Dummit & Foote Ch. 1.7: Group Actions

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Apr. 2023

1. (4/27/23)

Let F be a field. Show that the multiplicative group of nonzero elements of F (denoted by $F^\times$) acts on the set F by $g \cdot a = ga$, where $g \in F^\times, a \in F$ and ga is the usual product in F of the two field elements.

Proof. To show that $F^\times$ acts on F, we must show that $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for all $g_1, g_2 \in F^\times, a \in F$, and $1 \cdot a = a$ for all $a \in F$.

First, let $g_1, g_2 \in F^\times$ and $a \in F$. By the definition of the action, $g_1 \cdot (g_2 \cdot a) = g_1 \cdot (g_2 a) = g_1 g_2 a$. By the associativity of multiplication, $g_1 g_2 a = (g_1 g_2) a$. Again by the action definition, this equals $(g_1 g_2) \cdot a$.

It follows directly from the field axiom of multiplicative identity that $1 \cdot a = a$ for all $a \in A$. Thus $F^\times$ acts on F by $g \cdot a = ga$. □

2. (4/27/23)

Show that the additive group $\mathbb{Z}$ acts on itself by $z \cdot a = z + a$ for all $z, a \in \mathbb{Z}$.

Proof. First, $z_1 \cdot (z_2 \cdot a) = z_1 \cdot (z_2 + a) = z_1 + z_2 + a = (z_1 + z_2) + a = (z_1 + z_2) \cdot a$.

Also, $0 \cdot a = 0 + a = a$ for all $a \in \mathbb{Z}$. Thus $\mathbb{Z}$ acts on itself by $z \cdot a = z + a$. □

3. (4/27/23)

Show that the additive group $\mathbb{R}$ acts on the x, y plane $\mathbb{R} \times \mathbb{R}$ by $r \cdot (x, y) = (x + ry, y)$.

Proof. First, $r_1 \cdot (r_2 \cdot (x, y)) = r_1 \cdot (x + r_2 y, y) = (x + r_2 y + r_1 y, y) = (x + (r_1 + r_2) y, y) = (r_1 + r_2) \cdot (x, y)$.

Also, $0 \cdot (x, y) = (x + 0y, y) = (x, y)$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$. Thus $\mathbb{R}$ acts on $\mathbb{R} \times \mathbb{R}$ by $r \cdot (x, y) = (x + ry, y)$. □

4. (4/27/23)

Let G be a group acting on a set A and fix some $a \in A$. Show that the following sets are subgroups of G:

(a) the kernel of the action,

Proof. The kernel of G is the set $\{g \in G \mid g \cdot a = a \text{ for all } a \in A\}$. It is closed under the binary operation of G: If g_1, g_2 are in the kernel, then $g_1 \cdot (g_2 \cdot a) = g_1 \cdot a = a$ for all $a \in A$. And, by definition of a group action, $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$, which implies that $(g_1 g_2) \cdot a = a$, so $g_1 g_2$ is in the kernel of G.

The kernel is also closed under inverses: Let g be in the kernel of G. Then $1 \cdot a = (g^{-1} g) \cdot a = g^{-1} \cdot (g \cdot a) = g^{-1} \cdot a$. By definition, $1 \cdot a = a$, so $g^{-1} \cdot a = a$ for all a, so g^{-1} is in the kernel. Thus the kernel of the action is a subgroup of G. $\square$

(b) $\{g \in G \mid ga = a\}$ — this subgroup is called the *stabilizer* of G.

Proof. The proof that this set of elements if a subgroup is identical to the one immediately above, but for a fixed a as opposed to all $a \in A$. $\square$