A Conjecture for Primitive Pythagorean Triples

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1 Definitions

After a few definitions we describe a conjecture regarding Primitive Pythagorean Triples. We use standard notation to indicate that the natural numbers are represented by the symbol $\mathbb{N}$ and we use the symbol $\mathbb{P}$ to denote the set of prime numbers.

Definition 1. *Pythagorean Triples* denoted, $\mathcal{T}$, are triplets of numbers representing the length of the sides of right triangles where all the sides are integers. Specifically,

$$\mathcal{T} \quad = \quad \{(x, y, z) \mid (x, y, z) \in \mathbb{N}^3 \ \wedge \ x^2 + y^2 = z^2\} \tag{1}$$

The usual definition.

Definition 2. *Primitive Pythagorean Triples* denoted, $\mathcal{T}_p$, are Pythagorean Triples which have no common factors. Specifically,

$$\mathcal{T}_p \quad = \quad \{(x, y, z) \mid (x, y, z) \in \mathbb{N}^3 \ \wedge \ \gcd(x, y) = 1 \ \wedge \ x^2 + y^2 = z^2\} \tag{2}$$

The usual definition.

Definition 3. *Primitive Pythagorean Hypotenuses* denoted, $\mathcal{H}$, defined by:

$$\mathcal{H} \quad = \quad \{z \mid \exists (x, y) \in \mathbb{N}^2, \ (x, y, z) \in \mathcal{T}_p\} \tag{3}$$

This represents the set of all possible lengths of hypotenuses of right triangles with integer sides.

Definition 4. *Pythagorean Primes* denoted, $\mathcal{H}_p$, defined by:

$$\mathcal{H}_p \quad = \quad \{z \mid \exists (x, y) \in \mathbb{N}^2, \ (x, y, z) \in \mathcal{T}_p \ \wedge \ z \in \mathbb{P}\} \tag{4}$$

This represents the set of all possible lengths of hypotenuses of right triangles with integer sides that are prime.

Note: $\mathcal{H}_p$ is a *proper* subset of the primes: $\mathcal{H}_p \subset \mathbb{P}$.

Note : If the sets $\mathcal{H}$ or $\mathcal{H}_p$ are treated as lists, they are indexed in sorted order.

Definition 5. *The Prime Parity* function, Π, is defined on any prime, p, with $p \geq 5$ by:

$$\Pi(p) = \begin{cases} 1 & \exists n \in \mathbb{N}, p = 6n+1 \\ -1 & \exists n \in \mathbb{N}, p = 6n-1 \end{cases} \tag{5}$$

Definition 6. *First Deviation* function, Δ_p, is defined by:[1]

$$\Delta_p(n) = \min_m \left\{ m \;\middle|\; n \leq \left(\sum_{i=1}^{m} \Pi(\mathcal{H}_p[i]) - \sum_{i=3}^{m+2} \Pi(\mathbb{P}[i]) \right) \right\} \tag{6}$$

Clearly, Δ_p is a non-decreasing function.

2 McIntire's Pythagorean Triple Conjectures

Define $\mathcal{H}_d$ and $\mathcal{H}_u$ by:

$$\mathcal{H}_d = \{z \mid \exists\,(x_1, x_2, y_1, y_2) \in \mathbb{N}^4,\ (x_1, y_1, z) \in \mathcal{T}_p \ \wedge\ (x_2, y_2, z) \in \mathcal{T}_p \ \wedge\ x_1 \neq x_2\} \tag{7}$$
$$\mathcal{H}_u = \{z \mid \exists\,(p, n) \in (\mathcal{H}_p, \mathbb{N}),\ z = p^n\} \tag{8}$$

It is a well established *conjecture* in the mathematical community that the cardinality of $\mathcal{H}_p$ is infinite: $|\mathcal{H}_p| = \infty$.

Conjecture 1. The set $\mathcal{H}_d$ has infinite cardinality:

1. $|\mathcal{H}_d| = \infty$.

Conjecture 2. The set $\mathcal{H}$ is partitioned by the sets $\mathcal{H}_d$ and $\mathcal{H}_u$:

1. $\mathcal{H} = \mathcal{H}_d \cup \mathcal{H}_u$;
2. $\emptyset = \mathcal{H}_d \cap \mathcal{H}_u$.

In simple terms the conjecture says that each element of the set of all primitive Pythagorean hypotenuses is either a power of a Pythagorean prime; or, is the length of the hypotenuse of more than one "primitive" right triangle.

Conjecture 3. Properties of the cumulative parity of $\mathcal{H}_p$ indicates that it is non-trivial:

[1] If for a given n, $\left\{ m \;\middle|\; n \leq \left(\sum_{i=1}^{m} \Pi(\mathcal{H}_p[i]) - \sum_{i=3}^{m+2} \Pi(\mathbb{P}[i]) \right) \right\} = \emptyset$, we set $\Delta_p(n) = \infty$.

1. $\lim\limits_{n \to \infty} \frac{\sum\limits_{i=1}^{n} \Pi(\mathcal{H}_p[i])}{n} = 0$;

2. $\forall n \in \mathbb{N}, \Delta_p(n) < \infty$;

3. $\lim\limits_{n \to \infty} \frac{\Delta_p(n)}{n^{\ln(2\pi)}} = 41$.

These last conjectures suggest that the Pythagorean primes are non-trivial in a way similar to the primes.

1. The first conjecture states that the $\mathcal{H}_p$ primes are "spread out" in a way similar to the set of all primes.

2. From numerical experiments, its seems that the cumulative parity of $\mathcal{H}_p$ is "on average" lower than the cumulative parity of all primes. The second and third conjectures suggest that there are non-trivial deviations from this rule. This "mixing" of the cumulative parity of $\mathcal{H}_p$ with that of $\mathbb{P}$ suggests that the Pythagorean primes are as "complex" as the set of all primes.