Dummit & Foote Ch. 1: Groups

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1. (11/14/22)

Let G be a group. Determine which of the following binary operations are associative:

a) The operation $\star$ on $\mathbb{Z}$ defined by $a \star b = a - b$:
Not associative. $3 \star (2 \star 1) = 3 - 1 = 2$ but $(3 \star 2) \star 1 = 3 - 2 = 1$.

b) The operation $\star$ on $\mathbb{R}$ defined by $a \star b = a + b + ab$:
Associative.

$$a \star (b \star c) = a \star (b + c + bc) = a + b + c + bc + ab + ac + abc = (a + b + ab) \star c = (a \star b) \star c$$

c) The operation $\star$ on $\mathbb{Q}$ defined by $a \star b = \frac{a+b}{5}$:
Not associative. $0 \star (1 \star 1) = 0 + 2/5 = 2/5$ but $(0 \star 1) \star 1 = 1/5 \star 1 = 6/5 * 1/5 = 6/25$.

d) The operation $\star$ on $\mathbb{Z} \times \mathbb{Z}$ defined by $(a, b) \star (c, d) = (ad + bc, bd)$:
Associative.

$$((a, b) \star (c, d)) \star (e, f) = (ad + bc, bd) \star (e, f) =$$
$$(adf + bcf + bde, bdf) = (a, b) \star (cf + de, df) = (a, b) \star ((c, d) \star (e, f)).$$

e) The operation $\star$ on $\mathbb{Q} - \{0\}$ defined by $a \star b = a/b$:
Not associative. $(1 \star 2) \star 3 = 1/6$ but $1 \star (2 \star 3) = 3/2$.

2. (11/14/22)

Decide which of the binary operations in the preceding exercise are commutative.

a) Not commutative. $1 - 2 = -1$ but $2 - 1 = 1$.

b) Commutative. $a \star b = a + b + ab = b + a + ba = b \star a$.

c) Commutative. $a \star b = \frac{a+b}{5} = \frac{b+a}{5} = b \star a$.

d) Commutative. $(a, b) \star (c, d) = (ad + bc, bd) = (cb + da, db) = (c, d) \star (a, b)$.

e) Not commutative. $1/2 = 1/2$ but $2/1 = 2$.

3. (11/16/22)

Prove that addition of residue classes in $\mathbb{Z}/n\mathbb{Z}$ is associative.

Proof. First, we will show that subtraction in $\mathbb{Z}/n\mathbb{Z}$ is well-defined. Given a representative element $\bar{a}$, $1 \leq \bar{a} \leq n-1$, the element $n-\bar{a}$ is $\bar{a}$'s inverse. $1 \leq n-\bar{a} \leq n-1$, so $n-\bar{a}$ is also a representative element. Also, $\bar{a}+(n-\bar{a}) = n \sim 0$. Thus, subtracting an element $\bar{a}$ from $\bar{b}$ is the same as adding $n-\bar{a}$ to $\bar{b}$, and so subtraction is well-defined.

Now, to show that addition is associative, let $\bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}/n\mathbb{Z}$. Suppose that $(\bar{a}+\bar{b})+\bar{c} = \bar{d}$ and $\bar{a}+(\bar{b}+\bar{c}) = \bar{e}$. Then:

$$\bar{d}-\bar{c} = \bar{a}+\bar{b} \Rightarrow \bar{a} = (\bar{d}-\bar{c})-\bar{b}$$

And:

$$\bar{e}-\bar{a} = \bar{b}+\bar{c} \Rightarrow \bar{e} = ((\bar{d}-\bar{c})-\bar{b})+\bar{b}+\bar{c} = \bar{d}-\bar{c}+\bar{c} = \bar{d}$$

Therefore $\bar{d} = \bar{e}$, so $(\bar{a}+\bar{b})+\bar{c} = \bar{a}+(\bar{b}+\bar{c})$. $\square$

4. (11/16/22)

Prove that multiplication of residue classes in $\mathbb{Z}/n\mathbb{Z}$ is associative.

Proof. Let $\bar{a}, \bar{b}, \bar{c} \in \mathbb{Z}/n\mathbb{Z}$. Then:

$$\overline{a}(\overline{b}\overline{c}) = \overline{a}(\overline{bc}) = \overline{a(bc)}$$

Since the latter expression involves arbitrary integers a, b, c whose representative elements in $\mathbb{Z}/n\mathbb{Z}$ are $\overline{a}, \overline{b}, \overline{c}$, we can use the associative property of standard multiplication:

$$\overline{a(bc)} = \overline{(ab)c} = (\overline{ab})\overline{c} = (\overline{a}\overline{b})\overline{c}$$

Therefore multiplication of residue classes is associative. $\square$

5. (11/16/22)

Prove for all $n > 1$ that $\mathbb{Z}/n\mathbb{Z}$ is not a group under multiplication of residue classes.

Proof. Let $\mathbb{Z}/n\mathbb{Z}$ with $n > 1$. The element 1 is the identity element, since (by multiplication of standard integers), $1 \cdot \bar{a} = \bar{a}$ for all $\bar{a} \in \mathbb{Z}/n\mathbb{Z}$. However, the element 0 has no inverse, since (again by standard multiplication), there is no element $\bar{a}$ such that $0 \cdot \bar{a} = 1$. Thus, $\mathbb{Z}/n\mathbb{Z}$ is not a group under multiplication. $\square$