Dummit & Foote Ch. 1.6: Homomorphisms and Isomorphisms

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1. (3/25/23)

Let $\varphi : G \to H$ be a homomorphism.

(a) Prove that $\varphi(x^n) = \varphi(x)^n$ for all $n \in \mathbb{Z}^+$.

Proof. By induction. When $n = 1, \varphi(x^1) = \varphi(x) = \varphi(x)^1$.

Suppose for some n, $\varphi(x^n) = \varphi(x)^n$. Then $\varphi(x^{n+1}) = \varphi(x^n x)$. By definition, because φ is a homomorphism from G to H, $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$. So $\varphi(x^n x) = \varphi(x^n)\varphi(x)$. By the induction hypothesis, $\varphi(x^n) = \varphi(x)^n$, so this equals $\varphi(x)^{n+1}$.

Therefore $\varphi(x^n) = \varphi(x)^n$ for all $n \in \mathbb{Z}^+$. □

(b) Do part (a) for $n = -1$ and deduce that $\varphi(x^n) = \varphi(x)^n$ for all $n \in \mathbb{Z}$.

This proof diverges slightly from the directions but arrives at the same result.

Note that, for all $x \in G$, $\varphi(x) = \varphi(1 \cdot x) = \varphi(1)\varphi(x)$. Therefore $\varphi(1) = 1$ (in H). Now $1 = \varphi(1) = \varphi(x^n \cdot x^{-n}) = \varphi(x^n)\varphi(x^{-n})$. From part a), this equals $\varphi(x)^n\varphi(x^{-n})$. Left-multiplying both sides by $\varphi(x)^{-n}$, we obtain $\varphi(x^{-n}) = \varphi(x)^{-n}$, as desired.

2. (3/26/23)

If $\varphi : G \to H$ is an isomorphism, prove that $|\varphi(x)| = |x|$ for all $x \in G$. Deduce that any two isomorphic groups have the same number of elements of order n for each $n \in \mathbb{Z}^+$.

Proof. Let $\varphi : G \to H$ be an isomorphism and let $x \in G$. If $|x|$ is finite, then, from 1.a), $\varphi(x^n) = \varphi(x)^n$ and (from 1.b) $\varphi(1) = \varphi(x^n) = \varphi(x)^n = 1 \in H$. The order of the element $\varphi(x)^n \in H$ is therefore at most n. Because φ is an

isomorphism, there is only one element whose image is 1, and that is $\varphi(1) = 1$. Therefore for no $m < n$ do we have $\varphi(x)^m = 1$, and so the order of $\varphi(x)$ is n.

Next, suppose that x has infinite order in G. Then $x^n \neq 1$ for all $n > 0$. Because φ is an isomorphism, we know that only $\varphi(1) = 1 \in H$. Therefore $\varphi(x^n) = \varphi(x)^n \neq 1$ for all $n > 0$. Therefore $|\varphi(x)| = \infty$.

This result is not necessarily true if φ is a homomorphism. For example, φ could send every element of G to the identity in H. (This is a homomorphism: $\varphi(x)\varphi(y) = 1 \cdot 1 = 1$ and $\varphi(x)\varphi(y) = \varphi(xy) = 1$.) Then for all $x \in G$, $|\varphi(x)| = 1$, regardless of the order of x. □

3. (3/27/23)

If $\varphi : G \to H$ is an isomorphism, prove that G is abelian if and only if H is abelian. If φ is a homomorphism, what additional conditions on φ (if any) are sufficient to ensure that if G is abelian, then so is H?

Proof. First, let G be an abelian group and $\varphi : G \to H$ be an isomorphism. Given arbitrary distinct elements of H, because φ is surjective, there are two distinct elements in G whose images are these elements in H. Let $\varphi(x), \varphi(y) \in H$ be distinct elements and $x, y \in G$. Then $\varphi(xy) = \varphi(x)\varphi(y)$. Also, because x and y commute, $\varphi(xy) = \varphi(yx) = \varphi(y)\varphi(x)$. Therefore $\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$, so H is an abelian group.

Next, let H be an abelian group. Again let $\varphi(x), \varphi(y) \in H$ and $x, y \in G$. Then $\varphi(x)\varphi(y) = \varphi(xy)$. Also, $\varphi(x)\varphi(y) = \varphi(y)\varphi(x) = \varphi(yx)$. So $\varphi(xy) = \varphi(yx)$. Because φ is one-to-one, this implies that $xy = yx$, and so G is an abelian group.

If φ is a homomorphism, then G being an abelian group does not imply that H is abelian. For example, H could be a non-abelian group and φ could send every element of G to the identity in H.

A sufficient condition for a homomorphism $\varphi : G \to H$ to ensure that if G is abelian, then so is H, is that φ is surjective. Then for all $h \in H$, $h = \varphi(x)$ for some $x \in G$ (possibly more than one x). Let $h_1, h_2 \in H$ with $h_1 = \varphi(x_1) = \varphi(x_2) = ...$ and $h_2 = \varphi(y_1) = \varphi(y_2) = ...$ and with $x_i, y_j \in G$. φ is a homomorphism, so for any i, j, $\varphi(x_i y_j) = \varphi(x_i)\varphi(y_j) = h_1 h_2$. Also, because G is abelian, $\varphi(x_i y_j) = \varphi(y_j x_i) = \varphi(y_j)\varphi(x_i) = h_2 h_1$. Therefore $h_1 h_2 = h_2 h_1$, so H is abelian. □

4. (3/27/23)

Prove that the multiplicative groups $\mathbb{R} - \{0\}$ and $\mathbb{C} - \{0\}$ are not isomorphic.

Proof. For any $x \in \mathbb{R} - \{0\}$, $x \neq \pm 1$, x has infinite order. The proof of this is as follows: Let $x \in \mathbb{R} - \{0, \pm 1\}$. If the absolute value of x is greater than 1, then the absolute value of x^n is greater than 1 for all n, and by induction x has infinite order. If the absolute value of x is less than 1, then the absolute value

of x^n is less than 1 for all n, and by induction x has infinite order. So 1 and -1 are the only elements of $\mathbb{R} - \{0\}$ with finite order.

In $\mathbb{C} - \{0\}$, i and $-i$ have order 4. From 2., isomorphic groups have the same number of elements of order n for each $n \in \mathbb{Z}^+$. However, $\mathbb{R} - \{0\}$ has no elements of order 4, and $\mathbb{C} - \{0\}$ has at least 2. Therefore they are not isomorphic. □

5. (3/27/23)

Prove that the additive groups $\mathbb{R}$ and $\mathbb{Q}$ are not isomorphic.

Proof. Given that $\mathbb{R}$ and $\mathbb{Q}$ do not have the same cardinality ($\mathbb{R}$ is uncountable while $\mathbb{Q}$ is countably infinite), there is no map $\varphi : \mathbb{Q} \to \mathbb{R}$ that is surjective. An isomorphism is a bijection that is necessarily surjective, and so the two groups are not isomorphic.

Alternatively, consider the homomorphism $\varphi : \mathbb{Q} \to \mathbb{R}$ defined by $\varphi(q) = q$. Such a map is injective but not surjective: There is no $q \in \mathbb{Q}$ with $\varphi(q) = \sqrt{2} \in \mathbb{R}$. If we attempt to make φ surjective by assigning $\varphi(q_1) = \sqrt{2}$ for some q_1, then q_1 now has no preimage in $\mathbb{Q}$, and so we must find a q_2 and assign $\varphi(q_2) = q_1$. However, now q_2 has no preimage. This process continues *ad infinitum*, and φ is forever not surjective. Therefore $\mathbb{R}$ and $\mathbb{Q}$ are not isomorphic. □

6. (3/27/23)

Prove that the additive groups $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic.

Proof. Consider a homomorphism $\varphi : \mathbb{Z} \to \mathbb{Q}$. For all $n \in \mathbb{Z}$, $\varphi(0) = \varphi(n + (-n)) = \varphi(n) + \varphi(-n)$. From 1.b), $\varphi(0) = 0$, so φ preserves inverses: $\varphi(-n) = -\varphi(n)$. That is, $\varphi(n) = q$ implies that $\varphi(-n) = -q$.

We also claim that, if $\varphi(1) = k$, then φ assigns all integers to their product with k in $\mathbb{Q}$. Since φ preserves inverses, we only have to show this for $n \in \mathbb{Z}^+$, by induction (base case given): Suppose that $\varphi(n) = kn$ for some $n \in \mathbb{Q}^+$. Then $\varphi(n+1) = \varphi(n) + \varphi(1) = kn + k = k(n+1)$, as desired. Therefore φ assigns all integers to their product with k in $\mathbb{Q}$.

But now it is impossible for φ to be surjective, because only integer multiples of k have preimages in $\mathbb{Z}$. For example, $k/2 \in \mathbb{Q}$ has no preimage. Therefore $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic. □

7. (3/27/23)

Prove that D_8 and Q_8 are not isomorphic.

Proof. $s, sr, sr^2, sr^3 \in D_8$ all have order 2. However, in Q_8, only -1 has order 2. From 2., isomorphic groups must have the same number of elements of each order. Therefore D_8 and Q_8 are not isomorphic. □

8. (3/28/23)

Prove that if $n \neq m$, S_n and S_m are not isomorphic.

Proof. Without loss of generality, let $n > m$. From Chapter 1.3, the order of a symmetric group S_n is $n!$. Then S_n contains $n!$ elements, and S_m contains $m!$ elements. It is trivial to show that $n > m \Rightarrow n! > m!$. Since the two groups do not have the same cardinality, there is no bijection between them. Thus S_n and S_m are not isomorphic. □

9. (3/28/23)

Prove that D_{24} and S_4 are not isomorphic.

Proof. D_{24} has 24 elements, and S_4 has 24 elements. They are both non-abelian. In order to prove that they are not isomorphic, then, let us consider the orders of each group's respective elements.

D_{24} has 13 elements of order 2: $\{sr^i \mid i \in \{0, ..., 11\}\}$ and r^6.

The elements of order 2 in S_4 are those permutations with cycle decompositions that are disjoint 2-cycles:
$\{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$. So there are 9 elements of order 2 in S_4.

Since D_{24} and S_4 do not have the same number of elements of order 2, they are not isomorphic. □

10. (3/31/23)

Fill in the details of the proof that the symmetric groups S_Δ and S_Ω are isomorphic if $|\Delta| = |\Omega|$ as follows: Let $\theta : \Delta \to \Omega$ be a bijection. Define

$$\varphi : S_\Delta \to S_\Omega \text{ by } \varphi(\sigma) = \theta \circ \sigma \circ \theta^{-1} \text{ for all } \sigma \in S_\Delta$$

and prove the following:

(a) φ is well-defined, that is, if σ is a permutation of Δ then $\theta \circ \sigma \circ \theta^{-1}$ is a permutation of Ω.

To show that φ is well-defined, we need to show that it assigns a given permutation of Δ to a unique permutation of Ω.

An arbitrary permutation σ is a bijection from Δ to itself. It is represented with a cycle decomposition that shows how it assigns a given element of Δ to another element. For σ and a given element s_1, we can say that σ assigns s_1 to $s_2 \in \Delta$.

Since Δ and Ω have the same cardinality, there exists a bijection θ between them, and we can say that θ assigns distinct $s_1, s_2 \in \Delta$ to distinct $t_1, t_2 \in \Omega$, respectively.

Now let us consider what happens when we apply φ to σ. By definition, $\varphi(\sigma) = \theta \circ \sigma \circ \theta^{-1}$. θ^{-1} is a bijection: $\Omega \to \Delta$, σ is a bijection: $\Delta \to \Delta$, and θ is a bijection: $\Delta \to \Omega$. Applying the compositions, we see that $\varphi(\sigma)$ is a map from $\Omega \to \Omega$ (not yet proven to be a bijection).

t_1 is an arbitrary element of Ω with preimage $s_1 \in \Delta$. Then:

$$\varphi(\sigma)(t_1) = \theta(\sigma(\theta^{-1}(t_1))) = \theta(\sigma(s_1)) = \theta(s_2) = t_2,$$

that is, $\varphi(\sigma)$ is a permutation of Ω that uniquely assigns t_1 to t_2. Therefore φ is well-defined.

(b) φ is a bijection from S_Δ onto S_Ω.

We have shown that φ is a well-defined map from S_Δ onto S_Ω. However, it remains to be shown that φ is a bijection.

To show that φ is invertible, define a map $\gamma : S_\Omega \to S_\Delta$, with $\gamma(\tau) = \theta^{-1} \circ \tau \circ \theta$ for $\tau \in \Omega$. The proof above suffices to show that γ is well-defined.

Consider what happens when we take $\gamma(\varphi(\sigma))$:

$$\gamma(\varphi(\sigma)) = \gamma(\theta \circ \sigma \circ \theta^{-1}) = \theta^{-1} \circ (\theta \circ \sigma \circ \theta^{-1}) \circ \theta = (\theta^{-1}\theta) \circ \sigma \circ (\theta^{-1}\theta) = \sigma.$$

That is, $\gamma(\varphi(\sigma)) = \sigma$ for all $\sigma \in S_\Delta$. Therefore $\gamma = \varphi^{-1}$. Since φ has a well-defined inverse, it is a bijection from S_Δ onto S_Ω.

(c) φ is a homomorphism, that is, $\varphi(\sigma \circ \tau) = \varphi(\sigma) \circ \varphi(\tau)$.

We apply the function compositions:

$$\varphi(\sigma \circ \tau) = (\theta \circ \sigma \circ \theta^{-1}) \circ (\theta \circ \tau \circ \theta^{-1}) = \theta \circ \sigma \circ (\theta^{-1} \circ \theta) \circ \tau \circ \theta^{-1} = \theta \circ \sigma \circ \tau \circ \theta^{-1} = \varphi(\sigma) \circ \varphi(\tau).$$

Thus φ is a homomorphism, and since it is also a bijection, the groups S_Δ and S_Ω are isomorphic.

11. (4/1/23)

Let A and B be groups. Prove that $A \times B \cong B \times A$.

Proof. Consider the map $\varphi : A \times B \to B \times A$ defined by $\varphi(a, b) = (b, a)$. φ is injective, since $\varphi(a_1, b_1) = \varphi(a_2, b_2) \Rightarrow (b_1, a_1) = (b_2, a_2) \Rightarrow a_1 = a_2$ and $b_1 = b_2$. φ is surjective, since for every $(b, a) \in B \times A$, there exists by definition $(a, b) \in A$ with $\varphi(a, b) = (b, a)$. Therefore φ is a bijection from $A \times B \to B \times A$.

φ is also a homomorphism: Let $(a_1, b_1), (a_2, b_2) \in A \times B$. Then:

$$\varphi((a_1, b_1)(a_2, b_2)) = \varphi(a_1a_2, b_1b_2) = \\ (b_1b_2, a_1a_2) = (b_1, a_1)(b_2, a_2) = \varphi(a_1, b_1)\varphi(a_2, b_2).$$

Since φ is a bijective homomorphism, it is an isomorphism, and so $A \times B \cong B \times A$. □

12. (4/5/23)

Let A, B, and C be groups and let $G = A \times B$ and $H = B \times C$. Prove that $G \times C \cong A \times H$.

Proof. Let $\varphi : G \times C \to A \times H$ defined by $\varphi \circ ((a, b), c) = (a, (b, c))$. We will show that φ is a bijective homomorphism, that is, an isomorphism, and thus that $G \times C \cong A \times H$.

To show that φ is injective, let $((a_1, b_1), c_1)$ and $((a_2, b_2), c_2) \in G \times C$, and suppose that applying φ to both gives the same element $(a, (b, c)) \in A \times H$. Then, by definition of φ, $a_1 = a$ and $a_2 = a$, so $a_1 = a_2$. The same logic shows that $b_1 = b_2$ and $c_1 = c_2$. Thus the two elements in G are in fact the same element, and therefore φ is injective.

To show that φ is surjective, let $(a, (b, c)) \in A \times H$. Then, by definition of φ, there exists $a, b, c \in A, B, C$, respectively, such that for $((a, b), c) \in G \times C$, $\varphi \circ ((a, b), c) = (a, (b, c))$. Therefore φ is surjective, and so it is a bijection.

Finally, let $((a_1, b_1), c_1)$ and $((a_2, b_2), c_2) \in G \times C$. Then:

$$\varphi \circ (((a_1, b_1), c_1)((a_2, b_2), c_2)) = \varphi \circ ((a_1a_2, b_1b_2), c_1c_2) = (a_1a_2, (b_1b_2, c_1c_2)) = \\ (a_1, (b_1, c_1))(a_2, (b_2, c_2)) = \varphi \circ ((a_1, b_1), c_1)\varphi \circ ((a_2, b_2), c_2).$$

Thus φ is an isomorphism from $G \times C \to A \times H$, and so $G \times C \cong A \times H$. □

13. (4/5/23)

Let G and H be groups and let $\varphi : G \to H$ be a homomorphism. Prove that the image of $\varphi, \varphi(G)$, is a subgroup of H. Prove that if φ is injective then $G \cong \varphi(G)$.

Proof. To prove that $\varphi(G)$ is a subgroup of H, we must show that it is closed under the binary operation of H and that it is closed under inverses (the other group properties follow from these). By definition, for $a, b \in G$, $\varphi(a), \varphi(b) \in H$. Since φ is a homomorphism, their product, $\varphi(a)\varphi(b) = \varphi(ab)$, is also an element of H. Thus $\varphi(G)$ is closed under the binary operation of H.

To show that $\varphi(G)$ is closed under inverses, let $a \in G$. From 1.b), $\varphi(1) = 1$. Also, $\varphi(1) = \varphi(aa^{-1}) = \varphi(a)\varphi(a^{-1})$. Therefore $\varphi(a^{-1}) = \varphi(a)^{-1}$, and so $\varphi(G)$ is closed under inverses. Thus it is a subgroup of H.

Now suppose that φ is injective. To prove that G is then isomorphic to $\varphi(G)$, we must show that φ is an isomorphism, that is, that it is also surjective onto $\varphi(G)$. By definition, since $\varphi(G) = \{h \in H \mid h = \varphi(g) \text{ for some } g \in G\}$, φ is surjective onto $\varphi(G)$. Thus φ is an isomorphism, and so $G \cong \varphi(G)$. □

14. (4/9/23)

Let G and H be groups and let $\varphi : G \to H$ be a homomorphism. Define the *kernel* of φ to be $\{g \in G \mid \varphi(g) = 1_H\}$ (so the kernel is the set of elements in G which map to the identity in H, i.e., is the fiber over the identity of H). Prove that the kernel of φ is a subgroup of G. Prove that φ is injective if and only if the kernel of φ is the identity subgroup of G.

Proof. To show that the kernel of φ is a subgroup of G, we need to show that it is closed under the binary operation of G and that it is closed under inverses.

Let g_1, g_2 be in the kernel of φ. Then $\varphi(g_1) = \varphi(g_2) = 1_H$. Since φ is a homomorphism, $1_H = \varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2)$. So the product $g_1 g_2$ of arbitrary elements in the kernel of G is also in the kernel of G. Thus the kernel of φ is closed under the binary operation of G.

From 1.b), $\varphi(1) = 1_H$. Also, $1_H = \varphi(1) = \varphi(gg^{-1}) = \varphi(g)\varphi(g^{-1}) = 1_H \cdot \varphi(g^{-1})$, which implies that $\varphi(g^{-1}) = 1_H$, so g^{-1} is also in the kernel of φ. The kernel of φ is closed under inverses, and thus it is a subgroup of G.

Now we will show that φ is injective if and only if the kernel of φ is $\{1_G\}$.

First, let φ be an injective homomorphism from G to H. So for any $g_1, g_2 \in G$, $\varphi(g_1) = \varphi(g_2)$ implies that $g_1 = g_2$. Let g be in the kernel of φ, so $\varphi(g) = 1_H$. Also, $\varphi(1_G) = 1_H$, which implies that $g = 1_G$, so the only unique element in the kernel of φ is 1_G.

Next, suppose the kernel of φ is $\{1_G\}$. So $g \in G, g \neq 1_G$ implies that $\varphi(g) \neq 1_H$. Suppose that $g_1, g_2 \in G, g_1, g_2 \neq 1_G$ such that $\varphi(g_1) = \varphi(g_2) = h \in H$. From 1.b), $\varphi(g_2) = h$ implies that $\varphi(g_2)^{-1} = \varphi(g_2^{-1}) = h^{-1}$. So $\varphi(g_1)\varphi(g_2^{-1}) = hh^{-1} = 1_H$. Because φ is a homomorphism, $\varphi(g_1 g_2^{-1}) = 1_H$. Because the only element in the kernel of φ is 1_G, we must have $g_1 g_2^{-1} = 1_G$, which implies that $g_1 = g_2$. Therefore φ is an injective map from G to H.

This completes the proof that φ is injective if and only if the kernel of φ is $\{1_G\}$. □

15. (4/9/23)

Define a map $\pi : \mathbb{R}^2 \to \mathbb{R}$ by $\pi((x, y)) = x$. Prove that π is a homomorphism and find the kernel of π.

Proof. To show that π is a homomorphism, let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. Then $\pi((x_1, y_1)(x_2, y_2)) = \pi((x_1 x_2, y_1 y_2)) = x_1 x_2$. Also, $\pi((x_1, y_1)) \cdot \pi((x_2, y_2)) = x_1 x_2$. Thus π is a homomorphism.

By definition, the kernel of π is the set $\{(x, y) \in \mathbb{R}^2 \mid \pi(x, y) = 1\}$. Now $\pi((x, y)) = 1$ if and only if $x = 1$. Note that $x = 1 \Rightarrow \pi((1, y)) = 1$ and $\pi((x, y)) = 1 \Rightarrow x = 1$. So the kernel of π is $\{(1, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$. □

16. (4/10/23)

Let A and B be groups and let G be their direct product, $A \times B$. Prove that the maps $\pi_1 : G \to A$ and $\pi_2 : G \to B$ defined by $\pi_1((a, b)) = a$ and $\pi_2((a, b)) = b$ are homomorphisms and find their kernels.

Proof. To show that π_1 and π_2 are homomorphisms, let $(a_1, b_1), (a_2, b_2) \in G$. Then $\pi_1((a_1, b_1)(a_2, b_2)) = \pi_1((a_1a_2, b_1, b_2)) = a_1a_2$ and $\pi_1((a_1, b_1)) \cdot \pi_1((a_2, b_2)) = a_1a_2$, so π_1 is a homomorphism. Similarly, $\pi_2((a_1, b_1)(a_2, b_2)) = b_1b_2 = \pi_2((a_1, b_1)) \cdot \pi_2((a_2, b_2))$, so it is also a homomorphism.

Now by identical proof to 15., the kernel of π_1 is $\{(1, b) \in G \mid b \in B\}$ and the kernel of π_2 is $\{(a, 1) \in G \mid a \in A\}$. □

17. (4/11/23)

Let G be any group. Prove that the map from G to itself defined by $g \mapsto g^{-1}$ is a homomorphism if and only if G is abelian.

Proof. First, let $\varphi : G \to G$ be defined by $\varphi(g) = g^{-1}$, and let G be abelian. Then for $g, h \in G$, $\varphi(gh) = (gh)^{-1}$. If we let $x = (gh)^{-1}$, then $ghx = 1 \Rightarrow hx = g^{-1} \Rightarrow x = h^{-1}g^{-1}$, so $\varphi(gh) = h^{-1}g^{-1}$. Also, $\varphi(g)\varphi(h) = g^{-1}h^{-1} = h^{-1}g^{-1}$ (since any elements of G commute), and thus φ is a homomorphism.

Next, let φ be the map $G \to G$ defined by $\varphi(g) = g^{-1}$. As above, $\varphi(g)\varphi(h) = g^{-1}h^{-1}$ and $\varphi(gh) = (gh)^{-1} = h^{-1}g^{-1}$. If φ is a homomorphism, then we must have $\varphi(g)\varphi(h) = \varphi(gh)$, that is, $g^{-1}h^{-1} = h^{-1}g^{-1}$. This implies that $(hg)^{-1}g = h^{-1} \Rightarrow (hg)^{-1}gh = 1 \Rightarrow gh = hg$, and so G is abelian.

Therefore the map from G to itself defined by $g \mapsto g^{-1}$ is a homomorphism if and only if G is abelian. □

18. (4/14/23)

Let G be any group. Prove that the map from G to itself defined by $g \mapsto g^2$ is a homomorphism if and only if G is abelian.

Proof. First, let $\varphi : G \to G$ be defined by $\varphi(g) = g^{-1}$, and let G be abelian. Then for $g, h \in G$, $\varphi(gh) = (gh)^2 = ghgh = gghh = g^2h^2 = \varphi(g)\varphi(h)$, so φ is a homomorphism.

Next, φ be the map from G to G defined by $\varphi(g) = g^2$, and suppose that φ is a homomorphism. Then $\varphi(g)\varphi(h) = \varphi(gh) \Rightarrow g^2h^2 = (gh)^2$. Then:

$$gghh = ghgh \Rightarrow ghh = hgh \Rightarrow gh = hg,$$

so G is abelian.

Therefore the map from G to itself defined by $g \mapsto g^2$ is a homomorphism if and only if G is abelian. $\square$

19. (4/14/23)

Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}^+\}$. Prove that for any fixed integer $k > 1$ the map from G to itself defined by $z \mapsto z^k$ is a surjective homomorphism but is not an isomorphism.

Proof. Using de Moivre's formula, we can rewrite any element $z \in G$ as $z = \cos\frac{2\pi m}{n} + i\sin\frac{2\pi m}{n}$ for a unique $m \in \{0, ..., n-1\}$. For conciseness, let us write $z = \operatorname{cis}\frac{2\pi m}{n}$.

The proof that the map is a homomorphism follows naturally from the fact that multiplication in $\mathbb{C}$ is commutative. Let φ_k be the map defined by $\varphi_k(z) = z^k$, then $\varphi_k(yz) = (yz)^k = y^k z^k = \varphi_k(y)\varphi_k(z)$.

To prove that the map is surjective, let $z \in G = \operatorname{cis}\frac{2\pi m}{n}$. Now de Moivre's formula states that, for a complex number $\operatorname{cis} x$, $(\operatorname{cis} x)^m = \operatorname{cis} mx$. So there exists a complex number $y = \operatorname{cis}\frac{2\pi m}{nk}$ with $y^k = (\operatorname{cis}\frac{2\pi m}{nk})^k = \operatorname{cis} k\frac{2\pi m}{nk} = \operatorname{cis}\frac{2\pi m}{n} = z$. And, we have $y \in G$, because $z^n = (y^k)^n = y^{kn} = 1$.

However, this map is not an isomorphism because it is not one-to-one. Since it is surjective, let $z \in G$ and let $y^k = z$. Now let $z = \operatorname{cis}\frac{2\pi m}{n}$ for a unique $m \in \{0, ..., n-1\}$. Then y is a complex number such that $y^k = \operatorname{cis}\frac{2\pi m}{n}$. From de Moivre's formula, y has the form $\operatorname{cis}\frac{2\pi m}{nk}$. Such a y is not uniquely defined. Consider $\operatorname{cis}\frac{2\pi m}{nk}$ and $\operatorname{cis}\frac{2\pi(m+n)}{nk}$ (the latter is distinct from the former whenever $n < k$, and we can always choose such a positive n for $k > 1$). It follows that, raised to the power k, they are $\operatorname{cis}\frac{2\pi m}{n}$ and $\operatorname{cis}\frac{2\pi(m+n)}{n} = \operatorname{cis}(\frac{2\pi m}{n} + 2\pi) = \operatorname{cis}\frac{2\pi m}{n}$, since sine and cosine have period 2π. That is, they are equal to each other. Thus, the map is not one-to-one, and is therefore not an isomorphism. $\square$

20. (4/21/23)

Let G be a group and let $\mathrm{Aut}(G)$ be the set of all isomorphisms from G onto G. Prove that $\mathrm{Aut}(G)$ is a group under function composition (called the *automorphism group* of G and the elements of $\mathrm{Aut}(G)$ are called *automorphisms* of G).

Proof. In order to prove that $\mathrm{Aut}(G)$ is a group, we must show that it is associative and closed under the binary operation of function composition, that it contains an identity element, and that each element has a unique inverse. Associativity is given by the definition of the operation of function composition.

$\mathrm{Aut}(G)$ contains an identity element: Let $e : G \to G$ such that $e(g) = g$ for all $g \in G$. e is a bijection and a homomorphism, thus an isomorphism, and so belongs to $\mathrm{Aut}(G)$. For any other isomorphism $f \in \mathrm{Aut}(G)$, note that

$(e \circ f)(g) = f(g) = f(e(g)) = (f \circ e)(g)$. Thus e is the identity element of $\mathrm{Aut}(G)$.

Next, $\mathrm{Aut}(G)$ is closed under function composition. Let $f, h \in \mathrm{Aut}(G)$ (to show that $f \circ h \in \mathrm{Aut}(G)$). We will show that $f \circ h$ is an isomorphism, and thus belongs to $\mathrm{Aut}(G)$. Let $g_1, g_2 \in G$. Then:

$$(f \circ h)(g_1 g_2) = f(h(g_1 g_2)) = f(h(g_1)h(g_2)) = f(h(g_1))f(h(g_2)) = (f \circ h)(g_1)(f \circ h)(g_2),$$

so $f \circ h$ is a homomorphism. We will next show that $f \circ h$ is both injective and surjective.

Injective: Let $g_1, g_2 \in G, g_1 \neq g_2$. Because h is injective, $h(g_1) \neq h(g_2)$. Because f is injective, $f(h(g_1)) \neq f(h(g_2)) \Rightarrow (f \circ h)(g_1) \neq (f \circ h)(g_2)$. Thus, $f \circ h$ is injective.

Surjective: Let $g_3 \in G$. Because f is surjective, there exists a $g_2 \in G$ such that $f(g_2) = g_3$. Because h is surjective, there exists a $g_1 \in G$ such that $h(g_1) = g_2$. Thus, $g_3 = f(g_2) = f(h(g_1)) = (f \circ h)(g_1)$. Therefore $f \circ h$ is surjective, and so it is an isomorphism and belongs to $\mathrm{Aut}(G)$.

Finally, to show that $\mathrm{Aut}(G)$ is closed under inverses, we simply note that, if $f \in \mathrm{Aut}(G)$, then f is a bijective map from G onto G, so it has a well-defined inverse f^{-1} that is also an isomorphism.

Thus $\mathrm{Aut}(G)$ is a group under function composition. □

21. (4/21/23)

Prove that for each fixed nonzero $k \in \mathbb{Q}$ the map from $\mathbb{Q}$ to itself defined by $q \mapsto kq$ is an automorphism of $\mathbb{Q}$.

Proof. Let $\varphi : \mathbb{Q} \to \mathbb{Q}$ be defined by $\varphi(q) = kq, k \in \mathbb{Q}, k \neq 0$. We will show that φ is a homomorphism and is both injective and surjective, thus an automorphism of $\mathbb{Q}$.

Let $r \in \mathbb{Q}$ and let $s = r/k$. Since $\mathbb{Q}$ is closed under division and $k \neq 0$, $s \in \mathbb{Q}$. Also, $\varphi(s) = ks = r$. Therefore φ is surjective.

Next, to show that φ is injective, let $r \in \mathbb{Q}$ with $\varphi(s_1) = \varphi(s_2) = r$. Then $ks_1 = ks_2 = r$. It follows that $r/k = s_1 = s_2$. Therefore φ is injective.

Finally, to show that φ is a homomorphism, let $s_1, s_2 \in \mathbb{Q}$. We note that $\varphi(s_1 + s_2) = k(s_1 + s_2) = ks_1 + ks_2 = \varphi(s_1) + \varphi(s_2)$. Therefore φ is a homomorphism.

This concludes the proof that φ is an automomorphism of $\mathbb{Q}$. □

22. (4/21/23)

Let A be an abelian group and fix some $k \in \mathbb{Q}$. Prove that the map $a \mapsto a^k$ is a homomorphism from A to itself. If $k = -1$ prove that this homomorphism is an automorphism of A.

Proof. Let $\varphi : A \to A$ be defined by $\varphi(a) = a^k, k \in \mathbb{Z}$. Let $a, b \in A$. Then $\varphi(ab) = (ab)^k = a^k b^k$ (from Ch. 1.1, exercise 24.), which equals $\varphi(a)\varphi(b)$. Therefore φ is a homomorphism from A to itself.

Now let us consider the case where $k = -1$. We will show that φ is now both injective and surjective, that is, an automorphism of A.

Let $a \in A$. Given that A is a group, a has an inverse a^{-1}, and $(a^{-1})^{-1} = a$, so $\varphi(a^{-1}) = a$. Thus φ is surjective. Also, from the uniqueness of inverses, a^{-1} is the only element in A for which $\varphi(a^{-1}) = (a^{-1})^{-1} = a$, so φ is injective.

Since φ is a bijective homomorphism, it is an automorphism of A. □

23. (4/21/23)

Let G be a finite group which possesses an automorphism σ such that $\sigma(g) = g$ if and only if $g = 1$. If σ^2 is the identity map from G to G, prove that G is abelian (such a σ is called *fixed point free* of order 2).

Proof. Let $g \in G, g \neq 1$, and let σ be an automorphism of G with $\sigma^2(g) = g$. Applying the inverse of σ to $\sigma(\sigma(g)) = g$ implies that $\sigma(g) = \sigma^{-1}(g)$.

Next, we see that $1 = \sigma(1) = \sigma(gg^{-1}) = \sigma(g)\sigma(g^{-1})$. Therefore $\sigma(g)^{-1} = \sigma(g^{-1})$. From above, then, we have $\sigma(g)^{-1} = \sigma(g)$. From 1.a), $\sigma(g)^{-1} = \sigma(g^{-1})$, so we have $\sigma(g^{-1}) = \sigma(g)$, and since σ is injective, this implies that $g = g^{-1}$.

Now let $g, h \in G$. $gh = g^{-1}h^{-1} = (hg)^{-1} = hg$, so G is an abelian group. □

24. (4/23/23)

Let G be a finite group and let x and y be distinct elements of order 2 in G that generate G. Prove that $G \cong D_{2n}$, where $n = |xy|$.

Proof. Given that x and y generate G and each have order 2 (that is, $x = x^{-1}$ and $y = y^{-1}$), we must have xy as a third distinct element, because $x \neq y \Rightarrow x \neq y^{-1} \Rightarrow xy \neq 1$. For brevity, let $z = xy$. Then x and z also generate G, because y can be written as $y = x^2y = x(xy) = xz$.

Also, note that $xz = xxy = y = yxx = y^{-1}x^{-1}x = (xy)^{-1}x = z^{-1}x$.

Now we can rewrite the given generators and relations $\langle x, y \mid x^2 = y^2 = 1\rangle$ with $\langle x, z \mid x^2 = z^n = 1, xz = z^{-1}x\rangle$.

Consider a map $\varphi : G \to D_{2n}$ defined on the new generators x and z by $\varphi(x) = s$ and $\varphi(z) = r$. The new generators and relations under φ become $\langle s, r \mid s^2 = r^n = 1, sr = r^{-1}s\rangle$, that is, the generators and relations that defined D_{2n} in Ch. 1.2. Given that the generators and relations are identical, the two groups are isomorphic. □

25. (4/26/23)

Let $n \in \mathbb{Z}^+$, let r and s be the generators of D_{2n} and let $\theta = 2\pi/n$.

(a) Prove that the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the matrix of linear transformation which rotates the x, y plane about the origin in a counterclockwise direction by θ radians.

Proof. Given that the unit vector in the x direction, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, rotated counterclockwise by θ radians is $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, and the unit vector in the y direction $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, rotated is $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$, consider the given 2×2 matrix applied to each of these basis vectors:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \text{ and } \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

This proves that the given matrix rotates the x, y plane counterclockwise about the origin by θ radians. □

(b) Prove that the map $\varphi : D_{2n} \to GL_2(\mathbb{R})$ defined on generators by

$$\varphi(r) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ and } \varphi(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

extends to a homomorphism of D_{2n} into $GL_2(\mathbb{R})$.

Proof. In order to show that φ is a homomorphism, it suffices to show that φ applied to the relations of D_{2n} hold in $GL_2(\mathbb{R})$. That is, given that $s^2 = r^n = 1$ and $sr = r^{-1}s$ in D_{2n}, we must show that $\varphi(s)^2 = \varphi(r)^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\varphi(s)\varphi(r) = \varphi(r)^{-1}\varphi(s)$ in $GL_2(\mathbb{R})$.

$\varphi(s)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the identity matrix in $GL_2(\mathbb{R})$.

We will next show that $\varphi(r)^n$ is also the identity matrix. We will first show that $\varphi(r)^k, k \in \{0, ..., n-1\}$, is the matrix $\begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$ by induction. The base case is obvious.

For some $k \in \{0, ..., n-1\}$, suppose that $\varphi(r)^k = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}$. Then:

$$\varphi(r)^{k+1} = \varphi(r)^k \varphi(r) = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} =$$
$$\begin{pmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & -\cos k\theta\sin\theta - \cos\theta\sin k\theta \\ \cos k\theta\sin\theta + \cos\theta\sin k\theta & \cos k\theta\cos\theta - \sin k\theta\sin\theta \end{pmatrix}.$$

From the trigonometric addition formulae, $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, the above is equal to $\begin{pmatrix}\cos(k+1)\theta & -\sin(k+1)\theta \\ \sin(k+1)\theta & \cos(k+1)\theta\end{pmatrix}$, which completes the proof by induction.

Then $\varphi(r)^n = \begin{pmatrix}\cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta\end{pmatrix} = \begin{pmatrix}\cos 2\pi & -\sin 2\pi \\ \sin 2\pi & \cos 2\pi\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$.

Finally, we show that the relation $\varphi(s)\varphi(r) = \varphi(r)^{-1}\varphi(s)$ holds.

We can find the inverse of a 2×2 matrix $\begin{pmatrix}a & b \\ c & d\end{pmatrix}$ by the formula $\frac{1}{ad-bc}\begin{pmatrix}d & -b \\ -c & a\end{pmatrix}$. The inverse $\varphi(r)^{-1}$, then, is: $\frac{1}{\sin^2\theta+\cos^2\theta}\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$. Then:

$$\varphi(s)\varphi(r) = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix} = \begin{pmatrix}\sin\theta & \cos\theta \\ \cos\theta & -\sin\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} = \varphi(r)^{-1}\varphi(s),$$

as desired. Since the generators and relations of D_{2n} hold under φ into $GL_2(\mathbb{R})$, φ is a homomorphism of D_{2n} into $GL_2(\mathbb{R})$. □

(c) Prove that the homomorphism φ in part (b) is injective.

Proof. In order to prove that $\varphi : D_{2n} \to GL_2(\mathbb{R})$ is injective, we must show that $\varphi(a) = \varphi(b) \Rightarrow a = b$ for all $a, b \in D_{2n}$. We will considerately three cases separately.

First, let $a = r^k$ and $b = r^m$ for $k, m \in \{0, ..., n-1\}$. Then:

$$\varphi(r^k) = \varphi(r^m) \Rightarrow \begin{pmatrix}\cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta\end{pmatrix} = \begin{pmatrix}\cos m\theta & \sin m\theta \\ -\sin m\theta & \cos m\theta\end{pmatrix} \Rightarrow$$
$$\cos k\theta = \cos m\theta \text{ and } \sin k\theta = \sin m\theta.$$

This implies that $k = m$, and so $a = b$.

Similarly, if $a = sr^k$ and $b = sr^m$, then we again have $\sin k\theta = \sin m\theta \Rightarrow k = m \Rightarrow a = b$.

For the third case (without loss of generality), let $a = sr^k$ and $b = r^m$. Then we have $\begin{pmatrix}\sin k\theta & \cos k\theta \\ \cos k\theta & -\sin k\theta\end{pmatrix} = \begin{pmatrix}\cos m\theta & \sin m\theta \\ -\sin m\theta & \cos m\theta\end{pmatrix}$. Then $\sin k\theta = -\sin k\theta$, which happens when $k\theta = 0, \pi$. Also, $\cos k\theta = -\cos k\theta$, which happens when $k\theta = \pi/2, 3\pi/2$. Therefore there are no k, m (and thus a, b) where this equality holds.

Therefore, in the only cases where we have $\varphi(a) = \varphi(b)$, it implies that $a = b$, and so φ is injective. □