Dummit & Foote Ch. 3.2: More on Cosets and Lagrange's Theorem

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Oct. 2023

Let G be a group.

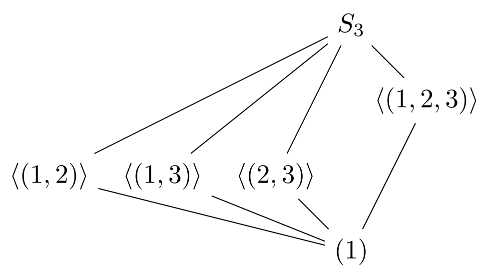
1. (10/1/23)

Which of the following are permissible orders of subgroups of a group of order 120: 1, 2, 5, 7, 9, 15, 60, 240? For each permissible order give the corresponding index.

Proof. From Lagrange's theorem, the order of a subgroup of a group of order 120 must divide 120. Then the permissible orders for subgroups are $1 = \frac{120}{120}$, $2 = \frac{120}{60}, 5 = \frac{120}{24}, 15 = \frac{120}{8}$, and $60 = \frac{120}{2}$. For each of these orders the index is given by the corresponding denumerator. $\square$

2. (10/2/23)

Prove that the lattice of subgroups of S_3 below is correct (i.e., prove that it contains all subgroups of S_3 and that their pairwise joins and intersections are correctly drawn).



Proof. The symmetric group S_3 contains 6 elements. By Lagrange's theorem, its proper subgroups must have order 2 or 3. Each of the subgroups in the lattice above have order 2 or 3, so there are no smaller or larger subgroups not depicted above.

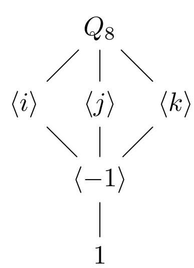
From Corollary 10, a subgroup of order 2 must be isomorphic to Z_2, that is, cyclic and generated by a single element of order 2. The three subgroups generated by the three elements of order 2 (the 2-cycles of S_3) are depicted above. Similarly, a subgroup of order 3 must be isomorphic to Z_3 and generated by a single element of order 3. The subgroup generated by $(1, 2, 3)$ contains $(1, 3, 2)$, so there is only a single subgroup of order 3.

Next, again by Lagrange's Theorem, a subgroup of two different containing groups must have an order that divides the order of both of the containing groups. First consider a subgroup of order 2 and a subgroup of order 3. Only 1 divides 2 and 3, so the intersection must be the identity. Similarly, if a subgroup of order 2 and a subgroup of order 3 are contained in a larger group, then that group's order must have both 2 and 3 as divisors. The smallest integer for which this is possible is 6, which is the order of all of S_3.

Finally, consider a pair of subgroups of order 2. Their intersection is either the identity or else they are the same subgroup. Their join must have even order, but 4 does not divide 6 and any larger even number exceeds the order of S_3. Thus their join is all of S_3. This concludes the proof that the lattice of subgroups of S_3 is correct. $\square$

3. (10/2/23)

Prove that the lattice of subgroups of Q_8 below is correct.



Proof. The group Q_8 has order $8 = 2^3$, so by Lagrange's theorem its proper subgroups must have order 2 or 4. We will start from the bottom and work toward the top: There is only one element of order 2 in Q_8, -1, and the cyclic subgroup generated by it is in the lattice.

For each of i, j, and k, $\langle -1 \rangle$ is contained in the subgroup generated by them (ex. $\langle i \rangle = \{\pm 1, \pm i\}$) and there are no intermediate subgroups, since there is no divisor of 4 that is strictly greater than 2. At this point, every element of Q_8 is represented, so there are no cyclic subgroups missing. We might ask if there is a subgroup of order 4 missing. If so, it cannot be cyclic, and from Ch. 1.1, Exercise 36, it must be isomorphic to V_4. However, V_4 contains three elements of order 2, and Q_8 only has one, so there is no subgroup of Q_8 isomorphic to V_4.

Finally, the join of any of the subgroups generated by i, j, or k must contain strictly more than 4 elements and its order must divide 8. Then any of their joins must have order 8, that is, be all of Q_8. $\square$

4. (10/3/23)

Show that if $|G| = pq$ for some primes p and q (not necessarily distinct) then either G is abelian or $Z(G) = 1$.

Proof. We will show, equivalently, that if $|Z(G)| > 1$, then G is abelian.

Let $x \in Z(G)$. From Corollary 9, the order of x divides $|G| = pq$. If $|x| = pq$, then $G = \langle x \rangle$ and so is abelian. Suppose without loss of generality that $|x| = p$. Now since the center of a group is a subgroup, we must have $\langle x \rangle \leq Z(G)$. If there exists a $y \in Z(G), y \notin \langle x \rangle$, then the order of $Z(G)$ exceeds p and must divide pq, then it must be all of G and hence G is abelian. So suppose $Z(G) = \langle x \rangle$.

The center of a group is normal in that group, so $G/Z(G)$ is well-defined. Since $|Z(G)| = p$, it has q cosets in G; that is, the quotient group $G/Z(G)$ has prime order q and is thus isomorphic to Z_q, hence cyclic. From Ch. 3.1, Exercise 36., G is thus abelian. $\square$

5. (10/4/23)

Let H be a subgroup of G and fix some element $g \in G$.

(a) Prove that gHg^{-1} is a subgroup of G of the same order as H.

Proof. By definition elements of gHg^{-1} can be written in the form ghg^{-1} for some $h \in H$, so let $gh_1g^{-1}, gh_2g^{-1} \in gHg^{-1}$. Then we have:

$$(gh_1g^{-1})(gh_2g^{-1})^{-1} = gh_1g^{-1}gh_2^{-1}g_1 = gh_1h_2^{-1}g^{-1} \in gHg^{-1},$$

so gHg^{-1} fulfills the subgroup criterion and is thus a subgroup of G.

Next, let $\varphi_g : H \to gHg^{-1}$ be defined by $\varphi_g(h) = ghg^{-1}$ for all $h \in H$. This map is injective by the cancellation laws: $gh_1g^{-1} = gh_2g^{-1}$ implies that $h_1 = h_2$. It is also surjective: Let $x \in gHg^{-1}$. By definition $x = ghg^{-1}$ for some $h \in H$, so $\varphi_g(h) = x$. Therefore φ_g is a bijection, and so H and gHg^{-1} have the same order. $\square$

(b) Deduce that if $n \in \mathbb{Z}^+$ and H is the unique subgroup of G of order n then $H \trianglelefteq G$.

Suppose that H is the unique subgroup of order n in G. Then for all $g \in G$, we must have $gHg^{-1} = H$ (it cannot be any other subgroup, because $|gHg^{-1}| = |H| = n$ and there is no other subgroup of order n in G). It follows that H is normal in G.

6. (10/4/23)

Let $H \leq G$ and let $g \in G$. Prove that if the right coset of Hg equals *some* left coset of H in G then it equals the left coset gH and g must be in $N_G(H)$.

Proof. Suppose $Hg = xH$ for some $x \in G$. Now $g \in Hg$, so we must also have $g \in xH$. Then $g = xh$ for some $h \in H$. It follows that $x = gh^{-1}$. So $Hg = xH = (gh^{-1})H = gH$, which in turns implies that $gHg^{-1} = H$. Therefore $g \in N_G(H)$. $\square$

7. (10/5/23)

Let $H \leq G$ and define a relation $\sim$ on G by $a \sim b$ if and only if $b^{-1}a \in H$. Prove that $\sim$ is an equivalence relation and describe the equivalence class of each $a \in G$. Use this to prove Proposition 4.

Proof. Let $a, b, c \in G$. We have $a \sim a$, because $a^{-1}a = 1 \in H$. If $a \sim b$, then we have $b^{-1}a \in H$. Now $b \sim a = a^{-1}b = (b^{-1}a)^{-1} \in H$, since H is closed under inverses, so $a \sim b$ implies that $b \sim a$ (and the logic holds in reverse). Finally, if $a \sim b$ and $b \sim c$, then $b^{-1}a, c^{-1}b \in H$. Then their product, $c^{-1}bb^{-1}a = c^{-1}a$, is an element of H, which implies $a \sim c$. The relation $\sim$ is reflexive, symmetric, and transitive, therefore it is an equivalence relation.

Let $a \in G$ and let b lie in the left coset aH, so $b = ah$ for some $h \in H$. Then $b^{-1}a = (ah)^{-1}a = h^{-1}a^{-1}a = h^{-1} \in H$, so $a \sim b$. This implies that aH is a subset of the equivalence class of a. And, if we have $a \sim b$, then $b^{-1}a \in H$, so $b^{-1}a = h$ for some $h \in H$. It follows that $b = ah^{-1} \in aH$, so the equivalence class of a is a subset of aH. Since each is contained in the other, the equivalence class of a under $\sim$ is the left coset aH.

Now Proposition 4 states that:

- The set of left cosets of H in G form a partition of G.
- For all $a, b \in G, aH = bH$ if and only if $b^{-1}a \in H$.
- In particular, $aH = bH$ if and only if a and b are representatives of the same coset.

Since the equivalence class of a under $\sim$ is exactly the left coset aH and equivalence classes partition a set, the left cosets of H in G partition G. The proof for the remaining items follows directly from the proof above that $a \sim b \iff b^{-1}a \in H \iff b \in aH$. $\square$

8. (10/6/23)

Prove that if H and K are finite subgroups of G whose orders are relatively prime then $H \cap K = 1$.

Proof. Let $H, K \leq G$ be finite subgroups whose orders are relatively prime. Let $x \in H \cap K$, so $x \in H$ and $x \in K$. From Corollary 9, the order of x divides the orders of both H and K. Since $|H|$ and $|K|$ are relatively prime, the order of x must be 1, therefore $x = 1$. It follows that $H \cap K = 1$. $\square$

9. (10/12/23)

This exercise outlines a proof of Cauchy's Theorem due to James McKay (*Another proof of Cauchy's group theorem*, Amer. Math. Monthly, 66(1959), p. 119). Let G be a finite group and let p be a prime dividing $|G|$. Let $\mathcal{S}$ denote the set of p-tuples of elements of G the product of whose coordinates is 1:

$$\mathcal{S} = \{(x_1, x_2, ..., x_p) \mid x_1 x_2 ... x_p = 1\}.$$

(a) Show that $\mathcal{S}$ has $|G|^{p-1}$ elements, hence has order divisible by p.

Proof. Construct an element of $\mathcal{S}$ coordinate by coordinate. There are $|G|$ choices for the first element x_1. There are again $|G|$ choices for the second element x_2. We proceed similarly until the final element, which must satisfy the constraint that the product of all coordinates is 1. Therefore the final element must be equal to $(x_1 x_2 ... x_{p-1})^{-1}$. We have freely chosen $p-1$ coordinates from among $|G|$ possibilities; therefore $|\mathcal{S}| = |G|^{p-1}$. $\square$

Define the relation $\sim$ on $\mathcal{S}$ by letting $\alpha \sim \beta$ if β is a cyclic permutation of α.

(b) Show that a cyclic permutation of $\mathcal{S}$ is again an element of $\mathcal{S}$.

Proof. Since $\alpha \sim \beta$ implies that β is a cyclic permutation of α, we have

$$\alpha = (x_1, x_2, ..., x_p) \Rightarrow \beta = (x_{1+n}, x_{2+n}, ..., x_{p+n}),$$

where the subscripts of elements of β are taken mod p (although wrapping from 1 to p, rather than 0 to $p-1$).

The product of the coordinates of α is:

$$\begin{aligned} 1 = \prod \alpha &= x_1 x_2 ... x_p \\ &= (x_1 ... x_n)(x_{n+1} ... x_p) \\ &= (x_{n+1} ... x_p)(x_1 ... x_n) \text{ (if } ab = 1\text{, then } ab = ba) \\ &= (x_{1+n} ... x_{p-n+n})(x_{(p-n+1)+n} ... x_{p+n}) \\ &= x_{1+n} ... x_{p+n} = \prod \beta, \end{aligned}$$

and so the product of β's coordinates is 1, making it an element of $\mathcal{S}$. $\square$

(c) Prove that $\sim$ is an equivalence relation on $\mathcal{S}$.

Proof. Let $\alpha, \beta, \gamma \in \mathcal{S}$. The relation $\sim$ is:

- Reflexive: Let $\alpha = (x_1, x_2, ..., x_p)$. Then $x_i = x_{i+0}$ for all coordinates x_i, so α is a cyclic permutation of itself, and therefore $\alpha \sim \alpha$.
- Symmetric: Let $\alpha \sim \beta$, α, β indexed by x, y respectively. Since β is a cyclic permutation of α, we have $y_i = x_{i+n}$ for all $i \in \{1, ..., p\}$ for some $n \in \mathbb{Z}$. It follows that $x_i = y_{i+(p-n)}$ (subscripts mod p wrapping from 1 to p), so α is also a cyclic permutation of β, and therefore $\beta \sim \alpha$.
- Transitive: Let $\alpha \sim \beta$ and $\beta \sim \gamma$, with α, β as above and γ indexed by z. We have $y_i = x_{i+n}$ and $z_i = y_{i+k}$ for some $k, n \in \mathbb{Z}$. It follows that $z_i = x_{i+k+n}$, which implies that γ is a cyclic permutation of α, so $\alpha \sim \gamma$.

Therefore $\sim$ is an equivalence relation on $\mathcal{S}$. $\square$

(d) Prove that an equivalence class contains a single element if and only if it is of the form $(x, x, ..., x)$ with $x^p = 1$.

Proof. First, let $\alpha = (x, ..., x)$ and let $\alpha \sim \beta$. Then β is a cyclic permutation of α. Since α consists of a single, repeated coordinate value, we must have $\beta = (x, ..., x) = \alpha$. Therefore the equivalence class of α consists only of itself.

Next, let $\alpha \in \mathcal{S}$ and suppose that the equivalence class of α under $\sim$ consists only of α. Suppose $\alpha = (x_1, x_2, ..., x_p)$. Let β be a cyclic permutation of α shifted by 1: $\beta = (x_2, x_3 ..., x_p, x_1)$. Now β is in the equivalence class of α, but we must have $\beta = \alpha$, so $x_{i+1} = x_i$ for all x_i. It follows that $x_2 = x_1, x_3 = x_2 = x_1$, and so every value is equal to x_1. Then we have $\alpha = (x_1, ..., x_1)$, which is of the form $(x, ..., x)$, and by definition we must have $x^p = 1$. $\square$

(e) Prove that every equivalence class has order 1 or p (this uses the fact that p is a *prime*). Deduce that $|G|^{p-1} = k + pd$, where k is the number of classes of size 1 and d is the number of classes of size p.

Proof. From (d), if $\alpha = (x, ..., x)$ for some $x \in G$, its equivalence class has order 1.

Let $\alpha = (x_1, x_2, ..., x_p)$. Then there are exactly p members in the equivalence class of α, and they are the cyclic permutations of α shifted by $0, 1, 2, ..., p-1$, respectively. For example, the n-th member of the equivalence class is $(x_{1+n}, x_{2+n}, ..., x_{p+n})$.

The equivalence classes of the elements of $\mathcal{S}$ partition $\mathcal{S}$. Suppose there are k equivalence classes of order 1, and d equivalence classes of order p. From (a), the order of $\mathcal{S}$ is $|G|^{p-1}$. Then we have $|G|^{p-1} = k + pd$. $\square$

(f) Since $\{(1,1,...,1)\}$ is an equivalence class of size 1, conclude from (e) that there must be a nonidentity element x in G with $x^p = 1$, i.e., G contains an element of order p.

Proof. From (e), we have $|G|^{p-1} = k + pd$ for some $k, d \geq 0$. From (a), p divides the order of $\mathcal{S} = |G|^{p-1}$, so we can write $ps = k+pd$ for some $s > 0$. Then $k = ps - pd = p(s-d)$, and so p divides k. Because p is prime, this implies that $k > 1$, so there are at least two elements whose equivalence classes have size 1. We already know that one is the identity; therefore there must be some element $\alpha \in \mathcal{S}, \alpha \neq (1,...,1)$ whose equivalence class under $\sim$ has size 1. From (d), $\alpha = (x,...,x)$ for some $x \in G$, and we thus have $x^p = 1$, which implies that $|x| = p$. □