

A Conjecture for Primitive Pythagorean Triples

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June 19, 2023

1 Definitions

After a few definitions we describe a conjecture regarding Primitive Pythagorean Triples. We use standard notation to indicate that the natural numbers are represented by the symbol \mathbb{N} and we use the symbol \mathbb{P} to denote the set of prime numbers.

Definition 1. *Pythagorean Triples* denoted, \mathcal{T} , are triplets of numbers representing the length of the sides of right triangles where all the sides are integers. Specifically,

$$\mathcal{T} = \{(x, y, z) \mid (x, y, z) \in \mathbb{N}^3 \wedge x^2 + y^2 = z^2\} \quad (1)$$

The usual definition.

Definition 2. *Primitive Pythagorean Triples* denoted, \mathcal{T}_p , are Pythagorean Triples which have no common factors. Specifically,

$$\mathcal{T}_p = \{(x, y, z) \mid (x, y, z) \in \mathbb{N}^3 \wedge \gcd(x, y) = 1 \wedge x^2 + y^2 = z^2\} \quad (2)$$

The usual definition.

Definition 3. *Primitive Pythagorean Hypotenuses* denoted, \mathcal{H} , defined by:

$$\mathcal{H} = \{z \mid \exists (x, y) \in \mathbb{N}^2, (x, y, z) \in \mathcal{T}_p\} \quad (3)$$

This represents the set of all possible lengths of hypotenuses of right triangles with integer sides.

Definition 4. *Pythagorean Primes* denoted, \mathcal{H}_p , defined by:

$$\mathcal{H}_p = \{z \mid \exists (x, y) \in \mathbb{N}^2, (x, y, z) \in \mathcal{T}_p \wedge z \in \mathbb{P}\} \quad (4)$$

This represents the set of all possible lengths of hypotenuses of right triangles with integer sides that are prime.

Note: \mathcal{H}_p is a *proper* subset of the primes: $\mathcal{H}_p \subset \mathbb{P}$.

Note : If the sets \mathcal{H} or \mathcal{H}_p are treated as lists, they are indexed in sorted order.

Definition 5. The *prime parity* function, Π , is defined on any prime, p , with $p \geq 5$ by:

$$\Pi(p) = \begin{cases} 1 & \exists n \in \mathbb{N}, p = 6n + 1 \\ -1 & \exists n \in \mathbb{N}, p = 6n - 1 \end{cases} \quad (5)$$

Definition 6. *First Deviation* function, Δ_p , is defined by:¹

$$\Delta_p(n) = \min_m \left\{ m \mid n \leq \left(\sum_{i=1}^m \Pi(\mathcal{H}_p[i]) - \sum_{i=3}^{m+2} \Pi(\mathbb{P}[i]) \right) \right\} \quad (6)$$

Clearly, Δ_p is a non-decreasing function.

2 McIntire's Pythagorean Triple Conjectures

Define \mathcal{H}_d and \mathcal{H}_u by:

$$\mathcal{H}_d = \{z \mid \exists (x_1, x_2, y_1, y_2) \in \mathbb{N}^4, (x_1, y_2, z) \in \mathcal{T}_p \wedge (x_2, y_2, z) \in \mathcal{T}_p \wedge x_1 \neq x_2\} \quad (7)$$

$$\mathcal{H}_u = \{z \mid \exists (p, n) \in (\mathcal{H}_p, \mathbb{N}), z = p^n\} \quad (8)$$

It has already been conjectured that the cardinality of \mathcal{H}_p is infinite: $|\mathcal{H}_p| = \infty$.

Conjecture 1. The the set \mathcal{H}_d has infinite cardinality:

1. $|\mathcal{H}_d| = \infty$.

Conjecture 2. The set \mathcal{H} is partitioned by the sets \mathcal{H}_d and \mathcal{H}_u :

1. $\mathcal{H} = \mathcal{H}_d \cup \mathcal{H}_u$;
2. $\emptyset = \mathcal{H}_d \cap \mathcal{H}_u$.

In simple terms the conjecture says that each element of the set of all primitive Pythagorean hypotenuses is either a power of a Pythagorean prime; OR, is the length of the hypotenuse of more than one "primitive" right triangle.

Conjecture 3. Properties of the cumulative parity of \mathcal{H}_p indicates that it is non-trivial:

¹If for a given n , $\left\{ m \mid n \leq \left(\sum_{i=1}^m \Pi(\mathcal{H}_p[i]) - \sum_{i=3}^{m+2} \Pi(\mathbb{P}[i]) \right) \right\} = \emptyset$, we set $\Delta_p(n) = \infty$.

1. $\lim_{n \rightarrow \infty} \frac{\sum_{i=1}^n \Pi(\mathcal{H}_p[i])}{n} = 0;$
2. $\forall n \in \mathbb{N}, \Delta_p(n) < \infty;$
3. $\lim_{n \rightarrow \infty} \frac{\Delta_p(n)}{n^{\ln(2\pi)}} = 41.$

These last conjectures suggest that the Pythagorean primes are non-trivial in a way similar to the primes.

1. The first conjecture states that the \mathcal{H}_p primes are "spread out" in a similar way to all primes.
2. Although it seems that the cumulative parity of \mathcal{H}_p is "on average" lower than the cumulative parity of all primes, the second and third conjectures say that there is non-trivial deviation from this rule. This "mixing" of the cumulative parity of \mathcal{H}_p with that of \mathbb{P} suggests that the Pythagorean primes are as "complex" as the set of all primes.