Homework 12
APPM 4600 Numerical Analysis, Fall 2025

Due date: Friday, December 5, before midnight, via Gradescope. **Instructor**: Prof. Becker
Revision date: 11/29/2025

Theme: Eigenvalues

Instructions Collaboration with your fellow students is OK and in fact recommended, although direct copying is not allowed. The internet is allowed for basic tasks (e.g., looking up definitions on wikipedia) but it is not permissible to search for proofs or to *post* requests for help on forums such as `http://math.stackexchange.com/` or to look at solution manuals. Please write down the names of the students that you worked with. Please also follow our AI policy.

An arbitrary subset of these questions will be graded.

Turn in a PDF (either scanned handwritten work, or typed, or a combination of both) to **Gradescope**, using the link to Gradescope from our Canvas page. Gradescope recommends a few apps for scanning from your phone; see the Gradescope HW submission guide.

We will primarily grade your written work, and computer source code is *not* necessary (and you can use any language you want). You may include it at the end of your homework if you wish (sometimes the graders might look at it, but not always; it will be a bit easier to give partial credit if you include your code). For nicely exporting code to a PDF, see the APPM 4600 HW submission guide FAQ.

Problem 1: Similar matrices Prove that $A = \begin{bmatrix} 4 & 2 & 0 \\ -1 & 3 & 8 \\ 7 & 7 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 & 8 \\ -4 & 2 & 1 \\ -8 & -6 & 2 \end{bmatrix}$ are not similar. *Hint: there are ways to do this that are very painless!*

Problem 2: Power method Define the matrix

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}.$$

a) On a computer, create a $n \times n$ matrix of this form for $n = 1000$ and then run 100,000 iterations or more of the power method (*Hint: use sparse matrices*). What is the dominant eigenvalue? *Please turn in your code too*

b) Plot the absolute value of the entries of the dominant eigenvector that you found. *You might be interested in looking at this for when you run only* 100*, or* 1000*, or* 10,000 *iterations, and see how it changes.*

c) Again for $n = 1000$, compare how long it takes your code to run 100,000 iterations of the power method with how long it it takes to call a standard eigenvalue computation on the dense version of A (i.e., in Python, `scipy.linalg.eigh(A.todense())`).

d) Repeat the timing comparison for $n = 3000$.

Problem 3: Eigenvalues of infinite dimensional linear operators

a) If $f \in C^3(\mathbb{R})$, show for any $x \in \mathbb{R}$ that

$$f''(x) = \lim_{h \to 0} \frac{-f(x-h) + 2f(x) - f(x+h)}{h^2}$$

b) If one sampled f on an equispaced grid $x_1, x_2, \ldots, x_n$ with grid spacing h to create the vector $\boldsymbol{f} \in \mathbb{R}^n$ with $f_i = f(x_i)$, show that $(A\boldsymbol{f})_i$ is an approximation of $h^2 f''(x_i)$ *where A is the matrix from Problem 2.*

c) Consider the linear operator $L[f] = f''$. It's a linear operator, so you might wonder if it has eigenvalues the same way that a matrix does.[1] Can you think of any functions f for which there is some λ so that $f'' = \lambda f$?

[1]There are some subtleties involved, and you should take APPM 5440 and 5450 if this kind of question interests you!