Dummit & Foote Ch. 3.4: Composition Series and the Hölder Program

Scott Donaldson

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1. (11/2/23)

Prove that if G is an abelian simple group then $G \cong Z_p$ for some prime p (do not assume G is a finite group).

Proof. Since G is simple, the only normal subgroups of G are 1 and G itself. However, since G is abelian, any subgroup of G must be normal, so it follows that G contains *no* subgroups other than 1 and itself.

If $x_1, x_2 \in G$ are distinct generators for G, then $\langle x_1 \rangle$ and $\langle x_2 \rangle$ would be distinct subgroups of G; therefore G is generated by a single element and is a cyclic group. Let us write $G = \langle x \rangle$. If G were infinite, then for any $n > 1$, $\langle x^n \rangle$ would be a distinct subgroup of G, so G must be finite.

Finally, if n divides $|G|$, then from Chapter 2, Theorem 7.(3), G contains a proper subgroup of order n. Therefore $|G|$ has no divisors other than 1 and itself, so we have $|G| = p$ for some prime p. We conclude that $G \cong Z_p$ for some prime p. □

2. (11/3/23)

Exhibit all 3 composition series for Q_8 and all 7 composition series for D_8. List the composition factors in each case.

The 3 composition series for Q_8 are:

1. $1 \leq \langle -1 \rangle \leq \langle i \rangle \leq Q_8$
2. $1 \leq \langle -1 \rangle \leq \langle j \rangle \leq Q_8$
3. $1 \leq \langle -1 \rangle \leq \langle k \rangle \leq Q_8$

In each series, each composition factor is isomorphic to Z_2 (thus each N_i is normal in N_{i+1}; since there is only one left coset it must equal the only right coset).

The 7 composition series for D_8 are:

1. $1 \leq \langle s \rangle \leq \langle s, r^2 \rangle \leq D_8$

2. $1 \leq \langle sr^2 \rangle \leq \langle s, r^2 \rangle \leq D_8$
3. $1 \leq \langle r^2 \rangle \leq \langle s, r^2 \rangle \leq D_8$
4. $1 \leq \langle r^2 \rangle \leq \langle r \rangle \leq D_8$
5. $1 \leq \langle r^2 \rangle \leq \langle sr, r^2 \rangle \leq D_8$
6. $1 \leq \langle sr \rangle \leq \langle sr, r^2 \rangle \leq D_8$
7. $1 \leq \langle sr^3 \rangle \leq \langle sr, r^2 \rangle \leq D_8$

Again each composition factor is isomorphic to Z_2.

3. (11/3/23)

Find a composition series for the quasidihedral group of order 16 (cf. Exercise 11, Section 2.5). Deduce that QD_{16} is solvable.

Solution. Recall that $QD_{16} = \langle \sigma, \tau \mid \sigma^8 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle$.

A composition series for QD_{16} is:

$$1 \leq \langle \sigma^4 \rangle \leq \langle \sigma^2 \rangle \leq \langle \sigma \rangle \leq QD_{16},$$

where each composition factor is isomorphic to Z_2. Since Z_2 is abelian, each composition factor is solvable, and so QD_{16} is solvable. □

4. (11/4/23)

Use Cauchy's Theorem and induction to show that a finite abelian group has a subgroup of order n for each positive divisor n of its order.

Proof. Let G be a finite abelian group. Let us suppose that, for all groups H, $|H| < |G|$, H has a subgroup of order n for each positive divisor n of its order.

Let p be a prime dividing $|G|$. From Cauchy's Theorem, there is an $x \in G$ with $|x| = p$. Since G is abelian, $\langle x \rangle$ is normal in G. So the quotient group $G/\langle x \rangle$ is well-defined and has order $|G|/p < |G|$, thus it has a subgroup of order n for each n dividing $|G|/p$.

Let n be a positive divisor of $|G|$. Since $|G| = p \cdot \frac{|G|}{p}$, n divides $\frac{|G|}{p}$. From the induction hypothesis, let $\overline{K}$ be a subgroup of $G/\langle x \rangle$ of order n. For each $\overline{k} \in \overline{K}, \overline{k} \neq \overline{1}$, we must have $k \notin \langle x \rangle$, or else we would have $\overline{k} = k \cdot \langle x \rangle = \langle x \rangle$. Then there is a bijection from $\overline{K}$ onto K given by $\overline{k} \mapsto k$. Thus K is a subgroup of G of order n. □

5. (11/7/23)

Prove that subgroups and quotient groups of a solvable group are solvable.

Proof. Let G be a solvable group. Then there exists a chain of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq ... \trianglelefteq G_s = G$$

such that G_{i+1}/G_i is abelian for each $i \in \{0, ..., s-1\}$.

Let $N \leq G$ and let G_i be the smallest subgroup in the above series such that $N \leq G_i$. Since $G_{i-1} \trianglelefteq G_i$, we have $G_i \leq N_G(G_{i-1})$ and so $N \leq N_G(G_{i-1})$. Then by the Diamond Isomorphism Theorem it follows that

$$NG_{i-1} \leq G_i, \; N \cap G_{i-1} \trianglelefteq N, \text{ and } NG_{i-1}/G_{i-1} \cong N/N \cap G_{i-1}.$$

Since the quotient group G_i/G_{i-1} is abelian, its subgroup NG_{i-1}/G_{i-1} is as well. Then, since $N/N \cap G_{i-1} \cong NG_{i-1}/G_{i-1}$, it follows that $N/N \cap G_{i-1}$ is abelian.

We can repeat the above process with $N \cap G_{i-1} \leq G_{i-1}$ to conclude that $N \cap G_{i-2} \trianglelefteq N \cap G_{i-1}$, with $N \cap G_{i-1}/N \cap G_{i-2}$ abelian. Continuing this way we produce the chain

$$1 = N \cap G_0 \trianglelefteq N \cap G_1 \trianglelefteq ... \trianglelefteq N \cap G_{i-1} \trianglelefteq N \cap G_i = N$$

where $N \cap G_{i+1}/N \cap G_i$ is abelian for $i \in \{0, ..., i-1\}$, which shows that N is solvable. $\square$

6. (11/9/23)

Prove part (1) of the Jordan-Hölder Theorem by induction on $|G|$.

Proof. Part (1) of the Jordan-Hölder Theorem states that if G is a finite group, $G \neq 1$, then G has a composition series. Suppose that for all groups H, $|H| < G$, H has a composition series.

If G is a simple group, then $1 \leq G$ is a composition series, since $G/1 \cong G$ is simple.

Therefore assume that G contains at least one proper normal subgroup N. Then we have $|N| < |G|$, so by assumption N has a composition series

$$1 = N_0 \leq N_1 \leq ... \leq N_{k-1} \leq N_k = N,$$

where the quotient group N_{i+1}/N_i is simple for $i \in \{0, ..., k-1\}$. And, the quotient group G/N has order $|G/N| = \frac{|G|}{|N|} < |G|$, so it also contains a composition series

$$N/N = G_0/N \leq G_1/N \leq ... \leq G_{m+1}/N \leq G_m/N = G,$$

where each $(G_{i+1}/N)/(G_i/N)$ is simple for $i \in \{0, ..., m-1\}$. By the Third Isomorphism Theorem, this implies that each G_{i+1}/G_i is simple.

We now have a chain

$$1 = N_0 \leq N_1 \leq ... \leq N_{k-1} \leq N_k = N = G_0 \leq G_1 \leq ... \leq G_{m+1} \leq G_m = G$$

where the quotient of each successive subgroup by the previous is a simple group. Thus it is a composition series for G. $\square$

7. (11/9/23)

If G is a finite group and $H \trianglelefteq G$ prove that there is a composition series for G, one of whose terms is H.

Proof. By the Jordan-Hölder Theorem, H and the quotient group G/H both have composition series. Then we can construct a chain (identical to the immediately above proof) such that

$$1 = H_0 \leq H_1 \leq ... \leq H_{k-1} \leq H_k = H = G_0 \leq G_1 \leq ... \leq G_{m+1} \leq G_m = G$$

is a composition series for G, one of whose terms is H. $\square$

8. (11/12/23)

Let G be a *finite* group. Prove that the following are equivalent:

(i) G is solvable

(ii) G has a chain of subgroups: $1 = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq ... \trianglelefteq H_s = G$ such that H_{i+1}/H_i is cyclic, $0 \leq i \leq s-1$

(iii) all composition factors of G are of prime order

(iv) G has a chain of subgroups: $1 = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq ... \trianglelefteq N_t = G$ such that each N_i is a normal subgroup of G and N_{i+1}/N is abelian, $0 \leq i \leq t-1$.

Proof. To show that (i) implies (ii), let G be a finite solvable group. Then there exists a chain

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq ... \trianglelefteq G_s = G$$

such that G_{i+1}/G_i is abelian for $0 \leq i \leq s-1$. If for some i, G_{i+1}/G_i is not simple, then there exists a proper normal subgroup H of G_{i+1} that contains but is not equal to G_i. Since $G_i \trianglelefteq G_{i+1}$, we also have $G_i \trianglelefteq H$, and since G_{i+1}/G_i is abelian, H/G_i is as well. So we can subdivide every link in the original chain to produce another chain:

$$1 = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq ... \trianglelefteq H_t = G,$$

where each H_{i+1}/H_i is an abelian simple group for $0 \leq i \leq t-1$. From Exercise 1, an abelian simple group is isomorphic to Z_p for some prime p. Therefore each quotient in the chain is cyclic.

Similarly (ii) implies (iii). If G has a chain of subgroups such that each quotient is cyclic, then each quotient is also abelian. If there is a quotient H_{i+1}/H_i that is composite, then we can find another proper normal subgroup K of H_{i+1} that is not equal to H_i. We continue to do this until the links in the chain are of prime order.

Next we show that (iii) implies (iv). All composition factors of G are of prime order, so they are all isomorphic to Z_p for some prime p, thus cyclic and abelian. Let M be a minimal nontrivial normal subgroup of G. From Exercise

7, there is a composition series of G that includes M. Let $N \trianglelefteq M$ be of prime index, so M/N is abelian. From Chapter 3.1, Exercise 40, it follows that, for all $x, y \in M$, the commutator element $x^{-1}y^{-1}xy$ lies in N.

We claim next that, for all $g \in G$, $gNg^{-1} \trianglelefteq M$. Let $x = gng^{-1} \in gNg^{-1}$. Then for all $h \in G$:

$$hxh^{-1} = hgng^{-1}h^{-1} = (hg)n(hg)^{-1} \in M \text{ (since } hg \in G \text{ and } n \in M),$$

which shows that $gNg^{-1} \trianglelefteq M$. Since $|gNg^{-1}| = |N|$, M/gNg^{-1} has the same prime order as M/N, and the quotient group is therefore abelian, so for all $g \in G, x, y \in M$, we have $x^{-1}y^{-1}xy \in gNg^{-1}$. It follows that $gNg^{-1} = N$ for all $g \in G$, and so $N \trianglelefteq G$, which contradicts M being a minimal normal subgroup. Therefore we must have $N = 1$. In turn, we conclude that $x^{-1}y^{-1}xy = 1$ for all $x, y \in M$, and so $xy = yx$, hence M is abelian.

Since $M \trianglelefteq G$, next let $\overline{G} = G/M$ and let $\overline{M_1} \in \overline{G}$ be a minimal nontrivial normal subgroup. Then $\overline{M_1} = M_1/M$ is an abelian quotient group. We continue inductively until we produce the chain

$$1 = M \trianglelefteq M_1 \trianglelefteq M_2 \trianglelefteq ... \trianglelefteq M_r = G,$$

where each M_i is normal in G and M_{i+1}/M_i is abelian, $0 \leq i \leq r-1$.

Finally, (iv) implies (i), for each M_{i+1}/M_i is already abelian, and so G is abelian. This concludes the proof that the four statements above are equivalent. $\square$

9. (11/21/23)

Prove the following special case of part (2) of the Jordan-Hölder Theorem: assume the finite group G has two composition series

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq ... \trianglelefteq N_r = G \text{ and } 1 = M_0 \trianglelefteq M_1 \trianglelefteq M_2 = G.$$

Show that $r = 2$ and that the list of composition factors is the same. [Use the Second Isomorphism Theorem.]

Proof. Consider M_1 and $N_{r-1} \trianglelefteq G$. From the Diamond Isomorphism Theorem, $N_{r-1} \cap M_1 \trianglelefteq M_1$. Since M_1 is simple, this implies that either $N_{r-1} \cap M_1 = M_1$ or $N_{r-1} \cap M_1 = 1$.

If $N_{r-1} \cap M_1 = M_1$, then $M_1 \leq N_{r-1}$, and specifically, $M_1 \trianglelefteq N_{r-1}$. By the Lattice Isomorphism Theorem, since $N_{r-1} \trianglelefteq G$, $N_{r-1}/M_1 \trianglelefteq G/M_1$. However, since G/M_1 is simple, this implies that either $N_{r-1} = M_1$ or $N_{r-1} = G$. In either case, this means that N_{r-1} is part of the same composition series as M_1.

Next, if $N_{r-1} \cap M_1 = 1$, then consider the product $N_{r-1}M_1$, which is a subgroup of G, and specifically a normal subgroup of G: Let $nm \in N_{r-1}M_1$ and consider the conjugate by any element of g:

$$gnmg^{-1} = gng^{-1} \cdot gmg^{-1},$$

the product of an element of N_{r-1} with an element of M_1 and therefore belonging to $N_{r-1}M_1$, which shows that $N_{r-1}M_1 \trianglelefteq G$. Similar to above, because G/M_1 is simple, $N_{r-1}M_1$ must be either M_1 or all of G. If $N_{r-1}M_1 = M_1$, then N_{r-1} is either 1 or M_1 itself, and so is again part of the same composition series.

However, if $N_{r-1}M_1 = G$ and N_{r-1} is not the trivial group nor M_1, then $1 = N_0 \trianglelefteq N_{r-1} \trianglelefteq N_r = G$ is a composition series with only two factors (so we conclude that $r = 2$). By the Diamond Isomorphism Theorem, we have $G/M_1 \cong N_{r-1}/1 \cong N_{r-1}$ and $G/N_{r-1} \cong M_1/1 \cong M_1$, so the list of composition factors is the same. $\square$