Dummit & Foote Ch. 3.4: Composition Series and the Hölder Program

Scott Donaldson

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1. (11/2/23)

Prove that if G is an abelian simple group then $G \cong Z_p$ for some prime p (do not assume G is a finite group).

Proof. Since G is simple, the only normal subgroups of G are 1 and G itself. However, since G is abelian, any subgroup of G must be normal, so it follows that G contains *no* subgroups other than 1 and itself.

If $x_1, x_2 \in G$ are distinct generators for G, then $\langle x_1 \rangle$ and $\langle x_2 \rangle$ would be distinct subgroups of G; therefore G is generated by a single element and is a cyclic group. Let us write $G = \langle x \rangle$. If G were infinite, then for any $n > 1$, $\langle x^n \rangle$ would be a distinct subgroup of G, so G must be finite.

Finally, if n divides $|G|$, then from Chapter 2, Theorem 7.(3), G contains a proper subgroup of order n. Therefore $|G|$ has no divisors other than 1 and itself, so we have $|G| = p$ for some prime p. We conclude that $G \cong Z_p$ for some prime p. $\square$