Dummit & Foote Ch. 3.5: Transpositions and the Alternating Group

Scott Donaldson

Dec. 2023

1. (12/6/23)

In Exercises 1 and 2 of Section 1.3 you were asked to find the cycle decompositions of some permutations. Write each of these permutations as a product of transpositions. Determine which of these is an even permutation and which is an odd permutation.

In Exercise 1,

$$\begin{aligned}
\sigma &= (1,3,5)(2,4) = (1,3)(1,5)(2,4), \text{ odd.}\\
\tau &= (1,5)(2,3), \text{ even.}\\
\sigma^2 &= (1,5,3) = (1,3)(1,5), \text{ even.}\\
\sigma\tau &= (2,5,3,4) = (2,4)(2,3)(2,5), \text{ odd.}\\
\tau^2\sigma &= (1,3,5)(2,4) = (1,5)(1,3)(2,4), \text{ odd.}
\end{aligned}$$

In Exercise 2,

$$\begin{aligned}
\sigma &= (1,13,5,10)(3,15,8)(4,14,11,7,12,9)\\
&= (1,10)(1,5)(1,13)(3,8)(3,15)(4,9)(4,12)(4,7)(4,11)(4,14), \text{ even.}\\
\tau &= (1,14)(2,9,15,13,4)(3,10)(5,12,7)(8,11)\\
&= (1,14)(2,4)(2,13)(2,15)(2,9)(3,10)(5,7)(5,12)(8,11), \text{ odd.}\\
\sigma^2 &= (1,5)(3,8,15)(4,11,12)(7,9,4)(10,13)\\
&= (1,15)(3,15)(3,8)(4,12)(4,11)(7,4)(7,9)(10,13), \text{ even.}\\
\sigma\tau &= (1,11,3)(2,4)(5,9,8,7,10,15)(13,14)\\
&= (1,3)(1,11)(2,4)(5,15)(5,10)(5,7)(5,8)(5,9)(13,14), \text{ odd.}\\
\tau\sigma &= (1,4)(2,9)(3,13,12,15,11,5)(8,10,14)\\
&= (1,4)(2,9)(3,5)(3,11)(3,15)(3,12)(3,13)(8,14)(8,10), \text{ odd.}\\
\tau^2\sigma &= (1,2,15,8,3,4,14,11,12,13,7,5,10)\\
&= (1,10)(1,5)(1,7)(1,13)(1,12)(1,11)(1,14)(1,4)(1,3)(1,8)(1,15)(1,2),\\
&\quad \text{even.}
\end{aligned}$$

2. (12/6/23)

Prove that σ^2 is an even permutation for every permutation σ.

Proof. We take as given the homomorphism $\epsilon : S_n \to \{\pm 1\}$ defined in this chapter, which determines the sign of every permutation $\sigma \in S_n$.

If σ is an even permutation, then $\epsilon(\sigma) = 1$. It follows that:

$$\epsilon(\sigma^2) = \epsilon(\sigma)\epsilon(\sigma) = 1 \cdot 1 = 1,$$

and so σ^2 is an even permutation.

If σ is an odd permutation, then $\epsilon(\sigma) = -1$. It follows that:

$$\epsilon(\sigma^2) = \epsilon(\sigma)\epsilon(\sigma) = -1 \cdot -1 = 1,$$

and so σ^2 is an even permutation.

Since for every $\sigma \in S_n$, σ is either an even or an odd permutation, this proves that σ^2 is an even permutation for every permutation σ. $\square$

3. (12/6/23)

Prove that S_n is generated by $\{(i, i+1) \mid 1 \leq i \leq n-1\}$.

Proof. Since any element of S_n may be written as a product of transpositions, it suffices to show that the set $\{(i, i+1) \mid 1 \leq i \leq n-1\}$ can generate any transposition. Writing an arbitrary transposition in S_n as $(i, i+a)$, we will prove this by strong induction on a (where $1 \leq a \leq n-i$).

The base case $a = 1$ is given, since $(i, i+1)$ is a member of the generating set for all $i \in \{1, ..., n-1\}$.

Next, suppose that for all $i \in \{1, ..., n-1\}$ and $a \in \{1, ..., n-i\}$, the transposition $(i, i+a-1)$ can be obtained from the generating set. So we have the transpositions $(i+a-1, i+a)$ (in the generating set) and $(i, i+a-1)$ (from the inductive hypothesis). Then:

$$(i+a-1, i+a)(i, i+a-1)(i+a-1, i+a) = (i, i+a),$$

so we can obtain the transposition $(i, i+a)$. This concludes the proof that the set $\{(i, i+1) \mid 1 \leq i \leq n-1\}$ can generate any transposition, and therefore generates all of S_n. $\square$

4. (12/7/23)

Show that $S_n = \langle (1, 2), (1, 2, 3, ..., n) \rangle$ for all $n \geq 2$.

Proof. Note that:

$$\begin{aligned}
&(1,2,3,...,n)(1,2)(1,2,3,...,n)^{-1} \\
&= (1,2,3,...,n)(1,2)(1,n,n-1,...,2) \\
&= (2,3),
\end{aligned}$$

and in general,

$$\begin{aligned}
&(1,2,3,...,n)(i,i+1)(1,2,3,...,n)^{-1} \\
&= (1,2,3,...,n)(i,i+1)(1,n,n-1,...,2) \\
&= (i+1,i+2)
\end{aligned}$$

for $1 \leq i \leq n-1$ (if $i = n-1$, then the resulting transposition is equal to $(1,n)$).

This shows that every transposition of adjacent integers can be obtained from $\langle(1,2),(1,2,3,...,n)\rangle$, and from the results of Exercise 3, it therefore generates all of S_n. □

5. (12/7/23)

Show that if p is prime, $S_p = \langle\sigma,\tau\rangle$ where σ is any transposition and τ is any p-cycle.

Proof. Let $\tau = (a_1,a_2,...,a_p)$ and $\sigma = (a_i,a_{i+k})$, where $1 \leq i < p$ and $i < k \leq p-i$. Note that:

$$\begin{aligned}
\tau\sigma\tau^{-1} &= (a_1,a_2,...,a_p)(a_i,a_{i+k}) \\
&\quad\ (a_1,a_p,a_{p-1},...,a_2) \\
&= (a_{i+1},a_{i+k+1}), \text{ and so:} \\
(\tau^2)\sigma(\tau^2)^{-1} = \tau(\tau\sigma\tau^{-1})\tau^{-1} &= (a_1,a_2,...,a_p)(a_{i+1},a_{i+k+1}) \\
&\quad\ (a_1,a_p,a_{p-1},...,a_2) \\
&= (a_{i+2},a_{i+k+2}), \text{ and in general:} \\
(\tau^n)\sigma(\tau^n)^{-1} = \tau((\tau^{n-1})\sigma(\tau^{n-1})^{-1})\tau^{-1} &= (a_1,a_2,...,a_p)(a_{i+n-1},a_{i+k+n-1}) \\
&\quad\ (a_1,a_p,a_{p-1},...,a_2) \\
&= (a_{i+n},a_{i+k+n}),
\end{aligned}$$

where all subscripts are taken mod p if they are greater than p.

Next, we define a set:

$$\begin{aligned}
\Sigma &= \{(\tau^n)\sigma(\tau^n)^{-1} \mid 0 \leq n < p\} \\
&= \{(a_j,a_{j+k}) \mid 1 \leq j \leq p\}.
\end{aligned}$$

Clearly Σ is generated by σ and τ. We claim that Σ generates any transposition of the form (a_j,a_{j+nk}), where $1 \leq j \leq p$, $n \geq 1$. We will show this by strong induction on n.

The base case $n = 1$ is given by the construction of Σ, since it contains all transpositions of the form (a_j, a_{j+k}).

Next, suppose that Σ can generate any transposition of the form (a_j, a_{j+mk}), where $1 \leq m < n$. Then:

$$\underbrace{(a_j, a_{j+(n-1)k})}_{m=n-1}\underbrace{(a_{j+(n-1)k}, a_{j+nk})}_{m=1}\underbrace{(a_j, a_{j+(n-1)k})}_{m=n-1} = (a_j, a_{j+nk}),$$

which shows that we can generate any transposition of the form (a_j, a_{j+nk}).

Now since p is prime, for any transposition (a_j, a_{j+q}), we can write $q = nk \mod p$ for some $n \geq 1$. Therefore Σ can generate any transposition in S_p, and it therefore generates all of S_p. $\square$

6. (12/7/23)

Show that $\langle (1,3), (1,2,3,4) \rangle$ is a proper subgroup of S_4. What is the isomorphism type of this subgroup?

Proof. First, we will define a map $\varphi : D_8 \to \langle (1,3), (1,2,3,4) \rangle$ and show that it is an isomorphism. Since the order of D_8 is strictly less than S_4, we will conclude that $\langle (1,3), (1,2,3,4) \rangle$ is a proper subgroup of S_4.

Define φ such that $\varphi(s) = (1,3)$ and $\varphi(r) = (1,2,3,4)$. We will first show that φ is a homomorphism. The orders of s and r hold under φ, since $s^2 = 1$ and $(1,3)^2 = (1)$, and $r^4 = 1$ and $(1,2,3,4)^4 = (1)$. Also, the relation in D_8 that $sr = r^{-1}s$ holds under φ:

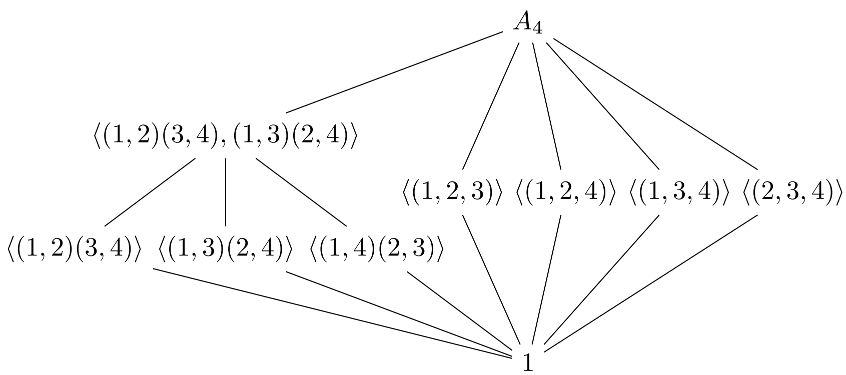
$$\varphi(s)\varphi(r) = (1,3)(1,2,3,4) = (1,2)(3,4) = (1,4,3,2)(1,3) = \varphi(r)^{-1}\varphi(s).$$

Since φ is defined on the generators of D_8 to the generators $(1,3)$ and $(1,2,3,4)$, φ is surjective.

We next show that $\langle (1,3), (1,2,3,4) \rangle$ contains 8 elements. The cyclic group generated by $(1,2,3,4)$ contains 4 elements. Its left and right cosets with $(1,3)$ are equal to each other, so there are therefore no other elements that can be generated. Since $|\langle (1,3), (1,2,3,4) \rangle| = |D_8|$ and there exists a surjective homomorphism between them, φ is necessarily an isomorphism, so $\langle (1,3), (1,2,3,4) \rangle \cong D_8$. We conclude that it is a proper subgroup of S_4. $\square$

8. (12/8/23)

Prove the lattice of subgroups of A_4 given in this text is correct.



Proof. The alternating group A_4 has order $|S_4|/2 = 12$. By Lagrange's Theorem, its proper subgroups must have order 2, 3, 4, or 6.

It contains no subgroups generated by a single transposition, e.g. $\langle(1,2)\rangle$, since these contain odd permutations. The other subgroups generated by an element of order 2 are all shown above.

The lattice also contains all subgroups generated by a single 3-cycle, e.g. $\langle(1,2,3)\rangle$. There might be a proper subgroup of order 6 containing one of these. However, as we will show in Exercises 14 and 15, the join of $\langle(1,2,3)\rangle$ with another 3-cycle or with a pair of disjoint transpositions produces all of A_4. Since there are no other permutations in A_4, this implies that there is no proper subgroup containing the cyclic group generated by a 3-cycle.

Finally, the join of two order 2 subgroups produces $\langle(1,2)(3,4),(1,3)(2,4)\rangle$. Since this subgroup is of index 3 in A_4, there are no other subgroups of A_4, and thus the lattice displayed above is correct and complete. $\square$

9. (12/8/23)

Prove that the (unique) subgroup of order 4 in A_4 is normal and is isomorphic to V_4.

Proof. From above, the subgroup $\langle(1,2)(3,4),(1,3)(2,4)\rangle$ is the only subgroup of order 4 in A_4. Its generators are both elements of order 2. Since the cyclic group Z_4 contains only one element of order 2, it is not isomorphic to Z_4. There are only two groups of order 4 up to isomorphism, and therefore it is isomorphic to V_4.

Next, it is normal in A_4. We consider the conjugate of its generators by (without loss of generality) the permutation $(1,2,3)$:

$$\begin{aligned}&(1,2,3)(1,2)(3,4)(1,3,2) = (1,4)(2,3), \text{ and}\\ &(1,2,3)(1,3)(2,4)(1,3,2) = (1,2)(3,4),\end{aligned}$$

both of which lie in $\langle(1,2)(3,4),(1,3)(2,4)\rangle$. Thus $\langle(1,2)(3,4),(1,3)(2,4)\rangle$ is normal in A_4. $\square$

10. (12/8/23)

Find a composition series for A_4. Deduce that A_4 is solvable.

Solution.

$$1 \leq \langle(1,2)(3,4)\rangle \leq \langle(1,2)(3,4),(1,3)(2,4)\rangle \leq A_4$$

is a composition series for A_4. The lower two quotient groups are isomorphic to Z_2, a simple group, and $|A_4 : \langle(1,2)(3,4),(1,3)(2,4)\rangle| = 3$, which implies that the last quotient group is isomorphic to Z_3, also simple. Since these quotient groups are also abelian, this implies that A_4 is solvable. □

11. (12/12/23)

Prove that S_4 has no subgroup isomorphic to Q_8.

Proof. Suppose that $A \leq S_4$ and that $\varphi : Q_8 \to A$ is an isomorphism. In Q_8, $|i| = 4$, so φ must assign i to a permutation whose cycle decomposition is a 4-cycle. Without loss of generality, suppose that $\varphi(i) = (1,2,3,4)$.

Because φ is injective, we cannot have $\varphi(j) = (1,2,3,4)$. Also, $(1,4,3,2) = (1,2,3,4)^{-1}$, and since $j \neq -i$, we cannot have $\varphi(j) = (1,4,3,2)$, so $\varphi(j)$ must equal some other 4-cycle in S_4. The remaining options are:

$$\varphi(j) = (1,2,4,3),(1,3,2,4),(1,3,4,2), \text{ or } (1,4,2,3).$$

Note that, in Q_8, $i^2 = j^2 = -1$. Under φ, we have $\varphi(i)^2 = (1,3)(2,4)$. However, for none of the remaining 4-cycles we might assign j to do we have $\varphi(j)^2 = (1,3)(2,4)$. Thus there is no element to which we can assign j and have φ be an isomorphism. Therefore there S_4 has no subgroup isomorphic to Q_8. □

12. (12/12/23)

Prove that A_n contains a subgroup isomorphic to S_{n-2} for each $n \geq 3$.

Proof. We define a map $\varphi : S_{n-2} \to A_n$ by:

$$\varphi(\sigma) = \begin{cases} \sigma & \text{if } \sigma \text{ is even} \\ \sigma \cdot (n-1,n) & \text{if } \sigma \text{ is odd} \end{cases}.$$

Now noting that $\frac{1}{2}n(n-1) > 1$ for all $n \leq 3$, we conclude that:

$$\frac{1}{2}n(n-1) > 1$$
$$\frac{1}{2}n(n-1)(n-2) \cdot ... \cdot 2 \cdot 1 > (n-2) \cdot ... \cdot 2 \cdot 1$$
$$\frac{1}{2}(n!) > (n-2)!$$
$$|A_n| > S_{n-2}.$$

Since the order of A_n is strictly greater than that of S_{n-2}, φ cannot be surjective. It is trivial to show that it is injective, and so if φ is a homomorphism, then its image is a proper subgroup of A_n isomorphic to S_{n-2}.

Let $\sigma_1, \sigma_2 \in S_{n-2}$ and consider the different cases:

- Both even permutations. Then $\sigma_1\sigma_2$ is even, so:

$$\varphi(\sigma_1\sigma_2) = \sigma_1\sigma_2 = \varphi(\sigma_1)\varphi(\sigma_2)$$

- Both odd permutations. Then $\sigma_1\sigma_2$ is even. Note that each $\sigma \in S_{n-2}$ is disjoint with the transposition $(n-1, n)$, and so commutes with it in A_n. Therefore:

$$\begin{aligned}
\varphi(\sigma_1)\varphi(\sigma_2) &= \sigma_1 \cdot (n-1, n) \cdot \sigma_2 \cdot (n-1, n) \\
&= \sigma_1\sigma_2(n-1, n)(n-1, n) \\
&= \sigma_1\sigma_2, \text{ and} \\
\varphi(\sigma_1\sigma_2) &= \sigma_1\sigma_2.
\end{aligned}$$

- One even, one odd. Let σ_1 be an even permutation and σ_2 be odd (and their product is odd). Then:

$$\begin{aligned}
\varphi(\sigma_1\sigma_2) &= \sigma_1\sigma_2 \cdot (n-1, n), \text{ and} \\
\varphi(\sigma_1)\varphi(\sigma_2) &= \sigma_1\sigma_2 \cdot (n-1, n).
\end{aligned}$$

This proves that φ is a homomorphism, and since it is injective but not surjective, its image is a subgroup of A_n that is isomorphic to S_{n-2}. □

13. (12/13/23)

Prove that every element of order 2 in A_n is the square of an element of order 4 in S_n. [An element of order 2 in A_n is a product of $2k$ commuting transpositions.]

Proof. From Chapter 1.3, Exercise 15, the order of a permutation is equal to the least common multiple of the lengths of cycles in its cycle decomposition. Therefore an element of order 2 in $A_n \leq S_n$ must have a cycle decomposition with only 2-cycles, that is, it must be the product of disjoint transpositions.

Let $\sigma \in A_n$ have order 2 with the cycle decomposition:

$$(a_1, b_1)(c_1, d_1)...(a_k, b_k)(c_k, d_k).$$

Then σ is the square of the permutation in S_n with the cycle decomposition:

$$(a_1, c_1, b_1, d_1)...(a_k, c_k, b_k, d_k).$$

Since all of these cycles are disjoint, the permutation has order 4, so every element of order 2 in A_n is the square of an element of order 4 in S_n. □

14. (12/13/23)

Prove that the subgroup of A_4 generated by any element of order 2 and any element of order 3 is all of A_4.

Proof. Without loss of generality, we consider the subgroups generated by an arbitrary element of order 3 and $(1,2)(3,4) \in A_4$. We claim that the product of $(1,2)(3,4)$ and σ, a 3-cycle is always another 3-cycle that is not the inverse of σ:

$$\begin{aligned}(1,2)(3,4) \cdot (1,2,3) &= (2,4,3)\\ (1,2)(3,4) \cdot (1,2,4) &= (2,3,4)\\ (1,2)(3,4) \cdot (1,3,4) &= (1,4,2)\\ (1,2)(3,4) \cdot (2,3,4) &= (1,2,4).\end{aligned}$$

For each of the four 3-cycles on the right-hand side of the equation, multiplying them on the left by $(1,2)(3,4)$ produces the 3-cycle on the left-hand side of the equation.

Now the generated subgroup contains the the identity, $(1,2)(3,4)$, and two distinct 3-cycles (as well as their inverses), for a total of 6 elements. From the table above, left-multiplying one of the inverses of the 3-cycles by $(1,2)(3,4)$ produces yet another 3-cycle, so the subgroup contains at least 7 elements. By Lagrange's Theorem, its order must divide $|A_4| = 12$. Therefore, its order must be 12, that is, all of A_4. □

15. (12/14/23)

Prove that if x and y are distinct 3-cycles in S_4 with $x \neq y^{-1}$, then the subgroup of S_4 generated by x and y is A_4.

Proof. Without loss of generality, let $x = (1,2,3)$. Then y may be:

$$(1,2,4), (1,4,2), (1,3,4), (1,4,3), (2,3,4), \text{ or } (2,4,3).$$

For all x and y, $\langle x, y\rangle = \langle x, y^{-1}\rangle$, so (for example) if we prove that $(1,2,3)$ and $(1,2,4)$ generate A_4, we conclude that $(1,2,3)$ and $(1,4,2) = (1,2,4)^{-1}$ do as well.

Consider the options for y:

- $y = (1,2,4)$: Then $xy = (1,2,3)(1,2,4) = (1,3)(2,4)$, so from Exercise 14, we generate all of A_4.
- $y = (2,3,4)$: Then $xy = (1,2,3)(2,3,4) = (1,2)(3,4)$, so from Exercise 14, we generate all of A_4.
- $y = (1,3,4)$: Then $xy = (1,2,3)(1,3,4) = (2,3,4)$, so from the above case, we generate all of A_4.

Thus any two distinct 3-cycles in S_4 that are not each other's inverse generate A_4. □

16. (12/15/23)

Let x and y be distinct 3-cycles in S_5 with $x \neq y^{-1}$.

(a) Prove that if x and y fix a common element of $\{1, ..., 5\}$ then $\langle x, y\rangle \cong A_4$.

Proof. Without loss of generality let $x = (1, 2, 3)$ and suppose that x and y both fix 5. The possible 3-cycles y may be either assign one element of $\{1, ..., 5\}$ to the same element or assign none of the elements to the same element. So we only need to consider $y = (1, 2, 4)$ (both assign 1 to 2) or $y = (1, 4, 2)$ (assign none to the same).

- $y = (1, 2, 4)$: Then $xy = (1, 2, 3)(1, 2, 4) = (1, 3)(2, 4)$, so from Exercise 14, they generate A_4.
- $y = (1, 4, 2)$: Then $x^{-1}y = (1, 3, 2)(1, 4, 2) = (1, 4)(2, 3)$, so from Exercise 14, they generate A_4.

If x and y do not fix 5, then for whichever element they both fix, we can map them to permutations in S_4 by decrementing the elements they each permute that are greater than the fixed element (e.g. if $x = (1, 3, 5), y = (1, 3, 4)$, then we map them to $(1, 2, 4), (1, 2, 3)$, respectively), so that the group generated by them is indeed A_4. $\square$

(b) Prove that if x and y do not fix a common element of $\{1, ..., 5\}$ then $\langle x, y\rangle = A_5$.

Proof. Without loss of generality, we need only consider the case $x = (1, 2, 3), y = (3, 4, 5)$ (all other cases have the same structure in that their respective cycle decompositions each share exactly one element of $\{1, ..., 5\}$).

Since x and y are both even permutations, they can only generate even permutations. We conclude that $\langle x, y\rangle \leq A_5$, so by Lagrange's Theorem its order must divide $|A_5| = 60$.

Note that:

$$\begin{aligned}
xy &= (1,2,3)(3,4,5) = (1,2,3,4,5),\\
yx &= (3,4,5)(1,2,3) = (1,2,4,5,3),\\
x^{-1}y &= (1,3,2)(3,4,5) = (1,3,4,5,2), \text{ and } (1,3,4,5,2)^{-1} = (1,2,5,4,3),\\
xy^{-1} &= (1,2,3)(3,5,4) = (1,2,3,5,4),\\
xyx &= (1,2,3)(3,4,5)(1,2,3) = (1,3,2,4,5), \text{ and}\\
&\qquad (1,3,2,4,5)^2 = (1,2,5,3,4),\\
yxy &= (3,4,5)(1,2,3)(3,4,5) = (1,2,3,5,4).
\end{aligned}$$

Consider the cyclic subgroup of S_5 generated by a 5-cycle. It contains 5 elements, but we ignore the identity since it is common to all. Then all the cyclic subgroups generated by the above 5-cycles contain a total of $6 \cdot 4 = 24$ non-identity elements (we know that they are all distinct since each can only contain one permutation beginning $(1, 2, ...)$, which is shown above).

Now $(1, 2, 3, 4, 5)(1, 2, 3, 5, 4) = (1, 3)(2, 4)$, and by Exercise 14, x together with this order 2 element generates A_4, which contains 12 elements.

So far we have seen how to produce at least $24 + 12 = 36$ distinct elements. By Lagrange's Theorem, since the order of this generated group must divide 60 yet is greater than 30, it must contain 60 elements, and therefore be all of S_5. $\square$