Dummit & Foote Ch. 1.3: Symmetric Groups

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1. (2/16/23)

- $\sigma : (1, 3, 5)(2, 4)$
- $\tau : (1, 5)(2, 3)$
- $\sigma^2 : (1, 5, 3)$
- $\sigma\tau : (2, 5, 3, 4)$
- $\tau\sigma : (1, 2, 4, 3)$
- $\tau^2\sigma : (1, 3, 5)(2, 4)$ (because $\tau^2 = 1$, so $\tau^2\sigma = \sigma$)

2. (2/16/23)

- $\sigma : (1, 13, 5, 10)(3, 15, 8)(4, 14, 11, 7, 12, 9)$
- $\tau : (1, 14)(2, 9, 15, 13, 4)(3, 10)(5, 12, 7)(8, 11)$
- $\sigma^2 : (1, 5)(3, 8, 15)(4, 11, 12)(7, 9, 4)(10, 13)$
- $\sigma\tau : (1, 11, 3)(2, 4)(5, 9, 8, 7, 10, 15)(13, 14)$
- $\tau\sigma : (1, 4)(2, 9)(3, 13, 12, 15, 11, 5)(8, 10, 14)$
- $\tau^2\sigma : (1, 2, 15, 8, 3, 4, 14, 11, 12, 13, 7, 5, 10)$

3. (2/16/23)

Compute the order of each of the permutations whose cycle decompositions were computed above.

1. $|\sigma| = 6$; $|\tau| = 2$; $|\sigma^2| = 3$; $|\sigma\tau| = 4$; $|\tau\sigma| = 4$; $|\tau^2\sigma| = 6$
2. $|\sigma| = 12$; $|\tau| = 30$; $|\sigma^2| = 6$; $|\sigma\tau| = 6$; $|\tau\sigma| = 6$; $|\tau^2\sigma| = 13$

4. (2/16/23)

Compute the order of each of the elements in the following groups:

(a) S_3

- (1): 1
- $(1, 2); (1, 3); (2, 3)$: 2
- $(1, 2, 3); (1, 3, 2)$: 3

(b) S_4

- (1): 1
- $(1, 2); (1, 3); (1, 4); (2, 3); (2, 4); (3, 4); (1, 2)(3, 4);$
  $(1, 3)(2, 4); (1, 4)(2, 3)$: 2
- $(1, 2, 3); (1, 3, 2); (1, 2, 4); (1, 4, 2); (1, 3, 4); (1, 4, 3);$
  $(2, 3, 4); (2, 4, 3)$: 3
- $(1, 2, 3, 4); (1, 4, 2, 3); (1, 3, 2, 4); (1, 3, 4, 2); (1, 4, 2, 3); (1, 4, 3, 2)$: 4

5. (2/16/23)

Find the order of $(1, 12, 8, 10, 4)(2, 13)(5, 11, 7)(6, 9)$.

Proof. The order of a permutation in a symmetric group is the least common multiple of its cycles. However, since we have not yet proven this, we will calculate the first few multiples of the permutation manually, and extrapolate from there. Let $\sigma = (1, 12, 8, 10, 4)(2, 13)(5, 11, 7)(6, 9)$.

$\sigma^2 = (1, 8, 4, 12, 10)(5, 7, 11)$.

$\sigma^3 = (1, 10, 12, 4, 8)(2, 13)(6, 9)$.

$\sigma^4 = (1, 4, 10, 8, 12)(5, 11, 7)$.

$\sigma^5 = (2, 13)(5, 7, 11)(6, 9)$.

From this pattern, we see that each constituent cycle vanishes from the cycle decomposition when the exponent is a multiple of the cycle's length. Thus, the order of σ is the least common multiple of the lengths of its cycles, which is $2 \cdot 3 \cdot 5 = 30$. □

6. (2/17/23)

Write out the cycle decomposition of each element of order 4 in S_4.

- $(1, 2, 3, 4)$
- $(1, 2, 4, 3)$
- $(1, 3, 2, 4)$

- $(1, 3, 4, 2)$
- $(1, 4, 2, 3)$
- $(1, 4, 3, 2)$

7. (2/20/23)

Write out the cycle decomposition of each element of order 2 in S_4.

- $(1, 2)$
- $(1, 3)$
- $(1, 4)$
- $(2, 3)$
- $(2, 4)$
- $(3, 4)$
- $(1, 2)(3, 4)$
- $(1, 3)(2, 4)$
- $(1, 4)(2, 3)$

8. (2/22/23)

Prove that if $\Omega = \{1, 2, 3...\}$ then S_Ω is an infinite group.

Proof. Let $\Omega = \{1, 2, 3...\}$. Consider the subset T consisting of all elements whose cycle decomposition is a single 2-cycle permuting $1 \in \Omega$, for example $(1, 2), (1, 10)$ but not $(2, 3)$.

There is a bijection $f : \mathbb{Z}^+ \to T$ defined by $f(n) = (1, n + 1)$. Because there is a bijection between these two sets, they have the same cardinality; that is, like $\mathbb{Z}^+$, T is infinite.

Because Ω contains a proper subset of infinite size, Ω has infinite elements and is therefore an infinite group.

$\square$

9. (2/22/23)

(a) Let σ be the 12-cycle $(1,2,3,4,5,6,7,8,9,10,11,12)$. For which positive integers i is σ^i also a 12-cycle?

- $\sigma^5 = (1,6,11,4,9,2,7,12,5,10,3,8)$
- $\sigma^7 = (1,8,3,10,5,12,7,2,9,4,11,6)$
- $\sigma^1 1 = (1,12,11,10,...,2)$

(b) Let τ be the 8-cycle $(1,2,3,4,5,6,7,8)$. For which positive integers i is τ^i also a 12-cycle?

- $\tau^3 = (1,4,7,2,5,8,3,6)$
- $\tau^5 = (1,6,3,8,5,2,7,4)$
- $\tau^7 = (1,8,7,6,5,4,3,2)$

(c) Let ω be the 14-cycle $(1,2,3,4,5,6,7,8,9,10,11,12,13,14)$. For which positive integers i is ω^i also a 12-cycle?

- $\omega^3 = (1,4,7,10,13,2,5,8,11,14,3,6,9,12)$
- $\omega^5 = (1,6,11,2,7,12,3,8,13,4,9,14,5,10)$
- $\omega^9 = (1,10,5,14,9,4,13,8,3,12,7,2,11,6)$
- $\omega^{11} = (1,12,9,6,3,14,11,8,5,2,13,10,7,4)$
- $\omega^{13} = (1,14,13,12,...,2)$

10. (2/23/23)

Prove that if σ is the m-cycle $(a_1, a_2, ..., a_m)$, then for all $i \in \{1,2,3,...,m\}$, $\sigma^i(a_k) = a_{k+i}$, where $k+i$ is replaced by its least residue mod m when $k+i > m$. Deduce that $|\sigma| = m$.

Proof. We will prove this by induction on i. For the base case, $i = 1$, we have, by definition, $\sigma^1 = \sigma$ and $\sigma(a_k) = a_{k+1}$ for $k < m$. For m, $\sigma(a_m) = a_1$, and since $1 = (m+1) \bmod m$, this holds for all $i \in \{1,2,3,...,m\}$.

For the induction case, suppose that for some $n \in \{1,2,3,...,m\}$, $\sigma^n(a_k) = a_{k+n}$ (where $k+n$ is assumed mod m) for all valid k. Consider $\sigma^{n+1} = \sigma^n\sigma^1 = \sigma^n\sigma$. For an arbitrary element a_k, then, $\sigma^{n+1}(a_k) = \sigma^n(\sigma(a_k)) = \sigma^n(a_{k+1})$ (by the base case), which equals a_{k+n+1} (by the induction hypothesis). Therefore, $\sigma^{n+1}(a_k) = a_{k+(n+1)}$.

Thus, by induction, $\sigma^i(a_k) = a_{k+i}$ for all $i \in \{1,2,3,...,m\}$. It follows that $\sigma^m(a_k) = a_{k+m} = a_k$ for all k, so $\sigma^m = 1$. Therefore, $|\sigma| = m$. $\square$