Notes on (Spanne, 1988)

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1 Brief overview of Spanne's derivation

Spanne assumes that N, the number of detected photons is an uncorrelated Gaussian process. The variance of the estimate of $\mu(x, y)$ is approximated by:

$$\text{var}(\mu(x, y)) = \left(\frac{\pi}{m}\right)^2 \sum_{n_\theta=0}^{m} \sum_{n_R=-L/2}^{L/2} \frac{g^2(n_R \Delta R - R')}{E\{N(R', n_\theta \Delta\theta)\}} \tag{1}$$

where:

- m : number of views/angles
- n_θ : angular sampling index
- n_R : sampling point index
- g : filter function
- $E\{\}$: denotes expectation
- R' : radial coordinate
- $\Delta\theta$: angular increment

Solving for the variance at an arbitrary pixel (x, y) would involve finding all rays traversing that pixel, so Spanne simplifies things by looking at a single pixel in the center of a circular object, for which $E\{N\}$ is equal for all angles. The variance then reduces to:

$$\begin{aligned}
\text{var}(\mu(0,0)) &= \left(\frac{\pi}{m}\right)^2 \sum_{n_\theta=0}^{m} \sum_{n_R=-L/2}^{L/2} \frac{g^2(n_R \Delta R)}{E\{N(0, n_\theta \Delta\theta)\}} \\
&= c_{filter} \left(\frac{\pi}{m}\right)^2 \sum_{n_\theta=0}^{m} \frac{1}{E\{N(0, n_\theta \Delta\theta)\}} \\
&= c_{filter} \left(\frac{\pi^2}{m}\right) \frac{1}{E\{N(0, n_\theta \Delta\theta)\}} \qquad (2)
\end{aligned}$$

where he has grouped the sum of squared filter terms into c_{filter} and exploited the fact that $E\{N(0, n_\theta \Delta\theta\}$ is independent of n_θ. Note also that the expectation for N is just the usual:

$$E\{N(0, n_\theta \Delta\theta)\} = N_0 \exp\left\{\sum_i \mu_i d_i\right\} \tag{3}$$

Spanne then defines the SNR as

$$\frac{S}{\sigma} = \frac{|\mu_1 - \mu_2|}{\sqrt{\mathrm{var}(\mu_1(0,0)) + \mathrm{var}(\mu_2(0,0)}} \tag{4}$$

where μ_1 represents the attenuation in the central voxel *without* any contrast material, and μ_2 represents the attenuation in the central voxel *with* added contrast material.

2 Applications to our model

Spanne includes a third material as a "shell" in his derivation. Ignoring that shell for our purposes, consider a homogeneous spherical object of diameter d, with an attenuation coefficient $\mu_1 \equiv \mu_{bg}$. Now, consider adding a small amount of contrast material to the central voxel, with voxel dimension r and attenuation coefficient μ_c, such that the total attenuation coefficient is now $\mu_2 \equiv \mu_{bg} + \mu_c$. The SNR can then be calculated from equations 2, 3 and 4 to be:

$$\left(\frac{S}{\sigma}\right)_{\text{Spanne}} = \sqrt{\frac{mN_0}{\pi^2 c_{filter}}} \frac{\mu_c}{\sqrt{\exp\{\mu_{bg} d\} + \exp\{\mu_c r + \mu_{bg} d\}}} \tag{5}$$

Compare this to our working model:

$$\left(\frac{S}{\sigma}\right)_{\text{Ours}} = \sqrt{N_0}\mu_c \sqrt{\exp\{-\mu_c r - \mu_{bg} d\}} \tag{6}$$

Besides the inclusion of energy-independent constants (including the reconstruction term, c_{filter}), the main difference is that our working model only incorporates the noise from the contrast-present case:

$$\sigma \propto \frac{1}{\bar{N}} \tag{7}$$

$$\bar{N} \equiv N_0 \exp\{-\mu_c r - \mu_{bg} d\} \tag{8}$$

whereas Spanne adds the variances from the contrast-present and contrast-absent cases in quadrature.