# Dummit & Foote Ch. 4.2: Groups Acting on Themselves by Left Multiplication — Cayley's Theorem

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Let $G$ be a group and let $H$ be a subgroup of $G$.

## 1. (2/12/24)

Let $G = \{1, a, b, c\}$ be the Klein 4-group whose group table is written out in Section 2.5.

(a) Label $1, a, b, c$ with the integers $1, 2, 4, 3$, respectively, and prove that under the left regular representation of $G$ into $S_4$ the nonidentity elements are mapped as follows:

$$a \mapsto (1\,2)(3\,4) \qquad b \mapsto (1\,4)(2\,3) \qquad c \mapsto (1\,3)(2\,4).$$

*Proof.* The left regular representation of $G$ into $S_4$ is the homomorphism $\varphi : G \to S_4$ defined by $\varphi(g) = \sigma_g$, where $\sigma_g : G \to G$ is the permutation of $G$ defined by $\sigma_g(x) = gx$ for all $x \in G$.

Each non-identity element maps the elements as follows:

$$\begin{array}{cccc} \sigma_a(1) = a1 = a & \sigma_a(a) = a^2 = 1 & \sigma_a(b) = ab = c & \sigma_a(c) = ac = b \\ \sigma_b(1) = b1 = b & \sigma_b(a) = ba = c & \sigma_b(b) = b^2 = 1 & \sigma_b(c) = bc = a \\ \sigma_c(1) = c1 = c & \sigma_c(a) = ca = b & \sigma_c(b) = cb = a & \sigma_c(c) = c^2 = 1. \end{array}$$

By the given labeling, this assigns the elements $a, b$, and $c$ to the pairs of 2-cycles shown above. $\square$

(b) Relabel $1, a, b, c$ as $1, 4, 2, 3$, respectively, and compute the image of each element of $G$ under the left regular representation of $G$ into $S_4$. Show that the image of $G$ in $S_4$ under this labeling is the same *subgroup* as the image of $G$ in part (a) (even though the nonidentity elements individually map to different permutations under the two different labelings).

*Proof.* Under this labeling, the elements $a, b$, and $c$ are mapped to the permutations $(1\,4)(2\,3), (1\,2)(3\,4)$, and $(1\,3)(2\,4)$, respectively. Although each element maps to a different permutation from part (a), the subgroup of $S_4$ is the same in both cases. $\square$

## 2. (2/12/24)

List the elements of $S_3$ as $1, (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)$ and label these with the integers $1, 2, 3, 4, 5, 6$, respectively. Exhibit the image of each element of $S_3$ under the left regular representation of $S_3$ into $S_6$.

*Solution.* First, consider the element $(1\,2)$. We see that:

$$\begin{aligned} (1\,2)1 = (1\,2) &\mapsto 2 & (1\,2)(1\,2) = 1 &\mapsto 1 \\ (1\,2)(2\,3) = (1\,2\,3) &\mapsto 5 & (1\,2)(1\,3) = (1\,3\,2) &\mapsto 6 \\ (1\,2)(1\,2\,3) = (2\,3) &\mapsto 3 & (1\,2)(1\,3\,2) = (1\,3) &\mapsto 4. \end{aligned}$$

So the left regular representation of $(1\,2)$ under the given labeling in $S_6$ is $(1\,2)(3\,4)(5\,6)$.

The left regular representations of the remaining elements are:

$$\begin{aligned} (2\,3) &\mapsto (1\,3)(2\,6)(4\,5) \\ (1\,3) &\mapsto (1\,4)(2\,5)(3\,6) \\ (1\,2\,3) &\mapsto (1\,5\,6)(2\,4\,3) \\ (1\,3\,2) &\mapsto (1\,6\,5)(2\,3\,4). \end{aligned}$$

$\square$

## 3. (2/12/24)

Let $r$ and $s$ be the usual generators for the dihedral group of order 8.

(a) List the elements of $D_8$ as $1, r, r^2, r^3, s, sr, sr^2, sr^3$ and label these with the integers $1, 2, ..., 8$, respectively. Exhibit the image of each element of $D_8$ under the left regular representation of $D_8$ into $S_8$.

$$\begin{aligned} 1 &\mapsto 1 \\ r &\mapsto (1\,2\,3\,4)(5\,8\,7\,6) \\ r^2 &\mapsto (1\,3)(2\,4)(5\,7)(6\,8) \\ r^3 &\mapsto (1\,4\,3\,2)(5\,6\,7\,8) \\ s &\mapsto (1\,5)(2\,6)(3\,7)(4\,8) \\ sr &\mapsto (1\,6)(2\,7)(3\,8)(4\,5) \\ sr^2 &\mapsto (1\,7)(2\,8)(3\,5)(4\,6) \\ sr^3 &\mapsto (1\,8)(2\,5)(3\,6)(4\,7) \end{aligned}$$

(b) Relabel this same list of elements of $D_8$ with the integers $1, 3, 5, 7, 2, 4, 6, 8$ respectively and recompute the image of each element of $D_8$ under the left regular representation with respect to this new labeling. Show that the two subgroups of $S_8$ obtained in parts (a) and (b) are different.

$$\begin{aligned}
1 &\mapsto 1 \\
r &\mapsto (1\,3\,5\,7)(2\,8\,6\,4) \\
r^2 &\mapsto (1\,5)(2\,6)(3\,7)(4\,8) \\
r^3 &\mapsto (1\,7\,5\,3)(2\,4\,6\,8) \\
s &\mapsto (1\,2)(3\,4)(5\,6)(7\,8) \\
sr &\mapsto (1\,4)(2\,7)(3\,6)(5\,8) \\
sr^2 &\mapsto (1\,6)(2\,5)(3\,8)(4\,7) \\
sr^3 &\mapsto (1\,8)(2\,3)(4\,5)(6\,7).
\end{aligned}$$

We see that the generators of the subgroups of $S_8$ in parts (a) and (b) are different, and so these are different subgroups of $S_8$.

## 4. (2/12/24)

Use the left regular representation of $Q_8$ to produce two elements of $S_8$ which generate a subgroup of $S_8$ isomorphic to the quaternion group $Q_8$.

*Proof.* We know that the elements $i$ and $j$ generate the quaternion group $Q_8$. Labeling the elements $1, -1, i, -i, j, -j, k, -k$ with $1, 2, ..., 8$ respectively, the elements $i$ and $j$ map to the following permutations in $S_8$:

$$\begin{aligned}
i &\mapsto (1\,3\,2\,4)(5\,7\,6\,8) \\
j &\mapsto (1\,5\,2\,6)(3\,8\,4\,7).
\end{aligned}$$

Since the left regular representation of $Q_8$ in $S_8$ is a homomorphism, these two permutations generate a subgroup of $S_8$ isomorphic to $Q_8$. $\square$

## 5. (2/12/24)

Let $r$ and $s$ be the usual generators for the dihedral group of order 8 and let $H = \langle s \rangle$. List the left cosets of $H$ in $D_8$ as $1H, rH, r^2H, r^3H$.

(a) Label these cosets with the integers $1, 2, 3, 4$, respectively. Exhibit the image of each element of $D_8$ under the representation $\pi_H$ of $D_8$ into $S_4$ obtained from the action of $D_8$ by left multiplication on the set of 4 left cosets of $H$ in $D_8$. Deduce that this representation is faithful (i.e., the elements of $S_4$ obtained form a subgroup isomorphic to $D_8$).

$$\begin{aligned} 1 &\mapsto 1 & s &\mapsto (2\,4) \\ r &\mapsto (1\,2\,3\,4) & sr &\mapsto (1\,4)(2\,3) \\ r^2 &\mapsto (1\,3)(2\,4) & sr^2 &\mapsto (1\,3) \\ r^3 &\mapsto (1\,4\,3\,2) & sr^3 &\mapsto (1\,2)(3\,4). \end{aligned}$$

Since each element of $D_8$ induces a unique permutation in $S_4$, the resulting image under the left regular representation is isomorphic to $D_8$, and so this representation is faithful.

(b) Repeat part (a) with the list of cosets relabeled by the integers $1, 3, 2, 4$, respectively. Show that the permutations obtained from this labeling form a subgroup of $S_4$ that is different from the subgroup obtained in part (a).

$$\begin{aligned} 1 &\mapsto 1 & s &\mapsto (3\,4) \\ r &\mapsto (1\,3\,2\,4) & sr &\mapsto (1\,4)(2\,3) \\ r^2 &\mapsto (1\,2)(3\,4) & sr^2 &\mapsto (1\,2) \\ r^3 &\mapsto (1\,4\,2\,3) & sr^3 &\mapsto (1\,3)(2\,4). \end{aligned}$$

Since the generators (the images of $r$ and $s$) of this subgroup of $S_4$ are different from those in part (a), this is a different subgroup from part (a).

(c) Let $K = \langle sr \rangle$, list the cosets of $K$ in $D_8$ as $1K, rK, r^2K, r^3K$, and label these with the integers $1, 2, 3, 4$. Prove that, with respect to this labeling, the image of $D_8$ under the representation $\pi_K$ obtained from left multiplication on the cosets of $K$ is the same *subgroup* of $S_4$ as in part (a) (even though the subgroups $H$ and $K$ are different and some of the elements of $D_8$ map to different permutations under the two homomorphisms).

*Proof.* Consider the images of the generators $r$ and $s$ under $\pi_K$:

$$\begin{aligned} r \cdot 1K &= rK & s \cdot 1K &= rK \\ r \cdot rK &= r^2K & s \cdot rK &= 1K \\ r \cdot r^2K &= r^3K & s \cdot r^2K &= r^3K \\ r \cdot r^3K &= 1K & s \cdot r^3K &= r^2K. \end{aligned}$$

So $r$ and $s$ map to $(1\,2\,3\,4)$ and $(1\,2)(3\,4) \in S_4$, respectively. These elements are both in the subgroup in part (a) above, and so they are the same subgroup, but the image of $s$ is different. $\square$

# 6. (2/15/24)

Let $r$ and $s$ be the usual generators for the dihedral group of order 8 and let $N = \langle r^2 \rangle$. List the left cosets of $N$ in $D_8$ as $1N, rN, sN$, and $srN$. Label these cosets with the integers $1, 2, 3, 4$ respectively. Exhibit the image of each element of $D_8$ under the representation $\pi_N$ of $D_8$ into $S_4$ obtained from the action of $D_8$ by left multiplication on the set of 4 left cosets of $N$ in $D_8$. Deduce that this representation is not faithful and prove that $\pi_N(D_8)$ is isomorphic to the Klein 4-group.

*Solution.*

$$\begin{aligned} 1 &\mapsto 1 & s &\mapsto (1\,3)(2\,4) \\ r &\mapsto (1\,2)(3\,4) & sr &\mapsto (1\,4)(2\,3) \\ r^2 &\mapsto 1 & sr^2 &\mapsto (1\,3)(2\,4) \\ r^3 &\mapsto (1\,2)(3\,4) & sr^3 &\mapsto (1\,4)(2\,3). \end{aligned}$$

The left regular representation assigns 1 and $r^2$ to the identity permutation, so this action is not faithful.

The image of $D_8$ under $\pi_N$ consists of the four permutations $1, (1\,2)(3\,4)$, $(1\,3)(2\,4)$, and $(1\,4)(2\,3)$. From Ch. 2.5, Exercise 10, this is isomorphic to the Klein 4-group $V_4$. □

## 7. (2/15/24)

Let $Q_8$ be the quaternion group of order 8.

(a) Prove that $Q_8$ is isomorphic to a subgroup of $S_8$.

*Proof.* From Exercise 4, $Q_8$ is isomorphic to

$$\langle (1\,3\,2\,4)(5\,7\,6\,8), (1\,5\,2\,6)(3\,8\,4\,7) \rangle \in S_8.$$

□

(b) Prove that $Q_8$ is not isomorphic to a subgroup of $S_n$ for any $n \leq 7$.

*Proof.* Let $A$ be a set with $|A| = n \leq 7$, let $a \in A$, and let $\cdot$ be the action of $Q_8$ on $A$. We attempt to find a subgroup of $S_n$ that is isomorphic to $Q_8$ by considering the permutation representations of the elements of $Q_8$. Now if $i \cdot a = j \cdot a$, then the permutation representations $\sigma_i$ and $\sigma_j$ are equal to each other, and so $Q_8$ is not isomorphic to the resulting subgroup of $S_n$. Further (without loss of generality), if $i \cdot a = -i \cdot a$, then:

$$i \cdot a = -i \cdot a \Rightarrow -i \cdot i \cdot a = -i \cdot -i \cdot a \Rightarrow a = -1 \cdot a,$$

and so the permutation representation of $-1$ is equal to the identity permutation, which implies that $Q_8$ is not isomorphic to the subgroup of $S_n$. Therefore the elements $\pm i, \pm j, \pm k$ must all assign $a$ to different elements. However, these 6 unique elements together with $a$ are at least all of $A$, and so we must have $-1 \cdot a = a$. Thus $Q_8$ is not isomorphic to a subgroup of $S_n$. □

## 9. (2/16/24)

Prove that if $p$ is a prime and $G$ is a group of order $p^a$ for some $a \in \mathbb{Z}^+$, then every subgroup of index $p$ is normal in $G$. Deduce that every group of order $p^2$ has a normal subgroup of order $p$.

*Proof.* Let $H$ be a subgroup of $G$ with $[G : H] = p$. Let $gH$ be the left coset of $H$ by some element $g \in G$.

Suppose that, for some $n < p$, $g^n \in H$ and let $h = g^n$. Since $p$ is a prime, there exists a positive integer $k$ such that $kn = 1 \pmod{p}$. Then $h^k = g^{kn} = g$, which implies that $g \in H$. We conclude that, if we restrict to $g \notin H$, then for all $n < p$, $g^n \notin H$. This implies that $\{H, gH, g^2H, ..., g^{p-1}H\}$ is a set of $p$ distinct cosets of $H$. Because the index of $H$ in $G$ is $p$, this must be all the cosets of $H$ in $G$, and so $g^p \in H$.

Now by the operation defined on left cosets of $H$ by $aH \cdot bH = (ab)H$, we see that this is isomorphic to the cyclic group $Z_p$. We conclude by Theorem 6(d) of Chapter 3.1 that $H$ is normal in $G$.

Further, if $|G| = p^2$, then by Cauchy's Theorem it contains an element of order $p$ which generates a subgroup of order $p$. This subgroup has index $p^2/p = p$, and so from above, is a normal subgroup of order $p$. □

## 14. (2/20/24)

Let $G$ be a finite group of composite order $n$ with the property that $G$ has a subgroup of order $k$ for each positive integer $k$ dividing $n$. Prove that $G$ is not simple.

*Proof.* By definition, a group $G$ is simple if it contains no proper normal subgroups other than 1 and $G$ itself. Therefore, it suffices to show that $G$ contains at least one proper normal subgroup.

Let $p$ be the smallest prime dividing $n$ and let $n = pk$. Then $G$ contains a subgroup of order $k$ that has index $n/k = p$. By Corollary 5, this subgroup is normal and, since it is a proper subgroup, $G$ is therefore not simple. □