

Dummit & Foote Ch. 3.1: Quotient Groups and Homomorphisms

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Let G and H be groups.

1. (9/1/23)

Let $\varphi : G \rightarrow H$ be a homomorphism and let $E \leq H$. Prove that $\varphi^{-1}(E) \leq G$ (i.e., the preimage or pullback of a subgroup under a homomorphism is a subgroup). If $E \trianglelefteq H$ prove that $\varphi^{-1}(E) \trianglelefteq G$. Deduce that $\ker \varphi \trianglelefteq G$.

Proof. Let $x, y \in \varphi^{-1}(E) \subseteq G$. Suppose that $\varphi(x) = a, \varphi(y) = b, a, b \in E \leq H$. Since φ is a homomorphism, we have $\varphi(y^{-1}) = \varphi(y)^{-1} = b^{-1}$. Then:

$$\varphi(xy^{-1}) = \varphi(x)\varphi(y^{-1}) = \varphi(x)\varphi(y)^{-1} = ab^{-1} \in E,$$

which implies that $xy^{-1} \in \varphi^{-1}(E)$. It follows that, by the subgroup criterion, $\varphi^{-1}(E) \leq G$.

Next, let $E \trianglelefteq H$ (to show that $\varphi^{-1}(E) \trianglelefteq G$). Again let $x \in \varphi^{-1}(E) \leq G$ and suppose $\varphi(x) = a$. Now for some $g \in G$ (not necessarily in $\varphi^{-1}(E)$), consider $\varphi(gxg^{-1})$. Suppose also that $\varphi(g) = h \in H$. Because E is normal in H and $a \in E$, we have $hah^{-1} \in E$. Then:

$$\varphi(gxg^{-1}) = \varphi(g)\varphi(x)\varphi(g^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1} = hah^{-1} \in E,$$

which implies that $gxg^{-1} \in \varphi^{-1}(E)$. Since the conjugate of any element of $\varphi^{-1}(E)$ by any other element of G lies in $\varphi^{-1}(E)$, we therefore conclude that $\varphi^{-1}(E) \trianglelefteq G$.

Finally, we note that $\ker \varphi = \{g \in G \mid \varphi(g) = 1_H\}$. Since the trivial subgroup consisting of the identity of H is normal (the conjugate of 1_H by any element of H is 1_H), we therefore have $\varphi^{-1}(\{1_H\}) = \ker \varphi \trianglelefteq G$. \square

2. (8/23/23)

Let $\varphi : G \rightarrow H$ be a homomorphism of groups with kernel K and let $a, b \in \varphi(G)$. Let $X \in G/K$ be the fiber above a and Y be the fiber above b , i.e.,

$X = \varphi^{-1}(a), Y = \varphi^{-1}(b)$. Fix an element $x \in X$ (so $\varphi(x) = a$). Prove that if $XY = Z$ in the quotient group G/K and z is any member of Z , then there is some $y \in Y$ such that $xy = z$.

Proof. We know that, for any $x \in X, y \in Y$, $\varphi(x) = a$ and $\varphi(y) = b$. Since φ is a homomorphism, it follows that $\varphi(xy) = \varphi(x)\varphi(y) = ab$, and so the image of any element of $XY = Z$ under φ is $ab \in H$.

Next, consider the element $x^{-1}z \in G$, as well as its image under φ . Since φ is a homomorphism, we have $\varphi(x^{-1}) = \varphi(x)^{-1}$. So $\varphi(x^{-1}z) = \varphi(x^{-1})\varphi(z) = \varphi(x)^{-1}\varphi(z) = a^{-1}ab = b$. The set Y consists of all elements of G whose image under φ is b , and so we must have $x^{-1}z \in Y$.

Now if we fix some element $x \in X$, then for any $z \in Z$, we have $x^{-1}z \in Y$ such that its product with x is z : $xx^{-1}z = z$. \square

3. (8/23/23)

Let A be an abelian group and let B be a subgroup of A . Prove that A/B is abelian. Give an example of a non-abelian group G containing a proper normal subgroup N such that G/N is abelian.

Proof. Because A is abelian, all subgroups of A are normal, so A/B is well-defined for every $B \leq A$.

Let $C, D \in A/B$ with $C = cB$ and $D = dB$ for some $c, d \in A$. Then:

$$CD = (cB)(dB) = (cd)B = (dc)B = (dB)(cB) = DC,$$

which implies that A/B is abelian.

Now if we let G be the dihedral group D_8 , then G is non-abelian. Let N be the cyclic subgroup generated by $r : \{1, r, r^2, r^3\}$. The only coset of N is sN ; together these two sets cover G . Then $G/N = \{N, sN\}$. There is only one group of order 2 up to isomorphism, and it is abelian. Thus G/N is abelian. \square

4. (8/23/23)

Prove that in the quotient group G/N , $(gN)^\alpha = (g^\alpha)N$ for all $\alpha \in \mathbb{Z}$.

Proof. We start by induction: In the base case, $\alpha = 1$, we have $(gN)^1 = gN = (g^1)N$. Next, suppose that for some $\alpha > 1$, we have $(gN)^\alpha = (g^\alpha)N$. Then:

$$(gN)^{\alpha+1} = (gN)^\alpha gN = g^\alpha N \cdot gN = (g^{\alpha+1})N,$$

as desired. We have now proven that $(gN)^\alpha = (g^\alpha)N$ for $\alpha \geq 1$.

Next, consider $(gN)^\alpha (gN)^{-\alpha}$, where $\alpha \geq 1$. In the quotient group G/N , for any subset $X \in G/N$, we must have $X^\alpha X^{-\alpha} = N$ (the identity of G/N), so $(gN)^\alpha (gN)^{-\alpha} = N$. From above, $(gN)^\alpha = (g^\alpha)N$, so $(g^\alpha)N \cdot (gN)^{-\alpha} = N$. Also, from the operation on left cosets, we know that $N = (g^\alpha)N \cdot (g^{-\alpha})N$.

Since both $(g^\alpha)N \cdot (gN)^{-\alpha} = N$ and $(g^\alpha)N \cdot (g^{-\alpha})N = N$, we must have $(gN)^{-\alpha} = (g^{-\alpha})N$. We have now proven for all nonzero integers.

Finally, we note that $(gN)^0 = N$ (the identity of G/N) and that $(g^0)N = eN = N$, so $(gN)^0 = (g^0)N$. This concludes the proof that $(gN)^\alpha = (g^\alpha)N$ for all $\alpha \in \mathbb{Z}$. \square

5. (8/23/23)

Use the preceding exercise to prove that the order of the element gN in G/N is n , where n is the smallest positive integer such that $g^n \in N$ (and gN has infinite order if no such positive integer exists). Give an example to show that the order of gN in G/N may be strictly smaller than the order of g in G .

Proof. Let $gN \in G/N$, and let n be the smallest positive integer such that $g^n \in N$. Suppose that $g^n = h \in N$.

From Exercise 4., $(gN)^n = (g^n)N = hN = N$ (because $h \in N$), so the order of gN must divide n .

Suppose (toward contradiction) that the order of gN is k , where $k < n$. Then $(gN)^k = (g^k)N = N$, which implies that g^k lies in N , contradicting our assumption that n is the smallest such positive integer. Therefore the order of gN is n .

If there is no positive integer n such that $g^n \in N$, then for all $k \in \mathbb{Z}^+$, we have $(gN)^k = (g^k)N \neq N$, so gN has infinite order.

As an example where $|gN| < |g|$, let $G = Z_9 = \langle x \rangle$ and let $N = \langle x^3 \rangle$. Because all cyclic groups are abelian, N is normal in G , and so G/N is well-defined. The quotient group G/N contains three elements: N, xN , and $(x^2)N$. The element $xN \in G/N$ has order 3: $(xN)^3 = (x^3)N = N$ (because $x^3 \in N$). However, the generating element $x \in G$ has order 9. \square

6. (8/24/23)

Define $\varphi : \mathbb{R}^\times \rightarrow \{\pm 1\}$ by letting $\varphi(x)$ be x divided by the absolute value of x . Describe the fibers of φ and prove that φ is a homomorphism.

Proof. We consider the two cases where $x < 0$ and $x > 0$ (0 is not an element of \mathbb{R}^\times). If $x > 0$, then $\varphi(x) = x/|x| = x/x = 1$. If $x < 0$, then $\varphi(x) = x/|x| = x/-x = -1$. Therefore the fiber above -1 is every negative real number and the fiber above 1 is every positive real number.

To show that φ is a homomorphism, we let $x, y \in \mathbb{R}^\times$ and again consider the different cases: Where x and y are both positive, where they are both negative, and where one is positive and the other negative.

If both x and y are positive, then $\varphi(x)\varphi(y) = 1 \cdot 1 = 1$ and $\varphi(xy) = \frac{xy}{|xy|} = \frac{xy}{xy} = 1$, so $\varphi(x)\varphi(y) = \varphi(xy)$.

If both x and y are negative, then $\varphi(x)\varphi(y) = -1 \cdot -1 = 1$ and $\varphi(xy) = \frac{xy}{|xy|} = \frac{xy}{xy} = 1$, so $\varphi(x)\varphi(y) = \varphi(xy)$.

Suppose x is positive and y is negative. Then $\varphi(x)\varphi(y) = 1 \cdot -1 = -1$ and $\varphi(xy) = \frac{xy}{|xy|} = \frac{xy}{-xy} = -1$, so $\varphi(x)\varphi(y) = \varphi(xy)$.

Thus, in every case of $x, y \in \mathbb{R}^\times$, we have $\varphi(x)\varphi(y) = \varphi(xy)$, and φ is thus a homomorphism. \square

7. (8/24/23)

Define $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}$ by $\pi((x, y)) = x + y$. Prove that π is a surjective homomorphism and describe the kernel and fibers of π geometrically.

Proof. First, to show that π is surjective, let $z \in \mathbb{R}$. Now $z = z + 0$, so $(z, 0)$ is an element of \mathbb{R}^2 such that $\pi((z, 0)) = z + 0 = z$.

Next, to show that π is a homomorphism, let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. We have $\pi((x_1, y_1) + (x_2, y_2)) = \pi((x_1 + x_2, y_1 + y_2)) = x_1 + x_2 + y_1 + y_2$, and $\pi((x_1, y_1)) + \pi((x_2, y_2)) = x_1 + y_1 + x_2 + y_2$. By the commutativity of addition in \mathbb{R} , these are equal to each other, and so π is a surjective homomorphism.

The kernel of π consists of all points $(x, y) \in \mathbb{R}^2$ such that $x + y = 0$, that is, the diagonal line running from the upper-left to the bottom-right of the Cartesian plane. Geometrically, the fibers of π are translations of this line, such that for any $z \in \mathbb{R}$, the fiber of π above z is the diagonal line intersecting both $(z, 0)$ and $(0, z)$. \square

8. (8/24/23)

Let $\varphi : \mathbb{R}^\times \rightarrow \mathbb{R}^\times$ be the map sending x to the absolute value of x . Prove that φ is a homomorphism and find the image of φ . Describe the kernel and the fibers of φ .

Proof. Let $x, y \in \mathbb{R}^\times$ (so $x \neq 0, y \neq 0$). If both x and y are positive or both are negative, then:

$$\varphi(xy) = |xy| = |x||y| = \varphi(x)\varphi(y),$$

and if x is positive and y is negative, then:

$$\varphi(xy) = |xy| = x(-y) = |x||y| = \varphi(x)\varphi(y),$$

so φ is a homomorphism.

The image of φ consists of every positive real number. The kernel of φ is the set $\{x \in \mathbb{R}^\times \mid |x| = 1\}$, that is, $\{\pm 1\}$. For a given element $z > 0$, the fiber of φ above z is the set $\{\pm z\}$. \square

9. (8/25/23)

Define $\varphi : \mathbb{C}^\times \rightarrow \mathbb{R}^\times$ by $\varphi(a + bi) = a^2 + b^2$. Prove that φ is a homomorphism and find the image of φ . Describe the kernel and the fibers of φ geometrically (as subsets of the plane).

Proof. To show that φ is a homomorphism, let $z_1 = a_1 + b_1i$, $z_2 = a_2 + b_2i \in \mathbb{C}^\times$. We calculate:

$$\begin{aligned}
\varphi(z_1 z_2) &= \varphi((a_1 + b_1i)(a_2 + b_2i)) \\
&= \varphi((a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i) \\
&= (a_1 a_2 - b_1 b_2)^2 + (a_1 b_2 + a_2 b_1)^2 \\
&= a_1^2 a_2^2 - 2a_1 a_2 b_1 b_2 + b_1^2 b_2^2 + a_1^2 b_2^2 + 2a_1 a_2 b_1 b_2 + a_2^2 b_1^2 \\
&= a_1^2 a_2^2 + b_1^2 b_2^2 + a_1^2 b_2^2 + a_2^2 b_1^2, \text{ and} \\
\varphi(z_1) \varphi(z_2) &= \varphi(a_1 + b_1i) \varphi(a_2 + b_2i) = (a_1^2 + b_1^2)(a_2^2 + b_2^2) \\
&= a_1^2 a_2^2 + b_1^2 b_2^2 + a_1^2 b_2^2 + a_2^2 b_1^2,
\end{aligned}$$

which proves that φ is a homomorphism.

The image of a complex number $a + bi$ under φ is $a^2 + b^2$, which is always non-negative because it is the sum of two non-negative numbers. Since both \mathbb{C}^\times and \mathbb{R}^\times exclude 0, the image of φ is therefore all positive real numbers.

The kernel of φ are those complex numbers whose image under φ is 1. Geometrically, φ is a map from a point in the complex plane to its length, or distance from zero. Therefore the kernel of φ is the unit circle in the complex plane. The fibers of a given positive real number x is the circle of radius x centered at the origin in the complex plane. \square

10. (8/28/23)

Let $\varphi : \mathbb{Z}/8\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z}$ by $\varphi(\bar{a}) = \bar{a}$. Show that this is a well-defined, surjective homomorphism and describe its fibers and kernel explicitly (showing that φ is well-defined involves the fact that \bar{a} has a different meaning in the domain and range of φ).

Proof. The map φ is well-defined because it assigns to each member of $\mathbb{Z}/8\mathbb{Z}$ a single, unique element of $\mathbb{Z}/4\mathbb{Z}$. Let $a \in \{0, \dots, 7\}$ be equal to $\bar{a} \bmod 8$. Then we have $\varphi(\bar{a}) = \varphi(a)$. Further, φ assigns each $a \in \{0, \dots, 7\}$ to $a \bmod 4$; that is, it assigns 0 and 4 to 0, 1 and 5 to 1, 2 and 6 to 2, and 3 and 7 to 3. This also shows that φ is surjective, since each $\bar{a} \in \mathbb{Z}/4\mathbb{Z}$ (represented by $a = \bar{a} \bmod 4$) has a preimage in $\mathbb{Z}/8\mathbb{Z}$.

The kernel of φ is $\{0, 4\} \leq \mathbb{Z}/8\mathbb{Z}$, and the fiber of any $a \in \mathbb{Z}/4\mathbb{Z}$ is the tuple $\{a, a + 4\}$. \square

11. (8/28/23)

Let F be a field and let $G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in F, ac \neq 0 \right\} \leq GL_2(F)$.

- (a) Prove that the map $\varphi : \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto a$ is a surjective homomorphism from G onto F^\times (recall that F^\times is the multiplicative group of nonzero elements in F). Describe the fibers and kernel of φ .

Proof. To show that φ is surjective, let $a \in F^\times$ (so $a \neq 0$). Then we have $\varphi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = a$, so φ is onto.

Next, to show that it is a homomorphism, we note that:

$$\varphi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} ad & ae + bf \\ 0 & cf \end{pmatrix}\right) = ad = \varphi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right)\varphi\left(\begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\right),$$

so φ is also a homomorphism.

The kernel of φ is $\left\{\begin{pmatrix} 1 & b \\ 0 & c \end{pmatrix} \mid b, c \in F, c \neq 0\right\}$, and the fiber of φ over a given element $a \in F^\times$ is $\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid b, c \in F, c \neq 0\right\}$. \square

- (b) Prove that the map $\psi : \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto (a, c)$ is a surjective homomorphism from G onto $F^\times \times F^\times$. Describe the fibers and kernel of ψ .

Proof. To show that ψ is surjective, let $(a, c) \in F^\times \times F^\times$ (so $a, c \neq 0$). Then we have $\psi\left(\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}\right) = (a, c)$, so ψ is onto.

Next, to show that it is a homomorphism, we note that:

$$\begin{aligned} \psi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\right) &= \psi\left(\begin{pmatrix} ad & ae + bf \\ 0 & cf \end{pmatrix}\right) = (ad, cf) \\ &= (a, c)(d, f) = \psi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right)\psi\left(\begin{pmatrix} d & e \\ 0 & f \end{pmatrix}\right), \end{aligned}$$

so ψ is also a homomorphism.

The kernel of ψ is the preimage of $(1, 1)$, that is, $\left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in F\right\}$, and the fiber of ψ over a given element $(a, c) \in F^\times \times F^\times$ is $\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid b \in F\right\}$. \square

- (c) Let $H = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in F\right\}$. Prove that H is isomorphic to the additive group F .

Proof. As usual, to show that H is isomorphic to the additive group F , we must show that there exists a bijective homomorphism $\varphi : H \rightarrow F$. Define φ by $\varphi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = b$. We will show that it is an isomorphism.

First, φ is injective: Suppose that $\varphi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = c$. Then we have $a = c$ and $b = c$, so the two matrices are the same, and φ is injective.

Next, φ is surjective: Let $b \in F$. Then we have $\varphi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = b$.

Finally, φ is a homomorphism:

$$\varphi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right) \varphi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \varphi\left(\begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}\right) = a+b = \varphi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right) + \varphi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right).$$

□

12. (8/30/23)

Let G be the additive group of real numbers, let H be the multiplicative group of complex numbers of absolute value 1 (the unit circle S^1 in the complex plane) and let $\varphi : G \rightarrow H$ be the homomorphism $\varphi : r \mapsto e^{2\pi ir}$. Draw the points on the real line which lie in the kernel of φ . Describe similarly the elements in the fibers of φ above the points -1 , i , and $e^{4\pi i/3}$ of H .

Proof. The kernel of φ is the set $\{r \in \mathbb{R} \mid e^{2\pi ir} = 1\}$. Recall that $e^{2\pi ir} = \cos 2\pi r + i \sin 2\pi r$, so the values of r for which $e^{2\pi ir} = 1$ are those where $\cos 2\pi r = 1$, that is, all of the integers.

We similarly obtain the fiber of φ above -1 by considering when $\cos 2\pi r = -1$, which occurs when $r = 1/2, 3/2, 5/2, \dots$, that is, $r \in \{n + \frac{1}{2} \mid n \in \mathbb{Z}\}$. For the fiber above i , we must have $\sin 2\pi r = 1$, which occurs when $r = 1/4, 5/4, 9/4, \dots$, that is, $r \in \{n + \frac{1}{4} \mid n \in \mathbb{Z}\}$. Finally, we have $4\pi/3 = \frac{2}{3} \cdot 2\pi$, so the fiber above $e^{4\pi i/3}$ is $\{n + \frac{2}{3} \mid n \in \mathbb{Z}\}$.

We can also write these as cosets of \mathbb{Z} , so the fibers are $\frac{1}{2} + \mathbb{Z}$, $\frac{1}{4} + \mathbb{Z}$, and $\frac{2}{3} + \mathbb{Z}$, respectively. □

13. (8/31/23)

Repeat the preceding exercise with the map φ replaced by the map $\varphi : r \mapsto e^{4\pi ir}$.

Proof. In this case, the kernel of φ consists of values of r for which $e^{4\pi ir} = 1 \Rightarrow \cos 4\pi r = 1$. The period is now halved, so this occurs when $r \in \{1/2, 1, 3/2, \dots\}$; the kernel is $\{\frac{n}{2} \mid n \in \mathbb{Z}\}$.

The fiber of φ above -1 has $\cos 4\pi r = -1$, when $r = 1/4, 3/4, 5/4, \dots$, that is, $r \in \{\frac{1}{4} + \frac{n}{2} \mid n \in \mathbb{Z}\}$. Above i , we have $\sin 4\pi r = 1$, so $r \in \{\frac{1}{8}, \frac{5}{8}, \dots\}$, and the fiber is $\{\frac{1}{8} + \frac{n}{2} \mid n \in \mathbb{Z}\}$. Finally, above $4\pi/3$, the fiber is $\{\frac{1}{3} + \frac{n}{2} \mid n \in \mathbb{Z}\}$.

If we denote the kernel in this exercise as $\frac{1}{2}\mathbb{Z}$, then as cosets, the fibers are $\frac{1}{4} + \frac{1}{2}\mathbb{Z}$, $\frac{1}{8} + \frac{1}{2}\mathbb{Z}$, and $\frac{1}{3} + \frac{1}{2}\mathbb{Z}$, respectively. \square

14. (8/31/23)

Consider the additive quotient group \mathbb{Q}/\mathbb{Z} .

- (a) Show that every coset of \mathbb{Z} in \mathbb{Q} contains exactly one representative $q \in \mathbb{Q}$ in the range $0 \leq q < 1$.

Proof. The rational numbers under addition constitutes an abelian group, so \mathbb{Z} is a normal subgroup of \mathbb{Q} , and \mathbb{Q}/\mathbb{Z} is therefore well-defined. The elements of the quotient group \mathbb{Q}/\mathbb{Z} are cosets of \mathbb{Z} in \mathbb{Q} , for example, \mathbb{Z} itself (the identity), as well as $\frac{1}{2} + \mathbb{Z}$, $\frac{7}{4} + \mathbb{Z}$, and so on.

Let $q + \mathbb{Z}$ be a coset of \mathbb{Z} (for arbitrary $q \in \mathbb{Q}$). If $q > 1$, then let $n \in \mathbb{Z}$ be the largest integer such that $q - n \geq 0$ (such an integer exists by the well-ordering property). Then $q - n$ is the unique representative for $q + \mathbb{Z}$ in the range $[0, 1)$, since $q - n - 1 < 0$ and $q - n + 1 > 1$. Similarly, if $q < 0$, there exists a unique n such that $0 \leq q + n < 1$. Finally, if $0 \leq q < 1$, then q itself is the unique representative for $q + \mathbb{Z}$ lying between 0 (inclusive) and 1 (exclusive). \square

- (b) Show that every element of \mathbb{Q}/\mathbb{Z} has finite order but that there are elements of arbitrarily large order.

Proof. Let $\frac{a}{b} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ (with $0 \leq \frac{a}{b} < 1$, as above, and suppose that $\frac{a}{b}$ is in lowest terms). Then we have:

$$\underbrace{\left(\frac{a}{b} + \mathbb{Z}\right) + \dots + \left(\frac{a}{b} + \mathbb{Z}\right)}_{b \text{ times}} = \underbrace{\left(\frac{a}{b} + \dots + \frac{a}{b}\right)}_{b \text{ times}} + \mathbb{Z} = a + \mathbb{Z} = \mathbb{Z},$$

so the order of $\frac{a}{b} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$ is at most b , and it therefore has finite order.

However, given a coset $\frac{1}{b} + \mathbb{Z}$ of order b , there always exists an element of higher order, for example $\frac{1}{b+1} + \mathbb{Z}$ and $\frac{1}{2b} + \mathbb{Z}$, which have order $b+1$ and $2b$, respectively. \square

- (c) Show that \mathbb{Q}/\mathbb{Z} is the torsion subgroup of \mathbb{R}/\mathbb{Z} .

Proof. Recall that the torsion subgroup of \mathbb{R}/\mathbb{Z} is the set of elements of \mathbb{R}/\mathbb{Z} of finite order (by Chapter 2.1, Exercise 6., this set is a subgroup when the parent group is abelian).

First, let $q + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$. Since rational numbers are also real numbers, $q + \mathbb{Z}$ also lies in \mathbb{R}/\mathbb{Z} . From 14.b), it has finite order. Therefore it is an element of the torsion subgroup of \mathbb{R}/\mathbb{Z} .

Next, let $x + \mathbb{Z}$ be an element of the torsion subgroup of \mathbb{R}/\mathbb{Z} . Suppose that $|x + \mathbb{Z}| = n < \infty$. Then we have:

$$\underbrace{(x + \mathbb{Z}) + \dots + (x + \mathbb{Z})}_{n \text{ times}} = \underbrace{(x + \dots + x)}_{n \text{ times}} + \mathbb{Z} = nx + \mathbb{Z} = \mathbb{Z},$$

which implies that nx is an integer. Suppose that $nx = m \in \mathbb{Z}$. Then $x = m/n$, and so we have $x \in \mathbb{Q}$, which implies that $x + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$.

Therefore, because inclusion in one implies inclusion in the other and vice-versa, these groups are equal. \square

- (d) Prove that \mathbb{Q}/\mathbb{Z} is isomorphic to the multiplicative group of roots of unity in \mathbb{C}^\times .

Proof. Let $\varphi : \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{C}^\times$ be defined by $\varphi(r + \mathbb{Z}) = e^{2\pi i r}$, where $0 \leq r < 1$. We will show that φ is a bijective homomorphism, and that the groups are thus isomorphic to each other.

First, to show that φ is a homomorphism, note that:

$$\begin{aligned} \varphi((q + \mathbb{Z}) + (r + \mathbb{Z})) &= \varphi((q + r) + \mathbb{Z}) = e^{2\pi i(q+r)}, \text{ and} \\ \varphi(q + \mathbb{Z})\varphi(r + \mathbb{Z}) &= e^{2\pi i q}e^{2\pi i r} = e^{2\pi i q + 2\pi i r} = e^{2\pi i(q+r)}, \end{aligned}$$

as desired.

Next, φ is one-to-one: Suppose $e^{2\pi i r} = \varphi(r + \mathbb{Z}) = \varphi(q + \mathbb{Z})$ for some $r, q \in [0, 1)$. In fact, there are many possible rational numbers fulfilling this if we open the range to all of \mathbb{Q} ; however, because the period of $e^{2\pi i r}$ is 1, there is only one unique value in the range $[0, 1)$, so we must have $r = q$. Therefore φ is injective.

Finally, φ is surjective: Let z be a root of unity with order n . Then z can be expressed as $e^{2\pi i t/n}$ for some $t \in \{0, 1, \dots, n-1\}$. By definition of φ , the rational number $t/n \in [0, 1)$ has $\varphi(t/n) = e^{2\pi i t/n} = z$. Thus φ is a bijective homomorphism, and so \mathbb{Q}/\mathbb{Z} is isomorphic to the roots of unity in \mathbb{C}^\times . \square

15. (9/1/23)

Prove that the quotient of a divisible abelian group by any proper subgroup is also divisible. Deduce that \mathbb{Q}/\mathbb{Z} is divisible.

Proof. Let A be a divisible abelian group and let B be a proper subgroup of A . Since A is abelian, all of its subgroups are normal, so the quotient group A/B is well-defined.

Let $aB \in A/B$ and let $k > 0$. Since A is divisible, there exists an $x \in A$ such that $x^k = a$. Then we have $aB = (x^k)B = (xB)^k$ for $xB \in A/B$, so aB has a k -th root in A/B . Therefore A/B is divisible.

Note that the rational numbers under addition form a divisible abelian group (from Ch. 2.4, Exercise 19.) and the integers are a proper subgroup of the rational numbers. It follows that the quotient group \mathbb{Q}/\mathbb{Z} is divisible. \square

16. (9/5/23)

Let G be a group, let N be a normal subgroup of G , and let $\overline{G} = G/N$. Prove that if $G = \langle x, y \rangle$ then $\overline{G} = \langle \overline{x}, \overline{y} \rangle$. Prove more generally that if $G = \langle S \rangle$ for any subset S of G then $\overline{G} = \langle \overline{S} \rangle$.

Proof. If $G = \langle x, y \rangle$, then we can write any element g as a finite product of x and y , say $g = x^{a_1}y^{b_1} \dots x^{a_n}y^{b_n}$. It follows that, for $\overline{g} \in \overline{G}$, we have:

$$\begin{aligned} \overline{g} = gN &= (x^{a_1}y^{b_1} \dots x^{a_n}y^{b_n})N = (x^{a_1})N(y^{b_1})N \dots (x^{a_n})N(y^{b_n})N = \\ &= (xN)^{a_1}(yN)^{b_1} \dots (xN)^{a_n}(yN)^{b_n} = \overline{x}^{a_1}\overline{y}^{b_1} \dots \overline{x}^{a_n}\overline{y}^{b_n}, \end{aligned}$$

that is, we can write \overline{g} as a finite product of $\overline{x}, \overline{y} \in \overline{G}$, and so $\overline{G} = \langle \overline{x}, \overline{y} \rangle$.

More generally, if $G = \langle S \rangle$, then any element g can be written as a finite product of elements of S , say $g = (s_1^{a_{11}} \dots s_n^{a_{n1}})(s_1^{a_{12}} \dots s_n^{a_{n2}}) \dots (s_1^{a_{1k}} \dots s_n^{a_{nk}})$. Then we have:

$$\overline{g} = gN = \left(\prod_{j=1}^k \left(\prod_{i=1}^n s_i^{a_{ij}} \right) \right) N = \prod_{j=1}^k \prod_{i=1}^n (s_i^{a_{ij}} N) = \prod_{j=1}^k \prod_{i=1}^n (s_i N)^{a_{ij}} = \prod_{j=1}^k \prod_{i=1}^n \overline{s}_i^{a_{ij}},$$

and so similar to above, this means that any element $\overline{g} = gN \in G/N$ can be written as a finite product of $\overline{s}_1, \overline{s}_2, \dots, \overline{s}_n$, and therefore $\overline{G} = \langle \overline{S} \rangle$. \square

17. (9/6/23)

Let G be the dihedral group of order 16: $G = \langle r, s \mid r^8 = s^2 = 1, rs = sr^{-1} \rangle$ and let $\overline{G} = G/\langle r^4 \rangle$ be the quotient of G by the subgroup generated by r^4 (this subgroup is the center of G , hence is normal).

(a) Show that the order of \overline{G} is 8.

The quotient group \overline{G} consists of cosets of the cyclic subgroup of G generated by r^4 , that is, cosets of $\{1, r^4\}$. For example, the coset $s\langle r^4 \rangle$ is $\{s, sr^4\}$. Notice that the coset for sr^4 is the same as for s , and because $\langle r^4 \rangle$ consists of two elements, for each element $x \in G$, there is another element whose coset is the same (namely xr^4). Thus the order of \overline{G} is $16/2 = 8$.

- (b) Exhibit each element of \overline{G} in the form $\overline{s}^a \overline{r}^b$, for some integers a and b .

The elements of \overline{G} are:

$$\begin{array}{ll} \overline{1} = \{1, r^4\} & \overline{s} = \{s, sr^4\} \\ \overline{r} = \{r, r^5\} & \overline{s} \cdot \overline{r} = \{sr, sr^5\} \\ \overline{r}^2 = \{r^2, r^6\} & \overline{s} \cdot \overline{r}^2 = \{sr^2, sr^6\} \\ \overline{r}^3 = \{r^3, r^7\} & \overline{s} \cdot \overline{r}^3 = \{sr^3, sr^7\} \end{array}$$

- (c) Find the order of each of the elements of \overline{G} exhibited in (b).

The orders of the elements of \overline{G} are: $\overline{1} : 1, \overline{r} : 4, \overline{r}^2 : 2, \overline{r}^3 : 4, \overline{s} : 2, \overline{s} \cdot \overline{r} : 2, \overline{s} \cdot \overline{r}^2 : 2, \overline{s} \cdot \overline{r}^3 : 2$.

- (d) Write each of the following elements of \overline{G} in the form $\overline{s}^a \overline{r}^b$, for some integers a and b as in (b):

- $\overline{r\overline{s}} = \overline{sr^7} = \overline{s} \cdot \overline{r}^3$
- $\overline{sr^{-2}s} = \overline{sr^6s} = \overline{ssr^2} = \overline{r}^2$
- $\overline{s^{-1}r^{-1}sr} = \overline{sr^7sr} = \overline{ssr\overline{r}} = \overline{r}^2$

- (e) Prove that $\overline{H} = \langle \overline{s}, \overline{r}^2 \rangle$ is a normal subgroup of \overline{G} and \overline{H} is isomorphic to the Klein 4-group. Describe the isomorphism type of the complete preimage of \overline{H} in G .

Proof. There is a clear isomorphism between \overline{G} and D_8 given by $\overline{x} \in \overline{G} \mapsto x \in D_8$. Because of this, we know that the elements \overline{s} and \overline{r} generate \overline{G} . Since we know the generators of both \overline{G} and \overline{H} , in order to test for normality, we only have to check that the conjugates of the generators of \overline{H} by the generators of \overline{G} are in \overline{H} .

Now powers of \overline{s} and \overline{r} commute with other powers of \overline{s} and \overline{r} , respectively, so we can proceed to:

$$\begin{aligned} \overline{r} \cdot \overline{s} \cdot \overline{r}^{-1} &= \overline{rsr^{-1}} = \overline{rsr^7} = \overline{sr^7r^7} = \overline{sr^{14}} = \overline{sr^6} = \overline{s} \cdot \overline{r}^2 \in \overline{H}, \text{ and} \\ \overline{s} \cdot \overline{r}^2 \cdot \overline{s} &= \overline{sr^2s} = \overline{ssr^6} = \overline{r^6} = \overline{r}^2 \in \overline{H}. \end{aligned}$$

This demonstrates that the conjugates of the generators of \overline{H} by the generators of \overline{G} lie in \overline{H} , and so $\overline{H} \trianglelefteq \overline{G}$.

The elements of \overline{H} are $\overline{1}, \overline{s}, \overline{r}^2$, and $\overline{s} \cdot \overline{r}^2$. Any other product of elements gives an element of \overline{H} . All of these elements have order 2, and so from Ch. 1.1, Exercise 36, $\overline{H} \cong V_4$.

The complete preimage of \overline{H} under the natural projection homomorphism $\pi(g) \mapsto \overline{g} = g\langle r^4 \rangle$ is the set $\{g \in G \mid \pi(g) \in \overline{H}\}$. The elements of G in the complete preimage of \overline{H} are $1, r^2, r^4, r^6, s, sr^2, sr^4$, and sr^6 . This set of elements is isomorphic to D_4 (given by $s, r^2 \in \pi^{-1}(\overline{H}) \mapsto s, r \in D_4$). \square

- (f) Find the center of \overline{G} and describe the isomorphism type of $\overline{H}/Z(\overline{G})$.

The center of \overline{G} consists of the elements of \overline{G} that commute with all other elements of \overline{G} . This is the subgroup $\langle \overline{r^2} \rangle$. Now the quotient group $\overline{H}/Z(\overline{G}) = \langle \overline{s}, \overline{r^2} \rangle / \langle \overline{r^2} \rangle$ consists of the cosets of $\langle \overline{r^2} \rangle$ in \overline{H} , that is, the elements $\langle \overline{r^2} \rangle, \overline{s}\langle \overline{r^2} \rangle$. We do not have $\overline{r^2}$ as a unique element in $\overline{H}/Z(\overline{G})$, because

$$\overline{r^2}\langle \overline{r^2} \rangle = \overline{r^2}\{\overline{1}, \overline{r^2}\} = \{\overline{r^2}, \overline{r^4}\} = \{\overline{1}, \overline{r^2}\} = \langle \overline{r^2} \rangle.$$

Similarly, $\overline{s} \cdot \overline{r^2} \notin \overline{H}/Z(\overline{G})$. Therefore it is isomorphic to the cyclic group Z_2 .

18. (9/10/23)

Let G be the quasidihedral group of order 16: $G = \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle$ and let $\overline{G} = G/\langle \sigma^4 \rangle$ be the quotient of G by the subgroup generated by $\langle \sigma^4 \rangle$ (this subgroup is the center of G , hence is normal).

- (a) Show that the order of \overline{G} is 8.

The elements of \overline{G} are the cosets of the subgroup generated by σ^4 . For example, for $\tau \in G$, the element $\overline{\tau} \in \overline{G} = \{\tau, \tau\sigma^4\}$. As with 17.a), there are two elements in this set, and the cosets of $\langle \sigma^4 \rangle$ partition G . Thus \overline{G} has $16/2 = 8$ elements.

- (b) Exhibit each element of \overline{G} in the form $\overline{\tau^a\sigma^b}$, for some integers a and b .

The elements of \overline{G} are:

$$\begin{array}{ll} \overline{1} = \{1, \sigma^4\} & \overline{\tau} = \{\tau, \tau\sigma^4\} \\ \overline{\sigma} = \{\sigma, \sigma^5\} & \overline{\tau} \cdot \overline{\sigma} = \{\tau\sigma, \tau\sigma^5\} \\ \overline{\sigma^2} = \{\sigma^2, \sigma^6\} & \overline{\tau} \cdot \overline{\sigma^2} = \{\tau\sigma^2, \tau\sigma^6\} \\ \overline{\sigma^3} = \{\sigma^3, \sigma^7\} & \overline{\tau} \cdot \overline{\sigma^3} = \{\tau\sigma^3, \tau\sigma^7\} \end{array}$$

- (c) Find the order of each of the elements of \overline{G} exhibited in (b).

The orders of the elements of \overline{G} are: $\overline{1} : 1, \overline{\sigma} : 4, \overline{\sigma^2} : 2, \overline{\sigma^3} : 4, \overline{\tau} : 2, \overline{\tau} \cdot \overline{\sigma} : 2, \overline{\tau} \cdot \overline{\sigma^2} : 2, \overline{\tau} \cdot \overline{\sigma^3} : 2$.

- (d) Write the following elements of \overline{G} in the form $\overline{\tau^a\sigma^b}$, for some integers a and b as in (b):

- $\overline{\sigma\tau} = \overline{\tau\sigma^3} = \overline{\tau} \cdot \overline{\sigma^3}$
- $\overline{\tau\sigma^{-2}\tau} = \overline{\tau\sigma^6\tau} = \overline{\tau\tau\sigma^{18}} = \overline{\sigma^2} = \overline{\sigma^2}$
- $\overline{\tau^{-1}\sigma^{-1}\tau\sigma} = \overline{\tau\sigma^7\tau\sigma} = \overline{\tau\tau\sigma^{21}\sigma} = \overline{\sigma^{22}} = \overline{\sigma^6} = \overline{\sigma^2}$

- (e) Prove that $\overline{G} \cong D_8$.

Proof. Let $\varphi : \overline{G} \rightarrow D_8$ be defined by $\varphi(\overline{\sigma}) = r$ and $\varphi(\overline{\tau}) = s$. Now $\overline{\sigma}$ and $\overline{\tau}$ are generators for \overline{G} , since (as shown above) every element can be written in the form $\overline{\tau}^a \overline{\sigma}^b$, for some integers a and b . Then φ is a map from \overline{G} to D_8 defined on the generators of \overline{G} to the generators of D_8 . Since both groups have the same cardinality, in order to show that φ is an isomorphism, it only remains to check that the relations of \overline{G} are the same as those in D_8 .

In D_8 , we have $s^2 = r^4 = 1$ and $rs = sr^{-1}$. In part (c) above, we computed the orders of $\overline{\tau}$ and $\overline{\sigma}$, which are 2 and 4, respectively, matching their counterparts in D_8 . Finally, we have $\overline{\sigma} \cdot \overline{\tau} = \overline{\sigma\tau} = \overline{\tau\sigma^3} = \overline{\tau} \cdot \overline{\sigma^3} = \overline{\tau} \cdot \overline{\sigma}^{-1}$, and so the relations hold. Thus $\overline{G} \cong D_8$. \square

19. (9/13/23)

Let G be the modular group of order 16: $G = \langle u, v \mid u^2 = v^8 = 1, vu = uv^5 \rangle$ and let $\overline{G} = G/\langle v^4 \rangle$ be the quotient of G by the subgroup generated by v^4 (this subgroup is contained in the center of G , hence is normal).

- (a) Show that the order of \overline{G} is 8.

The elements of \overline{G} are the cosets of the subgroup generated by v^4 . For example, for $u \in G$, the element $\overline{u} \in \overline{G} = \{u, uv^4\}$. As with 17.a), there are two elements in this set, and the cosets of $\langle v^4 \rangle$ partition G . Thus \overline{G} has $16/2 = 8$ elements.

- (b) Exhibit each element of \overline{G} in the form $\overline{u}^a \overline{v}^b$, for some integers a and b .

The elements of \overline{G} are:

$$\begin{array}{ll} \overline{1} = \{1, v^4\} & \overline{u} = \{u, uv^4\} \\ \overline{v} = \{v, v^5\} & \overline{u} \cdot \overline{v} = \{uv, uv^5\} \\ \overline{v}^2 = \{v^2, v^6\} & \overline{u} \cdot \overline{v}^2 = \{uv^2, uv^6\} \\ \overline{v}^3 = \{v^3, v^7\} & \overline{u} \cdot \overline{v}^3 = \{uv^3, uv^7\} \end{array}$$

- (c) Find the order of each of the elements of \overline{G} exhibited in (b).

The orders of the elements of \overline{G} are: $\overline{1} : 1, \overline{v} : 4, \overline{v}^2 : 2, \overline{v}^3 : 4, \overline{u} : 2, \overline{u} \cdot \overline{v} : 4, \overline{u} \cdot \overline{v}^2 : 2, \overline{u} \cdot \overline{v}^3 : 4$.

- (d) Write each of the following elements of \overline{G} in the form $\overline{u}^a \overline{v}^b$, for some integers a and b as in (b):

- $\overline{vu} = \overline{uv^5} = \overline{u} \cdot \overline{v}$
- $\overline{uv^{-2}u} = \overline{uv^6u} = \overline{uuv^{30}} = \overline{v^{30}} = \overline{v^6} = \overline{v}^2$
- $\overline{u^{-1}v^{-1}uv} = \overline{uv^7uv} = \overline{uuv^{35}v} = \overline{v^{36}} = \overline{v^4} = \overline{1}$

- (e) Prove that \overline{G} is abelian and is isomorphic to $Z_2 \times Z_4$.

Proof. From part (d) above, we deduced that $\overline{vu} = \overline{uv^5} = \overline{uv}$. Since the generators of \overline{G} commute, \overline{G} is an abelian group.

For clarity, let us write the elements of $Z_2 \times Z_4$ as (u^k, v^j) , with $k \in \{0, 1\}$ and $j \in \{0, 1, 2, 3\}$. Then $(u, 1)$ and $(1, v)$ are generators of $Z_2 \times Z_4$.

Now let $\varphi : \overline{G} \rightarrow Z_2 \times Z_4$ be defined on generators \overline{u} and \overline{v} by $\varphi(\overline{u}) = (u, 1)$ and $\varphi(\overline{v}) = (1, v)$. As above, since φ is a map from \overline{G} to $Z_2 \times Z_4$, two groups of equal order, and φ is defined on and to the generators of each, respectively, we only have to check that the relations hold.

In \overline{G} , we have $\overline{u}^2 = 1$, and in $Z_2 \times Z_4$, we have $\varphi(\overline{u})^2 = (u, 1)^2 = (u^2, 1) = (1, 1)$, the identity of $Z_2 \times Z_4$. Also, we have $\overline{v}^4 = 1$ and $\varphi(\overline{v})^4 = (1, v)^4 = (1, v^4) = (1, 1)$. Since \overline{G} and $Z_2 \times Z_4$ are both abelian, there are no other relations we need to check. We conclude that φ is an isomorphism, and that the two groups are isomorphic. \square

20. (9/14/23)

Let $G = \mathbb{Z}/24\mathbb{Z}$ and let $\tilde{G} = G/\langle \overline{12} \rangle$, where for each integer a we simplify notation by writing \tilde{a} as \tilde{a} .

- (a) Show that $\tilde{G} = \{\tilde{0}, \tilde{1}, \dots, \tilde{11}\}$.

Now \tilde{G} consists of the cosets of $\langle \overline{12} \rangle = \{0, 12\}$ in $\mathbb{Z}/24\mathbb{Z}$, for example, $\tilde{4} = 4 + \{0, 12\} = \{4, 16\}$ and $\tilde{21} = 21 + \{0, 12\} = \{21, 33\} = \{9, 21\} = \tilde{9}$. For each $n \in \{0, \dots, 11\}$, the element $n + 12 \in \mathbb{Z}/24\mathbb{Z}$ has the same coset as n , since $n + 12 \cong n \pmod{12}$. Thus the elements of \tilde{G} are:

$$\begin{array}{lll} \tilde{0} = \{0, 12\} & \tilde{4} = \{4, 16\} & \tilde{8} = \{8, 20\} \\ \tilde{1} = \{1, 13\} & \tilde{5} = \{5, 17\} & \tilde{9} = \{9, 21\} \\ \tilde{2} = \{2, 14\} & \tilde{6} = \{6, 18\} & \tilde{10} = \{10, 22\} \\ \tilde{3} = \{3, 15\} & \tilde{7} = \{7, 19\} & \tilde{11} = \{11, 23\} \end{array}$$

- (b) Find the order of each element of \tilde{G} .

$$\begin{array}{lll} \tilde{0} : 1 & \tilde{4} : 3 & \tilde{8} : 3 \\ \tilde{1} : 12 & \tilde{5} : 12 & \tilde{9} : 4 \\ \tilde{2} : 6 & \tilde{6} : 2 & \tilde{10} : 6 \\ \tilde{3} : 4 & \tilde{7} : 12 & \tilde{11} : 12 \end{array}$$

- (c) Prove that $\tilde{G} \cong \mathbb{Z}/12\mathbb{Z}$. (Thus $(\mathbb{Z}/24\mathbb{Z})/(12\mathbb{Z}/24\mathbb{Z}) \cong \mathbb{Z}/12\mathbb{Z}$, just as if we inverted and cancelled the 24's.)

Proof. From Ch. 2.3, Theorem 4, $\mathbb{Z}/n\mathbb{Z}$ is another presentation of the unique cyclic group of order n . It suffices, then, to prove that \tilde{G} is cyclic in order to show that it is isomorphic to $\mathbb{Z}/12\mathbb{Z}$.

We claim that $\tilde{1}$ is a generator for \tilde{G} . For any element $\tilde{a} \in \tilde{G}$ ($0 \leq a < 12$), we can write:

$$\begin{aligned}\tilde{a} &= \{a, a + 12\} = a + \{0, 12\} = \underbrace{(1 + \dots + 1)}_{a \text{ times}} + \{0, 12\} \\ &= \underbrace{(1 + \{0, 12\}) + \dots + (1 + \{0, 12\})}_{a \text{ times}} = \underbrace{\tilde{1} + \dots + \tilde{1}}_{a \text{ times}} \\ &= a \cdot \tilde{1},\end{aligned}$$

and so any element of \tilde{G} is generated from $\tilde{1}$. Thus \tilde{G} is isomorphic to the cyclic group of order 12, which is isomorphic to $\mathbb{Z}/12\mathbb{Z}$. \square

22. (9/14/23)

- (a) Prove that if H and K are normal subgroups of G then their intersection $H \cap K$ is also a normal subgroup of G .

Proof. Let H and K be normal subgroups of G . Let $h \in H \cap K$, so $h \in H$ and $h \in K$. Since both H and K are normal, we have $ghg^{-1} \in H$ and $ghg^{-1} \in K$ for all $g \in G$. It follows that $ghg^{-1} \in H \cap K$ for all $g \in G$. Therefore $H \cap K$ is a normal subgroup of G . \square

- (b) Prove that the intersection of an arbitrary nonempty collection of normal subgroups of a group is a normal subgroup (do not assume the collection is countable).

Proof. Let \mathcal{H} be a nonempty collection of normal subgroups of G . Consider $\bigcap_{H \in \mathcal{H}} H = \{h \in G \mid h \in H \text{ for all } H \in \mathcal{H}\}$. From Ch. 2.1, Exercise 10., we know that \mathcal{H} is itself a subgroup of G . We will show that in this case it is normal in G .

Let $h \in \bigcap_{H \in \mathcal{H}} H$. Then for all $H \in \mathcal{H}$, we have $h \in H$. Since each H is normal in G , we have $ghg^{-1} \in H$ for all $g \in G, H \in \mathcal{H}$. It follows that $ghg^{-1} \in \bigcap_{H \in \mathcal{H}} H$, and therefore $\bigcap_{H \in \mathcal{H}} H$ is normal in G . \square

23. (9/16/23)

Prove that the join of any nonempty collection of normal subgroups of a group is a normal subgroup.

Proof. Let \mathcal{H} be a nonempty collection of subgroups of G and let $\langle \mathcal{H} \rangle$ be their join.

Let $h \in \langle \mathcal{H} \rangle$. Then h can be written as a finite product of elements, say h_1, h_2, \dots, h_n , where each h_i is an element of a corresponding normal subgroup $H_i \in \mathcal{H}$. We write this product:

$$h = (h_1^{a_{11}} \dots h_n^{a_{n1}})(h_1^{a_{12}} \dots h_n^{a_{n2}}) \dots (h_1^{a_{1k}} \dots h_n^{a_{nk}}) = \prod_{j=1}^k \prod_{i=1}^n h_i^{a_{ij}}.$$

Since each h_i belongs to a normal subgroup H_i of G , we have $gh_i g^{-1} \in H_i$ for all $g \in G$. It follows that, for any $m > 0$, we have $gh_i^m g^{-1} \in H_i$ (because $(gh_i g^{-1})^m = gh_i^m g^{-1}$). Now note that, since $(ga_1 g^{-1})(ga_2 g^{-1}) \dots (ga_n g^{-1}) = g(a_1 a_2 \dots a_n) g^{-1}$, the product of conjugates of the constituent elements of h is equal to the conjugate of the product of those elements:

$$\prod_{j=1}^k \prod_{i=1}^n gh_i^{a_{ij}} g^{-1} = g \left(\prod_{j=1}^k \prod_{i=1}^n h_i^{a_{ij}} \right) g^{-1} = ghg^{-1}.$$

The left-hand side of the equation is the product of conjugates of elements h_i that each belong to the corresponding normal subgroup H_i . Therefore the product is an element of the join $\langle \mathcal{H} \rangle$. Since it is equal to the right-hand side, the conjugate of h by any element $g \in G$, we must have $ghg^{-1} \in \langle \mathcal{H} \rangle$ for all $g \in G$. Thus the join of any nonempty collection of normal subgroups of a group is a normal subgroup. \square

24. (9/16/23)

Prove that if $N \trianglelefteq G$ and H is any subgroup of G then $N \cap H \trianglelefteq H$.

Proof. Let $N \trianglelefteq G$, $H \leq G$, and let $n \in N \cap H$, $h \in H$. Consider the conjugate element hnh^{-1} .

Since N is normal in G and $h \in H \Rightarrow h \in G$, we have $hnh^{-1} \in N$.

Since H is a subgroup of G , it is closed and closed under inverses. Also, $n \in N \cap H \Rightarrow n \in H$, so the product hnh^{-1} lies in H . We have both $hnh^{-1} \in N$ and $hnh^{-1} \in H$, so $hnh^{-1} \in N \cap H$.

So the conjugate of any element of $N \cap H$ by any element of H is again an element of $N \cap H$. Therefore $N \cap H$ is normal in H . \square

25. (9/17/23)

- (a) Prove that a subgroup N of G is normal if and only if $gNg^{-1} \subseteq G$ for all $g \in G$.

Proof. Recall that N is defined to be normal in G if $gNg^{-1} = N$ for all $g \in G$. Now if $N \trianglelefteq G$, then clearly $gNg^{-1} \subseteq N$, since $gNg^{-1} = N$.

Suppose that $gNg^{-1} \subseteq N$ for all $g \in G$. Let $x \in N, g \in G$. The conjugate of x by g^{-1} , $g^{-1}x(g^{-1})^{-1}$, must lie in N . Let us write $g^{-1}x(g^{-1})^{-1} = n \in N$. Then we have:

$$x = gg^{-1}xgg^{-1} = g(g^{-1}x(g^{-1})^{-1})g^{-1} = gng^{-1},$$

and so $x \in gNg^{-1}$. This implies that $N \subseteq gNg^{-1}$. Therefore $gNg^{-1} = N$ for all $g \in G$, and so $N \trianglelefteq G$. \square

- (b) Let $G = GL_2(\mathbb{Q})$, let N be the subgroup of upper triangular matrices with integer entries and 1's on the diagonal, and let g be the diagonal matrix with entries 2, 1. Show that $gNg^{-1} \subseteq N$ but g does *not* normalize N .

Proof. Let $N = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, where $n \in \mathbb{Z}$ and let $g = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. The inverse element $g^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$.

Then we have:

$$gNg^{-1} = \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} = \left\{ \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}.$$

Since $2n \in \mathbb{Z}$ for all $n \in \mathbb{Z}$, we have $gNg^{-1} \subseteq N$. However, there is no $n \in \mathbb{Z}$ such that $g \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. In order for g to normalize N , we must have $gNg^{-1} = N$. Therefore g does not normalize N . \square

26. (9/18/23)

Let $a, b \in G$.

- (a) Prove that the conjugate of the product of a and b is the product of the conjugate of a and the conjugate of b . Prove that the order of a and the order of any conjugate of a are the same.

Proof. Let $g \in G$. Then:

$$g(ab)g^{-1} = gabg^{-1} = gag^{-1}gbg^{-1} = (gag^{-1})(gbg^{-1}),$$

as desired.

Next, we show that $a^n = 1$ if and only if $(gag^{-1})^n = 1$. If $a^n = 1$, then we have $(gag^{-1})^n = ga^n g^{-1} = gg^{-1} = 1$. And, if $(gag^{-1})^n = 1$, then we have $ga^n g^{-1} = 1$. Left multiplying by g^{-1} and right-multiplying by g , we obtain $a^n = 1$. Therefore the order of a is equal to the order of any conjugate of a . \square

- (b) Prove that the conjugate of a^{-1} is the inverse of the conjugate of a .

Proof. We can see that:

$$(gag^{-1})(ga^{-1}g^{-1}) = gag^{-1}ga^{-1}g^{-1} = gaa^{-1}g^{-1} = gg^{-1} = 1,$$

and so the conjugate of a^{-1} is the inverse of the conjugate of a . \square

- (c) Let $N = \langle S \rangle$ for some subset S of G . Prove that $N \trianglelefteq G$ if $gSg^{-1} \subseteq N$ for all $g \in G$.

Proof. Let $x \in N$. Since $N = \langle S \rangle$, we can write x as a finite product of elements of S : $x = (s_1^{a_{11}} \dots s_n^{a_{n1}})(s_1^{a_{12}} \dots s_n^{a_{n2}}) \dots (s_1^{a_{1k}} \dots s_n^{a_{nk}})$. Now for each s_i^{ij} , we have $gs_i^{ij}g^{-1} \in N$ (since $gSg^{-1} \subseteq N$). Therefore $gxg^{-1} = g\left(\prod_{j=1}^k \prod_{i=1}^n s_i^{a_{ij}}\right)g^{-1} = \prod_{j=1}^k \prod_{i=1}^n (gs_i^{a_{ij}}g^{-1})$ lies in N (for all $g \in G$), since it is a finite product of elements of N . Thus $N \trianglelefteq G$. \square

- (d) Deduce that if N is the cyclic group $\langle x \rangle$, then N is normal in G if and only if for each $g \in G$, $gxg^{-1} = x^k$ for some $k \in \mathbb{Z}$.

If $N = \langle x \rangle$ is normal in G , then for all $g \in G$, we have $gNg^{-1} = N$, which implies that $gxg^{-1} \in N$. Since all elements of N can be written as x^k for some $k \in \mathbb{Z}$, we have $gxg^{-1} = x^k$.

Conversely, if for all $g \in G$, we have $gxg^{-1} = x^k$ for some $k \in \mathbb{Z}$, then we clearly have $gxg^{-1} \in N$, which implies that $gNg^{-1} \subseteq N$. From Exercise 25. above, this implies that $N \trianglelefteq G$.

Therefore $N \trianglelefteq G$ if and only for each $g \in G$, $gxg^{-1} = x^k$ for some $k \in \mathbb{Z}$.

- (e) Let n be a positive integer. Prove that the subgroup N of G generated by all the elements of G of order n is a normal subgroup of G .

Proof. Let $S \subseteq G$ be the subset of elements of order n in G and let $N = \langle S \rangle$. For each $x \in N$, x can be written as a finite product of elements of S : $x = (s_1^{a_{11}} \dots s_n^{a_{n1}})(s_1^{a_{12}} \dots s_n^{a_{n2}}) \dots (s_1^{a_{1k}} \dots s_n^{a_{nk}})$, where $|s_i| = n$ for each $s_i \in S$. From part (a) above, the conjugate of any element has the same order as the element itself, so $|gs_i g^{-1}| = n$ for each $s_i \in S$, $g \in G$. Then $gs_i g^{-1} \in S \Rightarrow gs_i g^{-1} \in N$, and it follows that:

$$gxg^{-1} = g\left(\prod_{j=1}^k \prod_{i=1}^n s_i^{a_{ij}}\right)g^{-1} = \prod_{j=1}^k \prod_{i=1}^n (gs_i^{a_{ij}}g^{-1})$$

is the product of a elements of N , and so belongs to N itself. Then $gxg^{-1} \in N$ for all $g \in G$, which implies that $gNg^{-1} \subseteq N$, and thus N is normal in G . \square