

# Charge Conservation and Higher Moments of Charge Fluctuations

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## I. INTRODUCTION

The fluctuation of conserved charges is a standard means by which to investigate and gauge phase transitions. At the critical point correlation lengths diverge, which results in peaks in charge fluctuations as one approaches the critical point. For systems with first-order phase transitions, fluctuations become infinite whenever one is in a region of phase separation, as the liquid and gas regions prefer to become infinite in extent when the surface energy is non-zero. The growth of fluctuations becomes increasingly dramatic as one considers progressively higher-order fluctuations. In a volume  $V$ , fluctuations of a charge  $Q$  can be defined as

$$\mathcal{M}_N \equiv \frac{1}{V} \langle (Q - \bar{Q})^N \rangle = \frac{1}{V} \sum_n P_n (n - \bar{n})^N, \quad (1)$$

when particles have unit charge. The measure is increasingly sensitive to the tails of the multiplicity distribution,  $P_n$ , as  $N$  increases. The free energy,  $F \sim a_2(n - \bar{n})^2 + a_3(n - \bar{n})^3 \dots$ , is minimized for  $n = \bar{n}$ , but the quadratic part vanishes at the critical point,  $a_2 \rightarrow 0$ , which allows the fluctuations to grow and be dominated by higher-order terms which manifest themselves in the tails of the distribution.

The properties of the QCD transition, deconfinement and the restoration of chiral symmetry, are not well understood at finite baryon density. There exists the possibility that this transition is a true phase transition, with a critical point at several times nuclear density and with a critical temperature close to the pion mass. If this is the case, it begs the question as to whether the conditions for phase separation or for critical phenomena can be reproduced in the laboratory. Heavy ion collisions at high energy, measured at the Relativistic Heavy Ion Collider (RHIC) or at the LHC, can produce mesoscopic regions at temperatures of a few hundred MeV, which is well above the expectations for a critical temperature, and densities of several times nuclear matter density.

High-energy heavy ion collisions are characterized by strong explosive collective flow. Measurement is confined to the outgoing asymptotic momenta, but because of strong flow, correlations in coordinate space manifest themselves as correlations in relative momentum. Thus, measurements of correlations binned by relative momen-

tum or charge fluctuations within some defined region of momentum space serve as surrogates for the corresponding observables in coordinate space. Indeed, measurements of charge and baryon number fluctuations have been performed at RHIC. By adjusting the beam energy of the colliding nuclei, experiments at RHIC have explored conditions at which novel phase phenomena might occur. Fluctuations of electric charge and baryon number have been especially popular. An initial scan of beam energies was rather inconclusive, but measurements with greatly improved statistics are currently being analyzed.

In addition to the finite system size (event multiplicities might number in the thousands), the novel states of matter created in heavy-ion collisions persist for  $\lesssim 10$  fm/c. This severely limits the degree to which phases can separate or to which critical fluctuations can grow. From the initial overlap of the colliding nuclei, until the mean time at which emitted particles experience their last collision, the collision lasts a few tens of fm/c. This limits the degree to which conserved charges can separate from one another. For example if a strange and an anti-strange quanta are produced together, their separation is limited by diffusion, and the enhancement of seeing a second charge of opposite sign having seen a first charge are known as charge-balance correlations.

The goal of this paper is to gauge the degree to which charge-balance correlations affect higher-order correlations. Such correlations need to be understood as potential sources of background before making firm arguments to have observed phenomena related to phase transitions. It is well known that charge-balance correlations are readily measurable at the two particle level,  $N = 2$  in Eq. (1), and that they explain the bulk of the  $N = 2$  fluctuation measurement. However, their impact on  $N = 3, 4$  fluctuations has not been investigated in great detail. For instance, the four-particle measure of correlation,

$$C_4 = \mathcal{M}_4 - 3\mathcal{M}_2^2, \quad (2)$$

which is based on cumulants, subtracts much of the contribution to the  $\mathcal{M}_4$  coming from purely two-particle correlations. However, charge conservation can involve multiple particles, and it is not clear the degree to which a cumulant-based measure, like the kurtosis, is affected by charge conservation. A simple model, where charge conservation is invoked through sampling a canonical ensemble, is presented in the next section. Although the

method is different, this is similar to ideas that have been pursued recently [1]. But with the method used here, events can be generated in a more efficient algorithm, which then enables the more systematic set of studies presented here.

After a brief review of cumulants and the definitions of skewness and kurtosis in Sec. ??, the method for exact solution of the canonical ensemble describing a multi-component, multi-charge hadron gas is presented in Sec. ?. The method includes all species and can account for Bose or Fermi statistics, but neglects interaction between hadrons. These techniques extend those used for canonical ensembles used to study isospin fluctuations of a hadron gas [2], nuclear fragmentation [3], the level density of a Fermi gas, and the effect of restricting a quark-gluon plasma (QGP) to having fixed charge, including being in an overall color singlet [4]. Exact methods for calculating correlations up to fourth-order are presented. Unfortunately, to include realistic acceptance effects and complex decays, the exact expressions are no longer viable. However, as shown in Sec. IV, the exact expressions show how sample events can be generated. An event, defined by a set of particles and momenta, are generated with perfect independence from one another, and perfectly reproduce the canonical expressions from Sec. III.

Conservation of charge is modeled by imagining independent sub-volume “patches” which individually conserve charge. The patch volume is set by how far charges have separated from another by the time chemical freeze-out occurs, just after hadronization. After this time, the volumes may grow and overlap, but the chemical makeup of the particles originating from each patch is assumed to be fixed. Sections ?? and ?? shows how fluctuation observables from charge conservation depend of patch size, measurement efficiency, and the chemical decoupling temperature by investigating systems with fixed efficiency. A very simplified system with a single charge, as already investigated in detail in [5], is reviewed in Sec. ??, while the full hadron gas, with three conserved charges, is presented in Sec. ?. The importance of Bose corrections for pions is also studied. Charge from the initial nuclei existed before the collision and might randomly populate the entire collision volume, whereas charge created from the breakup of color flux tubes, hadronization and decays, should be conserved locally. Thus, the effect of random placing some of the initial charge across the entire collision volume is also investigated. A more realistic picture of the acceptance and efficiency is presented in Sec. VII. A blast-wave model, which is a thermal model superimposed onto a simplified model of collective flow, is combined with a sampling of a canonical ensemble of a given patch. The resulting particles are then analyzed using a simulation of the acceptance and efficiency of the STAR Collaboration at RHIC. Results are compared to measurements at STAR to discern whether charge conservation effects might be responsible for a significant part of the observed correlation. Results are then summarized in Sec. VIII.

## II. CUMULANTS, SKEWNESS AND KURTOSIS

To keep the more manuscript self-contained, a brief review of cumulants and the definition of skewness and kurtosis is presented here.

Cumulants of a charge distribution are defined by

$$C_1 = \langle Q \rangle, \quad (3)$$

$$C_2 = \langle (Q - \bar{Q})^2 \rangle, \quad (4)$$

$$C_3 = \langle (Q - \bar{Q})^3 \rangle, \quad (5)$$

$$C_4 = \langle (Q - \bar{Q})^4 \rangle - 3C_2^2, \quad (6)$$

where  $Q$  is the net charge. Here,  $Q$  might refer to baryon number, to strangeness, or to the electric charge measured in units of  $e$ .

Rather than showing the cumulants  $C_n$ , ratios are presented to help minimize trivial dependences on system size. The skewness,  $S$ , is a measure of the third moment,

$$S = \frac{C_3}{C_2^{3/2}}. \quad (7)$$

This definition has the advantage in being dimensionless, but it does not become independent of volume in the limit of large volumes. Thus, it is more common to consider the ratio

$$S\sigma = S\sqrt{C_2} = \frac{C_3}{C_2}, \quad (8)$$

which becomes an intensive measure in the limit of larger volumes. The kurtosis is a measure of four-particle correlations,

$$K = \frac{C_4}{C_2^2}, \quad (9)$$

but instead of  $K$ , one typically chooses

$$K\sigma^2 = \frac{C_4}{C_2^2}, \quad (10)$$

to find an intensive measure of the fluctuation, at least for large volumes.

The measures  $S\sigma$  and  $K\sigma^2$  approach simple values in the limit that the distributions would be Poissonian. For Poissonian emission the observation of a charge in one region of momentum space is uncorrelated with the emission into any other space. Thus, particles are correlated only with themselves. If charges appear only in integral positive units, one can apply the usual expression for the Poissonian moments where the mean is  $\eta$ ,

$$C_1 = \bar{n} = \eta \quad (11)$$

$$C_2 = \langle (n - \bar{n})^2 \rangle = \eta,$$

$$C_3 = \langle (n - \bar{n})^3 \rangle = \eta = C_1,$$

$$C_4 = \langle (n - \bar{n})^4 \rangle - 3C_2^2 = \eta.$$

If there exist both positive and negative charges, the distribution of the net charge can be derived by convoluting

the two distributions. Convoluting two Poissonians results in a Skellam distribution. If the mean number of positives is  $\eta_+$  and the mean number of negatives is  $\eta_-$ , the distribution of net charge for a Skellam distribution,  $Q = n_+ - n_-$ , yields the following cumulants

$$\begin{aligned} C_1 &= \eta_+ - \eta_-, \\ C_2 &= \eta_+ + \eta_-, \\ C_3 &= \eta_+ - \eta_- = C_1, \\ C_4 &= \eta_+ + \eta_- = C_2. \end{aligned} \quad (12)$$

Thus, if charges are produced in an uncorrelated fashion in increments of either  $\pm 1$ , the skewness and kurtosis become

$$\begin{aligned} S\sigma &= \frac{C_3}{\sigma^2} = \frac{C_1}{\sigma^2}, \\ K\sigma^2 &= \frac{C_4}{\sigma^2} = 1, \end{aligned} \quad (13)$$

where  $\sigma^2 \equiv \langle (Q - \bar{Q})^2 \rangle = \eta_+ + \eta_-$ . It might have made more sense to consider  $C_3/C_1$  rather than  $S\sigma$ , as that ratio would be unity in the limit that emission is uncorrelated. However, because experimental analyses have often concentrated on the two ratios above, higher order correlations in this paper will also be presented in terms of these ratios.

Moments depend on the efficiency  $\alpha$  with which particles are measured. In the limit of vanishing efficiency all distributions of positives or of negatives tend to become Poissonian, and the distribution of the net charge will thus become a Skellam. This can be understood by having the probability of observing a positive charge being  $\alpha \rightarrow 0$  and the probability of observing two charges being proportional to  $\alpha^2$  being negligible. This assumption would fall through if all positive charges were measured in pairs, i.e. multiple charges on each measured particle, but for final-state hadrons the charges are only  $\pm 1$ . As  $\alpha \rightarrow 0$ , the moments are dominated by the probability of observing either zero charges or a single charge, and the moments for counting particles of a given charge are

$$\langle n^m \rangle = \alpha, \quad (14)$$

for all  $m > 0$ . It is then easy to see that  $\langle (n - \bar{n})^m \rangle = \alpha$ , which quickly leads to the result that as  $K\sigma^2 = C_3/C_1 = 1$ .

If the net charge is fixed, a non-perfect efficiency is required to produce fluctuations. For fixed charge, the efficiency divides into two sets, the measured and non-measured. Each set fluctuates equally, but oppositely, relative to the mean. Thus, all the even moments of net charge will have an even reflection symmetry about an efficiency of  $1/2$ , and the odd moments will have an odd symmetry,

$$C_n(1/2 + \delta\alpha) = \begin{cases} C_n(1/2 - \delta\alpha), & n = 2, 4, 6 \dots \\ -C_n(1/2 - \delta\alpha), & n = 3, 5, 7 \dots \end{cases} \quad (15)$$

### III. RECURSIVE TECHNIQUES FOR GENERATING CANONICAL PARTITION FUNCTIONS

For non-interacting particles the canonical partition function can be calculated exactly, or at least to the level that all partitions of  $A \leq A_{\max}$  hadrons are taken into account, with the exact solution being reached with  $A_{\max} = \infty$ . For our case, we conserve three quantities: the electric charge  $Q$ , the baryon number  $B$  and the strangeness  $S$ . For states  $i$  with energies  $E_i$ , the partition function,

$$Z(Q, B, S) = \sum_{i, Q_i=Q, B_i=B, S_i=S} e^{-\beta E_i}, \quad (16)$$

where  $Q_i, B_i$  and  $S_i$  are the discrete values of the conserved quantities for the state  $i$ , can be calculated recursively. The function  $Z_A(Q, B, S)$  refers to the subset of states with  $A$  hadrons,

$$Z(Q, B, S) = \sum_{A \geq 0} Z_A(Q, B, S). \quad (17)$$

The recursive procedure begins with

$$Z_{A=0}(0, 0, 0) = 1, \quad (18)$$

is the canonical partition function of the vacuum. The contribution for a given  $A$ ,  $Z_A(Q, B, S)$ , can be written as

$$Z_A(Q, B, S) = \frac{1}{A} \sum_h z_h Z_{A-1}(Q - q_h, B - b_h, S - s_h), \quad (19)$$

where  $z_h$  is the single-particle partition function for hadron species  $h$ , which has charges  $q_h, b_h$  and  $s_h$ . This was proved in [2], and can be understood by realizing that one can count all the ways to arrange  $A$  hadrons with a given charge by considering all the ways to arrange one hadron multiplied by all the ways to arrange the remaining hadrons. To avoid double counting, a factor of  $1/A$  is applied. For a fixed charge the probability to have  $A$  hadrons is,

$$P(A) = \frac{Z_A(Q, B, S)}{\sum_A Z_A(Q, B, S)} = \frac{Z_A(Q, B, S)}{Z(Q, B, S)}. \quad (20)$$

In practice, the sum over  $A$  is cut off at some  $A_{\max}$ , but in our studies here that cutoff is made large enough that contributions to  $Z$  for  $A > A_{\max}$  are negligible. Thus, once builds the partition function from  $A = 0$  to  $A_{\max}$  one has the partition function for all  $Q, B, S$ .

Once the partition function is calculated one can also calculate the multiplicities and moments of observing specific species. For example, the multiplicity of species  $h$  in a system with charge  $Q, B, S$  is

$$\langle N_h \rangle = z_h \frac{Z(Q - q_h, B - b_h, S - s_h)}{Z(Q, B, S)}. \quad (21)$$

This also provides expressions for the various charges, e.g.,

$$\langle Q \rangle = \sum_h q_h z_h \frac{Z(Q - q_h, B - b_h, S - s_h)}{Z(Q, B, S)}. \quad (22)$$

Spectra can also be calculated

$$\frac{dN_h}{d^3p} = \frac{(2s_h + 1)\Omega}{(2\pi\hbar)^3} e^{-E_h(p)} \frac{Z(Q - q_h, B - b_h, S - s_h)}{Z(Q, B, S)}. \quad (23)$$

Higher moments can also be extracted,

$$\begin{aligned} \langle N_h N_{h'} \rangle &= \delta_{hh'} z_h \frac{Z(Q - q_h, B - b_h, S - s_h)}{Z(Q, B, S)} \\ &+ z_h z_{h'} \frac{Z(Q - q_h - q_{h'}, B - b_h - b_{h'}, S - s_h - s_{h'})}{Z(Q, B, S)}. \end{aligned} \quad (24)$$

It is straightforward to extend this expression to the fluctuation of charges.

These expressions can also be extended to consider non-additive conservation laws. Net isospin conservation of a hadron gas was invoked in [3], i.e. restricting the states to being in an iso-singlet. For quark-gluon states were restricted to being in both an iso-singlet and a color singlet in [4]. For Bose corrections,

$$Z_A(B, Q, S) = \frac{1}{A} \sum_{h,n} z_{h,n} Z_{A-1}(Q - nq_h, B - nb_h, S - ns_h), \quad (25)$$

$$z_{h,n} = (2s + 1) \sum_p e^{-nE_p/T}.$$

For the conditions of a hadron gas in high-energy heavy-ion collisions only pions are significantly affected by quantum degeneracy.

Complicated experimental acceptances are difficult to incorporate into expressions for the moments. It is then easiest to generate entire events via Monte Carlo, and filter the events through the acceptance. The Monte Carlo procedure involves choosing a hadron proportional the number of ways the system might have such hadron, i.e. a product of the partition function of the individual hadron multiplied by the partition function of the remainder. The procedure becomes:

1. For Charge  $Q, B, S$ , choose the number of hadrons  $A$  proportional to  $Z_A(Q, B, S)/Z_A(Q, B, S)$ .
2. Choose a hadron  $h$  proportional to the probability  $z_h Z_{A-1}(Q - q_h, B - b_h, S - s_h)/Z_A(Q, B, S)$ .
3. Choose the momentum proportional to the thermal weight  $e^{-E_p/T}$ .
4. Repeat (2) but with  $A, Q, B, S$  being replaced by  $A - 1, Q - q_h, B - b_h, S - s_h$ . The procedure is finished when  $A = 0$ .

Bose effects for pions can be included by altering the second and third steps above. In addition to choosing a hadron, one might also consider adding a 2-pion, 3-pion or  $n$ -pion state. Simply treat the  $n$ -pion state as if it were a resonance with charges  $nq_h, nq_b, nq_s$ , but with a partition function  $z_{h,n}$  described above. If one chooses to make such a “particle”, the  $n$  identical particles are given the same momentum and spin projection, using a Boltzmann distribution with a temperature  $T/n$ . To be more realistic the momenta should be smeared by some momenta  $\approx \hbar/R$ , where  $R$  is a characteristic size of the system. If Fermi effects were to be included, one could calculate the partition function by adding a factor  $(-1)^{n-1}$  to each term in the sum described in Eq. (25). However, because some of the weights are negative, Monte Carlo procedures can become problematic. Fortunately, for the systems considered here are extremely hot, and degeneracy of fermions is negligible.

Storing the partition function can require substantial memory for larger  $A_{\max}$  because the indices  $Q, B$  and  $S$  must also vary over a range of order  $\pm A_{\max}$ , so memory usage roughly scales with  $A_{\max}^4$ . Because one is usually interested in calculations with total charge near zero, one can ignore partition functions for charges that cannot couple back to zero charge. Once  $A$  exceeds  $A_{\max}/2$ , the calculations here cutoff values of  $Q, B$  and  $S$  that could not ultimately affect the  $Q = B = S = 0$  partition function for  $A = A_{\max}$ . Even with this savings, partition functions with  $A_{\max} = 250$  could require approximately 13 GB of memory, which would then require on the order of 10 minutes to calculate on a single processor. For  $A_{\max} = 125$  less than a GB of memory was needed and partition functions could be calculated in less than a minute. For hadron gases at temperatures of 150 MeV,  $A_{\max} = 250$  was readily sufficient for patch volumes  $\lesssim 700 \text{ fm}^3$ . If multiple patch volumes are to be explored for the same temperature, computational time can also be saved by realizing that the partition functions scale as  $\Omega^A$ . Thus, if one performs a calculation for some initial volume  $\Omega_0$ , scaling can provide results for new volumes.

#### IV. GENERATION OF UNCORRELATED SAMPLE EVENTS

#### V. SINGLE-CHARGE MODEL WITH UNIFORM MEASUREMENT EFFICIENCY

For context, we first review an extremely simplified model of a system with only one conserved charge, and where the species all have charge of either  $-1, 0$  or  $+1$ . Further, Bose and Fermi degeneracy is ignored. The expressions shown here were worked out analytically in [1]. In addition to the results from the canonical ensemble, a system of uncorrelated neutral particles that decays to charged particles is also presented to illustrate the difference in charge conservation manifests itself in fluctuation



observables depending on whether the charges arise directly from an equilibrated system, or whether they arise from the decays of uncorrelated neutral particles.

As before, the partition functions are labeled both the charge  $Q$  and by the net number of hadrons, in this case charged hadrons. Because particles have charges only of  $\pm 1$ , the partition function for  $z = Z_{A=1, Q=1} = Z_{A=1, Q=-1}$  forms the basis of all other partition functions. From the methods of the previous section

$$Z_{A,Q} = \frac{z}{A} [Z_{A-1, Q-1} + Z_{A-1, Q+1}]. \quad (26)$$

This relation rapidly reproduces the partition function for all  $A$  and  $Q$  for  $A$  of the order of a few hundred.

As an aside we point out that the partition functions from this method agree perfectly with those generated by considering equilibrated reaction rates. For a given charge  $Q$ , if  $Q$  is even only even values of  $A$  are allowed, and if  $Q$  is odd, only odd values of  $A$  are possible. If there are reactions that annihilate hadrons, proportional to the density of positives multiplied by the density of negatives, and thermal rate at which pairs are created, which is depends only on temperature, then at equilibrium,

$$\alpha \frac{(A-Q)(A+Q)}{4} Z_{A,Q} = \beta Z_{A-2,Q}. \quad (27)$$

Here, annihilation is described by the constant  $\alpha$  which depends on the product of the number of positives,  $(A+Q)/2$ , and the number of negatives  $(A-Q)/2$ . Beginning with

$$Z_{A,Q=\pm A} = \frac{z^A}{A!}, \quad (28)$$

which gives the  $Z_{A,Q}$  for the lowest possible  $A$  given a specific  $Q$ , one can generate the partition function for all higher values of  $A$  using Eq. (27). If the ratio  $\beta/\alpha$  is chosen to be  $z$ ,

$$Z_{A,Q} = \begin{cases} \frac{z^A}{[(A-Q)/2]![(A+Q)/2]!}, & A-Q \text{ is even,} \\ 0, & A-Q \text{ is odd.} \end{cases} \quad (29)$$

This expression satisfies the recurrence relation from the previous section, Eq. (19). The probability of having  $A$  charged particles for fixed charge  $Q$  becomes,

$$P(A|Q) = \frac{Z_{A,Q}}{\sum_A Z_{A,Q}}, \quad (30)$$

$$= \frac{z^A}{[(A/2)!]^2} \left\{ \sum_A \frac{z^A}{[(A/2)!]^2} \right\}^{-1}, \quad Q=0. \quad (31)$$

Decays produce equal amounts of positive and negative charge, but if the decays occur after chemical freezeout, there is no balancing back reaction. This leads to a qualitatively different multiplicity distribution than that of the equilibrated system of charges described in Eq. (29). For example if one has uncorrelated emission of neutral

FIG. 1. Blah blah...

particles that decay to form charged pairs, the probability of having  $A_0$  decaying particles is Poissonian, and the probability of having  $A$  final particles is

$$P(A|Q=0) = P_0(A/2) = \frac{\eta^n}{(A/2)!} e^{-\bar{A}/2}, \quad A=0, 2, 4 \quad (32)$$

## VI. HADRON GAS WITH UNIFORM EFFICIENCY

Canonical partition functions depend on the temperature  $T$ , the volume  $V$ , the conserved charges  $\vec{Q} = (Q, B, S)$ , and whether or not Bose corrections are included. Of course, charges don't fluctuate in a canonical ensemble if all the particles are measured, so to understand moments of the charge distribution, acceptance and efficiency must be included. Thus, in addition to  $T, V$  and  $\vec{Q}$ , one needs to understand how the acceptance and efficiency change the result. In this section, to better understand how fluctuations are affected by the parameters above, a grossly simplified acceptance and efficiency is employed. Each particle has a probability  $\alpha$  of being observed, regardless of its momentum.

First, we set the net charges to zero, and study the effects of changing  $T, V$  and  $\alpha$ . Ensembles of particles are generated according to the Monte Carlo procedure described in Sec. ?? according to  $T$  and  $V$ . Unstable particles are then decayed with the relative momenta of the particles determined by the invariant mass of the decaying particles. For these calculations all unstable resonances are produced according to their pole mass. Each particle is then randomly kept or ignored with probability  $\alpha$ .

The dependence on the efficiency  $\alpha$  is illustrated in Fig. 1. The volume was fixed at  $V = 100 \text{ fm}^3$ , the temperature was set to  $T = 150 \text{ MeV}$ , with all the charges set to zero. The results have a mirror symmetry about  $\alpha = 0.5$  as expected from the discussion surrounding Eq. (15), and the kurtosis approaches the Skellam limit as  $\alpha$  nears either zero or unity. The kurtosis dips significantly compared to that of a Skellam distribution for  $\alpha \sim 1/2$ .

## VII. BLAST WAVE MODEL AND COMPARISON TO STAR RESULTS

## VIII. SUMMARY

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