Real Symmetric Matrices

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1 Introduction

This notes is about *real symmetric matrices* and its applications in TCS.

2 Eigenvalue decomposition

In general, the eigenvalues of a square matrix can be complex numbers; It is known that the eigenvalues and eigenvectors are reals.

Lemma 2.1. *The eigenvalues and eigenvectors of any real symmetric matrix are reals.*

Proof. Let λ and x be an eigenvalue and its corresponding eigenvector of A. Then $Ax = \lambda x$. Let $\bar{a}$ be the conjugate of a. Then $A\bar{x} = \bar{\lambda}\bar{x}$. We have

$$\lambda \bar{x}^T x = \bar{x}^T A x = \bar{\lambda} \bar{x}^T x,$$

which implies that $\lambda = \bar{\lambda}$. Therefore, $A\bar{x} = \lambda\bar{x}$, and thus $A(x + \bar{x}) = \lambda(x + \bar{x})$. As a result, $y = x + \bar{x}$ is a real eigenvector corresponding to λ. $\square$

Lemma 2.2. *If λ_1 and λ_2 are two distinct eigenvalues of A, then their corresponding eigenvector are orthogonal.*

Proof. We consider $v_1^T A v_2$.

$$v_1^T A v_2 = v_1^T \lambda_2 v_2 = \lambda_2 v_1^T v_2.$$

We also have

$$v_1^T A v_2 = v_1^T A^T v_2 = (Av_1)^T v_2 = \lambda_1 v_1^T v_2.$$

Since we assume $\lambda_1 \neq \lambda_2$, it follows that $v_1^T v_2 = 0$. $\square$

The fundamental theorem of real symmetric matrices is the spectral theorem.

Theorem 2.3. *For any real symmetric matrix A, we decompose it into $A = U\Sigma U^T$ such that U is orthonormal and Σ is nonnegative real diagonal matrix. Moreover, the columns of U are eigenvectors and the diagonal entries are corresponding eigenvalues.*

Proof. We present an analytical proof here. Let

$$v_1 = \arg\max_{x:\|x\|_2=1} x^T A x. \tag{1}$$

We will show that v_1 is an eigenvector of A corresponding to the eigenvalue of $v_1^T A v_1$. Note that, by compactness of unit circle, i.e. $\{x \mid \|x\|_2 = 1\}$, v_1 exists. Applying the method of Lagrange multipliers to the above optimization problem (1), we have $2Av_1 = \lambda v_1$ (setting the derivatives of the Lagrange function to zero, where the symmetric property of A is used.), i.e., v_1 is an eigenvector with eigenvalue λ.

Now we consider the subspace $V = \{v \mid v \perp v_1\}$. Using the same argument on V (replacing $\mathbb{R}^n$), we can prove that the vector v_2 such that

$$v_2 = \arg\max_{x:\|x\|_2=1 \text{ and } x\perp v_1} x^T A x \tag{2}$$

is also an eigenvector of A. By the same method, we can find n orthonormal eigenvectors of A. Let V be the matrix consists of $v_1, \cdots, v_n$ as columns, then $V^T A V = \Sigma$, where Σ is a diagonal matrix with eigenvalues on the diagonal. $\square$

3 Singular value decomposition

4 Finding ℓ_2 heavy hitters

First I will discuss a recent work by Braverman et.al. [1].

$$A \lesssim B$$

References

[1] V. Braverman, S. R. Chestnut, N. Ivkin, J. Nelson, D. P. Woodruff, and Z. Wang. Bptree: an ℓ_2 heavy hitters algorithm using constant memory. *arXiv preprint arXiv:1603.00759*, 2016.