2 = \int_{-\pi}^{\pi} f(x)^2\,dx \,, \quad \langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$$

$$\langle \cos(mx), \cos(nx)\rangle = \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx$$

$$= \begin{cases} 0 \,; & m \neq n \\ 2\pi \,; & m = n = 0 \\ \pi \,; & m = n > 0 \end{cases}$$

$$\langle \sin(mx), \sin(nx)\rangle = \begin{cases} 0 \,; & m \neq n \\ \pi \,; & m = n > 0 \end{cases}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \,, \quad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx$$

Decay of coeffs $\sim$ smoothness f.

$f^{(k-1)}$ [illegible] $f^{(k)}$ [illegible]

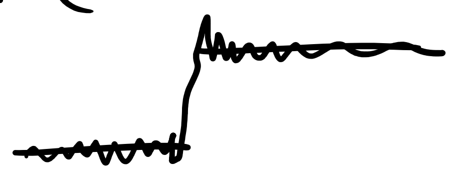
$f \in C^{(k-1)}$ and $f^{(k)}$ is cont. except at a discrete set of jumps.

then $$|a_n|, |b_n| \leq \frac{C}{n^k}$$

- $f(x) = |x|$ $\quad b_k = 0$

  $a_0 = \pi/2 \quad a_k = \frac{2}{\pi}\left(\frac{1}{k^2}\left((-1)^k - 1\right)\right)$

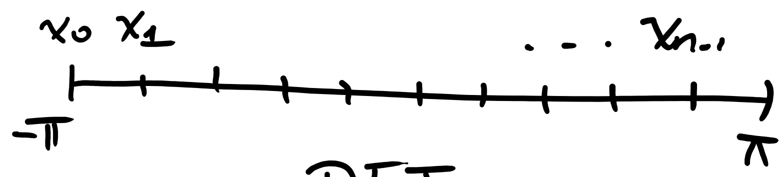
- $f(x) = \begin{cases} 1 ; & x \geq 0 \\ -1 ; & x < 0 \end{cases}$



$b_k = \frac{2}{\pi}\left(\frac{1-(-1)^k}{k}\right)$ Gibbs-phenom.

$\boxed{\text{EXTRA}}$

↳ f sample it at n equispaced nodes in $[-\pi, \pi)$



$$\{f(x_i)\}_{i=0}^{n-1} \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftarrows}} \{a_0, a_k, b_k\}$$

- Instead of

e^{ikx}

0 Instead of

cos(kx)

sin(kx) $\longrightarrow$ e^{ikx}

$= \underline{\cos(kx) + i\sin(kx)}$