Dummit & Foote Ch. 3.1: Quotient Groups and Homomorphisms

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Let G and H be groups.

1. (9/1/23)

Let $\varphi : G \to H$ be a homomorphism and let $E \leq H$. Prove that $\varphi^{-1}(E) \leq G$ (i.e., the preimage or pullback of a subgroup under a homomorphism is a subgroup). If $E \trianglelefteq H$ prove that $\varphi^{-1}(E) \trianglelefteq G$. Deduce that $\ker \varphi \trianglelefteq G$.

Proof. Let $x, y \in \varphi^{-1}(E) \subseteq G$. Suppose that $\varphi(x) = a, \varphi(y) = b, a, b \in E \leq H$. Since φ is a homomorphism, we have $\varphi(y^{-1}) = \varphi(y)^{-1} = b^{-1}$. Then:

$$\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1}) = \varphi(x)\varphi(y)^{-1} = ab^{-1} \in E,$$

which implies that $xy^{-1} \in \varphi^{-1}(E)$. It follows that, by the subgroup criterion, $\varphi^{-1}(E) \leq G$.

Next, let $E \trianglelefteq H$ (to show that $\varphi^{-1}(E) \trianglelefteq G$). Again let $x \in \varphi^{-1}(E) \leq G$ and suppose $\varphi(x) = a$. Now for some $g \in G$ (not necessarily in $\varphi^{-1}(E)$), consider $\varphi(gxg^{-1})$. Suppose also that $\varphi(g) = h \in H$. Because E is normal in H and $a \in E$, we have $hah^{-1} \in E$. Then:

$$\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1} = hah^{-1} \in E,$$

which implies that $gxg^{-1} \in \varphi^{-1}(E)$. Since the conjugate of any element of $\varphi^{-1}(E)$ by any other element of G lies in $\varphi^{-1}(E)$, we therefore conclude that $\varphi^{-1}(E) \trianglelefteq G$.

Finally, we note that $\ker \varphi = \{g \in G \mid \varphi(g) = 1_H\}$. Since the trivial subgroup consisting of the identity of H is normal (the conjugate of 1_H by any element of H is 1_H), we therefore have $\varphi^{-1}(\{1_H\}) = \ker \varphi \trianglelefteq G$. $\square$

2. (8/23/23)

Let $\varphi : G \to H$ be a homomorphism of groups with kernel K and let $a, b \in \varphi(G)$. Let $X \in G/K$ be the fiber above a and Y be the fiber above b, i.e.,

$X = \varphi^{-1}(a), Y = \varphi^{-1}(b)$. Fix an element $x \in X$ (so $\varphi(x) = a$). Prove that if $XY = Z$ in the quotient group G/K and z is any member of Z, then there is some $y \in Y$ such that $xy = z$.

Proof. We know that, for any $x \in X, y \in Y$, $\varphi(x) = a$ and $\varphi(y) = b$. Since φ is a homomorphism, it follows that $\varphi(xy) = \varphi(x)\varphi(y) = ab$, and so the image of any element of $XY = Z$ under φ is $ab \in H$.

Next, consider the element $x^{-1}z \in G$, as well as its image under φ. Since φ is a homomorphism, we have $\varphi(x^{-1}) = \varphi(x)^{-1}$. So $\varphi(x^{-1}z) = \varphi(x^{-1})\varphi(z) = \varphi(x)^{-1}\varphi(z) = a^{-1}ab = b$. The set Y consists of all elements of G whose image under φ is b, and so we must have $x^{-1}z \in Y$.

Now if we fix some element $x \in X$, then for any $z \in Z$, we have $x^{-1}z \in Y$ such that its product with x is z: $xx^{-1}z = z$. $\square$

3. (8/23/23)

Let A be an abelian group and let B be a subgroup of A. Prove that A/B is abelian. Give an example of a non-abelian group G containing a proper normal subgroup N such that G/N is abelian.

Proof. Because A is abelian, all subgroups of A are normal, so A/B is well-defined for every $B \leq A$.

Let $C, D \in A/B$ with $C = cB$ and $D = dB$ for some $c, d \in A$. Then:

$$CD = (cB)(dB) = (cd)B = (dc)B = (dB)(cB) = DC,$$

which implies that A/B is abelian.

Now if we let G be the dihedral group D_8, then G is non-abelian. Let N be the cyclic subgroup generated by $r : \{1, r, r^2, r^3\}$. The only coset of N is sN; together these two sets cover G. Then $G/N = \{N, sN\}$. There is only one group of order 2 up to isomorphism, and it is abelian. Thus G/N is abelian. $\square$

4. (8/23/23)

Prove that in the quotient group G/N, $(gN)^\alpha = (g^\alpha)N$ for all $\alpha \in \mathbb{Z}$.

Proof. We start by induction: In the base case, $\alpha = 1$, we have $(gN)^1 = gN = (g^1)N$. Next, suppose that for some $\alpha > 1$, we have $(gN)^\alpha = (g^\alpha)N$. Then:

$$(gN)^{\alpha+1} = (gN)^\alpha gN = g^\alpha N \cdot gN = (g^{\alpha+1})N,$$

as desired. We have now proven that $(gN)^\alpha = (g^\alpha)N$ for $\alpha \geq 1$.

Next, consider $(gN)^\alpha(gN)^{-\alpha}$, where $\alpha \geq 1$. In the quotient group G/N, for any subset $X \in G/N$, we must have $X^\alpha X^{-\alpha} = N$ (the identity of G/N), so $(gN)^\alpha(gN)^{-\alpha} = N$. From above, $(gN)^\alpha = (g^\alpha)N$, so $(g^\alpha)N \cdot (gN)^{-\alpha} = N$. Also, from the operation on left cosets, we know that $N = (g^\alpha)N \cdot (g^{-\alpha})N$.

Since both $(g^\alpha)N \cdot (gN)^{-\alpha} = N$ and $(g^\alpha)N \cdot (g^{-\alpha})N = N$, we must have $(gN)^{-\alpha} = (g^{-\alpha})N$. We have now proven for all nonzero integers.

Finally, we note that $(gN)^0 = N$ (the identity of G/N) and that $(g^0)N = eN = N$, so $(gN)^0 = (g^0)N$. This concludes the proof that $(gN)^\alpha = (g^\alpha)N$ for all $\alpha \in \mathbb{Z}$. □

5. (8/23/23)

Use the preceding exercise to prove that the order of the element gN in G/N is n, where n is the smallest positive integer such that $g^n \in N$ (and gN has infinite order if no such positive integer exists). Give an example to show that the order of gN in G/N may be strictly smaller than the order of g in G.

Proof. Let $gN \in G/N$, and let n be the smallest positive integer such that $g^n \in N$. Suppose that $g^n = h \in N$.

From Exercise 4., $(gN)^n = (g^n)N = hN = N$ (because $h \in N$), so the order of gN must divide n.

Suppose (toward contradiction) that the order of gN is k, where $k < n$. Then $(gN)^k = (g^k)N = N$, which implies that g^k lies in N, contradicting our assumption that n is the smallest such positive integer. Therefore the order of gN is n.

If there is no positive integer n such that $g^n \in N$, then for all $k \in \mathbb{Z}^+$, we have $(gN)^k = (g^k)N \neq N$, so gN has infinite order.

As an example where $|gN| < |g|$, let $G = Z_9 = \langle x \rangle$ and let $N = \langle x^3 \rangle$. Because all cyclic groups are abelian, N is normal in G, and so G/N is well-defined. The quotient group G/N contains three elements: N, xN, and $(x^2)N$. The element $xN \in G/N$ has order 3: $(xN)^3 = (x^3)N = N$ (because $x^3 \in N$). However, the generating element $x \in G$ has order 9. □

6. (8/24/23)

Define $\varphi : \mathbb{R}^\times \to \{\pm 1\}$ by letting $\varphi(x)$ be x divided by the absolute value of x. Describe the fibers of φ and prove that φ is a homomorphism.

Proof. We consider the two cases where $x < 0$ and $x > 0$ (0 is not an element of $\mathbb{R}^\times$). If $x > 0$, then $\varphi(x) = x/|x| = x/x = 1$. If $x < 0$, then $\varphi(x) = x/|x| = x/-x = -1$. Therefore the fiber above -1 is every negative real number and the fiber above 1 is every positive real number.

To show that φ is a homomorphism, we let $x, y \in \mathbb{R}^\times$ and again consider the different cases: Where x and y are both positive, where they are both negative, and where one is positive and the other negative.

If both x and y are positive, then $\varphi(x)\varphi(y) = 1 \cdot 1 = 1$ and $\varphi(xy) = \frac{xy}{|xy|} = \frac{xy}{xy} = 1$, so $\varphi(x)\varphi(y) = \varphi(xy)$.

If both x and y are negative, then $\varphi(x)\varphi(y) = -1 \cdot -1 = 1$ and $\varphi(xy) = \frac{xy}{|xy|} = \frac{xy}{xy} = 1$, so $\varphi(x)\varphi(y) = \varphi(xy)$.

Suppose x is positive and y is negative. Then $\varphi(x)\varphi(y) = 1 \cdot -1 = -1$ and $\varphi(xy) = \frac{xy}{|xy|} = \frac{xy}{-xy} = -1$, so $\varphi(x)\varphi(y) = \varphi(xy)$.

Thus, in every case of $x, y \in \mathbb{R}^\times$, we have $\varphi(x)\varphi(y) = \varphi(xy)$, and φ is thus a homomorphism. $\square$

7. (8/24/23)

Define $\pi : \mathbb{R}^2 \to \mathbb{R}$ by $\pi((x, y)) = x + y$. Prove that π is a surjective homomorphism and the describe the kernel and fibers of π geometrically.

Proof. First, to show that π is surjective, let $z \in \mathbb{R}$. Now $z = z + 0$, so $(z, 0)$ is an element of $\mathbb{R}^2$ such that $\pi((z, 0)) = z + 0 = z$.

Next, to show that π is a homomorphism, let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. We have $\pi((x_1, y_1) + (x_2, y_2)) = \pi((x_1 + x_2, y_1 + y_2)) = x_1 + x_2 + y_1 + y_2$, and $\pi((x_1, y_1)) + \pi((x_2, y_2)) = x_1 + y_1 + x_2 + y_2$. By the commutativity of addition in $\mathbb{R}$, these are equal to each other, and so π is a surjective homomorphism.

The kernel of π consists of all points $(x, y) \in \mathbb{R}^2$ such that $x + y = 0$, that is, the diagonal line running from the upper-left to the bottom-right of the Cartesian plane. Geometrically, the fibers of π are translations of this line, such that for any $z \in \mathbb{R}$, the fiber of π above z is the diagonal line intersecting both $(z, 0)$ and $(0, z)$. $\square$

8. (8/24/23)

Let $\varphi : \mathbb{R}^\times \to \mathbb{R}^\times$ be the map sending x to the absolute value of x. Prove that φ is a homomorphism and find the image of φ. Describe the kernel and the fibers of φ.

Proof. Let $x, y \in \mathbb{R}^\times$ (so $x \neq 0, y \neq 0$). If both x and y are positive or both are negative, then:

$$\varphi(xy) = |xy| = |x||y| = \varphi(x)\varphi(y),$$

and if x is positive and y is negative, then:

$$\varphi(xy) = |xy| = x(-y) = |x||y| = \varphi(x)\varphi(y),$$

so φ is a homomorphism.

The image of φ consists of every positive real number. The kernel of φ is the set $\{x \in \mathbb{R}^\times \mid |x| = 1\}$, that is, $\{\pm 1\}$. For a given element $z > 0$, the fiber of φ above z is the set $\{\pm z\}$. $\square$

9. (8/25/23)

Define $\varphi : \mathbb{C}^\times \to \mathbb{R}^\times$ by $\varphi(a + bi) = a^2 + b^2$. Prove that φ is a homomorphism and find the image of φ. Describe the kernel and the fibers of φ geometrically (as subsets of the plane).

Proof. To show that φ is a homomorphism, let $z_1 = a_1 + b_1 i, z_2 = a_2 + b_2 i \in \mathbb{C}^\times$. We calculate:

$$\begin{aligned}
\varphi(z_1 z_2) &= \varphi((a_1 + b_1 i)(a_2 + b_2 i)) \\
&= \varphi((a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i) \\
&= (a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2 \\
&= a_1^2 a_2^2 - 2a_1 a_2 b_1 b_2 + b_1^2 b_2^2 + a_1^2 b_2^2 + 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2 \\
&= a_1^2 a_2^2 + b_1^2 b_2^2 + a_1^2 b_2^2 + a_2^2 b_1^2, \text{ and} \\
\varphi(z_1)\varphi(z_2) &= \varphi(a_1 + b_1 i)\varphi(a_2 + b_2 i) = (a_1^2 + b_1^2)(a_2^2 + b_2^2) \\
&= a_1^2 a_2^2 + b_1^2 b_2^2 + a_1^2 b_2^2 + a_2^2 b_1^2,
\end{aligned}$$

which proves that φ is a homomorphism.

The image of a complex number $a + bi$ under φ is $a^2 + b^2$, which is always non-negative because it is the sum of two non-negative numbers. Since both $\mathbb{C}^\times$ and $\mathbb{R}^\times$ exclude 0, the image of φ is therefore all positive real numbers.

The kernel of φ are those complex numbers whose image under φ is 1. Geometrically, φ is a map from a point in the complex plane to its length, or distance from zero. Therefore the kernel of φ is the unit circle in the complex plane. The fibers of a given positive real number x is the circle of radius x centered at the origin in the complex plane. □

10. (8/28/23)

Let $\varphi : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ by $\varphi(\overline{a}) = \overline{a}$. Show that this is a well-defined, surjective homomorphism and describe its fibers and kernel explicitly (showing that φ is well-defined involves the fact that $\overline{a}$ has a different meaning in the domain and range of φ).

Proof. The map φ is well-defined because it assigns to each member of $\mathbb{Z}/8\mathbb{Z}$ a single, unique element of $\mathbb{Z}/4\mathbb{Z}$. Let $a \in \{0, ...7\}$ be equal to $\overline{a}$ mod 8. Then we have $\varphi(\overline{a}) = \varphi(a)$. Further, φ assigns each $a \in \{0, ...7\}$ to a mod 4; that is, it assigns 0 and 4 to 0, 1 and 5 to 1, 2 and 6 to 2, and 3 and 7 to 3. This also shows that φ is surjective, since each $\overline{a} \cong \mathbb{Z}/4\mathbb{Z}$ (represented by $a = \overline{a}$ mod 4) has a preimage in $\mathbb{Z}/8\mathbb{Z}$.

The kernel of φ is $\{0, 4\} \leq \mathbb{Z}/8\mathbb{Z}$, and the fiber of any $a \in \mathbb{Z}/4\mathbb{Z}$ is the tuple $\{a, a + 4\}$. □

11. (8/28/23)

Let F be a field and let $G = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F, ac \neq 0\} \leq GL_2(F)$.

(a) Prove that the map $\varphi : \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto a$ is a surjective homomorphism from G onto $F^\times$ (recall that $F^\times$ is the multiplicative group of nonzero elements in F). Describe the fibers and kernel of φ.

Proof. To show that φ is surjective, let $a \in F^\times$ (so $a \neq 0$). Then we have $\varphi(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) = a$, so φ is onto.

Next, to show that it is a homomorphism, we note that:

$$\varphi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}) = \varphi(\begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}) = ad = \varphi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix})\varphi(\begin{pmatrix} d & e \\ 0 & f \end{pmatrix}),$$

so φ is also a homomorphism.

The kernel of φ is $\{\begin{pmatrix} 1 & b \\ 0 & c \end{pmatrix} \mid b, c \in F, c \neq 0\}$, and the fiber of φ over a given element $a \in F^\times$ is $\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid b, c \in F, c \neq 0\}$. $\square$

(b) Prove that the map $\psi : \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto (a, c)$ is a surjective homomorphism from G onto $F^\times \times F^\times$. Describe the fibers and kernel of ψ.

Proof. To show that ψ is surjective, let $(a, c) \in F^\times \times F^\times$ (so $a, c \neq 0$). Then we have $\psi(\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}) = (a, c)$, so ψ is onto.

Next, to show that it is a homomorphism, we note that:

$$\begin{aligned} \psi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}) &= \psi(\begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}) = (ad, cf) \\ &= (a, c)(d, f) = \psi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix})\psi(\begin{pmatrix} d & e \\ 0 & f \end{pmatrix}), \end{aligned}$$

so ψ is also a homomorphism.

The kernel of ψ is the preimage of $(1, 1)$, that is, $\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in F\}$, and the fiber of ψ over a given element $(a, c) \in F^\times \times F^\times$ is $\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid b \in F\}$. $\square$

(c) Let $H = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in F\}$. Prove that H is isomorphic to the additive group F.

Proof. As usual, to show that H is isomorphic to the additive group F, we must show that there exists a bijective homomorphism $\varphi : H \to F$. Define φ by $\varphi(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}) = b$. We will show that it is an isomorphism.

First, φ is injective: Suppose that $\varphi(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}) = \varphi(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}) = c$. Then we have $a = c$ and $b = c$, so the two matrices are the same, and φ is injective.

Next, φ is surjective: Let $b \in F$. Then we have $\varphi(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}) = b$.

Finally, φ is a homomorphism:

$$\varphi(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}) = \varphi(\begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}) = a+b = \varphi(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix})+\varphi(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}).$$

□

12. (8/30/23)

Let G be the additive group of real numbers, let H be the multiplicative group of complex numbers of absolute value 1 (the unit circle S^1 in the complex plane) and let $\varphi : G \to H$ be the homomorphism $\varphi : r \mapsto e^{2\pi i r}$. Draw the points on the real line which lie in the kernel of φ. Describe similarly the elements in the fibers of φ above the points -1, i, and $e^{4\pi i/3}$ of H.

Proof. The kernel of φ is the set $\{r \in \mathbb{R} \mid e^{2\pi i r} = 1\}$. Recall that $e^{2\pi i r} = \cos 2\pi r + i \sin 2\pi r$, so the values of r for which $e^{2\pi i r} = 1$ are those where $\cos 2\pi r = 1$, that is, all of the integers.

We similarly obtain the fiber of φ above -1 by considering when $\cos 2\pi r = -1$, which occurs when $r = 1/2, 3/2, 5/2, ...$, that is, $r \in \{n + \frac{1}{2} \mid n \in \mathbb{Z}\}$. For the fiber above i, we must have $\sin 2\pi r = 1$, which occurs when $r = 1/4, 5/4, 9/4, ...$, that is, $r \in \{n + \frac{1}{4} \mid n \in \mathbb{Z}\}$. Finally, we have $4\pi/3 = \frac{2}{3} \cdot 2\pi$, so the fiber above $e^{4\pi i/3}$ is $\{n + \frac{2}{3} \mid n \in \mathbb{Z}\}$.

We can also write these as cosets of $\mathbb{Z}$, so the fibers are $\frac{1}{2} + \mathbb{Z}$, $\frac{1}{4} + \mathbb{Z}$, and $\frac{2}{3} + \mathbb{Z}$, respectively. □

13. (8/31/23)

Repeat the preceding exercise with the map φ replaced by the map $\varphi : r \mapsto e^{4\pi i r}$.

Proof. In this case, the kernel of φ consists of values of r for which $e^{4\pi i r} = 1 \Rightarrow \cos 4\pi r = 1$. The period is now halved, so this occurs when $r \in \{1/2, 1, 3/2, ...\}$; the kernel is $\{\frac{n}{2} \mid n \in \mathbb{Z}\}$.

The fiber of φ above -1 has $\cos 4\pi r = -1$, when $r = 1/4, 3/4, 5/4, ...$, that is, $r \in \{\frac{1}{4} + \frac{n}{2} \mid n \in \mathbb{Z}\}$. Above i, we have $\sin 4\pi r = 1$, so $r \in \{\frac{1}{8}, \frac{5}{8}, ...\}$, and the fiber is $\{\frac{1}{8} + \frac{n}{2} \mid n \in \mathbb{Z}\}$. Finally, above $4\pi/3$, the fiber is $\{\frac{1}{3} + \frac{n}{2} \mid n \in \mathbb{Z}\}$.

If we denote the kernel in this exercise as $\frac{1}{2}\mathbb{Z}$, then as cosets, the fibers are $\frac{1}{4} + \frac{1}{2}\mathbb{Z}$, $\frac{1}{8} + \frac{1}{2}\mathbb{Z}$, and $\frac{1}{3} + \frac{1}{2}\mathbb{Z}$, respectively. □

14. (8/31/23)

Consider the additive quotient group $\mathbb{Q}/\mathbb{Z}$.

(a) Show that every coset of $\mathbb{Z}$ in $\mathbb{Q}$ contains exactly one representative $q \in \mathbb{Q}$ in the range $0 \leq q < 1$.

Proof. The rational numbers under addition constitutes an abelian group, so $\mathbb{Z}$ is a normal subgroup of $\mathbb{Q}$, and $\mathbb{Q}/\mathbb{Z}$ is therefore well-defined. The elements of the quotient group $\mathbb{Q}/\mathbb{Z}$ are cosets of $\mathbb{Z}$ in $\mathbb{Q}$, for example, $\mathbb{Z}$ itself (the identity), as well as $\frac{1}{2} + \mathbb{Z}$, $\frac{7}{4} + \mathbb{Z}$, and so on.

Let $q + \mathbb{Z}$ be a coset of $\mathbb{Z}$ (for arbitrary $q \in \mathbb{Q}$). If $q > 1$, then let $n \in \mathbb{Z}$ be the largest integer such that $q - n \geq 0$ (such an integer exists by the well-ordering property). Then $q - n$ is the unique representative for $q + \mathbb{Z}$ in the range $[0, 1)$, since $q-n-1 < 0$ and $q-n+1 > 1$. Similarly, if $q < 0$, there exists a unique n such that $0 \leq q+n < 1$. Finally, if $0 \leq q < 1$, then q itself is the unique representative for $q + \mathbb{Z}$ lying between 0 (inclusive) and 1 (exclusive). □

(b) Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order but that there are elements of arbitrarily large order.

Proof. Let $\frac{a}{b} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ (with $0 \leq \frac{a}{b} < 1$, as above, and suppose that $\frac{a}{b}$ is in lowest terms). Then we have:

$$\underbrace{(\frac{a}{b} + \mathbb{Z}) + ... + (\frac{a}{b} + \mathbb{Z})}_{b \text{ times}} = \underbrace{(\frac{a}{b} + ... + \frac{a}{b})}_{b \text{ times}} + \mathbb{Z} = a + \mathbb{Z} = \mathbb{Z},$$

so the order of $\frac{a}{b} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ is at most b, and it therefore has finite order.

However, given a coset $\frac{1}{b} + \mathbb{Z}$ of order b, there always exists an element of higher order, for example $\frac{1}{b+1} + \mathbb{Z}$ and $\frac{1}{2b} + \mathbb{Z}$, which have order $b+1$ and $2b$, respectively. □

(c) Show that $\mathbb{Q}/\mathbb{Z}$ is the torsion subgroup of $\mathbb{R}/\mathbb{Z}$.

Proof. Recall that the torsion subgroup of $\mathbb{R}/\mathbb{Z}$ is the set of elements of $\mathbb{R}/\mathbb{Z}$ of finite order (by Chapter 2.1, Exercise 6., this set is a subgroup when the parent group is abelian).

First, let $q + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$. Since rational numbers are also real numbers, $q + \mathbb{Z}$ also lies in $\mathbb{R}/\mathbb{Z}$. From 14.b), it has finite order. Therefore it is an element of the torsion subgroup of $\mathbb{R}/\mathbb{Z}$.

Next, let $x + \mathbb{Z}$ be an element of the torsion subgroup of $\mathbb{R}/\mathbb{Z}$. Suppose that $|x + \mathbb{Z}| = n < \infty$. Then we have:

$$\underbrace{(x + \mathbb{Z}) + ... + (x + \mathbb{Z})}_{n \text{ times}} = \underbrace{(x + ... + x)}_{n \text{ times}} + \mathbb{Z} = nx + \mathbb{Z} = \mathbb{Z},$$

which implies that nx is an integer. Suppose that $nx = m \in \mathbb{Z}$. Then $x = m/n$, and so we have $x \in \mathbb{Q}$, which implies that $x + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$.

Therefore, because inclusion in one implies inclusion in the other and vice-versa, these groups are equal. $\square$

(d) Prove that $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the multiplicative group of roots of unity in $\mathbb{C}^\times$.

Proof. Let $\varphi : \mathbb{Q}/\mathbb{Z} \to \mathbb{C}^\times$ be defined by $\varphi(r+\mathbb{Z}) = e^{2\pi i r}$, where $0 \le r < 1$. We will show that φ is a bijective homomorphism, and that the groups are thus isomorphic to each other.

First, to show that φ is a homomorphism, note that:

$$\varphi((q + \mathbb{Z}) + (r + \mathbb{Z})) = \varphi((q + r) + \mathbb{Z}) = e^{2\pi i(q+r)}, \text{ and}$$
$$\varphi(q + \mathbb{Z})\varphi(r + \mathbb{Z}) = e^{2\pi i q}e^{2\pi i r} = e^{2\pi i q + 2\pi i r} = e^{2\pi i(q+r)},$$

as desired.

Next, φ is one-to-one: Suppose $e^{2\pi i r} = \varphi(r + \mathbb{Z}) = \varphi(q + \mathbb{Z})$ for some $r, q \in [0, 1)$. In fact, there are many possible rational numbers fulfilling this if we open the range to all of $\mathbb{Q}$; however, because the period of $e^{2\pi i r}$ is 1, there is only one unique value in the range $[0, 1)$, so we must have $r = q$. Therefore φ is injective.

Finally, φ is surjective: Let z be a root of unity with order n. Then z can be expressed as $e^{2\pi i t/n}$ for some $t \in \{0, 1, ..., n - 1\}$. By definition of φ, the rational number $t/n \in [0, 1)$ has $\varphi(t/n) = e^{2\pi i t/n} = z$. Thus φ is a bijective homomorphism, and so $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the roots of unity in $\mathbb{C}^\times$. $\square$

15. (9/1/23)

Prove that the quotient of a divisible abelian group by any proper subgroup is also divisible. Deduce that $\mathbb{Q}/\mathbb{Z}$ is divisible.

Proof. Let A be a divisible abelian group and let B be a proper subgroup of A. Since A is abelian, all of its subgroups are normal, so the quotient group A/B is well-defined.

Let $aB \in A/B$ and let $k > 0$. Since A is divisible, there exists an $x \in A$ such that $x^k = a$. Then we have $aB = (x^k)B = (xB)^k$ for $xB \in A/B$, so aB has a k-th root in A/B. Therefore A/B is divisible.

Note that the rational numbers under addition form a divisible abelian group (from Ch. 2.4, Exercise 19.) and the integers are a proper subgroup of the rational numbers. It follows that the quotient group $\mathbb{Q}/\mathbb{Z}$ is divisible. □

16. (9/5/23)

Let G be a group, let N be a normal subgroup of G, and let $\overline{G} = G/N$. Prove that if $G = \langle x, y \rangle$ then $\overline{G} = \langle \overline{x}, \overline{y} \rangle$. Prove more generally that if $G = \langle S \rangle$ for any subset S of G then $\overline{G} = \langle \overline{S} \rangle$.

Proof. If $G = \langle x, y \rangle$, then we can write any element g as a finite product of x and y, say $g = x^{a_1}y^{b_1}...x^{a_n}y^{b_n}$. It follows that, for $\overline{g} \in \overline{G}$, we have:

$$\overline{g} = gN = (x^{a_1}y^{b_1}...x^{a_n}y^{b_n})N = (x^{a_1})N(y^{b_1})N...(x^{a_n})N(y^{b_n})N = (xN)^{a_1}(yN)^{b_1}...(xN)^{a_n}(yN)^{b_n} = \overline{x}^{a_1}\overline{y}^{b_1}...\overline{x}^{a_n}\overline{y}^{b_n},$$

that is, we can write $\overline{g}$ as a finite product of $\overline{x}, \overline{y} \in \overline{G}$, and so $\overline{G} = \langle \overline{x}, \overline{y} \rangle$.

More generally, if $G = \langle S \rangle$, then any element g can be written as a finite product of elements of S, say $g = (s_1^{a_{11}}...s_n^{a_{n1}})(s_1^{a_{12}}...s_n^{a_{n2}})...(s_1^{a_{1k}}...s_n^{a_{nk}})$. Then we have:

$$\overline{g} = gN = \Big(\prod_{j=1}^{k}(\prod_{i=1}^{n} s_i^{a_{ij}})\Big)N = \prod_{j=1}^{k}\prod_{i=1}^{n}(s_i^{a_{ij}}N) = \prod_{j=1}^{k}\prod_{i=1}^{n}(s_iN)^{a_{ij}} = \prod_{j=1}^{k}\prod_{i=1}^{n}\overline{s_i}^{a_{ij}},$$

and so similar to above, this means that any element $\overline{g} = gN \in G/N$ can be written as a finite product of $\overline{s_1}, \overline{s_2}, ..., \overline{s_n}$, and therefore $\overline{G} = \langle \overline{S} \rangle$. □

17. (9/6/23)

Let G be the dihedral group of order 16: $G = \langle r, s \mid r^8 = s^2 = 1, rs = sr^{-1} \rangle$ and let $\overline{G} = G/\langle r^4 \rangle$ be the quotient of G by the subgroup generated by r^4 (this subgroup is the center of G, hence is normal).

(a) Show that the order of $\overline{G}$ is 8.

The quotient group $\overline{G}$ consists of cosets of the cyclic subgroup of G generated by r^4, that is, cosets of $\{1, r^4\}$. For example, the coset $s\langle r^4 \rangle$ is $\{s, sr^4\}$. Notice that the coset for sr^4 is the same as for s, and because $\langle r^4 \rangle$ consists of two elements, for each element $x \in G$, there is another element whose coset is the same (namely xr^4). Thus the order of $\overline{G}$ is $16/2 = 8$.

(b) Exhibit each element of $\overline{G}$ in the form $\overline{s}^a\overline{r}^b$, for some integers a and b.

The elements of $\overline{G}$ are:

$$\begin{aligned} \overline{1} &= \{1, r^4\} & \overline{s} &= \{s, sr^4\} \\ \overline{r} &= \{r, r^5\} & \overline{s}\cdot\overline{r} &= \{sr, sr^5\} \\ \overline{r}^2 &= \{r^2, r^6\} & \overline{s}\cdot\overline{r}^2 &= \{sr^2, sr^6\} \\ \overline{r}^3 &= \{r^3, r^7\} & \overline{s}\cdot\overline{r}^3 &= \{sr^3, sr^7\} \end{aligned}$$

(c) Find the order of each of the elements of $\overline{G}$ exhibited in (b).

The orders of the elements of $\overline{G}$ are: $\overline{1} : 1, \overline{r} : 4, \overline{r}^2 : 2, \overline{r}^3 : 4, \overline{s} : 2,$ $\overline{s}\cdot\overline{r} : 2, \overline{s}\cdot\overline{r}^2 : 2, \overline{s}\cdot\overline{r}^3 : 2.$

(d) Write each of the following elements of $\overline{G}$ in the form $\overline{s}^a\overline{r}^b$, for some integers a and b as in (b):

- $\overline{rs} = \overline{sr^7} = \overline{s}\cdot\overline{r}^3$
- $\overline{sr^{-2}s} = \overline{sr^6s} = \overline{ssr^2} = \overline{r}^2$
- $\overline{s^{-1}r^{-1}sr} = \overline{sr^7sr} = \overline{ssrr} = \overline{r}^2$

(e) Prove that $\overline{H} = \langle\overline{s}, \overline{r}^2\rangle$ is a normal subgroup of $\overline{G}$ and $\overline{H}$ is isomorphic to the Klein 4-group. Describe the isomorphism type of the complete preimage of $\overline{H}$ in G.

Proof. There is a clear isomorphism between $\overline{G}$ and D_8 given by $\overline{x} \in \overline{G} \mapsto x \in D_8$. Because of this, we know that the elements $\overline{s}$ and $\overline{r}$ generate $\overline{G}$. Since we know the generators of both $\overline{G}$ and $\overline{H}$, in order to test for normality, we only have to check that the conjugates of the generators of $\overline{H}$ by the generators of $\overline{G}$ are in $\overline{H}$.

Now powers of $\overline{s}$ and $\overline{r}$ commute with other powers of $\overline{s}$ and $\overline{r}$, respectively, so we can proceed to:

$$\overline{r}\cdot\overline{s}\cdot\overline{r}^{-1} = \overline{rsr^{-1}} = \overline{rsr^7} = \overline{sr^7r^7} = \overline{sr^{14}} = \overline{sr^6} = \overline{s}\cdot\overline{r}^2 \in \overline{H}, \text{ and}$$
$$\overline{s}\cdot\overline{r}^2\cdot\overline{s} = \overline{sr^2s} = \overline{ssr^6} = \overline{r^6} = \overline{r}^2 \in \overline{H}.$$

This demonstrates that the conjugates of the generators of $\overline{H}$ by the generators of $\overline{G}$ lie in $\overline{H}$, and so $\overline{H} \trianglelefteq \overline{G}$.

The elements of $\overline{H}$ are $\overline{1}, \overline{s}, \overline{r}^2,$ and $\overline{s}\cdot\overline{r}^2$. Any other product of elements gives an element of $\overline{H}$. All of these elements have order 2, and so from Ch. 1.1, Exercise 36, $\overline{H} \cong V_4$.

The complete preimage of $\overline{H}$ under the natural projection homomorphism $\pi(g) \mapsto \overline{g} = g\langle r^4\rangle$ is the set $\{g \in G \mid \pi(g) \in \overline{H}\}$. The elements of G in the complete preimage of $\overline{H}$ are $1, r^2, r^4, r^6, s, sr^2, sr^4,$ and sr^6. This set of elements is isomorphic to D_4 (given by $s, r^2 \in \pi^{-1}(\overline{H}) \mapsto s, r \in D_4$). $\square$

(f) Find the center of $\overline{G}$ and describe the isomorphism type of $\overline{H}/Z(\overline{G})$.

The center of $\overline{G}$ consists of the elements of $\overline{G}$ that commute with all other elements of $\overline{G}$. This is the subgroup $\langle \overline{r}^2 \rangle$. Now the quotient group $\overline{H}/Z(\overline{G}) = \langle \overline{s}, \overline{r}^2 \rangle / \langle \overline{r}^2 \rangle$ consists of the cosets of $\langle \overline{r}^2 \rangle$ in $\overline{H}$, that is, the elements $\langle \overline{r}^2 \rangle, \overline{s}\langle \overline{r}^2 \rangle$. We do not have $\overline{r}^2$ as a unique element in $\overline{H}/Z(\overline{G})$, because

$$\overline{r}^2 \langle \overline{r}^2 \rangle = \overline{r}^2 \{\overline{1}, \overline{r}^2\} = \{\overline{r}^2, \overline{r}^4\} = \{\overline{1}, \overline{r}^2\} = \langle \overline{r}^2 \rangle.$$

Similarly, $\overline{s} \cdot \overline{r}^2 \notin \overline{H}/Z(\overline{G})$. Therefore it is isomorphic to the cyclic roup Z_2.

18. (9/10/23)

Let G be the quasidihedral group of order 16: $G = \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle$ and let $\overline{G} = G/\langle \sigma^4 \rangle$ be the quotient of G by the subgroup generated by $\langle \sigma^4 \rangle$ (this subgroup is the center of G, hence is normal).

(a) Show that the order of $\overline{G}$ is 8.

The elements of $\overline{G}$ are the cosets of the subgroup generated by σ^4. For example, for $\tau \in G$, the element $\overline{\tau} \in \overline{G} = \{\tau, \tau\sigma^4\}$. As with 17.a), there are two elements in this set, and the cosets of $\langle \sigma^4 \rangle$ partition G. Thus $\overline{G}$ has $16/2 = 8$ elements.

(b) Exhibit each element of $\overline{G}$ in the form $\overline{\tau}^a \overline{\sigma}^b$, for some integers a and b.

The elements of $\overline{G}$ are:

$$\begin{aligned}
\overline{1} &= \{1, \sigma^4\} & \overline{\tau} &= \{\tau, \tau\sigma^4\} \\
\overline{\sigma} &= \{\sigma, \sigma^5\} & \overline{\tau} \cdot \overline{\sigma} &= \{\tau\sigma, \tau\sigma^5\} \\
\overline{\sigma}^2 &= \{\sigma^2, \sigma^6\} & \overline{\tau} \cdot \overline{\sigma}^2 &= \{\tau\sigma^2, \tau\sigma^6\} \\
\overline{\sigma}^3 &= \{\sigma^3, \sigma^7\} & \overline{\tau} \cdot \overline{\sigma}^3 &= \{\tau\sigma^3, \tau\sigma^7\}
\end{aligned}$$

(c) Find the order of each of the elements of $\overline{G}$ exhibited in (b).

The orders of the elements of $\overline{G}$ are: $\overline{1} : 1, \overline{\sigma} : 4, \overline{\sigma}^2 : 2, \overline{\sigma}^3 : 4, \overline{\tau} : 2,$
$\overline{\tau} \cdot \overline{\sigma} : 2, \overline{\tau} \cdot \overline{\sigma}^2 : 2, \overline{\tau} \cdot \overline{\sigma}^3 : 2.$

(d) Write the following elements of $\overline{G}$ in the form $\overline{\tau}^a \overline{\sigma}^b$, for some integers a and b as in (b):

- $\overline{\sigma\tau} = \overline{\tau\sigma^3} = \overline{\tau} \cdot \overline{\sigma}^3$
- $\overline{\tau\sigma^{-2}\tau} = \overline{\tau\sigma^6\tau} = \overline{\tau\tau\sigma^{18}} = \overline{\sigma^2} = \overline{\sigma}^2$
- $\overline{\tau^{-1}\sigma^{-1}\tau\sigma} = \overline{\tau\sigma^7\tau\sigma} = \overline{\tau\tau\sigma^{21}\sigma} = \overline{\sigma^{22}} = \overline{\sigma^6} = \overline{\sigma}^2$

(e) Prove that $\overline{G} \cong D_8$.

Proof. Let $\varphi : \overline{G} \to D_8$ be defined by $\varphi(\overline{\sigma}) = r$ and $\varphi(\overline{\tau}) = s$. Now $\overline{\sigma}$ and $\overline{\tau}$ are generators for $\overline{G}$, since (as shown above) every element can be written in the form $\overline{\tau}^a\overline{\sigma}^b$, for some integers a and b. Then φ is a map from $\overline{G}$ to D_8 defined on the generators of $\overline{G}$ to the generators of D_8. Since both groups have the same cardinality, in order to show that φ is an isomorphism, it only remains to check that the relations of $\overline{G}$ are the same as those in D_8.

In D_8, we have $s^2 = r^4 = 1$ and $rs = sr^{-1}$. In part (c) above, we computed the orders of $\overline{\tau}$ and $\overline{\sigma}$, which are 2 and 4, respectively, matching their counterparts in D_8. Finally, we have $\overline{\sigma} \cdot \overline{\tau} = \overline{\sigma\tau} = \overline{\tau\sigma^3} = \overline{\tau} \cdot \overline{\sigma^3} = \overline{\tau} \cdot \overline{\sigma}^{-1}$, and so the relations hold. Thus $\overline{G} \cong D_8$. ☐

19. (9/13/23)

Let G be the modular group of order 16: $G = \langle u, v \mid u^2 = v^8 = 1, vu = uv^5\rangle$ and let $\overline{G} = G/\langle v^4\rangle$ be the quotient of G by the subgroup generated by v^4 (this subgroup is contained in the center of G, hence is normal).

(a) Show that the order of $\overline{G}$ is 8.

The elements of $\overline{G}$ are the cosets of the subgroup generated by v^4. For example, for $u \in G$, the element $\overline{u} \in \overline{G} = \{u, uv^4\}$. As with 17.a), there are two elements in this set, and the cosets of $\langle v^4\rangle$ partition G. Thus $\overline{G}$ has $16/2 = 8$ elements.

(b) Exhibit each element of $\overline{G}$ in the form $\overline{u}^a\overline{v}^b$, for some integers a and b.

The elements of $\overline{G}$ are:

$$\begin{aligned} \overline{1} &= \{1, v^4\} & \overline{u} &= \{u, uv^4\} \\ \overline{v} &= \{v, v^5\} & \overline{u} \cdot \overline{v} &= \{uv, uv^5\} \\ \overline{v}^2 &= \{v^2, v^6\} & \overline{u} \cdot \overline{v}^2 &= \{uv^2, uv^6\} \\ \overline{v}^3 &= \{v^3, v^7\} & \overline{u} \cdot \overline{v}^3 &= \{uv^3, uv^7\} \end{aligned}$$

(c) Find the order of each of the elements of $\overline{G}$ exhibited in (b).

The orders of the elements of $\overline{G}$ are: $\overline{1} : 1, \overline{v} : 4, \overline{v}^2 : 2, \overline{v}^3 : 4, \overline{u} : 2,$
$\overline{u} \cdot \overline{v} : 4, \overline{u} \cdot \overline{v}^2 : 2, \overline{u} \cdot \overline{v}^3 : 4.$

(d) Write each of the following elements of $\overline{G}$ in the form $\overline{u}^a\overline{v}^b$, for some integers a and b as in (b):

- $\overline{vu} = \overline{uv^5} = \overline{u} \cdot \overline{v}$
- $\overline{uv^{-2}u} = \overline{uv^6u} = \overline{uuv^{30}} = \overline{v^{30}} = \overline{v^6} = \overline{v}^2$
- $\overline{u^{-1}v^{-1}uv} = \overline{uv^7uv} = \overline{uuv^{35}v} = \overline{v^{36}} = \overline{v^4} = \overline{1}$

(e) Prove that $\overline{G}$ is abelian and is isomorphic to $Z_2 \times Z_4$.

Proof. From part (d) above, we deduced that $\overline{v}\overline{u} = \overline{uv^5} = \overline{u}\overline{v}$. Since the generators of $\overline{G}$ commute, $\overline{G}$ is an abelian group.

For clarity, let us write the elements of $Z_2 \times Z_4$ as (u^k, v^j), with $k \in \{0,1\}$ and $j \in \{0,1,2,3\}$. Then $(u,1)$ and $(1,v)$ are generators of $Z_2 \times Z_4$.

Now let $\varphi : \overline{G} \to Z_2 \times Z_4$ be defined on generators $\overline{u}$ and $\overline{v}$ by $\varphi(\overline{u}) = (u,1)$ and $\varphi(\overline{v}) = (1,v)$. As above, since φ is a map from $\overline{G}$ to $Z_2 \times Z_4$, two groups of equal order, and φ is defined on and to the generators of each, respectively, we only have to check that the relations hold.

In $\overline{G}$, we have $\overline{u}^2 = 1$, and in $Z_2 \times Z_4$, we have $\varphi(\overline{u})^2 = (u,1)^2 = (u^2,1) = (1,1)$, the identity of $Z_2 \times Z_4$. Also, we have $\overline{v}^4 = 1$ and $\varphi(\overline{v})^4 = (1,v)^4 = (1,v^4) = (1,1)$. Since $\overline{G}$ and $Z_2 \times Z_4$ are both abelian, there are no other relations we need to check. We conclude that φ is an isomorphism, and that the two groups are isomorphic. $\square$

20. (9/14/23)

Let $G = \mathbb{Z}/24\mathbb{Z}$ and let $\widetilde{G} = G/\langle\overline{12}\rangle$, where for each integer a we simplify notation by writing $\widetilde{\overline{a}}$ as $\widetilde{a}$.

(a) Show that $\widetilde{G} = \{\widetilde{0}, \widetilde{1}, ..., \widetilde{11}\}$.

Now $\widetilde{G}$ consists of the cosets of $\langle\overline{12}\rangle = \{0,12\}$ in $\mathbb{Z}/24\mathbb{Z}$, for example, $\widetilde{4} = 4 + \{0,12\} = \{4,16\}$ and $\widetilde{21} = 21 + \{0,12\} = \{21,33\} = \{9,21\} = \widetilde{9}$. For each $n \in \{0,...,11\}$, the element $n + 12 \in \mathbb{Z}/24\mathbb{Z}$ has the same coset as n, since $n + 12 \cong n \pmod{12}$. Thus the elements of $\widetilde{G}$ are:

$$\begin{array}{lll} \widetilde{0} = \{0,12\} & \widetilde{4} = \{4,16\} & \widetilde{8} = \{8,20\} \\ \widetilde{1} = \{1,13\} & \widetilde{5} = \{5,17\} & \widetilde{9} = \{9,21\} \\ \widetilde{2} = \{2,14\} & \widetilde{6} = \{6,18\} & \widetilde{10} = \{10,22\} \\ \widetilde{3} = \{3,15\} & \widetilde{7} = \{7,19\} & \widetilde{11} = \{11,23\} \end{array}$$

(b) Find the order of each element of $\widetilde{G}$.

$$\begin{array}{lll} \widetilde{0} : 1 & \widetilde{4} : 3 & \widetilde{8} : 3 \\ \widetilde{1} : 12 & \widetilde{5} : 12 & \widetilde{9} : 4 \\ \widetilde{2} : 6 & \widetilde{6} : 2 & \widetilde{10} : 6 \\ \widetilde{3} : 4 & \widetilde{7} : 12 & \widetilde{11} : 12 \end{array}$$

(c) Prove that $\widetilde{G} \cong \mathbb{Z}/12\mathbb{Z}$. (Thus $(\mathbb{Z}/24\mathbb{Z})/(12\mathbb{Z}/24\mathbb{Z}) \cong \mathbb{Z}/12\mathbb{Z}$, just as if we inverted and cancelled the $24\mathbb{Z}$'s.)

Proof. From Ch. 2.3, Theorem 4, $\mathbb{Z}/n\mathbb{Z}$ is another presentation of the unique cyclic group of order n. It suffices, then, to prove that $\widetilde{G}$ is cyclic in order to show that it is isomorphic to $\mathbb{Z}/12\mathbb{Z}$.

We claim that $\widetilde{1}$ is a generator for $\overline{G}$. For any element $\widetilde{a} \in \widetilde{G}$ $(0 \leq a < 12)$, we can write:

$$\begin{aligned}\widetilde{a} &= \{a, a+12\} = a + \{0,12\} = (\underbrace{1 + ... + 1}_{a \text{ times}}) + \{0,12\} \\ &= \underbrace{(1 + \{0,12\}) + ... + (1 + \{0,12\})}_{a \text{ times}} = \underbrace{\widetilde{1} + ... + \widetilde{1}}_{a \text{ times}} \\ &= a \cdot \widetilde{1},\end{aligned}$$

and so any element of $\widetilde{G}$ is generated from $\widetilde{1}$. Thus $\widetilde{G}$ is isomorphic to the cyclic group of order 12, which is isomorphic to $\mathbb{Z}/12\mathbb{Z}$. □

22. (9/14/23)

(a) Prove that if H and K are normal subgroups of G then their intersection $H \cap K$ is also a normal subgroup of G.

Proof. Let H and K be normal subgroups of G. Let $h \in H \cap K$, so $h \in H$ and $h \in K$. Since both H and K are normal, we have $ghg^{-1} \in H$ and $ghg^{-1} \in K$ for all $g \in G$. It follows that $ghg^{-1} \in H \cap K$ for all $g \in G$. Therefore $H \cap K$ is a normal subgroup of G. □

(b) Prove that the intersection of an arbitrary nonempty collection of normal subgroups of a group is a normal subgroup (do not assume the collection is countable).

Proof. Let $\mathcal{H}$ be a nonempty collection of normal subgroups of G. Consider $\bigcap_{H \in \mathcal{H}} = \{h \in G \mid h \in H \text{ for all } H \in \mathcal{H}\}$. From Ch. 2.1, Exercise 10., we know that $\mathcal{H}$ is itself a subgroup of G. We will show that in this case it is normal in G.

Let $h \in \bigcap_{H \in \mathcal{H}}$. Then for all $H \in \mathcal{H}$, we have $h \in H$. Since each H is normal in G, we have $ghg^{-1} \in H$ for all $g \in G, H \in \mathcal{H}$. It follows that $ghg^{-1} \in \bigcap_{H \in \mathcal{H}}$, and therefore $\bigcap_{H \in \mathcal{H}}$ is normal in G. □

23. (9/16/23)

Prove that the join of any nonempty collection of normal subgroups of a group is a normal subgroup.

Proof. Let $\mathcal{H}$ be a nonempty collection of subgroups of G and let $\langle\mathcal{H}\rangle$ be their join.

Let $h \in \langle\mathcal{H}\rangle$. Then h can be written as a finite product of elements, say $h_1, h_2, ..., h_n$, where each h_i is an element of a corresponding normal subgroup $H_i \in \mathcal{H}$. We write this product:

$$h = (h_1^{a_{11}}...h_n^{a_{n1}})(h_1^{a_{12}}...h_n^{a_{n2}})...(h_1^{a_{1k}}...h_n^{a_{nk}}) = \prod_{j=1}^{k}\prod_{i=1}^{n} h_i^{a_{ij}}.$$

Since each h_i belongs to a normal subgroup H_i of G, we have $gh_ig^{-1} \in H_i$ for all $g \in G$. It follows that, for any $m > 0$, we have $gh_i^k g^{-1} \in H_i$ (because $(gh_ig^{-1})^k = gh_ig^{-1}$). Now note that, since $(ga_1g^{-1})(ga_2g^{-1})...(ga_ng^{-1}) = g(a_1a_2...a_n)g^{-1}$, the product of conjugates of the constituent elements of h is equal to the conjugate of the product of those elements:

$$\prod_{j=1}^{k}\prod_{i=1}^{n} gh_i^{a_{ij}}g^{-1} = g\Big(\prod_{j=1}^{k}\prod_{i=1}^{n} h_i^{a_{ij}}\Big)g^{-1} = ghg^{-1}.$$

The left-hand side of the equation is the product of conjugates of elements h_i that each belong to the corresponding normal subgroup H_i. Therefore the product is an element of the join $\langle\mathcal{H}\rangle$. Since it is equal to the right-hand side, the conjugate of h by any element $g \in G$, we must have $ghg^{-1} \in \langle\mathcal{H}\rangle$ for all $g \in G$. Thus the join of any nonempty collection of normal subgroups of a group is a normal subgroup. □

24. (9/16/23)

Prove that if $N \trianglelefteq G$ and H is any subgroup of G then $N \cap H \trianglelefteq H$.

Proof. Let $N \trianglelefteq G$, $H \leq G$, and let $n \in N \cap H$, $h \in H$. Consider the conjugate element hnh^{-1}.

Since N is normal in G and $h \in H \Rightarrow h \in G$, we have $hnh^{-1} \in N$.

Since H is a subgroup of G, it is closed and closed under inverses. Also, $n \in N \cap H \Rightarrow n \in H$, so the product hnh^{-1} lies in H. We have both $hnh^{-1} \in N$ and $hnh^{-1} \in H$, so $hnh^{-1} \in N \cap H$.

So the conjugate of any element of $N \cap H$ by any element of H is again an element of $N \cap H$. Therefore $N \cap H$ is normal in H. □

25. (9/17/23)

(a) Prove that a subgroup N of G is normal if and only if $gNg^{-1} \subseteq G$ for all $g \in G$.

Proof. Recall that N is defined to be normal in G if $gNg^{-1} = N$ for all $g \in G$. Now if $N \trianglelefteq G$, then clearly $gNg^{-1} \subseteq N$, since $gNg^{-1} = N$.

Suppose that $gNg^{-1} \subseteq N$ for all $g \in G$. Let $x \in N, g \in G$. The conjugate of x by g^{-1}, $g^{-1}x(g^{-1})^{-1}$, must lie in N. Let us write $g^{-1}x(g^{-1})^{-1} = n \in N$. Then we have:

$$x = gg^{-1}xgg^{-1} = g\big(g^{-1}x(g^{-1})^{-1}\big)g^{-1} = gng^{-1},$$

and so $x \in gNg^{-1}$. This implies that $N \subseteq gNg^{-1}$. Therefore $gNg^{-1} = N$ for all $g \in G$, and so $N \trianglelefteq G$. □

(b) Let $G = GL_2(\mathbb{Q})$, let N be the subgroup of upper triangular matrices with integer entries and 1's on the diagonal, and let g be the diagonal matrix with entries 2, 1. Show that $gNg^{-1} \subseteq N$ but g does *not* normalize N.

Proof. Let $N = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, where $n \in \mathbb{Z}$ and let $g = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, with inverse $g^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$.

Then we have:

$$gNg^{-1} = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} = \left\{ \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}.$$

Since $2n \in \mathbb{Z}$ for all $n \in \mathbb{Z}$, we have $gNg^{-1} \subseteq N$. However, there is no $n \in \mathbb{Z}$ such that $g \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. In order for g to normalize N, we must have $gNg^{-1} = N$. Therefore g does not normalize N. □

26. (9/18/23)

Let $a, b \in G$.

(a) Prove that the conjugate of the product of a and b is the product of the conjugate of a and the conjugate of b. Prove that the order of a and the order of any conjugate of a are the same.

Proof. Let $g \in G$. Then:

$$g(ab)g^{-1} = gabg^{-1} = gag^{-1}gbg^{-1} = (gag^{-1})(gbg^{-1}),$$

as desired.

Next, we show that $a^n = 1$ if and only if $(gag^{-1})^n = 1$. If $a^n = 1$, then we have $(gag^{-1})^n = ga^ng^{-1} = gg^{-1} = 1$. And, if $(gag^{-1})^n = 1$, then we have $ga^ng^{-1} = 1$. Left multiplying by g^{-1} and right-multiplying by g, we obtain $a^n = 1$. Therefore the order of a is equal to the order of any conjugate of a. □

(b) Prove that the conjugate of a^{-1} is the inverse of the conjugate of a.

Proof. We can see that:

$$(gag^{-1})(ga^{-1}g^{-1}) = gag^{-1}ga^{-1}g^{-1} = gaa^{-1}g^{-1} = gg^{-1} = 1,$$

and so the conjugate of a^{-1} is the inverse of the conjugate of a. $\square$

(c) Let $N = \langle S \rangle$ for some subset S of G. Prove that $N \trianglelefteq G$ if $gSg^{-1} \subseteq N$ for all $g \in G$.

Proof. Let $x \in N$. Since $N = \langle S \rangle$, we can write x as a finite product of elements of S: $x = (s_1^{a_{11}}...s_n^{a_{n1}})(s_1^{a_{12}}...s_n^{a_{n2}})...(s_1^{a_{1k}}...s_n^{a_{nk}})$. Now for each s_i^{ij}, we have $gs_i^{ij} \in N$ (since $gSg^{-1} \subseteq N$). Therefore $gxg^{-1} = g\Big(\prod_{j=1}^k \prod_{i=1}^n s_i^{a_{ij}}\Big)g^{-1} = \prod_{j=1}^k \prod_{i=1}^n (gs_i^{a_{ij}}g^{-1})$ lies in N (for all $g \in G$), since it is a finite product of elements of N. Thus $N \trianglelefteq G$. $\square$

(d) Deduce that if N is the cyclic group $\langle x \rangle$, then N is normal in G if and only if for each $g \in G$, $gxg^{-1} = x^k$ for some $k \in \mathbb{Z}$.

If $N = \langle x \rangle$ is normal in G, then for all $g \in G$, we have $gNg^{-1} = N$, which implies that $gxg^{-1} \in N$. Since all elements of N can be written as x^k for some $k \in \mathbb{Z}$, we have $gxg^{-1} = x^k$.

Conversely, if for all $g \in G$, we have $gxg^{-1} = x^k$ for some $k \in \mathbb{Z}$, then we clearly have $gxg^{-1} \in N$, which implies that $gNg^{-1} \subseteq N$. From Exercise 25. above, this implies that $N \trianglelefteq G$.

Therefore $N \trianglelefteq G$ if and only for each $g \in G, gxg^{-1} = x^k$ for some $k \in \mathbb{Z}$.

(e) Let n be a positive integer. Prove that the subgroup N of G generated by all the elements of G of order n is a normal subgroup of G.

Proof. Let $S \subseteq G$ be the subset of elements of order n in G and let $N = \langle S \rangle$. For each $x \in N$, x can be written as a finite product of elements of S: $x = (s_1^{a_{11}}...s_n^{a_{n1}})(s_1^{a_{12}}...s_n^{a_{n2}})...(s_1^{a_{1k}}...s_n^{a_{nk}})$, where $|s_i| = n$ for each $s_i \in S$. From part (a) above, the conjugate of any element has the same order as the element itself, so $|gs_ig^{-1}| = n$ for each $s_i \in S$, $g \in G$. Then $gs_ig^{-1} \in S \Rightarrow gs_ig^{-1} \in N$, and it follows that:

$$gxg^{-1} = g\Big(\prod_{j=1}^{k}\prod_{i=1}^{n} s_i^{a_{ij}}\Big)g^{-1} = \prod_{j=1}^{k}\prod_{i=1}^{n}(gs_i^{a_{ij}}g^{-1})$$

is the product of a elements of N, and so belongs to N itself. Then $gxg^{-1} \in N$ for all $g \in G$, which implies that $gNg^{-1} \subseteq N$, and thus N is normal in G. $\square$

27. (9/18/23)

Let N be a *finite* subgroup of a group G. Show that $gNg^{-1} \subseteq N$ if and only if $gNg^{-1} = N$. Deduce that $N_G(N) = \{g \in G \mid gNg^{-1} \subseteq N\}$.

Proof. Let $g \in G$. Now if $gNg^{-1} = N$, then clearly $gNg^{-1} \subseteq N$. So let us consider the case where $gNg^{-1} \subseteq N$.

Let $\varphi : N \to gNg^{-1}$ be defined by $\varphi(x) = gxg^{-1}$ for $x \in N$. We will show that φ is a bijection, which implies that its domain and range have equal cardinality.

To prove that φ is injective, let $x, y \in N$ and suppose that $\varphi(x) = \varphi(y)$. Then:

$$gxg^{-1} = gyg^{-1} \Rightarrow gx = gy \Rightarrow x = y,$$

so φ is one-to-one.

Next, let $z \in gNg^{-1}$. Since $gNg^{-1} = \{gxg^{-1} \mid x \in G\}$, there exists some $y \in N$ such that $\varphi(y) = z$, so φ is surjective. Therefore it is a bijection, and so $|N| = |gNg^{-1}|$.

Recall that the normalizer $N_G(N)$ is defined to be the subgroup $\{g \in G \mid gNg^{-1} = N\}$. From above, when N is finite, this is equal to $\{g \in G \mid gNg^{-1} \subseteq N\}$. □

28. (9/19/23)

Let N be a *finite* subgroup of a group G and assume $N = \langle S \rangle$ for some subset S of G. Prove that an element $g \in G$ normalizes N if and only if $gSg^{-1} \subseteq N$.

Proof. First, let $g \in G$ normalize N. Then $gNg^{-1} = N$. Since $N = \langle S \rangle$, we must have $S \subseteq N$, and so $gSg^{-1} \subseteq gNg^{-1} = N$.

Next, let $gSg^{-1} \subseteq N$ and let $n \in N$. We can write n as a product of elements of s as in Exercises 16., 23., and 26.(a) above. For convenience, let us write $n = \prod s_i^{ij}$. Then:

$$gng^{-1} = g(\prod s_i^{ij})g^{-1} = \prod (gs_i^{ij}g^{-1}),$$

which is the product of elements of N and so lies in N. We then have $gNg^{-1} \subseteq N$. From 27., this implies that $gNg^{-1} = N$, and so g normalizes N. □

29. (9/21/23)

Let N be a *finite* subgroup of G and suppose $G = \langle T \rangle$ and $N = \langle S \rangle$ for some subsets S and T of G. Prove that N is normal in G if and only if $tSt^{-1} \subseteq N$ for all $t \in T$.

Proof. First, let $N \trianglelefteq G$. Then, from Exercise 27., $gNg^{-1} \subseteq N$ for all $g \in G$. Now since $T \subseteq G$ and $S \subseteq N$, this implies that $tst^{-1} \in N$ for all $t \in T, s \in S$, and so $tSt^{-1} \subseteq N$ for all $t \in T$.

Next, let $tSt^{-1} \subseteq N$ for all $t \in T$. We will first show that we must have $tNt^{-1} \subseteq N$ for all $t \in T$, and that this subsequently implies that $gNg^{-1} \subseteq N$ for all $g \in G$. As above, let us write $n \in N = \prod s_i^{ij}$, and let $t \in T$. Then:

$$tnt^{-1} = t(\prod s_i^{ij})t^{-1} = \prod(ts_i^{ij}t^{-1}),$$

which is the product of elements of N and so lies in N. We then have $tNt^{-1} \subseteq N$.

Next, let $g \in G$. Let us write g as the product of elements of T, $g = (t_1^{11}...t_m^{1m})(t_1^{21}...t_m^{2m})...(t_1^{p1}...t_m^{pm})$. Then we have:

$$\begin{aligned} gng^{-1} &= (t_1^{11}...t_m^{1m})...(t_1^{p1}...t_m^{pm})(\prod s_i^{ij})((t_1^{11}...t_m^{1m})...(t_1^{p1}...t_m^{pm}))^{-1} \\ &= t_1^{11}t_2^{12}...t_m^{pm}(\prod s_i^{ij})(t_m^{pm})^{-1}...(t_2^{12})^{-1}(t_1^{11})^{-1} \\ &= t_1^{11}(t_2^{12}(...(t_m^{pm}(\prod s_i^{ij})t_m^{-pm})...)t_2^{-12})t_1^{-11} \\ &= \prod(t_1^{11}(t_2^{12}(...(t_m^{pm}s_i^{ij}t_m^{-pm})...)t_2^{-12})t_1^{-11}). \end{aligned}$$

Now the inner-most conjugate, $t_m^{pm}s_i^{ij}t_m^{-pm}$, is an element of N. Evaluating from the parentheses outward, each conjugate is of the form $t_a^{ab}s_i^{ij}t_a^{-ab}$, that is, always an element of N. Therefore we have $gng^{-1} \in N$ for all $g \in G, n \in N$, and so $gNg^{-1} \subseteq N$, which implies that $N \trianglelefteq G$. □

30. (9/21/23)

Let $N \leq G$ and let $g \in G$. Prove that $gN = Ng$ if and only if $g \in N_G(N)$.

Proof. Recall that $N_G(N)$, the normalizer of N in G, is $\{g \in G \mid gNg^{-1} = N\}$.

First, let $g \in N_G(N)$ (to show that $gN = Ng$). It follows that $gNg^{-1} = N$. Let $y \in gN$. Since $gN = \{gn \mid n \in N\}$, we have $y = gx$ for some $x \in N$. From Chapter 2.2, the normalizer of N is a subgroup of G, and so is closed under inverses, so we also have $g^{-1} \in N_G(N)$, and so $g^{-1}N(g^{-1})^{-1} = N$. It follows that $x = g^{-1}z(g^{-1})^{-1}$ for some (unique) $z \in N$. Then $y = gx = g(g^{-1}z(g^{-1})^{-1}) = zg$, so we have $y \in Ng$. This proves that $gN \subseteq Ng$. The proof showing that $Ng \subseteq gN$ is structurally identical (let $y \in Ng \Rightarrow y = xg$, etc.), and so we have $gN = Ng$.

Next, let $gN = Ng$ (to show that $gNg^{-1} = N$). Let $y \in N$. Then $yg = gx$ for some $x \in N$. So $y = gxg^{-1}$, which implies that $y \in gNg^{-1}$, and so $N \subseteq gNg^{-1}$.

Similarly, let $y \in gNg^{-1}$, so $y = gxg^{-1}$ for some $x \in N$. Since $gN = Ng$, we know that $gx = zg$ for some $z \in N$. Then $y = gxg^{-1} = zgg^{-1} = z \in N$, so $gNg^{-1} \subseteq N$. Thus $gNg^{-1} = N$, so g is in the normalizer of N. □

31. (9/22/23)

Prove that if $H \leq G$ and N is a normal subgroup of H then $H \leq N_G(N)$. Deduce that $N_G(N)$ is the largest subgroup of G in which N is normal (i.e., is the join of all subgroups H for which $N \trianglelefteq H$).

Proof. Let $H \leq G$, $N \trianglelefteq H$, and let $h \in H$. Since N is normal in H, we have $hNh^{-1} = N$. Recall that $N_G(N) = \{g \in G \mid gNg^{-1} = N\}$. It follows that $h \in N_G(N)$, and so $H \subseteq N_G(N)$. Since both are subgroups of G, more specifically, we have $H \leq N_G(N)$.

This implies that any subgroup in which N is normal is a subgroup of $N_G(N)$, and so $N_G(N)$ is the largest subgroup of G in which N is normal. In particular, if $N_G(N) \leq K$ for some subgroup K of G, then N is not normal in K, since otherwise we would have $K \leq N_G(N)$. $\square$