Dummit & Foote Ch. 3.3: The Isomorphism Theorems

Scott Donaldson

Oct. 2023

Let G be a group.

1. (10/20/23)

Let F be a finite field of order q and let $n \in \mathbb{Z}^+$. Prove that
$|GL_n(F) : SL_n(F)| = q - 1$.

Proof. Define a map $\varphi : GL_n(F) \to F^\times$ by $\varphi(A) = \det A$ for all $A \in GL_n(F)$. From Ch. 3.1, Exercise 35., φ is a surjective homomorphism with $\ker \varphi = SL_n(F)$.

From Corollary 17, we have:

$$\begin{aligned} |GL_n(F) : \ker \varphi| &= |\varphi(GL_n(F))|, \text{ which implies that} \\ |GL_n(F) : SL_n(F)| &= \underbrace{|F^\times|}_{\varphi \text{ is surjective}} = q - 1, \end{aligned}$$

as desired. □

3. (10/26/23)

Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either

(i) $K \leq H$ or

(ii) $G = HK$ and $|K : K \cap H| = p$.

Proof. Suppose that $H \trianglelefteq G$ with $|G : H| = |G/H| = p$, where p is a prime. Suppose additionally that $K \leq G$ and $K \nleq H$.

Now let $g \in G$. Clearly g belongs to the left coset gH, which we denote $\overline{g} \in G/H$. Since G/H has order p, it is cyclic, and so is generated by any non-identity element (that is, any coset of H other than itself). So $\overline{g}$ generates G/H. Similarly, for any $k \in K, k \notin H$, $\overline{k}$ generates G/H. Therefore $\overline{g} = \overline{k}$ for some g, k, which implies that $g \in kH$. It follows that $g \in KH$, so $G \leq KH$. Since G is closed, we must have $G = KH = HK$.

From the Diamond Isomorphism Theorem, we have $HK/H \cong K/H \cap K$. Since $HK = G$, it follows that $|G : H| = |K : H \cap K|$, and so $|K : K \cap H| = p$. $\square$

4. (10/27/23)

Let C be a normal subgroup of the group A and let D be a normal subgroup of the group B. Prove that $(C \times D) \trianglelefteq (A \times B)$ and $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.

Proof. Let $(c, d) \in C \times D$. Consider the conjugate of (c, d) by $(a, b) \in A \times B$:

$$(a, b)(c, d)(a, b)^{-1} = (a, b)(c, d)(a^{-1}, b^{-1}) = (aca^{-1}, bdb^{-1}).$$

Because $C \trianglelefteq A$, the first coordinate is an element of C, and similarly the second is an element of D. Therefore the conjugate element lies in $C \times D$, and it follows that $(C \times D) \trianglelefteq (A \times B)$.

Next, to show that $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$, define a map $\varphi : (A \times B)/(C \times D) \to (A/C) \times (B/D)$ by $\varphi(\overline{(a, b)}) = (\overline{a}, \overline{b})$. We see that this map is a homomorphism:

$$\begin{aligned}\varphi(\overline{(a_1, b_1)}\,\overline{(a_2, b_2)}) = \varphi(\overline{(a_1a_2, b_1b_2)}) &= (\overline{a_1a_2}, \overline{b_1b_2}) \\ &= (\overline{a_1}, \overline{b_1})(\overline{a_2}, \overline{b_2}) = \varphi(\overline{(a_1, b_1)})\varphi(\overline{(a_2, b_2)}).\end{aligned}$$

It is also surjective by definition, since $(\overline{a}, \overline{b}) = \varphi(\overline{(a, b)})$ is an arbitrary element of $(A/C) \times (B/D)$ with a preimage in $(A \times B)/(C \times D)$.

Finally, it is injective. Let $\varphi(\overline{(a_1, b_1)}) = \varphi(\overline{(a_2, b_2)})$. Then $(\overline{a_1}, \overline{b_1}) = (\overline{a_2}, \overline{b_2})$, so we have $\overline{a_1} = \overline{a_2}$ and $\overline{b_1} = \overline{b_2}$. Since $\overline{a_1} = \overline{a_2}$ implies $(\overline{a_1, x}) = (\overline{a_2, x})$ for all $\overline{x} \in B/D$ and vice-versa, we then have $(\overline{a_1, b_1}) = (\overline{a_2, b_2})$, and so φ is one-to-one.

Thus φ is an isomorphism, which concludes the proof that $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$. $\square$

5. (10/27/23)

Let QD_{16} be the quasidihedral group described in Exercise 11 of Section 2.5. Prove that $\langle \sigma^4 \rangle$ is normal in QD_{16} and use the Lattice Isomorphism Theorem to draw the lattice of subgroups of $QD_{16}/\langle \sigma^4 \rangle$. Which group of order 8 has the same lattice as this quotient? Use generators and relations for $QD_{16}/\langle \sigma^4 \rangle$ to decide the isomorphism type of this group.

Solution. Consider the subgroup $\langle \sigma^4 \rangle$ in QD_{16}. To prove that it is normal, it suffices to check that the conjugates of σ^4 by the generators of QD_{16} lie in $\langle \sigma^4 \rangle$. Now powers of σ commute, so we only need to check $\tau\sigma^4\tau^{-1}$:

$$\tau\sigma^4\tau^{-1} = \tau\sigma^4\tau = \tau\tau\sigma^{12} = \sigma^{12} = \sigma^4 \in \langle \sigma^4 \rangle,$$

so $\langle \sigma^4 \rangle \trianglelefteq QD_{16}$.

Now from the Lattice Isomorphism Theorem, the lattice of subgroups of $QD_{16}/\langle \sigma^4 \rangle$ corresponds to the lattice of subgroups of QD_{16} containing $\langle \sigma^4 \rangle$:

$$\begin{tikzcd}
 & & {\begin{array}{c} QD_{16}/\langle \sigma^4 \rangle = \\ \overline{QD_{16}} \end{array}} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & \\
 & \langle \overline{\sigma^2, \tau} \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \langle \overline{\sigma} \rangle \arrow[d, dash] & \langle \overline{\sigma^2, \tau\sigma} \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle \overline{\tau\sigma^2} \rangle \arrow[drr, dash] & \langle \overline{\tau} \rangle \arrow[dr, dash] & \langle \overline{\sigma^2} \rangle \arrow[d, dash] & \langle \overline{\tau\sigma} \rangle \arrow[dl, dash] & \langle \overline{\tau\sigma^3} \rangle \arrow[dll, dash] \\
 & & \langle \overline{\sigma^4} \rangle = \overline{1} & &
\end{tikzcd}$$

Next, consider the generators and relations for $\overline{QD_{16}}$:

$$\overline{QD_{16}} = \langle \overline{\sigma}, \overline{\tau} \mid \overline{\sigma}^4 = \overline{\tau}^2 = \overline{1}, \overline{\sigma\tau} = \overline{\tau\sigma^3} = \overline{\tau} \cdot \overline{\sigma}^{-1} \rangle.$$

The right-most equation among the relations: $\overline{\tau\sigma^3} = \overline{\tau} \cdot \overline{\sigma}^{-1}$ shows that the generators and relations of this quotient group are identical to those of D_8, mapping $s \in D_8$ to $\overline{\tau} \in \overline{QD_{16}}$ and $r \in D_8$ to $\overline{\sigma} \in \overline{QD_{16}}$. Thus we have $QD_{16}/\langle \sigma^4 \rangle \cong D_8$. □

6. (10/28/23)

Let $M = \langle v, u \rangle$ be the modular group of order 16 described in Exercise 14 of Section 2.5. Prove that $\langle v^4 \rangle$ is normal in M and use the Lattice Isomorphism Theorem to draw the lattice of subgroups of $M/\langle v^4 \rangle$. Which group of order 8 has the same lattice as this quotient? Use generators and relations for $M/\langle v^4 \rangle$ to decide the isomorphism type of this group.

Solution. Recall that the modular group of order 16 is defined as:

$$M = \langle v, u \mid u^2 = v^8 = 1, vu = uv^5 \rangle.$$

As above, to show that $\langle v^4 \rangle$ is normal in M, it suffices to show that the conjugate uv^4u^{-1} lies in $\langle v^4 \rangle$:

$$uv^4u^{-1} = uv^4u = uuv^{20} = v^4 \in \langle v^4 \rangle,$$

so $\langle v^4 \rangle \trianglelefteq M$.

From the Lattice Isomorphism Theorem, the lattice of subgroups of $M/\langle v^4 \rangle$ corresponds to the lattice of subgroups of M containing $\langle v^4 \rangle$:

$$
\begin{tikzcd}[column sep=small]
 & & M/\langle v^4\rangle = \overline{M} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & \\
 & \langle \overline{v^2, u} \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \langle \overline{v} \rangle \arrow[d, dash] & \langle \overline{v^2, uv} \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle \overline{uv^2} \rangle \arrow[drr, dash] & \langle \overline{u} \rangle \arrow[dr, dash] & \langle \overline{v^2} \rangle \arrow[d, dash] & \langle \overline{uv} \rangle \arrow[dl, dash] & \langle \overline{uv^3} \rangle \arrow[dll, dash] \\
 & & \langle \overline{v^4} \rangle = \overline{1} & &
\end{tikzcd}
$$

Next, consider the generators and relations for $M/\langle v^4\rangle$:

$$M/\langle v^4\rangle = \langle \overline{v}, \overline{u} \mid \overline{v}^4 = \overline{u}^2 = \overline{1}, \overline{vu} = \overline{uv^5} = \overline{uv} \rangle.$$

The right-most equation shows that this is an abelian group. Consider the presentation for $Z_2 \times Z_4$ given by $\langle x, y \mid x^2 = y^4 = 1, xy = yx \rangle$. Mapping $\overline{u} \in M/\langle v^4\rangle$ to $x \in Z_2 \times Z_4$ and $\overline{v} \in M/\langle v^4\rangle$ to $y \in Z_2 \times Z_4$, we obtain an isomorphism. Therefore $M/\langle v^4\rangle \cong Z_2 \times Z_4$. $\square$

7. (10/28/23)

Let M and N be normal subgroups of G such that $G = MN$. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$.

Proof. Define a map $\varphi : G/(M \cap N) \to (G/M) \times (G/N)$ by $\varphi(\overline{g}) = (gM, gN)$. We see that φ is a homomorphism:

$$\begin{aligned}\varphi(\overline{g} \cdot \overline{h}) = \varphi(\overline{gh}) = ((gh)M, (gh)N) = (gM \cdot hM, gN \cdot hN)& \\ = (gM, gN)(hM, hN) = \varphi(\overline{g})\varphi(\overline{h}).&\end{aligned}$$

It is also injective. Let $\varphi(\overline{g}) = \varphi(\overline{h})$, so $(gM, gN) = (hM, hN)$, which implies that $gM = hM$ and $gN = hN$. Now let $x \in M \cap N$, so $x \in M$ and $x \in N$. Then $gx \in hM$ (because $gM = hM$) and $gx \in hN$ (because $gN = hN$), so $gx \in hM \cap hN = h(M \cap N)$. The same logic shows that $hx \in g(M \cap N)$, and it follows that $g(M \cap N) = h(M \cap N) \Rightarrow \overline{g} = \overline{h}$, which proves that φ is injective.

Finally, φ is surjective. Let (gM, hN) be an element of $(G/M) \times (G/N)$. Since $G = MN = \{mn \mid m \in M, n \in N\}$, we can write:

$$\begin{aligned}(gM, hN) = \underbrace{((m_1n_1)M, (m_2n_2)N)}_{\text{for some } m_1, m_2 \in M,\, n_1, n_2 \in N}& \\ = (n_1M, m_2N) = ((m_2n_1)M, (m_2n_1)N) = \varphi(\overline{m_2n_1}).&\end{aligned}$$

Thus φ is an isomorphism, and so $G/(M \cap N) \cong (G/M) \times (G/N)$. $\square$

8. (10/29/23)

Let p be a prime and let G be the group of p-power roots of 1 in $\mathbb{C}$ (cf. Exercise 18, Section 2.4). Prove that the map $z \mapsto z^p$ is a surjective homomorphism. Deduce that G is isomorphic to a proper quotient of itself.

Proof. Recall that $G = \{z \in \mathbb{C} \mid z^{p^n} = 1 \text{ for some } n \in \mathbb{Z}^+\}$. Define a map $\varphi : G \to G$ by $\varphi(z) = z^p$ for all $z \in G$. Since $\varphi(z_1 z_2) = (z_1 z_2)^p = z_1^p z_2^p = \varphi(z_1)\varphi(z_2)$, it is a homomorphism.

To show that φ is surjective, let $y \in G$. In particular, let $y = e^{\frac{2\pi i}{p^n}}$ for some $n \in \mathbb{Z}^+$. Let $z = y^{1/p}$. Then $\varphi(z) = z^p = (y^{1/p})^p = y$. And, because $z = y^{1/p} = e^{\frac{2\pi i}{p^n} \cdot \frac{1}{p}} = e^{\frac{2\pi i}{p^{n+1}}}$, we have $z^{p^{n+1}} = 1$, so $z \in G$. Therefore φ is also surjective.

By the 1st Isomorphism Theorem, $\ker \varphi \trianglelefteq G$ and $G/\ker\varphi \cong \varphi(G)$. Now the kernel of φ is the set of those $z \in G$ such that $z^p = 1$, that is, $\langle e^{\frac{2\pi i}{p}} \rangle$. So $G/\ker\varphi$ is a proper quotient of G. Since φ is surjective, $\varphi(G) = G$, and therefore G is isomorphic to a proper quotient of itself. $\square$

9. (10/29/23)

Let p be a prime and let G be a group of order $p^a m$, where p does not divide m. Assume P is a subgroup of G of order p^a and N is a normal subgroup of G of order $p^b n$, where p does not divide n. Prove that $|P \cap N| = p^b$ and $|PN/N| = p^{a-b}$. (The subgroup P of G is called a *Sylow p-subgroup* of G. This exercise shows that the intersection of any Sylow p-subgroup of G with a normal subgroup N is a Sylow p-subgroup of N.)

Proof. Since $P \cap N \leq P$ and $P \cap N \leq N$, the order of $P \cap N$ must divide $|P| = p^a$ and $|N| = p^b n$. And since the order of N divides the order of G, we must have $b \leq a$. Therefore $|P \cap N| = p^c$ for some $c \leq b \leq a$.

Suppose that $c < a$. From the Diamond Isomorphism Theorem, $P \cap N \trianglelefteq P$, so we have:

$$|P : P \cap N| = |P/P \cap N| = \frac{|P|}{|P \cap N|} = \frac{p^a}{p^c} = p^{a-c}.$$

The Diamond Isomorphism Theorem also states that $PN/N \cong P/P \cap N$, which implies that $\frac{|PN|}{|N|} = p^{a-c}$. We know that the order of N is $p^b n$, and so $|PN| = p^{a-c} p^b n = p^{a+b-c} n$. Now note that $a + b - c > a$, which contradicts the order of N dividing the order of G. Therefore we must have $c = b$, and so $|P \cap N| = p^b$. Again, because $PN/N \cong P/P \cap N$, we obtain $|PN/N| = p^{a-b}$. $\square$

10. (11/2/23)

Generalize the preceding exercise as follows. A subgroup H of a finite group G is called a *Hall subgroup* of G if its index is relatively prime to its order:

$(|G:H|,|H|)=1$. Prove that if H is a Hall subgroup of G and $N \trianglelefteq G$, then $H \cap N$ is a Hall subgroup of N and HN/N is a Hall subgroup of G/N.

Proof. We wish to first show that $H \cap N$ is a Hall subgroup of N, that is, that $|H \cap N|$ and $|N : H \cap N|$ are relatively prime.

Toward contradiction, let $x > 1$ divide both $|H \cap N|$ and $|N : H \cap N|$. Since $H \cap N \leq H$, x also divides $|H|$. By Proposition 13,

$$|HN| = \frac{|H||N|}{|H \cap N|} = |H|\frac{|N|}{|H \cap N|} = |H||N : H \cap N|.$$

By Corollary 15, HN is a subgroup of G, so $|HN|$ divides $|G| = |H|\frac{|G|}{|H|}$. Since $|H|$ and $|G:H| = \frac{|G|}{|H|}$ are relatively prime, $|N : H \cap N|$ divides $\frac{|G|}{|H|}$. But if x divides $|H|$, it does not divide $\frac{|G|}{|H|}$, and therefore also does not divide $|N : H \cap N|$, a contradiction. So we must have $x = 1$, which implies that $|H \cap N|$ and then $|N : H \cap N|$ are relatively prime, and $H \cap N$ is a Hall subgroup of N.

Next, consider $HN/N \leq G/N$. We wish to show that HN/N is a Hall subgroup of G/N, that is, that $|HN/N|$ and $|G/N : HN/N|$ are relatively prime. Again suppose that $x > 1$ divides both $|HN/N|$ and $|G/N : HN/N|$. From the Diamond Isomorphism Theorem, $HN/N \cong H/H \cap N$, so x divides $|H/H \cap N| = \frac{|H|}{|H \cap N|}$, and therefore x divides the order of H. And, from the Third Isomorphism Theorem, $(G/N)/(HN/N) \cong G/HN$, so:

$$\begin{aligned} |G/N : HN/N| = |G : HN| = \frac{|G|}{|HN|} &= |G|\frac{|H \cap N|}{|H||N|} \\ &= \frac{|G|}{|H|} \cdot \frac{|H \cap N|}{|N|} = \frac{|G|}{|H|} \cdot \frac{1}{|N : H \cap N|}. \end{aligned}$$

By assumption, x divides this, and so it also divides $\frac{|G|}{|H|} = |G : H|$. However, because H is a Hall subgroup of G, this is a contradiction if $x > 1$. Therefore we must have $x = 1$, so HN/N is a Hall subgroup of G/N. $\square$