Lecture Notes

——School of Science in NJUPT

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September 1, 2023

Reading List

This is the Refernces and Textbooks.

- **Complex Analysis** by *Lars V. Ahlfors*
- **Concise Complex Analysis** by *Sheng Gong*
- **Real And Complex Analysis** by *W.Ruding*
- **Complex Analysis** by *Kunihiko Kodaira*
- **Visual Complex Analysis** by *Needham T.*

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Contents

Chapter 1

Complex numbers

Here

1.1 Section

Definition 1.1.1: The title

Let $\mathcal{O}$ be a finite union of semisimple conjugacy classes in $M(F)$. Define

$$D_{\text{geom},-}(\tilde{M},\mathcal{O}) := \left\{D \in D_{-}(\tilde{M}) : \tilde{\gamma} \in \operatorname{Supp}(D) \implies \gamma_{\text{ss}} \in \mathcal{O}\right\},$$

$$D_{\text{geom},-}(\tilde{M}) := \bigoplus_{\substack{\mathcal{O}\subset M(F)\\ \text{ss. conj. class}}} D_{\text{geom},-}(\tilde{M},\mathcal{O}) \subset D_{-}(\tilde{M}),$$

$$D_{\text{unip},-}(\tilde{M}) := D_{\text{geom},-}(\tilde{M},\{1\}).$$

Counting remarks with respect to the section:

***Remark* 1.** Counter is okay

Counting remarks with respect to the theorem:

***Remark* 2.** **Problem:** counter is missing and there is not even a break line.

Axiom 1: Axiom of choice

For any set X of nonempty Sets, there exists a choice function f that is defined on X and maps each set of X to an element of that set.

Note.

Theorem 1.1.0.1: Inhomogeneous Linear

Let f be a function whose derivative exists in every point, then f is a continuous function.

Proof. To prove it by contradiction try and assume that the statement is false, proceed from there and at some point you will arrive to a contradiction. ∎

Theorem 1.1.0.2

Given $\mathbf{G}^! \in \mathscr{E}(\tilde{G})$, there exists a linear map

$$\begin{aligned} \mathscr{T} = \mathscr{T}_{\mathbf{G}^!, \tilde{G}} : \mathscr{I}_{-}(\tilde{G}) \otimes \mathrm{mes}(G) &\longrightarrow S\mathscr{I}(G^!) \otimes \mathrm{mes}(G^!) \\ f &\longmapsto f^{G^!} \end{aligned}$$

such that for all $\delta \in \Sigma_{G\text{-reg}}(G^!)$, we have

$$\sum_{\delta \in \Gamma_{\mathrm{reg}}(G)} \Delta_{\mathbf{G}^!, \tilde{G}}(\delta, \tilde{\gamma}) f_{\tilde{G}}(\tilde{\gamma}) = f^{G^!}(\delta)$$

where $\tilde{\gamma} \in \mathbf{p}^{-1}(\gamma)$ is arbitrary, with the aforementioned convention on Haar measures.

When F is Archimedean, $\mathscr{T}$ is continuous and it restricts to

$$\mathscr{I}_{-}(\tilde{G}, \tilde{K}) \otimes \mathrm{mes}(G) \to S\mathscr{I}(G^!, K^!) \otimes \mathrm{mes}(G^!),$$

where $K \subset G(F)$ and $K^! \subset G^!(F)$ are maximal compact subgroups.

Proof. This is [**?**, Théorème 5.20]. First, it reduces to the case $\mathbf{G}^! \in \mathscr{E}_{\mathrm{ell}}(\tilde{G})$. The continuity in the Archimedean case is addressed in [**?**, §7.1], which is based on the works of Adams and Renard. The $\tilde{K} \times \tilde{K}$-finite transfer in the Archimedean case is [**?**, Theorem 7.4.5]. ∎

Theorem 1.1.0.3

Fix $n \in \mathbb{Z}_{\geq 1}$. Let W (resp. V) be a $2n$-dimensional symplectic F-vector space (resp. the quadratic F-vector space giving rise to the split $\mathrm{SO}(2n+1)$). There is a natural bijection $P \leftrightarrow P_{\mathrm{SO}}$ (resp. $M \leftrightarrow M_{\mathrm{SO}}$) between conjugacy classes of parabolic subgroups (resp. Levi subgroups) of $\mathrm{Sp}(W)$ and $\mathrm{SO}(V, q)$, such that if $M \simeq \prod_{i \in I} \mathrm{GL}(n_i) \times \mathrm{Sp}(W^\flat)$, then $M_{\mathrm{SO}} \simeq \prod_{i \in I} \mathrm{GL}(n_i) \times \mathrm{SO}(V^\flat, q^\flat)$, where $(n_i)_{i \in I} \in \mathbb{Z}_{\geq 1}^I$ satisfies

$$\frac{1}{2} \dim W^\flat = n - \sum_{i \in I} n_i = \frac{1}{2}\left(\dim V^\flat - 1\right).$$

The groups $W(M)$ and $W(M_{\mathrm{SO}})$ are also identified under this bijection: as groups of outer automorphisms of $\prod_{i\in I}\mathrm{GL}(n_i)$, both are generated by the transpose-inverse of $\mathrm{GL}(n_i)$ for various $i \in I$, together with permutations of factors of the same size.

The maximal tori T and T_{SO} are canonically isomorphic with respect to the given bases, compatibly with the identification of their Weyl groups $\mathfrak{S}_n \ltimes \{\pm 1\}^n$.

Proof. Exercise. ■

Lemma 1

For all $\phi \in T^{\mathscr{E}}(\tilde{G})$ and $f \in \mathscr{I}_{--}(\tilde{G}) \otimes \mathrm{mes}(G)$, we have

$$\mathscr{T}^{\mathscr{E}}(f)(\phi) = \sum_{\tau \in T_{-}(\tilde{G})/\mathbb{S}^1} \Delta(\phi, \tau) f_{\tilde{G}}(\tau).$$

Proof. The non-Archimedean case is the main result of [**?**]. Consider the case $F = \mathbb{R}$ next.

Write $f_{\tilde{G}}(\pi) := \Theta_\pi(f_{\tilde{G}})$ for each $\pi \in \Pi_{\mathrm{temp},-}(\tilde{G})$. The local character relation of [**?**, Theorem 7.4.3] yields a function $\Delta_{\mathrm{spec}} : T^{\mathscr{E}}(\tilde{G}) \times \Pi_{\mathrm{temp},-}(\tilde{G}) \to \{\pm 1\}$ satisfying

$$\mathscr{T}^{\mathscr{E}}(f)(\phi) = \sum_{\pi \in \Pi_{\mathrm{temp},-}(\tilde{G})} \Delta_{\mathrm{spec}}(\phi, \pi) f_{\tilde{G}}(\pi) \tag{1.1}$$

for all ϕ and $f \in \mathscr{I}_{--}(\tilde{G}) \otimes \mathrm{mes}(G)$, and $\Delta_{\mathrm{spec}}(\cdot, \pi)$ (resp. $\Delta_{\mathrm{spec}}(\phi, \cdot)$) has finite support for each π (resp. for each ϕ). These properties characterize Δ_{spec}.

Choose a representative in $T_{-}(\tilde{G})$ for every class in $T_{-}(\tilde{G})/\mathbb{S}^1$. By the theory of R-groups, $T_{-}(\tilde{G})/\mathbb{S}^1$ gives a basis of $D_{\mathrm{temp},-}(\tilde{G}) \otimes \mathrm{mes}(G)^\vee$: specifically, we may write

$$\Theta_\tau = \sum_\pi \mathrm{mult}(\tau : \pi) \Theta_\pi, \quad \Theta_\pi = \sum_\tau \mathrm{mult}(\pi : \tau) \Theta_\tau$$

for all $\tau \in T_{-}(\tilde{G})/\mathbb{S}^1$ with its representative and $\pi \in \Pi_{\mathrm{temp},-}(\tilde{G})$, for uniquely determined coefficients $\mathrm{mult}(\cdots)$. Switching between bases, (1.1) uniquely determines

$$\Delta^\circ : T^{\mathscr{E}}(\tilde{G}) \times T_{-}(\tilde{G}) \to \mathbb{S}^1$$

such that

$$\Delta^\circ(\phi, z\tau) = z\Delta^\circ(\phi, \tau), \quad z \in \mathbb{S}^1,$$
$$\mathscr{T}^{\mathscr{E}}(f)(\phi) = \sum_{\tau \in T_{-}(\tilde{G})/\mathbb{S}^1} \Delta^\circ(\phi, \tau) f_{\tilde{G}}(\tau)$$

for all f. Specifically, $\Delta^\circ(\phi, \tau) = \sum_{\pi \in \Pi_{\mathrm{temp},-}(\tilde{G})} \Delta_{\mathrm{spec}}(\phi, \pi) \mathrm{mult}(\pi : \tau)$ for all $\tau \in T_{-}(\tilde{G})/\mathbb{S}^1$.

Our goal is thus to show

$$\Delta^\circ(\phi, \tau) = \Delta(\phi, \tau), \quad (\phi, \tau) \in T^{\mathscr{E}}(\tilde{G}) \times T_{-}(\tilde{G}). \tag{1.2}$$

The first step is to reduce to the elliptic setting. We say $\pi \in \Pi_{\mathrm{temp},-}(\tilde{G})$ is *elliptic* if Θ_π is not identically zero on $\Gamma_{\mathrm{reg,ell}}(\tilde{G})$. In [**?**, Definition 7.4.1] one defined a subset $\Pi_{2\uparrow,-}(\tilde{G})$ of $\Pi_{\mathrm{temp},-}(\tilde{G})$. All $\pi \in \Pi_{2\uparrow,-}(\tilde{G})$ are elliptic. Indeed, by [**?**, Remark 7.5.1] π is a non-degenerate limit of discrete series in the sense of Knapp–Zuckerman, and such representations are known to be elliptic; see *loc. cit.* for the relevant references.

By [**?**, Proposition 5.4.4], $T_{\mathrm{ell},-}(\tilde{G})/\mathbb{S}^1$ gives a basis for the space spanned by the characters of all elliptic π.

Let $\phi \in T^{\mathscr{E}}(\tilde{G})$. Take $M \in \mathscr{L}(M_0)$ and $\mathbf{M}^! \in \mathscr{E}_{\mathrm{ell}}(\tilde{M})$ such that ϕ comes from $\phi_{M^!} \in \Phi_{\mathrm{bdd},2}(M^!)$ up to $W^G(M)$. Denote the factors relative to $\tilde{M}$ as $\Delta^{\tilde{M}}$, etc. By [**?**, Theorem 7.4.3],

$$\Delta_{\mathrm{spec}}(\phi, \pi) = \sum_{\pi_M \in \Pi_{2\uparrow,-}(\tilde{M})} \Delta^{\tilde{M}}_{\mathrm{spec}}(\phi_{M^!}, \pi_M) \mathrm{mult}(I_{\tilde{P}}(\pi_M) : \pi)$$

for all $\pi \in \Pi_{\mathrm{temp},-}(\tilde{G})$, where $P \in \mathscr{P}(M)$ and $\mathrm{mult}(I_{\tilde{P}}(\pi_M) : \pi)$ denotes the multiplicity of π in $I_{\tilde{P}}(\pi_M)$.

We claim that

$$\Delta^\circ(\phi, \tau) = \sum_{\substack{\tau_M \in T_{\mathrm{ell},-}(\tilde{M}) \\ \tau_M \mapsto \tau}} \Delta^{\tilde{M},\circ}(\phi_{M^!}, \tau_M). \tag{1.3}$$

Note that the sum is actually over an orbit $W^G(M)\tau_M$ if such a τ_M exists. We may assume τ is the representative of some element from $T_-(\tilde{G})/\mathbb{S}^1$; we also choose representatives in $T_-(\tilde{M})$ for $T_-(\tilde{M})/\mathbb{S}^1$ compatibly with induction. We have

$$\begin{aligned}\Delta^\circ(\phi, \tau) = \sum_{\pi \in \Pi_{\mathrm{temp},-}(\tilde{G})} & \Delta_{\mathrm{spec}}(\phi, \pi) \mathrm{mult}(\pi : \tau) \\ &= \sum_{\pi} \sum_{\pi_M \in \Pi_{2\uparrow,-}(\tilde{M})} \sum_{\tau_M \in T_{\mathrm{ell},-}(\tilde{M})/\mathbb{S}^1} \\ &\qquad \cdot \mathrm{mult}(\tau_M : \pi_{\tilde{M}}) \mathrm{mult}(I_{\tilde{P}}(\pi_{\tilde{M}}) : \pi) \mathrm{mult}(\pi : \tau) \Delta^{\tilde{M},\circ}(\phi_{M^!}, \tau_M).\end{aligned}$$

Given τ_M, the sum over (π, π_M) of the triple products of $\mathrm{mult}(\cdots)$ is readily seen to be 1 if $\tau_M \mapsto \tau$, otherwise it is zero. This proves (1.3).

Observe that (1.3) take the same form as the induction formula in Definition **??**. In order to prove (1.2), we may assume $\phi \in T^{\mathscr{E}}_{\mathrm{ell}}(\tilde{G})$, in which case $\Delta^\circ(\phi, \tau) = 0$ unless $\tau \in T_{\mathrm{ell},-}(\tilde{G})$, and ditto for $\Delta(\phi, \tau)$. It is thus legitimate to take $f \in \mathscr{I}_{--,\mathrm{cusp}}(\tilde{G}) \otimes \mathrm{mes}(G)$ in the characterization of $\Delta^\circ(\phi, \cdot)$. All in all, $\Delta^\circ(\phi, \cdot)$ and $\Delta(\phi, \cdot)$ have the same characterization, whence (1.2).

The case $F = \mathbb{C}$ is even simpler because it reduces to the case of split maximal tori via parabolic induction: see [**?**, §7.6]. ■

Corollary 1

Let $i_{M^!}^{G^![s]}(\epsilon[s])$ be as in (**??**). We have

$$i_{M^!}\left(\tilde{G}, G^![s]\right) i_{M^!}^{G^![s]}(\epsilon[s]) \cdot \|\check{\beta}\| = \left(Z_{\tilde{M}^\vee}^{\hat{\alpha}^*} : Z_{\underline{\tilde{M}}^\vee}^{\circ}\right)^{-1} i_{\underline{M}^!}\left(\tilde{G}, G^![s]\right) \cdot \|\check{\alpha}\|.$$

Proof. Use [**?**, Lemma 1.1] to obtain the natural surjection $Z^\circ_{\tilde{M}^\vee}/Z^\circ_{\tilde{G}^\vee} \twoheadrightarrow Z_{(M^!)^\vee}/Z_{G^![s]^\vee}$. Denote its kernel as K_1. One readily checks that $|K_1|^{-1} = i_{M^!}\left(\tilde{G}, G^![s]\right)$ as in [**?**, p.230 (2)].

We contend that the image of $Z^{\hat{\alpha}^*}_{\tilde{M}^\vee}/Z^\circ_{\tilde{G}^\vee}$ is contained in $Z^{\check{\beta}}_{(M^!)^\vee}/Z_{G^![s]^\vee}$. When α is short, $\check{\alpha}$ transports to $\check{\beta}$ under $T^! \simeq T$; in this case $Z^{\hat{\alpha}^*}_{\tilde{M}^\vee}/Z^\circ_{\tilde{G}^\vee}$ is actually the preimage of $Z^{\check{\beta}}_{(M^!)^\vee}/Z_{G^![s]^\vee}$. When α is long, $\check{\alpha}$ transports to $\frac{1}{2}\check{\beta}$, and the containment is clear.

Set $K_1' := K_1 \cap (Z^{\hat{\alpha}^*}_{\tilde{M}^\vee}/Z^\circ_{\tilde{G}^\vee})$; we have just seen that $K_1 = K_1'$ if α is short. From these and Lemma **??**, we obtain the following commutative diagram of abelian groups, with exact rows:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & K_3 \arrow[r] \arrow[d] & Z^\circ_{\underline{\tilde{M}}^\vee}/Z^\circ_{\tilde{G}^\vee} \arrow[r] \arrow[d] & Z_{(\underline{M}^!)^\vee}/Z_{G^![s]^\vee} \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K_1' \arrow[r] \arrow[d] & Z^{\hat{\alpha}^*}_{\tilde{M}^\vee}/Z^\circ_{\tilde{G}^\vee} \arrow[r] \arrow[d] & Z^{\check{\beta}}_{(M^!)^\vee}/Z_{G^![s]^\vee} \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K_2 \arrow[r] \arrow[d] & Z^{\hat{\alpha}^*}_{\tilde{M}^\vee}/Z^\circ_{\underline{\tilde{M}}^\vee} \arrow[r] \arrow[d] & Z^{\check{\beta}}_{(M^!)^\vee}/Z_{(\underline{M}^!)^\vee} \arrow[r] \arrow[d] & C_2 \arrow[r] & 1 \\
 & 1 & 1 & 1 & &
\end{tikzcd}$$

where K_2, K_3 (resp. C_1, C_2) are defined to be the kernels (resp. cokernels); they are all finite. The second and the third columns are readily seen to be exact, hence so is the first column by the Snake Lemma.

Next, observe that $|K_3|^{-1} = i_{\underline{M}^!}(\tilde{G}, G^![s])$ as in the case of $|K_1|^{-1}$. Hence

$$\begin{aligned}
i_{M^!}(\tilde{G}, G^![s]) &= |K_1|^{-1} = |K_1'|^{-1}(K_1 : K_1')^{-1} \\
&= |K_2|^{-1}|K_3|^{-1}(K_1 : K_1')^{-1} \\
&= |K_2|^{-1} i_{\underline{M}^!}(\tilde{G}, G^![s])(K_1 : K_1')^{-1}.
\end{aligned}$$

It remains to prove that

$$|K_2|(K_1 : K_1') = \left(Z^{\hat{\alpha}^*}_{\tilde{M}^\vee} : Z^\circ_{\underline{\tilde{M}}^\vee}\right) i_{M^!}^{G^![s]}(\epsilon[s]) \cdot \frac{\|\check{\beta}\|}{\|\check{\alpha}\|}.$$

Using the third row of the diagram and (**??**), we see $|K_2| = \left(Z^{\hat{\alpha}^*}_{\tilde{M}^\vee} : Z^\circ_{\underline{\tilde{M}}^\vee}\right) i^{G^![s]}_{M^!}(\epsilon[s])|C_2|$, thus we are reduced to proving

$$|C_2|(K_1 : K_1') = \frac{\|\check{\beta}\|}{\|\check{\alpha}\|}.$$

When α is short, we have seen that $K_1 = K_1'$, $\|\check{\alpha}\| = \|\check{\beta}\|$ whilst $C_2 = \{1\}$ (upon replacing $\tilde{G}$ by $\underline{\tilde{M}}$). The required equality follows at once.

Hereafter, suppose that α is long. Write $\tilde{G} = \prod_{i\in I} \mathrm{GL}(n_i) \times \widetilde{\mathrm{Sp}}(2n)$. Without loss of generality, we may express $\alpha = 2\epsilon_i$ under the usual basis for $\mathrm{Sp}(2n)$. The index i must fall under a GL-factor of M that embeds into $\mathrm{Sp}(2n)$. Moreover, $\check{\alpha} = \check{\epsilon}_i$ and $\check{\beta} = 2\check{\epsilon}_i$. It is clear that

$$Z^\circ_{\tilde{M}^\vee} = Z^\circ_{(M^!)^\vee}, \quad Z^{\hat{\alpha}^*}_{\tilde{M}^\vee} = Z^\circ_{\underline{\tilde{M}}^\vee} = Z^\circ_{(\underline{M}^!)^\vee},$$
$$(K_1 : K_1') = \left(Z^\circ_{(M^!)^\vee} \cap Z_{G^![s]^\vee} : Z^\circ_{(\underline{M}^!)^\vee} \cap Z_{G^![s]^\vee}\right).$$

Thus it remains to verify in this case that

$$\left(Z^{\check{\beta}}_{(M^!)^\vee} : Z_{(\underline{M}^!)^\vee}\right)\left(Z^\circ_{(M^!)^\vee} \cap Z_{G^![s]^\vee} : Z^\circ_{(\underline{M}^!)^\vee} \cap Z_{G^![s]^\vee}\right) = 2. \tag{1.4}$$

Observe that (1.4) involves only the objects on the endoscopic side. We may write

$$G^![s] = \prod_{i\in I} \mathrm{GL}(n_i) \times \mathrm{SO}(2n'+1) \times \mathrm{SO}(2n''+1), \quad n' + n'' = n.$$

The first step is to reduce to the case $I = \emptyset$ and $n'' = 0$. Indeed, $M^!$ and $\underline{M}^!$ decompose accordingly, and the construction of $\underline{M}^!$ takes place inside either $\mathrm{SO}(2n'+1)$ or $\mathrm{SO}(2n''+1)$, on which β lives. Hence we may rename $G^![s]$ to $G^!$ and assume $G^! = \mathrm{SO}(2n+1)$.

Accordingly, we can write $M^! = \prod_{j=1}^k \mathrm{GL}(m_j) \times \mathrm{SO}(2m+1)$ where $m \in \mathbb{Z}_{\geq 0}$, such that $\check{\beta}$ factors through the dual of $\mathrm{GL}(m_1)$, so that $\underline{M}^!$ is obtained by merging $\mathrm{GL}(m_1)$ with $\mathrm{SO}(2m'+1)$ to form a larger Levi subgroup of $G^!$.

- Suppose $m = 0$, then the first index in (1.4) is 1 since

$$Z^{\check{\beta}}_{(M^!)^\vee} = \{\pm 1\} \times \prod_{j\geq 2} \mathbb{C}^\times = Z_{(\underline{M}^!)^\vee}.$$

  On the other hand, $Z^\circ_{(M^!)^\vee} \cap Z_{(G^!)^\vee} = Z_{(G^!)^\vee} = \{\pm 1\}$ and $Z^\circ_{(\underline{M}^!)^\vee} \cap Z_{(G^!)^\vee} = \{1\}$, so the second index is 2. Hence (1.4) is verified.

- Suppose $m \geq 1$, then

$$Z^{\check{\beta}}_{(M^!)^\vee} = \{\pm 1\} \times \prod_{j\geq 2} \mathbb{C}^\times \times \{\pm 1\},$$

whilst $Z_{(\underline{M}^!)^\vee}$ has only one $\{\pm 1\}$-factor diagonally embedded, so the first index is 2. On the other hand,

$$Z^\circ_{(M^!)^\vee} = \prod_{j\geq 1} \mathbb{C}^\times \times \{1\}$$

intersects trivially with $Z_{(G^!)^\vee} \simeq \{\pm 1\}$, so the second index is 1. Again, (1.4) is verified.

Summing up, the case of long roots is completed. ■

Definition 1.1.2:

In view of Proposition **??**, we may define the *collective geometric transfer* $\mathcal{T}^{\mathcal{E}}$ as

$$\begin{array}{ccc} \mathcal{I}_{-}(\tilde{G}) \otimes \mathrm{mes}(G) & \xrightarrow{\mathcal{T}^{\mathcal{E}}} & \mathcal{I}^{\mathcal{E}}(\tilde{G}) \\ \cup & & \cup \\ \mathcal{I}_{-,\mathrm{cusp}}(\tilde{G}) \otimes \mathrm{mes}(G) & \xrightarrow[\mathcal{T}^{\mathcal{E}}_{\mathrm{cusp}}]{} & \mathcal{I}^{\mathcal{E}}_{\mathrm{cusp}}(\tilde{G}) \end{array}$$

mapping f to $\left(\mathcal{T}_{\mathbf{G}^!,\tilde{G}}(f)\right)_{\mathbf{G}^! \in \mathcal{E}_{\mathrm{ell}}(\tilde{G})}$. When F is Archimedean, it is continuous and restricts to

$$\mathcal{T}_{\mathbf{G}^!,\tilde{G}} : \mathcal{I}_{-}(\tilde{G}, \tilde{K}) \otimes \mathrm{mes}(G) \to \mathcal{I}^{\mathcal{E}}(\tilde{G}, \tilde{K});$$

ditto for the case with subscripts "cusp" (see Theorem **??**).

Taking transpose yields the collective transfer of distributions

$$\check{\mathcal{T}}^{\mathcal{E}} : \bigoplus_{\mathbf{G}^! \in \mathcal{E}_{\mathrm{ell}}(\tilde{G})} SD(G^!) \otimes \mathrm{mes}(G^!)^\vee \to D_{-}(\tilde{G}) \otimes \mathrm{mes}(G)^\vee.$$

These notions extend immediately to groups of metaplectic type.

Example 1

This is an example.

Hint. There is No solution.

Solution. test. ■

Proposition 1: Dirichlet BVP on the Upper Half Plane.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and both $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$ exist and are finite. Then $u : \mathbb{H} \to \mathbb{R}$ define by the integral

$$u(z) = \mathrm{Re}(\frac{1}{\pi i}) \int_{-\infty}^{+\infty} \frac{f(t)}{t-z} \mathrm{d}t,$$

is the unique solution to the Dirichlet BVP:

$$\nabla^2 u = 0 \quad in \quad \mathbb{H} \quad u = f \quad on \quad \partial\mathbb{H}$$

Proof. For any $z_0 \in \mathbb{H}$, consider the biholomorphism

$$\psi(z) = \frac{z - z_0}{z - \bar{z}_0}$$

which maps $\mathbb{H}$ onto $\mathbb{D}, (-\infty, +\infty)$ onto $\partial\mathbb{D}\backslash\{1\}$, and z_0 to 0 . Then $\psi \circ f$ is continuous on $\partial\mathbb{D}\backslash\{1\}$ and bounded on $\partial\mathbb{D}$. By Theorem 4.79 and the remark after it, there exists a unique bounded harmonic function $v : \mathbb{D} \to \mathbb{R}$ such that $v = \psi \circ f$ on $\partial\mathbb{D}\backslash\{1\}$. Hence $u := \psi^{-1} \circ v$ is the unique solution to the Dirichlet BVP on $\mathbb{H}$. Now we turn to the computation of the explicit formula of the solution. By mean value property we have

$$v(0) = \frac{1}{2\pi} \int_0^{2\pi} v\left(e^{i\theta}\right) d\theta$$

Since ψ maps $(-\infty, +\infty)$ onto $\partial\mathbb{D}\backslash\{1\}$,

$$e^{i\theta} = \frac{t - z_0}{t - \bar{z}_0} \Rightarrow \theta = -i\log\left(\frac{t - z_0}{t - \bar{z}_0}\right)$$

Take the differential:

$$d\theta = -i\frac{t - \bar{z}_0}{t - z_0} \cdot \left(-\frac{\bar{z}_0 - z_0}{(t - \bar{z}_0)^2}\right) dt = i\frac{\bar{z}_0 - z_0}{(t - z_0)(t - \bar{z}_0)} dt = \frac{2\operatorname{Im} z_0}{t^2 - 2t\operatorname{Re} z_0 + |z_0|^2} dt = \operatorname{Re}\left(\frac{2}{i(t - z_0)}\right) dt$$

Since $v(0) = u(z_0)$, we have

$$u(z_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) \operatorname{Re}\left(\frac{2}{i(t - z_0)}\right) dt = \operatorname{Re}\left(\frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{f(t)}{t - z} dt\right)$$

, Therefore the proposition is proved.

■

Property 1.1.1

If $x \in$ open set V then $\exists\, \delta > 0$ such that $B_\delta(x) \subset V$

***Claim* 1.** THIS IS CLAIM.

Exercise 1.1.1

This is The Exercise of the Textbook

***Claim* 2.** the propostion is Trival.

Proof of claim. start. ■

Chapter 2

Chapter

Here

2.1 Section

Definition 2.1.1: The title

Let $\mathscr{O}$ be a finite union of semisimple conjugacy classes in $M(F)$. Define

$$D_{\mathrm{geom},-}(\tilde{M},\mathscr{O}) := \left\{D \in D_-(\tilde{M}) : \tilde{\gamma} \in \mathrm{Supp}(D) \implies \gamma_{\mathrm{ss}} \in \mathscr{O}\right\},$$

$$D_{\mathrm{geom},-}(\tilde{M}) := \bigoplus_{\substack{\mathscr{O}\subset M(F)\\ \text{ss. conj. class}}} D_{\mathrm{geom},-}(\tilde{M},\mathscr{O}) \subset D_-(\tilde{M}),$$

$$D_{\mathrm{unip},-}(\tilde{M}) := D_{\mathrm{geom},-}(\tilde{M},\{1\}).$$