An Alternative Proof of an Elementary Property of Pythagorean Triples

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Sep 22, 2023

1 Introduction

When I was a freshmen in college I was briefly interested in the idea of using the areas of geometric figures to produce interesting formulas. The example that inspired me was the way you can derive the Pythagorean formula by examining the area of a square sliced into a few pieces.

I tried to do something similar with a more complicated geometric figure and I came up with figure (1).

By inscribing a circle in a *right* triangle, one can decompose the triangle into two pairs of right triangles and a square. The fact that the sub-triangles are right triangles follows using the fact that any radial segment of a circle is perpendicular to its associated tangent line.

From the figure, the lower right four sided sub-figure has equal sides, r. It is a square and not a rhombus as we see that three of its interior angles are $90°$; and, as the number of angles in a 4 sided polygon must sum to $360°$, the last angle must also be $90°$. Consequently, the four sided figure is a square.

Below we find a well known relation between the sides, a, b, and c of the original triangle when the sides are integers in reduced form. This is done by investigating the fact that the area of these component triangles and square must be the same as the area of the original triangle.

2 A Basic Property of Primitive Pythagorean Triples

Writing the equivalence of the area of the figure and its parts we have:

$$\overbrace{r^2}^{\text{Square}} + \overbrace{2\frac{r(a-r)}{2}}^{\text{Two lower triangles}} + \overbrace{2\frac{r(b-r)}{2}}^{\text{Two upper triangles}} = \overbrace{\frac{a\,b}{2}}^{\text{Original triangle}} \tag{1}$$

Simplifying, this is:

$$r^2 + r(a-r) + r(b-r) = \frac{a\,b}{2} \tag{2}$$

One can solve the quadratic for r or read off from the figure that $(a-r)+(b-r)=c$. Solving this last equation for r gives:

$$r = \frac{a+b-c}{2} \tag{3}$$

We now consider primitive Pythagorean triangles – right triangles with integer sides which have no common factor. In particular, we are interested in so-called primitive Pythagorean triples, (a, b, c) – the lengths of the sides of primitive Pythagorean triangles.

What can we say about the even or oddness of the sides when the sides have no common factors?

Since we know that $a^2 + b^2 = c^2$, we can reduce the combinations to the following:[1]

[1] We list a few facts to explain this and what follows next:

- c is even AND both a and b are odd;
- c is odd AND one and only one of either a or b is odd.

Regardless of which choice occurs, equation (3) implies that the radius of the circle is an integer.

Using this fact and equation (2) we see that the quantity, $\frac{a\,b}{2}$, must be an integer. This knowledge eliminates the first choice; namely, that both a and b are odd. Consequently, one of the sides is even. Let's say that it is b.

Finally, regardless of whether r is even or odd, the left hand side of (2) is even. This implies that $\frac{a\,b}{2}$ is even. Again since, a is odd, $b/2$ must be divisible by 2.

Finally, we have deduced the properties below for primitive Pythagorean triples, (a, b, c).

- One of the legs is odd; the other leg is even; and the hypotenuse is odd.
- The even leg is divisible by 4.
- The radius of the inscribed circle in the associated right triangle is an integer.

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1. $E \pm E = E$; $E \times E = E$;
2. $E \pm O = O$; $E \times O = E$;
3. $O \pm O = E$; $O \times O = O$.

These equations are to be read: An even number plus (or minus) and even number is an even number. Likewise, an even number plus (or minus) an odd number is an odd number, etc.

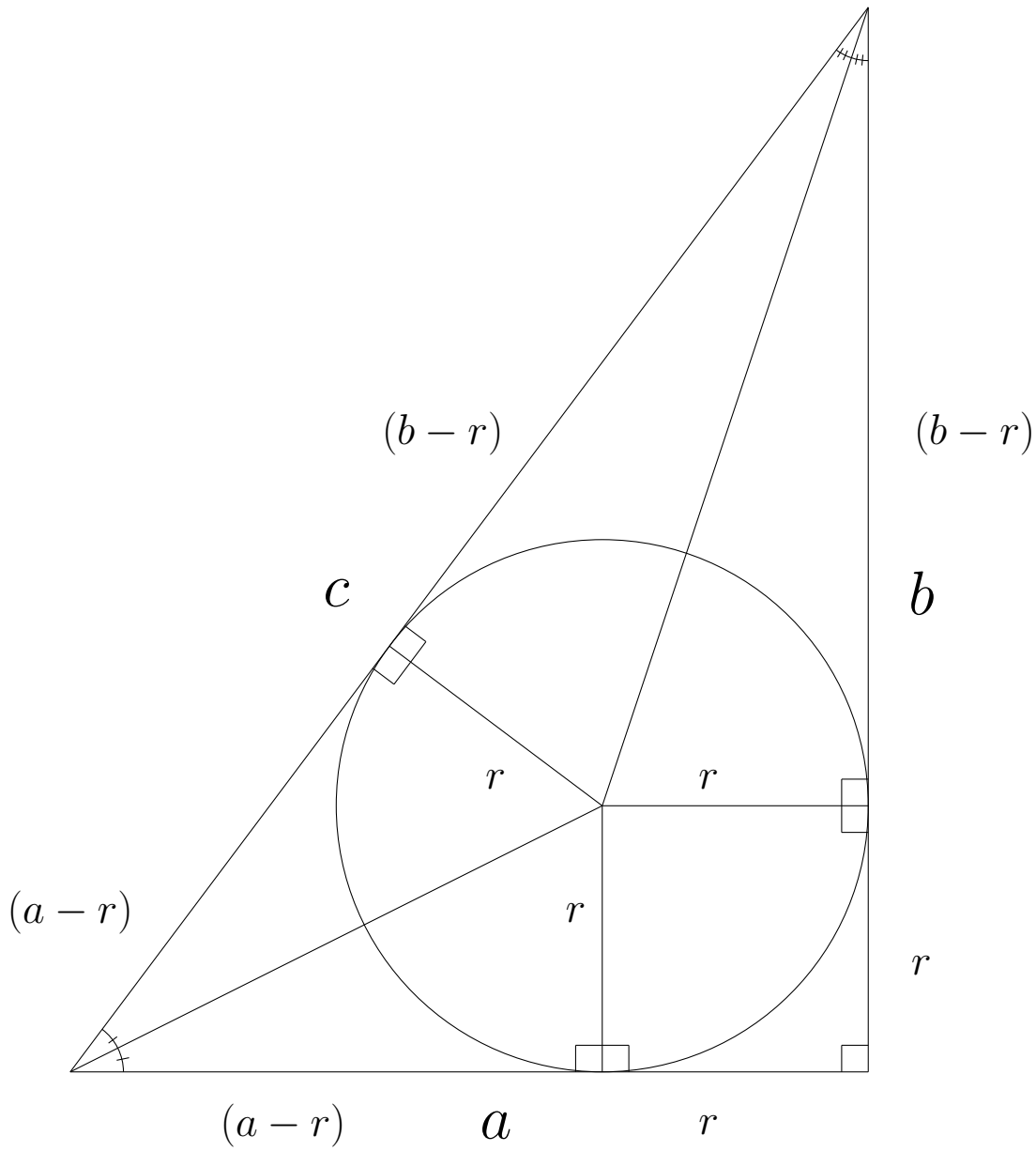

Figure 1: Dissected Right Triangle