Dummit & Foote Ch. 3.2: More on Cosets and Lagrange's Theorem

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Oct. 2023

Let G be a group.

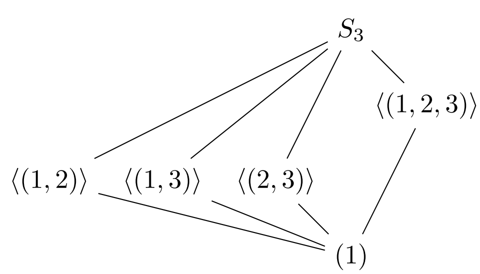
1. (10/1/23)

Which of the following are permissible orders of subgroups of a group of order 120: 1, 2, 5, 7, 9, 15, 60, 240? For each permissible order give the corresponding index.

Proof. From Lagrange's theorem, the order of a subgroup of a group of order 120 must divide 120. Then the permissible orders for subgroups are $1 = \frac{120}{120}$, $2 = \frac{120}{60}, 5 = \frac{120}{24}, 15 = \frac{120}{8}$, and $60 = \frac{120}{2}$. For each of these orders the index is given by the corresponding denumerator. $\square$

2. (10/2/23)

Prove that the lattice of subgroups of S_3 below is correct (i.e., prove that it contains all subgroups of S_3 and that their pairwise joins and intersections are correctly drawn).



Proof. The symmetric group S_3 contains 6 elements. By Lagrange's theorem, its proper subgroups must have order 2 or 3. Each of the subgroups in the lattice above have order 2 or 3, so there are no smaller or larger subgroups not depicted above.

From Corollary 10, a subgroup of order 2 must be isomorphic to Z_2, that is, cyclic and generated by a single element of order 2. The three subgroups generated by the three elements of order 2 (the 2-cycles of S_3) are depicted above. Similarly, a subgroup of order 3 must be isomorphic to Z_3 and generated by a single element of order 3. The subgroup generated by $(1, 2, 3)$ contains $(1, 3, 2)$, so there is only a single subgroup of order 3.

Next, again by Lagrange's Theorem, a subgroup of two different containing groups must have an order that divides the order of both of the containing groups. First consider a subgroup of order 2 and a subgroup of order 3. Only 1 divides 2 and 3, so the intersection must be the identity. Similarly, if a subgroup of order 2 and a subgroup of order 3 are contained in a larger group, then that group's order must have both 2 and 3 as divisors. The smallest integer for which this is possible is 6, which is the order of all of S_3.

Finally, consider a pair of subgroups of order 2. Their intersection is either the identity or else they are the same subgroup. Their join must have even order, but 4 does not divide 6 and any larger even number exceeds the order of S_3. Thus their join is all of S_3. This concludes the proof that the lattice of subgroups of S_3 is correct. $\square$

3. (10/2/23)

Prove that the lattice of subgroups of Q_8 below is correct.

$$
\begin{tikzcd}[arrows=dash]
 & Q_8 \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle i \rangle \arrow[dr] & \langle j \rangle \arrow[d] & \langle k \rangle \arrow[dl] \\
 & \langle -1 \rangle \arrow[d] & \\
 & 1 &
\end{tikzcd}
$$

Proof. The group Q_8 has order $8 = 2^3$, so by Lagrange's theorem its proper subgroups must have order 2 or 4. We will start from the bottom and work toward the top: There is only one element of order 2 in Q_8, -1, and the cyclic subgroup generated by it is in the lattice.

For each of i, j, and k, $\langle -1 \rangle$ is contained in the subgroup generated by them (ex. $\langle i \rangle = \{\pm 1, \pm i\}$) and there are no intermediate subgroups, since there is no divisor of 4 that is strictly greater than 2. At this point, every element of Q_8 is represented, so there are no cyclic subgroups missing. We might ask if there is a subgroup of order 4 missing. If so, it cannot be cyclic, and from Ch. 1.1, Exercise 36, it must be isomorphic to V_4. However, V_4 contains three elements of order 2, and Q_8 only has one, so there is no subgroup of Q_8 isomorphic to V_4.

Finally, the join of any of the subgroups generated by i, j, or k must contain strictly more than 4 elements and its order must divide 8. Then any of their joins must have order 8, that is, be all of Q_8. □

4. (10/3/23)

Show that if $|G| = pq$ for some primes p and q (not necessarily distinct) then either G is abelian or $Z(G) = 1$.

Proof. We will show, equivalently, that if $|Z(G)| > 1$, then G is abelian.

Let $x \in Z(G)$. From Corollary 9, the order of x divides $|G| = pq$. If $|x| = pq$, then $G = \langle x \rangle$ and so is abelian. Suppose without loss of generality that $|x| = p$. Now since the center of a group is a subgroup, we must have $\langle x \rangle \leq Z(G)$. If there exists a $y \in Z(G), y \notin \langle x \rangle$, then the order of $Z(G)$ exceeds p and must divide pq, then it must be all of G and hence G is abelian. So suppose $Z(G) = \langle x \rangle$.

The center of a group is normal in that group, so $G/Z(G)$ is well-defined. Since $|Z(G)| = p$, it has q cosets in G; that is, the quotient group $G/Z(G)$ has prime order q and is thus isomorphic to Z_q, hence cyclic. From Ch. 3.1, Exercise 36., G is thus abelian. □

5. (10/4/23)

Let H be a subgroup of G and fix some element $g \in G$.

(a) Prove that gHg^{-1} is a subgroup of G of the same order as H.

Proof. By definition elements of gHg^{-1} can be written in the form ghg^{-1} for some $h \in H$, so let $gh_1g^{-1}, gh_2g^{-1} \in gHg^{-1}$. Then we have:

$$(gh_1g^{-1})(gh_2g^{-1})^{-1} = gh_1g^{-1}gh_2^{-1}g_1 = gh_1h_2^{-1}g^{-1} \in gHg^{-1},$$

so gHg^{-1} fulfills the subgroup criterion and is thus a subgroup of G.

Next, let $\varphi_g : H \to gHg^{-1}$ be defined by $\varphi_g(h) = ghg^{-1}$ for all $h \in H$. This map is injective by the cancellation laws: $gh_1g^{-1} = gh_2g^{-1}$ implies that $h_1 = h_2$. It is also surjective: Let $x \in gHg^{-1}$. By definition $x = ghg^{-1}$ for some $h \in H$, so $\varphi_g(h) = x$. Therefore φ_g is a bijection, and so H and gHg^{-1} have the same order. □

(b) Deduce that if $n \in \mathbb{Z}^+$ and H is the unique subgroup of G of order n then $H \trianglelefteq G$.

Suppose that H is the unique subgroup of order n in G. Then for all $g \in G$, we must have $gHg^{-1} = H$ (it cannot be any other subgroup, because $|gHg^{-1}| = |H| = n$ and there is no other subgroup of order n in G). It follows that H is normal in G.

6. (10/4/23)

Let $H \leq G$ and let $g \in G$. Prove that if the right coset of Hg equals *some* left coset of H in G then it equals the left coset gH and g must be in $N_G(H)$.

Proof. Suppose $Hg = xH$ for some $x \in G$. Now $g \in Hg$, so we must also have $g \in xH$. Then $g = xh$ for some $h \in H$. It follows that $x = gh^{-1}$. So $Hg = xH = (gh^{-1})H = gH$, which in turns implies that $gHg^{-1} = H$. Therefore $g \in N_G(H)$. $\square$

7. (10/5/23)

Let $H \leq G$ and define a relation $\sim$ on G by $a \sim b$ if and only if $b^{-1}a \in H$. Prove that $\sim$ is an equivalence relation and describe the equivalence class of each $a \in G$. Use this to prove Proposition 4.

Proof. Let $a, b, c \in G$. We have $a \sim a$, because $a^{-1}a = 1 \in H$. If $a \sim b$, then we have $b^{-1}a \in H$. Now $b \sim a = a^{-1}b = (b^{-1}a)^{-1} \in H$, since H is closed under inverses, so $a \sim b$ implies that $b \sim a$ (and the logic holds in reverse). Finally, if $a \sim b$ and $b \sim c$, then $b^{-1}a, c^{-1}b \in H$. Then their product, $c^{-1}bb^{-1}a = c^{-1}a$, is an element of H, which implies $a \sim c$. The relation $\sim$ is reflexive, symmetric, and transitive, therefore it is an equivalence relation.

Let $a \in G$ and let b lie in the left coset aH, so $b = ah$ for some $h \in H$. Then $b^{-1}a = (ah)^{-1}a = h^{-1}a^{-1}a = h^{-1} \in H$, so $a \sim b$. This implies that aH is a subset of the equivalence class of a. And, if we have $a \sim b$, then $b^{-1}a \in H$, so $b^{-1}a = h$ for some $h \in H$. It follows that $b = ah^{-1} \in aH$, so the equivalence class of a is a subset of aH. Since each is contained in the other, the equivalence class of a under $\sim$ is the left coset aH.

Now Proposition 4 states that:

- The set of left cosets of H in G form a partition of G.
- For all $a, b \in G, aH = bH$ if and only if $b^{-1}a \in H$.
- In particular, $aH = bH$ if and only if a and b are representatives of the same coset.

Since the equivalence class of a under $\sim$ is exactly the left coset aH and equivalence classes partition a set, the left cosets of H in G partition G. The proof for the remaining items follows directly from the proof above that $a \sim b \iff b^{-1}a \in H \iff b \in aH$. $\square$

8. (10/6/23)

Prove that if H and K are finite subgroups of G whose orders are relatively prime then $H \cap K = 1$.

Proof. Let $H, K \leq G$ be finite subgroups whose orders are relatively prime. Let $x \in H \cap K$, so $x \in H$ and $x \in K$. From Corollary 9, the order of x divides the orders of both H and K. Since $|H|$ and $|K|$ are relatively prime, the order of x must be 1, therefore $x = 1$. It follows that $H \cap K = 1$. $\square$

9. (10/12/23)

This exercise outlines a proof of Cauchy's Theorem due to James McKay (*Another proof of Cauchy's group theorem*, Amer. Math. Monthly, 66(1959), p. 119). Let G be a finite group and let p be a prime dividing $|G|$. Let $\mathcal{S}$ denote the set of p-tuples of elements of G the product of whose coordinates is 1:

$$\mathcal{S} = \{(x_1, x_2, ..., x_p) \mid x_1 x_2 ... x_p = 1\}.$$

(a) Show that $\mathcal{S}$ has $|G|^{p-1}$ elements, hence has order divisible by p.

Proof. Construct an element of $\mathcal{S}$ coordinate by coordinate. There are $|G|$ choices for the first element x_1. There are again $|G|$ choices for the second element x_2. We proceed similarly until the final element, which must satisfy the constraint that the product of all coordinates is 1. Therefore the final element must be equal to $(x_1 x_2 ... x_{p-1})^{-1}$. We have freely chosen $p-1$ coordinates from among $|G|$ possibilities; therefore $|\mathcal{S}| = |G|^{p-1}$. $\square$

Define the relation $\sim$ on $\mathcal{S}$ by letting $\alpha \sim \beta$ if β is a cyclic permutation of α.

(b) Show that a cyclic permutation of $\mathcal{S}$ is again an element of $\mathcal{S}$.

Proof. Since $\alpha \sim \beta$ implies that β is a cyclic permutation of α, we have

$$\alpha = (x_1, x_2, ..., x_p) \Rightarrow \beta = (x_{1+n}, x_{2+n}, ..., x_{p+n}),$$

where the subscripts of elements of β are taken mod p (although wrapping from 1 to p, rather than 0 to $p-1$).

The product of the coordinates of α is:

$$\begin{aligned} 1 = \prod \alpha &= x_1 x_2 ... x_p \\ &= (x_1 ... x_n)(x_{n+1} ... x_p) \\ &= (x_{n+1} ... x_p)(x_1 ... x_n) \text{ (if } ab = 1\text{, then } ab = ba) \\ &= (x_{1+n} ... x_{p-n+n})(x_{(p-n+1)+n} ... x_{p+n}) \\ &= x_{1+n} ... x_{p+n} = \prod \beta, \end{aligned}$$

and so the product of β's coordinates is 1, making it an element of $\mathcal{S}$. $\square$

(c) Prove that $\sim$ is an equivalence relation on $\mathcal{S}$.

Proof. Let $\alpha, \beta, \gamma \in \mathcal{S}$. The relation $\sim$ is:

- Reflexive: Let $\alpha = (x_1, x_2, ..., x_p)$. Then $x_i = x_{i+0}$ for all coordinates x_i, so α is a cyclic permutation of itself, and therefore $\alpha \sim \alpha$.
- Symmetric: Let $\alpha \sim \beta$, α, β indexed by x, y respectively. Since β is a cyclic permutation of α, we have $y_i = x_{i+n}$ for all $i \in \{1, ..., p\}$ for some $n \in \mathbb{Z}$. It follows that $x_i = y_{i+(p-n)}$ (subscripts mod p wrapping from 1 to p), so α is also a cyclic permutation of β, and therefore $\beta \sim \alpha$.
- Transitive: Let $\alpha \sim \beta$ and $\beta \sim \gamma$, with α, β as above and γ indexed by z. We have $y_i = x_{i+n}$ and $z_i = y_{i+k}$ for some $k, n \in \mathbb{Z}$. It follows that $z_i = x_{i+k+n}$, which implies that γ is a cyclic permutation of α, so $\alpha \sim \gamma$.

Therefore $\sim$ is an equivalence relation on $\mathcal{S}$. $\square$

(d) Prove that an equivalence class contains a single element if and only if it is of the form $(x, x, ..., x)$ with $x^p = 1$.

Proof. First, let $\alpha = (x, ..., x)$ and let $\alpha \sim \beta$. Then β is a cyclic permutation of α. Since α consists of a single, repeated coordinate value, we must have $\beta = (x, ..., x) = \alpha$. Therefore the equivalence class of α consists only of itself.

Next, let $\alpha \in \mathcal{S}$ and suppose that the equivalence class of α under $\sim$ consists only of α. Suppose $\alpha = (x_1, x_2, ..., x_p)$. Let β be a cyclic permutation of α shifted by 1: $\beta = (x_2, x_3 ..., x_p, x_1)$. Now β is in the equivalence class of α, but we must have $\beta = \alpha$, so $x_{i+1} = x_i$ for all x_i. It follows that $x_2 = x_1, x_3 = x_2 = x_1$, and so every value is equal to x_1. Then we have $\alpha = (x_1, ..., x_1)$, which is of the form $(x, ..., x)$, and by definition we must have $x^p = 1$. $\square$

(e) Prove that every equivalence class has order 1 or p (this uses the fact that p is a *prime*). Deduce that $|G|^{p-1} = k + pd$, where k is the number of classes of size 1 and d is the number of classes of size p.

Proof. From (d), if $\alpha = (x, ..., x)$ for some $x \in G$, its equivalence class has order 1.

Let $\alpha = (x_1, x_2, ..., x_p)$. Then there are exactly p members in the equivalence class of α, and they are the cyclic permutations of α shifted by $0, 1, 2, ..., p-1$, respectively. For example, the n-th member of the equivalence class is $(x_{1+n}, x_{2+n}, ..., x_{p+n})$.

The equivalence classes of the elements of $\mathcal{S}$ partition $\mathcal{S}$. Suppose there are k equivalence classes of order 1, and d equivalence classes of order p. From (a), the order of $\mathcal{S}$ is $|G|^{p-1}$. Then we have $|G|^{p-1} = k + pd$. $\square$

(f) Since $\{(1, 1, ..., 1)\}$ is an equivalence class of size 1, conclude from (e) that there must be a nonidentity element x in G with $x^p = 1$, i.e., G contains an element of order p.

Proof. From (e), we have $|G|^{p-1} = k + pd$ for some $k, d \geq 0$. From (a), p divides the order of $\mathcal{S} = |G|^{p-1}$, so we can write $ps = k+pd$ for some $s > 0$. Then $k = ps - pd = p(s - d)$, and so p divides k. Because p is prime, this implies that $k > 1$, so there are at least two elements whose equivalence classes have size 1. We already know that one is the identity; therefore there must be some element $\alpha \in \mathcal{S}, \alpha \neq (1, ..., 1)$ whose equivalence class under $\sim$ has size 1. From (d), $\alpha = (x, ..., x)$ for some $x \in G$, and we thus have $x^p = 1$, which implies that $|x| = p$. $\square$

10. (11/2/23)

Suppose H and K are subgroups of finite index in the (possibly infinite) group G with $|G : H| = m$ and $|G : K| = n$. Prove that l.c.m.$(m, n) \leq |G : H \cap K| \leq mn$. Deduce that if m and n are relatively prime then $|G : H \cap K| = |G : H| \cdot |G : K|$.

Proof. Let $g \in G$. Now $H \cap K$ is a subgroup of G, so the left cosets of it partition G. Consider the left coset:

$$g(H \cap K) = \{gx \mid x \in H \cap K\} = \{gx \mid x \in H\} \cap \{gx \mid x \in K\} = gH \cap gK.$$

Since $|G : H| = m$, there are m unique left cosets of H in G, and similarly there are n unique left cosets of K in G. Then there are most mn unique intersections of a left coset of H with a left coset of K. It follows that there are at most mn left cosets of $H \cap K$ in G, and so $|G : H \cap K| \leq mn$.

Since we now know that $H \cap K$ has finite index in G, it must also have finite index in H and K, respectively. Let $|H : H \cap K| = r$. Then there are r unique cosets of $H \cap K$ in H and m cosets of H in G. We have:

$$\begin{aligned}
H &= \bigcup_{i=1}^{r} h_i(H \cap K) \text{ for some } h_1, ..., h_r \in H, \text{ and} \\
G &= \bigcup_{j=1}^{m} g_j H \text{ for some } g_1, ..., g_m \in G, \text{ therefore} \\
G &= \bigcup_{j=1}^{m} g_j \Big(\bigcup_{i=1}^{r} h_i(H \cap K) \Big),
\end{aligned}$$

a partition of G into mr unique cosets of $H \cap K$, so the index of H in G divides the index of $H \cap K$ in G. An identical proof shows the same is true for K. Since m and n divide $|G : H \cap K|$, it must be no less than the least common multiple of the two. Therefore l.c.m.$(m, n) \leq |G : H \cap K| \leq mn$.

Note that if m and n are relatively prime, then their least common multiple is their product, in which case $|G : H \cap K| = |G : H| \cdot |G : K|$. $\square$

11. (11/2/23)

Let $H \leq K \leq G$. Prove that $|G : H| = |G : K| \cdot |K : H|$ (do not assume G is finite).

Proof. The proof in Exercise 10 above generalizes to this case. Since we can partition G into $|G : K|$ cosets of K and K into $|K : H|$ cosets of H, there are $|G : K| \cdot |K : H|$ unique cosets of H in G, and so $|G : H| = |G : K| \cdot |K : H|$. □

12. (10/16/23)

Let $H \leq G$. Prove that the map $x \mapsto x^{-1}$ sends each left coset of H in G onto a right coset of H and gives a bijection between the set of left cosets and the set of right cosets of H in G (hence the number of left cosets of H in G equals the number of right cosets).

Proof. Let $\varphi : G \to G$ be defined by $\varphi(x) = x^{-1}$ for all $x \in G$. Consider:

$$\varphi(xH) = \{\varphi(xh) \mid h \in H\} = \{(xh)^{-1} \mid h \in H\} = \{h^{-1}x^{-1} \mid h \in H\} = Hx^{-1},$$

so φ maps left cosets of H onto right cosets of H.

Further, considering φ as a map from left cosets of H to right cosets of H, it is a bijection.

Toward injectivity, suppose that $\varphi(xH) = \varphi(yH)$ for some $x, y \in G$, and let $z \in xH$. Then $\varphi(z) = z^{-1} = hy^{-1}$, because $z \in xH$ and $\varphi(xH) = \varphi(yH)$. Inverting both sides, we obtain $z = (hy^{-1})^{-1} = yh^{-1} \in yH$, and so $xH \subseteq yH$. The same logic shows that $yH \subseteq xH$, so we must have $xH = yH$, and therefore φ is injective.

It is also surjective: Letting Hx be a right coset of H, by definition we have $\varphi(x^{-1}H) = Hx$. It is therefore a bijection, and so there are an equal number of left cosets and right cosets of H in G. □

13. (10/16/23)

Fix any labelling of the vertices of a square and use this to identify D_8 as a subgroup of S_4. Prove that the elements of D_8 and $\langle(1, 2, 3)\rangle$ do not commute in S_4.

Proof. Label the vertices of a square starting at the upper-left corner and going clockwise 1, 2, 3, 4. We can assign to the generators r, s of D_8 the permutations $(1, 2, 3, 4), (2, 4) \in S_4$, respectively.

To show that the elements of D_8 and $\langle(1, 2, 3)\rangle$ do not commute, we note that:

$$(1, 2, 3) \cdot s = (1, 2, 3)(2, 4) = (1, 2, 4, 3), \text{ and}$$
$$s \cdot (1, 2, 3) = (2, 4)(1, 2, 3) = (1, 4, 2, 3),$$

so s does not commute with $(1,2,3) \in S_4$. Therefore D_8 and $\langle (1,2,3) \rangle$ are not commuting subgroups of S_4. $\square$

14. (10/17/23)

Prove that S_4 does not have a normal subgroup of order 8 or a normal subgroup of order 3.

Proof. From Corollary 10, a subgroup of order 3 is isomorphic to Z_3, hence cyclic. So, without loss of generality, consider $\langle (1,2,3) \rangle \leq S_4$. Consider the conjugate of $(1,2,3)$ by $(1,2)(3,4)$:

$$(1,2)(3,4) \cdot (1,2,3) \cdot (1,2)(3,4) = (1,4,2),$$

which is not an element of $\langle (1,2,3) \rangle$. Therefore there is an element of S_4 that does not normalize $\langle (1,2,3) \rangle$ and, by isomorphism, any subgroup of order 3, so S_4 does not contain any normal subgroups of order 3.

Next, let $X \leq S_4$ with $|X| = 8$ and suppose that $X \trianglelefteq S_4$. From Cauchy's Theorem, X contains an element of order 2, which may be either a single 2-cycle or a pair of disjoint 2-cycles. We will consider each case individually:

- Without loss of generality, suppose that $(1,2) \in X$. Because X is normal in S_4, the conjugate element $(1,2,3) \cdot (1,2) \cdot (1,3,2) = (2,3)$ must lie in X. Because X is closed, the product $(1,2) \cdot (2,3) = (1,2,3)$ must lie in X, a contradiction since (from Corollary 9) a subgroup of order 8 contains no elements of order 3. Thus X is not normal in S_4.

- Similarly, suppose that $(1,2)(3,4) \in X$. Again, the conjugate $(1,2,3) \cdot (1,2)(3,4) \cdot (1,3,2) = (1,4,3,2)$ must lie in X. So the product $(1,2)(3,4) \cdot (1,4,3,2) = (1,3)$ must lie in X. Then, since X contains a 2-cycle, it must contain an element of order 3, a contradiction. Thus X is again not normal in S_4.

This concludes the proof that S_4 contains no normal subgroups of order 8 or order 3. $\square$

15. (10/19/23)

Let $G = S_n$ and for fixed $i \in \{1, 2, ..., n\}$ let G_i be the stabilizer of i. Prove that $G \cong S_{n-1}$.

Proof. From Ch. 2.2, Exercise 8., we have defined a bijection $\varphi : G_i \to S_{n-1}$ defined on a permutation $\sigma \in G_i$ and an element $m \in \{1, 2, ..., n\}$ it permutes:

$$\varphi(\sigma)(m) = \begin{cases} \sigma(m) & \text{if } \sigma(m) \leq i \\ \sigma(m) - 1 & \text{if } \sigma(m) > i \end{cases}.$$

For $\sigma_1, \sigma_2 \in G_i$ and $m \in \{1, ..., n\}$, let $\sigma_2(m) = k$ and $\sigma_1(k) = p$. Let us consider the different cases for k and p.

1. $k \leq i, p \leq i$. Then:

$$(\sigma_1 \circ \sigma_2)(m) = \sigma_1(\sigma_2(m)) = \sigma_1(k) = p \leq i, \text{ which implies that}$$
$$\varphi(\sigma_1 \circ \sigma_2)(m) = (\sigma_1 \circ \sigma_2)(m) = p.$$

   Also:

$$\begin{aligned} &\sigma_2(m) = k \leq i \Rightarrow \varphi(\sigma_2(m)) = \sigma_2(m) = k, \text{ and} \\ &\sigma_1(k) = p \leq i \Rightarrow \varphi(\sigma_1(k)) = \sigma_1(k) = p, \text{ so} \end{aligned}$$
$$\begin{aligned} (\varphi(\sigma_1) \circ \varphi(\sigma_2))(m) &= \varphi(\sigma_1)\big(\varphi(\sigma_2)(m)\big) \\ &= \varphi(\sigma_1)\big(\sigma_2(m)\big) \\ &= \varphi(\sigma_1)(k) = \sigma_1(k) = p, \end{aligned}$$

   thus $\varphi(\sigma_1 \circ \sigma_2) = \varphi(\sigma_1) \circ \varphi(\sigma_2)$.

2. $k > i, p \leq i$. As above, we have $(\sigma_1 \circ \sigma_2)(m) = p \leq i$, which implies that $\varphi(\sigma_1 \circ \sigma_2)(m) = p$. Also:

$$\sigma_2(m) = k > i \Rightarrow \varphi(\sigma_2)(m) = \sigma_2(m) - 1 = k - 1.$$

   Now note that, in the permutation $\varphi(\sigma_1)$, all values greater than or equal to i have been decremented by 1, so we have $\varphi(\sigma_1)(k-1) = \sigma_1(k) = p$. It follows that:

$$\begin{aligned} (\varphi(\sigma_1) \circ \varphi(\sigma_2))(m) &= \varphi(\sigma_1)\big(\varphi(\sigma_2)(m)\big) \\ &= \varphi(\sigma_1)(k-1) \\ &= \sigma_1(k) = p, \end{aligned}$$

   thus $\varphi(\sigma_1 \circ \sigma_2) = \varphi(\sigma_1) \circ \varphi(\sigma_2)$.

3. $k \leq i, p > i$. Then $(\sigma_1 \circ \sigma_2)(m) = p > i$, which implies that $\varphi(\sigma_1 \circ \sigma_2)(m) = (\sigma_1 \circ \sigma_2)(m) = p - 1$. As in the first case, $\varphi(\sigma_2)(m) = \sigma_2(m) = k$. So:

$$\begin{aligned} (\varphi(\sigma_1) \circ \varphi(\sigma_2))(m) &= \varphi(\sigma_1)\big(\varphi(\sigma_2)(m)\big) \\ &= \varphi(\sigma_1)(k) \\ &= \sigma_1(k) - 1 = p - 1, \end{aligned}$$

   thus $\varphi(\sigma_1 \circ \sigma_2) = \varphi(\sigma_1) \circ \varphi(\sigma_2)$.

4. $k > i, p > i$. As above, we have $\varphi(\sigma_1 \circ \sigma_2)(m) = p - 1$. As in the second case, we have $\varphi(\sigma_2)(m) = k - 1$; however, $\sigma_1(k) = p > i$, so $\varphi(\sigma_1(k-1)) = \sigma_1(k) - 1 = p - 1$. Then:

$$\begin{aligned} (\varphi(\sigma_1) \circ \varphi(\sigma_2))(m) &= \varphi(\sigma_1)\big(\varphi(\sigma_2)(m)\big) \\ &= \varphi(\sigma_1)(k-1) \\ &= \sigma_1(k-1) - 1 = p - 1, \end{aligned}$$

   thus $\varphi(\sigma_1 \circ \sigma_2) = \varphi(\sigma_1) \circ \varphi(\sigma_2)$.

This exhaustively shows that for all $\sigma_1, \sigma_2 \in G_i$, the equation $\varphi(\sigma_1 \circ \sigma_2) = \varphi(\sigma_1) \circ \varphi(\sigma_2)$ holds in S_{n-1}. Thus φ is an isomorphism, and so $G_i \cong S_{n-1}$. $\square$

16. (10/19/23)

Use Lagrange's Theorem in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ to prove *Fermat's Little Theorem*: if p is a prime then $a^p \equiv a \pmod{p}$ for all $a \in \mathbb{Z}$.

Proof. Recall that the order of the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ is equal to the number of positive integers n for which $n < p$ and n is relatively prime to p. Since p is prime, this is $p - 1$.

For any $\overline{a} \in (\mathbb{Z}/p\mathbb{Z})^\times$, the order of $\overline{a}$ must divide $p - 1$, and in particular, we have $\overline{a}^{p-1} = 1$. It follows that $\overline{a}^p = \overline{a}$. If $\overline{a} \in (\mathbb{Z}/p\mathbb{Z})^\times$ is a representative of some $a \in \mathbb{Z}$, we then conclude that $a^p \equiv a \pmod{p}$. $\square$

17. (10/19/23)

Let p be a prime and let n be a positive integer. Find the order of $\overline{p}$ in $(\mathbb{Z}/(p^n - 1)\mathbb{Z})^\times$ and deduce that $n \mid \varphi(p^n - 1)$ (here φ is Euler's function).

Proof. The order of $(\mathbb{Z}/(p^n - 1)\mathbb{Z})^\times$ is equal to the number of positive integers k for which $k < p^n - 1$ and k is relatively prime to $p^n - 1$, that is, $\varphi(p^n - 1)$.

Now $p^n = (p^n - 1) + 1 \equiv 1 \pmod{p^n - 1}$. For all non-negative $k < n$, we have $p^k < p^n$, so n is the smallest positive integer for which $p^n \equiv 1 \pmod{p^n - 1}$, which implies that $|\overline{p}| = n$. It follows that n divides $\varphi(p^n - 1)$, the order of $(\mathbb{Z}/(p^n - 1)\mathbb{Z})^\times$. $\square$

18. (11/3/23)

Let G be a finite group, let H be a subgroup of G and let $N \trianglelefteq G$. Prove that if $|H|$ and $|G : N|$ are relatively prime then $H \leq N$.

Proof. Toward contradiction, suppose that there exists an $h \in H, h \notin N$. The cyclic group $\langle hN \rangle$ is a subgroup of G/N, so its order divides $|G/N| = |G : N|$. Also, because for all $i, j \in \{0, ..., |h| - 1\}$, $h^i N = h^j N$ implies $h^i = h^j$,$\langle hN \rangle$ has order equal to $|h|$, so $|h|$ divides $|G : N|$. Now since $|H|$ and $|G : N|$ are relatively prime and $|h|$ divides both, we must have $|h| = 1$, which implies that h is the identity, and so lies in N, a contradiction.

Therefore for all $h \in H$, we must have $h \in N$, and so $H \leq N$. $\square$

19. (3/22/24)

Prove that if N is a normal subgroup of the finite group G and $(|N|, |G : N|) = 1$ then N is the unique subgroup of G of order $|N|$.

Proof. Suppose that $|N| = k$ and $|G| = mk$ with k, m relatively prime. Let $A \leq G$ and suppose that $|A| = |N| = k$.

Since $A \leq N_G(N) = G$, AN is a subgroup of G. Then $|AN|$ must divide mk. Since m and k are relatively prime, $|AN|$ divides only one of either m or k. Also, since $N \leq AN \leq G$, $|N| = k$ divides $|AN|$. We cannot have k dividing $|AN|$ and $|AN|$ dividing m, therefore $|AN|$ both divides and is divided by k, so it must be equal to k.

Now if there exists $a \in A$ such that $a \notin N$, then we would have $|AN| > k$. However, because $|AN| = k$, we must therefore have $A = N$. We conclude that N is the unique subgroup of G of order k. $\square$

20. (3/21/24)

If A is an abelian group with $A \trianglelefteq G$ and B is any subgroup of G prove that $A \cap B \trianglelefteq AB$.

Proof. Given $x \in A \cap B$, $g \in AB$, it suffices to show that $gxg^{-1} \in A \cap B$, or equivalenty that $gxg^{-1} \in A$ and $gxg^{-1} \in B$. Because $x \in A$ and $A \trianglelefteq G$ (therefore $A \trianglelefteq AB$) we already have $gxg^{-1} \in A$.

To show that gxg^{-1} also lies in B, from Corollary 15, we note that since $B \leq N_G(A) = G$, AB is a subgroup of G. And from Corollary 14, since AB is a subgroup, it follows that $AB = BA$. Write $g = ba$ for some $a \in A, b \in B$. Then:

$$gxg^{-1} = (ba)x(ba)^{-1} = baxa^{-1}b^{-1} = \underbrace{baa^{-1}xb^{-1}}_{A \text{ is abelian and } x \in A} = \underbrace{bxb^{-1} \in B}_{x \in B},$$

and so $gxg^{-1} \in B$. Therefore $gxg^{-1} \in A \cap B$ for all $x \in A \cap B$, $g \in AB$, and so $A \cap B \trianglelefteq AB$. $\square$

21. (3/19/24)

Prove that $\mathbb{Q}$ has no proper subgroups of finite index. Deduce that $\mathbb{Q}/\mathbb{Z}$ has no proper subgroups of finite index.

Proof. We will first prove the more general case that a divisible abelian group has no proper subgroups of finite index, and then deduce that both $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ have no proper subgroups of finite index.

Let A be a divisible abelian group and let $B \leq A$. Since A is abelian, all subgroups are normal, so the quotient group A/B is well-defined. From Chapter 3.1, Exercise 15, A/B is also divisible. From Chapter 2.4, Exercise 19(b), no finite groups are divisible, so $|A : B| = |A/B| = \infty$.

Since $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are both divisible abelian groups, they therefore have no proper subgroups of finite index. $\square$

22. (3/22/24)

Use Lagrange's Theorem in the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$ to prove *Euler's Theorem*: $a^{\varphi(n)} \equiv 1 \mod n$ for every integer a relatively prime to n, where φ denote's Euler's φ-function.

Proof. Given $n \geq 2$, the order of $(\mathbb{Z}/n\mathbb{Z})^\times$ is $\varphi(n)$, the number of positive integers less than or equal to n and relatively prime to n.

Let $a \in \mathbb{Z}$ and consider $\overline{a} \in (\mathbb{Z}/n\mathbb{Z})^\times$. From Corollary 9, $\overline{a}^{|(\mathbb{Z}/n\mathbb{Z})^\times|} = \overline{a}^\varphi(n) = \overline{1}$. Therefore $a^{\varphi(n)} \equiv 1 \mod n$. □

23. (3/25/24)

Determine the last two digits of $3^{3^{100}}$. [Determine 3^{100} mod $\varphi(100)$ and use the previous exercise.]

Solution. First, we note that $\varphi(100)$ — the number of integers relatively prime to and less than 100 — is 40 (this includes all and only numbers ending in 1, 3, 7, or 9). Next, note that $3^1 = 3, 3^2 = 9, 3^3 = 27,$ and $3^4 = 81 \equiv 1 \bmod 40$. It follows that $3^{100} = 3^{25\cdot 4} \equiv 1 \bmod 40$; that is, $3^{100} = 40n + 1$ for some $n \in \mathbb{Z}^+$.

Now from the previous exercise, we know that $3^{\varphi(100)} = 3^{40} \equiv 1 \bmod 100$. From above, we have $3^{100} \equiv 1 \bmod 40$. Therefore:

$$3^{3^{100}} = 3^{40n+1} = 3^{40n} \cdot 3 \equiv (1 \cdot 3) \bmod 100 = 3 \bmod 100.$$

We conclude that $3^{3^{100}}$ ends in 03. □