1 Sequences

Definition 1.1 (Sequence) *A sequence is a function $f : \mathbb{N}^+ \to \mathbb{R}$ that maps positive natural numbers to real numbers, written as $(a_n)_{n\geq 1}$ where $a_n = f(n)$.*

Definition 1.2 (Arithmetic Sequence) *An arithmetic sequence is the sequence $f : \mathbb{N}^+ \to \mathbb{R}$ defined by*

$$f : n \mapsto \begin{cases} a_1, & n = 1 \\ a_1 + (n-1)d, & otherwise \end{cases}$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

Definition 1.3 (Geometric Sequence) *An geometric sequence is the sequence $f : \mathbb{N}^+ \to \mathbb{R}$ defined by*

$$f : n \mapsto ar^{n-1}$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

Definition 1.4 (Fibonacci Sequence) *An fibonacci sequence is the sequence $f : \mathbb{N}^+ \to \mathbb{R}$ defined by*

$$f : n \mapsto \begin{cases} 0, & n = 1 \\ 1, & n = 2 \\ f(n-1) + f(n-2) & n \geq 3 \end{cases}$$

Definition 1.5 (Monotonic) *A sequence is increasing if $a_{n+1} \geq a_n$ for $n \geq 1$ and decreasing if $a_{n+1} \leq a_n$ for $n \geq 1$. A sequence is monotonic if it is either increasing or decreasing.*

Theorem 1.6 (Triangle Inequality and Reverse)

$$|a + b| \leq |a| + |b| \ AND \ |a - b| \geq ||a| - |b||$$

Definition 1.7 (Cauchy Sequences) *A sequence is a Cauchy sequence iff for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n, m > N$ we have $|a_n - a_m| < \epsilon$.*

- *Every sequence that converges to a real number is a Cauchy sequence.*

Definition 1.8 (Completeness) *A subset $A \subseteq \mathbb{R}$ is complete iff any Cauchy sequence in A converges to a limit in A.*

- *$\mathbb{Q}$ is not complete but $\mathbb{R}$ is complete.*

Definition 1.9 (Subsequence) *If $f : \mathbb{N} \to \mathbb{R}$ is a sequence $(a_n)_{n\geq 1}$ and M is an infinite subset of $\mathbb{N}$, then $g : M \to \mathbb{R}$ is a subsequence $(a_{n_i})_{i\geq 1}$ of f.*

- *Any subsequence converges to the limit of the sequence.*
- *Any sequence of real numbers has a monotonic subsequence.*

Theorem 1.10 (Order Theory) *Let $X \subseteq \mathbb{R}$ and l, u, s and $i \in \mathbb{R}$*

1. *u is an **upper bound** of X if $x \leq u$ for all $x \in X$*
2. *l is an **lower bound** of X if $l \leq x$ for all $x \in X$*
3. *s is the **supremum** (least upper bound) of X if $s \leq u$ for all u of X*
4. *i is the **infimum** (greatest lower bound) of X if $l \leq i$ for all l of X*
5. *X is **bounded above** if X has an upper bound*
6. *X is **bounded below** if X has a lower bound*
7. *X is **bounded** if X has an upper and lower bound*

Theorem 1.11 (Dedekind-completeness of $\mathbb{R}$) *Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum (least upper bound).*

Theorem 1.12 (Fundamental Theorem of Analysis)
If $(a_n)_{n\geq 1}$ is increasing and bounded above, then $s = sup\{a_n | n \geq 1\}$ exists and is the limit of $(a_n)_{n\geq 1}$.
If $(a_n)_{n\geq 1}$ is decreasing and bounded below, then $i = inf\{a_n | n \geq 1\}$ exists and is the limit of $(a_n)_{n\geq 1}$.

Theorem 1.13 (Bolzano-Weierstrass Theorem) *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

1.1 Convergence Tests for Sequences

Definition 1.14 (Convergence to a limit) *Let $(a_n)_{n\geq 1}$ be a sequence which*

converges to a limit l in $\mathbb{R}$ *iff for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n > N$ we have $|a_n - l| < \epsilon$, for any $\epsilon > 0$.*

converges to $+\infty$ *iff for all r in $\mathbb{R}$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n > r$*

converges to $-\infty$ *iff for all r in $\mathbb{R}$ there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $a_n < r$*

written as $\lim_{n\to\infty} a_n = x$ or $(a_n)_{n\geq 1} \to x$, where x is the unique limit

diverges *if it does not converge to a real number, ∞ or $-\infty$*

Corollary 1.15 (Common convergent sequences)

$$(\frac{1}{n^c})_{n\geq 1} \to 0, \text{ when } c > 0$$

$$(\frac{1}{c^n})_{n\geq 1} \to 0, \text{ when } |c| > 1 \ OR \ (c^n)_{n\geq 1} \to 0, \text{ when } |c| < 1$$

$$(\frac{1}{n!})_{n\geq 1} \to 0$$

$$(\frac{1}{\log n})_{n\geq 1} \to 0$$

Theorem 1.16 (Limits of combination of sequences)
Given $(a_n)_{n\geq 1} \to a$ and $(b_n)_{n\geq 1} \to b$ with a real constant λ

$$(\lambda a_n)_{n\geq 1} \to \lambda a$$

$$(a_n + b_n)_{n\geq 1} \to a + b$$

$$(a_n b_n)_{n\geq 1} \to ab$$

$$(\frac{a_n}{b_n})_{n\geq 1} \to \frac{a}{b} \text{ provided } b \neq 0$$

Theorem 1.17 (Sandwich Theorem) *Let $(l_n)_{n\geq 1} \to l$ and $(u_n)_{n\geq 1} \to l$ for some real number l. If for $(a_n)_{n\geq 1}$ we have some $N \in \mathbb{N}$ such that $l_n \leq a_n \leq u_n$ for all $n \geq N$, then $(a_n)_{n\geq 1} \to l$ as well.*

Theorem 1.18 (Ratio Test for Sequences) *Let $c \in \mathbb{R}$ such that $0 \leq c \leq 1$. If there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|\frac{a_{n+1}}{a_n}| \leq c$, then $(a_n)_{n\geq 1} \to 0$.*

Theorem 1.19 (Limit Ratio Test for Sequences) *If $|\frac{a_{n+1}}{a_n}| \to r$ and $r < 1$, then $(a_n)_{n\geq 1} \to 0$.*

Definition 2.1 (Neighbourhood) *A set $A \subseteq \mathbb{R}$ is called a neighbourhood of a if there exists an open interval I where $a \in I \subseteq A$*

- *An open interval is a neighbourhood of each of its points*

Definition 2.2 (Accumulation Point) *A real number ξ is an accumulation point of a set $A \subseteq R$ if every neighbourhood of ξ contains an infinite number of members of A.*

Definition 2.3 (Limit of a Function) *$f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$, has a limit $l \in \mathbb{R}$*

at the accumulation point x_0 of A *if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in A$ and $|x - x_0| < \delta$, then $|f(x) - l| < \epsilon$*

as x approaches $+\infty$ *if for all $\epsilon > 0$ there exists c such that if $x > c$ then $|f(x) - l| < \epsilon$*

as x approaches $-\infty$ *if for all $\epsilon > 0$ there exists c such that if $x < c$ then $|f(x) - l| < \epsilon$*

written as $\lim_{x\to x_0} f(x) = l$ or $f(x) \to l$ as $x \to x_0$, where l is the unique limit

- *Let $f : I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ is an open interval and x_0 is an accumulation point of I, then $lim_{x\to x_0} f(x) = l \in \overline{\mathbb{R}}$ iff for all sequences of points of I with $(y_n)_{n\geq 1} \to x_0$ we have $lim_{n\to\infty} f(y_n) = l$*

Theorem 2.4 (Limits of combination of sequences)
Given $f, g : A \to \mathbb{R}$ have limits $k, l \in \mathbb{R}$ at accumulation point x_0 of A,

$$f \pm g \text{ has limit } k \pm l \text{ at } x_0$$

$$fg \text{ has limit } kl \text{ at } x_0$$

$$\frac{f}{g} \text{ has limit } \frac{k}{l} \text{ at } x_0 \text{ if } l \neq 0$$

Lemma 2.5 (Axiom of Choice) *For any collection χ of nonempty sets, there exists a choice function f that maps each set of χ to an element of that set.*

Lemma 2.6 (Axiom of Countable Choice) *Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of nonempty sets, then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in S_n$ for all $n \in \mathbb{N}$.*

Definition 2.7 (Continuity of functions) *$f : [a, b] \to \mathbb{R}$, where $x \in \mathbb{R}$, is*

continuous at $x_0 \in$ [A, B] *iff $\lim_{x\to x_0} f(x) = f(x_0)$ OR for every $\epsilon > 0$ there exists $\delta > 0$ such that for all x, if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$.*

continuous in [A, B] *iff f is continuous at all $x_0 \in [A, B]$*

Theorem 2.8 (Combination of continuous functions)
Given $f, g : A \to \mathbb{R}$ are continuous at x_0,

$$f \pm g \text{ is continuous at } x_0$$

$$fg \text{ is continuous at } x_0$$

$$\frac{f}{g} \text{ is continuous at } x_0 \text{ if } g(x_0) \neq 0$$

Theorem 2.9 (Composition of continuous functions) *If g is continuous at x_0 and f is continuous at $g(x_0)$, then $f \circ g$ is continuous at x_0. Note that f need not be continuous at x_0.*

Theorem 2.10 (Maxima and Minima) *If $f : [a, b] \to \mathbb{R}$ with $a, b \in \mathbb{R}$, then there exists $r, s \in [a, b]$ such that $f(r) = sup_{x\in[a,b]} f(x)$ and $f(s) = inf_{x\in[a,b]} f(x)$.*

- *A continuous function on a closed bounded interval is bounded and attains their supremum and infimum*

Theorem 2.11 (Intermediate Value Theorem) *If $f : [a, b] \to \mathbb{R}$ is continuous with $s \in \mathbb{R}$ such that $f(a) < s < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = s$.*

Definition 2.12 (Uniform Continuity) *$f : A \to \mathbb{R}$ is uniformly continuous if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, x_0 \in A$ we have $|f(x) - f(x_0)| < \epsilon$ if $|x - x_0| < \delta$. In other words, δ is independent of x_0 and only dependent on ϵ.*

Theorem 2.13 *If $f : [a, b] \to \mathbb{R}$, for $a, b \in \mathbb{R}$, is continuous then it is uniformly continuous on $[a, b]$.*

3 Integration

Definition 3.1 (Partition) *A partition P of $[a, b]$ is given by the finite set*

$$P = \{r_i : 0 \leq i \leq n-1, a = r_0, b = r_n, r_i < r_{i+1}\}$$

subinterval *of P is a closed interval $[r_i, r_{i+1}]$ for $0 \leq i \leq n-1$*

norm *of P is the largest length of the subintervals in P, or $\|P\| = max\{r_{i+1} - r_1 : 0 \leq i \leq n-1\}$*

*P_2 **refines** P_1 if $P_1 \subset P_2$*

Definition 3.2 (Sums) *Given $f : [a, b] \to \mathbb{R}$ and a partition P of $[a, b]$, the **Lower Sum** L(f,P) and **Upper Sum** U(f,P) are defined as*

$$L(f, P) = \sum_{i=0}^{n-1} (r_{i+1} - r_i) \times \inf_{x\in[r_i, r_{i+1}]} f(x), \ U(f, P) = \sum_{i=0}^{n-1} (r_{i+1} - r_i) \times \sup_{x\in[r_i, r_{i+1}]} f(x)$$

Riemann Sum *for P for any choice of $s_i \in [r_i, r_{i+1}]$ for $0 \leq i \leq n-1$ is*

$$S(f, P, (s_i)_{0\leq i\leq n-1}) = \sum_{i=0}^{n-1} (r_{i+1} - r_i) \times f(s_i)$$

- *$L(f, P) \leq S(f, P, (s_i)_{0\leq i\leq n-1}) \leq U(f, P)$*
- *If $P_1 \subset P_2$, then $L(f, P_1) \leq L(f, P_2) \leq U(f, P_2) \leq U(f, P_1)$*

Definition 3.3 (Integrals) ***Lower*** *and* ***Upper*** *integrals of $f : [a, b] \to \mathbb{R}$ are*

$$\underline{\int_a^b} f(x)\, dx = \sup_P L(f, P), \ \overline{\int_a^b} f(x)\, dx = \inf_P U(f, P)$$

Riemann Integral *$\int_a^b f(x)\, dx$ exists if $\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx$.*

$f : [a, b] \to \mathbb{R}$ is Riemann integrable with Riemann integral $c \in \mathbb{R}$ iff

- *for each $\epsilon > 0$ there exists a partition P of $[a, b]$ with $c - L(f, P) < \epsilon$ and $U(f, P) - c < \epsilon$*
- *for each $\epsilon > 0$ there exists a $\delta > 0$ such that for all partitions P of $[a, b]$ with $\|P\| < \delta$ we have $|S(f, P, (s_i)_{0\leq i\leq n-1})| < \epsilon$*

Theorem 3.4 *A bounded function $f : [a, b] \to \mathbb{R}$ with a countable set of discontinuities on $[a, b]$ is Riemann integrable.*

- *If f is continuous on $[a, b]$ then the Riemann integral $\int_a^b f(x)\, dx$ exists*

Theorem 3.5 (Properties of Riemann Integrals)

$$\int_a^b sf(x) + tg(x)\, dx = s\int_a^b f(x)\, dx + t\int_a^b g(x)\, dx, \text{ if f and g are integrable}$$

$$\int_a^b c\, dx = c(b - a), \text{ where } c \in \mathbb{R}$$

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx, \text{ if the integrals exists for } a < b < c$$

$$\text{If } f(x) \geq 0 \text{ for } x \in [a, b], \text{ then } \int_a^b f(x)\, dx \geq 0 \text{ if the integral exists}$$

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx \text{ if the integral exists}$$

Definition 3.6 (Improper Riemann Integral) *$f : [a, b) \to \mathbb{R}$ has improper Riemann integral (integral converges) if $\lim_{x\to b} \int_a^x f(x)\, dx \in \mathbb{R}$ exists.*
If the limit does not exists or is $\pm\infty$, the integral diverges.

4 Series

Definition 4.1 (Series) *Series are formal infinite sums of real numbers $\sum_{i=1}^\infty a_i$.*

Remark 4.1.1 *We can associate $\sum_{i=1}^\infty a_i$ to $(S_n)_{n\geq 1}$ where for each $n \geq 1$, S_n is defined as the partial sum $\sum_{i=1}^n a_i$.*

Definition 4.2 (Convergence) *The series $\sum_{i=1}^\infty a_i$*

converges to $l \in \mathbb{R}$ *iff $(S_n)_{n\geq 1}$ has limit $l \in \mathbb{R}$ OR $\sum_{i=N}^\infty a_i$ converges for any $N \in \mathbb{N}$*

diverges *if it does not converges to some $l \in \mathbb{R}$*

Theorem 4.3 (Increasing and bounded above) *When $a_i \geq 0$ for all $i \geq 1$, $(S_n)_{n\geq 1}$ is increasing. If $(S_n)_{n\geq 1}$ is also bounded above, by the Fundamental Theorem of Analysis, $\sum_{i=1}^\infty a_i$ converges to a limit.*

Definition 4.4 (Permutation) *A permutation π over the natural numbers $\mathbb{N}$ is a function $\pi : \mathbb{N} \to \mathbb{N}$ that has an inverse (injective & surjective).*

Definition 4.5 (Unconditional Convergence) *A series $\sum_{i=1}^\infty a_i$ is unconditionally convergent iff it converges and the permuted series $\sum_{i=1}^\infty a_{\pi(i)}$ converges to the same limit for all permutations π.*

Definition 4.6 (Absolute Convergence) *A series $\sum_{i=1}^\infty a_i$ is absolutely convergent iff the corresponding series $\sum_{i=1}^\infty |a_i|$ converges. Absolute convergence implies unconditional convergence.*

Definition 4.7 (Limit Superior) *The limit superior of $(a_n)_{n\geq 1}$ is the limit of $(b_n)_{n\geq 1} \in \overline{\mathbb{R}}$,where $b_n = sup\{a_m | m \geq n\}$, denoted $\limsup_{n\to\infty} a_n$*

Definition 4.8 (Limit Inferior) *The limit inferior of $(a_n)_{n\geq 1}$ is the limit of $(c_n)_{n\geq 1} \in \overline{\mathbb{R}}$,where $c_n = inf\{a_m | m \geq n\}$, denoted $\liminf_{n\to\infty} a_n$*

Lemma 4.9 (Known Divergent & Convergent Series)

Geometric series $\sum_{n=1}^\infty x^n \to \frac{x}{1-x}$ *for all $x \in \mathbb{R}$ with $|x| < 1$*

Inverse squares series $\sum_{n=1}^\infty \frac{1}{n^2} \to \frac{\pi^2}{6}$

$\frac{1}{n^c}$ **series** $\sum_{n=1}^\infty \frac{1}{n^c}$ *converges for all $c \in \mathbb{R}$ with $c > 1$*

Harmonic series $\sum_{n=1}^\infty \frac{1}{n}$ *diverges*

Harmonic primes $\sum_{p:prime}^\infty \frac{1}{p}$ *diverges*

Geometric series $S = \sum_{n=1}^\infty x^n$ *diverges for $|x| \geq 1$*

4.1 Convergence Tests for Series

Theorem 4.10 *If $\lim_{n\to\infty} a_n \neq 0$, $\sum_{i=1}^\infty a_i$ diverges.*

Theorem 4.11 (Comparison Test) *Let $\lambda > 0$ and $N \in \mathbb{N}$.*
If $a_i \leq \lambda c_i$ for all $i > N$ for some convergent series $\sum_{i=1}^\infty c_i$, $\sum_{i=1}^\infty a_i$ converges.
If $a_i \geq \lambda d_i$ for all $i > N$ for some divergent series $\sum_{i=1}^\infty d_i$, $\sum_{i=1}^\infty a_i$ diverges.

Theorem 4.12 (Limit Comparison Test)
If $\lim_{i\to\infty} \frac{a_i}{c_i} \in \mathbb{R}$ exists for some convergent series $\sum_{i=1}^\infty c_i$, $\sum_{i=1}^\infty a_i$ converges.
If $\lim_{i\to\infty} \frac{d_i}{a_i} \in \mathbb{R}$ exists for some divergent series $\sum_{i=1}^\infty d_i$, $\sum_{i=1}^\infty a_i$ diverges.

Theorem 4.13 (D'Alembert's Ratio Test) *Let $N \in \mathbb{N}$*
If there exists $k \in \mathbb{R}$ with $k < 1$ such that $\frac{a_{i+1}}{a_i} \leq k$ for all $i \geq N$, then $\sum_{i=1}^\infty a_i$ converges.
If $\frac{a_{i+1}}{a_i} \geq 1$ for all $i \geq N$, then $\sum_{i=1}^\infty a_i$ diverges.

Theorem 4.14 (D'Alembert's Limit Ratio Test) *If $\lim_{i\to\infty} \frac{a_{i+1}}{a_i}$ exists,*
if $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} < 1$, then $\sum_{i=1}^\infty a_i$ converges.
if $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} > 1$, then $\sum_{i=1}^\infty a_i$ diverges.
else $\lim_{i\to\infty} \frac{a_{i+1}}{a_i} = 1$, the test is inconclusive.

Theorem 4.15 (Integral Test) *Let $f : \mathbb{R} \to \mathbb{R}^+$ be continuous, decreasing and positive on $[N, \infty)$, where $N \in \mathbb{Z}$, with $a_n = f(n)$ for all $n \in \mathbb{N}$.*
If $\int_N^\infty f(x)\, dx$ converges, then $\sum_{i=1}^\infty a_i$ converges.
If $\int_N^\infty f(x)\, dx$ diverges, then $\sum_{i=1}^\infty a_i$ diverges.

Theorem 4.16 (Absolute Value Comparison Test) *Let $(b_n)_{n\geq 1}$ be a non-negative sequence such that $\sum_{i=1}^\infty b_i$ converges, and $(a_n)_{n\geq 1}$ be a sequence such that $|a_i| \leq b_i$ for all $i \geq 1$. Then $\sum_{i=1}^\infty a_i$ converges.*

Theorem 4.17 (Limit Absolute Value Ratio Test)
Let $\sum_{i=1}^\infty a_i$ with $a_i \neq 0$ for $i \geq 1$,
if $\lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| < 1$, then $\sum_{i=1}^\infty a_i$ converges and absolutely converges.
if $\lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| > 1$, then $\sum_{i=1}^\infty a_i$ diverges.
else $\lim_{i\to\infty} \left|\frac{a_{i+1}}{a_i}\right| = 1$, the test is inconslusive.

Theorem 4.18 (n^{th} Root Test) *Consider $\sum_{i=1}^\infty a_i$,*
if $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$, $\sum_{i=1}^\infty a_i$ converges and absolutely converges.
if $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} > 1$, $\sum_{i=1}^\infty a_i$ diverges.
else $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = 1$, the test is inconclusive.

$$\int f \cdot g'\, dx = f \cdot g - \int f' \cdot g\, dx$$

5 Differentiation

Definition 5.1 *Let $f : \mathbb{R} \to \mathbb{R}$, $x \in \mathbb{R}$ and $h > 0$.*

Newton's Difference Quotient at x for f *is given by*

$$\frac{\Delta f(x)}{\Delta(x)} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

f is differentiable at x *iff $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists as a real number and has the same value for all ways where $h \to 0$*

derivative of f at x *equals to this limit if it exists, denoted as $f'(x)$ or $\frac{dy}{dx}$*

Theorem 5.2 (Properties of derivatives)
Given $f, g : (a, b) \to \mathbb{R}$ be two functions,

1. *Polynomials have derivatives at all points*
2. *If f is differentiable at x, f is continuous at x*
3. *If f is differentiable in (a,b), $f'(x_0) = 0$ for any point x_0 where f is maximum or minimum*
4. *If f and g are differentiable at x, $f \cdot g$ is differentiable under* ***product rule***
$$(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$
5. *If f and g are differentiable at g(x) and x respectively, $f \circ g$ is differentiable under* ***product rule***
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$
6. *Differentiation is a* ***linear function****, where for all f and g differentiable at x and for all $a, b \in \mathbb{R}$*
$$(a \cdot f + b \cdot g)'(x) = a \cdot f'(x) + b \cdot g'(x)$$

Theorem 5.3 (Rolle's Theorem) *If $f : [a, b] \to \mathbb{R}$ is continuous and $f : (a, b) \to \mathbb{R}$ is differentiable with $f(a) = f(b)$, there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Theorem 5.4 (Mean Value Theorem) *If $f : [a, b] \to \mathbb{R}$ is continuous and $f : (a, b) \to \mathbb{R}$ is differentiable, there exists $c \in (a, b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.*

- *If x is close to x_0, $f(x) = f(x_0) + f'(x_0)(x - x_0) + E$, where E is small*

Theorem 5.5 (Taylor's Theorem) *If f is n times differentiable in (a, b) with $x_0 \in (a, b)$, then for any $x \in (a, b)$ we have*

$$f(x) = f(x_0) + \frac{1}{1!}(x - x_0)f'(x_0) + ... + \frac{1}{n!}(x - x_0)^n f^{(n)}(x_0) + E_n$$

where $E_n = \frac{f^{(n+1)}(x^*)}{(n+1)!}(x - x_0)^{n+1}$ is the ***Lagrange error term*** *with x^* between x and x_0.*

Theorem 5.6 (L'Hospital's Rule) *Suppose $f, g : (a, b) \to \mathbb{R}$ have derivatives $f', g' : (a, b) \to \mathbb{R}$ that are continuous in (a, b).*

- *If $f(c) = g(c) = 0$ for some $c \in (a, b)$ and $g'(c) \neq 0$,*
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$
- *If $\lim_{x\to c}|f(x)| = \lim_{x\to c}|g(x)| = \infty$ for some $c \in (a, b)$*
$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$$
- *It can be extended to $x \to \infty$ by restricting $f, g : [0, \infty) \to [0, \infty)$*
$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{y\to 0} \frac{f(\frac{1}{y})}{g(\frac{1}{y})}$$

Theorem 5.7 (Fundamental Theorem of Calculus) *If $f : [a, b] \to \mathbb{R}$ is continuous and $F : [a, b] \to \mathbb{R}$ is defined by $F(x) = \int_a^x f(t)\,dt$, then F is uniformly continuous on [a, b] and $F'(x) = f(x)$ for $x \in (a, b)$.*

Corollary 5.8 (Change of variable) *Let $g : [a, b] \to [c, d]$ be a differentiable function with $g' : [a, b] \to \mathbb{R}$ and $f : [c, d] \to \mathbb{R}$ be a continuous function with $y = g(x)$.*

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy$$

6 Power Series

Definition 6.1 *A power series is a series of the form $\sum_{i=o}^{\infty} a_i \cdot (x - c)^i$ where x is a variable $\in \mathbb{R}$, c is a constant $\in \mathbb{R}$ and $(a_n)_{n\geq 0} \subseteq \mathbb{R}$.*

- *Polynomials are power series where $c = 0$ and there exists N such that $a_n = 0$ for all $n \geq N$. They converge for all $x \in \mathbb{R}$.*

Definition 6.2 (Radius of Convergence) *Let $c \in \mathbb{R}$ and $(a_n)_{n\geq 0} \subseteq \mathbb{R}$. $\sum_{i=o}^{\infty} a_i \cdot (x - c)^i$ has a radius of convergence $r \in [0, \infty) \cup \{\infty\}$ such that:*

1. *If $r \neq \infty$, then $\sum_{i=o}^{\infty} a_i \cdot (x - c)^i$ converges for all $x \in \mathbb{R}$ when $|x - c| < r$ and diverges for all $x \in \mathbb{R}$ when $|x - c| > r$.*
2. *If $r = \infty$, then $\sum_{i=o}^{\infty} a_i \cdot (x - c)^i$ converges for all $x \in \mathbb{R}$*

given by $r^{-1} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$.

Theorem 6.3 (Ratio Test) *Suppose $(\frac{|a_{n+1}|}{|a_n|})_{n\geq 1}$ has a limit $l \in \mathbb{R}$, then setting $l < 1$ and rewriting to $|x| < r$ gives the radius of convergence of any $\sum_{i=o}^{\infty} a_i \cdot (x - c)^i$.*

Definition 6.4 (Smoothness) *$f : \mathbb{R} \to \mathbb{R}$ is smooth at x_0 if for all $k \geq 1$ the k^{th} derivative exists at x_0.*

Definition 6.5 (Analytical) *Given $f : \mathbb{R} \to \mathbb{R}$, if the power series has the same outputs as f within the radius of convergence, f is a real analytical function.*

- *Not every smooth real function is analytical.*

Definition 6.6 (Maclaurin Series) *$\sum_{i=o}^{\infty} a_i \cdot (x - c)^i$ is called the Maclaurin Series when $c = 0$, or $f(x) = \sum_{i=o}^{\infty} a_i \cdot x^i$.*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n, \text{ with } r^{-1} = \limsup_{n\to\infty} \left|\frac{f^{(n)}(0)}{n!}\right|^{\frac{1}{n}}$$

Definition 6.7 (Taylor Series)

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x - c)^n$$

Theorem 6.8 *Within the radius of convergence, $\sum_{i=o}^{\infty} a_i \cdot (x - c)^i$ is continuous and can be differentiated and integrated term by term.*

7 Numerical Methods

Theorem 7.1 (Newton's Method) *Approximates $f(x) = 0$*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The rate of convergence is at least quadratic if

1. *$f'(x) \neq 0$ for $x \in I$ where $I = [\alpha - r, \alpha + r]$ for some $r \geq |\alpha - x_0|$*
2. *$f''(x)$ is continuous in I*
3. *x_0 is sufficiently close to α*

Theorem 7.2 (Relaxed Newton's Method) *For some $0 < \gamma \leq 1$,*

$$x_{n+1} = x_n - \gamma \frac{f(x_n)}{f'(x_n)}$$

Theorem 7.3 (Secant Method) *Reduces calculation of derivatives*

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

Theorem 7.4 (Gradient Descent) *Minimises $f(x)$ using a small enough η*

$$x_{n+1} = x_n - \eta f'(x_n)$$

Theorem 7.5 (Euler's Method) *Approximates the solution to $x = x_0 + h$, where h is small, in differential equations. Given $y' = f(x, y)$ and $y(x_0) = y_0$,*

$$x_{n+1} = x_0 + nh,\ y_{n+1} = y_n + hf(x_n, y_n), \text{ for } n = 0, 1, ...$$

Theorem 7.6 (Heun's Method) *A predictor-corrector method to modify Euler's Method. First approximate $y(x_{n+1})$ with $y^*_{n+1} = y_n + hf(x_n, y_n)$.*

$$y_{n+1} = y_n + \frac{1}{2}h[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})]$$

Theorem 7.7 (Runge-Kutta Method of Order Four) *Uses Simpson's Rule*

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = hf(x_n, y_n), \quad k_2 = hf(x_n + \tfrac{h}{2}, y_n + \tfrac{k_1}{2})$$
$$k_3 = hf(x_n + \tfrac{h}{2}, y_n + \tfrac{k_2}{2}), \quad k_4 = hf(x_{n+1}, y_n + k_3)$$

8 Metric Spaces

Definition 8.1 (Sequence of Functions) *Let $I \subseteq \mathbb{R}$. For each $n \in \mathbb{N}$, $f_n(x) : I \to \mathbb{R}$. Then (f_n) is a sequence of functions on I.*

Definition 8.2 (Pointwise Convergence) *(f_n) converges pointwise to f iff $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in I$, written as $\lim_{n\to\infty} f_n = f$ pointwise.*

Definition 8.3 (Uniform Convergence) *(f_n) converges uniformly to f iff $\lim_{n\to\infty} \sup\{|f_n(x) - f(x)|\,|x \in I\} = 0$, written as $\lim_{n\to\infty} f_n = f$ uniformly.*

- *Every uniformly convergent sequence is pointwise convergent to the same limiting function.*
- *The pointwise limit of a sequence of continuous functions may be a discontinuous function but only if the sequence is not uniformly convergent.*

Definition 8.4 (Ring) *A ring is a set R equipped with 2 binary operators + (addition) and $\cdot$ (multiplication) satisfying the ring axioms.*

1. *R is an* ***abelian group*** *under addition, meaning for all $a \in R$*
   (a) *+ is associative*
   (b) *+ is commutative*
   (c) *There exists $0 \in R$ which is the additive identity $(a + 0 = a)$*
   (d) *-a is the additive inverse of a $(a + (-a) = 0)$*
2. *R is a* ***monoid*** *under multiplication, meaning for all $a \in R$*
   (a) *$\cdot$ is associative*
   (b) *There exists $1 \in R$ which is the multiplicative identity $(a \cdot 1 = 1 \cdot a = a)$*
3. *Multiplication is* ***distributive*** *to addition, meaning for all $a, b, c \in R$*
   (a) *$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ (left associative)*
   (b) *$(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ (right associative)*

Definition 8.5 (Fields) *Fields are commutative rings with identity $(1 \neq 0)$ in which every nonzero element has a multiplicative inverse.*

Definition 8.6 (Distance) *$d : X \times X \to \mathbb{R}$ is a distance function or metric on the underlying set X if*

1. *$d(x, y) = 0$ iff $x = y$ (identity of indiscernibles)*
2. *$d(x, y) = d(y, x)$ (symmetry)*
3. *$d(x, z) \leq d(x, y) + d(y, z)$ (subadditivity or triangle inequality)*

Definition 8.7 (Metric Space) *An ordered pair (X, d) consisting of a nonempty set X and distance function d on X is a metric space*

- *If (X,d) is a metric space and $S \subset X$, (S,d) is a metric subspace of (X,d)*

Definition 8.8 (Limit) *Let (X,d) be a metric space and $(x_n)_{n\geq 1}$ be a sequence of points in X, a point $x \in X$ is the limit of $(x_n)_{n\geq 1}$ if $\lim_{n\to\infty} d(x, x_n) = 0$*

Definition 8.9 (Open Ball) *Let (X,d) be a metric space, $a \in X$ and $\delta > 0$. The subset of X containing all points $x \in X$ such that $d(a, x) < \delta$ is called the open ball about a of radius delta, denoted by $B(a; \delta)$.*

Definition 8.10 (Neighbourhood) *Let (X,d) be a metric space and $a \in X$. A subset N of X is a neighbourhood of a if there exists $\delta > 0$ such that $B(a; \delta) \subseteq N$. The* ***complete system of neighbourhoods*** *of the point a $\mathcal{N}_a$ is the collection of all neighbourhoods of a. For all a and any neighbourhood N of a,*

1. *there exists at least one neighbourhood of a*
2. *$a \in N$*
3. *if $N' \supseteq N$, then N' is a neighbourhood of a*
4. *if M is another neighbourhood of a, $N \cap M$ is also a neighbourhood of a*
5. *there exists a neighbourhood O of a such that $O \subseteq N$ such that O is a neighbourhood of each of its points*

Lemma 8.11 (First Axiom of Countability) *Let (X,d) be a metric space. For every $a \in X$, there is a sequence of neighbourhoods of a $(O_n)_{n\geq 1}$ such that N contains at least one neighbourhood of this sequence.*

Lemma 8.12 (Hausdorff Axiom) *For every pair of distinct points x,y of (X,d), there is a neighbourhood M of X and N of Y sucht that $M \cap N = \varnothing$.*

Theorem 8.13 *Let (X,d) be a metric space and $a \in X$. For each $\delta > 0$, the open ball $B(a; \delta)$ is a neighbourhood of each of its points.*

Definition 8.14 (Open Set) *A subset O of a metric space (X,d) is open if O is a neighbourhood of each of its points.*

- *O is an open set iff it is a union of open balls*

1. *$\varnothing$ and X is open*
2. *The union and intersection of open sets is open*

Definition 8.15 (Function Composition) *Let (X,d), (Y,d') and (Z,d") be metric spaces. Also let $f : X \to Y$ be continuous at $a \in X$ and $g : Y \to Z$ be continuous at $f(a) \in Y$. Then $g \circ f : X \to Z$ is continuous at $a \in X$.*

Definition 8.16 (Topology) *An ordered pair (X, τ) consisting of a set X and a collection τ of subsets of X satisying the following axioms:*

1. *$\varnothing$ and X belongs to τ*
2. *any arbitrary union of members of τ belong to τ*
3. *the intersection of any finite members of τ belongs to τ*

with the elements of τ called ***open sets*** *and τ is called a topology on X. A subset $C \subseteq X$ is said to be closed in (X, τ) iff its complement $X \backslash C$ is open.*

Definition 8.17 (Homeomorphism) *(X, τ) and (X', τ') are homeomorphic if there exists mutually inverse continuous functions $f : X \to X'$ and $g : X' \to X$. f and g then define a homeomorphism between (X, τ) and (X', τ').*

- *Homeomorphisms form an equivalence relation on the class of topological spaces. The resulting equivalence classes are* ***homeomorphism classes****.*

Definition 8.18 (Embedding) *A map f between (X, τ) and (X', τ') is a topological embedding if f yields a homeomorphism between X and f(X), where f(X) carries the relative topology inherited from X', denoted by $X \hookrightarrow X'$.*

- *X can be treated as a subspace of X'*

Definition 8.19 (Compactness) *(X, τ) is compact if each of its open covers has a finite subcover, that is for any collection C of open subsets of X such that $X = \bigcup_{x\in C} x$ there exists a finite subset $F \subseteq C$ such that $X = \bigcup_{x\in F} x$.*

Theorem 8.20 (Heine-Borel Theorem) *For any subset A of a Euclidean space, A is compact iff A is closed and bounded.*

- *Closed intervals are compact*

Definition 8.21 (Continuity)
(Metric Spaces) *$f : (X, d) \to (Y, d')$ is continuous at $a \in X$ if*

- ***(Epsilon-Delta)*** *for any $\epsilon > 0$, there exists $\delta > 0$ such that*
  - *$d'(f(x), f(a)) < \epsilon$ whenever $x \in X$ and $d(x, a) < \delta$*
  - *$f(B(a; \delta)) \subseteq B(f(a); \epsilon)$ OR $B(a; \delta) \subseteq f^{-1}(B(f(a); \epsilon))$*
- ***(Neighbourhood)*** *for any neighbourhood M of f(a)*
  - *there exists a corresponding neighbourhood N of a such that $f(N) \subseteq M$ OR $N \subseteq f^{-1}(M)$*
  - *$f^{-1}(M)$ is a neighbourhood of a*
- ***(Open Set)*** *for any open set O of Y, the subset $f^{-1}(O)$ is an open subset of X*

(Topology) *$f : (X, \tau) \to (X', \tau')$ is continuous at $x \in X$ if for any neighbourhood G of f(x), where $G \in \tau'$, $f^{-1}(G)$ is a neighbourhood of x.*

9 Deep Learning and Multivariate Chain Rule

Definition 9.1 (Perceptron) *The function $f : \boldsymbol{x} \mapsto \sigma(\boldsymbol{w}^\intercal \boldsymbol{x} + b)$ where w is the vector of weights, b the scalar bias (offset), and the activation function σ is the Heaviside step function.*

$$\sigma(v) = \begin{cases} 0, & v < 0 \\ 1, & v \geq 0 \end{cases}$$

function	w_1	w_2	b
$\neg$	-1		0
$\wedge$	0.5	0.5	-1
$\vee$	1	1	-1

Definition 9.2 (Loss Function) *Let $\hat{y}$ be the output of $\sigma(\boldsymbol{w}^\intercal \boldsymbol{x} + b)$ and y the true value (target). The* ***zero-one loss function*** *is defined as*

$$l_{0-1}(y, \hat{y}) = \begin{cases} 0, & \hat{y} \neq y \\ 1, & \hat{y} = y \end{cases}$$

which is a piecewise constant function of the weights and bias. Hence we use the ***surrogate loss function***

$$l_{SE}(y, \hat{y}) = \frac{1}{2}(y - \hat{y})^2$$

Theorem 9.3 (Gradient Descent)

$$\boldsymbol{w}_{n+1} = \boldsymbol{w}_n - \gamma \frac{\partial l_{SE}}{\partial \boldsymbol{w}} = \boldsymbol{w}_n - \gamma(\hat{y} - y)\boldsymbol{x}$$

$$b_{n+1} = b_n - \gamma \frac{\partial l_{SE}}{\partial b} = b_n - \gamma(\hat{y} - y)$$

Definition 9.4 (Multilayer Perceptron) *or feedforward network*

$$\hat{\boldsymbol{Y}}(\boldsymbol{X}) := \boldsymbol{F}_{\boldsymbol{W},\boldsymbol{b}}(\boldsymbol{X}) = \left(\boldsymbol{f}^{(L)}_{\boldsymbol{W}^{(L)},\boldsymbol{b}^{(L)}} \circ ... \circ \boldsymbol{f}^{(1)}_{\boldsymbol{W}^{(1)},\boldsymbol{b}^{(1)}}\right)(\boldsymbol{X})$$

- *If we take σ to be the identity function in each layer, the MLP becomes a* ***linear regression***

Theorem 9.5 (Chain Rule)

$$(f \circ g)' = (f' \circ g) \cdot g', \text{ in Lagrange's notation}$$

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}, \text{ in Leibniz's notation}$$

Theorem 9.6 (Multivariate Chain Rule) *Let $g : \mathbb{R}^p \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$, with $\boldsymbol{b} = g(\boldsymbol{a})$ and $\boldsymbol{c} = f(\boldsymbol{b})$ where $\boldsymbol{a} \in \mathbb{R}^p$, $\boldsymbol{b} \in \mathbb{R}^n$ and $\boldsymbol{c} \in \mathbb{R}^m$.*

$$\frac{\partial c_i}{\partial a_j} = \sum_{k=1}^{n} \frac{\partial c_i}{\partial b_k} \cdot \frac{\partial b_k}{\partial a_j}$$

Maclaurin expansion:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \quad (|x| < 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots \quad \text{(all } x\text{)}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots \quad \text{(all } x\text{)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^r x^{2r}}{(2r)!} + \ldots \quad \text{(all } x\text{)}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{r+1} x^r}{r} + \ldots \quad (-1 < x \leq 1)$$