

Category Theory

- Idea:
- like sociology, not neuroscience
 - ideal for semantics
 - way of thinking
 - translate between fields

Def: a category consist of

- objects A, B, C, \dots
- morphisms $f, g, h, \dots : A \rightarrow B$ for every two objects A, B
- if $f: A \rightarrow B$ and $g: B \rightarrow C$, a morphism $g \circ f : A \rightarrow C$
- for every object A , a morphism $\text{id}_A : A \rightarrow A$

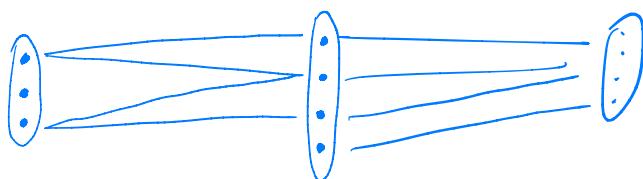
such that:

- associativity: $h \circ (g \circ f) = (h \circ g) \circ f$
- identity: $\text{id}_B \circ f = f = f \circ \text{id}_A$ for every $f: A \rightarrow B$

Examples: ① Set: obj: sets A, B, C, \dots
arr: functions $A \xrightarrow{f} B$

② Rel: obj: sets A, B, C, \dots
arr: relations $R \subseteq A \times B$

$$A \xrightarrow{R} B \xrightarrow{S = id_C} C = B$$



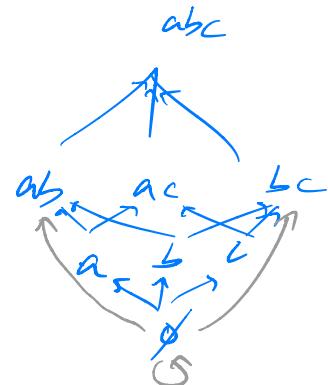
$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B : (a, b) \in R \text{ and } (b, c) \in S\}$$

PFn: arr: partial functions $A \rightarrow B$

③ if (P, \leq) is partially ordered set

obj: $p \in P$

arr: $p \rightarrow q$ iff $p \leq q$



④ Poset: obj: partially ordered set P, Q, \sim

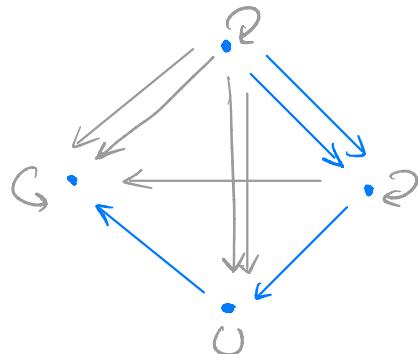
arr: monotone functions $f: P \rightarrow Q$

$$p \leq p' \implies f(p) \leq f(p')$$

⑤ if $G = (V, E)$ directed graph

obj: $v \in V$

arr: $v \rightarrow w$ are paths $v \rightarrow w$



⑥ Graph: obj: directed graphs

arr: $(V, E) \rightarrow (V', E')$ graph homomorphisms

$f: V \rightarrow V'$

$(v, u) \in E \implies (fv, fu) \in E'$

e.g. $\text{id}: V \rightarrow V$

⑦ simply typed λ -calculus:

obj: types

arr: $A \rightarrow B$ are $\beta\eta$ -equivalence classes of terms of type B with one free var of type A

⑧ Haskell: obj: Haskell types

arr $A \rightarrow B$ are closed Haskell expressions
of type $A \rightarrow B$

comp: $g \circ f = (\lambda x \rightarrow g(f x))$

but: $\text{undefined} \circ \text{id} = \lambda x. \text{undefined} \neq \text{undefined}$

solution? equate $f, g: A \rightarrow B$ when $f x = g x$
for all $x: A$ but need operational
semantics

Def: if \mathcal{C}, \mathcal{D} are categories, a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of:

- an object $FA \in \mathcal{D}$ for each object $A \in \mathcal{C}$
- a morphisms $Ff: FA \rightarrow FB$ in \mathcal{D} for each morphism $A \xrightarrow{f} B$ in \mathcal{C}

such that:

- $F(g \circ f) = Fg \circ Ff$
- $F(id_A) = id_{FA}$

Examples: ③ if P, Q are posets

and $f: P \rightarrow Q$ monotone function

regard as categories \underline{P} and \underline{Q}

can make functor $f: \underline{P} \rightarrow \underline{Q}$

$$p \mapsto f(p)$$

$$p \leq p' \Rightarrow f(p) \leq f(p')$$

⑤ If $G = (V, E)$ and $G' = (V', E')$ directed graphs

can regard as categories \underline{G} , \underline{G}' ,

if $f: G \rightarrow G'$ graph homomorphism,

can regard as functor $\underline{G} \longrightarrow \underline{G}'$
 $v \mapsto f(v)$

$$\begin{pmatrix} v_1 \\ \downarrow \\ v_2 \\ \vdots \\ \downarrow \\ v_n \end{pmatrix} \longmapsto \begin{pmatrix} f(v_1) \\ \downarrow \\ f(v_2) \\ \vdots \\ \downarrow \\ f(v_n) \end{pmatrix}$$

(6)

$$\begin{array}{ccc}
 \text{Set} & \xrightarrow{\Gamma} & \text{Rel} \\
 A & \longmapsto & A \\
 (A \sqsubseteq B) & \longmapsto & \{(a, f(a)) \in A \times B \mid a \in A\}
 \end{array}$$

$$\begin{array}{ccc}
 \text{Rel} & \xrightarrow{\mathcal{P}} & \text{Set} \\
 A & \longmapsto & \mathcal{P}A \\
 (R \subseteq A \times B) & \longmapsto & \left(\begin{array}{l} \mathcal{P}A \rightarrow \mathcal{P}B \\ u \mapsto \{b \in B \mid \exists a \in u : (a, b) \in R\} \end{array} \right)
 \end{array}$$

⑨ $\text{Cat} : \begin{matrix} \text{obj} : \text{categories } \mathcal{C}, \mathcal{D}, \dots \\ \text{arr} : \text{functors } \mathcal{C} \rightarrow \mathcal{D} \end{matrix}$

④ $\begin{matrix} \text{Posets} & \longrightarrow & \text{Cat} \\ P & \longmapsto & \underline{P} \\ f & \longmapsto & \underline{f} \end{matrix}$

⑥ $\begin{matrix} \text{Graph} & \xrightarrow{\text{Path}} & \text{Cat} \\ G & \longmapsto & \text{Path}(G) \\ f & \longmapsto & \text{Path}(f) \end{matrix}$

Universal properties

Def: an object $A \in \mathcal{C}$ is terminal if (strictly)
for any object $B \in \mathcal{C}$ there is a unique morphism $B \xrightarrow{!} A$.

- Ex:
- Set: any singleton set
 - Rel: the empty set
 - Poset: any singleton poset

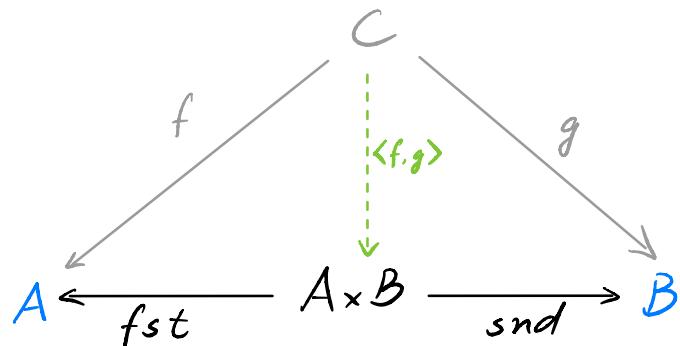
universal property

Def: a morphism $f: A \rightarrow B$ is an isomorphism if
there is $g: B \rightarrow A$ such that $g \circ f = id_A$ and $f \circ g = id_B$

Lem: terminal objects are unique up to isomorphism.

Pf: If A, B terminal, then $\begin{array}{ccc} id_A & \longrightarrow & B \\ \longleftarrow & & \supset id_B \end{array}$

Def: If $A, B \in \mathcal{C}$, a product of A and B is an object $A \times B$ together with map $A \xleftarrow{\pi_A} A \times B \xrightarrow{\pi_B} B$ such that for every object C and maps $A \xleftarrow{f} C \xrightarrow{g} B$ there is unique map $\langle f, g \rangle: C \rightarrow A \times B$ s.t. $f = \text{fst} \circ \langle f, g \rangle$ and $g = \text{snd} \circ \langle f, g \rangle$



Ex:

- Set: cartesian product of sets is a (categorical) product
- Rel: disjoint union of sets is a (categorical) product

"limit"
↓
"cone"
↓
"diagram"

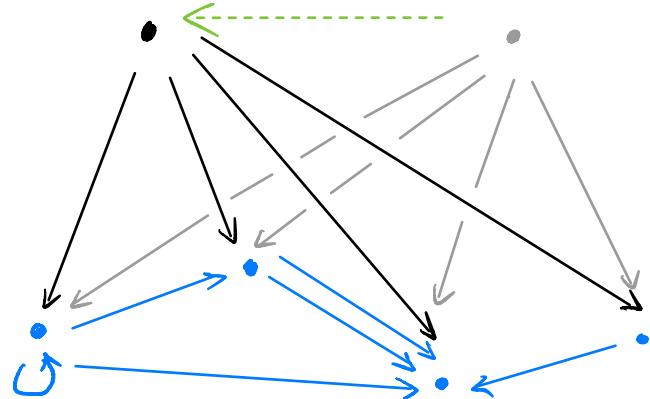
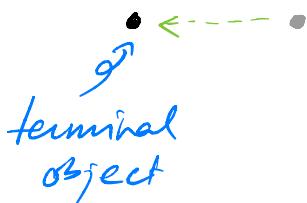
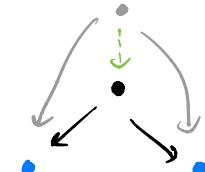


diagram = \emptyset

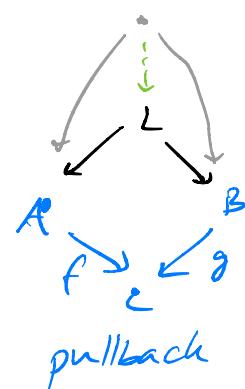


diagram



product

in Set: $L = \{(a, b) \in A \times B \mid f a = g b\}$



pullback

Def : If $A, B \in \mathcal{C}$, an exponential object is an object $B^A \in \mathcal{C}$

$$\begin{array}{ccc} C & C \times A & \\ \downarrow f & \downarrow & \searrow ev' \\ B^A & B^A \times A & \xrightarrow{ev} B \end{array}$$

Ex: - in Set, $B^A = \{f: A \rightarrow B\}$

- in Graph, $H^G = \{f: (\bullet \rightarrow \bullet) \times G \rightarrow H \text{ graph homomorph}\}$

Def: a category is cartesian closed if it has
a terminal object, all (binary) products, and all
exponential objects



if A, B are objects
then $A \times B$ exists
and B^A exists

Thm: $\text{Cart} \iff \text{Lambda}$

of

obj: cartesian closed cat's

arr: functions

in

models of simply-typed λ -calculus

Cat : obj: categories is cartesian closed:
arr: functors if \mathcal{C}, \mathcal{D} are categories.

$\mathcal{C} \times \mathcal{D}$? obj: (A, B) where $A \in \mathcal{C}, B \in \mathcal{D}$
arr $(A, B) \rightarrow (A', B')$ are pairs (f, g) where $f: A \rightarrow A'$ in \mathcal{C}
 $g: B \rightarrow B'$ in \mathcal{D}

$\mathcal{D}^{\mathcal{C}} = [\mathcal{C}, \mathcal{D}]$? obj: functors $F: \mathcal{C} \rightarrow \mathcal{D}$
arr $F \rightarrow G$: natural transformations, i.e. collection
 $\alpha_A: FA \rightarrow GA$ for each $A \in \mathcal{C}$ such that

$[\mathcal{C}, \text{Set}]$ \leftarrow presheaf

$$\begin{array}{ccc}
 A & & \\
 \downarrow f & & \\
 B & & \\
 \end{array}
 \quad
 \begin{array}{ccc}
 FA & \xrightarrow{\alpha_A} & GA \\
 \downarrow Ff & & \downarrow Gf \\
 FB & \xrightarrow{\alpha_B} & GB
 \end{array}$$

Def1: a monad ("standard construction," "triple") consists of

- a functor $T: \mathcal{C} \rightarrow \mathcal{C}$

- a natural transformation $\eta_A: A \rightarrow TA$ — "unit"

$\mu_A: T^2A \rightarrow TA$ — "multiplication" "join"

such that

$$\begin{array}{ccc} TA & \xrightarrow{T\eta_A} & T^2A \\ \downarrow \eta_{TA} & \nearrow & \downarrow \mu_A \\ T^2A & \xrightarrow{\mu_A} & TA \end{array}$$

"unit laws"

$$\begin{array}{ccc} T^2(TA) = T^3A & \xrightarrow{T\mu_A} & T^2A \\ \downarrow \mu_{TA} & & \downarrow \mu_A \\ T^2A & \xrightarrow{\mu_A} & TA \end{array}$$

"associative law"

Def 2: a monad ("Kleisli triple") consists of

- an object $T \in \mathcal{C}$ for every object $A \in \mathcal{C}$ a "type constructor"
- morphisms return: $A \rightarrow T(A)$ a "type converter"
- a morphism bind($-$, f) = ($- \gg f$): $TA \rightarrow TB$ for each $f: A \rightarrow TB$ a "combinator"

such that:

$$\text{return}(x) \gg f = f(x)$$

$$t \gg \text{return} = t \quad \text{where } t: TA$$

$$t \gg (\lambda x \rightarrow (f(x) \gg g)) = ((t \gg f) \gg g)$$

Def 1 = def 2: $\Rightarrow:$ $\text{bind}(x, f) = \mu_{Tf}(x)$

$\eta = \text{return}$

$$\Leftarrow: \mu = \text{bind}(-, \text{id}_{TA})$$

Remark: do notation: $\text{do } a \leftarrow b \dots = b \gg (\lambda a \rightarrow \dots)$

Example ① Maybe: $\mathcal{C} = \text{Set}$

$$T(A) = A + 1$$

$$T(f) = f + 1$$

$$\begin{array}{ccc} \gamma_A: A & \longrightarrow & A + 1 \\ a & \longmapsto & a \end{array}$$

$$\begin{array}{ccc} \mu_A: (A + 1) + 1 & \longrightarrow & A + 1 \\ a & \longmapsto & a \\ * & \longmapsto & * \\ * & \longmapsto & * \end{array}$$

② Exception: $\mathcal{C} = \text{Set}$ — work in any category with coproducts

$$T = (-) + E$$
 — fixed set of 'exceptions'

$$\gamma_A(a) = a$$

$$\begin{array}{ccc} \mu_A: (A + E) + E & \longrightarrow & A + E \\ a & \longmapsto & a \\ e & \longmapsto & e \\ e & \longmapsto & e \end{array}$$

③ Reader: $TA = A^I \rightarrow$ fixed obj of 'inputs'

$$\frac{A \xrightarrow{\eta_A} A^I}{A \times I \xrightarrow{\text{fst}} A}$$

$$\frac{(A^I)^I \xrightarrow{\mu_A} A^I}{(A^I)^I \times I \longrightarrow A}$$

eg. in set: $\mu_A(\varphi)(i) = \varphi^{(i)}(i)$

$\downarrow \text{id}_{A^I} \times \text{id}_I$ $\uparrow \text{ev}$

$$(A^I)^I \times (I \times I) \xrightarrow{\text{ev} \times \text{id}_I} A^I \times I$$

④ Writer: $TA = A \times O \rightarrow$ fixed object of 'outputs' that is a semigroup

$$O \times O \xrightarrow{c} O \quad 1 \xrightarrow{e} O$$

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & A \times O \\ & \searrow & \nearrow \text{id}_A \times e \\ A \times I & & \end{array}$$

$$\begin{array}{ccc} (A \times O) \times O & \xrightarrow{\mu_A} & A \times O \\ & \searrow & \nearrow \text{id}_A \times c \\ A \times (O \times O) & & \end{array}$$

⑤ State : $\tau A = (A \times S)^S$ where S fixed object of 'States'

$$\frac{A \xrightarrow{\eta_A} (A \times S)^S}{A \times S \xrightarrow{id_{A \times S}} A \times S}$$

$$\frac{(A \times S)^S \times_S S \xrightarrow{\mu_A} (A \times S)^S}{((A \times S)^S \times_S S)^S \times_S S \xrightarrow{\quad\quad\quad A \times S \quad\quad\quad} (A \times S)^S \times_S S}$$

⑥ Nondeterminism

$\mathcal{C} = \text{Set}$

$TA = \wp A$ List A

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \wp A \\ a & \longmapsto & \{a\} \end{array}$$

$$\begin{array}{ccc} \wp \wp A & \xrightarrow{\mu_A} & \wp A & \text{concat} \\ A & \longmapsto & \{ \langle a, t \rangle \mid \exists u : a \in u \} \end{array}$$

⑦ Probability

$\mathcal{C} = \text{Set}$

$TA = \{ p : A \rightarrow [0,1] \mid \text{supp}(p) \text{ finite and } \sum_{a \in A} p(a) = 1 \}$

$$\begin{array}{ccc} \eta_A : A & \longrightarrow & TA \\ a & \longmapsto & \lambda b : A . [\begin{matrix} 0 & \text{if } a \neq b \\ 1 & \text{if } a = b \end{matrix}] \end{array} \quad \text{Dirac distribution}$$

$$\begin{array}{ccc} \mu_A : TA & \longrightarrow & TA \\ p & \longmapsto & \lambda a : A . \sum_{p \in TA} P(p) \cdot p(a) \end{array}$$

"pure" computations
↓



"effectful computations"
↓

Prop: If $T: \mathcal{C} \rightarrow \mathcal{C}$ is a monad, then there is a category $\text{Kl}(T) = \mathcal{C}_T$
with obj: $A \in \mathcal{C}$ same as in \mathcal{C}
arr $A \rightarrow B$: are morphisms $f: A \rightarrow T(B)$ in \mathcal{C}

"Kleisli category of T "

Pf: What is composition?

given $A \xrightarrow{f} T(B)$ and $B \xrightarrow{\delta} T(C)$ in \mathcal{C} ,

want $A \xrightarrow{f} TB \xrightarrow{Tg} T(C) \xrightarrow{\mu_C} T(C)$ in \mathcal{C}

What are identities?

want $A \xrightarrow{\eta_A} TA$ in \mathcal{C}

Ex ① $\text{Kl}(\text{Maybe})$:
 ||
 Obj: sets A, B, C
 $\text{arr } A \rightarrow B$ are functions $A \rightarrow B + 1$
 can be thought of as partial functions $A \rightarrow B$

② $\text{Kl}(\text{Powerset})$:
 ||
 Rel
 $\text{arr } A \rightarrow B$ are functions $A \xrightarrow{f} \mathcal{P}B$
 can be thought of as relations $R \subseteq A \times B$
$$R = \{(a, b) \mid b \in f(a)\}$$

$$f(a) = \{b \mid (a, b) \in R\}$$

Def: an (Eilenberg-Moore) algebra of a monad $T: \mathcal{C} \rightarrow \mathcal{C}$ consists of:

- an object $A \in \mathcal{C}$ \Leftarrow "carrier object"
- a morphism $TA \xrightarrow{a} A$ \Leftarrow "structure map"

such that

$$\begin{array}{ccc} T^2A & \xrightarrow{\quad Ta \quad} & TA \\ \downarrow \mu_A & & \downarrow a \\ TA & \xrightarrow{\quad a \quad} & A \end{array}$$

$$A \xrightarrow{\eta_A} TA \quad \begin{matrix} \nearrow \\ \searrow \end{matrix} \quad \downarrow a \quad \downarrow A$$

Prop: there is a category $EM(T) = \mathcal{C}^T$ with

obj: EM -algebras

and $(A, a) \rightarrow (B, b)$ are $f: A \rightarrow B$ in \mathcal{C} s.t.

$$\begin{array}{ccc} TA & \xrightarrow{\quad Tf \quad} & TB \\ \downarrow a & & \downarrow b \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

Ex: $EM(\text{Powerset}) = \text{Complete Lattices}$ \Leftarrow obj: poset (P, \leq) s.t. each subset has a least upper bound $a(P)$
arr: functions that preserve \vee

