

# Dummit & Foote Ch. 4.1: Group Actions and Permutation Representations

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Let  $G$  be a group and  $A$  be a nonempty set.

## 1. (12/24/23)

Let  $G$  act on the set  $A$ . Prove that if  $a, b \in A$  and  $b = g \cdot a$  for some  $g \in G$ , then  $G_b = gG_ag^{-1}$  ( $G_a$  is the stabilizer of  $a$ ). Deduce that if  $G$  acts transitively on  $A$  then the kernel of the action is  $\bigcap_{g \in G} gG_ag^{-1}$ .

*Proof.* We will show first that  $G_b$ , the stabilizer of  $b$ , is contained in  $gG_ag^{-1}$ , and then show the converse, which proves that they are equal.

Let  $x \in G_b$ , so  $x \cdot b = b$ . Then:

$$\begin{aligned} x \cdot g \cdot a &= g \cdot a \quad (b = g \cdot a) \\ (gg^{-1}) \cdot (xg) \cdot a &= g \cdot a \quad (gg^{-1} = 1, 1 \cdot a = a) \\ g \cdot (g^{-1}xg) \cdot a &= g \cdot a \\ (g^{-1}xg) \cdot a &= a, \end{aligned}$$

which implies that  $g^{-1}xg \in G_a$ , and therefore  $x \in gG_ag^{-1}$ , so  $G_b \subseteq gG_ag^{-1}$ .

The converse, that  $gG_ag^{-1} \subseteq G_b$ , can be shown by following the above proof in reverse (that is, let  $x \in gG_ag^{-1}$ , so  $g^{-1}xg \in G_a$ , which implies that  $(g^{-1}xg) \cdot a = a$ , and each assertion holds from bottom to top). Since each is contained in the other, we have  $G_b = gG_ag^{-1}$ .

Now we already know that the kernel of the group action of  $G$  on  $A$  is the intersection of the stabilizers of all the elements of  $A$ , that is,  $\bigcap_{b \in A} G_b$ . If  $G$  acts transitively on  $A$ , fixing  $a \in A$ , then for all  $b \in A$ , we can write  $b = g \cdot a$  for some  $g \in G$ , which from above implies that  $G_b = gG_ag^{-1}$ . We deduce that the kernel can be expressed in terms of a fixed element  $a$ , namely:

$$\bigcap_{b \in A} G_b = \bigcap_{b \in A} \underbrace{gG_ag^{-1}}_{b=g \cdot a} = \bigcap_{g \in G} gG_ag^{-1}.$$

We know that  $\bigcap_{g \in G} gG_ag^{-1}$  intersects all of the same conjugates as does  $\bigcap_{b \in A}$ , since  $G$  acts transitively on  $A$ . And, since  $b = g \cdot a \Rightarrow G_b = gG_ag^{-1}$ , it intersects no conjugates not represented by  $G_b$  for all  $b \in A$ .  $\square$

## 2. (1/2/24)

Let  $G$  be a *permutation group* on the set  $A$  (i.e.,  $G \leq S_A$ ), let  $\sigma \in G$  and let  $a \in A$ . Prove that  $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$ . Deduce that if  $G$  acts transitively on  $A$  then

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = 1.$$

*Proof.* We first show that  $\sigma G_a \sigma^{-1} \subseteq G_{\sigma(a)}$ , and then show the converse. To begin, let  $\tau \in G_a$  and consider  $\sigma \tau \sigma^{-1} \in \sigma G_a \sigma^{-1}$ . We note that:

$$(\sigma \tau \sigma^{-1})(\sigma(a)) = (\sigma \tau \sigma^{-1} \sigma)(a) = (\sigma \tau)(a) = \underbrace{\sigma(\tau(a))}_{\tau \in G_a \Rightarrow \tau(a)=a} = \sigma(a),$$

and so  $\sigma \tau \sigma^{-1}$  stabilizes  $\sigma(a)$ , which implies that  $\sigma G_a \sigma^{-1} \subseteq G_{\sigma(a)}$ .

For the converse, let  $\tau \in G$  and suppose that  $\sigma \tau \sigma^{-1} \in G_{\sigma(a)}$ . Then:

$$\begin{aligned} (\sigma \tau \sigma^{-1})(\sigma(a)) &= \sigma(a) \\ (\sigma \tau \sigma^{-1} \sigma)(a) &= \sigma(a) \\ (\sigma \tau)(a) &= \sigma(a) \\ \sigma(\tau(a)) &= \sigma(a) \\ \tau(a) &= a, \end{aligned}$$

so  $\tau$  is in the stabilizer of  $a$ , which implies that  $\sigma \tau \sigma^{-1} \in \sigma G_a \sigma^{-1}$ , and so  $G_{\sigma(a)} \subseteq \sigma G_a \sigma^{-1}$ .

This concludes the proof that  $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$ .

Now if  $G$  acts transitively on  $A$ , then there is only one orbit; that is, given some  $a \in A$ , for all  $b \in A$ , there is a  $\sigma \in G$  such that  $b = \sigma(a)$ .

From above, we conclude:

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = \bigcap_{\sigma \in G} G_{\sigma(a)} = \bigcap_{a \in A} G_a \text{ (because } G \text{ acts transitively on } A),$$

and since the only permutation that fixes every element of  $A$  is the identity, this intersection consists therefore only the identity permutation.  $\square$

## 3. (1/2/24)

Assume that  $G$  is an abelian, transitive subgroup of  $S_A$ . Show that  $\sigma(a) \neq a$  for all  $\sigma \in G - \{1\}$  and all  $a \in A$ . Deduce that  $|G| = |A|$ . [Use the preceding exercise.]

*Proof.* Suppose that  $\sigma_1$  fixes  $a$ , so  $\sigma_1(a) = a$ , and let  $\sigma_2(a) = b$ . Then:

$$\begin{aligned} (\sigma_1 \circ \sigma_2)(a) &= \sigma_1(\sigma_2(a)) = \sigma_1(b), \text{ and} \\ (\sigma_2 \circ \sigma_1)(a) &= \sigma_2(\sigma_1(a)) = \sigma_2(a). \end{aligned}$$

Since  $G$  is abelian, these must be equal, and so  $\sigma_1(b) = \sigma_2(a) = b$ . Then  $\sigma_1$  also fixes  $b$ .

Since  $G$  is transitive, for every  $b \in A$ , there exists a  $\sigma \in G$  such that  $\sigma(b) = a$ , which implies that  $\sigma_1$  fixes every element of  $A$  and is therefore the identity. Thus the identity is the only element of  $G$  for which  $\sigma(a) = a$ ; equivalently,  $\sigma(a) \neq a$  for all  $\sigma \in G - \{1\}$  and all  $a \in A$ .

Now let  $A = \{1, \dots, n\}$ . Since  $G$  is transitive, it must contain at least  $n$  permutations. For all  $i \in A$ , define  $\sigma_i$  such that  $\sigma_i(1) = i$  (with  $\sigma_1$  the identity permutation). Suppose that  $\tau$  is another permutation in  $G$ . Since  $A$  only contains  $n$  elements, we must have  $\tau(1) = i$  for some  $i \in A$ , so  $\tau(1) = \sigma_i(1)$ . Then:

$$\begin{aligned}(\tau \circ \sigma_i)(1) &= \tau(\sigma_i(1)) = \tau(i), \text{ and} \\(\sigma_i \circ \tau)(1) &= \sigma_i(\tau(1)) = \sigma_i(i).\end{aligned}$$

Since  $G$  is abelian, these are equal, so  $\tau(i) = \sigma_i(i)$ . It follows that, if  $j = \tau(i) = \sigma_i(i)$ , then  $\tau(j) = \sigma_i(j)$ , and so on for every element which  $\sigma_i$  permutes. Therefore  $\tau = \sigma_i$ , so  $G$  contains exactly  $n$  permutations. We conclude that  $|G| = |A|$ .  $\square$

#### 4. (1/3/24)

Let  $S_3$  act on the set  $\Omega$  of ordered pairs:  $\{(i, j) \mid 1 \leq i, j \leq 3\}$  by  $\sigma((i, j)) = (\sigma(i), \sigma(j))$ . Find the orbits of  $S_3$  on  $\Omega$ . For each  $\sigma \in S_3$  find the cycle decomposition of  $\sigma$  under this action (i.e., find its cycle decomposition when  $\sigma$  is considered as an element of  $S_9$  — first fix a labelling of these nine ordered pairs). For each orbit  $\mathcal{O}$  of  $S_3$  acting on these nine points pick some  $a \in \mathcal{O}$  and find the stabilizer of  $a$  in  $S_3$ .

*Solution.* The elements  $(1, 1)$ ,  $(2, 2)$ , and  $(3, 3)$  all belong to the same orbit. We see that  $(12) \cdot (1, 1) = (2, 2)$  and  $(23) \cdot (2, 2) = (3, 3)$ . Further, since for any  $(i, i) \in \Omega$ , the action of  $\sigma \in S_3$  on it results in an element with the same coordinates, there is no  $(i, j) \in \Omega$  with  $i \neq j$  such that  $\sigma((i, i)) = (i, j)$ . Therefore these three elements constitute one orbit.

The other orbit consists of the remaining six elements. Beginning with  $(1, 2)$ , we have:

$$\begin{aligned}(123)(1, 2) &= (2, 3), \text{ then} \\(123)(2, 3) &= (3, 1), \\(12)(3, 1) &= (3, 2), \\(123)(3, 2) &= (1, 3), \text{ and finally} \\(123)(1, 3) &= (2, 1).\end{aligned}$$

Conversely to the first orbit, for no  $(i, j) \in \Omega$  with  $i \neq j$  do we have  $\sigma((i, j)) = (i, i)$ . Thus these are the two disjoint orbits of  $S_3$  on  $\Omega$ .

Next, let us label the elements of  $\Omega$ :

$$\begin{array}{lll} (1, 1) \rightarrow 1 & (1, 2) \rightarrow 2 & (1, 3) \rightarrow 3 \\ (2, 1) \rightarrow 4 & (2, 2) \rightarrow 5 & (2, 3) \rightarrow 6 \\ (3, 1) \rightarrow 7 & (3, 2) \rightarrow 8 & (3, 3) \rightarrow 9 \end{array}$$

Then we can describe the cycle decomposition of each permutation of  $S_3$  by how it acts on these elements:

$$\begin{array}{l} 1 \rightarrow 1 \\ (1\ 2) \rightarrow (1\ 5)(2\ 4)(3\ 6)(7\ 8) \\ (1\ 3) \rightarrow (1\ 9)(2\ 8)(3\ 7)(4\ 6) \\ (2\ 3) \rightarrow (2\ 3)(4\ 7)(5\ 9)(6\ 8) \\ (1\ 2\ 3) \rightarrow (1\ 5\ 9)(2\ 6\ 7)(3\ 4\ 8) \\ (1\ 3\ 2) \rightarrow (1\ 9\ 5)(2\ 7\ 6)(3\ 8\ 4) \end{array}$$

For the first orbit, let us choose the point  $(1, 1)$  and find its stabilizer. The permutations in  $S_3$  that fix this element must fix 1. Obviously this includes the identity. Neither of the 3-cycles fix 1. Only one of the 2-cycles,  $(2\ 3)$ , fixes 1. Therefore the stabilizer of  $(1, 1)$  is the subgroup  $\{1, (2\ 3)\}$ .

For the second orbit, we choose  $(1, 2)$ . Since all non-identity permutations of  $S_3$  reassign either 1 or 2 (or both), the stabilizer consists of only the identity.  $\square$

## 5. (1/3/24)

For each of parts (a) and (b) repeat the preceding exercise but with  $S_3$  acting on the specified set:

(a) the set of 27 triples  $\{(i, j, k) \mid 1 \leq i, j, k \leq 3\}$

- The orbit of any point  $(i, j, k)$  consists of those other points whose coordinates are equal if  $i, j$ , or  $k$  are equal or not if they are not. For example, the orbit of  $(1, 1, 2)$  contains all those points whose first and second coordinates are the same and whose third is different, that is:

$$(1, 1, 3), (2, 2, 1), (2, 2, 3), (3, 3, 1), (3, 3, 2)$$

(It is simple to find a 2-cycle that acts on  $(1, 1, 2)$  to send it to any other point in this orbit.)

For each point where at least one of the coordinates differs from the others (of the type  $(i, i, j)$ ,  $(i, j, i)$ ,  $(j, i, i)$ , or  $(i, j, k)$ ), its orbit contains the 6 points of the same type. The three points  $(1, 1, 1)$ ,  $(2, 2, 2)$ , and  $(3, 3, 3)$  are all in the same remaining orbit.

- Next, we label the 27 triples:

$$\begin{array}{llll}
(1, 1, 1) \rightarrow 1 & (1, 1, 2) \rightarrow 2 & (1, 1, 3) \rightarrow 3 & (1, 2, 1) \rightarrow 4 \\
(1, 2, 2) \rightarrow 5 & (1, 2, 3) \rightarrow 6 & (1, 3, 1) \rightarrow 7 & (1, 3, 2) \rightarrow 8 \\
(1, 3, 3) \rightarrow 9 & (2, 1, 1) \rightarrow 10 & (2, 1, 2) \rightarrow 11 & (2, 1, 3) \rightarrow 12 \\
(2, 2, 1) \rightarrow 13 & (2, 2, 2) \rightarrow 14 & (2, 2, 3) \rightarrow 15 & (2, 3, 1) \rightarrow 16 \\
(2, 3, 2) \rightarrow 17 & (2, 3, 3) \rightarrow 18 & (3, 1, 1) \rightarrow 19 & (3, 1, 2) \rightarrow 20 \\
(3, 1, 3) \rightarrow 21 & (3, 2, 1) \rightarrow 22 & (3, 2, 2) \rightarrow 23 & (3, 2, 3) \rightarrow 24 \\
(3, 3, 1) \rightarrow 25 & (3, 3, 2) \rightarrow 26 & (3, 3, 3) \rightarrow 27 & 
\end{array}$$

So we can describe each permutation in  $S_3$  by how it acts on these elements:

$$\begin{array}{l}
1 \rightarrow 1 \\
(1\ 2) \rightarrow (1\ 14)(2\ 13)(3\ 15)(4\ 11)(5\ 10)(6\ 12)(7\ 17) \\
\quad (8\ 16)(9\ 18)(19\ 23)(20\ 22)(21\ 24)(25\ 26) \\
(1\ 3) \rightarrow (1\ 27)(2\ 26)(3\ 25)(4\ 24)(5\ 23)(6\ 22)(7\ 21) \\
\quad (8\ 20)(9\ 19)(10\ 18)(11\ 17)(12\ 16)(13\ 15) \\
(2\ 3) \rightarrow (2\ 3)(4\ 7)(5\ 9)(6\ 8)(10\ 19)(11\ 21)(12\ 20) \\
\quad (13\ 25)(14\ 27)(15\ 26)(16\ 22)(17\ 24)(18\ 23) \\
(1\ 2\ 3) \rightarrow (1\ 14\ 27)(2\ 15\ 25)(3\ 13\ 26)(4\ 17\ 21)(5\ 18\ 19) \\
\quad (6\ 16\ 20)(7\ 11\ 24)(8\ 12\ 22)(9\ 10\ 23) \\
(1\ 3\ 2) \rightarrow (1\ 27\ 14)(2\ 25\ 15)(3\ 26\ 13)(4\ 21\ 17)(5\ 19\ 18) \\
\quad (6\ 20\ 16)(7\ 24\ 11)(8\ 22\ 12)(9\ 23\ 10)
\end{array}$$

- Finally, for each orbit, we choose an element from it and find its stabilizer:

$$(1, 1, 1) \text{ has stabilizer } \{1, (2, 3)\}.$$

For each remaining orbit, since an element from has at least two coordinates that are different, each non-identity element of  $S_3$  reassigns it, so the stabilizer of all elements from the other orbits is simply the identity.

(b) the set  $\mathcal{P}(\{1, 2, 3\}) - \{\emptyset\}$  of all 7 nonempty subsets of  $\{1, 2, 3\}$ .

- There are three orbits, which partition the power set by how many elements each child set contains (ex. one orbit consists of the three singleton sets, namely  $\{\{1\}, \{2\}, \{3\}\}$ ).
- We label each element:

$$\begin{array}{lll}
\{1\} \rightarrow 1 & \{2\} \rightarrow 2 & \{3\} \rightarrow 3 \\
\{1, 2\} \rightarrow 4 & \{1, 3\} \rightarrow 5 & \{2, 3\} \rightarrow 6 \\
\{1, 2, 3\} \rightarrow 7
\end{array}$$

So we can describe each permutation in  $S_3$  by how it acts on these elements:

$$\begin{aligned} 1 &\rightarrow 1 \\ (1\ 2) &\rightarrow (1\ 2)(5\ 6) \\ (1\ 3) &\rightarrow (1\ 3)(4\ 6) \\ (2\ 3) &\rightarrow (2\ 3)(4\ 5) \\ (1\ 2\ 3) &\rightarrow (1\ 2\ 3)(4\ 6\ 5) \\ (1\ 3\ 2) &\rightarrow (1\ 3\ 2)(4\ 5\ 6) \end{aligned}$$

- Finally, for each orbit, we choose an element from it and find its stabilizer. In the orbit of singletons, the stabilizer of  $\{1\}$  is  $\{1, (2\ 3)\}$ . In the orbit of doubles, the stabilizer of  $\{1, 2\}$  is  $\{1, (1\ 2)\}$ . In the orbit that consists only of the triple  $\{1, 2, 3\}$ , the stabilizer is all of  $S_3$ .