Dummit & Foote Ch. 3.3: The Isomorphism Theorems

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Let G be a group.

1. (10/20/23)

Let F be a finite field of order q and let $n \in \mathbb{Z}^+$. Prove that $|GL_n(F) : SL_n(F)| = q - 1$.

Proof. Define a map $\varphi : GL_n(F) \to F^\times$ by $\varphi(A) = \det A$ for all $A \in GL_n(F)$. From Ch. 3.1, Exercise 35., φ is a surjective homomorphism with $\ker \varphi = SL_n(F)$.

From Corollary 17, we have:

$$
\begin{aligned}
|GL_n(F) : \ker \varphi| &= |\varphi(GL_n(F))|, \text{ which implies that} \\
|GL_n(F) : SL_n(F)| &= \underbrace{|F^\times|}_{\varphi \text{ is surjective}} = q - 1,
\end{aligned}
$$

as desired. □

3. (10/26/23)

Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either

(i) $K \leq H$ or

(ii) $G = HK$ and $|K : K \cap H| = p$.

Proof. Suppose that $H \trianglelefteq G$ with $|G : H| = |G/H| = p$, where p is a prime. Suppose additionally that $K \leq G$ and $K \nleq H$.

Now let $g \in G$. Clearly g belongs to the left coset gH, which we denote $\overline{g} \in G/H$. Since G/H has order p, it is cyclic, and so is generated by any non-identity element (that is, any coset of H other than itself). So $\overline{g}$ generates G/H. Similarly, for any $k \in K, k \notin H$, $\overline{k}$ generates G/H. Therefore $\overline{g} = \overline{k}$ for some g, k, which implies that $g \in kH$. It follows that $g \in KH$, so $G \leq KH$. Since G is closed, we must have $G = KH = HK$.

From the Diamond Isomorphism Theorem, we have $HK/H \cong K/H \cap K$. Since $HK = G$, it follows that $|G : H| = |K : H \cap K|$, and so $|K : K \cap H| = p$. $\square$

4. (10/27/23)

Let C be a normal subgroup of the group A and let D be a normal subgroup of the group B. Prove that $(C \times D) \trianglelefteq (A \times B)$ and $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.

Proof. Let $(c, d) \in C \times D$. Consider the conjugate of (c, d) by $(a, b) \in A \times B$:

$$(a, b)(c, d)(a, b)^{-1} = (a, b)(c, d)(a^{-1}, b^{-1}) = (aca^{-1}, bdb^{-1}).$$

Because $C \trianglelefteq A$, the first coordinate is an element of C, and similarly the second is an element of D. Therefore the conjugate element lies in $C \times D$, and it follows that $(C \times D) \trianglelefteq (A \times B)$.

Next, to show that $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$, define a map $\varphi : (A \times B)/(C \times D) \to (A/C) \times (B/D)$ by $\varphi(\overline{(a, b)}) = (\overline{a}, \overline{b})$. We see that this map is a homomorphism:

$$\begin{aligned}\varphi(\overline{(a_1, b_1)}\,\overline{(a_2, b_2)}) = \varphi(\overline{(a_1a_2, b_1b_2)}) &= (\overline{a_1a_2}, \overline{b_1b_2}) \\ &= (\overline{a_1}, \overline{b_1})(\overline{a_2}, \overline{b_2}) = \varphi(\overline{(a_1, b_1)})\varphi(\overline{(a_2, b_2)}).\end{aligned}$$

It is also surjective by definition, since $(\overline{a}, \overline{b}) = \varphi(\overline{(a, b)})$ is an arbitrary element of $(A/C) \times (B/D)$ with a preimage in $(A \times B)/(C \times D)$.

Finally, it is injective. Let $\varphi(\overline{(a_1, b_1)}) = \varphi(\overline{(a_2, b_2)})$. Then $(\overline{a_1}, \overline{b_1}) = (\overline{a_2}, \overline{b_2})$, so we have $\overline{a_1} = \overline{a_2}$ and $\overline{b_1} = \overline{b_2}$. Since $\overline{a_1} = \overline{a_2}$ implies $\overline{(a_1, x)} = \overline{(a_2, x)}$ for all $\overline{x} \in B/D$ and vice-versa, we then have $\overline{(a_1, b_1)} = \overline{(a_2, b_2)}$, and so φ is one-to-one.

Thus φ is an isomorphism, which concludes the proof that $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$. $\square$