# Dummit & Foote Ch. 2.2: Centralizers and Normalizers, Stabilizers and Kernels

Scott Donaldson

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## 1. (6/5/23)

Prove that $C_G(A) = \{g \in G \mid g^{-1}ag = a \text{ for all } a \in A\}$.

*Proof.* By definition, $C_G(A) = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$ (that is, it is the set of elements of $G$ that commute with all elements of $A$).

Let $g \in C_G(A), a \in A$. Then $gag^{-1} = a$, which implies that $ga = ag$, and so left-multiplying by $g^{-1}$ we obtain $a = g^{-1}ag$. Therefore, equivalently, $C_G(A)$ is the set of elements $g \in G$ such that $g^{-1}ag = a$ for all $a \in A$. □

## 2. (6/5/23)

Prove that $C_G(Z(G)) = G$ and deduce that $N_G(Z(G)) = G$.

*Proof.* Recall that $Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$. Let $z \in Z(G)$, so $z$ commutes with every element of $G$.

Also recall that $C_G(A) = \{g \in G \mid gag^{-1} = a \text{ for all } a \in A\}$. When $A = Z(G)$, then every element of $g$ commutes with every element of $A$. Therefore for all $g \in G$, $g \in C_G(Z(G))$. Thus $C_G(Z(G)) = G$.

Note that, since $C_G(A) \leq N_G(A)$ for all subsets $A$, we must have $G = C_G(Z(G)) \leq N_G(Z(G))$. Since there is no greater set of elements, we also have $N_G(Z(G)) = G$. □

## 3. (6/8/23)

Prove that if $A$ and $B$ are subsets of $G$ with $A \subseteq B$ then $C_G(B)$ is a subgroup of $C_G(A)$.

*Proof.* Let $a \in A$ and $g \in C_G(B)$. Then $g$ commutes with every element of $b$, that is, $gb = bg \Rightarrow gbg^{-1} = b$ for all $b \in B$. Since $A \subseteq B$, we also have $gag^{-1} = a$ for all $a \in A$. Therefore $g \in C_G(A)$, which implies that $C_G(B) \subseteq C_G(A)$.

From the introduction to this chapter, centralizers are subgroups, so both $C_G(B) \leq G$ and $C_G(A) \leq G$. Since $C_G(B)$ is contained within $C_G(A)$ and both are subgroups of $G$, $C_G(B)$ must be closed within $C_G(A)$ and closed under inverses within $C_G(A)$, so it is also a subgroup of $C_G(A)$. $\square$