Dummit & Foote Ch. 2.4: Subgroups Generated by Subsets of a Group

Scott Donaldson

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1. (7/13/23)

Prove that if H is a subgroup of G then $\langle H \rangle = H$.

Proof. Let $H \leq G$. To show that $\langle H \rangle = H$, we must show that each is contained in the other. By definition, $H \subseteq \langle H \rangle$, so it remains to be proven that $\langle H \rangle \subseteq H$.

Let $h \in \langle H \rangle$. Recall that:

$$\langle H \rangle = \bigcap_{\substack{H \subseteq K \\ K \leq G}} K,$$

that is, for all subset $K \leq G$ with $H \subseteq K$, we have $h \in K$. In particular, since H is a subgroup of G, we have $h \in H$, since $H \leq G$ and $H \subseteq H$. Therefore $\langle H \rangle \subseteq H$, and it follows that $\langle H \rangle = H$. □

2. (7/17/23)

Prove that if A is a subset of B then $\langle A \rangle \leq \langle B \rangle$. Give an example where $A \subseteq B$ with $A \neq B$ but $\langle A \rangle = \langle B \rangle$.

Proof. Let G be a group and let $A \subseteq B \subseteq G$. Recall that one definition of $\langle A \rangle$ is the set of all finite words of elements and inverses of elements of A, that is, every element of $\langle A \rangle$ can be written $a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_n^{\varepsilon_n}$, where $n \in \mathbb{Z}, n \geq 0$ and $a_i \in A, \varepsilon_i = \pm 1$ for each i. Since A is a subset of B, $a_i \in A \Rightarrow a_i \in B$, and so each element $a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_n^{\varepsilon_n} \in \langle A \rangle$ is also in $\langle B \rangle$. Therefore $\langle A \rangle \leq \langle B \rangle$.

Now let $G = \mathbb{Z}/3\mathbb{Z}$, $A = \{1\}$, and $B = \{0, 1\}$. Then we have $A \subseteq B$ with $A \neq B$ but $\langle A \rangle = \langle B \rangle = G$. □

3. (7/17/23)

Prove that if H is an abelian subgroup of G then $\langle H, Z(G) \rangle$ is abelian. Give an explicit example of an abelian subgroup H of a group G such that $\langle H, C_G(H) \rangle$ is not abelian.

Proof. Let G be a group and let H be an abelian subgroup of G. Recall that $Z(G) = \{g \in G \mid xg = gx \text{ for all } x \in G\}$, that is, the set of elements of G that commute with every element of G. We will show that $\langle H, Z(G)\rangle$ is an abelian subgroup of G.

First, we will show that the product of any two elements commutes with both elements. Let $a, b \in G$ be commuting elements. Then:

$$(ab)a = aba = aab = a(ab), \text{ and } (ab)b = abb = bab = b(ab),$$

as desired.

Now the generated subgroup $\langle H, Z(G)\rangle$ is constructed from finite words of elements and inverses of elements from H and $Z(G)$. Since H is an abelian subgroup and elements of $Z(G)$ (and therefore their inverses) commute with every element of G (and therefore H), it follows that every element in $\langle H, Z(G)\rangle$ is a product of commuting elements. Every such element therefore commutes with every other element in H and $Z(G)$, as well as any other product of elements of H and $Z(G)$. Thus $\langle H, Z(G)\rangle$ is an abelian subgroup of G.

However, it does not follow that $\langle H, C_G(H)\rangle$ is an abelian subgroup of G. Let $G = D_8$ and $H = \{1, r^2\}$. The centralizer of H in G is all of G, since every element of H commutes with every other element of G (that is, $H = Z(G)$). Then the generated subgroup $\langle H, C_G(H)\rangle = \langle H, G\rangle = G$, which is non-abelian. $\square$