

# Dummit & Foote Ch. 2.3: Cyclic Groups and Cyclic Subgroups

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## 1. (6/18/23)

Find all subgroups of  $Z_{45} = \langle x \rangle$ , giving a generator for each. Describe the containments between these subgroups.

*Proof.* The subgroups of  $Z_{45} = \langle x \rangle$  are those cyclic groups generated by  $x^n$ , where  $n$  divides 45. These are:

- $\langle 1 \rangle = \{1\}$ , the trivial subgroup
- $\langle x^{15} \rangle = \{1, x^{15}, x^{30}\} \cong \mathbb{Z}/3\mathbb{Z}$
- $\langle x^9 \rangle = \{1, x^9, x^{18}, x^{27}, x^{36}\} \cong \mathbb{Z}/5\mathbb{Z}$
- $\langle x^5 \rangle = \{1, x^5, x^{10}, x^{15}, x^{20}, x^{25}, x^{30}, x^{35}, x^{40}\} \cong \mathbb{Z}/9\mathbb{Z}$
- $\langle x^3 \rangle = \{1, x^3, x^6, \dots, x^{39}, x^{42}\} \cong \mathbb{Z}/15\mathbb{Z}$
- $\langle x \rangle = Z_{45}$  itself

Among these subgroups, we have  $\langle 1 \rangle$  contained within every other subgroup, as well as  $\langle x^{15} \rangle \leq \langle x^5 \rangle$ ,  $\langle x^{15} \rangle \leq \langle x^3 \rangle$ , and  $\langle x^9 \rangle \leq \langle x^3 \rangle$ .  $\square$

## 2. (6/19/23)

If  $x$  is an element of the finite group  $G$  and  $|x| = |G|$ , prove that  $G = \langle x \rangle$ . Give an explicit example to show that this result need not be true if  $G$  is an infinite group.

*Proof.* Let  $|x| = |G| = n < \infty$ . By definition,  $G$  is closed, so it contains all powers of  $x : 1, x, x^2, \dots, x^{n-1}$ . These are exactly  $n$  elements, so  $G$  contains no other elements. It is therefore generated by  $x$ , that is,  $G = \langle x \rangle$ .

However, if  $G$  is an infinite group and  $x \in G$  with  $|x| = \infty$ , then this is not necessarily the case. For example, if  $G = \mathbb{Z}$  and  $x = 2$ , then  $x$  generates all even integers in  $\mathbb{Z}$ , but does not generate the element 5.  $\square$

### 3. (6/19/23)

Find all generators for  $\mathbb{Z}/48\mathbb{Z}$ .

*Proof.* From Proposition 6., the generators for  $\mathbb{Z}/48\mathbb{Z}$  are those positive integers  $n < 48$  for which  $n$  is relatively prime to 48. These are: 1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, and 47.  $\square$

### 4. (6/19/23)

Find all generators for  $\mathbb{Z}/202\mathbb{Z}$ .

*Proof.* As above, the generators for  $\mathbb{Z}/202\mathbb{Z}$  are those positive integers  $n < 202$  for which  $n$  is relatively prime to 202. The integer 202 only has two divisors greater than 1, namely 2 and 101. Therefore the generators of  $\mathbb{Z}/202\mathbb{Z}$  are every odd positive integer less than 202 except for 101.  $\square$

### 5. (6/19/23)

Find the number of generators for  $\mathbb{Z}/49000\mathbb{Z}$ .

*Proof.* We are concerned with the number of integers  $n$  between 0 and 48999 for which  $n$  is relatively prime to 49000. It will be helpful to write 49000 uniquely as the product of primes:  $2^3 \cdot 5^3 \cdot 7^2$ .

Let us first consider the generators for  $\mathbb{Z}/49000\mathbb{Z}$  between 0 and 69, that is, all the numbers that are relatively prime to 49000 between 0 and 69: 1, 3, 9, 11, 13, 17, 19, 23, 27, 29, 31, 33, 37, 39, 41, 43, 47, 51, 53, 57, 59, 61, 67, and 69. There are 24 such generators.

Next, we show that, for any  $n \in \{0, \dots, 48999\}$ , the greatest common divisor of  $n$  and 49000 is equal to the greatest common divisor of  $n \bmod 70$  and 49000. This is because 70 is equal to the product of the bases of the prime factors of 49000:  $70 = 2 \cdot 5 \cdot 7$ . So for any  $n$ , we have  $n = m + 70k = m + (2 \cdot 5 \cdot 7)k$ , where  $m \in \{0, \dots, 69\}$  and  $k \geq 0$ . Suppose that  $m$  is *not* in the list of the above generators (that is, that the greatest common divisor of  $m$  and 49000 is greater than 1). Then either 2, 5, or 7 divides  $m$  (otherwise  $m$  would be relatively prime to 49000). Without loss of generality, suppose that 2 divides  $m$ , and write  $m = 2p$ . We can then rewrite  $n$  as:

$$n = m + (2 \cdot 5 \cdot 7)k = 2p + (2 \cdot 5 \cdot 7)k = 2(p + (5 \cdot 7)k),$$

that is, 2 divides  $n$ , so it is not relatively prime to 49000 (similarly, if 5 or 7 divide  $m$ , then 5 or 7 also divide  $n$ , respectively). It follows that the generators for  $\mathbb{Z}/49000\mathbb{Z}$  between 0 and 69 repeat  $(\bmod 70)$  over the rest of 49000. Since  $49000/70 = 700$ , there are thus  $700 \cdot 24 = 16800$  generators for  $\mathbb{Z}/49000\mathbb{Z}$ .  $\square$

## 6. (6/20/23)

In  $\mathbb{Z}/48\mathbb{Z}$  write out all elements of  $\langle \bar{a} \rangle$  for every  $\bar{a}$ . Find all inclusions between subgroups in  $\mathbb{Z}/48\mathbb{Z}$ .

- Subgroup of order 48:  $\langle \bar{1} \rangle = \langle \bar{5} \rangle = \langle \bar{7} \rangle = \langle \bar{11} \rangle = \langle \bar{13} \rangle = \langle \bar{17} \rangle = \langle \bar{19} \rangle = \langle \bar{23} \rangle = \langle \bar{25} \rangle = \langle \bar{29} \rangle = \langle \bar{31} \rangle = \langle \bar{35} \rangle = \langle \bar{37} \rangle = \langle \bar{41} \rangle = \langle \bar{43} \rangle = \langle \bar{47} \rangle$ .
- Subgroup of order 24:  $\langle \bar{2} \rangle = \langle \bar{10} \rangle = \langle \bar{14} \rangle = \langle \bar{22} \rangle = \langle \bar{26} \rangle = \langle \bar{34} \rangle = \langle \bar{38} \rangle = \langle \bar{46} \rangle$ .
- Subgroup of order 16:  $\langle \bar{3} \rangle = \langle \bar{9} \rangle = \langle \bar{15} \rangle = \langle \bar{21} \rangle = \langle \bar{27} \rangle = \langle \bar{33} \rangle = \langle \bar{39} \rangle = \langle \bar{45} \rangle$ .
- Subgroup of order 12:  $\langle \bar{4} \rangle = \langle \bar{20} \rangle = \langle \bar{28} \rangle = \langle \bar{44} \rangle$ .
- Subgroup of order 8:  $\langle \bar{6} \rangle = \langle \bar{18} \rangle = \langle \bar{30} \rangle = \langle \bar{42} \rangle$ .
- Subgroup of order 6:  $\langle \bar{8} \rangle = \langle \bar{40} \rangle$ .
- Subgroup of order 4:  $\langle \bar{12} \rangle = \langle \bar{36} \rangle$ .
- Subgroup of order 3:  $\langle \bar{16} \rangle = \langle \bar{32} \rangle$ .
- Subgroup of order 2:  $\langle \bar{24} \rangle$ .
- Subgroup of order 1, the trivial subgroup:  $\{0\}$ .

Among these subgroups, all contain the trivial subgroup. The subgroups of order 2 and 3 are distinct, but both are contained in the subgroup of order 6. The subgroup of order 2 is also contained in the subgroup of order 4. The subgroups of order 4 and 6 are both contained in the subgroup of order 12. The subgroup of order 4 is also contained in the subgroup of order 8. The subgroups of order 8 and 12 are both contained in the subgroup of order 24. The subgroup of order 8 is also contained in the subgroup of order 16.

## 7. (6/22/23)

Let  $Z_{48} = \langle x \rangle$  and use the isomorphism  $\mathbb{Z}/48\mathbb{Z} \cong Z_{48}$  given by  $\bar{1} \mapsto x$  to list all subgroups of  $Z_{48}$  as computed in the preceding exercise.

- Subgroup of order 48:  $\{1, x, x^2, \dots, x^{47}\}$ .
- Subgroup of order 24:  $\{1, x^2, x^4, \dots, x^{46}\}$ .
- Subgroup of order 16:  $\{1, x^3, x^6, \dots, x^{45}\}$ .
- Subgroup of order 12:  $\{1, x^4, x^8, \dots, x^{44}\}$ .
- Subgroup of order 8:  $\{1, x^6, x^{12}, x^{18}, x^{24}, x^{30}, x^{36}, x^{42}\}$ .
- Subgroup of order 6:  $\{1, x^8, x^{16}, x^{24}, x^{32}, x^{40}\}$ .
- Subgroup of order 4:  $\{1, x^{12}, x^{24}, x^{36}\}$ .
- Subgroup of order 3:  $\{1, x^{16}, x^{32}\}$ .
- Subgroup of order 2:  $\{1, x^{24}\}$ .
- Subgroup of order 1, the trivial subgroup:  $\{1\}$ .

## 8. (6/23/23)

Let  $Z_{48} = \langle x \rangle$ . For which integers  $a$  does the map  $\varphi_a$  defined by  $\varphi_a : \bar{1} \mapsto x^a$  extend to an *isomorphism* from  $\mathbb{Z}/48\mathbb{Z}$  onto  $Z_{48}$ ?

*Proof.* We will show that  $\varphi_a$  is an isomorphism from  $\mathbb{Z}/48\mathbb{Z}$  onto  $Z_{48}$  if and only if  $a \in \mathbb{Z}$  is relatively prime to 48.

First, let  $m, n \in \mathbb{Z}/48\mathbb{Z}$ . Then  $\varphi_a(m)\varphi_a(n) = (x^a)^m(x^a)^n = (x^a)^{m+n} = \varphi_a(m+n)$ . So  $\varphi_a$  is a homomorphism.

Next,  $\varphi_a$  is one-to-one. Let  $\varphi_a(n) = \varphi_a(m)$  for  $m, n \in \mathbb{Z}/48\mathbb{Z}$ . Then  $(x^a)^m = (x^a)^n \Rightarrow x^{am} = x^{an}$ , and so  $am = an \pmod{48}$ . Since  $a$  is relatively prime to 48, we must therefore have  $m = n$ , and it follows that  $\varphi_a$  is injective. (Note, however, that if  $k > 1$  divides both  $a$  and 48, then  $am = an$  does not imply that  $m = n$ , and  $\varphi_a$  is therefore not injective. For example, if  $a = 14$ , then  $\varphi_a(7) = (x^{14})^7 = x^{98} = x^2$  and  $\varphi_a(31) = (x^{14})^{31} = x^{434} = x^2$ ).

Finally,  $\varphi_a$  is onto. Let  $x^b \in Z_{48}$ . Suppose there exists some  $n \in \mathbb{Z}/48\mathbb{Z}$  such that  $\varphi_a(n) = x^b$ , that is,  $(x^a)^n = x^b$ . Then we must have  $an = b \pmod{48}$ . Since  $a$  is relatively prime to 48, any integer between 0 and 47 can be written as  $an$  for some  $n \in \mathbb{Z}/48\mathbb{Z}$ , and so  $\varphi_a$  is onto.

Thus for  $a$  relatively prime to 48,  $\varphi_a : \bar{1} \mapsto x^a$  is an isomorphism from  $\mathbb{Z}/48\mathbb{Z}$  onto  $Z_{48}$ .  $\square$

## 9. (7/2/23)

Let  $Z_{36} = \langle x \rangle$ . For which integers  $a$  does the map  $\varphi_a$  defined by  $\varphi_a : \bar{1} \mapsto x^a$  extend to a *well defined homomorphism* from  $\mathbb{Z}/48\mathbb{Z}$  onto  $Z_{36}$ ? Can  $\varphi_a$  ever be a surjective homomorphism?

*Proof.* We will show that  $\varphi_a : \mathbb{Z}/48\mathbb{Z} \rightarrow Z_{36}$  is a well defined homomorphism if and only if  $a$  is a multiple of 3.

For  $\varphi_a$  to be a homomorphism, we must have  $\varphi_a(b)\varphi_a(c) = \varphi_a(b+c)$  for all  $b, c \in \mathbb{Z}/48\mathbb{Z}$ . Now  $\varphi_a(b)\varphi_a(c) = (x^a)^b(x^a)^c = (x^a)^{b+c} = x^{a(b+c)}$  and  $\varphi_a(b+c) = (x^a)^{b+c} = x^{a(b+c)}$ . Superficially these appear identical already. However, note that in  $\varphi_a(b)\varphi_a(c)$  we compute  $ab + ac \pmod{36}$ , while in  $\varphi_a(b+c)$  we first take  $b+c \pmod{48}$  before then computing  $a(b+c)$ . That is,  $a$  must satisfy

$$a(b+c \pmod{48}) \pmod{36} = a(b+c) \pmod{36}$$

for all  $b, c \in \mathbb{Z}/48\mathbb{Z}$ . If  $b+c < 48$ , then the two are equal for all  $a \in \mathbb{Z}$ . So suppose that  $b+c \geq 48$ . Then  $b+c \pmod{48} = b+c-48$ , so we must have

$$a(b+c-48) \pmod{36} = a(b+c) \pmod{36}$$

$$ab + ac - 48a \pmod{36} = ab + ac \pmod{36}$$

$$-48a \pmod{36} = 0 \pmod{36}$$

$$-48a \cong 36 \Rightarrow 48a \cong 36,$$

that is,  $a$  is some integer which, when multiplied by 48, results in a multiple of 36. Writing 48 as the product of its prime factors gives  $2^4 \cdot 3$ , while  $36 = 2^2 \cdot 3^2$ . Note that 36 has one more factor of 3, and so when  $a$  is a multiple of 3,  $48a$  will be a multiple of 36. Only these values satisfy the exponents in the equation above, and thus  $\varphi_a$  is a homomorphism if and only if  $a$  is a multiple of 3.

It is not possible for  $\varphi_a$  to be a surjective homomorphism. Because  $a$  must be a multiple of 3, we have  $\varphi_a(1) = x^a = x^{3n} = (x^3)^n$  for some  $n \in \mathbb{Z}$ . In turn,  $\varphi_a$  generates only the values  $\varphi_a(2) = (x^6)^n, \varphi_a(3) = (x^9)^n, \dots$ , that is, it only generates powers of  $x^3$  in  $Z_{36}$ . By counterexample, there is no value in  $\mathbb{Z}/48\mathbb{Z}$  whose image under  $\varphi_a$  is  $x$ , and so  $\varphi_a$  cannot be surjective.  $\square$

## 10. (7/2/23)

What is the order of  $\overline{30}$  in  $\mathbb{Z}/54\mathbb{Z}$ ? Write out all the elements and their orders in  $\langle \overline{30} \rangle$ .

*Proof.* First, the group  $\langle \overline{30} \rangle$  (ordered by multiples of  $\overline{30}$ ) consists of the elements  $\{0, 30, 6, 36, 12, 42, 18, 48, 24\}$ . This implies that the order of  $\overline{30} = |\langle \overline{30} \rangle| = 9$ .

The orders of each of the elements of  $\langle \overline{30} \rangle$  are:

- 0: 1
- 6: 9
- 12: 9
- 18: 3
- 24: 9
- 30: 9
- 36: 3
- 42: 9
- 48: 9

$\square$

## 11. (7/2/23)

Find all cyclic subgroups of  $D_8$  Find a proper subgroup of  $D_8$  which is not cyclic.

*Proof.* Recall that  $D_8 = \{1, r, r^2, r^3, s, sr, sr^2, sr^3\}$ . A cyclic subgroup of  $D_8$  must be generated by one element, so it cannot contain both  $s$  and a multiple of  $r$ . Therefore the cyclic subgroups of  $D_8$  are:

- $\langle 1 \rangle = \{1\}$
- $\langle r \rangle = \langle r^3 \rangle = \{1, r, r^2, r^3\}$

- $\langle r^2 \rangle = \{1, r^2\}$
- $\langle s \rangle = \{1, s\}$

The group  $D_8$  also contains as a subgroup  $\{1, r^2, s, sr^2\}$ , which is generated by the two elements  $r^2$  and  $s$ , and is therefore not cyclic.  $\square$

## 12. (7/2/23)

Prove that the following groups are *not* cyclic:

- (a)  $Z_2 \times Z_2$

*Proof.* This group consists of the elements  $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ . So each non-identity element has order 2, and there is no element of order 4 (the size of the group). Therefore it is not generated by any single element, and so it is not a cyclic group.  $\square$

- (b)  $Z_2 \times \mathbb{Z}$

*Proof.* Now  $Z_2 \times \mathbb{Z} = \{(a, b) \mid a = 0 \text{ or } 1, b \in \mathbb{Z}\}$ . So a generating element must be of the form  $(0, b)$  or  $(1, b)$ . Elements of the form  $(0, b)$  can only generate  $(0, 2b), (0, 3b), \dots$  but never  $(1, nb)$ , so a generating element must be of the form  $(1, b)$ . Multiples of  $(1, b)$  include  $(0, 2b), (1, 3b), (0, 4b), \dots$ , that is,  $(0, nb)$  and  $(1, mb)$  for even  $n$  and odd  $m$ , respectively. However, then this element cannot generate  $(1, nb)$ , and so it is not a generating element. Since both candidates fail to generate the group, it is not cyclic.  $\square$

- (c)  $\mathbb{Z} \times \mathbb{Z}$

*Proof.* Similar to  $Z_2 \times \mathbb{Z}$ , consider a generating element of  $\mathbb{Z} \times \mathbb{Z}$ ,  $(a, b)$ . Multiples of this element include  $(2a, 2b), (3a, 3b), \dots$ , that is,  $(na, nb)$  for  $n \in \mathbb{Z}$ . However, this element cannot generate  $(a, nb)$  (where  $n \neq 1$ ), and so it is not a generating element. Since all elements of  $\mathbb{Z} \times \mathbb{Z}$  are of this form, there is no generating element, and so the group is not cyclic.  $\square$