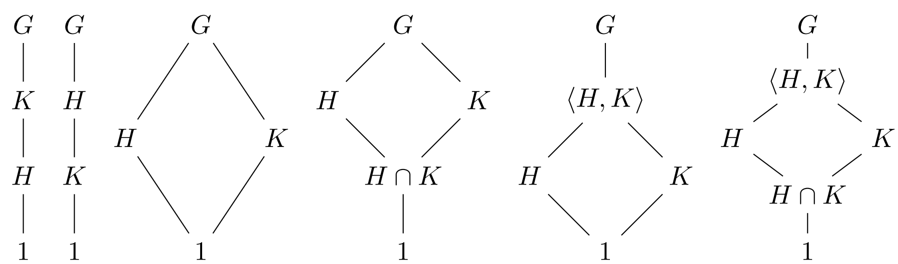
# Dummit & Foote Ch. 2.5: The Lattice of Subgroups of a Group

Scott Donaldson

Aug. 2023

## 1. (8/11/23)

Let $H$ and $K$ be subgroups of $G$. Exhibit all possible sublattices which show only $G$, 1, $H$, $K$, and their joins and intersections. What distinguishes the different drawings?




The left two lattices show the group structure when either $H \leq K$ or $K \leq H$ (they omit any subgroups of the smaller of the two, as well as any containing subgroups between the larger and $G$).

The next lattice shows the group structure when $H$ and $K$ are not comparable, their intersection consists only of the identity, and their join is all of $G$. The final three lattices show the cases where $H \cap K$ is a subgroup not equal to the identity, where $\langle H, K\rangle$ is a subgroup not equal to $G$, and where both of these occur.

## 2. (8/11/23)

In each of (a) to (d) list all subgroups of $D_{16}$ that satisfy the given condition.

(a) Subgroups that are contained in $\langle sr^2, r^4\rangle$
$\{1\}, \langle sr^6\rangle, \langle sr^2\rangle, \langle r^4\rangle$, $\langle sr^2, r^4\rangle$

(b) Subgroups that are contained in $\langle sr^7, r^4 \rangle$

$\{1\}, \langle sr^3 \rangle, \langle sr^7 \rangle, \langle r^4 \rangle, \langle sr^7, r^4 \rangle$

(c) Subgroups that contain $\langle r^4 \rangle$

$\langle r^4 \rangle, \langle sr^2, r^4 \rangle, \langle s, r^4 \rangle, \langle r^2 \rangle, \langle sr^3, r^4 \rangle, \langle sr^5, r^4 \rangle, \langle s, r^2 \rangle, \langle r \rangle, \langle sr, r^2 \rangle, D_{16}$

(d) Subgroups that contain $\langle s \rangle$

$\langle s \rangle, \langle s, r^4 \rangle, \langle s, r^2 \rangle, \langle D_{16} \rangle$

## 3. (8/11/23)

Show that the subgroup $\langle s, r^2 \rangle$ of $D_8$ is isomorphic to $V_4$.

*Proof.* The subgroup $\langle s, r^2 \rangle$ of $D_8$ contains the elements $\{1, s, r^2, sr^2\}$. There is no element in this group of order 4. From Ch. 1.1, Exercise 36, there is only one unique group of order 4 with no element of order 4, the Klein group $V_4$. Thus $\langle s, r^2 \rangle$ is isomorphic to $V_4$. $\square$

## 4. (8/14/23)

Use the given lattice to find all pairs of elements that generate $D_8$.

*Proof.* Since $D_8$ is generated by $\langle s, r \rangle$, it suffices to find pairs of elements that generate $s$ and $r$. These pairs of elements are:

- $\langle s, r \rangle$ (trivial)
- $\langle s, r^3 \rangle$ $(r = (r^3)^3)$
- $\langle s, sr \rangle$ $(r = s \cdot sr)$
- $\langle s, sr^3 \rangle$ $(r = s \cdot (sr^3)^3)$
- $\langle sr, r \rangle$ $(s = r \cdot sr)$
- $\langle sr, r^2 \rangle$ $(r^3 = r^2 \cdot sr, r = (r^3)^3, s = r \cdot sr)$
- $\langle sr, r^3 \rangle$ $(r = (r^3)^3, s = r \cdot sr)$
- $\langle sr^2, r \rangle$ $(s = sr^2 \cdot r^2)$
- $\langle sr^2, r^3 \rangle$ $(r = (r^3)^3, s = sr^2 \cdot r^2)$
- $\langle sr^2, sr^3 \rangle$ $(r = sr^2 \cdot sr^3, s = sr^2 \cdot r^2)$
- $\langle sr^3, r \rangle$ $(s = sr^3 \cdot r)$
- $\langle sr^3, r^3 \rangle$ $(s = r^3 \cdot sr^3, r = s \cdot sr^3)$

$\square$

## 5. (8/14/23)

Use the given lattice to find all elements $x \in D_{16}$ such that $D_{16} = \langle x, s \rangle$.

*Proof.* The element $x \in D_{16}$ generates $D_{16}$ together with $s$ if $r$ can be expressed as a product of $s$ and $x$:

- $x = r$ (trivial)
- $x = r^3$ ($r = (r^3)^3$)
- $x = r^5$ ($r = (r^5)^5$)
- $x = r^7$ ($r = (r^7)^7$)
- $x = sr$ ($r = s \cdot sr$)
- $x = sr^3$ ($r^3 = s \cdot sr^3$, $r = (r^3)^3$)
- $x = sr^5$ ($r^5 = s \cdot sr^5$, $r = (r^5)^5$)
- $x = sr^7$ ($r^7 = s \cdot sr^7$, $r = (r^7)^7$)

$\square$