Gaussian Elimination and LU

Wednesday, November 13, 2024 10:48 AM

Class 34: November 13, 2024

Recall: Last class, we reviewed key concepts in linear algebra that will be useful in our discussions for this last section of material. Those are:

- Vector space and subspaces
- Linear combinations and linear dependence / independence
- Bases and dimension
- **Range R(A) and rank r(A**) and their relationship to the **existence** of solutions of Ax=b.
- **Nullspace N(A) and nullity v(A)** and their relationship to **uniqueness** of solutions of Ax=b.
- Rank and nullity theorem, and what can happen for overdetermined, underdetermined and determined (square) systems of linear equations.
- How we can test whether an n x n matrix A is invertible.

We then discussed the differences between **direct solvers** and **iterative solvers**. Today we will be discussing our first topic: Gaussian Elimination and its associated matrix factorization, the LU decomposition. This is the first and prime example of a **direct solver**.

$\underset{n\times n}{A}\vec{x} = \vec{b}$ "ELIMINATION METHOD"
$\hookrightarrow$ DIRECT METHOD.

Special case where it is not needed:

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \\ 0 & a_{22} & a_{23} & \cdots & \\ 0 & 0 & a_{33} & \cdots & \\ \vdots & \vdots & 0 & \ddots & \\ \vdots & \vdots & \vdots & \boxed{a_{n-1,n-1}} & \\ 0 & 0 & 0 & 0 & a_{nn} \end{bmatrix}}_{\text{upper triangular}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

SOLVE:

1. $x_n = \frac{1}{a_{nn}} \cdot b_n$. $\to$ 1 div.
2. $x_{n-1} = \frac{1}{a_{n-1,n-1}} \Big[b_{n-1} \underset{1\,\text{sum}}{-} \overbrace{a_{n-1,n}\, x_n}^{1\text{ mult}} \Big]$

$\vdots$

$$x_j = \frac{1}{a_{jj}} \Big[b_j - \sum_{k>j} a_{jk} x_k \Big]$$

"back substitution"

A upper triangular - back sub.
A lower triangular - forward sub.

OP COUNT:

$2(1+2+3+\ldots+n-1) + n$

$$2(\underbrace{1+2+3+\dots+n-1}) + n$$

$$\cancel{2}\cdot\frac{n(n-1)}{\cancel{2}} = n(n-1)+n \to \boxed{O(n^2)}.$$

3 x 3 example:

$$\begin{array}{l} R_2 \leftarrow R_2 - \frac{2}{4}R_1 \\ R_3 \leftarrow R_3 - \frac{1}{4}R_1 \end{array} \begin{bmatrix} 4 & -1 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 11 \end{bmatrix}$$

"Elementary row operations"

- multiply R_i by $c \neq 0$.
- permute R_i and R_j.
- $\boxed{R_j \leftarrow R_j + aR_i}$ $\leftarrow$

$$R_3 \leftarrow R_3 - \left(\frac{9/4}{11/2}\right)R_2 \quad \left[\begin{array}{ccc|c} 4 & -1 & 1 & 8 \\ 0 & \boxed{11/2} & 3/2 & -1 \\ 0 & 9/4 & 15/4 & 9 \end{array}\right]$$

augmented matrix

$$\left[\begin{array}{ccc|c} 4 & -1 & 1 & 8 \\ 0 & 11/2 & 3/2 & -1 \\ 0 & 0 & 69/22 & 207/22 \end{array}\right]$$

$$x_3 = \frac{207}{69} = 3$$

$$x_2 = \frac{2}{11}\left(-1 - \frac{3}{2}\cdot 3\right) = -1$$

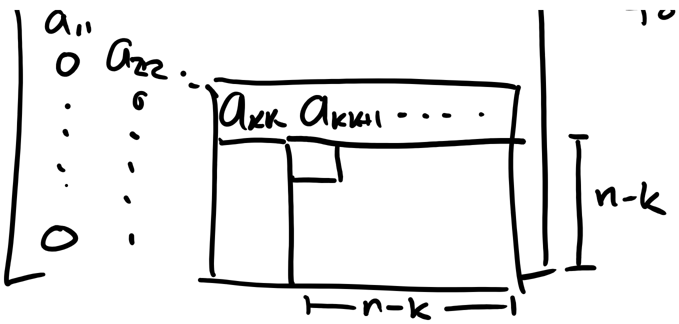
$$x_1 = \frac{1}{4}(8 - x_3 + x_2) = 1$$

SOL $x = (1, -1, 3)$.

$$\begin{bmatrix} a_{11} & & & \\ 0 & a_{22} & \ddots & \\ \vdots & & & \end{bmatrix}$$

for $k = 1$ to $n-1$:

for j from $k+1$ to n



for j from $k+1$ to n

$$\begin{cases} R_j = R_j - \left(\frac{a_{jk}}{a_{kk}}\right) R_k \\ b_j = b_j - \left(\frac{a_{jk}}{a_{kk}}\right) b_k. \end{cases}$$

```
for k = 1 to n-1:
    for j from k+1 to n:
        m_jk = a_jk / a_kk; %compute multiplier.
        A(j, k+1:n) = A(j : k+1:n)
                      - m_jk A(K, k+1:n);
        A(j,k) = 0
        b_j = b_j - m_jk b_k;
```

$$2\left[(n-1)^2 + (n-2)^2 + (n-3)^2 + \dots + 1\right]$$

$$2\left[\frac{(n-1)(n)(2n-1)}{6}\right] \to \frac{4}{6}n^3 + \dots$$

$$O(n^3),$$

$\boxed{0}$ $\{Ax = b_1,\ Ax = b_2,\ Ax = b_3,\ \dots\}$

$$Ax = b \longrightarrow U\vec{x} = \vec{c}$$

$$R_j = R_j - m_{jk} R_k$$

$$\begin{array}{cc} & \\ j & \end{array}\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & -m_{jk} & & 1 \\ & & & & & 1 \\ & & & & & & \ddots \end{bmatrix} \to \text{lower triangular}$$

$$j \begin{bmatrix} & \boxed{-m_{jk}} & 1 & & \\ & & & 1 & \\ & & & & \ddots & 1 \end{bmatrix}$$

$$\mathbb{L}_k = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & -m_{k+1,k} & 1 & \\ & & & \vdots & & \ddots \\ & & & -m_{nk} & & & 1 \end{bmatrix} \quad \mathbb{L}_k^{-1} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & m_{k+1,k} & 1 & \\ & & & \vdots & & \ddots \\ & & & m_{nk} & & & 1 \end{bmatrix}$$

$$\mathbb{U} = (\mathbb{L}_{n-1} \cdots \mathbb{L}_2 \mathbb{L}_1) A, \quad \vec{c} = (\mathbb{L}_{n-1} \cdots \mathbb{L}_2 \mathbb{L}_1) \vec{b}$$

$$A = \underbrace{(\mathbb{L}_1^{-1} \mathbb{L}_2^{-1} \cdots \mathbb{L}_{n-1}^{-1})}_{\mathbb{L}} \mathbb{U}$$

$$A x = b \longrightarrow \mathbb{L} (\overbrace{\mathbb{U} x}^{y}) = b$$

$$y = \text{forward\_sub}(\mathbb{L}, b) \quad O(n^2)$$

$$x = \text{back\_sub}(\mathbb{U}, y) \quad O(n^2)$$