Defining Fractal Dimension

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Introduction

In this paper we examine a notion of the size of a set called *fractal dimension* that enables one to more finely distinguish the density of a given set than the usual dimensional classification of $0, 1, 2, 3, \ldots$.[1] Such a notion is motivated by sets that, while constructed in one of the given dimensions $0, 1, 2, 3, \ldots$; and which are robust with respect to structure, nevertheless have zero "volume" in the ambient space containing the set. Such sets beg for a more nuanced "size" metric.

The sets that we are interested in are non-physical sets that have the property that they retain structure no matter how much they are magnified. We call this class of sets *fractal sets*, or *fractals*.

A common place example that resembles a fractal is the coast line of a beach. If one were to take a map of the coast line and blow it up again and again, one would find that it retains structure in the sense that the expanded maps would not "straighten out". Because this is a physical example, the complicated structure would fade after a finite number of expansions. A true fractal, say representing a coast line, would retain complicated structure no matter how many times a piece of it was magnified. Because of this, the length of any section of this idealized coast line is infinite. Another "natural" fractal is the surface of a idealized sponge. While defined in a three dimensional world, one can construct such a sponge so that it retains complicated structure no matter how many times it is magnified. In addition, such a constructed sponge could have an infinite surface area and also have 0 volume.

It would be convenient if there was a way to measure the density of an idealized coast line to indicate that it is, in some sense, larger than a one dimensional object; or a way to show that the idealized surface of a sponge is larger than two dimensions. This would allow us to compare the "size" of such sets in a more precise way.

This is what the fractal dimension of a set tries to address. In the case of the coast line, it will give us a dimension between one and two – where the coast line "naturally" lives. The same with the surface of the aforementioned sponge, there will be a dimension between two and three where it naturally lives. In each case, the fractal dimension is not an actual dimension, it is a way to quantify the density of such complex sets. Yet, it should also "agree" with the usual dimensional size of an object if it is not a fractal. That is, the fractal dimension of a line segment should be 1; the fractal dimension of a triangle sitting in three dimensions should be 2, etc.

When studying any new phenomenon, it is natural to first examine the simplest examples. The simplest such fractals are ones which have a *self similarity* property; namely, they have the property that if one magnifies a piece of the fractal by a certain amount, the original fractal is recovered. This regular structure makes them easier to study mathematically than a general fractal. However, before we start with more specific examples of fractals, we need to understand what one might mean by fractal dimension.

Volume of an n-Dimensional Sphere

[1] The zero dimension of a set is the number of points in that set.

The n-dimensional volume of any sphere is easy to compute if one knows its surface area. Since the surface area of an n-dimensional sphere with radius, r, is r^{n-1} times the surface of the unit sphere, it suffices to know the surface area of the unit sphere. In what follows we will refer to an n-dimensional sphere of radius, r, as $B_n(r)$, and B_n as the specific n-dimensional sphere of radius 1. We will use the notation, $S_n(r)$ and $V_n(r)$ for the surface area and volume of $B_n(r)$ respectively.

The relationship between $V_n(r)$ and $S_n(1)$ follows from the following formula:

$V_n(r) = \int_{\partial B_n} \int_0^r \tau^{n-1}\, dS\, d\tau = \left(\int_{\partial B_n} dS\right) \left(\int_0^r \tau^{n-1}\, d\tau\right)$

The iterated integrals on the right hand side of the above equation can be easily computed, yielding: $V_n(r) = S_n(1)\frac{r^n}{n}$.

What remains now is to find a formula for $S_n(r)$. This can be done by expressing an easily computable n-dimensional integral in R^n as n iterated integrals. And more importantly, this integral may be represented as a *different* integral involving the surface are of B_n.

$$\int_{-\infty}^{\infty} e^{-(x_1^2+x_2^2+\cdots+x_n^2)}\, dx_1\, dx_2\, \cdots\, dx_n = \int_{\partial B_n} \int_0^{\infty} e^{-r^2} r^{n-1}\, dS\, dr$$

The left side of the above can be written as an iterated integral of identical one dimensional integrals. The right side of the equation is the same integral in "spherical coordinates": this is a convenient choice as the integrand is spherically symmetric. The left side simplifies to $\int_{-\infty}^{\infty} e^{-x^2}\, dx$ raised to the nth power; which is $\pi^{n/2}$. Substituting $t = r^2$ in the inner integral on the right hand side we have:

$$\pi^{n/2} = \frac{1}{2} \int_{\partial B_n} \int_0^{\infty} e^{-t} t^{\frac{n}{2}-1}\, dt\, dS$$

which may be written:[2]

$$2\pi^{n/2} = S_n(1)\Gamma(n/2)$$

Or,

$$S_n(1) = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

Consequently,[3]

$$S_n(r) = \frac{2\pi^{n/2}}{\Gamma(n/2)} r^{n-1}$$

And so the volume of an n-dimensional sphere, $B_n(r)$, of radius r is:

$$V_n(r) = \int_{B_n} d\mu = \int_{\partial B_n} \int_0^r \tau^{n-1}\, d\tau\, dS = \left(\frac{r^n}{n}\right) S_n(1)$$

Which is

$$V_n(r) = \frac{2\pi^{n/2}}{n\,\Gamma(n/2)} r^n$$

Fractal Dimension and Associated Volume of a Set

Although we derived the formula for the n-dimensional volume of a sphere. The formula may take non-integral values. Armed with this formula, we make the following definition.

[2] The function $\Gamma(n)$ is called the Gamma function and has the property: $\forall n \in \mathcal{R}^+, \Gamma(n+1) = n\Gamma(n)$. It can be thought of as an extension of the integer factorial function extended to reals as: $\forall n \in \mathcal{Z}^+ : \Gamma(n+1) = n!$.

[3] The surface area of an n-dimensional sphere scales by the radius to the $(n-1)^{\text{th}}$ power.

Definition. *For any $\alpha \in R^+$, the α-dimensional volume, V_α, of an n-dimensional sphere of radius r is defined to be:*[4]

$$V_\alpha(r) = \frac{2\pi^{\alpha/2}}{\alpha\,\Gamma(\alpha/2)} r^\alpha$$

This is a mathematical construct, not a physical one; there is no claim here of a physical fractional dimension. Also, note that the value is independent of the ambient space of the ball, R^n. That is, α can be either larger or smaller than n.

Given that we have defined the fractional dimensional volume of a sphere in "dimension" α we wish to extend this to a definition of the fractional dimensional volume of *any* set.

Definition. *The α-dimensional ($\alpha \in R^+$) covering volume of a set $S \in R^n$, $\mathcal{V}_{\alpha,\epsilon}(S)$, with parameter, $\epsilon \in R^+$, is defined as*[5]

$$\mathcal{V}_{\alpha,\epsilon}(S) = \inf_{\{O_i\}_{i=1}^N} \left\{ \sum_{i=1}^N V_\alpha\left(\mathrm{Diam}(O_i)/2\right) \; : S \subseteq \bigcup_{i=1}^N O_i \;\wedge\; \left(\operatorname*{Max}_{i \in [1,N]} \mathrm{Diam}(O_i) \right) \geq \epsilon \right\}$$

Roughly speaking, for any given finite covering, $\{O_i\}_{i=1}^N$, and value ϵ, we estimate the α-dimensional volume of the set, S, by summing up the α-dimensional volumes of the collection of spheres, $\{P_i\}_{i=1}^N$, where each P_i is the smallest sphere containing O_i. In this case the radius of a given P_i is $\mathrm{Diam}(O_i)/2$. The α covering volume of a set, S, is the infimum over all such covers. The only restriction of the covering sets is that they are finite and the corresponding spheres have radius no smaller than ϵ.

The intent of this definition is to give a practicable upper estimate of the eventual α volume of a set. We would also like that the above would give finer and finer estimates of the α volume with smaller ϵ yielding smaller estimates.

This is indeed the case:

$$\mathcal{V}_{\alpha,\epsilon_1}(S) \leq \mathcal{V}_{\alpha,\epsilon_2}(S) \qquad \text{whenever } \epsilon_1 \leq \epsilon_2$$

This follows as the set of coverings used to compute $\mathcal{V}_{\alpha,\epsilon_1}(S)$ includes the set of all coverings used to compute $\mathcal{V}_{\alpha,\epsilon_2}(S)$. Therefore, the infimum over all coverings associated with ϵ_1 cannot be larger than the infimum associated with ϵ_2.

Using the covering volume, we may define the fractional dimensional volume by taking a limit of increasingly refined covers.

Definition. *The α-dimensional ($\alpha \in R^+$) volume of a set $S \in R^n$, $\mathcal{V}_\alpha(S)$, is defined as*[6]

$$\mathcal{V}_\alpha(S) = \lim_{\epsilon \to 0^+} \mathcal{V}_{\alpha,\epsilon}(S)$$

It is not immediately clear how the α volumes of a set behave as the dimension changes. It turns out that these volumes are non-increasing with increasing α. This follows from the following two facts:

1. For $r \leq 0.1$, the α-dimensional volume, $V_\alpha(r)$, of a sphere monotonically decreases with α.
2. This means that for a given bounded set S and for any fixed $\epsilon \leq 0.1$, $\mathcal{V}_{\alpha,\epsilon}(S)$ is non-increasing in α.
3. If for some $\alpha_0 \in R^+$, $\mathcal{V}_{\alpha_0}(S) < \infty$, then $\mathcal{V}_\alpha(S)$ is monotonically decreasing for $\alpha \geq \alpha_0$.

While these statements define the α volume for any given set, S, they do not tell us the "natural" fractal volume of S. The following definition does precisely that.

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4 *The set R^+ is defined by: $R^+ = \{x \mid x \in R \;\wedge\; x \geq 0\}$.*

5 *The diameter of a set is defined by:* $\mathrm{Diam}(S) = \sup\{|x - y| : x, y \in S\}$.

6 *This definition is well defined, as the limit of a monotonic sequence exists, provided we acknowledge that it may be infinite.*

Definition. *The* **fractal dimension** *of a set* $S \in R^m$ *is defined as* $\alpha^* = \inf\{\alpha : \mathcal{V}_\alpha(S) = 0\}$*; the associated* **fractal volume** *of* S *is defined to be* $\mathcal{V}_{\alpha^*}(S)$.

Roughly speaking, this says that the fractal dimension of a set S is the "smallest" α such that the α-dimensional volume of S is non-zero.

Note that the fractal dimension of a set, S, does not depend on the ambient space S sits in.

Computing the Fractal Dimension of a Simple Set

The fractal dimension of a set S should produce the usual answer when S is a non-fractal set. We show this for the case of a straight line.

Example: ***A line segment in two dimensions.***

We wish to find the fractal dimension and fractal volume of the line segment

$$L = \{t\mathbf{x} + (1-t)\mathbf{y} : t \in [0,1], \mathbf{x}, \mathbf{y} \ \in \ R^2\}$$

While this line segment sits in the two dimensional plane, we wish to find its "natural" dimension and associated volume.

For a given n, consider the sets $\{O_{n,i}\}_{i=0}^{n-1}$, with $O_{n,i} \ = \ [\frac{i}{n}\mathbf{x} + (1-\frac{i}{n})\mathbf{y}, \ \frac{i+1}{n}\mathbf{x} + (1-\frac{i+1}{n})\mathbf{y}]$. The collection of line segments, $O_{n,i}$, act as a cover for L; that is, $L = \bigcup_{i=0}^{n-1} O_{n,i}$. For a given α the covering volume of L with $\epsilon = \frac{1}{n}\|\mathbf{x}-\mathbf{y}\|$ is bounded above by

$$\sum_{i=0}^{n-1} \mathrm{V}_\alpha(\mathrm{Diam}(O_{n,i})/2) = \sum_{i=0}^{n-1} \mathrm{V}_\alpha(\|\mathbf{x}-\mathbf{y}\|/2n) = \sum_{i=0}^{n-1} \frac{2\pi^{\alpha/2}}{\alpha\Gamma(\alpha/2)} \frac{\|\mathbf{x}-\mathbf{y}\|^\alpha}{(2n)^\alpha} = \frac{2\pi^{\alpha/2}\|\mathbf{x}-\mathbf{y}\|^\alpha}{2^\alpha\alpha\Gamma(\alpha/2)} \frac{n}{n^\alpha}$$

As n increases, the covering becomes more refined and the α-dimensional volume is dominated by the term $\frac{n}{n^\alpha}$. Clearly, this term tends to zero as n tends to infinity for $\alpha > 1$. This implies $\mathcal{V}_\alpha(L) = 0$ for $\alpha > 1$. The smallest α that yields a non-zero value is $\alpha = 1$. In this case $\mathcal{V}_1(L) = \left.\frac{2\pi^{\alpha/2}\|\mathbf{x}-\mathbf{y}\|^\alpha}{2^\alpha\alpha\Gamma(\alpha/2)}\right|_{\alpha=1} = \|\mathbf{x}-\mathbf{y}\|$. Therefore, the fractal dimension of the line segment is 1 and its fractal volume is the length of the line segment, $\|\mathbf{x}-\mathbf{y}\|$. This, of course, coincides with what we would previously regard as the dimension and "volume" of a line segment.

Note that the answer would stay the same if the line segment was embedded in R^n (for any $n \in \mathcal{Z}^+$) rather than R^2.

This result is true for any "ordinary" set: its fractal dimension is its "ordinary" dimension. This means that we can think of fractal dimension as a *generalization* of dimension. It is instructive to repeat the above analysis and compute the fractal dimension and fractal volume of a simple fractal.

The Cantor Set: A Simple Fractal Set

As mentioned at the beginning of this paper, the simplest fractals to study are the ones with regular behavior; in particular, fractals with a self similarity property are more mathematically tractable. Perhaps the simplest fractal of this type is the Cantor set. It is a self-similar set formed by doing the following: Take the unit interval, [0,1], and remove the middle third, (1/3, 2/3). This leaves the intervals, [0,1/3] and [2/3,1]. Now remove the middle third of each of these two intervals, this gives four intervals. If we repeat this process indefinitely with the remaining intervals, the resulting set is called the Cantor set, C. See the appendix for a graphical depiction of the construction process . The Cantor set has the property that $3 * ([0, 1/3] \cap C) = C$; that is, it is a self-similar fractal.

The one dimensional length of the Cantor set is easy to compute, it must be one (the length of the interval $[0, 1]$) less the sum of all the lengths of the intervals removed. This is computed as $1 - \sum_{n=1}^{\infty}(2^{n-1}/3^n) = 1 - \frac{1}{2}\sum_{n=1}^{\infty}(2/3)^n = 0$. Since the Cantor set has zero length, perhaps its zero dimensional "length" is a more natural description of its size. But its zero dimensional length is equal to the number of points in the Cantor set; and, it turns out that this is infinite. What's more startling is that, in a sense that can be made precise, there are as many points in the Cantor set as there are in the interval $[0, 1]$. So, although the Cantor set has no length, it is comparable in complexity to $[0, 1]$. This suggests that it might have a natural dimension between zero and one.

The Fractal Dimension of the Cantor Set and its Volume

We compute the fractal dimension of the Cantor set. To do so, consider a refinement of coverings of the Cantor set, C_n, where C_n is the union of the set of 2^n intervals that remain after the n^{th} stage of the Cantor construction process. Each of these intervals has length $(1/3)^n$. Clearly, $C \subseteq C_n$, and $C = \bigcap\limits_{n=1}^{\infty} C_n$.

We label $I_{n,i}$ as the i^{th} interval in the set C_n. Each of these intervals has a diameter, $\epsilon_n = \left(\frac{1}{3}\right)^n$. The collection, $\bigcup\limits_{i=1}^{2^n} I_{n,i}$, provides an approximation to $\mathcal{V}_{\alpha,\epsilon_n}(C)$:

$$\mathcal{V}_{\alpha,\epsilon_n}(C) \leq \sum_{i=1}^{2^n} \mathrm{V}_\alpha(\mathrm{Diam}(I_{n,i})/2)$$

As $\mathrm{Diam}(I_{n,i}) = \left(\frac{1}{3}\right)^n$, this becomes

$$\mathcal{V}_{\alpha,\epsilon_n}(C) \leq \sum_{i=1}^{2^n} \frac{2\pi^{\alpha/2}}{\alpha\Gamma(\alpha/2)2^\alpha 3^{n\alpha}} \approx \frac{2^{1-\alpha}\pi^{\alpha/2}}{\alpha\Gamma(\alpha/2)} \sum_{i=1}^{2^n} \frac{1}{3^{n\alpha}}$$

Or,

$$\mathcal{V}_{\alpha,\epsilon_n}(C) \leq \frac{2^{1-\alpha}\pi^{\alpha/2}}{\alpha\Gamma(\alpha/2)} \left(\frac{2}{3^\alpha}\right)^n$$

If $\frac{2}{3^\alpha} < 1$ then, in the limit as $n \to \infty$ this limit will be zero. Certainly, this is the case for $\alpha \geq 1$; in which case

$$\mathcal{V}_\alpha(C) = \lim_{n\to\infty} \mathcal{V}_{\alpha,\epsilon_n}(C) \leq \lim_{n\to\infty} \frac{2^{1-\alpha}\pi^{\alpha/2}}{\alpha\Gamma(\alpha/2)} \left(\frac{2}{3^\alpha}\right)^n = 0$$

The smallest value of α that gives a non-zero $\mathcal{V}_\alpha(C)$ is α such that $\frac{2}{3^\alpha} = 1$. This occurs when $\alpha^* = \ln 2/\ln 3$. Therefore, the fractal dimension of the Cantor set is $\alpha^* = \ln 2/\ln 3 \approx 0.63093$. Its fractal volume in this dimension is $\mathcal{V}_{\alpha^*}(C) = \frac{2^{1-\alpha^*}\pi^{\alpha^*/2}}{\alpha^*\Gamma(\alpha^*/2)} \approx 1.03505$.

Conclusion

We have shown that there are sets with sufficient complexity that the usual categorization of dimension as a measure of size is lacking. We have further shown that the notion of fractal dimension, as an extension to the usual notion of dimension, provides a more discriminating view of the size of a set.

While physical examples like a coast line or the surface of a sponge do not really have complexity on an infinite scale, we idealize them as such with fractals. Just as "dimensionless" points are an idealization useful to describe space, fractals are also a useful idealization. As with all such mathematical abstractions, they allows us to view and compute what they model in a simpler way.

Appendix A: Examples of Self-Similar Fractals

We briefly describe two *self-similar* fractal sets. The first set lives in 1 dimension, but its fractal dimension is between 0 and 1... The set is called the Cantor set, named after the mathematician Georg Cantor. This set may be created by creating intermediary sets $C_0, C_1, \ldots$ and setting the Cantor set, C, to be the intersection of these sets:

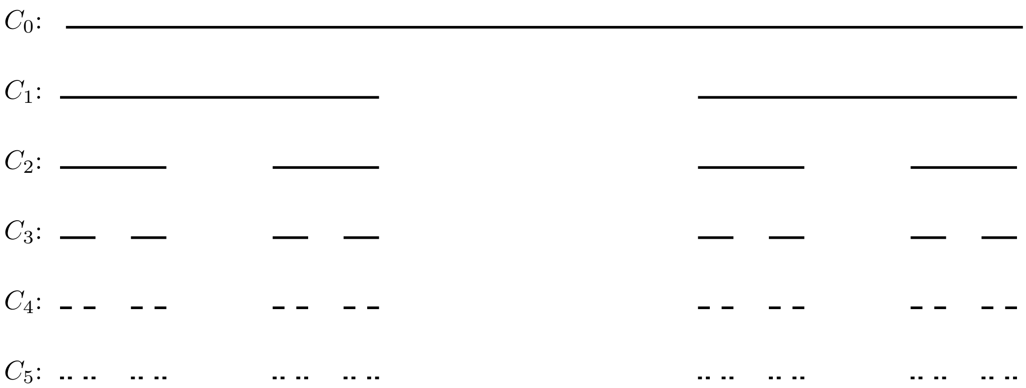
$$C = \bigcap_{n=0}^{\infty} C_n$$

The intermediate sets have the property that $\forall n, C_{n+1} \subseteq C_n$. These sets are defined by:

$$C_0 = [0, 1]$$

$C_{n+1} =$ For each contiguous segment of the set C_n, remove the open interval that is its middle third.

As the sets, C_n, are nested, each successive set is an improved approximation to the Cantor set. The first 6 intermediate sets are graphed below:



The second self-similar set lives in 2 dimensions while its fractal dimension is between 1 and 2. The set is "best" described in a recursive way as a scaled version of the lower triangular portion of itself.

Similar to the graphical display of the Cantor set construction, below we show a graph of 8 "levels" of its construction without going into construction details.