Dummit & Foote Ch. 7.1: Introduction to Rings

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Jul. 2024

Let R be a ring with 1.

1. (7/1/24)

Show that $(-1)^2 = 1$ in $\mathbb{R}$.

Proof. We have:

$$(-1) + (-1)^2 = \underbrace{(-1)(1)}_{\text{identity}} + (-1)(-1) = \underbrace{(-1)(1 + (-1))}_{\text{distribution}} = (-1)\underbrace{(0)}_{\text{inverses}} = 0,$$

and therefore, since $(-1) + (-1)^2 = 0$, $(-1)^2 = 1$. □

2. (7/1/24)

Prove that if u is a unit in R then so is $-u$.

Proof. Recall that u is a unit in R if there exists some $v \in R$ such that $uv = vu = 1$.

Now:

$$\begin{aligned}(-u)(v) &= -(uv) = -1, \text{ which implies that}\\ (-u)(v)(-1) &= (-1)^2 = 1, \text{ so}\\ (-u)(-v) &= 1,\end{aligned}$$

which implies that $-u$ is also a unit in R. □

7. (7/5/24)

The *center* of a ring R is $\{z \in R \mid zr = rz \text{ for all } r \in R\}$ (i.e., is the set of all elements which commute with every element of R). Prove that the center of a ring is a subring that contains the identity. Prove that the center of a division ring is a field.

Proof. Let $a, b \in R$ be in the center of R and let $x \in R$. Then:

$$(a-b)x = ax - bx = xa - xb = x(a-b),$$

so $a - b$ is in the center of R. And, since a and b both commute with x, we have $(ab)x = abx = xab = x(ab)$, so ab lies in the center of R as well. Since by definition 1 commutes with every element of R, the center of R is a subring of R containing the identity.

If R is a division ring, then every element in its center (except 0) has a multiplicative inverse (is a unit). Every element in its center also commutes with every other element. A field is a commutative ring where every nonzero element is a unit; therefore the center of a division ring is a field. $\square$