A Conjecture for Primitive Pythagorean Triples

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July 8, 2023

1 Overview

We list a few conjectures regarding the set, $\mathcal{H}$, of primitive Pythagorean hypotenuses. This is the set constructed by taking the "hypotenuse" value from all primitive Pythagorean triples – corresponding to the lengths of the sides of right triangles with integer sides with no common factors.

A related set, $\mathcal{H}_p$, is the subset of $\mathcal{H}$ that are primes. That is, primitive Pythagorean hypotenuses that are prime.

Below we list the main conjectures informally, listing them in order of complexity:

1. **Structure Theorem of $\mathcal{H}$:** Each element of $\mathcal{H}$ can be written as a *unique* product of powers of elements from $\mathcal{H}_p$.
2. **Partition of $\mathcal{H}$:** $\mathcal{H}$ can be *partitioned* into two disjoint infinite sets: the set of all powers of Pythagorean primes; and the set of all hypotenuses that come from more than one primitive Pythagorean triple.
3. $\mathcal{H}_p$ **is complex:** The set of Pythagorean primes, $\mathcal{H}_p$, is infinite; *similar*; and as *complex* as the primes when measured using certain metrics.

2 Definitions

After a few definitions we describe a conjecture regarding Primitive Pythagorean Triples. We use standard notation to indicate that the natural numbers are represented by the symbol $\mathbb{N}$ and we use the symbol $\mathbb{P}$ to denote the set of prime numbers.

Definition 1. *Pythagorean Triples* denoted, $\mathcal{T}$, are triplets of numbers representing the length of the sides of right triangles where all the sides are integers. Specifically,

$$\mathcal{T} \quad = \quad \{(x,y,z) \mid (x,y,z) \in \mathbb{N}^3 \ \wedge \ x^2+y^2=z^2\} \tag{1}$$

The usual definition.

Definition 2. *Primitive Pythagorean Triples* denoted, $\mathcal{T}_p$, are Pythagorean Triples which have no common factors. Specifically,

$$\mathcal{T}_p \quad = \quad \{(x,y,z) \mid (x,y,z) \in \mathbb{N}^3 \ \wedge \ \gcd(x,y)=1 \ \wedge \ x^2+y^2=z^2\} \tag{2}$$

The usual definition.

Definition 3. *Primitive Pythagorean Hypotenuses* denoted, $\mathcal{H}$, defined by:

$$\mathcal{H} \quad = \quad \{z \mid \exists (x,y) \in \mathbb{N}^2,\ (x,y,z) \in \mathcal{T}_p\} \tag{3}$$

This represents the set of all possible hypotenuses of primitive Pythagorean triples.

Definition 4. *Pythagorean Primes* denoted, $\mathcal{H}_p$, defined by:

$$\mathcal{H}_p \quad = \quad \{z \mid \exists (x,y) \in \mathbb{N}^2,\ (x,y,z) \in \mathcal{T}_p \ \wedge \ z \in \mathbb{P}\} \tag{4}$$

This represents the set of all possible lengths of hypotenuses of right triangles with integer sides that are prime.

Definition 5. *Duplicate Primitive Hypotenuses* denoted $\mathcal{H}_d$, defined by:

$$\mathcal{H}_d \quad = \quad \{z \mid \exists\, (x_1,x_2,y_1,y_2) \in \mathbb{N}^4,\ (x_1,y_1,z) \in \mathcal{T}_p \ \wedge \ (x_2,y_2,z) \in \mathcal{T}_p \ \wedge \ x_1 \neq x_2\} \tag{5}$$

This represents the set of all primitive hypotenuses that can be found in more than one primitive Pythagorean triple.

Definition 6. *Primitive Hypotenuses Powers* denoted $\mathcal{H}_u$, defined by:

$$\mathcal{H}_u \quad = \quad \{z \mid \exists\, (p,n) \in (\mathcal{H}_p, \mathbb{N}),\ z = p^n\} \tag{6}$$

This represents the set of all powers of Pythagorean primes.

Definition 7. *The Prime Parity* function, Π, is defined on any prime, p, with $p \geq 5$ by:

$$\Pi(p) = \begin{cases} 1 & \exists n \in \mathbb{N}, p = 6n+1 \\ -1 & \exists n \in \mathbb{N}, p = 6n-1 \end{cases} \tag{7}$$

Definition 8. *First Deviation* function, Δ_p, is defined by:[1]

$$\Delta_p(n) = \min_m \left\{ m \;\middle|\; n \leq \left(\sum_{i=1}^{m} \Pi(\mathcal{H}_p[i]) - \sum_{i=3}^{m+2} \Pi(\mathbb{P}[i]) \right) \right\} \tag{8}$$

Notes:

[1] If for a given n, $\left\{ m \;\middle|\; n \leq \left(\sum_{i=1}^{m} \Pi(\mathcal{H}_p[i]) - \sum_{i=3}^{m+2} \Pi(\mathbb{P}[i]) \right) \right\} = \emptyset$, we set $\Delta_p(n) = \infty$.

- The subscript u in the name $\mathcal{H}_u$ is not yet justified.
- $\mathcal{H}_p$ is a *proper* subset of the primes: $\mathcal{H}_p \subset \mathbb{P}$.
- If the sets above are treated as lists, they are indexed in sorted order.
- Clearly, Δ_p is a non-decreasing function.

3 McIntire's Pythagorean Triple Conjectures

It is a well established *conjecture* in the mathematical community that the cardinality of $\mathcal{H}_p$ is infinite: $|\mathcal{H}_p| = \infty$. It is also known that all prime factors, p, of elements in $\mathcal{H}$ have the property that $p = 1 \bmod 4$.

Conjecture 1. The set $\mathcal{H}_p = \{p \mid p \in \mathbb{P} \ \wedge \ p = 1 \bmod 4\}$

Immediate Corollary from this conjecture: The set $\mathcal{H}$ has a unique factorization in $\mathcal{H}_p$. That is, each element of $\mathcal{H}$ can be uniquely written as a product of powers from $\mathcal{H}_p$:

$$\forall h \in \mathcal{H},\ \exists! N \in \mathbb{N},\ \exists!(\mathbf{p}, \mathbf{k}) \in \left(\mathcal{H}_p^N, \mathbb{N}^N\right) : h = \prod_{i=1}^{N} p_i^{k_i}. \tag{9}$$

Conjecture 2. The set $\mathcal{H}$ is partitioned by the sets $\mathcal{H}_d$ and $\mathcal{H}_u$:

1. $\mathcal{H} = \mathcal{H}_d \cup \mathcal{H}_u$;
2. $\emptyset = \mathcal{H}_d \cap \mathcal{H}_u$;
3. $|\mathcal{H}_d| = \infty$.

In simple terms the conjecture says that each element of the set of all primitive Pythagorean hypotenuses is the union of two disjoint infinite sets: the set of powers of a Pythagorean primes; and the set of all hypotenuses that come from more than one primitive Pythagorean triple.

Conjecture 3. Properties of the cumulative parity of $\mathcal{H}_p$ indicates that it is non-trivial:

1. $\lim\limits_{n\to\infty} \frac{\sum_{i=1}^{n} \Pi(\mathcal{H}_p[i])}{n} = 0$;
2. $\forall n \in \mathbb{N}, \Delta_p(n) < \infty$;
3. $\lim\limits_{n\to\infty} \frac{\Delta_p(n)}{n^{\ln(2\pi)}} = 41$.

These last conjectures suggest that the Pythagorean primes are non-trivial in a way similar to the primes.

1. The first conjecture states that the $\mathcal{H}_p$ primes are "spread out" in a way similar to the set of all primes.

2. From numerical experiments, its seems that the cumulative parity of $\mathcal{H}_p$ is "on average" lower than the cumulative parity of all primes. The second and third conjectures suggest that there are non-trivial deviations from this rule. This "mixing" of the cumulative parity of $\mathcal{H}_p$ with that of $\mathbb{P}$ suggests that the Pythagorean primes are as "complex" as the set of all primes.