Dummit & Foote Ch. 7.1: Introduction to Rings

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Let R be a ring with 1.

1. (7/1/24)

Show that $(-1)^2 = 1$ in $\mathbb{R}$.

Proof. We have:

$$(-1) + (-1)^2 = \underbrace{(-1)(1)}_{\text{identity}} + (-1)(-1) = \underbrace{(-1)(1 + (-1))}_{\text{distribution}} = (-1)\underbrace{(0)}_{\text{inverses}} = 0,$$

and therefore, since $(-1) + (-1)^2 = 0$, $(-1)^2 = 1$. $\square$

2. (7/1/24)

Prove that if u is a unit in R then so is $-u$.

Proof. Recall that u is a unit in R if there exists some $v \in R$ such that $uv = vu = 1$.

Now:

$$\begin{aligned}(-u)(v) &= -(uv) = -1, \text{ which implies that} \\ (-u)(v)(-1) &= (-1)^2 = 1, \text{ so} \\ (-u)(-v) &= 1,\end{aligned}$$

which implies that $-u$ is also a unit in R. $\square$

7. (7/5/24)

The *center* of a ring R is $\{z \in R \mid zr = rz \text{ for all } r \in R\}$ (i.e., is the set of all elements which commute with every element of R). Prove that the center of a ring is a subring that contains the identity. Prove that the center of a division ring is a field.

Proof. Let $a, b \in R$ be in the center of R and let $x \in R$. Then:

$$(a-b)x = ax - bx = xa - xb = x(a-b),$$

so $a-b$ is in the center of R. And, since a and b both commute with x, we have $(ab)x = abx = xab = x(ab)$, so ab lies in the center of R as well. Since by definition 1 commutes with every element of R, the center of R is a subring of R containing the identity.

If R is a division ring, then every element in its center (except 0) has a multiplicative inverse (is a unit). Every element in its center also commutes with every other element. A field is a commutative ring where every nonzero element is a unit; therefore the center of a division ring is a field. $\square$

8. (7/9/24)

Describe the center of the Hamilton Quaternions $\mathbb{H}$. Prove that $\{a+bi \mid a, b \in \mathbb{R}\}$ is a subring of $\mathbb{H}$ which is a field but is not contained in the center of $\mathbb{H}$.

Proof. Let $a+bi+cj+dk$ $(a, b, c, d \in \mathbb{R})$ lie in the center of $\mathbb{H}$. It must commute with i $(= 0 + 1i + 0j + 0k)$. Then:

$$\begin{aligned}
(a+bi+cj+dk)i &= ai + bi^2 + cji + dki \\
&= -b + ai + dj - ck, \text{ and} \\
i(a+bi+cj+dk) &= ai + bi^2 + cij + dik \\
&= -b + ai - dj + ck.
\end{aligned}$$

If these are equal, then we have:

$$\begin{aligned}
-b + ai + dj - ck &= -b + ai - dj + ck \\
dj - ck &= -dj + ck \\
2dj &= 2ck \\
dj &= ck,
\end{aligned}$$

and since $c, d \in \mathbb{R}$, there are no non-zero values of c, d such that $dj = ck$. Thus we must have $c = d = 0$.

Repeating the above steps for the product of $a + bi + cj + dk$ and j or k, we see that b must also be 0.

Now because real coefficients of i, j, k commute, a may take any value, and so the center of $\mathbb{H}$ consists of the real numbers (that is, quaternions of the form $a + 0i + 0j + 0k$).

Consider the subset $\{a + bi \mid a, b \in \mathbb{R}\}$. Let $a + bi, c + di$ be two elements of this subset. We see that $(a+bi)-(c+di) = (a-c)+(b-d)i$ and $(a+bi)(c+di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$. Since this subset is closed under subtraction and multiplication, it is a subring of $\mathbb{H}$. However, since it includes elements with non-zero i components, it is not contained in the center of $\mathbb{H}$. $\square$