Conjectures for Primitive Pythagorean Triples

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1 Overview

We list a few conjectures regarding the set, $\mathcal{H}$, of primitive Pythagorean hypotenuses. This is the set constructed by taking the "hypotenuse" value from all primitive Pythagorean triples – corresponding to the lengths of the sides of right triangles with integer sides with no common factors.

A related set, $\mathcal{H}_p$, is the subset of $\mathcal{H}$ that are primes. That is, primitive Pythagorean hypotenuses that are prime.

Below we list the main conjectures informally in order of complexity:

1. **Partition of $\mathcal{H}$:** $\mathcal{H}$ can be *partitioned* into two disjoint infinite sets: the set of all powers of Pythagorean primes; and the set of all hypotenuses that come from more than one primitive Pythagorean triple.
2. $\mathcal{H}_p$ **is complex:** The set of Pythagorean primes, $\mathcal{H}_p$, is infinite; *similar*; and as *complex* as the primes when measured using certain metrics.

2 Definitions

After a few definitions we describe a conjecture regarding Primitive Pythagorean Triples. We use standard notation, associating the natural numbers with the symbol $\mathbb{N}$[1] and we use the symbol $\mathbb{P}$ to denote the set of prime numbers.

[1] We assume that $\mathbb{N}$ does *not* include 0.

Definition 1. *Pythagorean Triples* denoted, $\mathcal{T}$, are triplets of numbers representing the length of the sides of right triangles where all the sides are integers. Specifically,

$$\mathcal{T} \quad = \quad \{(x,y,z) \mid (x,y,z) \in \mathbb{N}^3 \ \wedge \ x^2+y^2=z^2\} \tag{1}$$

The usual definition.

Definition 2. *Primitive Pythagorean Triples* denoted, $\mathcal{T}_p$, are Pythagorean Triples which have no common factors. Specifically,

$$\mathcal{T}_p \quad = \quad \{(x,y,z) \mid (x,y,z) \in \mathbb{N}^3 \ \wedge \ \gcd(x,y)=1 \ \wedge \ x^2+y^2=z^2\} \tag{2}$$

The usual definition.

Definition 3. *Primitive Pythagorean Hypotenuses* denoted, $\mathcal{H}$, defined by:

$$\mathcal{H} \quad = \quad \{z \mid \exists (x,y) \in \mathbb{N}^2,\ (x,y,z) \in \mathcal{T}_p\} \tag{3}$$

This represents the set of all possible hypotenuses of primitive Pythagorean triples.

Definition 4. *Pythagorean Primes* denoted, $\mathcal{H}_p$, defined by:

$$\mathcal{H}_p \quad = \quad \{z \mid \exists (x,y) \in \mathbb{N}^2,\ (x,y,z) \in \mathcal{T}_p \ \wedge \ z \in \mathbb{P}\} \tag{4}$$

This represents the set of all possible lengths of hypotenuses of right triangles with integer sides that are prime.

Definition 5. *Duplicate Primitive Hypotenuses* denoted $\mathcal{H}_d$, defined by:

$$\mathcal{H}_d \quad = \quad \{z \mid \exists\, (x_1,x_2,y_1,y_2) \in \mathbb{N}^4,\ (x_1,y_1,z) \in \mathcal{T}_p \ \wedge \ (x_2,y_2,z) \in \mathcal{T}_p \ \wedge \ x_1 \neq x_2\} \tag{5}$$

This represents the set of all primitive hypotenuses that can be found in more than one primitive Pythagorean triple.

Definition 6. *Primitive Hypotenuses Powers* denoted $\mathcal{H}_u$, defined by:

$$\mathcal{H}_u \quad = \quad \{z \mid \exists\, (p,n) \in (\mathcal{H}_p, \mathbb{N}),\ z = p^n\} \tag{6}$$

This represents the set of all powers of Pythagorean primes.

Definition 7. *The Prime Parity* function, Π, is defined on any prime, p, with $p \geq 5$ by:

$$\Pi(p) = \begin{cases} 1 & \exists n \in \mathbb{N}, p = 6n+1 \\ -1 & \exists n \in \mathbb{N}, p = 6n-1 \end{cases} \tag{7}$$

Definition 8. *First Deviation* function, Δ_p, is defined by:[2]

$$\Delta_p(n) = \min_m \left\{ m \ \middle|\ n \leq \left(\sum_{i=1}^{m} \Pi(\mathcal{H}_p[i]) - \sum_{i=3}^{m+2} \Pi(\mathbb{P}[i]) \right) \right\} \tag{8}$$

[2]If for a given n, $\left\{ m \ \middle|\ n \leq \left(\sum_{i=1}^{m} \Pi(\mathcal{H}_p[i]) - \sum_{i=3}^{m+2} \Pi(\mathbb{P}[i]) \right) \right\} = \emptyset$, we set $\Delta_p(n) = \infty$.

Notes:

- The subscript u in the name $\mathcal{H}_u$ is not yet justified.
- $\mathcal{H}_p$ is a *proper* subset of the primes: $\mathcal{H}_p \subset \mathbb{P}$.
- If the sets above are treated as lists, they are indexed in sorted order.
- Clearly, Δ_p is a non-decreasing function.

First we list a few facts and state a couple of known theorems before proceeding:

1. The Pythagorean primes are of the form $4n+1$. This is a special case of a more general theorem proven by Pierre de Fermat, known as Fermat's theorem on sums of two squares. According to this theorem, an odd prime number p can be expressed as the sum of two squares if and only if p is congruent to 1 modulo 4, i.e., $p = 4n + 1$ for some positive integer n.
2. Moreover, the distribution of primes of the form $4n + 1$ has been studied extensively. The German mathematician Peter Gustav Lejeune Dirichlet proved that there are infinitely many primes of the form $4n + 1$, a result known as Dirichlet's theorem on arithmetic progressions. This theorem, which generalizes the case of Pythagorean primes, states that for any two positive coprime numbers a and d, there are infinitely many primes of the form $a + nd$, where n is a non-negative integer.
3. In the case of Pythagorean primes, we can take a = 1 and d = 4 to get the form 4n + 1. Dirichlet's theorem thus guarantees that there are infinitely many Pythagorean primes. This is a profound result, as it extends the prime number theorem, which states that there are infinitely many prime numbers, to arithmetic progressions.

From these results we get the following structure theorem for Pythagorean hypotenuses:

Theorem 1. *The set* $\mathcal{H}_p = \{p \mid p \in \mathbb{P} \ \wedge \ p = 1 \bmod 4\}$

Theorem 2. *The set $\mathcal{H}$ has a unique factorization in $\mathcal{H}_p$. That is, each element of $\mathcal{H}$ can be uniquely written as a product of powers from $\mathcal{H}_p$:*

$$\forall h \in \mathcal{H},\ \exists! N \in \mathbb{N},\ \exists! (\mathbf{p}, \mathbf{k}) \in \left(\mathcal{H}_p^N, \mathbb{N}^N\right) : h = \prod_{i=1}^{N} p_i^{k_i}. \tag{9}$$

3 McIntire's Pythagorean Triple Conjectures

Conjecture 1. The set $\mathcal{H}$ is partitioned by the sets $\mathcal{H}_d$ and $\mathcal{H}_u$:

1. $\mathcal{H} = \mathcal{H}_d \cup \mathcal{H}_u$;
2. $\emptyset = \mathcal{H}_d \cap \mathcal{H}_u$;
3. $|\mathcal{H}_d| = \infty$.

In simple terms the conjecture says that each element of the set of all primitive Pythagorean hypotenuses is the union of two disjoint infinite sets: the set of powers of a Pythagorean primes; and the set of all hypotenuses that come from more than one primitive Pythagorean triple.

Conjecture 2. Properties of the cumulative parity of $\mathcal{H}_p$ indicates that it is non-trivial:

1. $\lim\limits_{n\to\infty} \frac{\sum\limits_{i=1}^{n} \Pi(\mathcal{H}_p[i])}{n} = 0$;
2. $\forall n \in \mathbb{N}, \Delta_p(n) < \infty$;
3. $\lim\limits_{n\to\infty} \frac{\Delta_p(n)}{n^{\ln(2\pi)}} = 41$.

These last conjectures suggest that the Pythagorean primes are non-trivial in a way similar to the primes.

1. The first conjecture states that the $\mathcal{H}_p$ primes are "spread out" in a way similar to the set of all primes.
2. From numerical experiments, its seems that the cumulative parity of $\mathcal{H}_p$ is "on average" lower than the cumulative parity of all primes. The second and third conjectures suggest that there are non-trivial deviations from this rule. This "mixing" of the cumulative parity of $\mathcal{H}_p$ with that of $\mathbb{P}$ suggests that the Pythagorean primes are as "complex" as the set of all primes.