Dummit & Foote Ch. 3.2: More on Cosets and Lagrange's Theorem

Scott Donaldson

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Let G be a group.

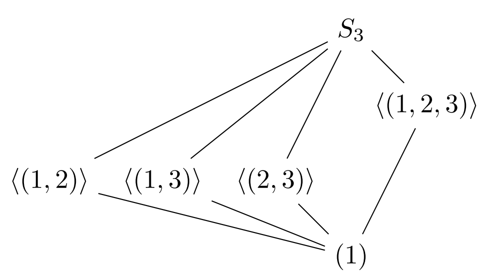
1. (10/1/23)

Which of the following are permissible orders of subgroups of a group of order 120: 1, 2, 5, 7, 9, 15, 60, 240? For each permissible order give the corresponding index.

Proof. From Lagrange's theorem, the order of a subgroup of a group of order 120 must divide 120. Then the permissible orders for subgroups are $1 = \frac{120}{120}$, $2 = \frac{120}{60}, 5 = \frac{120}{24}, 15 = \frac{120}{8}$, and $60 = \frac{120}{2}$. For each of these orders the index is given by the corresponding denumerator. $\square$

2. (10/2/23)

Prove that the lattice of subgroups of S_3 below is correct (i.e., prove that it contains all subgroups of S_3 and that their pairwise joins and intersections are correctly drawn).



Proof. The symmetric group S_3 contains 6 elements. By Lagrange's theorem, its proper subgroups must have order 2 or 3. Each of the subgroups in the lattice above have order 2 or 3, so there are no smaller or larger subgroups not depicted above.

From Corollary 10, a subgroup of order 2 must be isomorphic to Z_2, that is, cyclic and generated by a single element of order 2. The three subgroups generated by the three elements of order 2 (the 2-cycles of S_3) are depicted above. Similarly, a subgroup of order 3 must be isomorphic to Z_3 and generated by a single element of order 3. The subgroup generated by $(1, 2, 3)$ contains $(1, 3, 2)$, so there is only a single subgroup of order 3.

Next, again by Lagrange's Theorem, a subgroup of two different containing groups must have an order that divides the order of both of the containing groups. First consider a subgroup of order 2 and a subgroup of order 3. Only 1 divides 2 and 3, so the intersection must be the identity. Similarly, if a subgroup of order 2 and a subgroup of order 3 are contained in a larger group, then that group's order must have both 2 and 3 as divisors. The smallest integer for which this is possible is 6, which is the order of all of S_3.

Finally, consider a pair of subgroups of order 2. Their intersection is either the identity or else they are the same subgroup. Their join must have even order, but 4 does not divide 6 and any larger even number exceeds the order of S_3. Thus their join is all of S_3. This concludes the proof that the lattice of subgroups of S_3 is correct. $\square$

3. (10/2/23)

Prove that the lattice of subgroups of Q_8 below is correct.

$$\begin{tikzcd}[arrows=dash]
 & Q_8 \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle i \rangle \arrow[dr] & \langle j \rangle \arrow[d] & \langle k \rangle \arrow[dl] \\
 & \langle -1 \rangle \arrow[d] & \\
 & 1 &
\end{tikzcd}$$

Proof. The group Q_8 has order $8 = 2^3$, so by Lagrange's theorem its proper subgroups must have order 2 or 4. We will start from the bottom and work toward the top: There is only one element of order 2 in Q_8, -1, and the cyclic subgroup generated by it is in the lattice.

For each of i, j, and k, $\langle -1 \rangle$ is contained in the subgroup generated by them (ex. $\langle i \rangle = \{\pm 1, \pm i\}$) and there are no intermediate subgroups, since there is no divisor of 4 that is strictly greater than 2. At this point, every element of Q_8 is represented, so there are no cyclic subgroups missing. We might ask if there is a subgroup of order 4 missing. If so, it cannot be cyclic, and from Ch. 1.1, Exercise 36, it must be isomorphic to V_4. However, V_4 contains three elements of order 2, and Q_8 only has one, so there is no subgroup of Q_8 isomorphic to V_4.

Finally, the join of any of the subgroups generated by i, j, or k must contain strictly more than 4 elements and its order must divide 8. Then any of their joins must have order 8, that is, be all of Q_8. $\square$