

# Dummit & Foote Ch. 1.2: Dihedral Groups

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## 1. (1/23/23)

Compute the order of each of the elements in the following groups:

(a)  $D_6$

- $r, r^2$ : 3
- $s, sr, sr^2$ : 2

(b)  $D_8$

- $r$ : 4
- $r^2$ : 2
- $r^3$ : 4
- $s, sr, sr^2, sr^3$ : 2

(c)  $D_{10}$

- $r, r^2, r^3, r^4$ : 5
- $s, sr, sr^2, sr^3, sr^4$ : 2

## 2. (1/23/23)

Use the generators and relations of  $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$  to show that if  $x$  is any element of  $D_{2n}$  which is not a power of  $r$ , then  $rx = xr^{-1}$ .

*Proof.* Let  $x \in D_{2n}$  such that  $x \neq r^k$  for all  $k \in \mathbb{Z}$ . Then, since all elements of  $D_{2n}$  can be written as a product of generators  $s$  and  $r$ , we must have  $x = sr^k$  for some  $k \in \{1, 2, \dots, n-1\}$ . Therefore:

$$rx = rsr^k = sr^{-1}r^k = sr^{k-1} = sr^k r^{-1} = xr^{-1},$$

as desired. □

### 3. (1/25/23)

Use the generators and relations above to show that every element of  $D_{2n}$  which is not a power of  $r$  has order 2. Deduce that  $D_{2n}$  is generated by the two elements  $s$  and  $sr$ , both of which have order 2.

*Proof.* Let  $sr^k \in D_{2n}$ .  $(sr^k)(sr^k) = s(r^k s)r^k = s(sr^{-k})r^k = ssr^{-k}r^k = 1 \cdot 1 = 1$ . Thus the order of elements of the form  $sr^k$ , that is, every element which is not a power of  $r$ , has order 2.

To show that  $D_{2n}$  is generated by  $s$  and  $sr$ , let  $r^k, sr^k \in D_{2n}$ . Now  $s \cdot sr = r$ , so  $(s \cdot sr)^k = r^k$ . To obtain  $sr^k$ , we simply left-multiply the previous by  $s$ :  $s(s \cdot sr)^k = sr^k$ . Thus every element of  $D_{2n}$  can be written as a product of  $s$  and  $sr$ , and so  $\langle s, sr \rangle$  is a generator for  $D_{2n}$ .  $\square$

### 4. (1/25/23)

If  $n = 2k$  is even and  $n \geq 4$ , show that  $z = r^k$  is an element of order 2 which commutes with all elements of  $D_{2n}$ . Show also that  $z$  is the only nonidentity element of  $D_{2n}$  which commutes with all elements of  $D_{2n}$ .

*Proof.* Let  $n = 2k, n \geq 4$ , and let  $z = r^k \in D_{2n}$ .  $z \cdot z = r^k r^k = r^{2k} = r^n = 1$ , so  $z$  has order 2.

Since  $r^k r^k = 1$ , it follows that  $r^k = r^{-k}$  (equivalently,  $z = z^{-1}$ ). Elements of the form  $r^m$  obviously commute with each other, so we only need to show that  $z = r^k$  commutes with elements of the form  $sr^m$ . Now:

$$\begin{aligned} r^k sr^m &= r^k r^{-m} s = r^{-k} r^{-m} s = r^{-k-m} s = (r^{k+m})^{-1} s = \\ &sr^{k+m} = sr^{m+k} = sr^m r^k, \end{aligned}$$

which shows that  $z = r^k$  commutes with elements of the form  $sr^m$ .

Finally, to show that  $z$  is the only nonidentity element which commutes with all elements, we will consider the possible separate cases of the forms of arbitrary elements of  $D_{2n}$ . Let  $a, b \in D_{2n}$ .

- Let  $a = r^m$ . From above,  $a$  commutes with all elements of the form  $r^p$ . Does  $a$  commute with elements of the form  $sr^p$ ?  $r^m sr^p = r^m r^{-p} s = r^{m-p} s$ . On the other hand, we have  $sr^p r^m = sr^{p+m} = r^{-p-m} s$ . These two are equal when  $m - p = -p - m$ , that is, when  $m = -m$  (in  $\mathbb{Z}/n\mathbb{Z}$ ). This only occurs when  $m = n/2 = k$ , and so  $z = r^k$  is the only element of the form  $r^m$  which commutes with all elements of  $D_{2n}$ .
- Let  $a = sr^m$ . As a counterexample, it suffices to show that there is at least one element of  $D_{2n}$  which  $a$  does not commute with:  $r$ .  $sr^m r = sr^{m+1}$ , while  $r sr^m = r r^{-m} s = r^{1-m} s = sr^{m-1}$ . Because  $n \geq 4$ , there are no values of  $m \in \mathbb{Z}/n\mathbb{Z}$  for which  $m + 1 = m - 1$ . Thus elements of the form  $sr^m$  do not commute in  $D_{2n}$ .

This completes the proof that  $z = r^k$  is the only nonidentity element of  $D_{2n}$  which commutes with all other elements.  $\square$

## 5. (1/26/23)

If  $n$  is odd and  $n \geq 3$ , show that the identity is the only element of  $D_{2n}$  which commutes with all elements of  $D_{2n}$ .

*Proof.* This proof is nearly identical to that of Exercise 4. above, only with  $n$  odd instead of even. The proof that elements of the form  $sr^m$  is the same as above. To show that elements of the form  $r^m$  do not commute, we again consider  $r^m sr^p$  and  $sr^p r^m$  and see that we must have  $m = -m$  (in  $\mathbb{Z}/n\mathbb{Z}$ ). Adding  $m$  to both sides, we must have  $2m = 0 \Rightarrow 2m = n$ . However, because  $n$  is odd, this does not occur, and so there are no nonidentity elements of  $D_{2n}$  which commute with all elements of  $D_{2n}$ .  $\square$

## 6. (1/26/23)

Let  $x, y$  be elements of order 2 in any group  $G$ . Prove that if  $t = xy$  then  $tx = xt^{-1}$  (so that if  $n = |xy| < \infty$  then  $x, t$  satisfy the same relations in  $G$  as  $s, r$  do in  $D_{2n}$ ).

*Proof.* Let  $x, y \in G, |x| = |y| = 2$  and let  $t = xy$ . From  $x^2 = y^2 = 1$ , we have  $x = x^{-1}$  and  $y = y^{-1}$ . Then:

$$t = xy \Rightarrow tx = xyx = x(y^{-1}x^{-1}) = x(xy)^{-1} = xt^{-1},$$

as desired.

If  $|xy| = |t| = n < \infty$ , then we have  $t^n = x^2 = 1, tx = xt^{-1}$ . These are the same relations in  $G$  for  $x, t$  as  $s, r$  do in  $D_{2n}$ .  $\square$

## 7. (1/26/23)

Show that  $\langle a, b \mid a^2 = b^2 = (ab)^n = 1 \rangle$  gives a presentation for  $D_{2n}$  in terms of the two generators  $a = s$  and  $b = sr$  of order 2 computed in Exercise 3 above.

*Proof.* First, we will show that the relations for  $r, s$  follow from the relations for  $a, b$ . Let  $a = s$ , so  $s^2 = 1$ . Let  $r = ab, sor^n = (ab)^n = 1$ . The orders of  $r$  and  $s$  are correct, but it remains to be shown that  $sr = r^{-1}s$ . Now  $r = ab = sb$ , so left-multiplying both sides by  $s$ , we obtain  $sr = b$ . Also,  $r^{-1}s = (ab)^{-1}a = b^{-1}a^{-1}a = b^{-1} = b$ . Thus  $sr = r^{-1}s$ , and so the relations for  $r, s$  can be derived from those for  $a, b$ .

Next, we will prove the converse, that the relations for  $a, b$  follow from those for  $r, s$ . Let  $a = s$ , so  $a^2 = 1$ . Let  $b = sr$ , so (from Exercise 3.)  $b^2 = (sr)^2 = 1$ .

It remains to be shown that  $(ab)^n = 1$ . Now  $ab = s(sr) = r$ , and  $r^n = 1$ , so  $(ab)^n = 1$ . Thus the relations for  $a, b$  can be derived from those for  $s, r$ .

Since each set of relations implies the other, they are identical, and thus present the same group, that is,  $D_{2n}$ .  $\square$

## 8. (1/26/23)

Find the order of the cyclic subgroup of  $D_{2n}$  generated by  $r$ .

*Proof.* Let  $R$  be the cyclic subgroup of  $D_{2n}$  generated by  $r$ , consisting of the elements  $\{1, r, r^2, \dots, r^{n-1}\}$ . Intuitively it contains  $n$  elements (half the order of  $D_{2n}$ ). If less, then some  $r^k, k \in \{0, 1, 2, \dots, n-1\}$  is excluded from the subgroup, contra the definition of  $R$ . If more, then for some element  $r^k$  we must have  $k > n$  (or else it would not be a unique element). However, since  $r^n = 1$ , we would then have  $r^k = r^{k-n}r^n = r^{k-n}$ . If  $k - n$  is still greater than  $n$ , we would continue this process until we arrive at a  $k - mn \in \{0, 1, 2, \dots, n-1\}$ . In either case,  $r^k$  is not unique. Therefore the order of  $R$  is exactly  $n$ .  $\square$

## 9. (2/2/23)

Let  $G$  be the group of rigid motions in  $\mathbb{R}^3$  of a tetrahedron. Show that  $|G| = 12$ .

*Proof.* Label the vertices of the tetrahedron 1, 2, 3, 4. It has six edges, each labeled by its vertices: 1-2, 1-3, 1-4, 2-3, 2-4, 3-4. A rigid motion maps one edge to another, in either orientation – that is, a rotation in  $\mathbb{R}^3$  could map 1-2 to 2-3, the identity would map 1-2 to itself, and a reflection might map 1-2 to 2-1 (swapping the positions of vertices 2 and 1).

If we consider that a motion might send one edge to six possible edges, each with two possible orientations (reflected or not), then there must be 12 unique rigid motions in  $\mathbb{R}^3$  of a tetrahedron.  $\square$

## 10. (2/2/23)

Let  $G$  be the group of rigid motions in  $\mathbb{R}^3$  of a cube. Show that  $|G| = 24$ .

*Proof.* Following the pattern of the proof in Exercise 9., there are twelve edges on a cube (labeled by pairs of eight vertices). So a motion might send one edge to twelve possible edges, each with two possible orientations. Thus there are 24 unique rigid motions in  $\mathbb{R}^3$  of a cube.  $\square$

## 11. (2/2/23)

Let  $G$  be the group of rigid motions in  $\mathbb{R}^3$  of an octahedron. Show that  $|G| = 24$ .

*Proof.* Like the cube, the octahedron has twelve edges, and therefore  $12 \cdot 2 = 24$  unique rigid motions.  $\square$

## **12. (2/2/23)**

Let  $G$  be the group of rigid motions in  $\mathbb{R}^3$  of a dodecahedron. Show that  $|G| = 60$ .

*Proof.*  $\square$