The mean-field contribution to the energy-momentum tensor,

$$
T_\kappa^{\mu\nu} = \kappa(\rho\partial_\sigma\rho)\Big\{\langle\partial^\mu u^\sigma\rangle u^\nu + \langle\partial^\nu u^\sigma\rangle u^\mu\Big\} + \kappa\langle\partial^\mu\rho\rangle\langle\partial^\nu\rho\rangle - \frac{\kappa}{2}\rho^3\Big\{u^\mu\partial^\nu\Big[\frac{\vec{\nabla}\cdot\vec{v}}{\rho}\Big] + u^\nu\partial^\mu\Big[\frac{\vec{\nabla}\cdot\vec{v}}{\rho}\Big]\Big\}
$$
$$
+ \quad \kappa\Big\{\frac{2}{3}\rho^{1/2}\partial^2(\rho^{3/2}) - \frac{1}{2}\rho^2(\vec{\nabla}\cdot\vec{v})^2 + \rho^2 u^\alpha\partial_\alpha\partial_\beta u^\beta\Big\}g^{\mu\nu}
$$
$$
- \quad \kappa\Big\{\frac{2}{3}\rho^{1/2}\partial^2(\rho^{3/2}) - \frac{3}{2}\rho^2(\vec{\nabla}\cdot\vec{v})^2\Big\}u^\mu u^\nu , \tag{1}
$$

with

$$
u^\mu = \gamma(1,\vec{v}) , \qquad \gamma = \frac{1}{\sqrt{1-v^2}} \tag{2}
$$

The rank-2 symmetric projection tensor is

$$
\langle A^\mu B^\nu\rangle = \frac{1}{2}\big\{\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}\big\}A_\alpha B_\beta, \qquad \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu. \tag{3}
$$

For a comoving frame $u^\mu = (1,\vec{0})$, the property of projection tensor,

$$
\langle A^0 B^\mu\rangle = \langle A^\mu B^0\rangle = 0, \tag{4}
$$

since $\Delta^{\mu 0} = 0$. So the projection tensor in the first two terms of $T_\kappa^{\mu\nu}$ only project out the vector component of the associated quantities.

The conservation equation of density ρ,

$$
D\rho = -\rho\vec{\nabla}\cdot\vec{v} . \tag{5}
$$

Using these properties, from Eq.(1),

$$
T_\kappa^{00} = \quad 0, \tag{6}
$$
$$
T_\kappa^{ij} = \quad \kappa(\partial^i\rho)(\partial^j\rho) - \kappa\Big\{\rho\nabla^2\rho + \frac{1}{2}(\vec{\nabla}\rho)^2\Big\}\delta^{ij} , \tag{7}
$$
$$
T_\kappa^{i0} = \quad -\frac{\kappa}{2}(\rho\partial_j\rho)\big\{\partial^i u^j + \partial^j u^i\big\} + \frac{\kappa}{2}(\vec{\nabla}\cdot\vec{v})\big\{\rho\partial^i\rho\big\} - \frac{\kappa}{2}\rho^2\big\{\partial^i(\vec{\nabla}\cdot\vec{v})\big\} . \tag{8}
$$

Furthermore, the additional contribution to the energy density due to (1),

$$
u_\mu u_\nu T_\kappa^{\mu\nu} = 0 . \tag{9}
$$

Important identities :

1.

$$
\begin{aligned}
\nabla^2\rho &= \quad \nabla_\mu\nabla^\mu\rho \\
&= \quad \partial^2\rho - D^2\rho + \frac{1}{\rho}(D\rho)^2 \\
&= \quad \partial^2\rho - D^2\rho + \rho(\vec{\nabla}\cdot\vec{v})^2 .
\end{aligned} \tag{10}
$$

Here, $\partial^2 = \partial^\mu\partial_\mu$, $\partial_\mu = u_\mu D + \nabla_\mu$, $D = u^\mu\partial_\mu$ and $\nabla_\mu = \Delta_{\mu\nu}\partial^\nu$.

2.

$$
(\vec{\nabla}\rho)^2 = (\nabla_\mu\rho)(\nabla^\mu\rho) = (\partial\rho)^2 - (D\rho)^2 , \qquad\qquad (\partial\rho)^2 = (\partial^\mu\rho)(\partial_\mu\rho) . \tag{11}
$$

3. With the help of Eq.(10), (11) and (5) we obtain the coefficient of the $g^{\mu\nu}$ term in Eq.(1),

$$
\begin{aligned}
\rho\nabla^2\rho + \frac{1}{2}(\vec{\nabla}\rho)^2 = \quad & \rho\partial^2\rho + \frac{1}{2}(\partial\rho)^2 - \rho D^2\rho + \frac{1}{2}(D\rho)^2 \\
= \quad & \frac{2}{3}\rho^{1/2}\partial^2(\rho^{3/2}) - \frac{1}{2}\rho^2(\vec{\nabla}\cdot\vec{v})^2 + \rho^2 u^\alpha\partial_\alpha\partial_\beta u^\beta \ ,
\end{aligned}
\tag{12}
$$

where $\frac{2}{3}\rho^{1/2}\partial^2(\rho^{3/2}) = \rho\partial^2\rho + \frac{1}{2}(\partial\rho)^2$ and in a comoving frame $u^\alpha\partial_\alpha\partial_\beta u^\beta = D(\vec{\nabla}\cdot\vec{v})$.

4. Similarly, the coefficient of the $u^\mu u^\nu$ term in Eq.(1) turns out to be,

$$
\rho D^2\rho - 2(D\rho)^2 + \rho\nabla^2\rho + \frac{1}{2}(\vec{\nabla}\rho)^2 = \frac{2}{3}\rho^{1/2}\partial^2(\rho^{3/2}) - \frac{3}{2}\rho^2(\vec{\nabla}\cdot\vec{v})^2 \ .
\tag{13}
$$