# Dummit & Foote Ch. 2.2: Centralizers and Normalizers, Stabilizers and Kernels

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Jun. 2023

## 1. (6/5/23)

Prove that $C_G(A) = \{g \in G \mid g^{-1}ag = a \text{ for all } a \in A\}$.

*Proof.* By definition, $C_G(A) = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$ (that is, it is the set of elements of $G$ that commute with all elements of $A$).

Let $g \in C_G(A), a \in A$. Then $gag^{-1} = a$, which implies that $ga = ag$, and so left-multiplying by $g^{-1}$ we obtain $a = g^{-1}ag$. Therefore, equivalently, $C_G(A)$ is the set of elements $g \in G$ such that $g^{-1}ag = a$ for all $a \in A$. $\square$

## 2. (6/5/23)

Prove that $C_G(Z(G)) = G$ and deduce that $N_G(Z(G)) = G$.

*Proof.* Recall that $Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$. Let $z \in Z(G)$, so $z$ commutes with every element of $G$.

Also recall that $C_G(A) = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$. When $A = Z(G)$, then every element of $g$ commutes with every element of $A$. Therefore for all $g \in G$, $g \in C_G(Z(G))$. Thus $C_G(Z(G)) = G$.

Note that, since $C_G(A) \leq N_G(A)$ for all subsets $A$, we must have $G = C_G(Z(G)) \leq N_G(Z(G))$. Since there is no greater set of elements, we also have $N_G(Z(G)) = G$. $\square$

## 3. (6/8/23)

Prove that if $A$ and $B$ are subsets of $G$ with $A \subseteq B$ then $C_G(B)$ is a subgroup of $C_G(A)$.

*Proof.* Let $a \in A$ and $g \in C_G(B)$. Then $g$ commutes with every element of $b$, that is, $gb = bg \Rightarrow gbg^{-1} = b$ for all $b \in B$. Since $A \subseteq B$, we also have $gag^{-1} = a$ for all $a \in A$. Therefore $g \in C_G(A)$, which implies that $C_G(B) \subseteq C_G(A)$.

From the introduction to this chapter, centralizers are subgroups, so both $C_G(B) \leq G$ and $C_G(A) \leq G$. Since $C_G(B)$ is contained within $C_G(A)$ and both are subgroups of $G$, $C_G(B)$ must be closed within $C_G(A)$ and closed under inverses within $C_G(A)$, so it is also a subgroup of $C_G(A)$. $\square$

## 4. (6/8/23)

For each of $S_3$, $D_8$, and $Q_8$ compute the centralizers of each element and find the center of each group.

### $S_3$

- $C_{S_3}((1)) = S_3$
- $C_{S_3}((1,2)) = \{(1),(1,2)\}$
- $C_{S_3}((1,3)) = \{(1),(1,3)\}$
- $C_{S_3}((2,3)) = \{(1),(2,3)\}$
- $C_{S_3}((1,2,3)) = C_{S_3}((1,3,2)) = \{(1),(1,2,3),(1,3,2)\}$

The center $Z(S_3)$ consists only of the identity permutation.

### $D_8$

- $C_{D_8}(1) = D_8$
- $C_{D_8}(r) = C_{D_8}(r^2) = C_{D_8}(r^3) = \{1, r, r^2, r^3\}$
- $C_{D_8}(s) = C_{D_8}(sr^2) = \{1, r^2, s, sr^2\}$
- $C_{D_8}(sr) = C_{D_8}(sr^3) = \{1, r^2, sr, sr^3\}$

The center $Z(D_8)$ is $\{1, r^2\}$.

### $Q_8$

- $C_{D_8}(1) = C_{D_8}(-1) = Q_8$
- $C_{D_8}(i) = C_{D_8}(-i) = \{1, -1, i, -i\}$
- $C_{D_8}(j) = C_{D_8}(-j) = \{1, -1, j, -j\}$
- $C_{D_8}(k) = C_{D_8}(-k) = \{1, -1, k, -k\}$

The center $Z(Q_8)$ is $\{1, -1\}$.