Spherical Harmonic Update

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1 From orientation data to SH

The following is adapted from [1]:

The spherical harmonics (SH) are defined as

$$Y_l^m(\theta,\phi) = N_l^m P_l^m(\cos\theta)e^{jm\phi}, \tag{1}$$

and form an orthonormal basis over $L_2(\mathbb{S}^2)$. Any square integrable function $f(\theta,\phi) \in L_2(\mathbb{S}^2)$ can be expressed as a linear combination of SH:

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm}Y_l^m(\theta,\phi), \tag{2}$$

with coefficients c_{lm} given by

$$c_{lm} = \int_{\mathbb{S}^2} f(\boldsymbol{w})\bar{Y}_l^m(\boldsymbol{w})\mathrm{d}\boldsymbol{w}, \tag{3}$$

where the overbar denotes conjugation and

$$\boldsymbol{w}(\theta,\phi) = [\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta]^T. \tag{4}$$

We are interested in determining the orientation distribution function (ODF) $f(\theta,\phi)$ from the voxel-wise principal orientation vectors determined from structure tensor analysis. In order to expand this function on the SH, we model each orientation vector as a 2D Dirac delta function δ on the sphere. That is, in a ROI containing K voxels, our ODF can be written as

$$f(\theta,\phi) = \frac{1}{K}\sum_{k=1}^{K}\delta(\theta-\theta_k)\delta(\phi-\phi_k). \tag{5}$$

If we substitute this into Eqn 3, then the sifting property of the Dirac delta function can be used, and the integral reduces to:

$$c_{lm} = \frac{1}{K}\sum_{k=1}^{K}\bar{Y}_l^m(\theta_k,\phi_k). \tag{6}$$

A SH approximation $\hat{f}(\theta, \phi)$ can then be determined to an arbitrary band-limit L_{max} using Eqn 2.

2 Simplifications in HARDI literature

Spherical harmonics are used extensively in the HARDI literature to represents ODFs [2]. Generally, the following simplifications are made

- Diffusion is symmetric about the origin, so odd-ordered SH components are assumed zero and ignored
- The diffusion-weighted signal and ODF are both real functions, so their SH representations exhibit conjugate symmetry

Accordingly, in the Dipy python package [3], only even-ordered spherical harmonic components are calculated for the various HARDI reconstruction methods, and the conjugate symmetry is exploited as follows:

$$Y_l^m(\theta, \phi)_{real} \equiv \begin{cases} \sqrt{2}\mathrm{Re}\left[Y_l^{|m|}(\theta, \phi)\right] & m < 0 \\ Y_l^0(\theta, \phi) & m = 0 \\ \sqrt{2}\mathrm{Im}\left[Y_l^m(\theta, \phi)\right] & m > 0 \end{cases} \tag{7}$$

3 Applications to xray ODFs

Figure 1 shows ODFs calculated with this method for both real xray data and a simulated phantom with three groups of orthogonal fibers, using both even-only and all SH components up to $L_{max} = 8$. The figures also lightly display the peak estimate as a thin red line. These ODFs were generated using $\sigma_d = 3\ \mu m$ and $\sigma_N = 6\ \mu m$ for the structure tensors. These values were shown to have the highest AUC value in classifying fibers using the fractional anisotropy metric. Further work will use the accuracy of the peak direction estimate to tune these and other parameters.

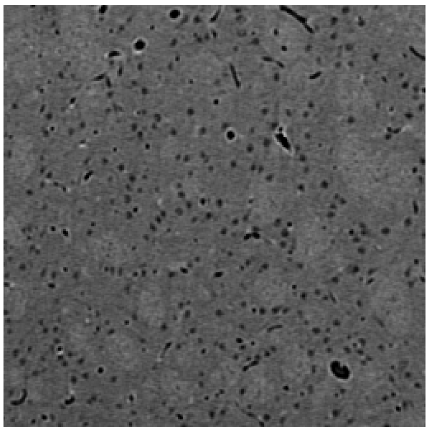
We see that overall, the two ODFs identify similar peaks for both the even and full SH representations. The symmetry is obviously lost when the odd components are used, however. This is most obvious in the XYZ phantom. In Figure 1f, we see that the structure tensor analysis resulted in few orientations in the -z and +x directions, while that symmetry is enforced with the even SH representation in 1e. For the real data, the overall ODF shapes are more similar, but the peak estimate is different.

References

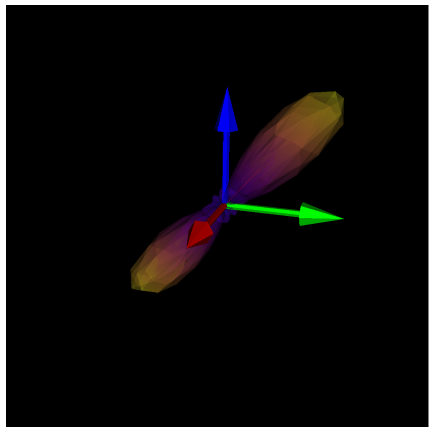
[1] A. Alimi, Y. Ussou, J. Pierre-Simon, G. Michalowicz, and R. Deriche, "An Analytical Fiber ODF Reconstruction in 3D Polarized Light Imaging," in *15th IEEE International Symposium on Biomedical Imaging (ISBI)*, (Washington, D.C.), pp. 1276–1279, 2018.

[2] J.-D. Tournier, F. Calamante, D. G. Gadian, and A. Connelly, "Direct estimation of the fiber orientation density function from diffusion-weighted MRI data using spherical deconvolution," *NeuroImage*, vol. 23, pp. 1176–1185, nov 2004.

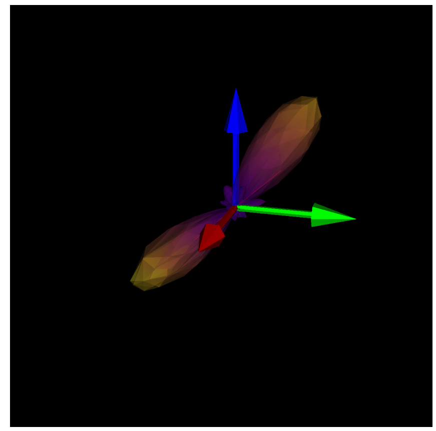
[3] E. Garyfallidis, M. Brett, B. Amirbekian, A. Rokem, S. van der Walt, M. Descoteaux, and I. Nimmo-Smith, "Dipy, a library for the analysis of diffusion MRI data," *Frontiers in Neuroinformatics*, vol. 8, p. 8, feb 2014.



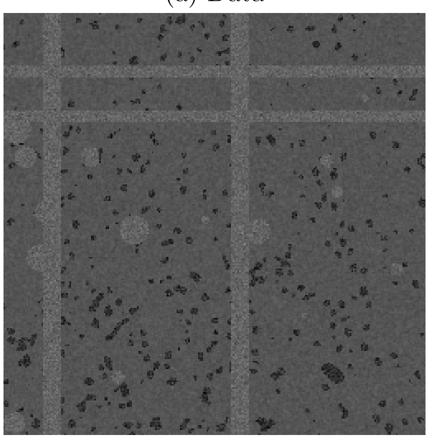
(a) Data



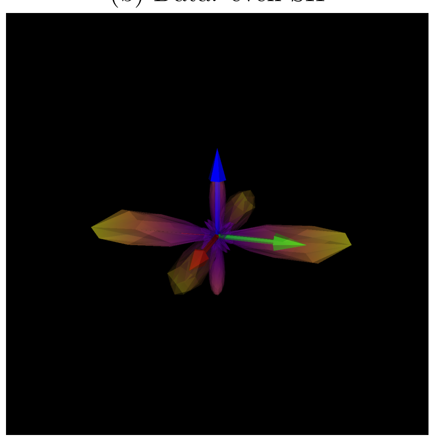
(b) Data: even SH



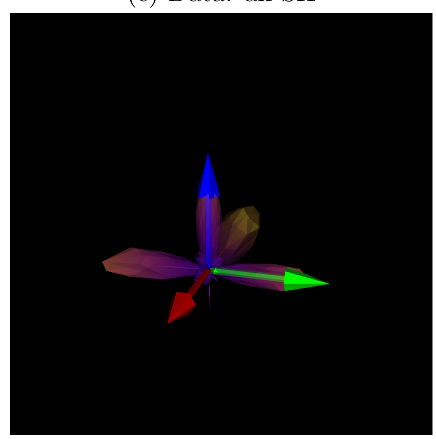
(c) Data: all SH



(d) XYZ Phantom



(e) XYZ Phantom: even SH



(f) XYZ Phantom: all ODF

Figure 1: ODFs for $L_{max} = 8$.