

Dummit & Foote Ch. 3.3: The Isomorphism Theorems

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Oct. 2023

Let G be a group.

1. (10/20/23)

Let F be a finite field of order q and let $n \in \mathbb{Z}^+$. Prove that $|GL_n(F) : SL_n(F)| = q - 1$.

Proof. Define a map $\varphi : GL_n(F) \rightarrow F^\times$ by $\varphi(A) = \det A$ for all $A \in GL_n(F)$. From Ch. 3.1, Exercise 35., φ is a surjective homomorphism with $\ker \varphi = SL_n(F)$.

From Corollary 17, we have:

$$\begin{aligned} |GL_n(F) : \ker \varphi| &= |\varphi(GL_n(F))|, \text{ which implies that} \\ |GL_n(F) : SL_n(F)| &= \underbrace{|F^\times|}_{\varphi \text{ is surjective}} = q - 1, \end{aligned}$$

as desired. □

3. (10/26/23)

Prove that if H is a normal subgroup of G of prime index p then for all $K \leq G$ either

- (i) $K \leq H$ or
- (ii) $G = HK$ and $|K : K \cap H| = p$.

Proof. Suppose that $H \trianglelefteq G$ with $|G : H| = |G/H| = p$, where p is a prime. Suppose additionally that $K \leq G$ and $K \not\leq H$.

Now let $g \in G$. Clearly g belongs to the left coset gH , which we denote $\bar{g} \in G/H$. Since G/H has order p , it is cyclic, and so is generated by any non-identity element (that is, any coset of H other than itself). So \bar{g} generates G/H . Similarly, for any $k \in K, k \notin H$, \bar{k} generates G/H . Therefore $\bar{g} = \bar{k}$ for

some g, k , which implies that $g \in kH$. It follows that $g \in KH$, so $G \leq KH$. Since G is closed, we must have $G = KH = HK$.

From the Diamond Isomorphism Theorem, we have $HK/H \cong K/H \cap K$. Since $HK = G$, it follows that $|G : H| = |K : H \cap K|$, and so $|K : K \cap H| = p$. \square

4. (10/27/23)

Let C be a normal subgroup of the group A and let D be a normal subgroup of the group B . Prove that $(C \times D) \trianglelefteq (A \times B)$ and $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$.

Proof. Let $(c, d) \in C \times D$. Consider the conjugate of (c, d) by $(a, b) \in A \times B$:

$$(a, b)(c, d)(a, b)^{-1} = (a, b)(c, d)(a^{-1}, b^{-1}) = (aca^{-1}, bdb^{-1}).$$

Because $C \trianglelefteq A$, the first coordinate is an element of C , and similarly the second is an element of D . Therefore the conjugate element lies in $C \times D$, and it follows that $(C \times D) \trianglelefteq (A \times B)$.

Next, to show that $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$, define a map $\varphi : (A \times B)/(C \times D) \rightarrow (A/C) \times (B/D)$ by $\varphi(\overline{(a, b)}) = (\overline{a}, \overline{b})$. We see that this map is a homomorphism:

$$\begin{aligned} \varphi(\overline{(a_1, b_1)}\overline{(a_2, b_2)}) &= \varphi(\overline{(a_1a_2, b_1b_2)}) = (\overline{a_1a_2}, \overline{b_1b_2}) \\ &= (\overline{a_1}, \overline{b_1})(\overline{a_2}, \overline{b_2}) = \varphi(\overline{(a_1, b_1)})\varphi(\overline{(a_2, b_2)}). \end{aligned}$$

It is also surjective by definition, since $(\overline{a}, \overline{b}) = \varphi(\overline{(a, b)})$ is an arbitrary element of $(A/C) \times (B/D)$ with a preimage in $(A \times B)/(C \times D)$.

Finally, it is injective. Let $\varphi(\overline{(a_1, b_1)}) = \varphi(\overline{(a_2, b_2)})$. Then $(\overline{a_1}, \overline{b_1}) = (\overline{a_2}, \overline{b_2})$, so we have $\overline{a_1} = \overline{a_2}$ and $\overline{b_1} = \overline{b_2}$. Since $\overline{a_1} = \overline{a_2}$ implies $(\overline{a_1}, x) = (\overline{a_2}, x)$ for all $x \in B/D$ and vice-versa, we then have $(\overline{a_1}, \overline{b_1}) = (\overline{a_2}, \overline{b_2})$, and so φ is one-to-one.

Thus φ is an isomorphism, which concludes the proof that $(A \times B)/(C \times D) \cong (A/C) \times (B/D)$. \square

5. (10/27/23)

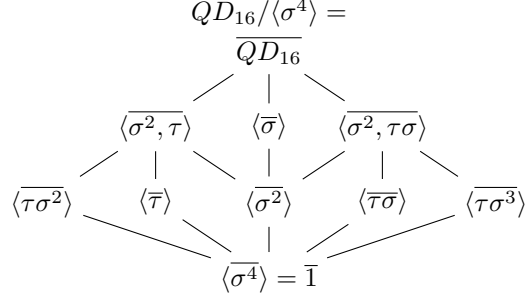
Let QD_{16} be the quasidihedral group described in Exercise 11 of Section 2.5. Prove that $\langle \sigma^4 \rangle$ is normal in QD_{16} and use the Lattice Isomorphism Theorem to draw the lattice of subgroups of $QD_{16}/\langle \sigma^4 \rangle$. Which group of order 8 has the same lattice as this quotient? Use generators and relations for $QD_{16}/\langle \sigma^4 \rangle$ to decide the isomorphism type of this group.

Solution. Consider the subgroup $\langle \sigma^4 \rangle$ in QD_{16} . To prove that it is normal, it suffices to check that the conjugates of σ^4 by the generators of QD_{16} lie in $\langle \sigma^4 \rangle$. Now powers of σ commute, so we only need to check $\tau\sigma^4\tau^{-1}$:

$$\tau\sigma^4\tau^{-1} = \tau\sigma^4\tau = \tau\tau\sigma^{12} = \sigma^{12} = \sigma^4 \in \langle \sigma^4 \rangle,$$

so $\langle \sigma^4 \rangle \trianglelefteq QD_{16}$.

Now from the Lattice Isomorphism Theorem, the lattice of subgroups of $QD_{16}/\langle \sigma^4 \rangle$ corresponds to the lattice of subgroups of QD_{16} containing $\langle \sigma^4 \rangle$:



Next, consider the generators and relations for $\overline{QD_{16}}$:

$$\overline{QD_{16}} = \langle \overline{\sigma}, \overline{\tau} \mid \overline{\sigma^4} = \overline{\tau^2} = \overline{1}, \overline{\sigma\tau} = \overline{\tau\sigma^3} = \overline{\tau} \cdot \overline{\sigma^{-1}} \rangle.$$

The right-most equation among the relations: $\overline{\tau\sigma^3} = \overline{\tau} \cdot \overline{\sigma^{-1}}$ shows that the generators and relations of this quotient group are identical to those of D_8 , mapping $s \in D_8$ to $\overline{\tau} \in \overline{QD_{16}}$ and $r \in D_8$ to $\overline{\sigma} \in \overline{QD_{16}}$. Thus we have $QD_{16}/\langle \sigma^4 \rangle \cong D_8$. \square

6. (10/28/23)

Let $M = \langle v, u \rangle$ be the modular group of order 16 described in Exercise 14 of Section 2.5. Prove that $\langle v^4 \rangle$ is normal in M and use the Lattice Isomorphism Theorem to draw the lattice of subgroups of $M/\langle v^4 \rangle$. Which group of order 8 has the same lattice as this quotient? Use generators and relations for $M/\langle v^4 \rangle$ to decide the isomorphism type of this group.

Solution. Recall that the modular group of order 16 is defined as:

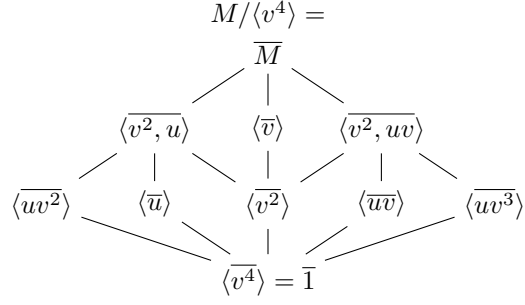
$$M = \langle v, u \mid u^2 = v^8 = 1, vu = uv^5 \rangle.$$

As above, to show that $\langle v^4 \rangle$ is normal in M , it suffices to show that the conjugate uv^4u^{-1} lies in $\langle v^4 \rangle$:

$$uv^4u^{-1} = uv^4u = uvv^{20} = v^4 \in \langle v^4 \rangle,$$

so $\langle v^4 \rangle \trianglelefteq M$.

From the Lattice Isomorphism Theorem, the lattice of subgroups of $M/\langle v^4 \rangle$ corresponds to the lattice of subgroups of M containing $\langle v^4 \rangle$:



Next, consider the generators and relations for $M/\langle v^4 \rangle$:

$$M/\langle v^4 \rangle = \langle \overline{v}, \overline{u} \mid \overline{v}^4 = \overline{u}^2 = \overline{1}, \overline{v}\overline{u} = \overline{uv^5} = \overline{uv} \rangle.$$

The right-most equation shows that this is an abelian group. Consider the presentation for $Z_2 \times Z_4$ given by $\langle x, y \mid x^2 = y^4 = 1, xy = yx \rangle$. Mapping $\overline{u} \in M/\langle v^4 \rangle$ to $x \in Z_2 \times Z_4$ and $\overline{v} \in M/\langle v^4 \rangle$ to $y \in Z_2 \times Z_4$, we obtain an isomorphism. Therefore $M/\langle v^4 \rangle \cong Z_2 \times Z_4$. \square