Dummit & Foote Ch. 3.3: The Isomorphism Theorems

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Let G be a group.

1. (10/20/23)

Let F be a finite field of order q and let $n \in \mathbb{Z}^+$. Prove that $|GL_n(F): SL_n(F)| = q - 1$.

Proof. Define a map $\varphi: GL_n(F) \to F^{\times}$ by $\varphi(A) = \det A$ for all $A \in GL_n(F)$. From Ch. 3.1, Exercise 35., φ is a surjective homomorphism with $\ker \varphi = SL_n(F)$.

From Corollary 17, we have:

$$|GL_n(F): \ker \varphi| = |\varphi(GL_n(F))|$$
, which implies that $|GL_n(F): SL_n(F)| = \underbrace{|F^{\times}|}_{\varphi \text{ is surjective}} = q - 1,$

as desired. \Box

3. (10/26/23)

Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either

- (i) $K \leq H$ or
- (ii) G = HK and $|K: K \cap H| = p$.

Proof. Suppose that $H \subseteq G$ with |G:H| = |G/H| = p, where p is a prime. Suppose additionally that $K \subseteq G$ and $K \nleq H$.

Now let $g \in G$. Clearly g belongs to the left coset gH, which we denote $\overline{g} \in G/H$. Since G/H has order p, it is cyclic, and so is generated by any non-identity element (that is, any coset of H other than itself). So \overline{g} generates G/H. Similarly, for any $k \in K, k \notin H$, \overline{k} generates G/H. Therefore $\overline{g} = \overline{k}$ for

some g, k, which implies that $g \in kH$. It follows that $g \in KH$, so $G \leq KH$. Since G is closed, we must have G = KH = HK.

From the Diamond Isomorphism Theorem, we have $HK/H \cong K/H \cap K$. Since HK = G, it follows that $|G:H| = |K:H \cap K|$, and so $|K:K \cap H| = p$. \square

4. (10/27/23)

Let C be a normal subgroup of the group A and let D be a normal subgroup of the group B. Prove that $(C \times D) \subseteq (A \times B)$ and $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.

Proof. Let $(c,d) \in C \times D$. Consider the conjugate of (c,d) by $(a,b) \in A \times B$:

$$(a,b)(c,d)(a,b)^{-1} = (a,b)(c,d)(a^{-1},b^{-1}) = (aca^{-1},bdb^{-1}).$$

Because $C \subseteq A$, the first coordinate is an element of C, and similarly the second is an element of D. Therefore the conjugate element lies in $C \times D$, and it follows that $(C \times D) \subseteq (A \times B)$.

Next, to show that $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$, define a map $\varphi: (A \times B)/(C \times D) \to (A/C) \times (B/D)$ by $\varphi((\overline{a}, \overline{b})) = (\overline{a}, \overline{b})$. We see that this map is a homomorphism:

$$\begin{split} \varphi((\overline{a_1,b_1})(\overline{a_2,b_2})) &= \varphi((\overline{a_1a_2,b_1b_2})) = (\overline{a_1}\overline{a_2},\overline{b_1b_2}) \\ &= (\overline{a_1},\overline{b_1})(\overline{a_2},\overline{b_2}) = \varphi((\overline{a_1,b_1}))\varphi((\overline{a_2,b_2})). \end{split}$$

It is also surjective by definition, since $(\overline{a}, \overline{b}) = \varphi((\overline{a}, \overline{b}))$ is an arbitrary element of $(A/C) \times (B/D)$ with a preimage in $(A \times B)/(C \times D)$.

Finally, it is injective. Let $\varphi((\overline{a_1}, \overline{b_1})) = \varphi((\overline{a_2}, \overline{b_2}))$. Then $(\overline{a_1}, \overline{b_1}) = (\overline{a_2}, \overline{b_2})$, so we have $\overline{a_1} = \overline{a_2}$ and $\overline{b_1} = \overline{b_2}$. Since $\overline{a_1} = \overline{a_2}$ implies $(\overline{a_1}, \overline{x}) = (\overline{a_2}, \overline{x})$ for all $\overline{x} \in B/D$ and vice-versa, we then have $(\overline{a_1}, \overline{b_1}) = (\overline{a_2}, \overline{b_2})$, and so φ is one-to-one.

Thus φ is an isomorphism, which concludes the proof that $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.

5. (10/27/23)

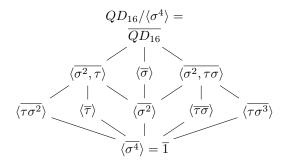
Let QD_{16} be the quasidihedral group described in Exercise 11 of Section 2.5. Prove that $\langle \sigma^4 \rangle$ is normal in QD_{16} and use the Lattice Isomorphism Theorem to draw the lattice of subgroups of $QD_{16}/\langle \sigma^4 \rangle$. Which group of order 8 has the same lattice as this quotient? Use generators and relations for $QD_{16}/\langle \sigma^4 \rangle$ to decide the isomorphism type of this group.

Solution. Consider the subgroup $\langle \sigma^4 \rangle$ in QD_{16} . To prove that it is normal, it suffices to check that the conjugates of σ^4 by the generators of QD_{16} lie in $\langle \sigma^4 \rangle$. Now powers of σ commute, so we only need to check $\tau \sigma^4 \tau^{-1}$:

$$\tau \sigma^4 \tau^{-1} = \tau \sigma^4 \tau = \tau \tau \sigma^{12} = \sigma^{12} = \sigma^4 \in \langle \sigma^4 \rangle,$$

so
$$\langle \sigma^4 \rangle \leq QD_{16}$$
.

Now from the Lattice Isomorphism Theorem, the lattice of subgroups of $QD_{16}/\langle \sigma^4 \rangle$ corresponds to the lattice of subgroups of QD_{16} containing $\langle \sigma^4 \rangle$:



Next, consider the generators and relations for $\overline{QD_{16}}$:

$$\overline{QD_{16}} = \langle \overline{\sigma}, \overline{\tau} \mid \overline{\sigma}^4 = \overline{\tau}^2 = \overline{1}, \overline{\sigma}\overline{\tau} = \overline{\tau}\overline{\sigma}^3 = \overline{\tau} \cdot \overline{\sigma}^{-1} \rangle.$$

The right-most equation among the relations: $\overline{\tau\sigma^3} = \overline{\tau} \cdot \overline{\sigma}^{-1}$ shows that the generators and relations of this quotient group are identical to those of D_8 , mapping $s \in D_8$ to $\overline{\tau} \in \overline{QD_{16}}$ and $r \in D_8$ to $\overline{\sigma} \in \overline{QD_{16}}$. Thus we have $QD_{16}/\langle \sigma^4 \rangle \cong D_8$.

6. (10/28/23)

Let $M=\langle v,u\rangle$ be the modular group of order 16 described in Exercise 14 of Section 2.5. Prove that $\langle v^4\rangle$ is normal in M and use the Lattice Isomorphism Theorem to draw the lattice of subgroups of $M/\langle v^4\rangle$. Which group of order 8 has the same lattice as this quotient? Use generators and relations for $M/\langle v^4\rangle$ to decide the isomorphism type of this group.

Solution. Recall that the modular group of order 16 is defined as:

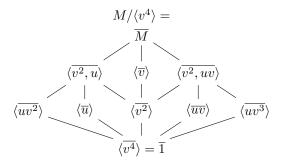
$$M = \langle v, u \mid u^2 = v^8 = 1, vu = uv^5 \rangle.$$

As above, to show that $\langle v^4 \rangle$ is normal in M, it suffices to show that the conjugate uv^4u^{-1} lies in $\langle v^4 \rangle$:

$$uv^4u^{-1} = uv^4u = uuv^{20} = v^4 \in \langle v^4 \rangle,$$

so
$$\langle v^4 \rangle \leq M$$
.

From the Lattice Isomorphism Theorem, the lattice of subgroups of $M/\langle v^4 \rangle$ corresponds to the lattice of subgroups of M containing $\langle v^4 \rangle$:



Next, consider the generators and relations for $M/\langle v^4 \rangle$:

$$M/\langle v^4 \rangle = \langle \overline{v}, \overline{u} \mid \overline{v}^4 = \overline{u}^2 = \overline{1}, \overline{vu} = \overline{uv^5} = \overline{uv} \rangle.$$

The right-most equation shows that this is an abelian group. Consider the presentation for $Z_2 \times Z_4$ given by $\langle x,y \mid x^2 = y^4 = 1, xy = yx \rangle$. Mapping $\overline{u} \in M/\langle v^4 \rangle$ to $x \in Z_2 \times Z_4$ and $\overline{v} \in M/\langle v^4 \rangle$ to $y \in Z_2 \times Z_4$, we obtain an isomorphism. Therefore $M/\langle v^4 \rangle \cong Z_2 \times Z_4$.

7. (10/28/23)

Let M and N be normal subgroups of G such that G = MN. Prove that $G/(M \cap N) \cong (G/M) \times (G/N)$.

Proof. Define a map $\varphi: G/(M \cap N) \to (G/M) \times (G/N)$ by $\varphi(\overline{g}) = (gM, gN)$. We see that φ is a homomorphism:

$$\begin{split} \varphi(\overline{g}\cdot\overline{h}) &= \varphi(\overline{gh}) = ((gh)M,(gh)N) = (gM\cdot hM,gN\cdot hN) \\ &= (gM,gN)(hM,hN) = \varphi(\overline{g})\varphi(\overline{h}). \end{split}$$

It is also injective. Let $\varphi(\overline{g}) = \varphi(\overline{h})$, so (gM,gN) = (hM,hN), which implies that gM = hM and gN = hN. Now let $x \in M \cap N$, so $x \in M$ and $x \in N$. Then $gx \in hM$ (because gM = hM) and $gx \in hN$ (because gN = hN), so $gx \in hM \cap hN = h(M \cap N)$. The same logic shows that $hx \in g(M \cap N)$, and it follows that $g(M \cap N) = h(M \cap N) \Rightarrow \overline{g} = \overline{h}$, which proves that φ is injective.

Finally, φ is surjective. Let (gM, hN) be an element of $(G/M) \times (G/N)$. Since $G = MN = \{mn \mid m \in M, n \in N\}$, we can write:

$$(gM, hN) = \underbrace{((m_1n_1)M, (m_2n_2)N)}_{\text{for some } m_1, m_2 \in M, n_1, n_2 \in N}$$

$$= (n_1M, m_2N) = ((m_2n_1)M, (m_2n_1)N) = \varphi(\overline{m_2n_1}).$$

Thus φ is an isomorphism, and so $G/(M \cap N) \cong (G/M) \times (G/N)$.