Munkres Ch. 2: Topological Spaces and Continuous Functions

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§12. Topological Spaces
§13. Basis for a Topology

1. (7/30/24)

Let X be a topological space; let A be a subset of X. Suppose that for each $x \in A$ there is an open set U containing x such that $U \subset A$. Show that A is open in X.

Proof. From the definition of an open set in a topological space, we know that an arbitrary union of open sets is again an open set. We will show that A is a union of open sets in X, and is therefore open.

For each $x \in A$, there exists an open set U_x containing x that is a subset of A. We claim that $A = \bigcup_{x \in A} U_x$.

Let $a \in A$. It is given that there exists a $U_a \subset A$ with $a \in U_x$. Therefore $a \in \bigcap_{x \in A} U_x$.

Conversely, let $a \in \bigcap_{x \in A} U_x$. Then a lies in some U_x such that $U_x \subset A$, and so $a \in A$. Thus $A = \bigcup_{x \in A} U_x$. $\square$

2. (7/30/24)

Consider the nine topologies on the set $X = \{a, b, c\}$ indicated in Example 1 of §12. Compare them; that is, for each pair of topologies, determine whether they are comparable, and if so, which is the finer.

Solution. Let the nine topologies on X be:

i)	$\{\varnothing, X\}$
ii)	$\{\varnothing, \{a\}, \{a,b\}, X\}$
iii)	$\{\varnothing, \{b\}, \{a,b\}, \{b,c\}, X\}$
iv)	$\{\varnothing, \{b\}, X\}$
v)	$\{\varnothing, \{a\}, \{b,c\}, X\}$
vi)	$\{\varnothing, \{b\}, \{c\}, \{a,b\}, \{b,c\}, X\}$
vii)	$\{\varnothing, \{a,b\}, X\}$
vii)	$\{\varnothing, \{a\}, \{b\}, \{a,b\}, X\}$
ix)	$\{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, X\}$

Then:

i) is coarser than every other topology; ix) is finer than every other topology;

ii) is finer than vii), is not comparable to iii), iv), v), and vi), and is coarser than viii);

iii) is finer than iv) and vii), and is not comparable to v), vi), and viii);

iv) is not comparable to v) and vii), and is coarser than vi) and viii);

v) is not comparable to vi), vii), or vii);

vi) is finer than vii) and is not comparable to viii); and

vii) is coarser than viii).

□

3. (7/31/24)

Show that the collection $\mathcal{T}_C$ given in Example 4 of §12 is a topology on the set X. Is the collection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

Proof. Let $\mathcal{T}_C = \{U \mid X - U \text{ is countable or is all of } X\}$. Now $\mathcal{T}_C$ certainly contains $\varnothing$ and X, since $X - \varnothing = X$ and $X - X = \varnothing$, a countable set.

We must next show that arbitrary unions and finite intersections of elements of $\mathcal{T}_C$ are again elements of $\mathcal{T}_C$.

Let $\{U_\alpha\}$ be an indexed family of nonempty elements of $\mathcal{T}_C$. To show that $\bigcup U_\alpha$ is in $\mathcal{T}_C$, we compute:

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha),$$

and since each $X - U_\alpha$ is countable, their intersection is also countable. Therefore an arbitrary union of elements of $\mathcal{T}_C$ lies in $\mathcal{T}_C$.

Finally, let $U_1, U_2, ..., U_n$ be nonempty elements of $\mathcal{T}_C$ and consider $\bigcup_{i=1}^n U_i$. We have:

$$X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n} (X - U_i),$$

which is a finite union of countable sets, and is therefore countable. Thus $\bigcup_{i=1}^n U_i$ is in $\mathcal{T}_C$, and so $\mathcal{T}_C$ is a topology on X.

Next we consider $\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$, and claim that it is not a topology on the set X. As a counterexample, suppose that

$$\begin{aligned} X &= \mathbb{R}, \\ U_a &= \{x \in X \mid x < 0\}, \text{ and} \\ U_b &= \{x \in X \mid x > 0\}. \end{aligned}$$

Then $U_a \cup U_b = \{x \in X \mid x < 0 \text{ or } x > 0\}$, that is, the nonzero real numbers, whose complement is $\{0\}$, a finite set, and therefore the union of U_a and U_b is not a member of $\mathcal{T}_\infty$. It is therefore not a topology on X. $\square$