

Dummit & Foote Ch. 4.3: Groups Acting on Themselves by Conjugation — The Class Equation

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Let G be a group.

1. (2/22/24)

Suppose G has a left action on a set A , denoted by $g \cdot a$ for all $g \in G$ and $a \in A$. Denote the corresponding right action on A by $a \cdot g$. Prove that the (equivalence) relations \sim and \sim' defined by

$$a \sim b \text{ if and only if } a = g \cdot b \text{ for some } g \in G$$

and

$$a \sim' b \text{ if and only if } a = b \cdot g \text{ for some } g \in G$$

are the same relation (i.e., $a \sim b$ if and only if $a \sim' b$).

Proof. To show that $a \sim b$ implies $a \sim' b$, we must show that, given a $g \in G$ with $a = g \cdot b$, there exists an $h \in G$ such that $a = b \cdot h$. By definition, the corresponding right action of a left action is specified to be $g \cdot x = x \cdot g^{-1}$ for all $g \in G$, $x \in A$. Letting $h = g^{-1}$, we have found an element where $a = g \cdot b = b \cdot h$, and so $a \sim' b$.

The proof for $a \sim' b$ implies $a \sim b$ is identical, letting $h = g^{-1}$ but with h acting on the left. \square

2. (2/22/24)

Find all conjugacy classes and their sizes in the following groups:

(a) D_8 :

$$\{1\}_1 \quad \{r^2\}_1 \quad \{r, r^3\}_2 \quad \{s, sr^2\}_2 \quad \{sr, sr^3\}_2$$

(b) Q_8 :

$$\{1\}_1 \quad \{-1\}_1 \quad \{\pm i\}_2 \quad \{\pm j\}_2 \quad \{\pm k\}_2$$

(c) A_4 :

$$\begin{aligned} \{1\}_1 \quad & \{(1\ 2\ 3), (1\ 3\ 4), (1\ 4\ 2), (2\ 4\ 3)\}_4 \quad \{(1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 3), (2\ 3\ 4)\}_4 \\ & \{(1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}_3 \end{aligned}$$

3. (2/22/24)

Find all the conjugacy classes and their sizes in the following groups:

(a) $Z_2 \times S_3$:

$$\begin{aligned} & \{(0, 1)\}_1 \quad \{(1, 1)\}_1 \quad \{(0, (1\ 2)), (0, (1\ 3)), (0, (2\ 3))\}_3 \\ & \{(1, (1\ 2)), (1, (1\ 3)), (1, (2\ 3))\}_3 \quad \{(0, (1\ 2\ 3)), (0, (1\ 3\ 2))\}_2 \\ & \{(1, (1\ 2\ 3)), (1, (1\ 3\ 2))\}_2 \end{aligned}$$

(b) $S_3 \times S_3$:

$$\begin{aligned} & \{(1, 1)\}_1 \quad \{(1, 2\text{-cycle})\}_3 \quad \{(2\text{-cycle}, 1)\}_3 \quad \{(1, 3\text{-cycle})\}_2 \quad \{(3\text{-cycle}, 1)\}_2 \\ & \{(2\text{-cycle}, 2\text{-cycle})\}_9 \quad \{(2\text{-cycle}, 3\text{-cycle})\}_6 \quad \{(3\text{-cycle}, 2\text{-cycle})\}_6 \\ & \{(3\text{-cycle}, 3\text{-cycle})\}_4 \end{aligned}$$

(c) $Z_3 \times A_4$ (using representatives from the conjugacy classes of A_4 above):

$$\begin{aligned} & \{(0, 1)\}_1 \quad \{(1, 1)\}_1 \quad \{(2, 1)\}_1 \\ & \{(0, \overline{(1\ 2\ 3)})\}_4 \quad \{(1, \overline{(1\ 2\ 3)})\}_4 \quad \{(2, \overline{(1\ 2\ 3)})\}_4 \\ & \{(0, \overline{(1\ 3\ 2)})\}_4 \quad \{(1, \overline{(1\ 3\ 2)})\}_4 \quad \{(2, \overline{(1\ 3\ 2)})\}_4 \\ & \{(0, \overline{(1\ 2)(3\ 4)})\}_3 \quad \{(1, \overline{(1\ 2)(3\ 4)})\}_3 \quad \{(2, \overline{(1\ 2)(3\ 4)})\}_3 \end{aligned}$$

4. (2/22/24)

Prove that if $S \subseteq G$ and $g \in G$ then $gN_g(S)g^{-1} = N_G(gSg^{-1})$ and $gC_g(S)g^{-1} = C_G(gSg^{-1})$.

Proof. Let $x \in N_G(S)$. So $xsx^{-1} \in S$ for all $s \in S$. Then

$$\begin{aligned} gxsx^{-1}g^{-1} & \in gSg^{-1} \\ gxx^{-1}gsg^{-1}gx^{-1}g^{-1} & \in gSg^{-1} \\ (gxx^{-1})gsg^{-1}(gx^{-1}g^{-1}) & \in gSg^{-1} \\ (gxx^{-1})gsg^{-1}(gxx^{-1})^{-1} & \in gSg^{-1}, \end{aligned}$$

which implies that $gxg^{-1} \in N_G(gSg^{-1})$, and so $gN_G(S)g^{-1} \subseteq N_G(gSg^{-1})$.

Conversely, let $x \in N_G(gSg^{-1})$. So $xsg^{-1}x^{-1} \in gSg^{-1}$ for all $s \in S$. Then

$$\begin{aligned} xsg^{-1}x^{-1} &\in gSg^{-1} \\ g^{-1}xsg^{-1}x^{-1} &\in Sg^{-1} \\ g^{-1}xsg^{-1}x^{-1}g &\in S \\ (g^{-1}xg)s(g^{-1}xg)^{-1} &\in S \\ g^{-1}xg &\in N_G(S) \\ x &\in gN_G(S)g^{-1}, \end{aligned}$$

which shows that $N_G(gSg^{-1}) \subseteq gN_G(S)g^{-1}$. This proves that $N_G(gSg^{-1}) = gN_G(S)g^{-1}$.

Next, let $x \in C_G(S)$. So $xs = sx$ for all $s \in S$. Then

$$\begin{aligned} xs &= sx \\ gsg^{-1} &= gsg^{-1} \\ gsg^{-1}gxg^{-1} &= gsg^{-1}gxg^{-1} \\ (gsg^{-1})(gxg^{-1}) &= (gsg^{-1})(gxg^{-1}), \end{aligned}$$

and so $gxg^{-1} \in C_G(gSg^{-1})$, which implies that $gC_G(S)g^{-1} \subseteq C_G(gSg^{-1})$. Finally, let $x \in C_G(gSg^{-1})$. So $x(gsg^{-1}) = (gsg^{-1})x$ for all $x \in S$. Then

$$\begin{aligned} xsg^{-1} &= gsg^{-1}x \\ g^{-1}xsg^{-1} &= sg^{-1}x \\ g^{-1}xgs &= sg^{-1}xg \\ (g^{-1}xg)s &= s(g^{-1}xg), \end{aligned}$$

which implies that $g^{-1}xg \in C_G(S)$, so $x \in gC_G(S)g^{-1}$. It follows that $C_G(gSg^{-1}) \subseteq gC_G(S)g^{-1}$, and therefore $gC_G(S)g^{-1} = C_G(gSg^{-1})$. \square

10. (2/28/24)

Let σ be the 5-cycle $(1\ 2\ 3\ 4\ 5)$ in S_5 . In each of (a) to (c) find an explicit element $\tau \in S_5$ which accomplishes the specified conjugation:

- (a) $\tau\sigma\tau^{-1} = \sigma^2 = (1\ 3\ 5\ 2\ 4)$. Let $\tau = (2\ 3\ 5\ 4)$. Then $\tau\sigma\tau^{-1} = (\tau(1)\tau(2)\tau(3)\tau(4)\tau(5)) = (1\ 3\ 5\ 2\ 4) = \sigma^2$.
- (b) $\tau\sigma\tau^{-1} = \sigma^{-1} = (1\ 5\ 4\ 3\ 2)$. Let $\tau = (2\ 5)(3\ 4)$. Then $\tau\sigma\tau^{-1} = \sigma^{-1}$.
- (c) $\tau\sigma\tau^{-1} = \sigma^{-2} = (1\ 4\ 2\ 5\ 3)$. Let $\tau = (2\ 4\ 5\ 3)$. Then $\tau\sigma\tau^{-1} = \sigma^{-2}$.

11. (2/28/24)

In each of (a) - (d) determine whether σ_1 and σ_2 are conjugate. If they are, give an explicit permutation τ such that $\tau\sigma_1\tau^{-1} = \sigma_2$.

- (a) $\sigma_1 = (12)(345)$ and $\sigma_2 = (123)(45)$. Both have cycle type 1, 1, 3 and so they are conjugate. Let $\tau = (14253)$. Then $\tau\sigma_1\tau^{-1} = \sigma_2$.
- (b) $\sigma_1 = (15)(372)(106811)$ and $\sigma_2 = (37510)(49)(13112)$. In S_{13} , both have cycle type 1, 1, 1, 1, 2, 3, 4 and so they are conjugate. Let $\tau = (14)(211103)(5967138)$. Then $\tau\sigma_1\tau^{-1} = \sigma_2$.
- (c) $\sigma_1 = (15)(372)(106811)$ and $\sigma_2 = \sigma_1^3 = (15)(101186)$. They do not have the same cycle type (σ_1 contains a 3-cycle that σ_2 does not), and so they are not conjugate.
- (d) $\sigma_1 = (13)(246)$ and $\sigma_2 = (35)(24)(56) = (24)(356)$. Let $\tau = (12345)$. Then $\tau\sigma_1\tau^{-1} = \sigma_2$.

13. (2/28/24)

Find all finite groups which have exactly two conjugacy classes.

Proof. Let G be a non-trivial finite group. Since the conjugacy class of 1 is $\{1\}$, if G has exactly two conjugacy classes, then every other element in G must have the same conjugacy class, namely $G - \{1\}$.

From Proposition 6, for any $g \in G$, the number of conjugates of g (i.e. the cardinality of the conjugacy class of g) is the index of the centralizer of g , $|G : C_G(g)|$. Therefore the size of the conjugacy class of g must divide the order of G .

Let $|G| = n$. Then the size of the conjugacy class of g is $|G - \{1\}| = n - 1$. This is only possible when $|G| = 2$, and so G must be the unique group of order two. \square