Dummit & Foote Ch. 4.2: Groups Acting on Themselves by Left Multiplication — Cayley's Theorem

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Let G be a group and let H be a subgroup of G.

1. (2/12/24)

Let $G = \{1, a, b, c\}$ be the Klein 4-group whose group table is written out in Section 2.5.

(a) Label $1, a, b, c$ with the integers $1, 2, 4, 3$, respectively, and prove that under the left regular representation of G into S_4 the nonidentity elements are mapped as follows:

$$a \mapsto (1\,2)(3\,4) \qquad b \mapsto (1\,4)(2\,3) \qquad c \mapsto (1\,3)(2\,4).$$

Proof. The left regular representation of G into S_4 is the homomorphism $\varphi : G \to S_4$ defined by $\varphi(g) = \sigma_g$, where $\sigma_g : G \to G$ is the permutation of G defined by $\sigma_g(x) = gx$ for all $x \in G$.

Each non-identity element maps the elements as follows:

$$\begin{array}{cccc} \sigma_a(1) = a1 = a & \sigma_a(a) = a^2 = 1 & \sigma_a(b) = ab = c & \sigma_a(c) = ac = b \\ \sigma_b(1) = b1 = b & \sigma_b(a) = ba = c & \sigma_b(b) = b^2 = 1 & \sigma_b(c) = bc = a \\ \sigma_c(1) = c1 = c & \sigma_c(a) = ca = b & \sigma_c(b) = cb = a & \sigma_c(c) = c^2 = 1. \end{array}$$

By the given labeling, this assigns the elements a, b, and c to the pairs of 2-cycles shown above. □

(b) Relabel $1, a, b, c$ as $1, 4, 2, 3$, respectively, and compute the image of each element of G under the left regular representation of G into S_4. Show that the image of G in S_4 under this labeling is the same *subgroup* as the image of G in part (a) (even though the nonidentity elements individually map to different permutations under the two different labelings).

Proof. Under this labeling, the elements a, b, and c are mapped to the permutations $(1\,4)(2\,3), (1\,2)(3\,4)$, and $(1\,3)(2\,4)$, respectively. Although each element maps to a different permutation from part (a), the subgroup of S_4 is the same in both cases. $\square$

2. (2/12/24)

List the elements of S_3 as $1, (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)$ and label these with the integers $1, 2, 3, 4, 5, 6$, respectively. Exhibit the image of each element of S_3 under the left regular representation of S_3 into S_6.

Solution. First, consider the element $(1\,2)$. We see that:

$$\begin{aligned} (1\,2)1 &= (1\,2) \mapsto 2 & (1\,2)(1\,2) &= 1 \mapsto 1 \\ (1\,2)(2\,3) &= (1\,2\,3) \mapsto 5 & (1\,2)(1\,3) &= (1\,3\,2) \mapsto 6 \\ (1\,2)(1\,2\,3) &= (2\,3) \mapsto 3 & (1\,2)(1\,3\,2) &= (1\,3) \mapsto 4. \end{aligned}$$

So the left regular representation of $(1\,2)$ under the given labeling in S_6 is $(1\,2)(3\,4)(5\,6)$.

The left regular representations of the remaining elements are:

$$\begin{aligned} (2\,3) &\mapsto (1\,3)(2\,6)(4\,5) \\ (1\,3) &\mapsto (1\,4)(2\,5)(3\,6) \\ (1\,2\,3) &\mapsto (1\,5\,6)(2\,4\,3) \\ (1\,3\,2) &\mapsto (1\,6\,5)(2\,3\,4). \end{aligned}$$

$\square$

3. (2/12/24)

Let r and s be the usual generators for the dihedral group of order 8.

(a) List the elements of D_8 as $1, r, r^2, r^3, s, sr, sr^2, sr^3$ and label these with the integers $1, 2, ..., 8$, respectively. Exhibit the image of each element of D_8 under the left regular representation of D_8 into S_8.

$$\begin{aligned} 1 &\mapsto 1 \\ r &\mapsto (1\,2\,3\,4)(5\,8\,7\,6) \\ r^2 &\mapsto (1\,3)(2\,4)(5\,7)(6\,8) \\ r^3 &\mapsto (1\,4\,3\,2)(5\,6\,7\,8) \\ s &\mapsto (1\,5)(2\,6)(3\,7)(4\,8) \\ sr &\mapsto (1\,6)(2\,7)(3\,8)(4\,5) \\ sr^2 &\mapsto (1\,7)(2\,8)(3\,5)(4\,6) \\ sr^3 &\mapsto (1\,8)(2\,5)(3\,6)(4\,7) \end{aligned}$$

(b) Relabel this same list of elements of D_8 with the integers $1, 3, 5, 7, 2, 4, 6, 8$ respectively and recompute the image of each element of D_8 under the left regular representation with respect to this new labeling. Show that the two subgroups of S_8 obtained in parts (a) and (b) are different.

$$\begin{aligned}
1 &\mapsto 1 \\
r &\mapsto (1\,3\,5\,7)(2\,8\,6\,4) \\
r^2 &\mapsto (1\,5)(2\,6)(3\,7)(4\,8) \\
r^3 &\mapsto (1\,7\,5\,3)(2\,4\,6\,8) \\
s &\mapsto (1\,2)(3\,4)(5\,6)(7\,8) \\
sr &\mapsto (1\,4)(2\,7)(3\,6)(5\,8) \\
sr^2 &\mapsto (1\,6)(2\,5)(3\,8)(4\,7) \\
sr^3 &\mapsto (1\,8)(2\,3)(4\,5)(6\,7).
\end{aligned}$$

We see that the generators of the subgroups of S_8 in parts (a) and (b) are different, and so these are different subgroups of S_8.

4. (2/12/24)

Use the left regular representation of Q_8 to produce two elements of S_8 which generate a subgroup of S_8 isomorphic to the quaternion group Q_8.

Proof. We know that the elements i and j generate the quaternion group Q_8. Labeling the elements $1, -1, i, -i, j, -j, k, -k$ with $1, 2, ..., 8$ respectively, the elements i and j map to the following permutations in S_8:

$$\begin{aligned}
i &\mapsto (1\,3\,2\,4)(5\,7\,6\,8) \\
j &\mapsto (1\,5\,2\,6)(3\,8\,4\,7).
\end{aligned}$$

Since the left regular representation of Q_8 in S_8 is a homomorphism, these two permutations generate a subgroup of S_8 isomorphic to Q_8. $\square$

5. (2/12/24)

Let r and s be the usual generators for the dihedral group of order 8 and let $H = \langle s \rangle$. List the left cosets of H in D_8 as $1H, rH, r^2H, r^3H$.

(a) Label these cosets with the integers $1, 2, 3, 4$, respectively. Exhibit the image of each element of D_8 under the representation π_H of D_8 into S_4 obtained from the action of D_8 by left multiplication on the set of 4 left cosets of H in D_8. Deduce that this representation is faithful (i.e., the elements of S_4 obtained form a subgroup isomorphic to D_8).

$$\begin{aligned} 1 &\mapsto 1 & s &\mapsto (2\,4) \\ r &\mapsto (1\,2\,3\,4) & sr &\mapsto (1\,4)(2\,3) \\ r^2 &\mapsto (1\,3)(2\,4) & sr^2 &\mapsto (1\,3) \\ r^3 &\mapsto (1\,4\,3\,2) & sr^3 &\mapsto (1\,2)(3\,4). \end{aligned}$$

Since each element of D_8 induces a unique permutation in S_4, the resulting image under the left regular representation is isomorphic to D_8, and so this representation is faithful.

(b) Repeat part (a) with the list of cosets relabeled by the integers $1, 3, 2, 4$, respectively. Show that the permutations obtained from this labeling form a subgroup of S_4 that is different from the subgroup obtained in part (a).

$$\begin{aligned} 1 &\mapsto 1 & s &\mapsto (3\,4) \\ r &\mapsto (1\,3\,2\,4) & sr &\mapsto (1\,4)(2\,3) \\ r^2 &\mapsto (1\,2)(3\,4) & sr^2 &\mapsto (1\,2) \\ r^3 &\mapsto (1\,4\,2\,3) & sr^3 &\mapsto (1\,3)(2\,4). \end{aligned}$$

Since the generators (the images of r and s) of this subgroup of S_4 are different from those in part (a), this is a different subgroup from part (a).

(c) Let $K = \langle sr \rangle$, list the cosets of K in D_8 as $1K, rK, r^2K, r^3K$, and label these with the integers $1, 2, 3, 4$. Prove that, with respect to this labeling, the image of D_8 under the representation π_K obtained from left multiplication on the cosets of K is the same *subgroup* of S_4 as in part (a) (even though the subgroups H and K are different and some of the elements of D_8 map to different permutations under the two homomorphisms).

Proof. Consider the images of the generators r and s under π_K:

$$\begin{aligned} r \cdot 1K &= rK & s \cdot 1K &= rK \\ r \cdot rK &= r^2K & s \cdot rK &= 1K \\ r \cdot r^2K &= r^3K & s \cdot r^2K &= r^3K \\ r \cdot r^3K &= 1K & s \cdot r^3K &= r^2K. \end{aligned}$$

So r and s map to $(1\,2\,3\,4)$ and $(1\,2)(3\,4) \in S_4$, respectively. These elements are both in the subgroup in part (a) above, and so they are the same subgroup, but the image of s is different. $\square$

6. (2/15/24)

Let r and s be the usual generators for the dihedral group of order 8 and let $N = \langle r^2 \rangle$. List the left cosets of N in D_8 as $1N, rN, sN$, and srN. Label these cosets with the integers $1, 2, 3, 4$ respectively. Exhibit the image of each element of D_8 under the representation π_N of D_8 into S_4 obtained from the action of D_8 by left multiplication on the set of 4 left cosets of N in D_8. Deduce that this representation is not faithful and prove that $\pi_N(D_8)$ is isomorphic to the Klein 4-group.

Solution.

$$\begin{aligned} 1 &\mapsto 1 & s &\mapsto (1\,3)(2\,4) \\ r &\mapsto (1\,2)(3\,4) & sr &\mapsto (1\,4)(2\,3) \\ r^2 &\mapsto 1 & sr^2 &\mapsto (1\,3)(2\,4) \\ r^3 &\mapsto (1\,2)(3\,4) & sr^3 &\mapsto (1\,4)(2\,3). \end{aligned}$$

The left regular representation assigns 1 and r^2 to the identity permutation, so this action is not faithful.

The image of D_8 under π_N consists of the four permutations $1, (1\,2)(3\,4)$, $(1\,3)(2\,4)$, and $(1\,4)(2\,3)$. From Ch. 2.5, Exercise 10, this is isomorphic to the Klein 4-group V_4. □

7. (2/15/24)

Let Q_8 be the quaternion group of order 8.

(a) Prove that Q_8 is isomorphic to a subgroup of S_8.

Proof. From Exercise 4, Q_8 is isomorphic to

$$\langle (1\,3\,2\,4)(5\,7\,6\,8), (1\,5\,2\,6)(3\,8\,4\,7) \rangle \in S_8.$$

□

(b) Prove that Q_8 is not isomorphic to a subgroup of S_n for any $n \leq 7$.

Proof. Let A be a set with $|A| = n \leq 7$, let $a \in A$, and let $\cdot$ be the action of Q_8 on A. We attempt to find a subgroup of S_n that is isomorphic to Q_8 by considering the permutation representations of the elements of Q_8. Now if $i \cdot a = j \cdot a$, then the permutation representations σ_i and σ_j are equal to each other, and so Q_8 is not isomorphic to the resulting subgroup of S_n. Further (without loss of generality), if $i \cdot a = -i \cdot a$, then:

$$i \cdot a = -i \cdot a \Rightarrow -i \cdot i \cdot a = -i \cdot -i \cdot a \Rightarrow a = -1 \cdot a,$$

and so the permutation representation of -1 is equal to the identity permutation, which implies that Q_8 is not isomorphic to the subgroup of S_n. Therefore the elements $\pm i, \pm j, \pm k$ must all assign a to different elements. However, these 6 unique elements together with a are at least all of A, and so we must have $-1 \cdot a = a$. Thus Q_8 is not isomorphic to a subgroup of S_n. □

9. (2/16/24)

Prove that if p is a prime and G is a group of order p^a for some $a \in \mathbb{Z}^+$, then every subgroup of index p is normal in G. Deduce that every group of order p^2 has a normal subgroup of order p.

Proof. Let H be a subgroup of G with $[G : H] = p$. Let gH be the left coset of H by some element $g \in G$.

Suppose that, for some $n < p$, $g^n \in H$ and let $h = g^n$. Since p is a prime, there exists a positive integer k such that $kn = 1 \pmod p$. Then $h^k = g^{kn} = g$, which implies that $g \in H$. We conclude that, if we restrict to $g \notin H$, then for all $n < p$, $g^n \notin H$. This implies that $\{H, gH, g^2H, ..., g^{p-1}H\}$ is a set of p distinct cosets of H. Because the index of H in G is p, this must be all the cosets of H in G, and so $g^p \in H$.

Now by the operation defined on left cosets of H by $aH \cdot bH = (ab)H$, we see that this is isomorphic to the cyclic group Z_p. We conclude by Theorem 6(d) of Chapter 3.1 that H is normal in G.

Further, if $|G| = p^2$, then by Cauchy's Theorem it contains an element of order p which generates a subgroup of order p. This subgroup has index $p^2/p = p$, and so from above, is a normal subgroup of order p. $\square$

10. (2/20/24)

Prove that every non-abelian group of order 6 has a nonnormal subgroup of order 2. Use this to classify groups of order 6.

Proof. Let G be a group of order 6. From Cauchy's Theorem, let $g, x \in G$ such that $|g| = 2$ and $|x| = 3$. If $gx = xg$, then $G \cong Z_2 \times Z_3 \cong Z_6$, and so G is abelian; therefore we must have $gx \neq xg$.

So G contains the elements $1, g, x, x^2, gx$, and xg. Consider the element gx^2. By the cancellation laws we can see that it is not equal to $1, g, x, x^2$, or gx. Therefore it must be equal to xg, so we can write $xg = gx^2$. We now see that $G \cong D_6 = \langle s, r \mid s^2 = r^3 = 1, rs = sr^2\rangle$. We note that $\langle s\rangle$ is a nonnormal subgroup of D_6, because $rsr^{-1} = rsr^2 = sr^2r^2 = sr \notin \langle s\rangle$.

Finally, we conclude that every group of order 6 is isomorphic to either the cyclic group or the dihedral group of order 6. $\square$

11. (2/20/24)

Let G be a finite group and let $\pi : G \to S_G$ be the left regular representation. Prove that if x is an element of G of order n and $|G| = mn$, then $\pi(x)$ is a product of m n-cycles.

Deduce that $\pi(x)$ is an odd permutation if and only if $|x|$ is even and $\dfrac{|G|}{|x|}$ is odd.

Proof. Let $x \in G$, $|x| = n$, $|G| = mn$, and consider the *right* cosets of the cyclic subgroup $\langle x \rangle$. There are m such cosets of $\langle x \rangle$; let $1, y_2, ..., y_m$ be representatives of the right cosets, so that $\{\langle x \rangle 1, \langle x \rangle y_2, ..., \langle x \rangle y_m\}$ forms a partition of G.

Now consider the cycle decomposition of $\pi(x)$. Certainly it contains at least one n-cycle, namely the cycle that contains the elements of the cyclic subgroup $\langle x \rangle$: $(1\, x\, x^2 \ldots x^{n-1})$. Every successive representative of the above right cosets of $\langle x \rangle$ also induces a (disjoint) n-cycle with the same order that we would list the elements of the coset, for example, $(y_2\, xy_2\, x^2 y_2 \ldots x^{n-1} y_2)$. Since there are m representatives each with a unique n-cycle, we conclude that the cycle decomposition of $\pi(x)$ consists of m n-cycles.

From Chapter 3.5, Proposition 25, a permutation is odd if and only if the number of cycles of even length in its cycle decomposition is odd. Therefore we conclude that $\pi(x)$ is odd if and only if $m = \dfrac{|G|}{|x|}$ (the number of cycles) is odd, and $n = |x|$ (the cycle length) is even. $\square$

12. (2/20/24)

Let G and π be as in the preceding exercise. Prove that if $\pi(G)$ contains an odd permutation then G has a subgroup of index 2.

Proof. Consider the homomorphism $\epsilon : \pi(G) \to \{\pm 1\}$ described in Chapter 3.5 which assigns a permutation to 1 if it is an even permutation and -1 if it is an odd permutation. Since $\pi(G)$ contains an odd permutation, ϵ is surjective, and so $\ker \pi(G) \neq \pi(G)$. Since ϵ is a homomorphism, there must be the same number of elements assigned to both 1 and -1. Therefore $|\pi(G) : \ker \pi(G)| = 2$, and so G contains a subgroup of index 2. $\square$

13. (2/20/24)

Prove that if $|G| = 2k$ where k is odd then G has a subgroup of index 2.

Proof. From Cauchy's Theorem, let $x \in G$ such that $|x| = 2$. From Exercise 11, since $|x| = 2$, even, and $\dfrac{|G|}{|x|} = \dfrac{2k}{2} = k$, odd, $\pi(x)$ is an odd permutation. From Exercise 12, since $\pi(G)$ contains an odd permutation, G contains a subgroup of index 2. $\square$

14. (2/20/24)

Let G be a finite group of composite order n with the property that G has a subgroup of order k for each positive integer k dividing n. Prove that G is not simple.

Proof. By definition, a group G is simple if it contains no proper normal subgroups other than 1 and G itself. Therefore, it suffices to show that G contains at least one proper normal subgroup.

Let p be the smallest prime dividing n and let $n = pk$. Then G contains a subgroup of order k that has index $n/k = p$. By Corollary 5, this subgroup is normal and, since it is a proper subgroup, G is therefore not simple. $\square$