Dummit & Foote Ch. 2.2: Centralizers and Normalizers, Stabilizers and Kernels

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1. (6/5/23)

Prove that $C_G(A) = \{g \in G \mid g^{-1}ag = a \text{ for all } a \in A\}$.

Proof. By definition, $C_G(A) = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$ (that is, it is the set of elements of G that commute with all elements of A).

Let $g \in C_G(A), a \in A$. Then $gag^{-1} = a$, which implies that $ga = ag$, and so left-multiplying by g^{-1} we obtain $a = g^{-1}ag$. Therefore, equivalently, $C_G(A)$ is the set of elements $g \in G$ such that $g^{-1}ag = a$ for all $a \in A$. $\square$

2. (6/5/23)

Prove that $C_G(Z(G)) = G$ and deduce that $N_G(Z(G)) = G$.

Proof. Recall that $Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$. Let $z \in Z(G)$, so z commutes with every element of G.

Also recall that $C_G(A) = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$. When $A = Z(G)$, then every element of g commutes with every element of A. Therefore for all $g \in G$, $g \in C_G(Z(G))$. Thus $C_G(Z(G)) = G$.

Note that, since $C_G(A) \leq N_G(A)$ for all subsets A, we must have $G = C_G(Z(G)) \leq N_G(Z(G))$. Since there is no greater set of elements, we also have $N_G(Z(G)) = G$. $\square$