Dummit & Foote Ch. 3.4: Composition Series and the Hölder Program

Scott Donaldson

Nov. 2023

1. (11/2/23)

Prove that if G is an abelian simple group then $G \cong Z_p$ for some prime p (do not assume G is a finite group).

Proof. Since G is simple, the only normal subgroups of G are 1 and G itself. However, since G is abelian, any subgroup of G must be normal, so it follows that G contains *no* subgroups other than 1 and itself.

If $x_1, x_2 \in G$ are distinct generators for G, then $\langle x_1 \rangle$ and $\langle x_2 \rangle$ would be distinct subgroups of G; therefore G is generated by a single element and is a cyclic group. Let us write $G = \langle x \rangle$. If G were infinite, then for any $n > 1$, $\langle x^n \rangle$ would be a distinct subgroup of G, so G must be finite.

Finally, if n divides $|G|$, then from Chapter 2, Theorem 7.(3), G contains a proper subgroup of order n. Therefore $|G|$ has no divisors other than 1 and itself, so we have $|G| = p$ for some prime p. We conclude that $G \cong Z_p$ for some prime p. $\square$

2. (11/3/23)

Exhibit all 3 composition series for Q_8 and all 7 composition series for D_8. List the composition factors in each case.

The 3 composition series for Q_8 are:

1. $1 \leq \langle -1 \rangle \leq \langle i \rangle \leq Q_8$
2. $1 \leq \langle -1 \rangle \leq \langle j \rangle \leq Q_8$
3. $1 \leq \langle -1 \rangle \leq \langle k \rangle \leq Q_8$

In each series, each composition factor is isomorphic to Z_2 (thus each N_i is normal in N_{i+1}; since there is only one left coset it must equal the only right coset).

The 7 composition series for D_8 are:

1. $1 \leq \langle s \rangle \leq \langle s, r^2 \rangle \leq D_8$

2. $1 \leq \langle sr^2 \rangle \leq \langle s, r^2 \rangle \leq D_8$
3. $1 \leq \langle r^2 \rangle \leq \langle s, r^2 \rangle \leq D_8$
4. $1 \leq \langle r^2 \rangle \leq \langle r \rangle \leq D_8$
5. $1 \leq \langle r^2 \rangle \leq \langle sr, r^2 \rangle \leq D_8$
6. $1 \leq \langle sr \rangle \leq \langle sr, r^2 \rangle \leq D_8$
7. $1 \leq \langle sr^3 \rangle \leq \langle sr, r^2 \rangle \leq D_8$

Again each composition factor is isomorphic to Z_2.

3. (11/3/23)

Find a composition series for the quasidihedral group of order 16 (cf. Exercise 11, Section 2.5). Deduce that QD_{16} is solvable.

Solution. Recall that $QD_{16} = \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle$.

A composition series for QD_{16} is:

$$1 \leq \langle \sigma^4 \rangle \leq \langle \sigma^2 \rangle \leq \langle \sigma \rangle \leq QD_{16},$$

where each composition factor is isomorphic to Z_2. Since Z_2 is abelian, each composition factor is solvable, and so QD_{16} is solvable. $\square$

4. (11/4/23)

Use Cauchy's Theorem and induction to show that a finite abelian group has a subgroup of order n for each positive divisor n of its order.

Proof. Let G be a finite abelian group. Let us suppose that, for all groups H, $|H| < |G|$, H has a subgroup of order n for each positive divisor n of its order.

Let p be a prime dividing $|G|$. From Cauchy's Theorem, there is an $x \in G$ with $|x| = p$. Since G is abelian, $\langle x \rangle$ is normal in G. So the quotient group $G/\langle x \rangle$ is well-defined and has order $|G|/p < |G|$, thus it has a subgroup of order n for each n dividing $|G|/p$.

Let n be a positive divisor of $|G|$. Since $|G| = p \cdot \frac{|G|}{p}$, n divides $\frac{|G|}{p}$. From the induction hypothesis, let $\overline{K}$ be a subgroup of $G/\langle x \rangle$ of order n. For each $\overline{k} \in \overline{K}, \overline{k} \neq \overline{1}$, we must have $k \notin \langle x \rangle$, or else we would have $\overline{k} = k \cdot \langle x \rangle = \langle x \rangle$. Then there is a bijection from $\overline{K}$ onto K given by $\overline{k} \mapsto k$. Thus K is a subgroup of G of order n. $\square$

5. (11/7/23)

Prove that subgroups and quotient groups of a solvable group are solvable.

Proof. Let G be a solvable group. Then there exists a chain of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq ... \trianglelefteq G_s = G$$

such that G_{i+1}/G_i is abelian for each $i \in \{0, ..., s-1\}$.

Let $N \leq G$ and let G_i be the smallest subgroup in the above series such that $N \leq G_i$. Since $G_{i-1} \trianglelefteq G_i$, we have $G_i \leq N_G(G_{i-1})$ and so $N \leq N_G(G_{i-1})$. Then by the Diamond Isomorphism Theorem it follows that

$$NG_{i-1} \leq G_i, \;\; N \cap G_{i-1} \trianglelefteq N, \text{ and } NG_{i-1}/G_{i-1} \cong N/N \cap G_{i-1}.$$

Since the quotient group G_i/G_{i-1} is abelian, its subgroup NG_{i-1}/G_{i-1} is as well. Then, since $N/N \cap G_{i-1} \cong NG_{i-1}/G_{i-1}$, it follows that $N/N \cap G_{i-1}$ is abelian.

We can repeat the above process with $N \cap G_{i-1} \leq G_{i-1}$ to conclude that $N \cap G_{i-2} \trianglelefteq N \cap G_{i-1}$, with $N \cap G_{i-1}/N \cap G_{i-2}$ abelian. Continuing this way we produce the chain

$$1 = N \cap G_0 \trianglelefteq N \cap G_1 \trianglelefteq ... \trianglelefteq N \cap G_{i-1} \trianglelefteq N \cap G_i = N$$

where $N \cap G_{i+1}/N \cap G_i$ is abelian for $i \in \{0, ..., i-1\}$, which shows that N is solvable. □

6. (11/9/23)

Prove part (1) of the Jordan-Hölder Theorem by induction on $|G|$.

Proof. Part (1) of the Jordan-Hölder Theorem states that if G is a finite group, $G \neq 1$, then G has a composition series. Suppose that for all groups H, $|H| < G$, H has a composition series.

If G is a simple group, then $1 \leq G$ is a composition series, since $G/1 \cong G$ is simple.

Therefore assume that G contains at least one proper normal subgroup N. Then we have $|N| < |G|$, so by assumption N has a composition series

$$1 = N_0 \leq N_1 \leq ... \leq N_{k-1} \leq N_k = N,$$

where the quotient group N_{i+1}/N_i is simple for $i \in \{0, ..., k-1\}$. And, the quotient group G/N has order $|G/N| = \frac{|G|}{|N|} < |G|$, so it also contains a composition series

$$N/N = G_0/N \leq G_1/N \leq ... \leq G_{m+1}/N \leq G_m/N = G,$$

where each $(G_{i+1}/N)/(G_i/N)$ is simple for $i \in \{0, ..., m-1\}$. By the Third Isomorphism Theorem, this implies that each G_{i+1}/G_i is simple.

We now have a chain

$$1 = N_0 \leq N_1 \leq ... \leq N_{k-1} \leq N_k = N = G_0 \leq G_1 \leq ... \leq G_{m+1} \leq G_m = G$$

where the quotient of each successive subgroup by the previous is a simple group. Thus it is a composition series for G. □