

Dummit & Foote Ch. 4.2: Groups Acting on Themselves by Left Multiplication — Cayley's Theorem

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Let G be a group and let H be a subgroup of G .

1. (2/12/24)

Let $G = \{1, a, b, c\}$ be the Klein 4-group whose group table is written out in Section 2.5.

- (a) Label $1, a, b, c$ with the integers $1, 2, 4, 3$, respectively, and prove that under the left regular representation of G into S_4 the nonidentity elements are mapped as follows:

$$a \mapsto (1\ 2)(3\ 4) \qquad b \mapsto (1\ 4)(2\ 3) \qquad c \mapsto (1\ 3)(2\ 4).$$

Proof. The left regular representation of G into S_4 is the homomorphism $\varphi : G \rightarrow S_4$ defined by $\varphi(g) = \sigma_g$, where $\sigma_g : G \rightarrow G$ is the permutation of G defined by $\sigma_g(x) = gx$ for all $x \in G$.

Each non-identity element maps the elements as follows:

$$\begin{aligned} \sigma_a(1) &= a1 = a & \sigma_a(a) &= a^2 = 1 & \sigma_a(b) &= ab = c & \sigma_a(c) &= ac = b \\ \sigma_b(1) &= b1 = b & \sigma_b(a) &= ba = c & \sigma_b(b) &= b^2 = 1 & \sigma_b(c) &= bc = a \\ \sigma_c(1) &= c1 = c & \sigma_c(a) &= ca = b & \sigma_c(b) &= cb = a & \sigma_c(c) &= c^2 = 1. \end{aligned}$$

By the given labeling, this assigns the elements a, b , and c to the pairs of 2-cycles shown above. \square

- (b) Relabel $1, a, b, c$ as $1, 4, 2, 3$, respectively, and compute the image of each element of G under the left regular representation of G into S_4 . Show that the image of G in S_4 under this labeling is the same *subgroup* as the image of G in part (a) (even though the nonidentity elements individually map to different permutations under the two different labelings).

Proof. Under this labeling, the elements a, b , and c are mapped to the permutations $(14)(23)$, $(12)(34)$, and $(13)(24)$, respectively. Although each element maps to a different permutation from part (a), the subgroup of S_4 is the same in both cases. \square

2. (2/12/24)

List the elements of S_3 as $1, (12), (23), (13), (123), (132)$ and label these with the integers $1, 2, 3, 4, 5, 6$, respectively. Exhibit the image of each element of S_3 under the left regular representation of S_3 into S_6 .

Solution. First, consider the element (12) . We see that:

$$\begin{aligned} (12)1 &= (12) \mapsto 2 & (12)(12) &= 1 \mapsto 1 \\ (12)(23) &= (123) \mapsto 5 & (12)(13) &= (132) \mapsto 6 \\ (12)(123) &= (23) \mapsto 3 & (12)(132) &= (13) \mapsto 4. \end{aligned}$$

So the left regular representation of (12) under the given labeling in S_6 is $(12)(34)(56)$.

The left regular representations of the remaining elements are:

$$\begin{aligned} (23) &\mapsto (13)(26)(45) \\ (13) &\mapsto (14)(25)(36) \\ (123) &\mapsto (156)(243) \\ (132) &\mapsto (165)(234). \end{aligned}$$

\square

3. (2/12/24)

Let r and s be the usual generators for the dihedral group of order 8.

- (a) List the elements of D_8 as $1, r, r^2, r^3, s, sr, sr^2, sr^3$ and label these with the integers $1, 2, \dots, 8$, respectively. Exhibit the image of each element of D_8 under the left regular representation of D_8 into S_8 .

$$\begin{aligned} 1 &\mapsto 1 \\ r &\mapsto (1234)(5876) \\ r^2 &\mapsto (13)(24)(57)(68) \\ r^3 &\mapsto (1432)(5678) \\ s &\mapsto (15)(26)(37)(48) \\ sr &\mapsto (16)(27)(38)(45) \\ sr^2 &\mapsto (17)(28)(35)(46) \\ sr^3 &\mapsto (18)(25)(36)(47) \end{aligned}$$

- (b) Relabel this same list of elements of D_8 with the integers 1, 3, 5, 7, 2, 4, 6, 8 respectively and recompute the image of each element of D_8 under the left regular representation with respect to this new labeling. Show that the two subgroups of S_8 obtained in parts (a) and (b) are different.

$$\begin{aligned}
1 &\mapsto 1 \\
r &\mapsto (1\ 3\ 5\ 7)(2\ 8\ 6\ 4) \\
r^2 &\mapsto (1\ 5)(2\ 6)(3\ 7)(4\ 8) \\
r^3 &\mapsto (1\ 7\ 5\ 3)(2\ 4\ 6\ 8) \\
s &\mapsto (1\ 2)(3\ 4)(5\ 6)(7\ 8) \\
sr &\mapsto (1\ 4)(2\ 7)(3\ 6)(5\ 8) \\
sr^2 &\mapsto (1\ 6)(2\ 5)(3\ 8)(4\ 7) \\
sr^3 &\mapsto (1\ 8)(2\ 3)(4\ 5)(6\ 7).
\end{aligned}$$

We see that the generators of the subgroups of S_8 in parts (a) and (b) are different, and so these are different subgroups of S_8 .

4. (2/12/24)

Use the left regular representation of Q_8 to produce two elements of S_8 which generate a subgroup of S_8 isomorphic to the quaternion group Q_8 .

Proof. We know that the elements i and j generate the quaternion group Q_8 . Labeling the elements $1, -1, i, -i, j, -j, k, -k$ with $1, 2, \dots, 8$ respectively, the elements i and j map to the following permutations in S_8 :

$$\begin{aligned}
i &\mapsto (1\ 3\ 2\ 4)(5\ 7\ 6\ 8) \\
j &\mapsto (1\ 5\ 2\ 6)(3\ 8\ 4\ 7).
\end{aligned}$$

Since the left regular representation of Q_8 in S_8 is a homomorphism, these two permutations generate a subgroup of S_8 isomorphic to Q_8 . \square

5. (2/12/24)

Let r and s be the usual generators for the dihedral group of order 8 and let $H = \langle s \rangle$. List the left cosets of H in D_8 as $1H, rH, r^2H, r^3H$.

- (a) Label these cosets with the integers 1, 2, 3, 4, respectively. Exhibit the image of each element of D_8 under the representation π_H of D_8 into S_4 obtained from the action of D_8 by left multiplication on the set of 4 left cosets of H in D_8 . Deduce that this representation is faithful (i.e., the

elements of S_4 obtained form a subgroup isomorphic to D_8).

$$\begin{array}{ll} 1 \mapsto 1 & s \mapsto (2\ 4) \\ r \mapsto (1\ 2\ 3\ 4) & sr \mapsto (1\ 4)(2\ 3) \\ r^2 \mapsto (1\ 3)(2\ 4) & sr^2 \mapsto (1\ 3) \\ r^3 \mapsto (1\ 4\ 3\ 2) & sr^3 \mapsto (1\ 2)(3\ 4). \end{array}$$

Since each element of D_8 induces a unique permutation in S_4 , the resulting image under the left regular representation is isomorphic to D_8 , and so this representation is faithful.

- (b) Repeat part (a) with the list of cosets relabeled by the integers 1, 3, 2, 4, respectively. Show that the permutations obtained from this labeling form a subgroup of S_4 that is different from the subgroup obtained in part (a).

$$\begin{array}{ll} 1 \mapsto 1 & s \mapsto (3\ 4) \\ r \mapsto (1\ 3\ 2\ 4) & sr \mapsto (1\ 4)(2\ 3) \\ r^2 \mapsto (1\ 2)(3\ 4) & sr^2 \mapsto (1\ 2) \\ r^3 \mapsto (1\ 4\ 2\ 3) & sr^3 \mapsto (1\ 3)(2\ 4). \end{array}$$

Since the generators (the images of r and s) of this subgroup of S_4 are different from those in part (a), this is a different subgroup from part (a).

- (c) Let $K = \langle sr \rangle$, list the cosets of K in D_8 as $1K, rK, r^2K, r^3K$, and label these with the integers 1, 2, 3, 4. Prove that, with respect to this labeling, the image of D_8 under the representation π_K obtained from left multiplication on the cosets of K is the same *subgroup* of S_4 as in part (a) (even though the subgroups H and K are different and some of the elements of D_8 map to different permutations under the two homomorphisms).

Proof. Consider the images of the generators r and s under π_K :

$$\begin{array}{ll} r \cdot 1K = rK & s \cdot 1K = rK \\ r \cdot rK = r^2K & s \cdot rK = 1K \\ r \cdot r^2K = r^3K & s \cdot r^2K = r^3K \\ r \cdot r^3K = 1K & s \cdot r^3K = r^2K. \end{array}$$

So r and s map to $(1\ 2\ 3\ 4)$ and $(1\ 2)(3\ 4) \in S_4$, respectively. These elements are both in the subgroup in part (a) above, and so they are the same subgroup, but the image of s is different. \square

6. (2/15/24)

Let r and s be the usual generators for the dihedral group of order 8 and let $N = \langle r^2 \rangle$. List the left cosets of N in D_8 as $1N, rN, sN$, and srN . Label these

cosets with the integers 1, 2, 3, 4 respectively. Exhibit the image of each element of D_8 under the representation π_N of D_8 into S_4 obtained from the action of D_8 by left multiplication on the set of 4 left cosets of N in D_8 . Deduce that this representation is not faithful and prove that $\pi_N(D_8)$ is isomorphic to the Klein 4-group.

Solution.

$$\begin{array}{ll} 1 \mapsto 1 & s \mapsto (13)(24) \\ r \mapsto (12)(34) & sr \mapsto (14)(23) \\ r^2 \mapsto 1 & sr^2 \mapsto (13)(24) \\ r^3 \mapsto (12)(34) & sr^3 \mapsto (14)(23). \end{array}$$

The left regular representation assigns 1 and r^2 to the identity permutation, so this action is not faithful.

The image of D_8 under π_N consists of the four permutations 1, $(12)(34)$, $(13)(24)$, and $(14)(23)$. From Ch. 2.5, Exercise 10, this is isomorphic to the Klein 4-group V_4 . \square

7. (2/15/24)

Let Q_8 be the quaternion group of order 8.

- (a) Prove that Q_8 is isomorphic to a subgroup of S_8 .

Proof. From Exercise 4, Q_8 is isomorphic to

$$\langle (1324)(5768), (1526)(3847) \rangle \in S_8.$$

\square

- (b) Prove that Q_8 is not isomorphic to a subgroup of S_n for any $n \leq 7$.

Proof. Let A be a set with $|A| = n \leq 7$, let $a \in A$, and let \cdot be the action of Q_8 on A . We attempt to find a subgroup of S_n that is isomorphic to Q_8 by considering the permutation representations of the elements of Q_8 .

Now if $i \cdot a = j \cdot a$, then the permutation representations σ_i and σ_j are equal to each other, and so Q_8 is not isomorphic to the resulting subgroup of S_n . Further (without loss of generality), if $i \cdot a = -i \cdot a$, then:

$$i \cdot a = -i \cdot a \Rightarrow -i \cdot i \cdot a = -i \cdot -i \cdot a \Rightarrow a = -1 \cdot a,$$

and so the permutation representation of -1 is equal to the identity permutation, which implies that Q_8 is not isomorphic to the subgroup of S_n . Therefore the elements $\pm i, \pm j, \pm k$ must all assign a to different elements. However, these 6 unique elements together with a are at least all of A , and so we must have $-1 \cdot a = a$. Thus Q_8 is not isomorphic to a subgroup of S_n . \square

9. (2/16/24)

Prove that if p is a prime and G is a group of order p^a for some $a \in \mathbb{Z}^+$, then every subgroup of index p is normal in G . Deduce that every group of order p^2 has a normal subgroup of order p .

Proof. Let H be a subgroup of G with $[G : H] = p$. Let gH be the left coset of H by some element $g \in G$.

Suppose that, for some $n < p$, $g^n \in H$ and let $h = g^n$. Since p is a prime, there exists a positive integer k such that $kn = 1 \pmod{p}$. Then $h^k = g^{kn} = g$, which implies that $g \in H$. We conclude that, if we restrict to $g \notin H$, then for all $n < p$, $g^n \notin H$. This implies that $\{H, gH, g^2H, \dots, g^{p-1}H\}$ is a set of p distinct cosets of H . Because the index of H in G is p , this must be all the cosets of H in G , and so $g^p \in H$.

Now by the operation defined on left cosets of H by $aH \cdot bH = (ab)H$, we see that this is isomorphic to the cyclic group Z_p . We conclude by Theorem 6(d) of Chapter 3.1 that H is normal in G .

Further, if $|G| = p^2$, then by Cauchy's Theorem it contains an element of order p which generates a subgroup of order p . This subgroup has index $p^2/p = p$, and so from above, is a normal subgroup of order p . \square