An Alternative Proof of an Elementary Property of Pythagorean Triples

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1 Introduction

When I was a freshmen in college I was briefly interested in the idea of using the areas of geometric figures to produce interesting formulas. The example that inspired me was the way one may derive the Pythagorean formula by examining a dissected rectangle composed of an inscribed square and triangles.

I tried to do something similar with a more complicated geometric figure and I came up with figure (1).

By inscribing a circle[1] in a *right* triangle, one can decompose the triangle into two pairs of right triangles and a square. The fact that the sub-triangles are right triangles follows using the fact that any radial segment of a circle is perpendicular to its associated tangent line.

From the figure, the lower right four sided sub-figure has equal sides, r. I claim that this sub-figure is a square and not a rhombus. This can be seen as three of its interior angles are

[1]This can be achieved in the following way. Take any point on the bisector of one of the angles formed with the hypotenuse and one of the legs. By symmetry there is a unique circle that touches the hypotenuse and the leg. It is clear that the radius of such circles grow as one moves out along the bisector. Consequently, there is a point on the bisector where the circle touches the opposite leg. The coordinate of this point (see the figure) is (a-r, r). Using the fact (see the figure) that $(a - r) + (b - r) = c$ we see that regardless of which leg we choice for this procedure, the value of r is the same as is the center of the corresponding circle. Thus an inscribed circle exists and is unique.

90°; and, as the number of interior angles in a 4 sided *convex* polygon must sum to 360°, the last interior angle must also be 90°. Consequently, the four sided figure is a square.

Below we find a well known relation between the sides, a, b, and c of the original triangle when the sides are integers in *reduced form*.[2] This is done by investigating the fact that the sum of the areas of these component triangles and square must be the same as the area of the original triangle.

2 A Basic Property of Primitive Pythagorean Triples

Writing the equivalence of the area of the figure and its parts we have:

$$\overbrace{r^2}^{\text{Square}} + \overbrace{2\frac{r(a-r)}{2}}^{\text{Two lower triangles}} + \overbrace{2\frac{r(b-r)}{2}}^{\text{Two upper triangles}} = \overbrace{\frac{a\,b}{2}}^{\text{Original triangle}} \tag{1}$$

Simplifying, this is:

$$r^2 + r(a-r) + r(b-r) = \frac{a\,b}{2} \tag{2}$$

One can solve the quadratic for r or read off from the figure that $(a-r)+(b-r)=c$. Solving this last equation for r gives:

$$r = \frac{a+b-c}{2} \tag{3}$$

We now consider primitive Pythagorean triangles – right triangles with positive integer sides in reduced form. In particular, we are interested in so-called primitive Pythagorean triples, (a,b,c) – the lengths of the sides of primitive Pythagorean triangles. Here c is the length of the hypotenuse, while a and b are the lengths of the "legs" of the associated Pythagorean triangle.

[2] A triple of integers, (a,b,c) is in reduced form if they have no common integer divisor(factor).

What can we say about the evenness or oddness of the sides when they have no common factor? The possibilities are:

- c is even AND *exactly* one of the following is true:
  - Both a and b are even;
  - Both a and b are odd;
  - One of either a or b is even and the other is odd.
- c is odd AND *exactly* one of the following is true:
  - Both a and b are even;
  - Both a and b are odd;
  - One of either a or b is even and the other is odd.

Since the triple, (a, b, c) is in reduced form we can eliminate the case of all three being even. By the Pythagorean formula we know that $a^2 + b^2 = c^2$. Using this we can reduce the combinations further to the following:[3]

- c is even AND both a and b are odd;
- c is odd AND one of either a or b is even and the other is odd.

Regardless of which choice occurs, equation (3) implies that the radius of the circle, r, is an integer.

[3] We list the results of all combinations of even and odd numbers with respect to: addition, subtraction, and multiplication:

1. $E \pm E = E$; $E \times E = E$;
2. $E \pm O = O$; $E \times O = E$;
3. $O \pm O = E$; $O \times O = O$.

These equations are to be read: An even integer plus (or minus) and even integer is an even integer. Likewise, an even integer plus (or minus) an odd integer is an odd integer, etc. These results are used throughout the rest of the document.

Using this fact and equation (2) we see that the quantity, $\frac{a\,b}{2}$, must be an integer; meaning, that the product of a, and b, $a\,b$ is even. This knowledge eliminates the first choice; namely, that both a and b are odd. Consequently, we know:

- c is odd AND one of either a or b is even and the other is odd.

Without loss of generality we assume that b is even; in which case, a is odd.

Now, regardless of whether r is even or odd, the left hand side of (2) is even. This implies that $\frac{a\,b}{2}$ is even. Since b is even, we can write this fraction as: $a\frac{b}{2}$. So we know that $a\frac{b}{2}$ (the product of two integers) is even. And we know that a is odd; therefore, $\frac{b}{2}$ is even. This means that b is divisible by 4.

In summary, we have deduced the following properties for primitive Pythagorean triangles whose side lengths are in reduced form.

- One of the legs is odd; the other leg is even; and the hypotenuse is odd.
- The even leg is divisible by 4.
- The radius of the inscribed circle in the Pythagorean triangle is a positive integer.

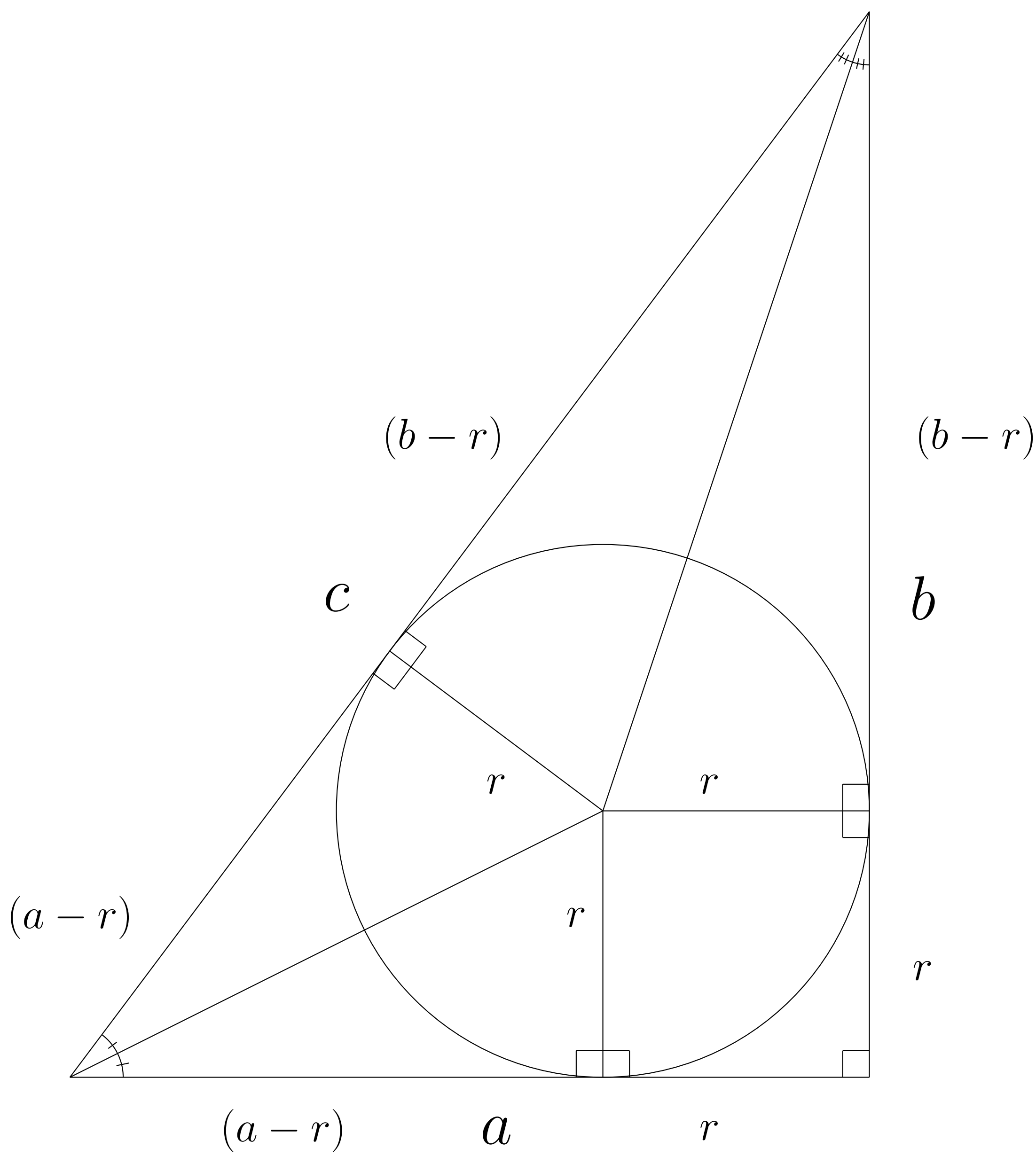

Figure 1: Dissected Right Triangle