Dummit & Foote Ch. 3.4: Composition Series and the Hölder Program

Scott Donaldson

Oct. 2023

1. (11/2/23)

Prove that if G is an abelian simple group then $G \cong Z_p$ for some prime p (do not assume G is a finite group).

Proof. Since G is simple, the only normal subgroups of G are 1 and G itself. However, since G is abelian, any subgroup of G must be normal, so it follows that G contains *no* subgroups other than 1 and itself.

If $x_1, x_2 \in G$ are distinct generators for G, then $\langle x_1 \rangle$ and $\langle x_2 \rangle$ would be distinct subgroups of G; therefore G is generated by a single element and is a cyclic group. Let us write $G = \langle x \rangle$. If G were infinite, then for any $n > 1$, $\langle x^n \rangle$ would be a distinct subgroup of G, so G must be finite.

Finally, if n divides $|G|$, then from Chapter 2, Theorem 7.(3), G contains a proper subgroup of order n. Therefore $|G|$ has no divisors other than 1 and itself, so we have $|G| = p$ for some prime p. We conclude that $G \cong Z_p$ for some prime p. $\square$

2. (11/3/23)

Exhibit all 3 composition series for Q_8 and all 7 composition series for D_8. List the composition factors in each case.

The 3 composition series for Q_8 are:

1. $1 \leq \langle -1 \rangle \leq \langle i \rangle \leq Q_8$
2. $1 \leq \langle -1 \rangle \leq \langle j \rangle \leq Q_8$
3. $1 \leq \langle -1 \rangle \leq \langle k \rangle \leq Q_8$

In each series, each composition factor is isomorphic to Z_2 (thus each N_i is normal in N_{i+1}; since there is only one left coset it must equal the only right coset).

The 7 composition series for D_8 are:

1. $1 \leq \langle s \rangle \leq \langle s, r^2 \rangle \leq D_8$

2. $1 \leq \langle sr^2 \rangle \leq \langle s, r^2 \rangle \leq D_8$
3. $1 \leq \langle r^2 \rangle \leq \langle s, r^2 \rangle \leq D_8$
4. $1 \leq \langle r^2 \rangle \leq \langle r \rangle \leq D_8$
5. $1 \leq \langle r^2 \rangle \leq \langle sr, r^2 \rangle \leq D_8$
6. $1 \leq \langle sr \rangle \leq \langle sr, r^2 \rangle \leq D_8$
7. $1 \leq \langle sr^3 \rangle \leq \langle sr, r^2 \rangle \leq D_8$

Again each composition factor is isomorphic to Z_2.

3. (11/3/23)

Find a composition series for the quasidihedral group of order 16 (cf. Exercise 11, Section 2.5). Deduce that QD_{16} is solvable.

Solution. Recall that $QD_{16} = \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle$.

A composition series for QD_{16} is:

$$1 \leq \langle \sigma^4 \rangle \leq \langle \sigma^2 \rangle \leq \langle \sigma \rangle \leq QD_{16},$$

where each composition factor is isomorphic to Z_2. Since Z_2 is abelian, each composition factor is solvable, and so QD_{16} is solvable. □

4. (11/4/23)

Use Cauchy's Theorem and induction to show that a finite abelian group has a subgroup of order n for each positive divisor n of its order.

Proof. Let G be a finite abelian group. Let us suppose that, for all groups H, $|H| < |G|$, H has a subgroup of order n for each positive divisor n of its order.

Let p be a prime dividing $|G|$. From Cauchy's Theorem, there is an $x \in G$ with $|x| = p$. Since G is abelian, $\langle x \rangle$ is normal in G. So the quotient group $G/\langle x \rangle$ is well-defined and has order $|G|/p < |G|$, thus it has a subgroup of order n for each n dividing $|G|/p$.

Let n be a positive divisor of $|G|$. Since $|G| = p \cdot \frac{|G|}{p}$, n divides $\frac{|G|}{p}$. From the induction hypothesis, let $\overline{K}$ be a subgroup of $G/\langle x \rangle$ of order n. For each $\overline{k} \in \overline{K}, \overline{k} \neq \overline{1}$, we must have $k \notin \langle x \rangle$, or else we would have $\overline{k} = k \cdot \langle x \rangle = \langle x \rangle$. Then there is a bijection from $\overline{K}$ onto K given by $\overline{k} \mapsto k$. Thus K is a subgroup of G of order n. □