Measure Theoretic Conditional Expectation in an Elementary Setting

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Date: Aug 2 2004

Overview

The measure theoretic approach to conditional expectation can be confusing when compared to the traditional approach – especially in a discrete setting. In what follows we go through a conditional expectation problem within a discrete and familiar setting in an attempt to reduce this confusion. In the process, we show explicitly that the conditional expectation function is non-measurable in the origin measure space.

Elementary Probability Example Using Measure Theory

Let $X = \{D_1, D_2, D_3, D_4, D_5, D_6\}$ and define a function P by $P(D_i) = \frac{1}{6}$, for $i \in \{1, 2, 3, 4, 5, 6\}$. The intent is that P will become a probability measure for the space we construct. Let $\mathcal{E} = 2^X$ be the sigma consisting of the power set of X. We extend P for every element in the sigma algebra. Since the sigma algebra consists of all sets we need an assignment for an arbitrary set, A. The assignment is $P(A) = \frac{|A|}{6}$; that is the cardinality of the set divided by 6. We now have a measure space; in fact, a probability space: $(P, X, \mathcal{E})$. Note that for a probability space we need an event space, X, a sigma algebra of sets (in the discrete case just an algebra), and a function P which takes elements of the sigma algebra to [0,1] with the property that

$$P\big(\bigcup_{i=1}^{N} A_i\big) = \sum_{i=1}^{N} P(A_i) \quad \text{when } A_k \cap A_j = \emptyset \quad k \neq j$$

We now consider a random variable from which we will get a sub-sigma algebra. Let

$$g(D_i) = \begin{cases} 0 & \text{if } i \text{ is even;} \\ 1 & \text{if } i \text{ is odd} \end{cases}$$

In a discrete space the sigma algebra generated from g is the algebra of sets generated from the sets: $g^{-1}(0), g^{-1}(1), g^{-1}(a)$, for $a \neq 0, 1$. It is not too hard to see that $\mathcal{F} = \{\emptyset, \{1, 3\}, \{2, 4, 6\}, \{1, 2, 3, 4, 5, 6\}\}$.

For a discrete space a function, f, is measurable with respect to a sigma algebra if $f^{-1}(a)$ is an element in the sigma algebra for all $a \in (-\infty, \infty)$. This has implications for the sigma algebra $\mathcal{F}$. The claim is that any function which is measurable over $\mathcal{F}$ has the property that f is constant on these sets $\{1, 3\}$ and $\{2, 4, 6\}$. That is, the minimal elements of the algebra, elements which have no non trivial subsets. To see this suppose that $f(1)$ differs from $f(3)$. Then $f^{-1}(f(1))$ is a non trivial subset of $\{1, 3\}$ as is $f^{-1}(f(3))$. These two sets must differ since $f(1)$ and $f(3)$ differ. This means that they are respectively $\{1\}$ and $\{3\}$. But these sets are not in our sigma algebra. Therefore, f must be constant on $\{1, 3\}$. In the same way it is the case that f is constant on the other minimal set $\{2, 4, 6\}$.

Notice that while any function over the measure space $(P, X, \mathcal{E})$ is measurable, we can write down a specific function that is non-measurable with respect to $\mathcal{F}$. We know that all we have to do is come up with a function that differs on either of the sets $\{1, 3\}$ or $\{2, 4, 6\}$. Let $f(D_i) = i$, for $i \in \{1, 2, 3, 4, 5, 6\}$, for instance.

Conditional Expectation

Given a probability space $(P, X, \mathcal{E})$, the conditional expectation of a measurable function f with respect to a sub-sigma algebra $\mathcal{F}$ is the unique $\mathcal{F}$ measurable function (random variable) labeled, $E[f/\mathcal{F}]$, such that

$$\int_\Lambda E[f/\mathcal{F}]\, dP = \int_\Lambda f\, dP \quad \forall \Lambda \in \mathcal{F} \tag{1}$$
$$\int_\Lambda E[f/\mathcal{F}]\, dP$$

That is, $E[f/\mathcal{F}]$ is a measurable function in the probability space $(P, X, \mathcal{F})$. Although it seems that f itself satisfies this equation you have to be careful. The function we are looking for must be measurable with respect to $\mathcal{F}$, and since $\mathcal{F}$ is a sub-sigma algebra of $\mathcal{E}$, it is quite possible that f is not $\mathcal{F}$ measurable. However, if f is measurable with respect to $\mathcal{F}$ then it is its own conditional expectation. We have the following basic facts about conditional expectation:

- $E[f/\mathcal{F}] = f$ when f is measurable with respect to $\mathcal{F}$;
- $E[E[f/\mathcal{F}]/\mathcal{F}] = E[f/\mathcal{F}]$;
- $E[(f - E[f/\mathcal{F}])/\mathcal{F}] = 0$.

Consider the function of the last section, $f(D_i) = i$, which is measurable in the space $(P, X, \mathcal{E})$. Let $\mathcal{F}$ be the sub-sigma algebra of the last section as well. We now compute the conditional expectation of f with respect to $\mathcal{F}$. Using (1) we choose two Λ's: $\Lambda_1 = \{1, 3\}$ and $\Lambda_2 = \{2, 4, 6\}$. We have

$$\int_{\Lambda_1} E[f/\mathcal{F}]\, dP = \int_{\Lambda_1} f\, dP \tag{2}$$

and

$$\int_{\Lambda_2} E[f/\mathcal{F}]\, dP = \int_{\Lambda_2} f\, dP \tag{3}$$

Since $E[f/\mathcal{F}]$ is constant on these set we have from (2)

$$E[f/\mathcal{F}](D_1) * P(\Lambda_1) = f(D_1) * P(D_1) + f(D_3) * P(D_3) = \frac{1}{6} + \frac{3}{6} = \frac{2}{3}$$

Since $P(\Lambda_1) = \frac{2}{6} = \frac{1}{3}$, we have that

$$E[f/\mathcal{F}](D_1) = E[f/\mathcal{F}](D_3) = 2$$

In the same way we can find the value of $E[f/\mathcal{F}]$ on the set $\{2, 4, 6\}$

$$E[f/\mathcal{F}](D_2) * P(\Lambda_2) = f(D_2) * P(D_2) + f(D_4) * P(D_4) + f(D_6) * P(6) = \frac{2}{6} + \frac{4}{6} + \frac{6}{6} = 2$$

Since $P(\Lambda_2) = \frac{3}{6} = \frac{1}{2}$, we have that

$$E[f/\mathcal{F}](D_2) = E[f/\mathcal{F}](D_4) = E[f/\mathcal{F}](D_6) = 4$$

We have now determined the value of $E[f/\mathcal{F}]$ for all values: $D_1 \ldots D_6$.