Dummit & Foote Ch. 2.5: The Lattice of Subgroups of a Group

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Aug. 2023

1. (8/11/23)

Let H and K be subgroups of G. Exhibit all possible sublattices which show only G, 1, H, K, and their joins and intersections. What distinguishes the different drawings?

$$\begin{tikzcd}[column sep=small]
G \arrow[d, dash] \\
K \arrow[d, dash] \\
H \arrow[d, dash] \\
1
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
G \arrow[d, dash] \\
H \arrow[d, dash] \\
K \arrow[d, dash] \\
1
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & G \arrow[dl, dash] \arrow[dr, dash] & \\
H \arrow[dr, dash] & & K \arrow[dl, dash] \\
 & 1 &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & G \arrow[dl, dash] \arrow[dr, dash] & \\
H \arrow[dr, dash] & & K \arrow[dl, dash] \\
 & H \cap K \arrow[d, dash] & \\
 & 1 &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & G \arrow[d, dash] & \\
 & \langle H, K \rangle \arrow[dl, dash] \arrow[dr, dash] & \\
H \arrow[dr, dash] & & K \arrow[dl, dash] \\
 & 1 &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
 & G \arrow[d, dash] & \\
 & \langle H, K \rangle \arrow[dl, dash] \arrow[dr, dash] & \\
H \arrow[dr, dash] & & K \arrow[dl, dash] \\
 & H \cap K \arrow[d, dash] & \\
 & 1 &
\end{tikzcd}$$

The left two lattices show the group structure when either $H \leq K$ or $K \leq H$ (they omit any subgroups of the smaller of the two, as well as any containing subgroups between the larger and G).

The next lattice shows the group structure when H and K are not comparable, their intersection consists only of the identity, and their join is all of G. The final three lattices show the cases where $H \cap K$ is a subgroup not equal to the identity, where $\langle H, K \rangle$ is a subgroup not equal to G, and where both of these occur.

2. (8/11/23)

In each of (a) to (d) list all subgroups of D_{16} that satisfy the given condition.

(a) Subgroups that are contained in $\langle sr^2, r^4 \rangle$

$\{1\}, \langle sr^6 \rangle, \langle sr^2 \rangle, \langle r^4 \rangle$

(b) Subgroups that are contained in $\langle sr^7, r^4 \rangle$

$\{1\}, \langle sr^3 \rangle, \langle sr^7 \rangle, \langle r^4 \rangle$

(c) Subgroups that contain $\langle r^4 \rangle$

$\langle sr^2, r^4 \rangle, \langle s, r^4 \rangle, \langle r^2 \rangle, \langle sr^3, r^4 \rangle, \langle sr^5, r^4 \rangle, \langle s, r^2 \rangle, \langle r \rangle, \langle sr, r^2 \rangle, D_{16}$

(d) Subgroups that contain $\langle s \rangle$

$\langle s, r^4 \rangle, \langle s, r^2 \rangle, \langle D_{16} \rangle$

3. (8/11/23)

Show that the subgroup $\langle s, r^2 \rangle$ of D_8 is isomorphic to V_4.

Proof. The subgroup $\langle s, r^2 \rangle$ of D_8 contains the elements $\{1, s, r^2, sr^2\}$. There is no element in this group of order 4. From Ch. 1.1, Exercise 36, there is only one unique group of order 4 with no element of order 4, the Klein group V_4. Thus $\langle s, r^2 \rangle$ is isomorphic to V_4. $\square$