Dummit & Foote Ch. 2.2: Centralizers and Normalizers, Stabilizers and Kernels

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1. (6/5/23)

Prove that $C_G(A) = \{g \in G \mid g^{-1}ag = a \text{ for all } a \in A\}$.

Proof. By definition, $C_G(A) = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$ (that is, it is the set of elements of G that commute with all elements of A).

Let $g \in C_G(A), a \in A$. Then $gag^{-1} = a$, which implies that $ga = ag$, and so left-multiplying by g^{-1} we obtain $a = g^{-1}ag$. Therefore, equivalently, $C_G(A)$ is the set of elements $g \in G$ such that $g^{-1}ag = a$ for all $a \in A$. □

2. (6/5/23)

Prove that $C_G(Z(G)) = G$ and deduce that $N_G(Z(G)) = G$.

Proof. Recall that $Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$. Let $z \in Z(G)$, so z commutes with every element of G.

Also recall that $C_G(A) = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$. When $A = Z(G)$, then every element of g commutes with every element of A. Therefore for all $g \in G$, $g \in C_G(Z(G))$. Thus $C_G(Z(G)) = G$.

Note that, since $C_G(A) \leq N_G(A)$ for all subsets A, we must have $G = C_G(Z(G)) \leq N_G(Z(G))$. Since there is no greater set of elements, we also have $N_G(Z(G)) = G$. □

3. (6/8/23)

Prove that if A and B are subsets of G with $A \subseteq B$ then $C_G(B)$ is a subgroup of $C_G(A)$.

Proof. Let $a \in A$ and $g \in C_G(B)$. Then g commutes with every element of b, that is, $gb = bg \Rightarrow gbg^{-1} = b$ for all $b \in B$. Since $A \subseteq B$, we also have $gag^{-1} = a$ for all $a \in A$. Therefore $g \in C_G(A)$, which implies that $C_G(B) \subseteq C_G(A)$.

From the introduction to this chapter, centralizers are subgroups, so both $C_G(B) \leq G$ and $C_G(A) \leq G$. Since $C_G(B)$ is contained within $C_G(A)$ and both are subgroups of G, $C_G(B)$ must be closed within $C_G(A)$ and closed under inverses within $C_G(A)$, so it is also a subgroup of $C_G(A)$. $\square$

4. (6/8/23)

For each of S_3, D_8, and Q_8 compute the centralizers of each element and find the center of each group.

S_3

- $C_{S_3}((1)) = S_3$
- $C_{S_3}((1,2)) = \{(1),(1,2)\}$
- $C_{S_3}((1,3)) = \{(1),(1,3)\}$
- $C_{S_3}((2,3)) = \{(1),(2,3)\}$
- $C_{S_3}((1,2,3)) = C_{S_3}((1,3,2)) = \{(1),(1,2,3),(1,3,2)\}$

The center $Z(S_3)$ consists only of the identity permutation.

D_8

- $C_{D_8}(1) = D_8$
- $C_{D_8}(r) = C_{D_8}(r^2) = C_{D_8}(r^3) = \{1,r,r^2,r^3\}$
- $C_{D_8}(s) = C_{D_8}(sr^2) = \{1,r^2,s,sr^2\}$
- $C_{D_8}(sr) = C_{D_8}(sr^3) = \{1,r^2,sr,sr^3\}$

The center $Z(D_8)$ is $\{1,r^2\}$.

Q_8

- $C_{D_8}(1) = C_{D_8}(-1) = Q_8$
- $C_{D_8}(i) = C_{D_8}(-i) = \{1,-1,i,-i\}$
- $C_{D_8}(j) = C_{D_8}(-j) = \{1,-1,j,-j\}$
- $C_{D_8}(k) = C_{D_8}(-k) = \{1,-1,k,-k\}$

The center $Z(Q_8)$ is $\{1,-1\}$.

5. (6/8/23)

In each of parts (a) through (c) show that for the specified group G and subgroup A of G, $C_G(A) = A$ and $N_G(A) = G$.

(a) $G = S_3$ and $A = \{(1), (1,2,3), (1,3,2)\}$.

Proof. From Exercise 4, we have $C_G((1,2,3)) = C_G((1,3,2)) = A$. No other non-identity permutation is in any of the centralizers of any element of A, therefore $C_G(A) = A$.

Next, consider $\sigma^{-1}(1,2,3)\sigma$ for some other permutation in S_3, for example $(1,2)(1,2,3)(1,2)$. This is equal to $(1,3,2)$, which is an element of A, so $(1,2)$ is in the normalizer of A. Since $C_G(A) \leq N_G(A)$ for all A, $A \subseteq N_G(A)$, and it follows that $N_G(A)$ consists of at least A and the element $(1,2)$. Then, because $N_G(A)$ is a subgroup, it is closed under permutation composition, and therefore must contain all elements of S_3. □

(b) $G = D_8$ and $A = \{1, s, r^2, sr^2\}$.

Proof. We know that $C_G(A)$ is a subgroup of G, and from Exercise 4, we have $A \leq C_G(A)$ (since A is commutative). Then $|C_G(A)| \geq 4$. By Lagrange's Theorem, the order of $C_G(A)$ divides the order of G, 8. Then we must have either $C_G(A) = A$ or $C_G(A) = G$. However, r is not in the centralizer of A, because $rsr^{-1} = rsr^3 = sr^{-1}r^3 = sr^2 \neq s$. Therefore $C_G(A) = A$.

When we consider the normalizer of A, note that $rsr^{-1} = sr^2 \in A$. Thus $N_G(A)$ is a subgroup of G that contains both A and the element r. By closing the subgroup, we obtain $N_G(A) = G$. □

(c) $G = D_{10}$ and $A = \{1, r, r^2, r^3, r^4\}$.

Proof. Since A consists only of powers of r, A is commutative, and so (as above) $A \leq C_G(A)$. The centralizer of A does not contain the element s, because $s^{-1}rs = srs = ssr^4 = r^4 \neq r$. Then we must have $|A| = 5 \leq |C_G(A)| \leq 9 = |G - \{s\}|$. Again by Lagrange's Theorem, the order of $C_G(A)$ must divide 10, and since it at least 5 and at most 9, it must be 5. Therefore $C_G(A) = A$.

When we consider the normalizer of A, note that $s^{-1}r^4s = r \in A$. Thus $N_G(A)$ is a subgroup of G that contains both A and the element s. By closing the subgroup, we obtain $N_G(A) = G$. □

6. (6/9/23)

Let H be a subgroup of the group G.

(a) Show that $H \leq N_G(H)$. Give an example to show that this is not necessarily true if H is not a subgroup.

Proof. Let $h_1, h_2 \in H$ (to show that $h_1 \in N_G(H)$). Because H is a subgroup of G, it is closed and closed under inverses, so $h_1 h_2 h_1^{-1} \in H$. So the conjugate of every element with every other element of H is in H, which implies that $H \leq N_G(H)$.

However, this does not follow if H is merely a subset of G. For example, let $G = D_6$ and $H = \{s, r\}$. Then $rsr^{-1} = sr^2r^2 = sr \notin H$, which implies that $r \notin H$. Therefore H is not contained within its normalizer. $\square$

(b) Show that $H \leq C_G(H)$ if and only if H is abelian.

Proof. First, let H be abelian and let $h_1, h_2 \in H$. Because H is abelian, we have $h_1 h_2 = h_2 h_1 \Rightarrow h_2 = h_1 h_2 h_1^{-1}$, so the conjugate of h_2 by h_1 is h_2. Thus the arbitrary element h_1 is in the centralizer of H, and so $H \leq C_G(H)$.

Next, let $H \leq C_G(H)$. Then for all $h_1, h_2 \in H$, $h_2 = h_1 h_2 h_1^{-1} \Rightarrow h_2 h_1 = h_1 h_2$, and so H is an abelian subgroup of G. $\square$