Dummit & Foote Ch. 4.2: Groups Acting on Themselves by Left Multiplication — Cayley's Theorem

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Let G be a group and let H be a subgroup of G.

1. (2/12/24)

Let $G = \{1, a, b, c\}$ be the Klein 4-group whose group table is written out in Section 2.5.

(a) Label $1, a, b, c$ with the integers $1, 2, 4, 3$, respectively, and prove that under the left regular representation of G into S_4 the nonidentity elements are mapped as follows:

$$a \mapsto (1\,2)(3\,4) \qquad b \mapsto (1\,4)(2\,3) \qquad c \mapsto (1\,3)(2\,4).$$

Proof. The left regular representation of G into S_4 is the homomorphism $\varphi : G \to S_4$ defined by $\varphi(g) = \sigma_g$, where $\sigma_g : G \to G$ is the permutation of G defined by $\sigma_g(x) = gx$ for all $x \in G$.

Each non-identity element maps the elements as follows:

$$\begin{array}{cccc} \sigma_a(1) = a1 = a & \sigma_a(a) = a^2 = 1 & \sigma_a(b) = ab = c & \sigma_a(c) = ac = b \\ \sigma_b(1) = b1 = b & \sigma_b(a) = ba = c & \sigma_b(b) = b^2 = 1 & \sigma_b(c) = bc = a \\ \sigma_c(1) = c1 = c & \sigma_c(a) = ca = b & \sigma_c(b) = cb = a & \sigma_c(c) = c^2 = 1. \end{array}$$

By the given labeling, this assigns the elements a, b, and c to the pairs of 2-cycles shown above. □

(b) Relabel $1, a, b, c$ as $1, 4, 2, 3$, respectively, and compute the image of each element of G under the left regular representation of G into S_4. Show that the image of G in S_4 under this labeling is the same *subgroup* as the image of G in part (a) (even though the nonidentity elements individually map to different permutations under the two different labelings).

Proof. Under this labeling, the elements a, b, and c are mapped to the permutations $(1\,4)(2\,3), (1\,2)(3\,4)$, and $(1\,3)(2\,4)$, respectively. Although each element maps to a different permutation from part (a), the subgroup of S_4 is the same in both cases. $\square$

2. (2/12/24)

List the elements of S_3 as $1, (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)$ and label these with the integers $1, 2, 3, 4, 5, 6$, respectively. Exhibit the image of each element of S_3 under the left regular representation of S_3 into S_6.

Solution. First, consider the element $(1\,2)$. We see that:

$$\begin{aligned} (1\,2)1 &= (1\,2) \mapsto 2 & (1\,2)(1\,2) &= 1 \mapsto 1 \\ (1\,2)(2\,3) &= (1\,2\,3) \mapsto 5 & (1\,2)(1\,3) &= (1\,3\,2) \mapsto 6 \\ (1\,2)(1\,2\,3) &= (2\,3) \mapsto 3 & (1\,2)(1\,3\,2) &= (1\,3) \mapsto 4. \end{aligned}$$

So the left regular representation of $(1\,2)$ under the given labeling in S_6 is $(1\,2)(3\,4)(5\,6)$.

The left regular representations of the remaining elements are:

$$\begin{aligned} (2\,3) &\mapsto (1\,3)(2\,6)(4\,5) \\ (1\,3) &\mapsto (1\,4)(2\,5)(3\,6) \\ (1\,2\,3) &\mapsto (1\,5\,6)(2\,4\,3) \\ (1\,3\,2) &\mapsto (1\,6\,5)(2\,3\,4). \end{aligned}$$

$\square$

3. (2/12/24)

Let r and s be the usual generators for the dihedral group of order 8.

(a) List the elements of D_8 as $1, r, r^2, r^3, s, sr, sr^2, sr^3$ and label these with the integers $1, 2, ..., 8$, respectively. Exhibit the image of each element of D_8 under the left regular representation of D_8 into S_8.

$$\begin{aligned} 1 &\mapsto 1 \\ r &\mapsto (1\,2\,3\,4)(5\,8\,7\,6) \\ r^2 &\mapsto (1\,3)(2\,4)(5\,7)(6\,8) \\ r^3 &\mapsto (1\,4\,3\,2)(5\,6\,7\,8) \\ s &\mapsto (1\,5)(2\,6)(3\,7)(4\,8) \\ sr &\mapsto (1\,6)(2\,7)(3\,8)(4\,5) \\ sr^2 &\mapsto (1\,7)(2\,8)(3\,5)(4\,6) \\ sr^3 &\mapsto (1\,8)(2\,5)(3\,6)(4\,7) \end{aligned}$$

(b) Relabel this same list of elements of D_8 with the integers $1, 3, 5, 7, 2, 4, 6, 8$ respectively and recompute the image of each element of D_8 under the left regular representation with respect to this new labeling. Show that the two subgroups of S_8 obtained in parts (a) and (b) are different.

$$\begin{aligned}
1 &\mapsto 1 \\
r &\mapsto (1\,3\,5\,7)(2\,8\,6\,4) \\
r^2 &\mapsto (1\,5)(2\,6)(3\,7)(4\,8) \\
r^3 &\mapsto (1\,7\,5\,3)(2\,4\,6\,8) \\
s &\mapsto (1\,2)(3\,4)(5\,6)(7\,8) \\
sr &\mapsto (1\,4)(2\,7)(3\,6)(5\,8) \\
sr^2 &\mapsto (1\,6)(2\,5)(3\,8)(4\,7) \\
sr^3 &\mapsto (1\,8)(2\,3)(4\,5)(6\,7).
\end{aligned}$$

We see that the generators of the subgroups of S_8 in parts (a) and (b) are different, and so these are different subgroups of S_8.

4. (2/12/24)

Use the left regular representation of Q_8 to produce two elements of S_8 which generate a subgroup of S_8 isomorphic to the quaternion group Q_8.

Proof. We know that the elements i and j generate the quaternion group Q_8. Labeling the elements $1, -1, i, -i, j, -j, k, -k$ with $1, 2, ..., 8$ respectively, the elements i and j map to the following permutations in S_8:

$$\begin{aligned}
i &\mapsto (1\,3\,2\,4)(5\,7\,6\,8) \\
j &\mapsto (1\,5\,2\,6)(3\,8\,4\,7).
\end{aligned}$$

Since the left regular representation of Q_8 in S_8 is a homomorphism, these two permutations generate a subgroup of S_8 isomorphic to Q_8. $\square$

5. (2/12/24)

Let r and s be the usual generators for the dihedral group of order 8 and let $H = \langle s \rangle$. List the left cosets of H in D_8 as $1H, rH, r^2H, r^3H$.

(a) Label these cosets with the integers $1, 2, 3, 4$, respectively. Exhibit the image of each element of D_8 under the representation π_H of D_8 into S_4 obtained from the action of D_8 by left multiplication on the set of 4 left cosets of H in D_8. Deduce that this representation is faithful (i.e., the elements of S_4 obtained form a subgroup isomorphic to D_8).

$$\begin{aligned} 1 &\mapsto 1 & s &\mapsto (2\,4) \\ r &\mapsto (1\,2\,3\,4) & sr &\mapsto (1\,4)(2\,3) \\ r^2 &\mapsto (1\,3)(2\,4) & sr^2 &\mapsto (1\,3) \\ r^3 &\mapsto (1\,4\,3\,2) & sr^3 &\mapsto (1\,2)(3\,4). \end{aligned}$$

Since each element of D_8 induces a unique permutation in S_4, the resulting image under the left regular representation is isomorphic to D_8, and so this representation is faithful.

(b) Repeat part (a) with the list of cosets relabeled by the integers $1, 3, 2, 4$, respectively. Show that the permutations obtained from this labeling form a subgroup of S_4 that is different from the subgroup obtained in part (a).

$$\begin{aligned} 1 &\mapsto 1 & s &\mapsto (3\,4) \\ r &\mapsto (1\,3\,2\,4) & sr &\mapsto (1\,4)(2\,3) \\ r^2 &\mapsto (1\,2)(3\,4) & sr^2 &\mapsto (1\,2) \\ r^3 &\mapsto (1\,4\,2\,3) & sr^3 &\mapsto (1\,3)(2\,4). \end{aligned}$$

Since the generators (the images of r and s) of this subgroup of S_4 are different from those in part (a), this is a different subgroup from part (a).

(c) Let $K = \langle sr \rangle$, list the cosets of K in D_8 as $1K, rK, r^2K, r^3K$, and label these with the integers $1, 2, 3, 4$. Prove that, with respect to this labeling, the image of D_8 under the representation π_K obtained from left multiplication on the cosets of K is the same *subgroup* of S_4 as in part (a) (even though the subgroups H and K are different and some of the elements of D_8 map to different permutations under the two homomorphisms).

Proof. Consider the images of the generators r and s under π_K:

$$\begin{aligned} r \cdot 1K &= rK & s \cdot 1K &= rK \\ r \cdot rK &= r^2K & s \cdot rK &= 1K \\ r \cdot r^2K &= r^3K & s \cdot r^2K &= r^3K \\ r \cdot r^3K &= 1K & s \cdot r^3K &= r^2K. \end{aligned}$$

So r and s map to $(1\,2\,3\,4)$ and $(1\,2)(3\,4) \in S_4$, respectively. These elements are both in the subgroup in part (a) above, and so they are the same subgroup, but the image of s is different. $\square$

6. (2/15/24)

Let r and s be the usual generators for the dihedral group of order 8 and let $N = \langle r^2 \rangle$. List the left cosets of N in D_8 as $1N, rN, sN$, and srN. Label these cosets with the integers $1, 2, 3, 4$ respectively. Exhibit the image of each element of D_8 under the representation π_N of D_8 into S_4 obtained from the action of D_8 by left multiplication on the set of 4 left cosets of N in D_8. Deduce that this representation is not faithful and prove that $\pi_N(D_8)$ is isomorphic to the Klein 4-group.

Solution.

$$\begin{aligned} 1 &\mapsto 1 & s &\mapsto (1\,3)(2\,4) \\ r &\mapsto (1\,2)(3\,4) & sr &\mapsto (1\,4)(2\,3) \\ r^2 &\mapsto 1 & sr^2 &\mapsto (1\,3)(2\,4) \\ r^3 &\mapsto (1\,2)(3\,4) & sr^3 &\mapsto (1\,4)(2\,3). \end{aligned}$$

The left regular representation assigns 1 and r^2 to the identity permutation, so this action is not faithful.

The image of D_8 under π_N consists of the four permutations $1, (1\,2)(3\,4)$, $(1\,3)(2\,4)$, and $(1\,4)(2\,3)$. From Ch. 2.5, Exercise 10, this is isomorphic to the Klein 4-group V_4. □

7. (2/15/24)

Let Q_8 be the quaternion group of order 8.

(a) Prove that Q_8 is isomorphic to a subgroup of S_8.

Proof. From Exercise 4, Q_8 is isomorphic to

$$\langle (1\,3\,2\,4)(5\,7\,6\,8), (1\,5\,2\,6)(3\,8\,4\,7) \rangle \in S_8.$$

□

(b) Prove that Q_8 is not isomorphic to a subgroup of S_n for any $n \leq 7$.

Proof. Let A be a set with $|A| = n \leq 7$, let $a \in A$, and let $\cdot$ be the action of Q_8 on A. We attempt to find a subgroup of S_n that is isomorphic to Q_8 by considering the permutation representations of the elements of Q_8. Now if $i \cdot a = j \cdot a$, then the permutation representations σ_i and σ_j are equal to each other, and so Q_8 is not isomorphic to the resulting subgroup of S_n. Further (without loss of generality), if $i \cdot a = -i \cdot a$, then:

$$i \cdot a = -i \cdot a \Rightarrow -i \cdot i \cdot a = -i \cdot -i \cdot a \Rightarrow a = -1 \cdot a,$$

and so the permutation representation of -1 is equal to the identity permutation, which implies that Q_8 is not isomorphic to the subgroup of S_n. Therefore the elements $\pm i, \pm j, \pm k$ must all assign a to different elements. However, these 6 unique elements together with a are at least all of A, and so we must have $-1 \cdot a = a$. Thus Q_8 is not isomorphic to a subgroup of S_n. □