Dummit & Foote Ch. 1.5: The Quaternion Group

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Mar. 2023

1. (3/24/23)

Compute the order of each of the elements of Q_8.

Element	1	-1	i	$-i$	j	$-j$	k	$-k$
Order	1	2	4	4	4	4	4	4

2. (3/24/23)

Write out the group tables for S_3, D_8 and Q_8.

1	$(1,2)$	$(1,3)$	$(2,3)$	$(1,2,3)$	$(1,3,2)$
$(1,2)$	1	$(1,3,2)$	$(1,2,3)$	$(2,3)$	$(1,3)$
$(1,3)$	$(1,2,3)$	1	$(1,3,2)$	$(1,2)$	$(2,3)$
$(2,3)$	$(1,3,2)$	$(1,2,3)$	1	$(1,3)$	$(1,2)$
$(1,2,3)$	$(1,3)$	$(2,3)$	$(1,2)$	$(1,3,2)$	1
$(1,3,2)$	$(2,3)$	$(1,2)$	$(1,3)$	1	$(1,2,3)$

1	r	r^2	r^3	s	sr	sr^2	sr^3
r	r^2	r^3	1	sr^3	s	sr	sr^2
r^2	r^3	1	r	sr^2	sr^3	s	sr
r^3	1	r	r^2	sr	sr^2	sr^3	s
s	sr	sr^2	sr^3	1	r	r^2	r^3
sr	sr^2	sr^3	s	r^3	1	r	r^2
sr^2	sr^3	s	sr	r^2	r^3	1	r
sr^3	s	sr	sr^2	r	r^2	r^3	1

1	-1	i	$-i$	j	$-j$	k	$-k$
-1	1	$-i$	i	$-j$	j	$-k$	k
i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	i	1	-1	$-k$	k	j	$-j$
j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	j	k	$-k$	1	-1	$-i$	i
k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	k	$-j$	j	i	$-i$	1	-1

3. (3/24/23)

Find a set of generators and relations for Q_8.

Proof. $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$.

Consider the presentation $\langle a, b, c \mid b^2 = c^2 = a, a^2 = 1, cb = abc \rangle$. If we replace the given generators with $a = -1, b = i, c = j$ (which satisfy the given relations), then we can show that every element of Q_8 is indeed generated: $1 = a^2, -1 = a, i = b, -i = ab, j = c, -j = ac, k = bc, -k = abc$. Thus, the presentation generates Q_8.

However, it remains to be shown that these generators and relations are not the presentation for some larger group that contains Q_8 as a subgroup. Consider an arbitrary product of a, b, c. a commutes with b:

$$b^2 = a \Rightarrow ab^2 = b^2a = 1 \Rightarrow (ab)b = b(ba) = 1 \Rightarrow b = (ab)^{-1} = (ba)^{-1} \Rightarrow ab = ba$$

and the same logic shows that a and c also commute. Therefore we can rewrite an arbitrary product of a, b, c so that all the a factors are at the start (ex. $b^{n_1}a^{n_2}c^{n_3}a^{n_4} = a^{n_2+n_4}b^{n_1}c^{n_3}$). Further, because $a^2 = 1$, this initial a^n can be reduced to either a (if n odd) or removed (if n even).

So any unique element generated must have the form $a^p b^{n_1} c^{n_2} ... b^{n_k} c^{n_{k+1}}$, $p \in \{0, 1\}$. Next, we can rewrite any arbitrary product of factors b and c by replacing any cb with abc and reducing so that the element has the form $a^p b^q c^r$. Further, while we already have $p \in \{0, 1\}$, we must also have $q, r \in \{0, 1\}$, because $b^2 = c^2 = a$. So the only elements that can be generated are $1, a, b, c, ab, ac, bc, abc$. By replacing a with -1, b with i, and c with j, we have $1, -1, i, -i, j, -j, ij, -ij$, and if we let $k = ij$, then we have precisely Q_8. $\square$