

# Measure Theoretic Conditional Expectation in an Elementary Setting

**Author:** *R. Scott McIntire*

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## Overview

The measure theoretic approach to conditional expectation can be confusing when compared to the traditional approach – especially in a discrete setting. In what follows we go through a conditional expectation problem within a discrete and familiar setting in an attempt to reduce this confusion. In the example shown, we show explicitly that the conditional expectation function is non-measurable in the origin measure space.

## Elementary Probability Example Using Measure Theory

Let  $X = \{D_1, D_2, D_3, D_4, D_5, D_6\}$  and define a function  $P$  by  $P(D_i) = \frac{1}{6}$ , for  $i \in \{1, 2, 3, 4, 5, 6\}$ . The intent is that  $P$  will become a probability measure for the space we construct. Let  $\mathcal{E} = 2^X$  be the sigma consisting of the power set of  $X$ . We extend  $P$  for every element in the sigma algebra. Since the sigma algebra consists of all sets we need an assignment for an arbitrary set,  $A$ . The assignment is  $P(A) = \frac{|A|}{6}$ ; that is the cardinality of the set divided by 6. We now have a measure space; in fact, a probability space:  $(P, X, \mathcal{E})$ . Note that for a probability space we need an event space,  $X$ , a sigma algebra of sets (in the discrete case just an algebra), and a function  $P$  which takes elements of the sigma algebra to  $[0,1]$  with the property that

$$P\left(\bigcup_{i=1}^N A_i\right) = \sum_{i=1}^N P(A_i) \quad \text{when } A_k \cap A_j = \emptyset \quad k \neq j$$

We now consider a random variable from which we will get a sub-sigma algebra. Let

$$g(D_i) = \begin{cases} 0 & \text{if } i \text{ is even;} \\ 1 & \text{if } i \text{ is odd} \end{cases}$$

In a discrete space the sigma algebra generated from  $g$  is the algebra of sets generated from the sets:  $g^{-1}(0), g^{-1}(1), g^{-1}(a)$ , for  $a \neq 0, 1$ . It is not too hard to see that  $\mathcal{F} = \{\emptyset, \{1, 3\}, \{2, 4, 6\}, \{1, 2, 3, 4, 5, 6\}\}$ .

For a discrete space a function,  $f$ , is measurable with respect to a sigma algebra if  $f^{-1}(a)$  is an element in the sigma algebra for all  $a \in (-\infty, \infty)$ . This has implications for the sigma algebra  $\mathcal{F}$ .

The claim is that any function which is measurable over  $\mathcal{F}$  has the property that  $f$  is constant on the sets  $\{1, 3\}$  and  $\{2, 4, 6\}$ . That is, the minimal elements of the algebra, elements which have no non trivial subsets. To see this suppose that  $f(1)$  differs from  $f(3)$ . Then  $f^{-1}(f(1))$  is a non trivial subset of  $\{1, 3\}$  as is  $f^{-1}(f(3))$ . These two sets must differ since  $f(1)$  and  $f(3)$  differ. This means that they are respectively

$\{1\}$  and  $\{3\}$ . But these sets are not in our sigma algebra. Therefore,  $f$  must be constant on  $\{1, 3\}$ . In the same way it is the case that  $f$  is constant on the other minimal set  $\{2, 4, 6\}$ .

Notice that while any function over the measure space  $(P, X, \mathcal{E})$  is measurable, we can write down a specific function that is non-measurable with respect to  $\mathcal{F}$ . We know that all we have to do is come up with a function that differs on either of the sets  $\{1, 3\}$  or  $\{2, 4, 6\}$ . Let  $f(D_i) = i$ , for  $i \in \{1, 2, 3, 4, 5, 6\}$ , for instance.

## Conditional Expectation

Given a probability space  $(P, X, \mathcal{E})$ , the conditional expectation of a measurable function  $f$  with respect to a sub-sigma algebra  $\mathcal{F}$  is the unique  $\mathcal{F}$  measurable function (random variable) labeled,  $E[f/\mathcal{F}]$ , such that

$$\int_{\Lambda} E[f/\mathcal{F}] dP = \int_{\Lambda} f dP \quad \forall \Lambda \in \mathcal{F} \quad (1)$$

That is,  $E[f/\mathcal{F}]$  is a measurable function in the probability space  $(P, X, \mathcal{F})$ . Although it seems that  $f$  itself satisfies this equation you have to be careful. The function we are looking for must be measurable with respect to  $\mathcal{F}$ , and since  $\mathcal{F}$  is a sub-sigma algebra of  $\mathcal{E}$ , it is quite possible that  $f$  is not  $\mathcal{F}$  measurable. However, if  $f$  is measurable with respect to  $\mathcal{F}$  then it is its own conditional expectation. We have the following basic facts about conditional expectation:

- $E[f/\mathcal{F}] = f$  when  $f$  is measurable with respect to  $\mathcal{F}$ ;
- $E[E[f/\mathcal{F}]/\mathcal{F}] = E[f/\mathcal{F}]$ ;
- $E[(f - E[f/\mathcal{F}])/\mathcal{F}] = 0$ .

Consider the function of the last section,  $f(D_i) = i$ , which is measurable in the space  $(P, X, \mathcal{E})$ . Let  $\mathcal{F}$  be the sub-sigma algebra of the last section as well. We now compute the conditional expectation of  $f$  with respect to  $\mathcal{F}$ . Using (1) we choose two  $\Lambda$ 's:  $\Lambda_1 = \{1, 3\}$  and  $\Lambda_2 = \{2, 4, 6\}$ . We have

$$\int_{\Lambda_1} E[f/\mathcal{F}] dP = \int_{\Lambda_1} f dP \quad (2)$$

and

$$\int_{\Lambda_2} E[f/\mathcal{F}] dP = \int_{\Lambda_2} f dP \quad (3)$$

Since  $E[f/\mathcal{F}]$  is constant on these sets we have from (2)

$$E[f/\mathcal{F}](D_1) * P(\Lambda_1) = f(D_1) * P(D_1) + f(D_3) * P(D_3) = \frac{1}{6} + \frac{3}{6} = \frac{2}{3}$$

Since  $P(\Lambda_1) = \frac{2}{6} = \frac{1}{3}$ , we have that

$$E[f/\mathcal{F}](D_1) = E[f/\mathcal{F}](D_3) = 2$$

In the same way we can find the value of  $E[f/\mathcal{F}]$  on the set  $\{2, 4, 6\}$

$$E[f/\mathcal{F}](D_2) * P(\Lambda_2) = f(D_2) * P(D_2) + f(D_4) * P(D_4) + f(D_6) * P(D_6) = \frac{2}{6} + \frac{4}{6} + \frac{6}{6} = 2$$

Since  $P(\Lambda_2) = \frac{3}{6} = \frac{1}{2}$ , we have that

$$E[f/\mathcal{F}](D_2) = E[f/\mathcal{F}](D_4) = E[f/\mathcal{F}](D_6) = 4$$

We have now determined the value of  $E[f/\mathcal{F}]$  for all values:  $D_1 \dots D_6$ .