Dummit & Foote Ch. 1.2: Dihedral Groups

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1. (1/23/23)

Compute the order of each of the elements in the following groups:

(a) D_6

- r, r^2: 3
- s, sr, sr^2: 2

(b) D_8

- r: 4
- r^2: 2
- r^3: 4
- s, sr, sr^2, sr^3: 2

(c) D_{10}

- r, r^2, r^3, r^4: 5
- s, sr, sr^2, sr^3, sr^4: 2

2. (1/23/23)

Use the generators and relations of $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$ to show that if x is any element of D_{2n} which is not a power of r, then $rx = xr^{-1}$.

Proof. Let $x \in D_{2n}$ such that $x \neq r^k$ for all $k \in \mathbb{Z}$. Then, since all elements of D_{2n} can be written as a product of generators s and r, we must have $x = sr^k$ for some $k \in \{1, 2, ..., n-1\}$. Therefore:

$$rx = rsr^k = sr^{-1}r^k = sr^{k-1} = sr^k r^{-1} = xr^{-1},$$

as desired. □

3. (1/25/23)

Use the generators and relations above to show that every element of D_{2n} which is not a power of r has order 2. Deduce that D_{2n} is generated by the two elements s and sr, both of which have order 2.

Proof. Let $sr^k \in D_{2n}$. $(sr^k)(sr^k) = s(r^k s)r^k = s(sr^{-k})r^k = ssr^{-k}r^k = 1 \cdot 1 = 1$. Thus the order of elements of the form sr^k, that is, every element which is not a power of r, has order 2.

To show that D_{2n} is generated by s and sr, let $r^k, sr^k \in D_{2n}$. Now $s \cdot sr = r$, so $(s \cdot sr)^k = r^k$. To obtain sr^k, we simply left-multiply the previous by s: $s(s \cdot sr)^k = sr^k$. Thus every element of D_{2n} can be written as a product of s and sr, and so $\langle s, sr \rangle$ is a generator for D_{2n}. $\square$