

# Dummit & Foote Ch. 2.4: Subgroups Generated by Subsets of a Group

Scott Donaldson

Jul. 2023

## 1. (7/13/23)

Prove that if  $H$  is a subgroup of  $G$  then  $\langle H \rangle = H$ .

*Proof.* Let  $H \leq G$ . To show that  $\langle H \rangle = H$ , we must show that each is contained in the other. By definition,  $H \subseteq \langle H \rangle$ , so it remains to be proven that  $\langle H \rangle \subseteq H$ .

Let  $h \in \langle H \rangle$ . Recall that:

$$\langle H \rangle = \bigcap_{\substack{H \subseteq K \\ K \leq G}} K,$$

that is, for all subset  $K \leq G$  with  $H \subseteq K$ , we have  $h \in K$ . In particular, since  $H$  is a subgroup of  $G$ , we have  $h \in H$ , since  $H \leq G$  and  $H \subseteq H$ . Therefore  $\langle H \rangle \subseteq H$ , and it follows that  $\langle H \rangle = H$ .  $\square$

## 2. (7/17/23)

Prove that if  $A$  is a subset of  $B$  then  $\langle A \rangle \leq \langle B \rangle$ . Give an example where  $A \subseteq B$  with  $A \neq B$  but  $\langle A \rangle = \langle B \rangle$ .

*Proof.* Let  $G$  be a group and let  $A \subseteq B \subseteq G$ . Recall that one definition of  $\langle A \rangle$  is the set of all finite words of elements and inverses of elements of  $A$ , that is, every element of  $\langle A \rangle$  can be written  $a_1^{\varepsilon_1} a_2^{\varepsilon_2} \dots a_n^{\varepsilon_n}$ , where  $n \in \mathbb{Z}, n \geq 0$  and  $a_i \in A, \varepsilon_i = \pm 1$  for each  $i$ . Since  $A$  is a subset of  $B$ ,  $a_i \in A \Rightarrow a_i \in B$ , and so each element  $a_1^{\varepsilon_1} a_2^{\varepsilon_2} \dots a_n^{\varepsilon_n} \in \langle A \rangle$  is also in  $\langle B \rangle$ . Therefore  $\langle A \rangle \leq \langle B \rangle$ .

Now let  $G = \mathbb{Z}/3\mathbb{Z}$ ,  $A = \{1\}$ , and  $B = \{0, 1\}$ . Then we have  $A \subseteq B$  with  $A \neq B$  but  $\langle A \rangle = \langle B \rangle = G$ .  $\square$

## 3. (7/17/23)

Prove that if  $H$  is an abelian subgroup of  $G$  then  $\langle H, Z(G) \rangle$  is abelian. Give an explicit example of an abelian subgroup  $H$  of a group  $G$  such that  $\langle H, C_G(H) \rangle$  is not abelian.

*Proof.* Let  $G$  be a group and let  $H$  be an abelian subgroup of  $G$ . Recall that  $Z(G) = \{g \in G \mid xg = gx \text{ for all } x \in G\}$ , that is, the set of elements of  $G$  that commute with every element of  $G$ . We will show that  $\langle H, Z(G) \rangle$  is an abelian subgroup of  $G$ .

First, we will show that the product of any two elements commutes with both elements. Let  $a, b \in G$  be commuting elements. Then:

$$(ab)a = aba = aab = a(ab), \text{ and } (ab)b = abb = bab = b(ab),$$

as desired.

Now the generated subgroup  $\langle H, Z(G) \rangle$  is constructed from finite words of elements and inverses of elements from  $H$  and  $Z(G)$ . Since  $H$  is an abelian subgroup and elements of  $Z(G)$  (and therefore their inverses) commute with every element of  $G$  (and therefore  $H$ ), it follows that every element in  $\langle H, Z(G) \rangle$  is a product of commuting elements. Every such element therefore commutes with every other element in  $H$  and  $Z(G)$ , as well as any other product of elements of  $H$  and  $Z(G)$ . Thus  $\langle H, Z(G) \rangle$  is an abelian subgroup of  $G$ .

However, it does not follow that  $\langle H, C_G(H) \rangle$  is an abelian subgroup of  $G$ . Let  $G = D_8$  and  $H = \{1, r^2\}$ . The centralizer of  $H$  in  $G$  is all of  $G$ , since every element of  $H$  commutes with every other element of  $G$  (that is,  $H = Z(G)$ ). Then the generated subgroup  $\langle H, C_G(H) \rangle = \langle H, G \rangle = G$ , which is non-abelian.  $\square$

## 4. (7/17/23)

Prove that if  $H$  is a subgroup of  $G$  then  $H$  is generated by the set  $H - \{1\}$ .

*Proof.* Let  $H \leq G$  and consider  $\langle H - \{1\} \rangle$ . If  $H = \{1\}$ , then  $H - \{1\} = \emptyset$ , and so by definition  $\langle H - \{1\} \rangle = \{1\} = H$ .

Suppose  $H \neq \{1\}$ . Then there exists some  $h \in H$  with  $h \neq 1$ . Since  $H$  is a subgroup, it is closed under inverses, so  $h^{-1} \in H$ . We generate  $\langle H - \{1\} \rangle$  by taking finite products of elements of  $H$ , and so  $hh^{-1} = 1 \in \langle H - \{1\} \rangle$ . Further, we cannot construct any element outside of  $H$  by taking products of elements of  $H$ , so we must therefore have  $\langle H - \{1\} \rangle = (H - \{1\}) \cup \{1\} = H$ .  $\square$