Dummit & Foote Ch. 3.1: Quotient Groups and Homomorphisms

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Let G and H be groups.

1. (8/21/23)

Let $\varphi : G \to H$ be a homomorphism and let $E \leq H$. Prove that $\varphi^{-1}(E) \leq G$ (i.e., the preimage or pullback of a subgroup under a homomorphism is a subgroup). If $E \trianglelefteq H$ prove that $\varphi^{-1}(E) \trianglelefteq G$. Deduce that $\ker \varphi \trianglelefteq G$.

Proof. Let $x, y \in \varphi^{-1}(E) \subseteq G$. Suppose that $\varphi(x) = a, \varphi(y) = b, a, b \in E \leq H$. Since φ is a homomorphism, we have $\varphi(y^{-1}) = \varphi(y)^{-1} = b^{-1}$. Then:

$$\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1}) = \varphi(x)\varphi(y)^{-1} = ab^{-1} \in E,$$

which implies that $xy^{-1} \in \varphi^{-1}(E)$. It follows that, by the subgroup criterion, $\varphi^{-1}(E) \leq G$. $\square$

2. (8/23/23)

Let $\varphi : G \to H$ be a homomorphism of groups with kernel K and let $a, b \in \varphi(G)$. Let $X \in G/K$ be the fiber above a and Y be the fiber above b, i.e., $X = \varphi^{-1}(a), Y = \varphi^{-1}(b)$. Fix an element $x \in X$ (so $\varphi(x) = a$). Prove that if $XY = Z$ in the quotient group G/K and z is any member of Z, then there is some $y \in Y$ such that $xy = z$.

Proof. We know that, for any $x \in X, y \in Y$, $\varphi(x) = a$ and $\varphi(y) = b$. Since φ is a homomorphism, it follows that $\varphi(xy) = \varphi(x)\varphi(y) = ab$, and so the image of any element of $XY = Z$ under φ is $ab \in H$.

Next, consider the element $x^{-1}z \in G$, as well as its image under φ. Since φ is a homomorphism, we have $\varphi(x^{-1}) = \varphi(x)^{-1}$. So $\varphi(x^{-1}z) = \varphi(x^{-1})\varphi(z) = \varphi(x)^{-1}\varphi(z) = a^{-1}ab = b$. The set Y consists of all elements of G whose image under φ is b, and so we must have $x^{-1}z \in Y$.

Now if we fix some element $x \in X$, then for any $z \in Z$, we have $x^{-1}z \in Y$ such that its product with x is z: $xx^{-1}z = z$. $\square$

3. (8/23/23)

Let A be an abelian group and let B be a subgroup of A. Prove that A/B is abelian. Give an example of a non-abelian group G containing a proper normal subgroup N such that G/N is abelian.

Proof. Because A is abelian, all subgroups of A are normal, so A/B is well-defined for every $B \leq A$.

Let $C, D \in A/B$ with $C = cB$ and $D = dB$ for some $c, d \in A$. Then:

$$CD = (cB)(dB) = (cd)B = (dc)B = (dB)(cB) = DC,$$

which implies that A/B is abelian.

Now if we let G be the dihedral group D_8, then G is non-abelian. Let N be the cyclic subgroup generated by $r : \{1, r, r^2, r^3\}$. The only coset of N is sN; together these two sets cover G. Then $G/N = \{N, sN\}$. There is only one group of order 2 up to isomorphism, and it is abelian. Thus G/N is abelian. $\square$

4. (8/23/23)

Prove that in the quotient group G/N, $(gN)^\alpha = (g^\alpha)N$ for all $\alpha \in \mathbb{Z}$.

Proof. We start by induction: In the base case, $\alpha = 1$, we have $(gN)^1 = gN = (g^1)N$. Next, suppose that for some $\alpha > 1$, we have $(gN)^\alpha = (g^\alpha)N$. Then:

$$(gN)^{\alpha+1} = (gN)^\alpha gN = g^\alpha N \cdot gN = (g^{\alpha+1})N,$$

as desired. We have now proven that $(gN)^\alpha = (g^\alpha)N$ for $\alpha \geq 1$.

Next, consider $(gN)^\alpha (gN)^{-\alpha}$, where $\alpha \geq 1$. In the quotient group G/N, for any subset $X \in G/N$, we must have $X^\alpha X^{-\alpha} = N$ (the identity of G/N), so $(gN)^\alpha (gN)^{-\alpha} = N$. From above, $(gN)^\alpha = (g^\alpha)N$, so $(g^\alpha)N \cdot (gN)^{-\alpha} = N$. Also, from the operation on left cosets, we know that $N = (g^\alpha)N \cdot (g^{-\alpha})N$. Since both $(g^\alpha)N \cdot (gN)^{-\alpha} = N$ and $(g^\alpha)N \cdot (g^{-\alpha})N = N$, we must have $(gN)^{-\alpha} = (g^{-\alpha})N$. We have now proven for all nonzero integers.

Finally, we note that $(gN)^0 = N$ (the identity of G/N) and that $(g^0)N = eN = N$, so $(gN)^0 = (g^0)N$. This concludes the proof that $(gN)^\alpha = (g^\alpha)N$ for all $\alpha \in \mathbb{Z}$. $\square$

5. (8/23/23)

Use the preceding exercise to prove that the order of the element gN in G/N is n, where n is the smallest positive integer such that $g^n \in N$ (and gN has infinite order if no such positive integer exists). Give an example to show that the order of gN in G/N may be strictly smaller than the order of g in G.

Proof. Let $gN \in G/N$, and let n be the smallest positive integer such that $g^n \in N$. Suppose that $g^n = h \in N$.

From Exercise 4., $(gN)^n = (g^n)N = hN = N$ (because $h \in N$), so the order of gN must divide n.

Suppose (toward contradiction) that the order of gN is k, where $k < n$. Then $(gN)^k = (g^k)N = N$, which implies that g^k lies in N, contradicting our assumption that n is the smallest such positive integer. Therefore the order of gN is n.

If there is no positive integer n such that $g^n \in N$, then for all $k \in \mathbb{Z}^+$, we have $(gN)^k = (g^k)N \neq N$, so gN has infinite order.

As an example where $|gN| < |g|$, let $G = Z_9 = \langle x \rangle$ and let $N = \langle x^3 \rangle$. Because all cyclic groups are abelian, N is normal in G, and so G/N is well-defined. The quotient group G/N contains three elements: N, xN, and $(x^2)N$. The element $xN \in G/N$ has order 3: $(xN)^3 = (x^3)N = N$ (because $x^3 \in N$). However, the generating element $x \in G$ has order 9. $\square$

6. (8/24/23)

Define $\varphi : \mathbb{R}^\times \to \{\pm 1\}$ by letting $\varphi(x)$ be x divided by the absolute value of x. Describe the fibers of φ and prove that φ is a homomorphism.

Proof. We consider the two cases where $x < 0$ and $x > 0$ (0 is not an element of $\mathbb{R}^\times$). If $x > 0$, then $\varphi(x) = x/|x| = x/x = 1$. If $x < 0$, then $\varphi(x) = x/|x| = x/-x = -1$. Therefore the fiber above -1 is every negative real number and the fiber above 1 is every positive real number.

To show that φ is a homomorphism, we let $x, y \in \mathbb{R}^\times$ and again consider the different cases: Where x and y are both positive, where they are both negative, and where one is positive and the other negative.

If both x and y are positive, then $\varphi(x)\varphi(y) = 1 \cdot 1 = 1$ and $\varphi(xy) = \frac{xy}{|xy|} = \frac{xy}{xy} = 1$, so $\varphi(x)\varphi(y) = \varphi(xy)$.

If both x and y are negative, then $\varphi(x)\varphi(y) = -1 \cdot -1 = 1$ and $\varphi(xy) = \frac{xy}{|xy|} = \frac{xy}{xy} = 1$, so $\varphi(x)\varphi(y) = \varphi(xy)$.

Suppose x is positive and y is negative. Then $\varphi(x)\varphi(y) = 1 \cdot -1 = -1$ and $\varphi(xy) = \frac{xy}{|xy|} = \frac{xy}{-xy} = -1$, so $\varphi(x)\varphi(y) = \varphi(xy)$.

Thus, in every case of $x, y \in \mathbb{R}^\times$, we have $\varphi(x)\varphi(y) = \varphi(xy)$, and φ is thus a homomorphism. $\square$

7. (8/24/23)

Define $\pi : \mathbb{R}^2 \to \mathbb{R}$ by $\pi((x, y)) = x + y$. Prove that π is a surjective homomorphism and the describe the kernel and fibers of π geometrically.

Proof. First, to show that π is surjective, let $z \in \mathbb{R}$. Now $z = z + 0$, so $(z, 0)$ is an element of $\mathbb{R}^2$ such that $\pi((z, 0)) = z + 0 = z$.

Next, to show that π is a homomorphism, let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. We have $\pi((x_1, y_1) + (x_2, y_2)) = \pi((x_1 + x_2, y_1 + y_2)) = x_1 + x_2 + y_1 + y_2$, and $\pi((x_1, y_1)) + \pi((x_2, y_2)) = x_1 + y_1 + x_2 + y_2$. By the commutativity of addition in $\mathbb{R}$, these are equal to each other, and so π is a surjective homomorphism.

The kernel of π consists of all points $(x, y) \in \mathbb{R}^2$ such that $x + y = 0$, that is, the diagonal line running from the upper-left to the bottom-right of the Cartesian plane. Geometrically, the fibers of π are translations of this line, such that for any $z \in \mathbb{R}$, the fiber of π above z is the diagonal line intersecting both $(z, 0)$ and $(0, z)$. $\square$