Dummit & Foote Ch. 3.3: The Isomorphism Theorems

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Oct. 2023

Let G be a group.

1. (10/20/23)

Let F be a finite field of order q and let $n \in \mathbb{Z}^+$. Prove that $|GL_n(F) : SL_n(F)| = q - 1$.

Proof. Define a map $\varphi : GL_n(F) \to F^\times$ by $\varphi(A) = \det A$ for all $A \in GL_n(F)$. From Ch. 3.1, Exercise 35., φ is a surjective homomorphism with $\ker \varphi = SL_n(F)$.

From Corollary 17, we have:

$$\begin{aligned} |GL_n(F) : \ker \varphi| &= |\varphi(GL_n(F))|, \text{ which implies that} \\ |GL_n(F) : SL_n(F)| &= \underbrace{|F^\times|}_{\varphi \text{ is surjective}} = q - 1, \end{aligned}$$

as desired. □

3. (10/26/23)

Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either

(i) $K \leq H$ or

(ii) $G = HK$ and $|K : K \cap H| = p$.

Proof. Suppose that $H \trianglelefteq G$ with $|G : H| = |G/H| = p$, where p is a prime. Suppose additionally that $K \leq G$ and $K \nleq H$.

Now let $g \in G$. Clearly g belongs to the left coset gH, which we denote $\overline{g} \in G/H$. Since G/H has order p, it is cyclic, and so is generated by any non-identity element (that is, any coset of H other than itself). So $\overline{g}$ generates G/H. Similarly, for any $k \in K, k \notin H$, $\overline{k}$ generates G/H. Therefore $\overline{g} = \overline{k}$ for some g, k, which implies that $g \in kH$. It follows that $g \in KH$, so $G \leq KH$. Since G is closed, we must have $G = KH = HK$.

From the Diamond Isomorphism Theorem, we have $HK/H \cong K/H \cap K$. Since $HK = G$, it follows that $|G : H| = |K : H \cap K|$, and so $|K : K \cap H| = p$. $\square$