Dummit & Foote Ch. 4.2: Groups Acting on Themselves by Left Multiplication — Cayley's Theorem

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Let G be a group and let H be a subgroup of G.

1. (2/12/24)

Let $G = \{1, a, b, c\}$ be the Klein 4-group whose group table is written out in Section 2.5.

(a) Label $1, a, b, c$ with the integers $1, 2, 4, 3$, respectively, and prove that under the left regular representation of G into S_4 the nonidentity elements are mapped as follows:

$$a \mapsto (1\,2)(3\,4) \qquad b \mapsto (1\,4)(2\,3) \qquad c \mapsto (1\,3)(2\,4).$$

Proof. The left regular representation of G into S_4 is the homomorphism $\varphi : G \to S_4$ defined by $\varphi(g) = \sigma_g$, where $\sigma_g : G \to G$ is the permutation of G defined by $\sigma_g(x) = gx$ for all $x \in G$.

Each non-identity element maps the elements as follows:

$$\begin{array}{cccc} \sigma_a(1) = a1 = a & \sigma_a(a) = a^2 = 1 & \sigma_a(b) = ab = c & \sigma_a(c) = ac = b \\ \sigma_b(1) = b1 = b & \sigma_b(a) = ba = c & \sigma_b(b) = b^2 = 1 & \sigma_b(c) = bc = a \\ \sigma_c(1) = c1 = c & \sigma_c(a) = ca = b & \sigma_c(b) = cb = a & \sigma_c(c) = c^2 = 1. \end{array}$$

By the given labeling, this assigns the elements a, b, and c to the pairs of 2-cycles shown above. □

(b) Relabel $1, a, b, c$ as $1, 4, 2, 3$, respectively, and compute the image of each element of G under the left regular representation of G into S_4. Show that the image of G in S_4 under this labeling is the same *subgroup* as the image of G in part (a) (even though the nonidentity elements individually map to different permutations under the two different labelings).

Proof. Under this labeling, the elements a, b, and c are mapped to the permutations $(1\,4)(2\,3), (1\,2)(3\,4)$, and $(1\,3)(2\,4)$, respectively. Although each element maps to a different permutation from part (a), the subgroup of S_4 is the same in both cases. $\square$