Unawareness Premia

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Abstract

This paper considers the effect on asset prices of investors contemplating the possible occurrence of unexpected and unprecedented events that they have no basis to evaluate. We build a Capital Asset Pricing Model (CAPM) where, in addition to regular risk, investors are aware that they are potentially unaware of some events. We show that when investors feel that there exist states about which they are unaware, asset prices contain an unawareness premium. A driving force is that the "risk free" asset is no longer considered to be truly risk free. We develop a methodology that enables us to estimate the systematic portion of the unawareness premium, and we estimate it using daily data from 1980 to 2021. This unawareness premium implies a theoretical motivation behind the correlation between estimated asset alphas and betas in the cross section. We find evidence in support of the hypothesis that unawareness, in addition to risk, is a determinant of expected equity returns. This additional factor adds insights into asset market behavior around market run ups like those during the dotcom boom and the pre-financial crisis market outperformance.

JEL classification: G41, D83, G12.
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1 Introduction

This paper considers the effect on asset prices of investors contemplating the possibility of unexpected and unprecedented events about which their tools for reasoning are limited. In other words, investors are cognizant of their potential ignorance, or aware of their potential unawareness, of such events. We build a Capital Asset Pricing Model (CAPM) where investors have such awareness of their potential unawareness of some events and investigate the interplay between risk and unawareness. We use the predictions from our CAPM model to develop a theoretically founded methodology that enables us to investigate empirically whether awareness of unawareness is reflected in market variables. We find evidence that unawareness is indeed relevant for asset pricing.

A recent unprecedented and unanticipated event was the COVID-19 pandemic, which had new and unexpected effects on individuals and businesses. Currently, there is debate about the unknown effects of the increasing use of artificial intelligence.[1] A popular term is that of a "Black Swan" event: an unpredictable event that is beyond what is normally expected (Taleb, 2010). Awareness of the possibility of such events has potential behavioral implications in many situations of uncertainty, including finance. If investors' expectations respond differently in the face of new and unfamiliar discoveries than to more familiar forms of resolutions of uncertainty, it raises the question whether awareness of the possibility of unexpected events constitutes a separate factor that is priced by the market.

Our CAPM model shows how accounting for unawareness and investors' cognizance thereof changes the risk-return relationship. The model predicts that asset prices contain an unawareness premium, in addition to the usual risk premium. We develop a technique that allows us to estimate the systematic part of this unawareness premium, termed the implied systematic unawareness premium, using standard returns data from a sample of individual

[1] Other situations are as wide-ranging as warfare, climate change, and R&D.

stocks.

In our CAPM model, investors believe that something completely unexpected may happen. There is a “normal” regime, which represents business as usual and an “unknown” regime which represents uncertainties about which investors have no experience. In the normal regime, investors believe that they know the distribution of returns (perhaps because they have observed historical data, or have received some other source of information). In the unknown regime, however, investors must rely on an entirely subjective perception about how assets will fare.

A driving force of our results is that because investors entertain that unanticipated and unprecedented events may occur, they do not consider the “risk-free” asset to be truly risk-free. While it is indeed risk-free in the normal regime, it is exposed to uncertainties in the unknown regime. Investors’ expectations about what will happen to the “risk-free” asset in the unknown regime is entirely subjective, just as it is the case for the risky assets. In unprecedented situations of surprise, investors may, for example, doubt that the government will honor their treasury bill payments. This lack of faith, that the “risk-free” asset is indeed risk-free, has behavioral implications.

Our theoretical model predicts that awareness of the potential presence of the unknown regime changes the observed risk-return relationship. Investors’ subjective attitudes toward the unexpected, i.e., how fearful or excited they are about the unknown, either dampens or amplifies how much excess return is required to compensate for increased risk in the normal regime. We obtain a formula for the security market line with unawareness that includes a *systematic unawareness premium*, abbreviated SUP. The SUP is the premium investors would be willing to pay, were it possible, to exchange the market portfolio that is exposed to the unknown uncertainties, for a market portfolio in a world where the probability of the unknown regime is zero. It can be interpreted as the premium investors would pay to live in a world where the “risk-free” asset is indeed risk-free.

Our main question is the empirical pricing question of whether unawareness, and cognizance thereof, affects stock market returns. In other words, do investors consider the degree of exposure to unawareness when they price financial assets. Our hypothesis is that the equity premium contains both a risk premium and an unawareness premium. We ask a number of questions under the umbrella of our main question. One is whether unawareness is a separate factor that is priced by the market. A second is, if so, how does this factor interact with other factors. Another is whether the SUP changes over time and if so, what the determinants of these changes are.

Our model builds on Karni and Vierø (2017), who provide a framework that allows decision makers to be aware that their awareness may expand. In other words, investors are aware that they may encounter surprise events. Karni and Vierø (2017) provide a generalised expected utility representation of preferences for such decision makers. The generalised expected utility representation extends the usual expected utility with an extra parameter that captures the decision maker's attitude towards the unknown. The higher the value of this parameter, the more excited is the decision maker about the unknown. Alternatively, the lower the value of the parameter, the more fearful is the decision maker towards the unknown.[2]

Our results show that we have identified a new pricing factor, the SUP, which is theoretically founded in the literature on awareness of unawareness. We compare our results to results in a paper on ambiguity and asset pricing by Brenner and Izhakian (2018) and also relate the SUP to other asset pricing factors. We do not find evidence that the SUP is a proxy for previously studied factors. We also show that the SUP varies substantially over time and relate the variation to macro events.

[2]Individuals' unawareness of some relevant aspects of a decision making situation, and their cognizance thereof, has been studied more broadly in decision theory as well as in some experimental work. Papers include Walker and Dietz (2011), Alon (2015), Grant and Quiggin (2015), Piermont (2017), Karni and Vierø (2013, 2015, 2017), Kochov (2018), Karni and Vierø (2021), Vierø (2021, 2022), Board and Chung (2007), Walker (2014), and Halpern and Rêgo (2009a,b). An overview is given by Schipper (2012).

While it still may be the workhorse in finance, several papers, far too many to mention in fact, have tested and rejected the standard CAPM (for a review see, e.g., Fama and French (2004)). This has been done by testing the implications of the CAPM, e.g., that expected returns on all assets are linearly related to their betas and that no other variable has (marginal) explanatory power, in cross-sectional or time-series regressions. More recently, a number of anomalies have been found that are inconsistent with the CAPM. Most notable, the literature has documented that there is a size effect and a value effect among other systematic pricing effects found empirically.[3]

An obvious criticism of the CAPM is that it is based on unrealistic assumptions. One key assumption is that investors only care about the mean and variance of one-period portfolio returns. Naturally, to explain the empirical contradictions of the CAPM more complicated asset pricing models have been considered. For example, the intertemporal capital asset pricing model (ICAPM) of Merton (1973) is a natural extension of the CAPM. It shows more generally that investors consider how their future wealth might vary with future state variables, and that they are concerned with the covariance of the portfolio returns with these state variables generating a multi-factor type model. While the ideal implementation of this model would be to directly specify the state variables that affect expected returns, Fama and French (1993) among others take a more indirect approach, and essentially argue that such anomalies can be used to create portfolios that proxy for unidentified state variables giving rise to, e.g., their famous three-factor model. As long as these portfolios are well diversified and sufficiently different from the market portfolio, adding them to the model is in the spirit of both the ICAPM and the Arbitrage Pricing Theory of Ross (1976).

Instead of adding additional portfolios other than the market to the model in the spirit

[3]Banz (1981) record that when stocks are sorted based on market capitalization, average returns are decreasing in market capitalization–contrary to the predictions of the CAPM. Rosenberg et al. (1985) show that book-to-market is a factor that has explanatory power in stock returns, even when controlling for a stock's β.

of, e.g., Fama and French (1993), Fama and French (2012), Fama and French (2015), and Carhart (1997), another branch of the literature has argued that the problem lies in the limits of covariance risk to fully represent systematic risk and not in the choice of the market or non-market portfolios. This was first proposed in Kraus and Litzenberger (1976) who introduced skewness into the CAPM and Dittmar (2002) who introduced the fourth moment, kurtosis, to the model.[4] The motivation for introducing higher order moments is twofold: 1) there is ample evidence that asset returns, both individually and at the portfolio level, exhibit non-normal distributions with (large) negative skewness and excess kurtosis, and 2) there is experimental evidence that investors like positive skewness and dislike extreme losses. Finally, one can combine these two approaches in models that extend the CAPM in both the moment and the factor dimension, see, e.g., Vendrame et al. (2016).

Theoretically, it is well known that the CAPM may hold conditionally, period by period, even though stocks are mispriced by the unconditional CAPM (e.g., Jensen, 1968; Dybvig and Ross, 1985; Jagannathan and Wang, 1996). This consideration has inspired a large literature using conditional formulations of the asset pricing models, for example with time-varying betas.[5] However, Lewellen and Nagel (2006) argue that time-variation in risk and expected returns cannot explain why the unconditional CAPM fails. They show that a stock's conditional alpha (or pricing error) might be zero, even when its unconditional alpha is not, if its beta changes through time and is correlated with the equity premium or with market volatility. However, they find that there is simply not enough variability to explain the large alphas found empirically.

Our model is theoretically founded in the literature on awareness of unawareness and

[4]Unlike the multi-factor single moment models, multi-moments models are appealing because they can be grounded in theory. However, other moment-based extensions, that are not directly based on utility maximization, have also been put forth, see, e.g., Ang et al. (2006).

[5]Studies on the conditional CAPM include Zhang (2005), Jagannathan and Wang (1996), Lettau and Ludvigson (2001), Santos and Veronesi (2006), Lustig and Van Nieuwerburgh (2005), Wang (2003), Adrian and Franzoni (2005), Ang and Chen (2006), and Petkova and Zhang (2005). Conditional specifications of moment-based models have also been considered, see e.g. Harvey and Siddique (2000).

is thus closer in spirit to the moment-based models than the factor based models above. Moreover, our model implies that alphas of individual stocks are non-zero and, consistent with the model, asset alphas could be time-varying as investors' awareness of unawareness and corresponding attitude towards the unknown changes. Finally, because of the particular structure of the model, all systematically important elements and parameters can be estimated with a simple two step method which is straightforward to implement using nothing but simple linear regressions on daily (excess) returns.

Thus, the model gives a theoretical justification for the empirical strategy that we employ. This empirical strategy is relatively simple – it involves calculating the cross-sectional covariance between alphas and betas. The important contribution of the model is that it allows us to interpret this calculation in a behaviorally meaningful way. That covariance is the premium (or discount) that traders assess to the possibility of the unknown state occuring.

Theoretical analyses of asset pricing under awareness of unawareness or unforeseen contingencies can be found in Kraus and Sagi (2006), Vierø (2021), and Madotto and Severino (2022). In Kraus and Sagi (2006), heterogeneous agents repeatedly trade Arrow securities. Unforeseen contingencies materialize in the form of taste shocks that vary over micro states, which are states that are not publicly observable and thus cannot be hedged. In Vierø (2021), a representative agent trades Lucas trees that may yield previously unknown payoffs. The agent uses her best subjective guess about the assets' future payoffs in unknown states. While the two above-mentioned papers derive pricing kernels in a dynamic context, our goal is different. Our goal is to provide an econometrically manageable model, based on which we can pursue an empirical investigation of unawareness as a pricing factor. Madotto and Severino (2022) consider a model of asset pricing with market makers à la Kyle (1985), where investors are unaware of some negative asset values and also aware of their unawareness. They focus on how unawareness affects information acquisition.

The paper is organized as follows. Section 2 introduces and analyses our theoretical model to arrive at a comparative statics analysis that forms the foundation for our empirical strategy. Section 3 explains our empirical strategy and presents the data. Section 4 presents the empirical results. Section 5 investigates the relationship between our SUP and other asset pricing factors. Section 6 concludes. The Appendix contains results that further motivates our empirical approach, by providing a Monte Carlo study that our strategy indeed correctly identifies the SUP.

2 A capital asset pricing model with unawareness

In our CAPM with unawareness, investors entertain the possibility that something completely unexpected may happen. As a result, investors operate with two regimes. There is a "normal" regime, which represents "business as usual." For the normal regime, investors can look at historical data to guide their expectations about the future. The "unknown" regime represents uncertainties with which investors have no experience. There may or may not, in fact, be such a regime. The important thing is that investors think there is. Investors assign probability p to the normal regime, and probability $1-p$ to the unknown regime.

There are N risky assets and one additional asset, which we will refer to as the zero-beta asset. In the normal regime, each risky asset i has a known mean and variance. Let $\boldsymbol{\mu}$ represent the vector of expected returns of the risky assets in the normal regime and let Σ be the covariance matrix for the normal regime. Let r_f denote the return on the zero-beta asset, and let $\boldsymbol{r_f} = r_f\mathbf{1}$ denote the constant vector of this return. The zero-beta asset is thus risk-free in the normal regime. Investors are assumed to have mean-variance preferences in the normal regime.

Consider an investor facing a portfolio decision. In the unknown regime, the investor cannot calculate the means, variances and covariances of his portfolio's assets. Instead, he only has a subjective assessment of each asset i's value in the unknown state. This value is not (cannot be) based on historical returns or other observations, since there are, by definition, no observations from the unknown regime. The investor's subjective value of asset i, referred to as his unawareness attitude toward asset i, is denoted u_i and is expressed in the same units as the expected return. A high value of u_i reflects a sense of the investor that asset i will do well in the unknown regime, while a low value reflects a sense that asset i will do poorly. Let $\boldsymbol{u}$ denote the vector of unawareness attitudes toward the risky assets. Let u_f denote the unawareness attitude toward the zero-beta asset, and let $\boldsymbol{u_f} = u_f\mathbf{1}$ denote the constant vector of this attitude.

Notice that since we do not require that $u_f = r_f$, the zero-beta asset is not truly risk-free. While it is indeed risk-free within the normal regime, it may be exposed to risk, should the unknown regime occur. However, this risk is uncorrelated with the market portfolio risk in the normal regime. Hence the name "zero-beta asset" rather than "risk-free asset." The feature that the "risk-free" asset is not truly risk-free is an important driver of our results.

Let x_i be the fraction of wealth invested in asset i and let $\boldsymbol{x}$ denote the vector of these portfolio weights. We model preferences based on Karni and Vierø (2017), where expected utility is generalized with the agent's attitude toward unawareness. Then, investors have the following generalized expected utility preferences, with mean-variance preferences in the normal regime:

$$U(x) = p\left(r_f + \boldsymbol{x'}(\boldsymbol{\mu} - \boldsymbol{r_f}) - \frac{A}{2}\boldsymbol{x'}\Sigma\boldsymbol{x}\right) + (1-p)\left(u_f + \boldsymbol{x'}(\boldsymbol{u} - \boldsymbol{u_f})\right). \tag{1}$$

The investor chooses a portfolio $\boldsymbol{x}$ to maximize the utility in Equation (1). The first order conditions imply an optimal portfolio of

$$\boldsymbol{x} = \frac{1}{A}\Sigma^{-1}\left(\boldsymbol{\mu} - \boldsymbol{r_f} + \frac{1-p}{p}(\boldsymbol{u} - \boldsymbol{u_f})\right). \tag{2}$$

We assume that all investors agree on $\boldsymbol{u}$ and u_f and are price takers. Let $\sigma_{im} = \Sigma_i \boldsymbol{x}$ denote the covariance in the known regime between the return to asset i and the rest of the risky portfolio.[6] Thus, Equation (2) implies that across assets i and j

$$\frac{\mu_i - r_f + \frac{1-p}{p}(u_i - u_f)}{\sigma_{im}} = \frac{\mu_j - r_f + \frac{1-p}{p}(u_j - u_f)}{\sigma_{jm}} = A. \tag{3}$$

This reflects that as usual the expected return to risk tradeoff must be the same for all assets. However, what constitutes the expected return is different than usual due to the presence of the unknown regime. The expected return for each asset depends on the subjective assessment of how that asset will fare in the unknown regime, compared to the zero-beta asset.

In equilibrium, all individuals have identical portfolios, so the portfolio weights $\boldsymbol{x}$ will be identical across individuals. Furthermore, net borrowing is zero, so $\mathbf{1}'\boldsymbol{x} = 1$. This implies that the expected return to risk tradeoff for the risky market portfolio as a whole is also identical across individuals. That is, letting $\mu_m = \boldsymbol{x}'\boldsymbol{\mu}$ denote the market-weighted average of returns in the known regime and σ_m^2 the variance of the market portfolio in the known regime, we have that

$$\frac{\mu_m - r_f + \frac{1-p}{p}\boldsymbol{x}'(\boldsymbol{u} - \boldsymbol{u_f})}{\sigma_m^2} = A. \tag{4}$$

Equations (3) and (4) give

$$\frac{\mu_i - r_f + \frac{1-p}{p}(u_i - u_f)}{\sigma_{im}} = \frac{\mu_m - r_f + \frac{1-p}{p}\boldsymbol{x}'(\boldsymbol{u} - \boldsymbol{u_f})}{\sigma_m^2} = A, \tag{5}$$

which has the traditional interpretation from the CAPM: the expected-return-to-risk tradeoff for each individual asset i must be equal to the expected-return-to-risk tradeoff for the market

[6] The inner product of the i-th row of Σ and $\boldsymbol{x}$ is by definition the covariance in the known regime between the return to asset i and the rest of the risky portfolio.

as a whole. Notice that the expected return for each asset depends on the unawareness attitude towards that asset, using the zero-beta asset as a benchmark. Likewise, the return for the market depends on the average unawareness attitude towards the unknown regime, again using the attitude toward the zero-beta asset as a benchmark.[7]

For what follows, let

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2}, \tag{6}$$

and

$$\overline{u} = \boldsymbol{x'}(\boldsymbol{u} - \boldsymbol{u_f}). \tag{7}$$

Hence, $\overline{u}$ is the market-weighted average of asset unawareness attitudes in the unknown regime, in excess of the zero-beta asset. It can be interpreted as the average optimism/pessimism about how risky assets will do relative to the zero-beta asset in the unknown regime. Using these notations, Equation (5) can be written as

$$\mu_i - r_f = \beta_i\left(\mu_m - r_f\right) + \beta_i\frac{1-p}{p}\overline{u} - \frac{1-p}{p}(u_i - u_f). \tag{8}$$

Equation (8) shows that with awareness of unawareness, an asset's return premium is determined by three distinct phenomena, represented by the three terms on the right hand side. The first of these terms, $\beta_i(\mu_m - r_f)$, is the typical risk term from the CAPM. It represents the risk in known states of asset i relative to the risk of the market. Plotted as a function of β, this risk traces out the traditional securities market line (SML).

[7] To understand the intuition here, suppose that *ceteris paribus*, u_i increases marginally, by Δ_i. This will increase the reward for holding asset i relative to the risk of holding that asset. It will also increase the reward to holding the market as a whole, but by a smaller amount, since the marginal change in the reward to the market is $\frac{1-p}{p}x_i\Delta_i$ and the marginal change in the reward to asset i is $\frac{1-p}{p}\Delta_i$. Here we assume that $(1-p)/p$ is relatively small. As such, if the LHS and RHS were equal before the change, then ignoring changes in composition of the market portfolio, they would now be unequal, with the LHS now bigger than the RHS, and both sides being larger than A. Thus, there will be a change in the composition of the market portfolio (since the LHS is greater than A) as well as a change in the risk associated with the market portfolio (since the RHS is larger than A). The change in the LHS occurs in the traditional way, with x_i increasing, which increases the risk associated with asset i.

The second determinant of the premium is systematic unawareness risk. The premium associated with this risk is given by the second term $\beta_i \frac{1-p}{p}\overline{u}$. We call the quantity $\frac{1-p}{p}\overline{u}$ the *systematic unawareness premium* (SUP) for asset i. This premium depends on two things. First, it depends on how likely the unknown regime is relative to the known regime. This is captured by the ratio $(1-p)/p$. The other is the average unawareness attitude towards risky assets in excess of the zero-beta asset, $\overline{u}$. Both the likelihood and the attitude are subjective and exogenous. Hence, with unawareness the equity premium to an asset i depends on the asset's exposure to systematic risk in known states as well as on its subjectively perceived exposure to the unknown regime.

The final determinant of the premium is an idiosyncratic unawareness premium. We assume that the unknown regime is sufficiently vague as to preclude the writing of contracts over this regime. Therefore, this form of idiosyncratic risk is not diversifiable.

The introduction of unawareness changes the systematic risk-return relationship. Recall that the perception of the rate of return in the unknown regime is purely subjective, reflecting the investor's subjective view about how risky assets will perform relative to the zero-beta asset in the unknown regime as well as how likely he thinks the unknown regime is. Both the subjective unawareness attitude and the subjective likelihood of the unknown regime are exogenous rather than determined in equilibrium. Therefore, only the expected rate of return in the normal regime, μ_i, adjusts as prices adjust in equilibrium. This rate of return μ_i in the normal regime is the one that is observed. In contrast, the expected average excess rate of return (compared to the zero-beta asset) in the unknown regime, or systematic unawareness premium, $\frac{1-p}{p}\overline{u}$, is not directly observable.

Define $\phi_i = u_i - \boldsymbol{x'u}$, which captures idiosyncratic exposure in the unknown regime over that of the market portfolio. Then Equation (8) can be written as

$$\mu_i - r_f = \beta_i (\mu_m - r_f) + (\beta_i - 1)\frac{1-p}{p}\overline{u} - \frac{1-p}{p}\phi_i. \tag{9}$$

If the investor has the same unawareness attitude toward all risky assets, then $\phi_i = 0$ for all i. We do not impose this *a priori*. In our empirical analysis, we do robustness checks regarding the idiosyncratic exposure by including dummy variables for sectors in our estimation procedure.

The first two terms on the right-hand side of Equation (9) define the security market line with unawareness, that is, the systematic risk-return relationship under unawareness:

$$\mu_i - r_f = \beta_i \left(\mu_m - r_f\right) + (\beta_i - 1)\frac{1-p}{p}\bar{u}. \tag{SMLU}$$

Equation (SMLU) shows that an asset's systematic return premium on the LHS consists of two components: A systematic risk component and a systematic unawareness component.

Taking the derivative of μ_i with respect to β_i in Equation (SMLU) yields

$$\frac{d\mu_i}{d\beta_i} = \mu_m - r_f + \frac{1-p}{p}\bar{u}.$$

If $\frac{1-p}{p}\bar{u}$ is positive, investors are more optimistic about risky assets (on average) than about the zero-beta asset with regard to uncertainties with which they have no experience. This could arise from investors losing confidence in the otherwise risk-free asset. Investors hence perceive there to be a downside risk to the zero-beta asset, and this risk is uncorrelated with the market portfolio. If an investor were to invest everything in the zero-beta asset, he would have full exposure to this downside risk. As a result, investors substitute toward the risky assets. Shifting more weight to the risky portfolio avoids more of the perceived downside risk, thus increasing expected returns more than the standard CAPM would predict. In this case, the security market line is steeper than in the traditional CAPM.

If $\beta_i \in (0, 1)$, the replicating portfolio is long in the zero-beta asset and thus perceived to be exposed to more downside risk than the normal regime suggests. Hence, investors require a lower rate of return for each particular $\beta_i \in (0,1)$. If $\beta_i > 1$, the replicating portfolio is short in the zero-beta asset and thus perceived to be exposed to more upside risk than the normal regime suggests. Hence, investors require a higher rate of return for each particular $\beta_i > 1$.

If $\frac{1-p}{p}\bar{u}$ is instead negative, investors are more pessimistic about the risky assets (on average) than about the zero-beta asset following uncertainties they have no experience with. The investors' subjective view is that the risky market portfolio will underperform relative to the zero-beta asset in the unknown state. In other words, investors perceive there to be an upside risk to the zero-beta asset, and this risk is uncorrelated with the market portfolio. When more weight is shifted to the risky portfolio the investor misses out on more of the perceived upside risk, thus increasing expected returns less than the standard CAPM would predict. In this case, the security market line is flatter than in the traditional CAPM.[8]

The security market line with unawareness in Equation (SMLU) has intercept $r_f - \frac{1-p}{p}\bar{u}$ and intersects the security market line from the traditional CAPM at $\beta = 1$. The change in intercept is due to the zero-beta asset not necessarily being truly risk free, since it may yield a different return in the unknown state than in the known state.

To set the stage for our empirical analysis, we elaborate on the intuition behind the systematic risk-return relationship with unawareness. Recall that an asset's overall expected rate of return is a weighted average of the rates of return in the known and unknown regimes, with the rate of return in the unknown regime being purely subjective and therefore not determined in equilibrium, while the expected rate of return in the normal regime, μ_i, adjusts as prices adjust in equilibrium. The rate of return μ_i in the normal regime is the one that is observed. In contrast, the subjective systematic unawareness premium, $\frac{1-p}{p}\bar{u}$, is not directly observable.

[8]If $\frac{1-p}{p}\bar{u} < -(\mu_i - r_f)$, i.e. very negative, the risk-return relationship changes sign, but we will leave that case out.

Figure 1: Systematic risk-return tradeoff by traditional CAPM and with unawareness; $\bar{u} > 0$

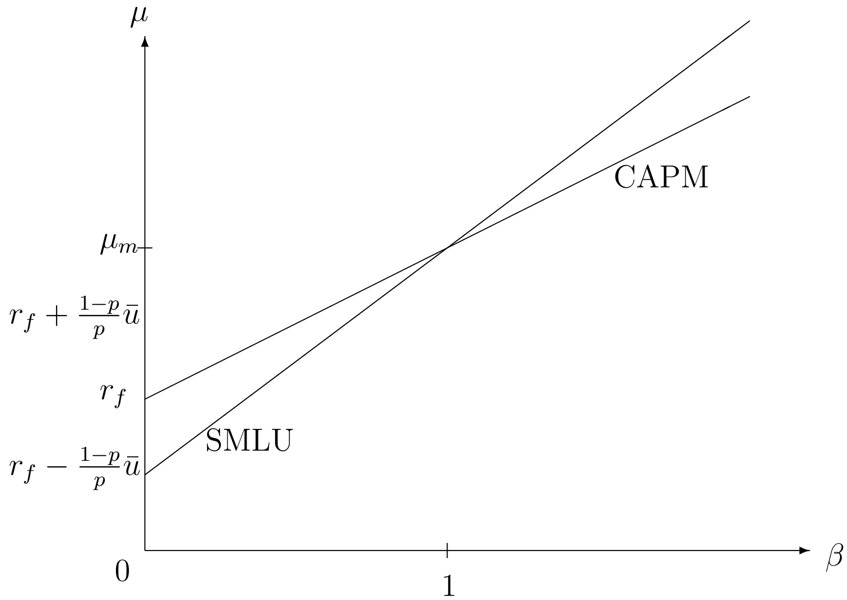
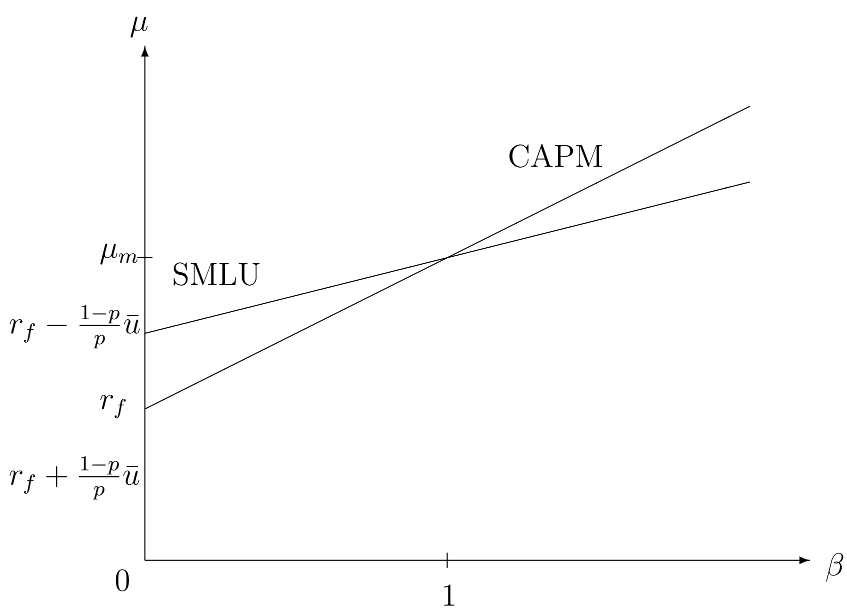


Figure 2: Systematic risk-return tradeoff by traditional CAPM and with unawareness; $\bar{u} < 0$



If $\bar{u} > 0$, investors are optimistic about the relative performance of risky assets in the unknown regime. It is as if investors perceive a positive constant to be added to the observed rates of return of all risky assets. In other words, the positive systematic unawareness premium is automatically included for all risky assets. Therefore, for each particular level of risk, risky assets are subjectively viewed to yield a higher rate of return than the observed rate of return. Thus, for stocks with $\beta > 1$, investors' expected return is higher than what the CAPM would predict given the stock's beta, suggesting that these stocks will appear to be overpriced (since the CAPM is what we observe).

Because the observed market risk premium is lower than the overall perceived equity premium, a larger adjustment of the observed premium is needed to adjust for risk and maintain the overall equilibrium trade-off. Since the part of the return that comes from the unknown state is exogenous, only the observed rate of return, μ_i, can adjust with the riskiness of an asset. The slope of the security market line with unawareness, SMLU, reflects the changed trade-off in the observed risk-return relationship. The security market line SMLU is steeper than in the traditional CAPM when $\bar{u} > 0$.

Figure 1 illustrates the observed risk-return relationship if there is no unawareness (curve labeled CAPM) and when there is unawareness (curve labeled SMLU) for $\bar{u} > 0$. If investors think that there is no unknown regime, i.e., there is no unawareness, the perceived equity premium equals the observed risk premium, while if there is unawareness, the perceived equity premium differs from the observed risk premium. Since the figure depicts the observed risk-return relationships, the curves intersect at $\beta = 1$.

If instead $\bar{u} < 0$, investors are pessimistic about the relative performance of risky assets in the unknown state. It is as if a negative constant is added to the observed rates of return of all risky assets. In other words, the (absolute value of the) systematic unawareness premium is automatically subtracted for all risky assets. For each particular level of risk, risky assets are subjectively viewed to yield a lower rate of return than the observed rate of return.

As a result, when $\bar{u} < 0$, smaller adjustments are needed in the observed risk premium in order for investors to take on further risk. The security market line SMLU, which depicts the observed risk-return tradeoff, is flatter than in the traditional CAPM when $\bar{u} < 0$. Figure 2 illustrates the observed risk-return relationship if there is no unawareness (curve labeled CAPM) and when there is unawareness for $\bar{u} < 0$. In this case, risky assets (with $\beta > 1$) will appear to be underpriced.

It is clear from the analysis above that the sign of $\bar{u}$, the performance of risky assets relative to the "risk free" (zero-beta) asset in the unknown state, is important. Due to the unknown regime, the zero-beta asset may not be truly risk free. Investors may perceive that whatever unknown uncertainties occur in the unknown regime also affects the zero-beta asset. Therefore, a positive $\bar{u}$ may be due to either high expectations about the performance of risky assets or low expectations about the performance of the zero-beta asset in the unknown regime. Consequently, a shock that changes investors' confidence in the "risk-free" asset in the unknown state changes the observed risk-return relationship.

This driving force from the zero-beta asset not being truly risk free is very different from the driving force in models of asset pricing under ambiguity. With ambiguity averse investors, there is inertia in portfolio adjustments because differently composed portfolios are evaluated with different probability distributions. As a result, aggregate risk is typically held by a subset of investors, who as a result require a higher premium than if aggregate risk was distributed on all investors.[9] This gives rise to an ambiguity premium on top of the standard risk premium. In contrast, in the current setting with unawareness, there is no inertia. Rather, it is the confidence in the risk-free asset in the unknown regime that changes the risk-return relationship, and, as we will explain in the next subsection, affects assets differently depending on their betas.

[9] See for example Easley and O'Hara (2009).

2.1 Comparative statics of changes in systematic unawareness

As a simple example, consider a market where initially individuals are pessimistic about risky assets relative to the zero-beta asset in states about which they are unaware, which implies that $\overline{u} < 0$. In this case, assets that are less risky than the market as a whole ($\beta_i < 1$) are underpriced relative to what the traditional CAPM would suggest and assets that are riskier than the market as a whole ($\beta_i > 1$) are overpriced relative to what the traditional CAPM would predict.

Suppose that due to a preference shock, possibly due to a change in policy, an economic shock, or some other reason, investors entertain a larger average unawareness premium in the unknown state so $\overline{u}$ increases. The increase in the average unawareness premium $\overline{u}$ will decrease the expected rate of return of assets i for which β_i is less than 1 and increase the expected rate of return of assets j for which $\beta_j > 1$.

The insight from this example holds regardless of the initial value of $\bar{u}$. Taking derivatives in (SMLU) gives $\frac{\partial \mu_i}{\partial \bar{u}} = (\beta_i - 1)\frac{1-p}{p}$, which is positive if $\beta_i > 1$ and negative if $\beta_i < 1$, as long as investors assign positive probability to the unknown scenario. On the other hand, $\frac{\partial \mu_i}{\partial \mu_m} = \beta_i > 0$ for all stocks with positive betas.

This implies that changes in the systematic unawareness premium have different effects on individual asset prices and expected returns than changes in the observed market risk premium $\mu_m - r_f$. Increases in the observed market risk premium increase expected returns (and decrease prices) for all assets, while changes in the average unawareness premium $\overline{u}$ affect assets differently depending on whether they are less risky ($\beta_i < 1$) or riskier ($\beta_i > 1$) than the market on average. These comparative statics results form the foundation for our empirical strategy.

2.2 Our hypotheses

We use the insights from our theoretical model to investigate the following hypotheses.

Hypothesis 1. Unawareness, in addition to risk, is a determinant of expected equity returns.

Hypothesis 2. The systematic unawareness premium varies over time.

Hypothesis 3. The systematic unawareness premium is a separate factor and not a proxy for previously studied factors.

3 Estimation method and data

Our empirical strategy centers around Equation (8). We propose a method to estimate the systematic unawareness premium $\frac{1-p}{p}\overline{u}$ using market data. By assumption, the regime about which investors are unaware has not been regularly observed. Thus, all previous observations are of the normal regime. Therefore, the observed average excess return from market data is a proxy (under rational expectations) for $\mu_i - r_f$.

Our estimation of the systematic unawareness premium proceeds in two steps. In the first step, we estimate a traditional single factor model of the form

$$\mu_i - r_f = \alpha_i + \beta_i(\mu_m - r_f), \tag{10}$$

for each asset in a cross section at a particular time t. Comparing coefficients in Equations (8) and (10), one observes that the β coefficient in each equation captures the same variation. At the same time, estimating α in Equation (10), captures variation in Equation (8) given by

$$\alpha_i = -\frac{1-p}{p}(u_i - u_f) + \beta_i \frac{1-p}{p}\overline{u}. \tag{11}$$

If we had data on α_i and β_i, we could estimate the term $\frac{1-p}{p}\overline{u}$ by regression in Equation (11).

Instead, in the second step, we take the estimates $\hat{\alpha}_i$ and $\hat{\beta}_i$ from estimation of Equation (10) and treat them as observations in a new regression given by

$$\hat{\alpha}_i = \gamma_0 + \gamma_1 \hat{\beta}_i + \epsilon_i. \tag{12}$$

The estimated coefficient $\hat{\gamma}_1$ then gives an estimate of the systematic market unawareness premium $\frac{1-p}{p}\overline{u}$.[10]

The Appendix contains results that further motivates our empirical approach, by providing a Monte Carlo study that our strategy indeed correctly identifies the systematic unawareness premium.

3.1 Data

To characterize the systematic unawareness premium we use daily returns for all CRSP tickers from January 1980 until December 2021 and estimate rolling αs and βs each end-of-month using the preceding year's worth of daily observations. Given this, our estimates of the SUP start at the beginning of 1981. At any time t, we exclude tickers that do not trade in the final month and those that have not traded for at least 100 days of the preceding year. The risk-free rate for these calculations is taken from Kenneth French's data library website and the aggregate index used is CRSP's value-weighted return with dividends.[11]

We obtain estimates of β_i and α_i for each ticker by calculating rolling estimates of the single factor model in Equation (10) using the one year window. We assign the estimates to the last month in that window. Using a trailing window is in line with the unawareness attitude being an expectation (about the unknown), since the trailing window looks at the

[10] We acknowledge that this estimation procedure is subject to measurement error in the way that has been well documented in the CAPM literature, including, for example Roll and Ross (1994).

[11] https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

past to estimate the expectation about the future. It would not make sense to use future observations to estimate an expectation about the future, because at the time of those observations, the future is already known.

We then use the resulting cross section of $\hat{\alpha}_i$s and $\hat{\beta}_i$s at time t, to estimate Equation (12) monthly. In this regression we include dummy variables by two-digit SIC classification whereby the constant term γ_0 is only restricted to be constant within industries but allowed to vary between industries. From the latter estimation, we obtain a time series of the *implied systematic unawareness premium* $\hat{\gamma}_{1t}$ that we call the SUP. The SUP is measured in the same units as the rate of return.

4 Empirical Results

Our fundamental hypothesis is that unawareness, in addition to risk, is a determinant of equity expected returns. Table 1 gives summary statistics for the SUP over this sample period compared to the monthly market excess return over the same period. As can be seen, the mean SUP value is -0.000191. Empirically, this says that estimated alphas and betas are, on average, negatively correlated. This implies that over the sample, the payoff to holding beta risk is slightly lower than the traditional CAPM would suggest. In the context of the model, this is interpreted as reflecting a mildly negative perception (on average) of the premium in the unaware state for high beta stocks. To give a sense of the relative importance of the SUP in the returns to the market, the estimated SUP is, on average, about 22% of the magnitude of the mean daily equity premium. The standard deviation of the SUP is about 5% of the standard deviation of the equity premium over this sample period. The correlation between the SUP and the realized equity premium is 0.162.

Figure 3 shows the implied systematic unawareness premium from 1981 to 2021. The figure shows that the implied systematic unawareness premium varies significantly and changes

Table 1: Estimated Systematic Unawareness Premium and realized market excess return

	SUP	$r_m - r_f$
Mean	-0.000191	0.000888
St. Dev.	0.000497	0.009486
Min	-0.001157	-0.066105
50%	-0.000295	0.000728
Max	0.002000	0.046561

Sample statistics for the SUP and the market excess return.

sign over time.

An increase in the SUP implies an increase in the systematic market risk premium. As such, the expected payoff to assets that are highly exposed to this source of risk–namely high beta assets–must be larger as seen in equation (8). Thus, these high beta assets are relatively underpriced and will have higher alphas when alpha is estimated in a way that does not compensate for the unaware state. This phenomenon leads to a higher observed correlation between α and β. If the change in the SUP arises because of a decline in u_f specifically, individuals may feel a general pessimism toward the risk-free asset in the states of the world about which they are unaware. This pessimism then leads to a similar increase in the correlation between observable asset αs and observable asset βs. Assets with high βs tend to have high αs in this setting because the decrease in u_f increases the perceived overall return to high beta assets which raises these assets' price.

Turning to the data, the large positive spike in the SUP around the 1994-95 turn of the year coincides with the Mexican Peso Crisis. The pairing of one of the first large-scale international financial crises with domestic concerns resulting from the crisis can reasonably have led to a loss of confidence in US treasury bonds relative to other equities.[12]

The large increase in the SUP surrounding the dotcom boom of the late 90s is consistent

[12] Contributing factors may be that the US president invoked emergency powers to extend a USD 20 Billion loan to Mexico, and there were concerns about a large decrease in exports and increased instability and illegal immigration.

with a world where traders are initially optimistic about the changes in technology bearing fruit in the dot com era. This generalized optimism improves the perceived risk premium (across aware and unaware states) and leads traders to price risky assets in such a way as to have increased alpha. These beliefs would lead to increasing correlation between alpha and beta over the period.

The run up in the SUP leading up to the financial crisis of 2008 is consistent with a story similar to that of the dot com boom. As the marginal trader becomes increasingly optimistic about the systematic portion of the unknown state in the run up to the crash, the SUP increases. The crash itself is consistent with a world where the marginal trader becomes pessimistic about the unknown state, leading to a decrease in the SUP and the market generally.

Our model suggests that the sentiment toward uncertainties estimated here, about which investors have no experience, may not map directly into real world activity since they represent an internal sense of the existence of unknown states of the world. However, since the theory is silent on how these internal feelings toward unknown states are formed, we explore the extent to which it is related to observable macro events. One possibility is that systematic unawareness is linked to U.S. transitions in presidential power. An individual's feelings towards the unknown may be affected by transitions in power because these transitions can lead to unknown unforeseen economic conditions that in turn lead to adverse outcomes.

Figure 4 shows the connection between the systematic unawareness premium and presidential administrations. There appears to consistently be an increase in the SUP at the time of the primaries, followed by a decrease around election time.

To summarize, our findings support Hypothesis 1, that unawareness, in addition to risk, is a determinant of expected equity returns, and Hypothesis 2, that the systematic unawareness premium varies over time.

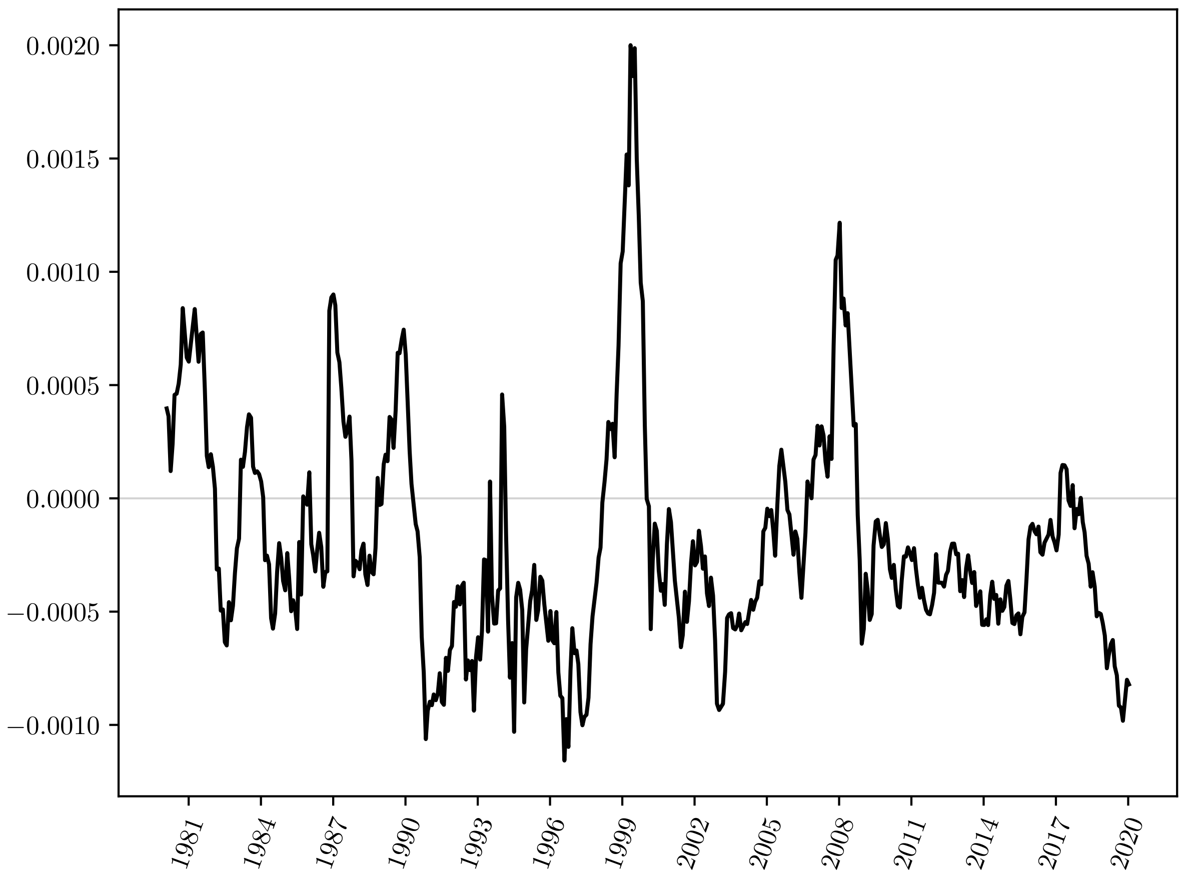


Figure 3: Estimated Systematic Unawareness Premium controlling for sector

The time series of the SUP (using the scale on the left hand side). The SUP is estimated by γ_1 in the regression in Equation (12). In this regression we include dummy variables by two-digit SIC classification whereby the constant term γ_0 is only restricted to be constant within industries but allowed to vary between industries.

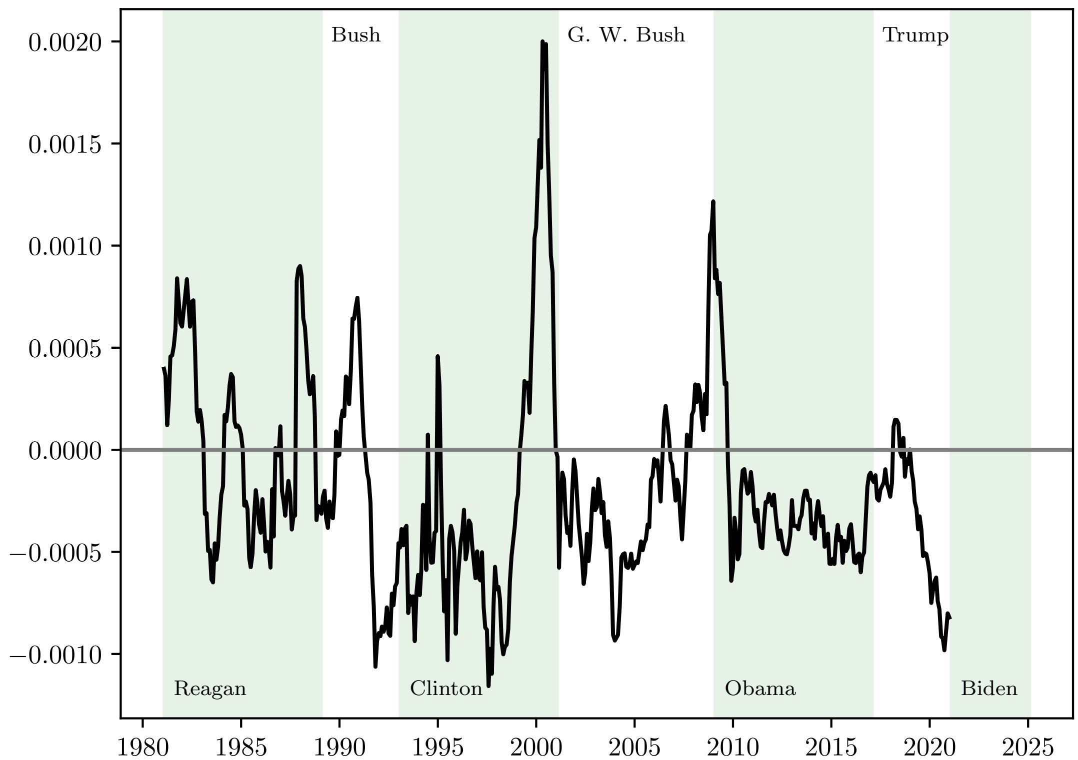


Figure 4: SUP with U.S. Presidential Administrations

The time series of the SUP (using the scale on the left hand side) with the U.S. presidential terms superimposed.

5 The systematic unawareness premium and other market risk factors

In this section we relate the systematic unawareness premium to other asset pricing factors that have been studied in the literature. The list of potential comparisons is long and we do not perform an exhaustive study of these. Instead we focus on two broad categories of interest: (1) the commonly studied factors in the Fama and French (2015) five-factor model together with momentum, and other measures of uncertainty that could be thought to be related to unawareness, including implied market volatility, as measured by the VIX, and ambiguity, as described in Brenner and Izhakian (2018).

We focus first on the ambiguity premium estimated in Brenner and Izhakian (2018). These authors estimate an uncertainty premium that includes a risk term and an ambiguity term. Figure 5 shows our implied SUP together with Brenner and Izhakian's ambiguity premium. The correlation over this sample period is $\rho = -0.048$ and there is little visual indication that these two factors are capturing the same variation. This is to be expected as the driving forces are different. In Brenner and Izhakian (2018), the driving force is ambiguity, or traders' attitudes toward very negative, but known outcomes, while in our model the driving forces are feelings towards states about which the person feels unaware, regardless of whether they are perceived to have negative outcomes, and the "risk-free" asset not being truly risk free.

Figure 6 shows the connection between the systematic unawareness premium as calculated here and the VIX index of implied volatility. As with the ambiguity premium displayed earlier, the method for calculating the SUP leads to a much slower moving number than the VIX, but there is higher correlation ($\rho = 0.31$) (in both level and magnitude) between the VIX and the SUP than between the measure of ambiguity and the SUP ($\rho = -0.048$). That being said, there are significant qualitative differences between these series.

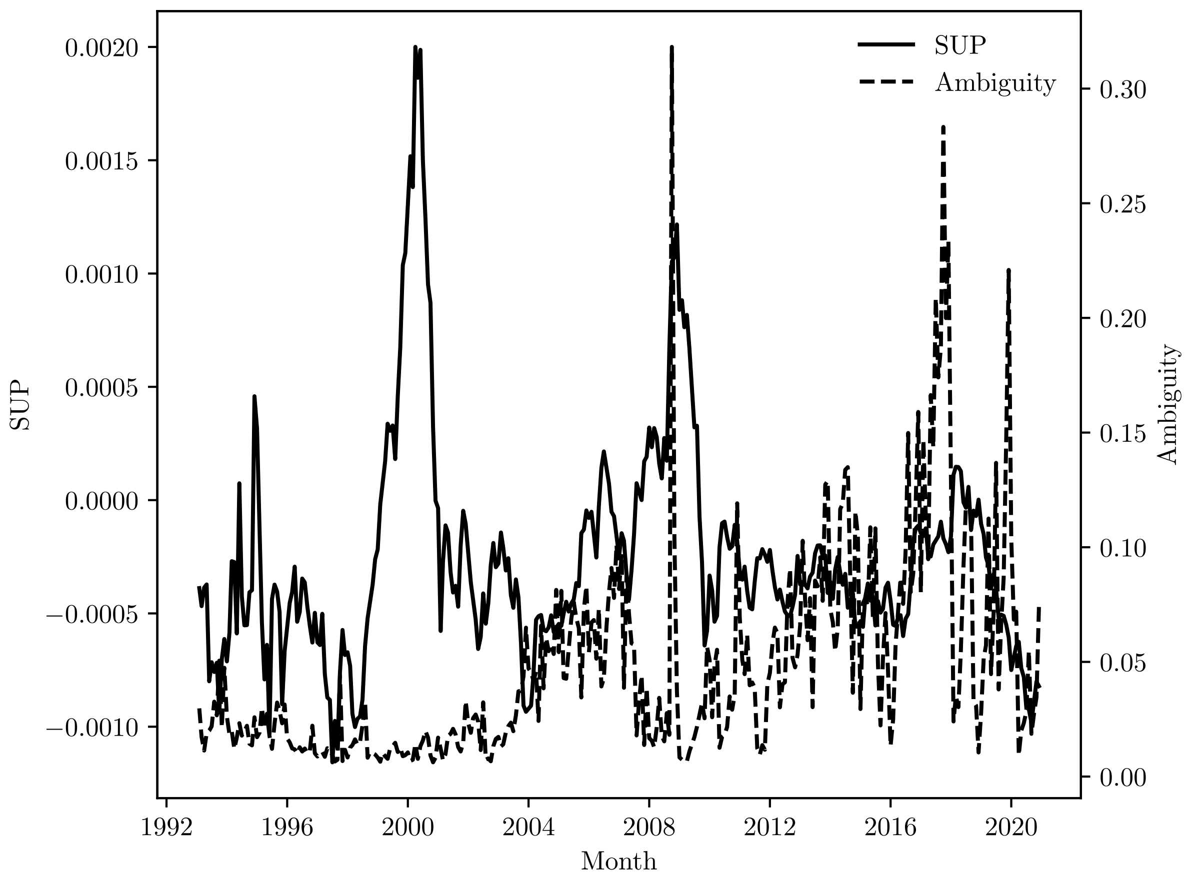


Figure 5: SUP with Ambiguity

The time series of the SUP (using the scale on the left hand side) together with the Amiguity premium (using the scale on the right hand side) from Brenner and Izhakian (2018).

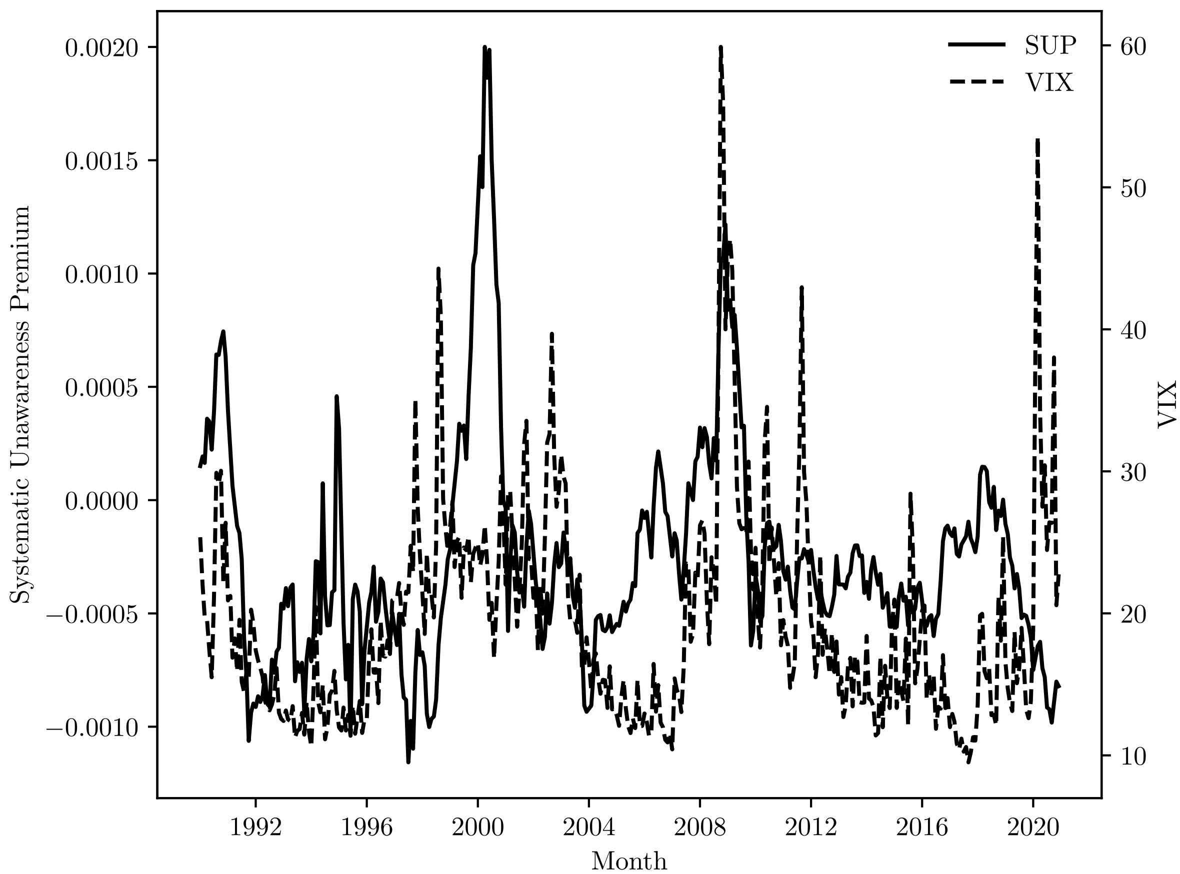


Figure 6: SUP with VIX

The time series of the SUP (using the scale on the left hand side) together with the VIX (using the scale on the right hand side).

Table 2: Systematic Unawareness and Asset Pricing Factors

	Coefficient	St. Error	t	$P > t$
Constant	-0.000598	0.000096	-6.207133	1.644467e-09
$r_m - r_f$	-0.000006	0.000007	-0.789449	4.304254e-01
SMB	0.000014	0.000010	1.324197	1.863680e-01
HML	-0.000008	0.000014	-0.561934	5.745484e-01
RMW	-0.000008	0.000013	-0.572287	5.675229e-01
CMA	0.000011	0.000017	0.650604	5.157619e-01
WML	-0.000010	0.000006	-1.670112	9.586004e-02
VIX	0.000017	0.000004	4.331941	1.972136e-05
AMB	0.000457	0.000611	0.747290	4.554288e-01

Results from a multivariate regression of SUP on other known risk factors. The factors are the 5 Fama-French factors along with momenturm (WML), the VIX and the Ambiguity Premium (AMB) from Brenner and Izhakian (2018). For each factor we report the estimated coefficient, the standard error, the t-statistic and the P-value. The variables that are significant at conventional significance levels are the constant and the VIX. The overall $R^2 = 0.0943$. The sample period is February 1993-December 2021, which is the intersection of the sample periods for which data is available for all factors.

To understand the collective role of the Fama-French factors and momentum, we regress the SUP on each of these factors as well as the ambiguity factor from Brenner and Izhakian (2018) and the VIX.[13] The results of this regression are given in Table 2. Together, the included factors explain only 9.43% of the variation in the SUP.[14]

Of these factors, the VIX is significant at conventional levels and momentum (WML) has a t statistic of 1.67, although taken together these factors explain only a small portion of the variation in the SUP. Overall, we do not find evidence that the SUP is a proxy for these previously studied factors. Thus, we also maintain our Hypothesis 3.

[13]The included Fama-French factors are the excess market return ($r_m - r_f$), small-cap minus big-cap (SMB), high minus low book-to-market ratios (HML), robust minus weak profitability (RMW), conservative minus aggressive firms (CMA). Additionally, we include momentum (WML).

[14]These series are not all available for the whole sample over which we have calculated the SUP.

6 Conclusion

In the CAPM, the introduction of unawareness and awareness thereof changes the systematic risk-return relationship. As a result, asset prices contain an unawareness premium in addition to the usual risk premium. A main driving force is that awareness of unawareness results in the risk-free asset not being perceived as being truly risk free.

We have developed a technique for estimating the systematic unawareness premium (SUP). Using this technique and standard CRSP data, we have investigated the empirical relevance of unawareness in the context of our CAPM. Our results suggest that unawareness is a separate factor that affects market prices and that it varies significantly over time.[15] The investigation reveals a possible relationship to presidential terms and suggests that the SUP is not a proxy for previously studied factors.

Further theoretical work is needed to understand what observable factors could lead to individuals forming beliefs about the unknown states of the world. In addition, further empirical work is needed to understand the interplay between the SUP and other factors that have been studied in the literature.

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[15] In a model that focuses on rare extreme events, Chollete, Jaffee and Mamun (2022) investigate regime switches in the relation between asset demand and stock return mean and volatility when there is a single risky asset. They find that allowing for a break point improves goodness of fit at the aggregate level. To the extent that extreme rare events can be interpreted similarly to our unknown regime, their results also highlight the relevance of the unknown regime.

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A Additional results

In this appendix we demonstrate with a Monte Carlo study that our strategy indeed correctly identifies the SUP. Our estimation of the systematic unawareness premium proceeds in two steps. In the first step, we estimate a traditional single factor model of the form

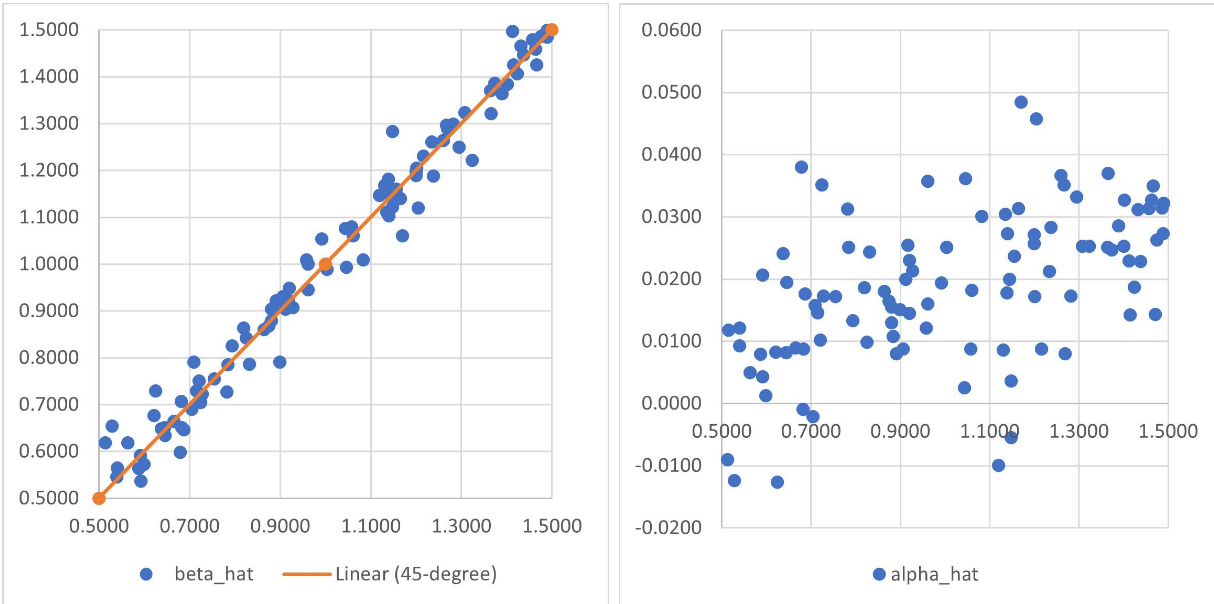
$$\mu_i - r_f = \alpha_i + \beta_i(\mu_m - r_f), \tag{13}$$

for each asset in a cross section at a particular time t. Comparing coefficients in this equation with that in Equation (8), we observe that the β coefficient in each equation captures the same variation. At the same time, estimating α in Equation (13), captures variation in Equation (8) given by

$$\alpha_i = -\frac{1-p}{p}(u_i - u_f) + \beta_i \frac{1-p}{p}\overline{u}, \tag{14}$$

which motivates our second step where we take the estimates $\hat{\alpha}_i$ and $\hat{\beta}_i$ from estimation of Equation (13) and treat them as observations in a new regression given by

$$\hat{\alpha}_i = \gamma_0 + \gamma_1 \hat{\beta}_i + \epsilon_i. \tag{15}$$



(a) Estimated $\hat{\beta}$ (b) Estimated $\hat{\alpha}$

Figure 7: Estimated $\hat{\beta}$ and $\hat{\alpha}$ against true β

The estimated β and α against the true β in a Monte Carlo simulation study with excess market return of 15% and volatility of 20%, with a SUP of 2%, and where the individual SUP constant is 1%. Each dot corresponds to one asset and is estimated with $1,000$ data points. The line in the left hand plot corresponds to the true 45-degree line.

The estimated coefficient $\hat{\gamma}_1$ then gives an estimate of the SUP.

However, this approach clearly relies on being able to identify correctly $\hat{\gamma}_1$ in Equation (15) when assuming that γ_0 is constant across observations, although it may in fact be observation-dependent as Equation (14) shows. To justify that this is indeed the case we conduct a small Monte Carlo study in which we simulate 100 returns of size 1,000 using Equation (8). We allow each of these stocks to have different β, which we randomly draw from a uniform distribution between 0.5 and 1.5, different level of idiosyncratic volatility, which we randomly draw from a uniform distribution between 10% and 35%, and to have different values for the term $\frac{1-p}{p}(u_i - u_f)$, which we randomly draw from a unform distribution between -0.5 and 0.5 and scale with a constant, $\overline{\text{SUP}}_i$ reflecting the average size.

We consider a simulation with excess market return of 15% and volatility of 20%, with a SUP of 2%, and where $\overline{\mathrm{SUP}}_i$ is 1%. Figure 7(a) shows a cross plot of the estimated β against the true β and shows that these align closely with the true values. Figure 7(b) shows a cross plot of the estimated α against the true β and shows that these vary significantly but do have some connection with each other. In this particular case the estimated value of the SUP is 1.8% with a standard deviation of 0.376%. The R^2 from the regression is 18.6% and with a t-stat of 4.74 the estimate of $\hat{\gamma}_1$ is clearly significantly different form zero but insignificantly different from the true value of 2% for the SUP with a t-stat of 0.59.