Dummit & Foote Ch. 1.4: Matrix Groups

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1. (3/16/23)

Prove that $|GL_2(\mathbb{F}_2)| = 6$.

Proof. Matrices in $GL_2(\mathbb{F}_2)$ have the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \{0, 1\}$. There are 16 possible matrices of this form (2 options for each entry over 4 entries, $2^4 = 16$).

From the definition of GL_2, we discount matrices with determinant 0. A 2×2 matrix has determinant 0 when $ad - bc = 0$, that is, $ad = bc$. This happens only when $ad = bc = 1$ or $ad = bc = 0$. There is only one matrix where $ad = bc = 1$, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Matrices with determinant 0 have one of a, d and b, c equal to 0. They are the matrices with all zero entries (1), with three zero entries (4), and with two zero entries (a and b, or a and c, or b and d, or c and d) (4).

This leaves us with $16 - 1 - 1 - 4 - 4 = 6$ matrices with nonzero determinants, so the order of $GL_2(\mathbb{F}_2) = 6$. $\square$

2. (3/16/23)

Write out all the elements of $GL_2(\mathbb{F}_2)$ and compute the order of each element.

- $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$: 1 (identity)
- $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: 2
- $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$: 2
- $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$: 3

- $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$: 3
- $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: 2

3. (3/16/23)

Show that $GL_2(\mathbb{F}_2)$ is non-abelian.

Proof. To prove that $GL_2(\mathbb{F}_2)$ is non-abelian, we need only show that it contains two non-commuting elements.
$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

However, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. These products are not equal, so $GL_2(\mathbb{F}_2)$ is non-abelian. $\square$