Dummit & Foote Ch. 1.7: Group Actions

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1. (4/27/23)

Let F be a field. Show that the multiplicative group of nonzero elements of F (denoted by $F^\times$) acts on the set F by $g \cdot a = ga$, where $g \in F^\times, a \in F$ and ga is the usual product in F of the two field elements.

Proof. To show that $F^\times$ acts on F, we must show that $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for all $g_1, g_2 \in F^\times, a \in F$, and $1 \cdot a = a$ for all $a \in F$.

First, let $g_1, g_2 \in F^\times$ and $a \in F$. By the definition of the action, $g_1 \cdot (g_2 \cdot a) = g_1 \cdot (g_2 a) = g_1 g_2 a$. By the associativity of multiplication, $g_1 g_2 a = (g_1 g_2) a$. Again by the action definition, this equals $(g_1 g_2) \cdot a$.

It follows directly from the field axiom of multiplicative identity that $1 \cdot a = a$ for all $a \in A$. Thus $F^\times$ acts on F by $g \cdot a = ga$. □

2. (4/27/23)

Show that the additive group $\mathbb{Z}$ acts on itself by $z \cdot a = z + a$ for all $z, a \in \mathbb{Z}$.

Proof. First, $z_1 \cdot (z_2 \cdot a) = z_1 \cdot (z_2 + a) = z_1 + z_2 + a = (z_1 + z_2) + a = (z_1 + z_2) \cdot a$.

Also, $0 \cdot a = 0 + a = a$ for all $a \in \mathbb{Z}$. Thus $\mathbb{Z}$ acts on itself by $z \cdot a = z + a$. □

3. (4/27/23)

Show that the additive group $\mathbb{R}$ acts on the x, y plane $\mathbb{R} \times \mathbb{R}$ by $r \cdot (x, y) = (x + ry, y)$.

Proof. First, $r_1 \cdot (r_2 \cdot (x, y)) = r_1 \cdot (x + r_2 y, y) = (x + r_2 y + r_1 y, y) =$
$(x + (r_1 + r_2) y, y) = (r_1 + r_2) \cdot (x, y)$.

Also, $0 \cdot (x, y) = (x + 0y, y) = (x, y)$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$. Thus $\mathbb{R}$ acts on $\mathbb{R} \times \mathbb{R}$ by $r \cdot (x, y) = (x + ry, y)$. □

4. (4/27/23)

Let G be a group acting on a set A and fix some $a \in A$. Show that the following sets are subgroups of G:

(a) the kernel of the action,

Proof. The kernel of G is the set $\{g \in G \mid g \cdot a = a \text{ for all } a \in A\}$. It is closed under the binary operation of G: If g_1, g_2 are in the kernel, then $g_1 \cdot (g_2 \cdot a) = g_1 \cdot a = a$ for all $a \in A$. And, by definition of a group action, $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$, which implies that $(g_1 g_2) \cdot a = a$, so $g_1 g_2$ is in the kernel of G.

The kernel is also closed under inverses: Let g be in the kernel of G. Then $1 \cdot a = (g^{-1}g) \cdot a = g^{-1} \cdot (g \cdot a) = g^{-1} \cdot a$. By definition, $1 \cdot a = a$, so $g^{-1} \cdot a = a$ for all a, so g^{-1} is in the kernel. Thus the kernel of the action is a subgroup of G. □

(b) $\{g \in G \mid ga = a\}$ — this subgroup is called the *stabilizer* of G.

Proof. The proof that this set of elements if a subgroup is identical to the one immediately above, but for a fixed a as opposed to all $a \in A$. □

5. (4/28/23)

Prove that the kernel of an action of the group G on the set A is the same as the kernel of the corresponding permutation representation $G \to S_A$.

Proof. Let φ be the permutation representation $G \to S_A$ corresponding to G acting on A. Let g be in the kernel of the action of G (to show that $\varphi(g)$ is in the kernel of φ). Then $g \cdot a = a$ for all $a \in A$. If σ_g is the permutation of S_A corresponding to g, then σ_g is the identity permutation, because $\sigma_g(a) = a$ for all $a \in A$. Thus $\sigma_g = \varphi(g)$ is in the kernel of φ.

Next, let $\varphi(g)$ be in the kernel of φ (to show that g is in the kernel of G). Then $\varphi(g)$ is the identity permutation, so $\varphi(g) \cdot a = \sigma_g(a) = a$ for all $a \in A$. Also, by definition, $\sigma_g(a) = g \cdot a$, so $g \cdot a = a$ for all $a \in A$. Thus g is in the kernel of the action of G.

Having shown that membership in one implies membership in the other, this proves that the kernel of G acting on A is thus equal to the kernel of the permutation representation $\varphi : G \to S_A$. □

6. (4/28/23)

Prove that a group G acts faithfully on a set A if and only if the kernel of the action is the set consisting of only the identity.

Proof. First, let G act on A. Suppose that G acts on A faithfully (to show that the kernel of the action of G is the set consisting of only the identity). Consider the permutation representation $\varphi : G \to S_A$. Since G acts on A faithfully, φ is injective (that is, $g_1, g_2 \in G$ induce different permutations $\varphi(g_1), \varphi(g_2)$). Thus the identity permutation $\varphi(1)$ is the only permutation that assigns a to a for all $a \in A$. From 5., the kernel of the action of G is the same as the kernel of φ, so the identity of G is the only element in the kernel of the action of G.

Next, suppose that the kernel of the action of $G = \{1\}$ (to show that G acts on A faithfully). Suppose for some $g_1, g_2 \in G$, we have $\varphi(g_1) = \varphi(g_2)$, that is, $\sigma_{g_1}(a) = \sigma_{g_2}(a)$ for all $a \in A$. Consider the permutation obtained by composing $\varphi(g_1)^{-1} \circ \varphi(g_2)$. Applying the resulting permutation to some $a \in A$ (and saying that $\sigma_{g_1}(a) = \sigma_{g_2}(a) = b$), we obtain $(\varphi(g_1)^{-1} \circ \varphi(g_2))(a) = \sigma_{g_1}^{-1}(\sigma_{g_2}(a)) = \sigma_{g_1}^{-1}(b) = a$. This implies that $\varphi(g_1)^{-1} \circ \varphi(g_2)$ is the identity permutation. Since φ is a homomorphism, $\varphi(g_1)^{-1} \circ \varphi(g_2) = \varphi(g_1^{-1}) \circ \varphi(g_2) = \varphi(g_1^{-1} g_2)$. However, because the kernel of the action of G is $\{1\}$, and from 5., the kernel of φ is also $\{1\}$, this implies that $g_1^{-1} g_2 = 1 \Rightarrow g_1 = g_2$. $\square$

7. (4/29/23)

Prove that the action of the multiplicative group $\mathbb{R}^\times$ on $\mathbb{R}^n$ defined by $\alpha \cdot (r_1, r_2, ..., r_n) = (\alpha r_1, \alpha r_2, ..., \alpha r_n)$ is faithful.

Proof. From 6., a group acts faithfully on a set if and only if the kernel of the action consists only of the group's identity. Therefore, to show that the given action of $\mathbb{R}^\times$ on $\mathbb{R}^n$ is faithful, it suffices to show that the kernel of the action is $\{1\}$.

By definition, the kernel of the action is the set of all $\alpha \in \mathbb{R}$ such that $\alpha \cdot (r_1, r_2, ..., r_n) = (r_1, r_2, ..., r_n)$ for all such elements of $\mathbb{R}^n$. By definition of the group action, then, for an element α of $\mathbb{R}^\times$ to be in the kernel of the action, we must have $\alpha r_1 = r_1, \alpha r_2 = r_2, ..., \alpha r_n = r_n$. The only element for which this holds is 1. Thus the kernel of the action is $\{1\}$, and so $\mathbb{R}^\times$ acts faithfully on $\mathbb{R}^n$. $\square$

8. (4/30/23)

Let A be a nonempty set and let k be a positive integer with $k \leq |A|$. The symmetric group S_A acts on B consisting of all subsets of A of cardinality k by $\sigma \cdot \{a_1, ..., a_k\} = \{\sigma(a_1), ..., \sigma(a_k)\}$.

(a) Prove that this is a group action.

Proof. The identity permutation acts on an arbitrary element of B by $(1) \cdot \{a_1, ..., a_k\} = \{a_1, ..., a_k\}$, as desired.

Further, $\sigma_1 \cdot (\sigma_2 \cdot \{a_1, ..., a_k\}) = \sigma_1 \cdot \{\sigma_2(a_1), ..., \sigma_2(a_k)\} = \{\sigma_1(\sigma_2(a_1)), ..., \sigma_1(\sigma_2(a_k))\} = \{(\sigma_1 \circ \sigma_2)(a_1), ..., (\sigma_1 \circ \sigma_2)(a_k)\} = (\sigma_1 \circ \sigma_2) \cdot \{a_1, ..., a_k\}$.

Together these two equations prove that this action of S_A on B is a group action. $\square$

(b) Describe exactly how the permutations $(1,2)$ and $(1,2,3)$ act on the six 2-element subsets of $\{1,2,3,4\}$.

- $(1,2)\cdot\{1,2\}=\{2,1\}=\{1,2\}$
- $(1,2)\cdot\{1,3\}=\{2,3\}$
- $(1,2)\cdot\{1,4\}=\{2,4\}$
- $(1,2)\cdot\{2,3\}=\{1,3\}$
- $(1,2)\cdot\{2,4\}=\{1,4\}$
- $(1,2)\cdot\{3,4\}=\{3,4\}$
- $(1,2,3)\cdot\{1,2\}=\{2,3\}$
- $(1,2,3)\cdot\{1,3\}=\{2,1\}=\{1,2\}$
- $(1,2,3)\cdot\{1,4\}=\{2,4\}$
- $(1,2,3)\cdot\{2,3\}=\{3,1\}=\{1,3\}$
- $(1,2,3)\cdot\{2,4\}=\{3,4\}$
- $(1,2,3)\cdot\{3,4\}=\{1,4\}$

9. (4/30/23)

Do both parts of the preceding exercise with "ordered k-tuples" in place of "k-element subsets," where the action on k-tuples is defined as above but with set braces replaced by parentheses (note that, for example, the 2-tuples $(1,2)$ and $(2,1)$ are different even though the sets $\{1,2\}$ and $\{2,1\}$ are the same).

(a) The proof is identical to that in 8., but with set braces replaced by parentheses. For the identity permutation, $(1)\cdot(a_1,...,a_k)=(a_1,...,a_k)$. Similarly for arbitrary σ_1,σ_2 and $(a_1,...,a_k)$, the logic holds.

(b) Describe exactly how the permutations $(1,2)$ and $(1,2,3)$ act on the twelve 2-element tuples of $(1,2,3,4)$.

- $(1,2)\cdot(1,2)=(2,1);(1,2)\cdot(2,1)=(1,2)$
- $(1,2)\cdot(1,3)=(2,3);(1,2)\cdot(3,1)=(3,2)$
- $(1,2)\cdot(1,4)=(2,4);(1,2)\cdot(4,1)=(4,2)$
- $(1,2)\cdot(2,3)=(1,3);(1,2)\cdot(3,2)=(3,1)$
- $(1,2)\cdot(2,4)=(1,4);(1,2)\cdot(4,2)=(4,1)$
- $(1,2)\cdot(3,4)=(3,4);(1,2)\cdot(4,3)=(4,3)$
- $(1,2,3)\cdot(1,2)=(2,3);(1,2,3)\cdot(2,1)=(3,2)$
- $(1,2,3)\cdot(1,3)=(2,1);(1,2,3)\cdot(3,1)=(1,2)$

- $(1,2,3) \cdot (1,4) = (2,4); (1,2,3) \cdot (4,1) = (4,2)$
- $(1,2,3) \cdot (2,3) = (3,1); (1,2,3) \cdot (3,2) = (1,3)$
- $(1,2,3) \cdot (2,4) = (3,4); (1,2,3) \cdot (4,2) = (4,3)$
- $(1,2,3) \cdot (3,4) = (1,4); (1,2,3) \cdot (4,3) = (4,1)$