Dummit & Foote Ch. 4.3: Groups Acting on Themselves by Conjugation — The Class Equation

Scott Donaldson

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Let G be a group.

1. (2/22/24)

Suppose G has a left action on a set A, denoted by $g \cdot a$ for all $g \in G$ and $a \in A$. Denote the corresponding right action on A by $a \cdot g$. Prove that the (equivalence) relations $\sim$ and $\sim'$ defined by

$$a \sim b \text{ if and only if } a = g \cdot b \text{ for some } g \in G$$

and

$$a \sim' b \text{ if and only if } a = b \cdot g \text{ for some } g \in G$$

are the same relation (i.e., $a \sim b$ if and only $a \sim' b$).

Proof. To show that $a \sim b$ implies $a \sim' b$, we must show that, given a $g \in G$ with $a = g \cdot b$, there exists an $h \in G$ such that $a = b \cdot h$. By definition, the corresponding right action of a left action is specified to be $g \cdot x = x \cdot g^{-1}$ for all $g \in G$, $x \in A$. Letting $h = g^{-1}$, we have found an element where $a = g \cdot b = b \cdot h$, and so $a \sim' b$.

The proof for $a \sim' b$ implies $a \sim b$ is identical, letting $h = g^{-1}$ but with h acting on the left. $\square$

2. (2/22/24)

Find all conjugacy classes and their sizes in the following groups:

(a) D_8:

$$\{1\}_1 \qquad \{r^2\}_1 \qquad \{r, r^3\}_2 \qquad \{s, sr^2\}_2 \qquad \{sr, sr^3\}_2$$

(b) Q_8:

$$\{1\}_1 \qquad \{-1\}_1 \qquad \{\pm i\}_2 \qquad \{\pm j\}_2 \qquad \{\pm k\}_2$$

(c) A_4:

$$\{1\}_1 \quad \{(1\,2\,3),(1\,3\,4),(1\,4\,2),(2\,4\,3)\}_4 \quad \{(1\,3\,2),(1\,2\,4),(1\,4\,3),(2\,3\,4)\}_4$$
$$\{(1\,2)(3\,4),(1\,3)(2\,4),(1\,4)(2\,3)\}_3$$

3. (2/22/24)

Find all the conjugacy classes and their sizes in the following groups:

(a) $Z_2 \times S_3$:

$$\{(0,1)\}_1 \quad \{(1,1)\}_1 \quad \{(0,(1\,2)),(0,(1\,3)),(0,(2\,3))\}_3$$
$$\{(1,(1\,2)),(1,(1\,3)),(1,(2\,3))\}_3 \quad \{(0,(1\,2\,3)),(0,(1\,3\,2))\}_2$$
$$\{(1,(1\,2\,3)),(1,(1\,3\,2))\}_2$$

(b) $S_3 \times S_3$:

$$\{(1,1)\}_1 \quad \{(1,\text{2-cycle})\}_3 \quad \{(\text{2-cycle},1)\}_3 \quad \{(1,\text{3-cycle})\}_2 \quad \{(\text{3-cycle},1)\}_2$$
$$\{(\text{2-cycle},\text{2-cycle})\}_9 \quad \{(\text{2-cycle},\text{3-cycle})\}_6 \quad \{(\text{3-cycle},\text{2-cycle})\}_6$$
$$\{(\text{3-cycle},\text{3-cycle})\}_4$$

(c) $Z_3 \times A_4$ (using representatives from the conjugacy classes of A_4 above):

$$\{(0,1)\}_1 \quad \{(1,1)\}_1 \quad \{(2,1)\}_1$$
$$\{(0,\overline{(1\,2\,3)})\}_4 \quad \{(1,\overline{(1\,2\,3)})\}_4 \quad \{(2,\overline{(1\,2\,3)})\}_4$$
$$\{(0,\overline{(1\,3\,2)})\}_4 \quad \{(1,\overline{(1\,3\,2)})\}_4 \quad \{(2,\overline{(1\,3\,2)})\}_4$$
$$\{(0,\overline{(1\,2)(3\,4)})\}_3 \quad \{(1,\overline{(1\,2)(3\,4)})\}_3 \quad \{(2,\overline{(1\,2)(3\,4)})\}_3$$

4. (2/22/24)

Prove that if $S \subseteq G$ and $g \in G$ then $gN_g(S)g^{-1} = N_G(gSg^{-1})$ and $gC_g(S)g^{-1} = C_G(gSg^{-1})$.

Proof. Let $x \in N_G(S)$. So $xsx^{-1} \in S$ for all $s \in S$. Then

$$\begin{aligned}
gxsx^{-1}g^{-1} &\in gSg^{-1} \\
gxg^{-1}gsg^{-1}gx^{-1}g^{-1} &\in gSg^{-1} \\
(gxg^{-1})gsg^{-1}(gx^{-1}g^{-1}) &\in gSg^{-1} \\
(gxg^{-1})gsg^{-1}(gxg^{-1})^{-1} &\in gSg^{-1},
\end{aligned}$$

which implies that $gxg^{-1} \in N_G(gSg^{-1})$, and so $gN_G(S)g^{-1} \subseteq N_G(gSg^{-1})$.

Conversely, let $x \in N_G(gSg^{-1})$. So $xgsg^{-1}x^{-1} \in gSg^{-1}$ for all $s \in S$. Then

$$\begin{aligned} xgsg^{-1}x^{-1} &\in gSg^{-1} \\ g^{-1}xgsg^{-1}x^{-1} &\in Sg^{-1} \\ g^{-1}xgsg^{-1}x^{-1}g &\in S \\ (g^{-1}xg)s(g^{-1}xg)^{-1} &\in S \\ g^{-1}xg &\in N_G(S) \\ x &\in gN_G(S)g^{-1}, \end{aligned}$$

which shows that $N_G(gSg^{-1}) \subseteq gN_G(S)g^{-1}$. This proves that $N_G(gSg^{-1}) = gN_G(S)g^{-1}$.

Next, let $x \in C_G(S)$. So $xs = sx$ for all $s \in S$. Then

$$\begin{aligned} xs &= sx \\ gsxg^{-1} &= gsxg^{-1} \\ gsg^{-1}gxg^{-1} &= gsg^{-1}gxg^{-1} \\ (gsg^{-1})(gxg^{-1}) &= (gsg^{-1})(gxg^{-1}), \end{aligned}$$

and so $gxg^{-1} \in C_G(gSg^{-1})$, which implies that $gC_G(S)g^{-1} \subseteq C_G(gSg^{-1})$. Finally, let $x \in C_G(gSg^{-1})$. So $x(gsg^{-1}) = (gsg^{-1})x$ for all $x \in S$. Then

$$\begin{aligned} xgsg^{-1} &= gsg^{-1}x \\ g^{-1}xgsg^{-1} &= sg^{-1}x \\ g^{-1}xgs &= sg^{-1}xg \\ (g^{-1}xg)s &= s(g^{-1}xg), \end{aligned}$$

which implies that $g^{-1}xg \in C_G(S)$, so $x \in gC_G(S)g^{-1}$. It follows that $C_G(gSg^{-1}) \subseteq gC_G(S)g^{-1}$, and therefore $gC_g(S)g^{-1} = C_G(gSg^{-1})$. $\square$

9. (3/7/24)

Show that $|C_{S_n}((1\,2)(3\,4))| = 8 \cdot (n-4)!$ for all $n \geq 4$. Determine the elements in this centralizer explicitly.

Proof. In S_4, the permutations that commute with $(1\,2)(3\,4)$ are the four elements of the cyclic subgroup generated by it, as well as the transpositions $(1\,2)$ and $(3\,4)$, and the 4-cycles $(1\,3\,2\,4)$ and $(1\,4\,2\,3)$.

Now let $n > 4$. Consider the product of one of the elements of $C_{S_4}((1\,2)(3\,4))$ with an element of S_n. If the permutation only acts on $1, 2, 3, 4$, then it is already in $C_{S_4}((1\,2)(3\,4))$. If the permutation only acts on $\{5, ..., n\}$ then it is disjoint with (thus commutes with) the permutations in $C_{S_4}((1\,2)(3\,4))$. Now $S_{\{5,...,n\}} \cong S_{n-4}$, therefore there are $(n-4)!$ such permutations. Since the product of any of these permutations with an element of $C_{S_4}((1\,2)(3\,4))$ must commute with $(1\,2)(3\,4)$, there are thus $8 \cdot (n-4)!$ elements in $C_{S_n}((1\,2)(3\,4))$. $\square$

10. (2/28/24)

Let σ be the 5-cycle $(1\,2\,3\,4\,5)$ in S_5. In each of (a) to (c) find an explicit element $\tau \in S_5$ which accomplishes the specified conjugation:

(a) $\tau\sigma\tau^{-1} = \sigma^2 = (1\,3\,5\,2\,4)$. Let $\tau = (2\,3\,5\,4)$. Then $\tau\sigma\tau^{-1} = (\tau(1)\,\tau(2)\,\tau(3)\,\tau(4)\,\tau(5)) = (1\,3\,5\,2\,4) = \sigma^2$.

(b) $\tau\sigma\tau^{-1} = \sigma^{-1} = (1\,5\,4\,3\,2)$. Let $\tau = (2\,5)(3\,4)$. Then $\tau\sigma\tau^{-1} = \sigma^{-1}$.

(c) $\tau\sigma\tau^{-1} = \sigma^{-2} = (1\,4\,2\,5\,3)$. Let $\tau = (2\,4\,5\,3)$. Then $\tau\sigma\tau^{-1} = \sigma^{-2}$.

11. (2/28/24)

In each of (a) - (d) determine whether σ_1 and σ_2 are conjugate. If they are, give an explicit permutation τ such that $\tau\sigma_1\tau^{-1} = \sigma_2$.

(a) $\sigma_1 = (1\,2)(3\,4\,5)$ and $\sigma_2 = (1\,2\,3)(4\,5)$. Both have cycle type $1, 1, 3$ and so they are conjugate. Let $\tau = (1\,4\,2\,5\,3)$. Then $\tau\sigma_1\tau^{-1} = \sigma_2$.

(b) $\sigma_1 = (1\,5)(3\,7\,2)(10\,6\,8\,11)$ and $\sigma_2 = (3\,7\,5\,10)(4\,9)(13\,11\,2)$. In S_13, both have cycle type $1, 1, 1, 1, 2, 3, 4$ and so they are conjugate. Let $\tau = (1\,4)(2\,11\,10\,3)(5\,9\,6\,7\,13\,8)$. Then $\tau\sigma_1\tau^{-1} = \sigma_2$.

(c) $\sigma_1 = (1\,5)(3\,7\,2)(10\,6\,8\,11)$ and $\sigma_2 = \sigma_1^3 = (1\,5)(10\,11\,8\,6)$. They do not have the same cycle type (σ_1 contains a 3-cycle that σ_2 does not), and so they are not conjugate.

(d) $\sigma_1 = (1\,3)(2\,4\,6)$ and $\sigma_2 = (3\,5)(2\,4)(5\,6) = (2\,4)(3\,5\,6)$. Let $\tau = (1\,2\,3\,4\,5)$. Then $\tau\sigma_1\tau^{-1} = \sigma_2$.

13. (2/28/24)

Find all finite groups which have exactly two conjugacy classes.

Proof. Let G be a non-trivial finite group. Since the conjugacy class of 1 is $\{1\}$, if G has exactly two conjugacy classes, then every other element in G must have the same conjugacy class, namely $G - \{1\}$.

From Proposition 6, for any $g \in G$, the number of conjugates of g (i.e. the cardinality of the conjugacy class of g) is the index of the centralizer of g, $|G : C_G(g)|$. Therefore the size of the conjugacy class of g must divide the order of G.

Let $|G| = n$. Then the size of the conjugacy class of g is $|G - \{1\}| = n - 1$. This is only possible when $|G| = 2$, and so G must be the unique group of order two. $\square$

17. (3/25/24)

Let A be a nonempty set and let X be any subset of A. Let

$$F(X) = \{a \in A \mid \sigma(a) = a \text{ for all } \sigma \in X\} \text{ — the \textit{fixed set} of } X.$$

Let $M(X) = A - F(X)$ be the elements which are *moved* by some element of X. Let $D = \{\sigma \in S_A \mid |M(\sigma)| < \infty\}$. Prove that D is a normal subgroup of S_A.

Proof. Let $\sigma \in D$. Then $M(\sigma)$ is a finite subset of A. Let $\tau \in S_A$ and consider $M(\tau\sigma\tau^{-1})$. If $|M(\tau\sigma\tau^{-1})|$ is also finite, then $\tau \in D$, and so $D \trianglelefteq S_A$. Now $|M(\sigma)| = |A - F(\sigma)| = |A| - |F(\sigma)|$, and $|A|$ is constant. Therefore, by proving that $|F(\sigma)| = |F(\tau\sigma\tau^{-1})|$, it follows that $|M(\sigma)| = |M(\tau\sigma\tau^{-1})|$, which proves that $D \trianglelefteq S_A$. We will now construct a bijection between $|F(\sigma)|$ and $|F(\tau\sigma\tau^{-1})|$.

Let $\varphi : F(\sigma) \to F(\tau\sigma\tau^{-1})$ be defined by $\varphi(a) = \tau(a)$. The map φ is well-defined, because:

$$a \in F(\sigma) \Rightarrow a = \sigma(a) \Rightarrow \tau(a) = \tau(\sigma(a)) = (\tau\sigma\tau^{-1})(\tau(a)),$$

which implies that $\varphi(a) = \tau(a) \in F(\tau\sigma\tau^{-1})$.

It is injective: Let $a, b \in F(\sigma)$ and suppose that $\varphi(a) = \varphi(b)$. Then $\tau(a) = \tau(b)$, and since τ is a permutation, by definition we have $a = b$.

Finally, φ is surjective. Let $b \in F(\tau\sigma\tau^{-1})$ (to show that there exists an $a \in F(\sigma)$ such that $\varphi(a) = b$). Let $a = \tau^{-1}(b)$. Then $b = \tau(a)$, so $\varphi(a) = b$. We show that $\sigma(a) = a$:

$$\begin{aligned} \tau\sigma\tau^{-1}(b) &= b \\ (\tau\sigma\tau^{-1})(\tau(a)) &= \tau(a) \\ \tau(\sigma(a)) &= \tau(a) \\ \sigma(a) &= a, \end{aligned}$$

which implies that $a \in F(\sigma)$, so φ is surjective. Therefore φ is a bijection between $F(\sigma)$ and $F(\tau\sigma\tau^{-1})$, which (as noted above), proves that D is a normal subgroup of S_A. $\square$