Munkres Ch. 2: Topological Spaces and Continuous Functions

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§12. Topological Spaces
§13. Basis for a Topology

1. (7/30/24)

Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Proof. From the definition of an open set in a topological space, we know that an arbitrary union of open sets is again an open set. We will show that A is a union of open sets in X, and is therefore open.

For each $x \in A$, there exists an open set U_x containing x that is a subset of A. We claim that $A = \bigcup_{x \in A} U_x$.

Let $a \in A$. It is given that there exists a $U_a \subset A$ with $a \in U_x$. Therefore $a \in \bigcap_{x \in A} U_x$.

Conversely, let $a \in \bigcap_{x \in A} U_x$. Then a lies in some U_x such that $U_x \subset A$, and so $a \in A$. Thus $A = \bigcup_{x \in A} U_x$. $\square$

2. (7/30/24)

Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.

Solution. Let the nine topologies on X be:

i)	$\{\varnothing, X\}$
ii)	$\{\varnothing, \{a\}, \{a,b\}, X\}$
iii)	$\{\varnothing, \{b\}, \{a,b\}, \{b,c\}, X\}$
iv)	$\{\varnothing, \{b\}, X\}$
v)	$\{\varnothing, \{a\}, \{b,c\}, X\}$
vi)	$\{\varnothing, \{b\}, \{c\}, \{a,b\}, \{b,c\}, X\}$
vii)	$\{\varnothing, \{a,b\}, X\}$
vii)	$\{\varnothing, \{a\}, \{b\}, \{a,b\}, X\}$
ix)	$\{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, X\}$

Then:

i) is coarser than every other topology; ix) is finer than every other topology;

ii) is finer than vii), is not comparable to iii), iv), v), and vi), and is coarser than viii);

iii) is finer than iv) and vii), and is not comparable to v), vi), and viii);

iv) is not comparable to v) and vii), and is coarser than vi) and viii);

v) is not comparable to vi), vii), or vii);

vi) is finer than vii) and is not comparable to viii); and

vii) is coarser than viii).

□