Dummit & Foote Ch. 4.3: Groups Acting on Themselves by Conjugation — The Class Equation

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Let G be a group.

1. (2/22/24)

Suppose G has a left action on a set A, denoted by $g \cdot a$ for all $g \in G$ and $a \in A$. Denote the corresponding right action on A by $a \cdot g$. Prove that the (equivalence) relations $\sim$ and $\sim'$ defined by

$$a \sim b \text{ if and only if } a = g \cdot b \text{ for some } g \in G$$

and

$$a \sim' b \text{ if and only if } a = b \cdot g \text{ for some } g \in G$$

are the same relation (i.e., $a \sim b$ if and only $a \sim' b$).

Proof. To show that $a \sim b$ implies $a \sim' b$, we must show that, given a $g \in G$ with $a = g \cdot b$, there exists an $h \in G$ such that $a = b \cdot h$. By definition, the corresponding right action of a left action is specified to be $g \cdot x = x \cdot g^{-1}$ for all $g \in G$, $x \in A$. Letting $h = g^{-1}$, we have found an element where $a = g \cdot b = b \cdot h$, and so $a \sim' b$.

The proof for $a \sim' b$ implies $a \sim b$ is identical, letting $h = g^{-1}$ but with h acting on the left. □

2. (2/22/24)

Find all conjugacy classes and their sizes in the following groups:

(a) D_8:

$$\{1\}_1 \qquad \{r^2\}_1 \qquad \{r, r^3\}_2 \qquad \{s, sr^2\}_2 \qquad \{sr, sr^3\}_2$$

(b) Q_8:

$$\{1\}_1 \qquad \{-1\}_1 \qquad \{\pm i\}_2 \qquad \{\pm j\}_2 \qquad \{\pm k\}_2$$

(c) A_4:

$$\{1\}_1 \quad \{(1\,2\,3),(1\,3\,4),(1\,4\,2),(2\,4\,3)\}_4 \quad \{(1\,3\,2),(1\,2\,4),(1\,4\,3),(2\,3\,4)\}_4$$
$$\{(1\,2)(3\,4),(1\,3)(2\,4),(1\,4)(2\,3)\}_3$$

3. (2/22/24)

Find all the conjugacy classes and their sizes in the following groups:

(a) $Z_2 \times S_3$:

$$\{(0,1)\}_1 \quad \{(1,1)\}_1 \quad \{(0,(1\,2)),(0,(1\,3)),(0,(2\,3))\}_3$$
$$\{(1,(1\,2)),(1,(1\,3)),(1,(2\,3))\}_3 \quad \{(0,(1\,2\,3)),(0,(1\,3\,2))\}_2$$
$$\{(1,(1\,2\,3)),(1,(1\,3\,2))\}_2$$

(b) $S_3 \times S_3$:

$$\{(1,1)\}_1 \quad \{(1,\text{2-cycle})\}_3 \quad \{(\text{2-cycle},1)\}_3 \quad \{(1,\text{3-cycle})\}_2 \quad \{(\text{3-cycle},1)\}_2$$
$$\{(\text{2-cycle},\text{2-cycle})\}_9 \quad \{(\text{2-cycle},\text{3-cycle})\}_6 \quad \{(\text{3-cycle},\text{2-cycle})\}_6$$
$$\{(\text{3-cycle},\text{3-cycle})\}_4$$

(c) $Z_3 \times A_4$ (using representatives from the conjugacy classes of A_4 above):

$$\{(0,1)\}_1 \quad \{(1,1)\}_1 \quad \{(2,1)\}_1$$
$$\{(0,\overline{(1\,2\,3)})\}_4 \quad \{(1,\overline{(1\,2\,3)})\}_4 \quad \{(2,\overline{(1\,2\,3)})\}_4$$
$$\{(0,\overline{(1\,3\,2)})\}_4 \quad \{(1,\overline{(1\,3\,2)})\}_4 \quad \{(2,\overline{(1\,3\,2)})\}_4$$
$$\{(0,\overline{(1\,2)(3\,4)})\}_3 \quad \{(1,\overline{(1\,2)(3\,4)})\}_3 \quad \{(2,\overline{(1\,2)(3\,4)})\}_3$$