Nuclear Physics **6** (1958) 440—442; © *North-Holland Publishing Co., Amsterdam*

SIMPLE EXPANSION FOR THE REGULAR COULOMB WAVE FUNCTION

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Received 7 January 1958

Abstract: The procedure employed by Shepanski and Butler [1]) for the expansion of the regular radial Coulomb wave function has been modified, leading to a simple expansion of the same rapidity of convergence.

1. Introduction

The Coulomb wave functions have been recently given much attention owing to their importance in nuclear scattering problems. Among various attempts to develop convenient formulae Shepanski and Butler [1]) reported an expansion for the regular radial Coulomb wave function as suggested by an iteration procedure on the wave equation. Although this expansion is more rapidly convergent than the usual expansion given by Yost, Wheeler and Breit [2]) yet it is more laborious in the evaluation of its terms. The object of this communication is to present an equally rapidly convergent expansion characterized by a simpler recurrence relationship between its terms.

2. Regular Solution of the Wave Equation

The regular radial Coulomb wave function, in standard notation [3]), is represented as follows:

$$F_L(\eta,\ \varrho) = C_L(\eta)\varrho^{L+1}\Phi_L(\eta,\ \varrho), \tag{1}$$

where

$$C_L(\eta) = \frac{2^L}{(2L+1)!}\left\{(1^2+\eta^2)(2^2+\eta^2)\dots(L^2+\eta^2)\frac{2\pi\eta}{e^{2\pi\eta}-1}\right\}^{\frac{1}{2}}$$

and $\Phi_L(\eta,\ \varrho)$ is the regular solution of the differential equation

$$\varrho\frac{d^2\Phi_L}{d\varrho^2} + (2L+2)\frac{d\Phi_L}{d\varrho} + (\varrho-2\eta)\Phi_L = 0. \tag{2}$$

According to Yost, Wheeler and Breit, the power series solution of (2) is given as

$$\Phi_L(\eta,\ \varrho) = 1+\frac{\eta\varrho}{L+1}+\frac{2\eta^2-L-1}{2(L+1)(2L+3)}\varrho^2+\dots. \tag{3}$$

It can be seen from (3) that, for small values of ϱ, the terms $\varrho d^2\Phi_L/d\varrho^2$ and $\varrho\Phi_L$ could be neglected in solving (2) for the first approximation. Thus,

$$\Phi_L{}^{(1)}(\eta,\ \varrho) = \exp\left(\frac{\eta\varrho}{L+1}\right). \tag{4}$$

This result and approximations of higher orders suggest an exact solution of (2) in the form

$$\Phi_L(\eta,\ \varrho) = e^{\eta\rho/(L+1)} \sum_{n=0}^{\infty} a_n \varrho^n. \tag{5}$$

On substituting (5) in (2), we find

$$a_0 = 1, \qquad a_2 = -\frac{\eta^2+(L+1)^2}{2(L+1)^2(2L+3)},$$

$$a_1 = 0, \qquad a_3 = \frac{\eta[\eta^2+(L+1)^2]}{3(L+1)^3(L+2)(2L+3)},$$

and in general, for $n \geqq 2$,

$$n(n+2L+1)a_n = -\frac{2\eta(n-1)}{L+1}a_{n-1} - \left[1+\left(\frac{\eta}{L+1}\right)^2\right]a_{n-2}. \tag{6}$$

It is evident that the present three-term recurrence relation, given in (6), is simpler than the five-term recurrence relation of the expansion of Shepanski and Butler. This was only a consequence of neglecting, in the first approximation, the term $\varrho\Phi_L$ in (2), which has been included in the work of the above mentioned authors. It should be noted that the present procedure does not spoil the rapidity of convergence of the resulting expansion. A comparison between the number of terms required in each expansion to evaluate $\Phi_L(\eta,\ \varrho)$ for different values of L and η is given in the following table; ϱ is chosen as unity to reduce the expansions to their coefficients.

L	η	$\Phi_L(\eta,\ 1)$	No. of Terms	
			Present	S. and B.
0	0	0.8415	4	4
2	0	0.9305	3	3
5	0	0.9622	3	3
0	1	2.0985	7	9
2	1	1.2937	5	5
5	1	1.1362	4	4
0	2	4.3653	10	12
2	2	1.7735	5	5
5	2	1.3388	5	4
2	3	2.4015	7	8
3	4	2.4624	6	6
4	5	2.5035	6	6
5	6	2.5331	5	5

From the table it is seen that both expansions have practically the same rapidity of convergence, which increases with increasing L. It should also be mentioned that for $\eta \leqq L+1$ the convergence is rapid, whereas it becomes slower as η have values greater than $L+1$. In order to explain this behaviour mathematically, let eq. (5) be expressed in the following form:

$$\Phi_L(\eta, \varrho) = e^{\eta\rho/(L+1)} \sum_{n=0}^{\infty} c_n \left[1+\left(\frac{\eta}{L+1}\right)^2\right]^{\frac{n}{2}} \varrho^n \tag{7}$$

where

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = -1/2(2L+3),$$

and in general

$$n(n+2L+1)c_n = -\alpha(n-1)c_{n-1} - c_{n-2}, \tag{8}$$

where

$$\alpha = 2 \bigg/ \left[1+\left(\frac{L+1}{\eta}\right)^2\right]^{\frac{1}{2}}.$$

Since $0 \leqq \alpha \leqq 2$, for any value of L and η, it follows that the rate of convergence of (7) is practically independent of the value of α. Hence it is obvious from (7) that the expansion can be conveniently used when η is not large with respect to L.

The writer would like to thank Professor Z. Morsi and Professor F. El Bedewi for their interest and discussion of the problem.

References

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