Dummit & Foote Ch. 4.1: Group Actions and Permutation Representations

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Let G be a group and A be a nonempty set.

1. (12/24/23)

Let G act on the set A. Prove that if $a, b \in A$ and $b = g \cdot a$ for some $g \in G$, then $G_b = gG_ag^{-1}$ (G_a is the stabilizer of a). Deduce that if G acts transitively on A then the kernel of the action is $\cap_{g \in G}\, gG_ag^{-1}$.

Proof. We will show first that G_b, the stabilizer of b, is contained in gG_ag^{-1}, and then show the converse, which proves that they are equal.

Let $x \in G_b$, so $x \cdot b = b$. Then:

$$\begin{aligned} x \cdot g \cdot a &= g \cdot a \ (b = g \cdot a) \\ (gg^{-1}) \cdot (xg) \cdot a &= g \cdot a \ (gg^{-1} = 1, 1 \cdot a = a) \\ g \cdot (g^{-1}xg) \cdot a &= g \cdot a \\ (g^{-1}xg) \cdot a &= a, \end{aligned}$$

which implies that $g^{-1}xg \in G_a$, and therefore $x \in gG_ag^{-1}$, so $G_b \subseteq gG_ag^{-1}$.

The converse, that $gG_ag^{-1} \subseteq G_b$, can be shown by following the above proof in reverse (that is, let $x \in gG_ag^{-1}$, so $g^{-1}xg \in G_a$, which implies that $(g^{-1}xg) \cdot a = a$, and each assertion holds from bottom to top). Since each is contained in the other, we have $G_b = gG_ag^{-1}$.

Now we already know that the kernel of the group action of G on A is the intersection of the stabilizers of all the elements of A, that is, $\cap_{b \in A}\, G_b$. If G acts transitively on A, fixing $a \in A$, then for all $b \in A$, we can write $b = g \cdot a$ for some $g \in G$, which from above implies that $G_b = gG_ag^{-1}$. We deduce that the kernel can be expressed in terms of a fixed element a, namely:

$$\cap_{b \in A}\, G_b = \cap_{b \in A}\, \underbrace{gG_ag^{-1}}_{b = g \cdot a} = \cap_{g \in G}\, gG_ag^{-1}.$$

We know that $\cap_{g \in G}\, gG_ag^{-1}$ intersects all of the same conjugates as does $\cap_{b \in A}$, since G acts transitively on A. And, since $b = g \cdot a \Rightarrow G_b = gG_ag^{-1}$, it intersects no conjugates not represented by G_b for all $b \in A$. $\square$

2. (1/2/24)

Let G be a *permutation group* on the set A (i.e., $G \leq S_A$), let $\sigma \in G$ and let $a \in A$. Prove that $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$. Deduce that if G acts transitively on A then

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = 1.$$

Proof. We first show that $\sigma G_a \sigma^{-1} \subseteq G_{\sigma(a)}$, and then show the converse. To begin, let $\tau \in G_a$ and consider $\sigma\tau\sigma^{-1} \in \sigma G_a \sigma^{-1}$. We note that:

$$(\sigma\tau\sigma^{-1})(\sigma(a)) = (\sigma\tau\sigma^{-1}\sigma)(a) = (\sigma\tau)(a) = \underbrace{\sigma(\tau(a)) = \sigma(a)}_{\tau \in G_a \Rightarrow \tau(a) = a},$$

and so $\sigma\tau\sigma^{-1}$ stabilizes $\sigma(a)$, which implies that $\sigma G_a \sigma^{-1} \subseteq G_{\sigma(a)}$.

For the converse, let $\tau \in G$ and suppose that $\sigma\tau\sigma^{-1} \in G_{\sigma(a)}$. Then:

$$\begin{aligned}
(\sigma\tau\sigma^{-1})(\sigma(a)) &= \sigma(a) \\
(\sigma\tau\sigma^{-1}\sigma)(a) &= \sigma(a) \\
(\sigma\tau)(a) &= \sigma(a) \\
\sigma(\tau(a)) &= \sigma(a) \\
\tau(a) &= a,
\end{aligned}$$

so τ is in the stabilizer of a, which implies that $\sigma\tau\sigma^{-1} \in \sigma G_a \sigma^{-1}$, and so $G_{\sigma(a)} \subseteq \sigma G_a \sigma^{-1}$.

This concludes the proof that $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$.

Now if G acts transitively on A, then there is only one orbit; that is, given some $a \in A$, for all $b \in A$, there is a $\sigma \in G$ such that $b = \sigma(a)$.

From above, we conclude:

$$\bigcap_{\sigma \in G} \sigma G_a \sigma^{-1} = \bigcap_{\sigma \in G} G_{\sigma(a)} = \bigcap_{a \in A} G_a \text{ (because } G \text{ acts transitively on } A\text{)},$$

and since the only permutation that fixes every element of A is the identity, this intersection consists therefore only the identity permutation. □

3. (1/2/24)

Assume that G is an abelian, transitive subgroup of S_A. Show that $\sigma(a) \neq a$ for all $\sigma \in G - \{1\}$ and all $a \in A$. Deduce that $|G| = |A|$. [Use the preceding exercise.]

Proof. Suppose that σ_1 fixes a, so $\sigma_1(a) = a$, and let $\sigma_2(a) = b$. Then:

$$\begin{aligned}
&(\sigma_1 \circ \sigma_2)(a) = \sigma_1(\sigma_2(a)) = \sigma_1(b), \text{ and} \\
&(\sigma_2 \circ \sigma_1)(a) = \sigma_2(\sigma_1(a)) = \sigma_2(a).
\end{aligned}$$

Since G is abelian, these must be equal, and so $\sigma_1(b) = \sigma_2(a) = b$. Then σ_1 also fixes b.

Since G is transitive, for every $b \in A$, there exists a $\sigma \in G$ such that $\sigma(b) = a$, which implies that σ_1 fixes every element of A and is therefore the identity. Thus the identity is the only element of G for which $\sigma(a) = a$; equivalently, $\sigma(a) \neq a$ for all $\sigma \in G - \{1\}$ and all $a \in A$.

Now let $A = \{1, ..., n\}$. Since G is transitive, it must contain at least n permutations. For all $i \in A$, define σ_i such that $\sigma_i(1) = i$ (with σ_1 the identity permutation). Suppose that τ is another permutation in G. Since A only contains n elements, we must have $\tau(1) = i$ for some $i \in A$, so $\tau(1) = \sigma_i(1)$. Then:

$$\begin{aligned}(\tau \circ \sigma_i)(1) &= \tau(\sigma_i(1)) = \tau(i), \text{ and} \\ (\sigma_i \circ \tau)(1) &= \sigma_i(\tau(1)) = \sigma_i(i).\end{aligned}$$

Since G is abelian, these are equal, so $\tau(i) = \sigma_i(i)$. It follows that, if $j = \tau(i) = \sigma_i(i)$, then $\tau(j) = \sigma_i(j)$, and so on for every element which σ_i permutes. Therefore $\tau = \sigma_i$, so G contains exactly n permutations. We conclude that $|G| = |A|$. $\square$

4. (1/3/24)

Let S_3 act on the set Ω of ordered pairs: $\{(i, j) \mid 1 \leq i, j \leq 3\}$ by $\sigma((i, j)) = (\sigma(i), \sigma(j))$. Find the orbits of S_3 on Ω. For each $\sigma \in S_3$ find the cycle decomposition of σ under this action (i.e., find its cycle decomposition when σ is considered as an element of S_9 — first fix a labelling of these nine ordered pairs). For each orbit $\mathcal{O}$ of S_3 acting on these nine points pick some $a \in \mathcal{O}$ and find the stabilizer of a in S_3.

Solution. The elements $(1, 1), (2, 2)$, and $(3, 3)$ all belong to the same orbit. We see that $(1\,2) \cdot (1, 1) = (2, 2)$ and $(2\,3) \cdot (2, 2) = (3, 3)$. Further, since for any $(i, i) \in \Omega$, the action of $\sigma \in S_3$ on it results in an element with the same coordinates, there is no $(i, j) \in \Omega$ with $i \neq j$ such that $\sigma((i, i)) = (i, j)$. Therefore these three elements constitute one orbit.

The other orbit consists of the remaining six elements. Beginning with $(1, 2)$, we have:

$$\begin{aligned}(1\,2\,3)(1, 2) &= (2, 3), \text{ then} \\ (1\,2\,3)(2, 3) &= (3, 1), \\ (1\,2)(3, 1) &= (3, 2), \\ (1\,2\,3)(3, 2) &= (1, 3), \text{ and finally} \\ (1\,2\,3)(1, 3) &= (2, 1).\end{aligned}$$

Conversely to the first orbit, for no $(i, j) \in \Omega$ with $i \neq j$ do we have $\sigma((i, j)) = (i, i)$. Thus these are the two disjoint orbits of S_3 on Ω.

Next, let us label the elements of Ω:

$$\begin{array}{lll} (1,1) \to 1 & (1,2) \to 2 & (1,3) \to 3 \\ (2,1) \to 4 & (2,2) \to 5 & (2,3) \to 6 \\ (3,1) \to 7 & (3,2) \to 8 & (3,3) \to 9 \end{array}$$

Then we can describe the cycle decomposition of each permutation of S_3 by how it acts on these elements:

$$\begin{aligned} 1 &\to 1 \\ (1\,2) &\to (1\,5)(2\,4)(3\,6)(7\,8) \\ (1\,3) &\to (1\,9)(2\,8)(3\,7)(4\,6) \\ (2\,3) &\to (2\,3)(4\,7)(5\,9)(6\,8) \\ (1\,2\,3) &\to (1\,5\,9)(2\,6\,7)(3\,4\,8) \\ (1\,3\,2) &\to (1\,9\,5)(2\,7\,6)(3\,8\,4) \end{aligned}$$

For the first orbit, let us choose the point $(1,1)$ and find its stabilizer. The permutations in S_3 that fix this element must fix 1. Obviously this includes the identity. Neither of the 3-cycles fix 1. Only one of the 2-cycles, $(2\,3)$, fixes 1. Therefore the stabilizer of $(1,1)$ is the subgroup $\{1, (2\,3)\}$.

For the second orbit, we choose $(1,2)$. Since all non-identity permutations of S_3 reassign either 1 or 2 (or both), the stabilizer consists of only the identity. $\square$