Dummit & Foote Ch. 2.1: Subgroups, Definition and Examples

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Let G be a group.

1. (5/22/23)

In each of (a) - (e) prove that the specified subset H is a subgroup of the given group G:

(a) $H =$ the set of complex numbers of the form $a + ai, a \in \mathbb{R}$, $G = \mathbb{C}$ (under addition)

Proof. Let $a + ai, b + bi \in H$. $(b + bi) + (-b - bi) = 0$, so the inverse of $b + bi$ is $-b - bi$.

Then $a + ai - b + bi = (a - b) + (a - b)i \in H$. By the subgroup criterion, H is a subgroup of G. □

(b) $H =$ the set of complex numbers of absolute value 1, i.e., the unit circle in the complex plane, $G = \mathbb{C}$ (under multiplication)

Proof. Let $a + bi, c + di \in H$. Since $|a + bi| = 1, \sqrt{a^2 + b^2} = 1$. The multiplicative inverse of a is $\frac{a-bi}{\sqrt{a^2+b^2}} = a - bi$. And the absolute value of $a - bi$ is $\sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = 1$. Thus H is closed under inverses.

Further, the product $(a + bi)(c + di) = ac - bd + (ad + bc)i$ has absolute value $\sqrt{(ac - bd)^2 + (ad + bc)^2}$. This simplifies to:

$$\sqrt{a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2} = \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2} = \sqrt{a^2(c^2 + d^2) + b^2(c^2 + d^2)} = \sqrt{(a^2 + b^2)(c^2 + d^2)} = \sqrt{a^2 + b^2}\sqrt{c^2 + d^2} = 1,$$

and so H is closed under multiplication. Thus it is a subgroup of G. □

(c) $H =$ for fixed $n \in \mathbb{Z}^+$ the set of rational numbers whose denominators divide n, $G = \mathbb{Q}$ (under addition)

Proof. Formally, $H = \{p/q \in \mathbb{Q} \mid q \text{ divides } n\}$. Let $p_1/q_1, p_2/q_2 \in H$. Since q_1, q_2 divide n, let $aq_1 = bq_2 = n$. Then $p_1/q_1 = ap_1/aq_1 = ap_1/n$ and $p_2/q_2 = bp_2/bq_2 = bp_2/n$. The additive inverse of $p_2/q_2 = bp_2/n$ is $-bp_2/n$. The sum $ap_1/n + (-bp_2/n) = (ap_1 - bp_2)/n$ has a denominator that divides n (or else simplifies to a denominator that divides n), and so it is an element of H. By the subgroup criterion, H is a subgroup of G. □

(d) $H =$ for fixed $n \in \mathbb{Z}^+$ the set of rational numbers whose denominators are relatively prime to n, $G = \mathbb{Q}$ (under addition)

Proof. As immediately above, let $p_1/q_1, p_2/q_2 \in H$. Let a be the greatest common divisor of q_1 and q_2, and let $q_1 = ar_1, q_2 = ar_2$. Since q_1, q_2 are relatively prime to n, so too are the corresponding divisors a, r_1, and r_2. Now the sum of the first element with the inverse of the second element is:

$$p_1/q_1 - p_2/q_2 = p_1/ar_1 - p_2/ar_2 = \frac{p_1r_2 - p_2r_1}{ar_1r_2},$$

and since the factors in the divisor are all relatively prime to n, so is their product, and so the result is an element of H. Thus by the subgroup criterion, H is a subgroup of G. □

(e) $H =$ the set of nonzero real numbers whose square is a rational number, $G = \mathbb{R}$ (under multiplication)

Proof. Let $x_1, x_2 \in H$, with $x_1^2 = p_1/q_1 \in \mathbb{Q}, x_2^2 = p_1/q_1 \in \mathbb{Q}$.
The multiplicative inverse of x_2 is $1/x_2$. Consider x_1/x_2. Now $(x_1/x_2)^2 = \frac{p_1/q_1}{p_2/q_2} = \frac{p_1}{q_1} \cdot \frac{q_2}{p_1} = \frac{p_1q_2}{p_2q_1} \in \mathbb{Q}$. Thus by the subgroup criterion, H is a subgroup of G. □

2. (5/22/23)

In each of (a) - (e) prove that the specified subset H is *not* a subgroup of the given group G:

(a) $H =$ the set of 2-cycles, $G = D_{2n}$ for $n \geq 3$

Proof. H is not closed. Let $\sigma_1 = (1,2), \sigma_2 = (2,3)$, then $\sigma_1\sigma_2 = (1,3,2)$, a 3-cycle and therefore not in H. □

(b) $H =$ the set of reflections, $G = D_{2n}$ for $n \geq 3$

Proof. Formally, $H = \{sr^k \in D_{2n} \mid 0 \leq k < n\}$. H is not closed. For example, $sr, sr^2 \in H$ but $sr^2 sr = sr^2 r^{-1} s = srs = ssr^{-1} = r^{-1} \notin H$. $\square$

(c) $H = \{x \in G \mid |x| = n\} \cup \{1\}$, G a group containing an element of order n where n is a composite integer greater than 1

Proof. By counterexample, let $G = \mathbb{Z}/8\mathbb{Z}$ under modular addition. Let $n = 8$. The elements 1 and 3 have order 8, so both are in H. However, their sum, 4, has order 2, and so is not an element of H. $\square$

(d) $H =$ the set of (positive and negative) odd integers together with 0, $G = \mathbb{Z}$

Proof. Let $k_1, k_2 \in H$. Since both are odd, there exist $n_1, n_2 \in \mathbb{Z}$ such that $k_1 = 2n_1 + 1$ and $k_2 = 2n_2 + 1$. Their sum, then, is $2n_1 + 1 + 2n_2 + 1 = 2n_1 + 2n_2 + 2 = 2(n_1 + n_2 + 1)$, which is an even integer, and so is not an element of H. $\square$

(e) $H =$ the set of real numbers whose square is a rational number, $G = \mathbb{R}$ (under addition)

Proof. By counterexample, consider $\sqrt{2}, \sqrt{3} \in H$. Their sum, $\sqrt{2} + \sqrt{3}$, when squared, is equal to $(\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{6} + 3 = 5 + 2\sqrt{6} \notin \mathbb{Q}$. Therefore H is not closed, and is not a subset of G. $\square$

3. (5/22/23)

Show that the following subsets of the dihedral group D_8 are actually subgroups:

(a) $\{1, r^2, s, sr^2\}$

Proof. For these 4 elements, we will exhaustively show that the subset fulfills the criteria for a subgroup of D_8.

Each element is its own inverse in D_8, so the set is closed under inverses.

It is also closed under the product of two elements. Considering only the non-trivial products, starting with r^2: $r^2 s = sr^{-2} = sr^2$ and $r^2 sr^2 = sr^{-2} r^2 = s$. For s: $ssr^2 = r^2$. Finally for sr^2: $sr^2 r^2 = s$; $sr^2 s = ssr^{-2} = r^2$. Since the subset of closed under inverses and the binary operation, it is a subgroup. $\square$

(b) $\{1, r^2, sr, sr^3\}$

Proof. Similar to above, each element is its own inverse. To show it is closed, then, starting with r^2: $r^2 sr = sr^{-2} r = sr^{-1} = sr^3$; $r^2 sr^3 = sr^{-2} r^3 = sr$. For sr: $srr^2 = sr^3$; $srsr^3 = ssr^{-1} r^3 = r^2$. Finally for sr^3: $sr^3 r^2 = sr^{-1} r^2 = sr$; $sr^3 sr = ssr^{-3} r = r^{-2} = r^2$. Thus it is a subgroup of D_8. $\square$

4. (5/22/23)

Give an explicit example of a group G and an infinite subset H of G that is closed under the group operation but is not a subgroup of G.

Proof. Let $G = \mathbb{Z}, H = \mathbb{Z}^+$. For any two $n, m \in H$, we have $n > 0$ and $m > 0$. Their sum, $n + m$, is also greater than zero, and so is an element of H. However, H does not contain the identity element 0 (as well as containing no additive inverses of any elements), and so is not a subgroup of G. $\square$

5. (5/22/23)

Prove that G cannot have a subgroup H with $|H| = n - 1$, where $n = |G| > 2$.

Proof. Let G be a finite group of order $n > 2$ and suppose (toward contradiction) that H is a subgroup of G with order $n - 1$. Since H is a subgroup, $1 \in H$. There is exactly one element of G that is not an element of H, and it is not the identity. Call that element g. Then g^{-1} must be an element of H. However, g^{-1} has no inverse in H, since by definition g is not in H. Therefore H cannot be a subgroup, contradicting the initial assumption that H is a subgroup of G with order $n - 1$. $\square$

6. (5/23/23)

Let G be an abelian group. Prove that $\{g \in G \mid |g| < \infty\}$ is a subgroup of G (called the *torsion subgroup* of G). Give an explicit example where this set is not a subgroup when G is non-abelian.

Proof. We will show that the given set is closed and closed under inverses, and is thus a subgroup of G.

First, let $g_1, g_2 \in G$ with $|g_1| = n$ and $|g_2| = m$. Let k be the least common multiple of n and m. Then $g_1^k = g_2^k = 1$. And, given that G is abelian, we have $g_1^k g_2^k = (g_1 g_2)^k = 1$. Thus the order of $g_1 g_2$ is finite, so the set is closed.

Next, it suffices to demonstrate that, for all $g \in G$ with $|g| = n$, we have $|g^{-1}| = n$, so the set is also closed under inverses and is thus a subgroup of G.

In the non-abelian group $G = GL_2(\mathbb{R})$, however, consider the two elements $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$. Each has order 2. However, their product is $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Multiplied by itself, this results in $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$, and in general, it can be proven through induction that $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}$; that is, it has infinite order. Thus the set of elements with finite order is not closed in $GL_2(\mathbb{R})$ and so it is not a subgroup. $\square$

7. (5/23/23)

Fix some $n \in \mathbb{Z}$ with $n > 1$. Find the torsion subgroup of $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$. Show that the set of elements infinite order together with the identity is *not* a subgroup of this direct product.

Proof. The torsion subgroup of $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$ is $\{(k, m) \mid |(k, m)| < \infty, k \in \mathbb{Z}, 0 \leq m < n\}$. Considering $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$ under addition, suppose that (k, m) has order p, so $p(k, m) = (0, 0)$. Then $pk = 0$ (in $\mathbb{Z}$) and $mk = 0$ (in $\mathbb{Z}/n\mathbb{Z}$). The only value of k that satisfies this in 0. Because $\mathbb{Z}/n\mathbb{Z}$ is finite, all values of m have finite order (that is, there exists a p for all $m \in \mathbb{Z}/n\mathbb{Z}$ such that $pm = 0$).

It follows that the torsion subgroup of $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$ is $\{(0, m) \mid 0 \leq m < n\}$.

Now let $A =$ the set of elements of infinite order together with identity, that is, $A = \{(k, m) \mid |(k, m)| = \infty\} \cup \{(0, 0)\}$. $(1, 1)$ and $(-1, 1)$ are both in A. However, their sum, $(0, 1)$, has finite order (it is in the torsion subgroup, above), and so A is not closed, and is therefore not a subgroup of $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$. □

8. (5/27/23)

Let H and K be subgroups of G. Prove that $H \cup K$ is a subgroup if and only if either $H \subseteq K$ or $K \subseteq H$.

Proof. First, to show that $H \cup K$ a subgroup implies that $H \subset K$, let $h \in H$ and $k \in K$. Because $H \cup K$ is a subgroup, it is closed, and since $h, k \in H \cup K$, it follows that $hk \in H \cup K$. Then either $hk \in H$ or $hk \in K$.

If $hk \in H$, then, since H is closed under inverses, $h^{-1} \in H$, we have $h^{-1}hk \in H \Rightarrow k \in H$. This implies that $K \subseteq H$. Or, if $hk \in K$, then similarly $hkk^{-1} = h \in K \Rightarrow H \subseteq K$.

Next, to show that one subgroup being contained in the other implies that their union is a subgroup, without loss of generality let $H \subseteq K$. In this case $H \subset K \Rightarrow H \cup K = K$, which by definition is a subgroup of G.

Thus $H \cup K$ is a subgroup if and only if $H \subseteq K$ or $K \subseteq H$. □

9. (5/27/23)

Let $G = GL_n(F)$, where F is any field. Define

$$SL_n(F) = \{A \in GL_n(F) \mid \det A = 1\}$$

(called the *special linear group*). Prove that $SL_n(F) \leq GL_n(F)$.

Proof. Clearly $SL_n(F)$ is a subset of $GL_n(F)$, since by definition $A \in SL_n(F)$ implies $A \in GL_n(F)$. It remains to be proven that $SL_n(F)$ is a subgroup, which we will show by the subgroup criterion.

Let $B \in SL_n(F)$. From elementary linear algebra, the determinant of the product of two matrices is equal to the product of the two determinants of each matrix. So $1 = \det I_n = \det B^{-1}B = \det B^{-1} \det B = \det B^{-1}$. Then the determinant of B's inverse is also 1, so $SL_n(F)$ is closed under inverses.

Next, let $A \in SL_n(F)$ and consider the product AB^{-1}. From above, the determinant of this matrix is equal to $\det A \det B^{-1} = 1 \cdot 1 = 1$. Thus $SL_n(F)$ is also closed under matrix multiplication, and so is a subgroup of $GL_n(F)$. □

10. (5/27/23)

(a) Prove that if H and K are subgroups of G then so is their intersection $H \cap K$.

Proof. Let $g_1, g_2 \in H \cap K$. Then $g_1 \in H, g_2 \in H, g_1 \in K,$ and $g_2 \in K$. It follows that the product $g_1 g_2$ is an element of both H and K, since both subgroups are closed. Thus $g_1 g_2 \in H \cap K$, and so $H \cap K$ is closed.

Similarly,

$$g \in H \cap K \Rightarrow g \in H, g \in K \Rightarrow g^{-1} \in H, g^{-1} \in K \Rightarrow g^{-1} \in H \cap K,$$

which shows that $H \cap K$ is also closed under inverses and is therefore a subgroup of G. □

(b) Prove that the intersection of an arbitrary nonempty collection of subgroups of G is again a subgroup of G (do not assume the collection is countable).

Proof. Let $\mathcal{H}$ be a nonempty collection of subgroups of G. Consider $\bigcap_{H \in \mathcal{H}} = \{h \in G \mid h \in H \text{ for all } H \in \mathcal{H}\}$. Let h_1, h_2 be in this subset. Then for all $H \in \mathcal{H}$, $h_1 \in H$ and $h_2 \in H$, so $h_1 h_2 \in H$. So this intersection is closed under the binary operation of G. Similarly, for all $H \in \mathcal{H}$, $h \in H \Rightarrow h^{-1} \in H$, and so it is also closed under inverses.

Thus an arbitrary nonempty collection of subgroups is a subgroup. □

11. (5/27/23)

Let A and B be groups. Prove that the following sets are subgroups of the direct product $A \times B$:

(a) $\{(a, 1) \mid a \in A\}$

Proof. Let $(a_1, 1), (a_2, 1)$ in the set. Then $(a_1, 1)(a_2, 1) = (a_1 a_2, 1)$. Now $a_1 a_2 \in A$, so this is closed. Also, given $(a, 1)$, $a \in A \Rightarrow a^{-1} \in A$, so $(a, 1)^{-1} = (a^{-1}, 1)$ is in the set. Since it is closed and closed under inverses, it is a subgroup. □

(b) $\{(1,b) \mid b \in B\}$

Proof. The proof is identical to the one above but using $b_1, b_2 \in B$ in place of $a_1, a_2 \in A$. $\square$

(c) $\{(a,a) \mid a \in A\}$, where here we assume $B = A$ (called the *diagonal subgroup*).

Proof. Let $(a_1, a_1), (a_2, a_2)$ in the set. Then $(a_1, a_1)(a_2, a_2) = (a_1a_2, a_1a_2)$. Since $a_1a_2 \in A$, it is closed. Also, $(a,a)^{-1} = (a^{-1}, a^{-1})$, so it is closed under inverses and is therefore a subgroup. $\square$

12. (5/27/23)

Let A be an abelian group and fix some $n \in \mathbb{Z}$. Prove that the following sets are subgroups of A:

(a) $B = \{a^n \mid a \in A\}$

Proof. Let $a, b \in A$. Then $a^n, b^n \in B$. From Ch. 1, Ex. 24, since A is abelian, $a^n b^n = (ab)^n$. Thus the product ab is an element of B, so it is closed.

Also, the inverse of a^n is a^{-n}, which is equal to $(a^{-1})^n$, so B is closed under inverses and is thus a subgroup of A. $\square$

(b) $B = \{a \in A \mid a^n = 1\}$.

Proof. Let $a, b \in B$. Then $a^n = b^n = 1$. Since A is abelian, we also have $1 = a^n b^n = (ab)^n$. Thus B is closed.

Also, $a^n = 1$ implies that its inverse, $a^{-n} = (a^{-1})^n = 1$, so B is closed under inverses and is a subgroup of A. $\square$

13. (5/27/23)

Let H be a subgroup of the additive group of rational numbers with the property that $1/x \in H$ for every nonzero element x of H. Prove that $H = \{0\}$ or $\mathbb{Q}$.

Proof. If $H = \{0\}$, then the definition trivially holds. Suppose that H contains at least one other element, $\frac{m}{n}$. Since H is closed under addition, $\underbrace{\frac{m}{n} + ... + \frac{m}{n}}_{n \text{ times}} = m \in H$. Then, by definition of H, $\frac{1}{m} \in H$. It follows that $\underbrace{\frac{1}{m} + ... + \frac{1}{m}}_{m \text{ times}} = 1 \in H$.

Now let $\frac{p}{q}$ be any rational number. We can show that it is contained in H by: $\underbrace{1 + ... + 1}_{q \text{ times}} = q \Rightarrow \frac{1}{q} \in H$. Then $\underbrace{\frac{1}{q} + ... + \frac{1}{q}}_{pq \text{ times}} = \frac{p}{q} \in H$. Thus, from its definition, if H contains any other rational number other than zero, it is itself the group of rational numbers under addition. □

14. (5/27/23)

Show that $\{x \in D_{2n} \mid x^2 = 1\}$ is not a subgroup of D_{2n} (here $n \geq 3$).

Proof. It suffices to show that the given subset is not closed. The elements s and sr in D_{2n} both have order 2 (by definition of generators, $s^2 = 1$ and $(sr)^2 = srsr = ssr^{-1}r = 1$). However, their product, $ssr = r$ has order $n > 2$. Thus the set is not closed and is therefore not a subgroup of D_{2n}. □

15. (5/27/23)

Let $H_1 \leq H_2 \leq ...$ be an ascending chain of subgroups of G. Prove that $\bigcup_{i=1}^{\infty} H_i$ is a subgroup of G.

Proof. Let $h_1, h_2 \in \bigcup_{i=1}^{\infty} H_i$. Then for some $n, m \in \mathbb{Z}^+$, $h_1 \in H_n, h_2 \in H_m$. Without loss of generality suppose that $n < m$. Then, since $H_n \subseteq H_m$, $h_1 \in H_m$, which implies that $h_1 h_2 \in H_m$. Thus $h_1 h_2 \in \bigcup_{i=1}^{\infty} H_i$, so the union is closed under the binary operation of G.

Next, note that for all $h \in \bigcup_{i=1}^{\infty} H_i$, $h \in H_n$ implies that $h^{-1} \in H_n$, and so $h^{-1} \in \bigcup_{i=1}^{\infty} H_i$; that is, the union is also closed under inverses.

Thus the union of an ascending chain of subgroups of G is also a subgroup of G. □

16. (6/4/23)

Let $n \in \mathbb{Z}^+$ and let F be a field. Prove that the set $\{(a_{ij}) \in GL_N(F) \mid a_{ij} = 0 \text{ for all } i > j\}$ is a subgroup of $GL_n(F)$ (called the group of *upper triangular* matrices).

Proof. First, to show that the set of upper triangular matrices is closed under matrix multiplication, let A and B be upper triangular matrices, and consider the i, j-th entry of AB, with $i > j$. The value is equal to $a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}$. For any $k \leq j$, $a_{ik} = 0$, so the product $a_{ik}b_{kj}$ is also zero. And, for $k > j$, $b_{kj} = 0$, so the product is again zero. Thus the sum equals zero, and so the i, j-th entry of AB is zero when $i > j$, so the product is an upper triangular matrix.

Next, we show that given an upper triangular matrix A, its inverse A^{-1} is also an upper triangular matrix. Toward contradiction, suppose that A^{-1}

is *not* upper triangular. Denote the entries of A^{-1} by c_{ij}. Then there exists an entry $c_{ij} \neq 0$ in A^{-1} with $i > j$. Consider the i, j-th entry of $A^{-1}A = c_{i1}a_{1j} + ... + c_{ij}a_{jj} + ... + c_{in}a_{nj}$. All of the products before $c_{ij}a_{jj}$ are zero, since the entries coming from A^{-1} are zero. All of the products after $c_{ij}a_{jj}$ are zero, since the entries coming from A are zero. However, the product $c_{ij}a_{jj}$ is not zero, since both c_{ij} (by assumption) and a_{jj} (by definition) are nonzero. It follows that the matrix product $A^{-1}A$ is not the identity matrix, and so A^{-1} is not A's inverse, a contradiction. Thus the inverse of A must be an upper triangular matrix, and so the set of upper triangular matrices is closed under inverses.

Having shown that it is closed under matrix multiplication and closed under inverses, the set of upper triangular matrices is a subgroup of $GL_n(F)$. □

17. (6/4/23)

Let $n \in \mathbb{Z}^+$ and let F be a field. Prove that the set $\{(a_{ij}) \in GL_N(F) \mid a_{ij} = 0 \text{ for all } i > j, \text{ and } a_{ii} = 1 \text{ for all } i\}$ is a subgroup of $GL_n(F)$.

Proof. This proof is similar to the preceding exercise. To show that the set is closed, let A and B be upper triangular matrices with all diagonal entries equal to 1. From 16., their product is also an upper triangular matrix, so we need only consider an arbitrary diagonal entry of AB. The value of the i, i-th entry is equal to $\underbrace{a_{i1}b_{1i} + ...}_{\text{entries from } A=0} + \underbrace{a_{ii}b_{ii}}_{1\cdot 1} + \underbrace{... + a_{in}b_{ni}}_{\text{entries from } B=0} = 1$. So the set is closed under matrix multiplication.

Next, from 16., we know that the inverse of a matrix A is upper triangular, so consider only its i, i-th entry. Suppose c_{ii} of A^{-1} is equal to $k \neq 1$. Then the i, i-th entry of the product $A^{-1}A$ equals $\underbrace{c_{i1}a_{1i} + ...}_{\text{entries from } A^{-1}=0} + \underbrace{c_{ii}a_{ii}}_{k\cdot 1} + \underbrace{... + c_{in}a_{ni}}_{\text{entries from } A=0} = k \neq 1$. It follows that this product is not the identity matrix, a contradiction. Thus the set is closed under inverses.

Having shown that it is closed under matrix multiplication and closed under inverses, the set of upper triangular matrices with diagonal entries equal to 1 is a subgroup of $GL_n(F)$. □