Propositional Logic

Binding Conventions (Strongest) $(\neg, \forall x, \exists x), \wedge, \vee, \rightarrow, \leftrightarrow$ (Weakest)

Principle Connective Connective at the root (top) of a formation tree. A formula with principle connective $\leftrightarrow$ is said to have the **logical form** $A \leftrightarrow B$.

Atomic Formula of the form $\top, \bot, p$ for an atom p.

Literal Formula that is atomic or negated-atomic.

Clause Disjunction of one or more literals.

Situation Determines whether each propositional atom is true or false.

Valid Argument Given formulas $A_1, A_2, \ldots, A_n, B$ an argument $A_1, A_2, \ldots, A_n \vDash B$ is valid if B is true in any situation in which $A_1, A_2, \ldots, A_n$ are all true. Here $\vDash$ denotes logical entailment.

Valid Formula A formula A is valid if it is true in every situation, i.e. $\vDash A$. A **tautology** is a valid propositional formula.

Satisfiable Formula True in at least one situation.

Equivalent Formulas True in exactly the same situations, i.e. $A \equiv B$.

Disjunctive Normal Form Formula as a disjunction of conjunctions of literals, not further simplifiable.

Conjunctive Normal Form Formula as a conjunction of disjunction of literals, not further simplifiable.

Theorem Formula that can be established by a given proof system, i.e. any A such that $\vdash A$. (Note that $\vdash$ is syntactic whilst $\vDash$ is semantic - $A_1, A_2, \ldots, A_n \vDash B$ means there is a proof of B starting with $A_1, A_2, \ldots, A_n$ as givens).

Soundness Any provable formula is valid, i.e. if $A_1, A_2, \ldots, A_n \vdash B$ then $A_1, A_2, \ldots, A_n \vDash B$.

Completeness Any valid formula can be proved, i.e. if $A_1, A_2, \ldots, A_n \vDash B$ then $A_1, A_2, \ldots, A_n \vdash B$.

Provably equivalent Show that A $\vdash$ B and B $\vdash$ A

Consistency A formula is consistent if $\nvdash \neg A$. So a formula is consistent if and only if it is satisfiable.

Signature Collection of constants and relation symbols and function symbols with specified arities.

Term For a signature L:

1. Any constant in L is an L-term.
2. Any variable is an L-term.
3. For an n-ary function symbol f in L and L-terms $t_1, t_2, \ldots, t_n$, $f(t_1, t_2, \ldots, t_n)$ is an L-term.

For an L-formula A and a variable x, $(\forall x A)$ and $(\exists x A)$ are L-formulas. $M, h \vDash \forall x A$ if $M, g \vDash A$ for every assignment g into M with $g =_x h$ and $M, h \vDash \exists x A$ if $M, g \vDash A$ for some assignment g into M with $g =_x h$. (The notation $g =_x h$ here means g agrees with h except perhaps on x).

Closed / Ground Term Does not involve a variable.

Bound Variable For a formula A and variable x, x is bound if it lies under a quantifier $\forall x$ or $\exists x$ in the formation tree of A.

Free Variable Variable which is not bound (this includes variables which do not appear in A!).

Sentence Formula with no free variables. (Does not require an assignment for evaluation).

Predicate Formula $R(t_1, t_2, \ldots, t_n)$ $\top, \bot$ $t_1 = t_2$ $(\neg A), (A \wedge B), (A \vee B), (A \rightarrow B), (A \leftrightarrow B)$ $(\forall x A)$ and $(\exists x A)$ are L-formulas.

Many-Sorted Predicate Logic

Term

1. Each variable and constant comes with a sort s. We indicate this as x : s
2. Each n-ary function symbol f comes with a template $f : (\mathrm{s}_1, \mathrm{s}_2, \ldots, \mathrm{s}_n) \rightarrow \mathrm{s}$.

Formula

1. Each n-ary relation symbol R comes with a template $R(\mathrm{s}_1, \mathrm{s}_2, \ldots, \mathrm{s}_n)$.
2. $t_1 = t_2$ is a formula if t_1, t_2 have the same sort.

It is polite to indicate the sort of a variable in $\forall, \exists$, e.g. $\forall x$: lecturer$\exists y$: Sun $(\text{bought}_{lecturer,Sun}(x, y))$.

Propositional Logic

argument validity	formula validity	satisfiability	equivalence
$\phi \vDash \psi$	$\phi \rightarrow \psi$ valid	$\phi \wedge \neg\psi$ unsatisfiable	$(\phi \rightarrow \psi) \equiv \top$
$\top \vDash \phi$	ϕ valid	$\neg\phi$ unsatisfiable	$\phi \equiv \top$
$\phi \nvDash \bot$	$\neg\phi$ not valid	ϕ satisfiable	$\phi \not\equiv \bot$
$\phi \vDash \psi$ and $\psi \vDash \phi$	$\phi \leftrightarrow \psi$ valid	$\phi \leftrightarrow \neg\psi$ unsatisfiable	$\phi \equiv \psi$

1. Take an arbitrary situation.
2. Prove that the formula is true in this situation. (Often this will require the law of excluded middle - argument by cases).

Predicate Logic To show the argument $A_1, A_2, \ldots, A_n \vDash B$ is valid:

1. Consider any M such that $M \vDash A_1, M \vDash A_2, \ldots, M \vDash A_n$.
2. Show $M \vDash B$, e.g.:
   (a) $M \vDash \forall x (B(x))$: Consider an arbitrary object a in dom(M). Show $M \vDash B(a)$.
   (b) $M \vDash \exists x (B(x))$: Consider any object b in dom(M). Show $M \vDash B(b)$.

Equivalences

$\neg$

1. $\neg\top \equiv \bot$
2. $\neg\bot \equiv \top$
3. $\neg\neg A \equiv A$
4. $\neg(A \wedge B) \equiv \neg A \vee \neg B$ (De Morgan)
5. $\neg(A \vee B) \equiv \neg A \wedge \neg B$ (De Morgan)

$\wedge$

1. $A \wedge B \equiv B \wedge A$ (Commutativity)
2. $A \wedge A \equiv A$ (Idempotence)
3. $A \wedge \top \equiv A$
4. $\bot \wedge A \equiv \neg A \wedge A \equiv \bot$
5. $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$ (Associativity)
6. $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ (Distributivity)
7. $A \wedge (A \vee B) \equiv A$ (Absorption)

$\vee$

1. $A \vee B \equiv B \vee A$ (Commutativity)
2. $A \vee A \equiv A$ (Idempotence)
3. $\top \vee A \equiv \neg A \vee A \equiv \top$
4. $A \vee \bot \equiv A$
5. $(A \vee B) \vee C \equiv A \vee (B \vee C)$ (Associativity)
6. $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ (Distributivity)
7. $A \vee (A \wedge B) \equiv A$ (Absorption)

$\rightarrow$

1. $A \rightarrow A \equiv \top$
2. $\top \rightarrow A \equiv A$
3. $A \rightarrow \top \equiv \top$
4. $\bot \rightarrow A \equiv \top$
5. $A \rightarrow \bot \equiv \neg A$
6. $A \rightarrow B \equiv \neg A \vee B \equiv \neg(A \wedge \neg B)$
7. $\neg(A \rightarrow B) \equiv A \wedge \neg B$

$\leftrightarrow$

1. $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A) \equiv (A \wedge B) \vee (\neg A \wedge \neg B) \equiv \neg A \leftrightarrow \neg B$
2. $\neg(A \leftrightarrow B) \equiv A \leftrightarrow \neg B \equiv \neg A \leftrightarrow B \equiv (A \wedge \neg B) \vee (\neg A \wedge B)$

$\forall, \exists$

1. $\forall x \forall y A \equiv \forall y \forall x A$
2. $\exists x \exists y A \equiv \exists y \exists x A$
3. $\neg\forall x A \equiv \exists x \neg A$
4. $\neg\exists x A \equiv \forall x \neg A$
5. $\forall x (A \wedge B) \equiv \forall x A \wedge \forall x B$
6. $\exists x (A \vee B) \equiv \exists x A \vee \exists x B$

For A in which x does not Occur Free:

1. $A \equiv \forall x A \equiv \exists x A$
2. $\exists x (A \wedge B) \equiv A \wedge \exists x B$
3. $\forall x (A \vee B) \equiv A \vee \forall x B$
4. $\exists x (A \rightarrow B) \equiv A \rightarrow \exists x B$
5. $\forall x (A \rightarrow B) \equiv A \rightarrow \forall x B$
6. $\exists x (B \rightarrow A) \equiv \forall x B \rightarrow A$*
7. $\forall x (B \rightarrow A) \equiv \exists x B \rightarrow A$*

* Watch out for these two cases!

Natural Deduction

Modus Tollens

1 $A \rightarrow B$
2 $\neg B$
3 $\neg A$ MT(1, 2)

$\vee$-Intro

1 A
2 $A \vee B$ $\wedge I(1)$
3 $B \vee A$ $\wedge I(1)$

$\vee$-Elim

1 $A \vee B$
2 A ass
3 C
4 B ass
5 C
6 C $\vee E(1,2,3,4,5)$

$\forall\rightarrow$ Elim

1 $P(c,d)$
2 $\forall x \forall y [P(x,y) \rightarrow Q(x)]$
3 $Q(c)$ $\forall\rightarrow E(1,2)$

$\rightarrow$-Intro

1 A ass
2 B
3 $A \rightarrow B$ $\rightarrow I(1,2)$

$\rightarrow$-Elim

1 $A \rightarrow B$
2 A
3 B $\rightarrow E(1,2)$

Proof by Contradiction

1 $\neg A$ ass
2 $\bot$
3 A $PC(1,2)$

$\bot$-Intro

1 A
2 $\neg A$
3 $\bot$ $\bot I(1,2)$

$\leftrightarrow$-Intro

1 $A \rightarrow B$
2 $B \rightarrow A$
3 $A \leftrightarrow B$ $\leftrightarrow I(1,2)$

$\leftrightarrow$-Elim

1 $A \leftrightarrow B$
2 A
3 B $\leftrightarrow E(1,2)$

$\neg$-Intro

1 A ass
2 $\bot$
3 $\neg A$ $\neg I(1,2)$

$\forall$-Intro

2 c $\forall I$ const
3 $A(c/x)$
4 $\forall x A$ $\forall I(1,2)$

$\neg\neg$-Elim

1 $\neg\neg A$
2 A $\neg\neg E(1)$

$\bot$-Elim

1 $\bot$
2 A $\bot E(1)$

$\exists$-Intro

1 $A(t/x)$
2 $\exists x A$ $\exists I(1)$

$\exists$-Elim

1 $\exists x A$
2 $A(c/x)$ ass
3 B
4 B $\exists E(1,2,3)$

$\forall$-Elim

1 $\forall x A$
2 $A(t/x)$ $\forall E(1)$

Substitution

1 $A(t/x)$
2 $t = u$
3 $A(u/x)$ sub(1, 2)

Symmetry

1 $c = d$
2 $d = c$ sym(1)

Reflexivity

1 $t = t$ refl

Excluded Middle

1 $A \vee \neg A$ lemma

Sets

Union: $A \cup B \triangleq \{x | x \in A \vee x \in B\}$.

Intersection: $A \cap B \triangleq \{x | x \in A \wedge x \in B\}$.

Difference: $A \backslash B \triangleq \{x | x \in A \wedge x \notin B\}$.

Symmetric Difference: $A \triangle B \triangleq (A \backslash B) \cup (B \backslash A)$

1. Idempotence $A \cup A = A$
2. Commutativity $A \cup B = B \cup A$
3. Associativity $A \cup (B \cup C) = (A \cup B) \cup C$
   $A \cap (B \cap C) = (A \cap B) \cap C$
4. Distributivity $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
5. Absorption $A \cup (A \cap B) = A$
   $A \cap (A \cup B) = A$

$$\wp\{a,b\} = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}$$
$$\wp\,\emptyset = \{\emptyset\}$$

1. *Subset*: $A \subseteq B \triangleq \forall x \in A\,(x \in B)$.
2. *Equality*: $A = B \triangleq A \subseteq B \wedge B \subseteq A$.

Theorem 2.26 (THE PIGEONHOLE PRINCIPLE) *If a set of n distinct objects is partitioned into k subsets, where $0 < k < n$, then at least one subset contains at least two elements.*

Relations

Identity $\mathrm{id}_A = \{\langle x, y\rangle \in A^2 | x = y\}$.

Composition For $R \subseteq A \times B, S \subseteq B \times C$:
$R \circ S \triangleq \{\langle a, c\rangle \in A \times C | \exists b \in B\,(a\,R\,b \wedge b\,R\,c)\}$.

Union: $R \cup S \triangleq \{\langle a, b\rangle \in A \times B | \langle a, b\rangle \in R \vee \langle a, b\rangle \in S\}$.

Intersection: $R \cap S \triangleq \{\langle a, b\rangle \in A \times B | \langle a, b\rangle \in R \wedge \langle a, b\rangle \in S\}$.

Complement: $\overline{R} \triangleq \{\langle a, b\rangle \in A \times B | \langle a, b\rangle \notin R\}$.

Inverse: $R^{-1} \triangleq \{\langle b, a\rangle \in A \times B | a\,R\,b\}$.

Proposition 3.9 1) If $R \subseteq A \times B$, then $Id_A \circ R = R = R \circ Id_B$.

2) Composition is associative: for arbitrary relations $R \subseteq A \times B$ and $S \subseteq B \times C$ and $T \subseteq C \times D$, we have $R \circ (S \circ T) = (R \circ S) \circ T$.

Transitive Closure *Transitive closure*: $a\,R^+\,b = \exists n \geq 1\,(a\,R^n\,b)$, i.e. $R^+ = \cup_{i \geq 1} R^i$. Contains at 'paths' in A through R. This is the smallest transitive relation containing R.

2) We write R^* for the reflexive and transitive closure of R.

3) The *transitive reduction* R^- of a transitive relation R is a smallest (it need not be unique) set S such that $S \subseteq R$, and $S^+ = R$. So

$$R^- = \{\langle a,b\rangle \in R \mid \neg\exists c \in A\,(a \neq c \wedge b \neq c \wedge \langle a,c\rangle \in R \wedge \langle c,b\rangle \in R)\}.$$

Proposition 3.12 Let R be a binary relation on A.

1) R is reflexive if and only if $Id_A \subseteq R$.

2) R is symmetric if and only if $R = R^{-1}$.

3) R is transitive if and only if $R \circ R \subseteq R$.

R is reflexive $\triangleq \forall x \in A\,(x\,R\,x)$

R is symmetric $\triangleq \forall x,y \in A\,(x\,R\,y \Rightarrow y\,R\,x)$

R is transitive $\triangleq \forall x,z \in A\,(\exists y \in A\,(x\,R\,y \wedge y\,R\,z) \Rightarrow x\,R\,z)$

Order

R is a pre-order: R is reflexive and transitive (so not necessarily symmetric).

R is anti-symmetric: $\forall x,y \in A\,(x\,R\,y \wedge y\,R\,x \Rightarrow x =_A y)$.

R is a partial order *relation*: R is an anti-symmetric pre-order (so is reflexive, transitive, and anti-symmetric).

R is irreflexive: $\forall a \in A\,(\neg(a\,R\,a))$.

R is a strict partial order *relation*: R is irreflexive and transitive.

R is a total order: A partial order that also satisfies: $\forall a,b \in A\,(a\,R\,b \vee b\,R\,a)$.

a is *minimal* $\triangleq \forall b \in A\,(b\,R\,a \Rightarrow b =_A a)$

a is *least* $\triangleq \forall b \in A\,(a\,R\,b)$

a is *maximal* $\triangleq \forall b \in A\,(a\,R\,b \Rightarrow a =_A b)$

a is *greatest* $\triangleq \forall b \in A\,(b\,R\,a)$

Definition 4.10 (WELL-FOUNDED PARTIAL ORDERS) A partial order $(A, \leq)$ is *well-founded* if it has no infinite decreasing chain of elements: that is, for every infinite sequence $a_1, a_2, a_3, \ldots$ of elements in A with $a_1 \geq a_2 \geq a_3 \geq \cdots$, there exists $m \in I\!N$ such that $m \geq 1$ and $a_n = a_m$ for every $n \geq m$.

Proposition 4.11 If two partial orders $(A, \leq_A)$ and $(B, \leq_B)$ are well-founded, then the lexicographical order $\leq_L$ on $A \times B$ (see Definition 4.3) is also well-founded.

Proposition 4.9 Let $(A, \leq)$ be a partial order.

1) If A has a least element, then it is a minimal element.

2) If A has a least element, then it is unique.

3) If A is finite and non-empty, then $(A, \leq)$ has a minimal element.

4) If $(A, \leq)$ is a total order, where A is finite and non-empty, then it has a least element.

Definition 5.2 Let $f : A \to B$ and $h : A \to B$. Then $f =_{A \to B} h \triangleq \forall a \in A\,(f(a) =_B h(a))$.

Definition 5.4 Let $f : A \to B$. For any $V \subseteq A$, we define the image of V under f to be

$$f[V] \triangleq \{b \in B \mid \exists a \in V\,(b = f(a))\}$$

Proposition 5.6 If $|A| = m$ and $|B| = n$, then $|B^A| = n^m$.

Functions

1. A *function* f from a set A to a set B, $f : A \to B$ is a relation $f \subseteq A \times B$ such that every element of A is related to one element in B.
2. A is the *domain* of f. 3. B is the *co-domain* of f.
4. Consider $f(a) = b$: a is the *pre-image* of b under f and b is the *image* of a under f. Every element of the domain has a single image but elements of the co-domain can have any number of pre-images.
5. An n-ary function is written $f(a_1, a_2, \ldots, a_n)$.
6. B^A denotes the set of all functions from A to B.
7. If $|A| = m$ and $|B| = n$, then $|B^A| = n^m$ or $(n+1)^m$ including partial functions.

Definition 5.7 (CHARACTERISTIC FUNCTION) *1)* Let A be a set. The *characteristic function of* $B \subseteq A$ is the function $\chi_B : A \to \{0,1\}$ defined as:

$$\chi_B(a) = \begin{cases} 1 & (a \in B) \\ 0 & (a \in A \setminus B) \end{cases}$$

2) The *characteristic function of* $B \subseteq A_1 \times \cdots \times A_n$ is the function $\chi_B : A_1 \times \cdots \times A_n \to \{0,1\}$ defined as:

$$\chi_B(a_1, \ldots, a_n) = \begin{cases} 1 & (\langle a_1, \ldots, a_n\rangle \in B) \\ 0 & (\langle a_1, \ldots, a_n\rangle \notin B) \end{cases}$$

Definition 5.8 A *partial function* f from a set A to a set B is a relation $f \subseteq A \times B$ such that just some elements of A are related to unique elements of B; more formally, it is a relation which satisfies only the first clause of Definition 5.1:

$$\forall a \in A, b_1, b_2 \in B[\langle a, b_1\rangle \in f \wedge \langle a, b_2\rangle \in f \Rightarrow b_1 = b_2]$$

Proposition 5.18 If $f : A \to B$ and $g : B \to C$ are bijections, then so is $g \circ f : A \to C$.

Definition 5.10

f is onto *(surjective)*: every element of B is in the image of f; that is:

$$\forall b \in B\,\exists a \in A\,(f(a) = b)$$

f is one-to-one *(injective)*: for each $b \in B$ there exists at most one $a \in A$ with $f(a) = b$; that is:

$$\forall a, a' \in A\,(f(a) = f(a') \Rightarrow a = a') \quad \forall a, a' \in A\,(a \neq a' \Rightarrow f(a) \neq f(a'))$$

f is bijective: f is both *one-to-one* and *onto*.

Definition 5.20 (INVERSE FUNCTION)

left inverse of f when $g \circ f = Id_A$: $\forall a \in A\,(g \circ f(a) = a)$

right inverse of f when $f \circ g = Id_B$: $\forall b \in B\,(f \circ g(b) = b)$

Proposition 5.22 Let $f : A \to B$ be a bijection, then f^{-1} (as relation) is a well-defined function, and is an inverse of f.

Proposition 5.23 Let $f : A \to B$. If f has an inverse g, then f is a bijection and the inverse is unique.

Theorem 5.14 ((DUAL) CANTOR-BERNSTEIN THEOREM) *If there exists functions $f : A \to B$ and $g : B \to A$, both injective or both surjective, then there exists a bijection $h : A \to B$.*

Definition 5.25 $A \approx B \triangleq \exists f : A \to B\,(f \textit{ is a bijection})$.

Corollary 5.26 If there exists functions $f : A \to B$ and $g : B \to A$, both injective or both surjective, then $A \approx B$.

Definition 5.32 (CARDINALITY) Given two (arbitrary) sets A and B, we say that A has the *same cardinality* as B, written $|A| = |B|$, whenever there exists a bijection between A and B, so when $A \approx B$.

$$|A| = |B| \triangleq A \approx B$$

Definition 6.1 (COUNTABILITY) A set A is *countable* if A is finite or $A \approx I\!N$.

Proposition 6.2 1) If $V \subseteq I\!N$, then V is countable.

2) Let A be a non-empty set. The statements i) A is countable; *ii)* there exists a surjection from $I\!N$ to A; *iii)* there exists an injection from A to $I\!N$, *are equivalent.*

Example 5.33 ($I\!N \approx I\!N^2$) We can build a bijection $f : I\!N \to I\!N^2$ as illustrated by the following diagram:

$$\begin{array}{cccccccccccc}
\langle 0,0\rangle & \longrightarrow & \langle 0,1\rangle & & \langle 0,2\rangle & \longrightarrow & \langle 0,3\rangle & & \langle 0,4\rangle & \longrightarrow & \langle 0,5\rangle & \cdots \\
\langle 1,0\rangle & & \langle 1,1\rangle & & \langle 1,2\rangle & & \langle 1,3\rangle & & \langle 1,4\rangle & & \langle 1,5\rangle & \cdots \\
\langle 2,0\rangle & & \langle 2,1\rangle & & \langle 2,2\rangle & & \langle 2,3\rangle & & \langle 2,4\rangle & & \langle 2,5\rangle & \cdots \\
\langle 3,0\rangle & & \langle 3,1\rangle & & \langle 3,2\rangle & & \langle 3,3\rangle & & \langle 3,4\rangle & & \langle 3,5\rangle & \cdots \\
\langle 4,0\rangle & & \langle 4,1\rangle & & \langle 4,2\rangle & & \langle 4,3\rangle & & \langle 4,4\rangle & & \langle 4,5\rangle & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots &
\end{array}$$

so $f(0) = \langle 0,0\rangle$, $f(1) = \langle 0,1\rangle$, $f(2) = \langle 1,0\rangle$, $f(3) = \langle 2,0\rangle$, $f(4) = \langle 1,1\rangle$, $f(5) = \langle 0,2\rangle$, $f(6) = \langle 0,3\rangle$, *etc.*. It is clear that f is a surjection, since all pairs will be visited; also, since all pairs are different, f is injective, even if we do not give a formal definition for $f(n)$.

We also have $[0,1] \approx I\!R$ via the surjections $f : [0,1] \to I\!R$ and $g : I\!R \to [0,1]$:

$$f(x) = \begin{cases} 0 & (x = 0) \\ 1 & (x = 1) \\ \tan(\pi \times (x - 1/2)) & (\textit{otherwise}) \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & (x \leq 0) \\ 1 & (x \geq 1) \\ x & (\textit{otherwise}) \end{cases}$$

Theorem 5.35 (CANTOR'S THEOREM) *For any set A, $A \not\approx \wp A$.*

Example 6.5 The set of finite subsets of $I\!N$, defined as $\{V \in \wp I\!N \mid \exists n \in I\!N\,(|V| = n)\}$, is countable. We define $f : \wp_f(I\!N \setminus \{0\}) \to I\!N$ by:

$$f(V) = 2^{v_1} \times 3^{v_2} \times 5^{v_3} \times 7^{v_4} \times \cdots \times p_n^{v_n} = \Pi_{i=1}^{n} p_i^{v_i}$$

(notice that we need to exclude 0 since it would not contribute to this product). Since each number has its unique decomposition as a product of prime numbers, it is straightforward to verify that if $V \neq V' \Rightarrow f(V) \neq f(V')$, so f is an injection. Then by Proposition 6.2, we know that $\wp_f(I\!N \setminus \{0\})$ is countable.

Example 6.6 ($\wp I\!N$ IS NOT COUNTABLE) *Example 6.8* ($I\!R \approx \wp I\!N$)

* *Example 6.7* ($I\!R$ IS NOT COUNTABLE)