Dummit & Foote Ch. 7.2: Polynomial Rings, Matrix Rings, and Group Rings

Scott Donaldson

Oct. 2024

Let R be a commutative ring with 1.

2. (10/7/24)

Let $p(x) = a_n x^n + a_{n-a} x^{n-1} + ... + a_1 x + a_0$ be an element of the polynomial ring $R[x]$. Prove that $p(x)$ is a zero divisor in $R[x]$ if and only if there is a nonzero $b \in R$ such that $bp(x) = 0$.

Proof. If there exists $b \in R$ such that $bp(x) = 0$, then the polynomial $b(x) = b \in R[x]$ is an element such that $b(x)p(x) = 0$, so $p(x)$ is a zero divisor.

Conversely, suppose that $p(x)$ is zero divisor in R. Let $g(x) = b_m x^m + b_{m-1} x^{m-1} + ... + b_0$ be a nonzero polynomial of minimal degree such that $p(x)g(x) = 0$. Then:

$$\begin{aligned} p(x)g(x) &= (a_n x^n + ... + a_0)(b_m x^m + ... + b_0) \\ &= a_n b_m x^{n+m} + ... + a_0 b_0 = 0, \end{aligned}$$

which implies that $a_n b_m = 0$.

Then $a_n g(x) = a_n b_m x^m + ... a_n b_0 = a_n b_{m-1} x^{m-1} + ... a_n b_0$ is a polynomial of degree $m-1$. And, because $p(x)g(x) = 0$, we have $a_n g(x)p(x) = 0$, contradicting $g(x)$ being a polynomial of minimal degree such that multiplying it by $p(x)$ is zero. Therefore we must have $a_n g(x) = 0$.

Now suppose inductively that $a_{n-i} g(x) = 0$ and $a_{n-k} g(x) = 0$ for some $i \in \{0, ..., n\}$ and all $k \leq i$. Given $p(x) = a_n x^n + ... a_0$, let us write $p_{n-i}(x) = a_{n-i} x^{n-i} + ... + a_0$. Then:

$$\begin{aligned} p(x)g(x) &= (a_n x^n + ... + a_0)g(x) \\ &= (a_n x^n + ... + a_{n-i} x^{n_i} + p_{n-i-1}(x))g(x) \\ &= p_{n-i-1}(x)g(x) \text{ (since } a_n g(x) = ... = a_{n-i} g(x) = 0) \\ &= a_{n-i-1} x^{n-i-1} g(x) + ... + a_0 g(x) = 0. \end{aligned}$$

It follows that the leading coefficient of $x^{n+m-i-1}$, $a_{n-i-1} b_m$, must equal zero. By induction, this implies that $a_i b_m = 0$ for all $i \in \{0, ..., n\}$; that is, $b_m p(x) = 0$ and so there exists a $b \in R$ such that $bp(x) = 0$. $\square$