Constructing Diversification Constraints

R. Scott McIntire

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1 Overview

Consider the function, $f : \mathbf{R}^n \mapsto \mathbf{R}$, defined by[1]

$$f(\mathbf{x}) \;=\; \sum_{i=1}^{k} x_{[i]} \tag{1}$$

From this we see that $f(\mathbf{x})$ is the value of the sum of the k^{th} largest values of its input $\mathbf{x}$. We are interested in optimization problems involving a vector $\mathbf{x}$ with the sum of the top values constrained by a given value; that is, such that $f(\mathbf{x}) \leq M$ for some value M. How could we do this? One way is to write down all possible combinations of the k elements of $\mathbf{x}$ and write a constraint that bounds their sum to be less than or equal to M. But the number of constraints that one has to write is $\binom{n}{k}$. This becomes large very quickly. In the next section we seek a way to represent f to reduce the number of constraints.[2]

This constraint often arises in the world of finance where one wants to create (or modify) a portfolio so that the new portfolio has the property that its top k securities (notionally) do not exceed some fixed percentage of its total notional.

[1] For a given vector, $\mathbf{x} \in \mathbf{R}^n$, create a vector, $\mathbf{x}'$, by *stably* sorting $\mathbf{x}$ from highest to lowest. Define the notation, $x_{[i]}$ by: $x_{[i]} \equiv x'_i$. Note: For the rest of this paper we assume that $k \in \mathbf{Z}^+$ with $k \leq n$.

[2] A bold font is used to denote vectors. We refer to a vector of 0s as $\mathbf{0}$, and likewise we refer to a vector of 1s as $\mathbf{1}$. The mathematical expression, $\mathbf{a} \preceq \mathbf{b}$, is to be interpreted as the statement that every element of the vector $\mathbf{a}$ is less than or equal to the corresponding element in the vector $\mathbf{b}$. Similarly, the expression, $\mathbf{a} \succeq \mathbf{b}$, is the statement that each element of $\mathbf{a}$ is greater than or equal to each corresponding element of $\mathbf{b}$.

2 The Function f as an Optimization Problem

We wish to characterize the value of f on a given vector, $\mathbf{x}$ as the result of an optimization. We claim that for a given $\mathbf{x}$, $f(\mathbf{x})$ is the value of a solution to the following constrained optimization problem:

$$\max_{\mathbf{y}\in\mathbf{Z}_2^n} \quad \mathbf{y}^T\mathbf{x} \tag{2}$$

$$\mathbf{y}^T\mathbf{1} \;=\; k \tag{3}$$

In English this says: "Take the maximum value of all possible sums of k values from $\mathbf{x}$." This is clearly just another way of restating $f(\mathbf{x})$. Although this optimization is succinct, there are still an exponential number of combinations to examine to find the optimal solution in this discrete setting.

claim: The solution to the above is the same as the solution to the *continuous* optimization problem:[3]

$$\max_{\mathbf{y}\in\mathbf{R}^n} \quad \mathbf{y}^T\mathbf{x} \tag{4}$$

$$\mathbf{0} \preceq \mathbf{y} \preceq \mathbf{1} \tag{5}$$

$$\mathbf{y}^T\mathbf{1} = k \tag{6}$$

proof: Since its domain, $\mathbf{R}^n$, includes $\mathbf{Z}_2^n$, the solution to the continuous problem is at least as large as the solution to the original discrete problem. We now show the reverse, by showing that for every optimal solution to the continuous problem, $\mathbf{y}^*$, there is a corresponding $\mathbf{y}'$ that lives in $\mathbf{Z}_2^n$, satisfying the constraint, (3), with (2) yielding the same value as the solution to the continuous problem.

To this end, let $\mathbf{y}^* \in \mathbf{R}^n$ be a solution to the continuous problem. If all of the values are integral, by (5), the only integral values are 0 or 1. In this case, we are done – take $\mathbf{y}'$ to be $\mathbf{y}^*$.

Otherwise, collect the components of $\mathbf{y}^*$ that are non-integral; that is, have values strictly between 0 and 1. By the constraint, (6), the sum of these non-integral components must sum to an integral value. Consequently, there must be more than one such component. It must also be the case that amongst these non-integral components the corresponding "x" values are the same. Otherwise, if two of these components had differing "x" values, then the "y"

[3]The solution to this problem exists as the supremum of a *continuous* objective function over a closed, bounded set in R^n – as determined by these particular constraints – is attained.

component with the smaller "x" value could shift some of its value to the "y" component with the larger "x" value in such a way as to preserve the constraints, (5) and (6). But then we would have a larger objective value than the supposed maximum – contradiction. Therefore, all of the "x" components associated with non-integral "y" values must have the same value. As previously stated, the sum of these non-integral values must sum to an integral value, which by necessity, is less than the number of non-integral components in $\mathbf{y}^*$. Let us call this number, j. Pick any j of these non-integral values, and let J be the set of their index values in $\mathbf{y}^*$. Let $\tilde{J}$ be the set of indices corresponding to the remaining non-integral values of $\mathbf{y}^*$.

Now, create a new vector $\mathbf{y}' \in \mathbf{Z}_2^n$ and set its values in the following way: Set the values of $\mathbf{y}'$ corresponding to the indices in J to be 1. Set the values of $\mathbf{y}'$ corresponding to the indices in $\tilde{J}$ to 0. Note that with this assignment, the sum of the elements of $\mathbf{y}^*$ with indices in $J \cup \tilde{J}$ is the same as the sum over $\mathbf{y}'$ with indices in $J \cup \tilde{J}$. Set the rest of the components of $\mathbf{y}'$ to the corresponding values from $\mathbf{y}^*$. As we have seen, these integral values can only be 0 or 1. This means that $\mathbf{y}'$ has the property that ${\mathbf{y}'}^T\mathbf{x} = k$. We have now shown that for any solution vector to the continuous problem $\mathbf{y}^*$, there is a corresponding vector $\mathbf{y}' \in \mathbf{Z}_2^n$ that is in the feasible set of the discrete problem and which yields the same value as the optimal value of the continuous problem. Therefore, the optimal solution to the discrete problem is at least as large as any optimal solution to the continuous problem. Consequently, the two problems are equivalent.

Note that the solution to the continuous problem is the same as the associated problem:

$$\min_{\mathbf{y}\in\mathbf{R}^n} \quad -\mathbf{y}^T\mathbf{x} \tag{7}$$

$$\mathbf{0} \preceq \mathbf{y} \preceq \mathbf{1} \tag{8}$$

$$\mathbf{y}^T\mathbf{1} = k \tag{9}$$

Treating $\mathbf{x}$ as a constant, this is a *convex* optimization problem.

3 A Dual Description of the Optimization

Since the problem described by equations $(7, 8, 9)$, is *convex*, the value of its solution is the same as the value of its associated *dual* problem.[4]

[4]Normally one needs to show that a convex problem satisfies Slater's condition in order to claim that it provides the same value as its associated dual. But this is not necessary when dealing with *linear* convex

To form the dual problem we need the Lagrangian, which is:[5]

$$L(\mathbf{y}, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \nu; \mathbf{x}) \;\; = \;\; -\mathbf{y}^T\mathbf{x} - \boldsymbol{\lambda}_1^T\mathbf{y} + \boldsymbol{\lambda}_2^T(\mathbf{y} - \mathbf{1}) + \nu(k - \mathbf{y}^T\mathbf{1}) \tag{10}$$

Here $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2 \in \mathbf{R}^n$ and $\nu \in R$ are the *dual* variables.

Set the function g:

$$g(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_1, \nu; \mathbf{x}) \;\; = \;\; \inf_{\mathbf{y}} \; L(\mathbf{y}, \boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \nu; \mathbf{x}) \tag{11}$$

Substituting for L this becomes

$$g(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_1, \nu; \mathbf{x}) \;\; = \;\; \inf_{\mathbf{y}} \; \left(\mathbf{y}^T \left(-\mathbf{x} - \boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 - \nu\mathbf{1}\right) - \mathbf{1}^T\boldsymbol{\lambda}_2 + \nu\, k\right) \tag{12}$$

The dual problem is then

$$\max_{\substack{\boldsymbol{\lambda}_2 \succeq \mathbf{0} \\ \boldsymbol{\lambda}_1 \succeq \mathbf{0} \\ \nu}} \; g(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \nu; \mathbf{x}) \tag{13}$$

Which is[6]

$$\max_{\substack{\boldsymbol{\lambda}_2 \succeq \mathbf{0} \\ \boldsymbol{\lambda}_1 \succeq \mathbf{0} \\ \nu}} \qquad -\mathbf{1}^T\boldsymbol{\lambda}_2 + \nu\, k \tag{14}$$

$$-\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2 - \nu\mathbf{1} = \mathbf{x} \tag{15}$$

This is equivalent to[7]

$$\max_{\substack{\boldsymbol{\lambda} \succeq \mathbf{0} \\ \nu}} \qquad \nu\, k - \mathbf{1}^T\boldsymbol{\lambda} \tag{16}$$

$$\mathbf{x} \preceq \boldsymbol{\lambda} - \nu\mathbf{1} \tag{17}$$

problems. Why, you may ask is it *linear*? The term $-\mathbf{y}^T\mathbf{x}$ is decidedly non-linear. It is linear if we treat $\mathbf{x}$ as a constant (we also use the term parameter).

[5]For the purposes of this optimization, $\mathbf{x}$ is seen as a parameter to the optimization – essentially a constant with respect to the optimization. As such, it is customary to treat such parameters as part of a function but separate them from the "true" list of variables of the problem with a semi-colon.

[6]Note that g is $-\infty$, unless the term that $\mathbf{y}$ is "dotting" is the zero vector. Consequently, the maximum must necessarily occur where the dotting vector is zero.

[7]We can remove n variables by eliminating $\boldsymbol{\lambda}_1$ from the equations while keeping the same number of inequalities. We can do this by realizing that $\boldsymbol{\lambda}_2 - \nu\mathbf{1} = \mathbf{x} + \boldsymbol{\lambda}_1$ expresses the same information as $\boldsymbol{\lambda}_2 - \nu\mathbf{1} \succeq \mathbf{x}$. Since there is now only one $\boldsymbol{\lambda}$, we relabel $\boldsymbol{\lambda}_2$ as $\boldsymbol{\lambda}$.

Since maximizing over ν or $-\nu$ is the same – as there are no restrictions on ν – we may replace ν with $-\nu$ in the last equations giving:

$$\max_{\substack{\boldsymbol{\lambda} \succeq \mathbf{0} \\ \nu}} \quad -\nu\, k - \mathbf{1}^T \boldsymbol{\lambda} \tag{18}$$

$$\mathbf{x} \preceq \boldsymbol{\lambda} + \nu \mathbf{1} \tag{19}$$

But this is the same as:

$$\min_{\boldsymbol{\lambda},\nu} \quad \nu\, k + \mathbf{1}^T \boldsymbol{\lambda} \tag{20}$$

$$\mathbf{x} \preceq \boldsymbol{\lambda} + \nu \mathbf{1} \tag{21}$$

$$\boldsymbol{\lambda} \succeq \mathbf{0} \tag{22}$$

4 Characterization of f as O(n) Linear Constraints

We now apply the results of the previous sections to the following problem: Add constraints to an optimization problem that includes a *variable* $\mathbf{x} \in \mathbf{R}^n$ so that $f(\mathbf{x}) \leq M$ for some value M.

We claim that one can bound $f(\mathbf{x})$ by adding two new variables, $\boldsymbol{\lambda} \in \mathbf{R}^n$ and $\nu \in R$, along with the following three constraints:

$$\begin{aligned} \nu\, k + \mathbf{1}^T \boldsymbol{\lambda} &\leq M & (23)\\ \mathbf{x} &\preceq \boldsymbol{\lambda} + \nu \mathbf{1} & (24)\\ \boldsymbol{\lambda} &\succeq \mathbf{0} & (25) \end{aligned}$$

Why? Because the expression $\nu\, k + \mathbf{1}^T \boldsymbol{\lambda}$ with the constraints (24) and (25) applied is *always* an upper bound to $f(\mathbf{x})$. Consequently, we have the inequality: $f(\mathbf{x}) \leq \nu\, k + \mathbf{1}^T \boldsymbol{\lambda} \leq M$. The only concern remaining is that there may be a gap between the values of $\nu\, k + \mathbf{1}^T \boldsymbol{\lambda}$ and $f(\mathbf{x})$ – making (23) too restrictive. But this is not the case as the minimum over all $\boldsymbol{\lambda}$ and ν (subject to (24) and (25)) is $f(\mathbf{x})$. Another way of saying this is that the parameters, $\boldsymbol{\lambda}$ and ν provide enough freedom to $\mathbf{x}$ so that $f(\mathbf{x})$ can be as close to M as one would like.

One might ask why we couldn't do exactly the same procedure with the "simpler" characterization we had with equations (7), (8), and (9), rather than going through so much effort with a dual problem. The same number of variables and the same number of constraints are required.

We could do that; however, note that once we take the characterization of a bound on a "given" $\mathbf{x}$ and place it into an optimization problem, that "given" $\mathbf{x}$ is no longer a constant. It, itself, is part of the optimization. And now, one of the inequalities that we would add would be related to (7); that is, the term: $-\mathbf{y}^T\mathbf{x}$. But this term involves the (now) *variable*, $\mathbf{x}$, and the new *variable*, $\mathbf{y}$, in a decidedly *non-linear* way. In fact, the term is not even *convex*.

In contrast to this "simpler" characterization and the original näive combinatorial approach, we have shown that by examining an associated dual problem, $f(\mathbf{x})$ can be bounded by adding $(n+1)$ new variables and $(2n+1)$ *linear* constraints.