

A Conjecture for Primitive Pythagorean Triples

R. Scott McIntire

July 8, 2023

1 Overview

We list a few conjectures regarding the set, \mathcal{H} , of primitive Pythagorean hypotenuses. This is the set constructed by taking the "hypotenuse" value from all primitive Pythagorean triples – corresponding to the lengths of the sides of right triangles with integer sides with no common factors.

A related set, \mathcal{H}_p , is the subset of \mathcal{H} that are primes. That is, primitive Pythagorean hypotenuses that are prime.

Below we list the main conjectures informally, listing them in order of complexity:

1. **Structure Theorem of \mathcal{H} :** Each element of \mathcal{H} can be written as a *unique* product of powers of elements from \mathcal{H}_p .
2. **Partition of \mathcal{H} :** \mathcal{H} can be *partitioned* into two disjoint infinite sets: the set of all powers of Pythagorean primes; and the set of all hypotenuses that come from more than one primitive Pythagorean triple.
3. **\mathcal{H}_p is complex:** The set of Pythagorean primes, \mathcal{H}_p , is infinite; *similar*; and as *complex* as the primes when measured using certain metrics.

2 Definitions

After a few definitions we describe a conjecture regarding Primitive Pythagorean Triples. We use standard notation to indicate that the natural numbers are represented by the symbol \mathbb{N} and we use the symbol \mathbb{P} to denote the set of prime numbers.

Definition 1. *Pythagorean Triples* denoted, \mathcal{T} , are triplets of numbers representing the length of the sides of right triangles where all the sides are integers. Specifically,

$$\mathcal{T} = \{(x, y, z) \mid (x, y, z) \in \mathbb{N}^3 \wedge x^2 + y^2 = z^2\} \quad (1)$$

The usual definition.

Definition 2. *Primitive Pythagorean Triples* denoted, \mathcal{T}_p , are Pythagorean Triples which have no common factors. Specifically,

$$\mathcal{T}_p = \{(x, y, z) \mid (x, y, z) \in \mathbb{N}^3 \wedge \gcd(x, y) = 1 \wedge x^2 + y^2 = z^2\} \quad (2)$$

The usual definition.

Definition 3. *Primitive Pythagorean Hypotenuses* denoted, \mathcal{H} , defined by:

$$\mathcal{H} = \{z \mid \exists (x, y) \in \mathbb{N}^2, (x, y, z) \in \mathcal{T}_p\} \quad (3)$$

This represents the set of all possible hypotenuses of primitive Pythagorean triples.

Definition 4. *Pythagorean Primes* denoted, \mathcal{H}_p , defined by:

$$\mathcal{H}_p = \{z \mid \exists (x, y) \in \mathbb{N}^2, (x, y, z) \in \mathcal{T}_p \wedge z \in \mathbb{P}\} \quad (4)$$

This represents the set of all possible lengths of hypotenuses of right triangles with integer sides that are prime.

Definition 5. *Duplicate Primitive Hypotenuses* denoted \mathcal{H}_d , defined by:

$$\mathcal{H}_d = \{z \mid \exists (x_1, x_2, y_1, y_2) \in \mathbb{N}^4, (x_1, y_1, z) \in \mathcal{T}_p \wedge (x_2, y_2, z) \in \mathcal{T}_p \wedge x_1 \neq x_2\} \quad (5)$$

This represents the set of all primitive hypotenuses that can be found in more than one primitive Pythagorean triple.

Definition 6. *Primitive Hypotenuses Powers* denoted \mathcal{H}_u , defined by:

$$\mathcal{H}_u = \{z \mid \exists (p, n) \in (\mathcal{H}_p, \mathbb{N}), z = p^n\} \quad (6)$$

This represents the set of all powers of Pythagorean primes.

Definition 7. *The Prime Parity* function, Π , is defined on any prime, p , with $p \geq 5$ by:

$$\Pi(p) = \begin{cases} 1 & \exists n \in \mathbb{N}, p = 6n + 1 \\ -1 & \exists n \in \mathbb{N}, p = 6n - 1 \end{cases} \quad (7)$$

Definition 8. *First Deviation* function, Δ_p , is defined by:¹

$$\Delta_p(n) = \min_m \left\{ m \mid n \leq \left(\sum_{i=1}^m \Pi(\mathcal{H}_p[i]) - \sum_{i=3}^{m+2} \Pi(\mathbb{P}[i]) \right) \right\} \quad (8)$$

Notes:

¹If for a given n , $\left\{ m \mid n \leq \left(\sum_{i=1}^m \Pi(\mathcal{H}_p[i]) - \sum_{i=3}^{m+2} \Pi(\mathbb{P}[i]) \right) \right\} = \emptyset$, we set $\Delta_p(n) = \infty$.

- The subscript u in the name \mathcal{H}_u is not yet justified.
- \mathcal{H}_p is a *proper* subset of the primes: $\mathcal{H}_p \subset \mathbb{P}$.
- If the sets above are treated as lists, they are indexed in sorted order.
- Clearly, Δ_p is a non-decreasing function.

3 McIntire's Pythagorean Triple Conjectures

It is a well established *conjecture* in the mathematical community that the cardinality of \mathcal{H}_p is infinite: $|\mathcal{H}_p| = \infty$. It is also known that all prime factors, p , of elements in \mathcal{H} have the property that $p \equiv 1 \pmod{4}$. The first conjecture – which must be suspected by the mathematical community – states that such primes constitute the set \mathcal{H}_p .

Conjecture 1. The set $\mathcal{H}_p = \{p \mid p \in \mathbb{P} \wedge p \equiv 1 \pmod{4}\}$

Immediate Corollary from this conjecture: The set \mathcal{H} has a unique factorization in \mathcal{H}_p . That is, each element of \mathcal{H} can be uniquely written as a product of powers from \mathcal{H}_p :

$$\forall h \in \mathcal{H}, \exists! N \in \mathbb{N}, \exists! (\mathbf{p}, \mathbf{k}) \in (\mathcal{H}_p^N, \mathbb{N}^N) : h = \prod_{i=1}^N p_i^{k_i}. \quad (9)$$

Conjecture 2. The set \mathcal{H} is partitioned by the sets \mathcal{H}_d and \mathcal{H}_u :

1. $\mathcal{H} = \mathcal{H}_d \cup \mathcal{H}_u$;
2. $\emptyset = \mathcal{H}_d \cap \mathcal{H}_u$;
3. $|\mathcal{H}_d| = \infty$.

In simple terms the conjecture says that each element of the set of all primitive Pythagorean hypotenuses is the union of two disjoint infinite sets: the set of powers of a Pythagorean primes; and the set of all hypotenuses that come from more than one primitive Pythagorean triple.

Conjecture 3. Properties of the cumulative parity of \mathcal{H}_p indicates that it is non-trivial:

1. $\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \Pi(\mathcal{H}_p[i])}{n} = 0$;
2. $\forall n \in \mathbb{N}, \Delta_p(n) < \infty$;
3. $\lim_{n \rightarrow \infty} \frac{\Delta_p(n)}{n^{\ln(2\pi)}} = 41$.

These last conjectures suggest that the Pythagorean primes are non-trivial in a way similar to the primes.

1. The first conjecture states that the \mathcal{H}_p primes are "spread out" in a way similar to the set of all primes.
2. From numerical experiments, it seems that the cumulative parity of \mathcal{H}_p is "on average" lower than the cumulative parity of all primes. The second and third conjectures suggest that there are non-trivial deviations from this rule. This "mixing" of the cumulative parity of \mathcal{H}_p with that of \mathbb{P} suggests that the Pythagorean primes are as "complex" as the set of all primes.