

Measure Theoretic Conditional Expectation in an Elementary Setting

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Overview

The measure theoretic approach to conditional expectation can be confusing when compared to the traditional approach – especially in a discrete setting. In what follows we go through a conditional expectation problem within a discrete and familiar setting in an attempt to reduce this confusion. In the example shown, we show explicitly that the conditional expectation function is non-measurable in the origin measure space.

Elementary Probability Example Using Measure Theory

Let $X = \{D_1, D_2, D_3, D_4, D_5, D_6\}$ and define a function P by $P(D_i) = \frac{1}{6}$, for $i \in \{1, 2, 3, 4, 5, 6\}$. The intent is that P will become a probability measure for the space we construct. Let $\mathcal{E} = 2^X$ be the sigma algebra consisting of the power set of X . We extend P for every element in the sigma algebra. Since the sigma algebra consists of all sets we need an assignment for an arbitrary set, A . The assignment is $P(A) = \frac{|A|}{6}$; that is the cardinality of the set divided by 6. We now have a measure space; in fact, a probability space: (P, X, \mathcal{E}) . Note that for a probability space we need an event space, X , a sigma algebra of sets (in the discrete case just an algebra), and a function P which takes elements of the sigma algebra to $[0,1]$ with the property that

$$P\left(\bigcup_{i=1}^N A_i\right) = \sum_{i=1}^N P(A_i) \quad \text{when } A_k \cap A_j = \emptyset \quad k \neq j$$

We now consider a random variable from which we will get a sub-sigma algebra. Let

$$g(D_i) = \begin{cases} 0 & \text{if } i \text{ is even;} \\ 1 & \text{if } i \text{ is odd} \end{cases}$$

In a discrete space the sigma algebra generated from g is the algebra of sets generated from the sets: $g^{-1}(0), g^{-1}(1), g^{-1}(a)$, for $a \neq 0, 1$. It is not too hard to see that $\mathcal{F} = \{\emptyset, \{1, 3\}, \{2, 4, 6\}, \{1, 2, 3, 4, 5, 6\}\}$.

For a discrete space a function, f , is measurable with respect to a sigma algebra if $f^{-1}(a)$ is an element in the sigma algebra for all $a \in (-\infty, \infty)$. This has implications for the sigma algebra \mathcal{F} .

The claim is that any function which is measurable over \mathcal{F} has the property that f is constant on the sets $\{1, 3\}$ and $\{2, 4, 6\}$. More generally, we claim that any measurable function f in \mathcal{F} must be constant on the *minimal* elements of the algebra – elements which have no non-trivial subsets.[†]

[†] By non-trivial subset we mean a subset that is not the empty set.

To see this suppose that $f(1)$ differs from $f(3)$. Then $\{1, 3\} \cap f^{-1}(f(1))$ is a non trivial subset of $\{1, 3\}$ as is $\{1, 3\} \cap f^{-1}(f(3))$. These two sets must differ since $f(1)$ and $f(3)$ differ. This means that they are respectively $\{1\}$ and $\{3\}$. Since \mathcal{F} is a sigma algebra and $\{1, 3\}$, $f^{-1}(f(1))$, and $f^{-1}(f(3))$ are sets in \mathcal{F} , any intersection of these sets is also in \mathcal{F} . But we have that $\{1\}$ and $\{3\}$ are sets that result from such intersections and consequently must be in \mathcal{F} . However, we know that they also are *not* in \mathcal{F} – contradiction. Therefore, our premise that $f(1)$ and $f(3)$ could take differing values is incorrect. Using the same argument one can show that f is constant on the other minimal set $\{2, 4, 6\}$.

Notice that while any function over the measure space (P, X, \mathcal{E}) is measurable, we can write down a specific function that is non-measurable with respect to \mathcal{F} . We know that all we have to do is come up with a function that differs on either of the sets $\{1, 3\}$ or $\{2, 4, 6\}$. For instance, the function: $f(D_i) = i$, for $i \in \{1, 2, 3, 4, 5, 6\}$, is a non-measurable function in \mathcal{F} .

Conditional Expectation

Given a probability space (P, X, \mathcal{E}) , the conditional expectation of a measurable function f with respect to a sub-sigma algebra \mathcal{F} is the unique \mathcal{F} measurable function (random variable) labeled, $\mathbb{E}[f/\mathcal{F}]$, such that

$$\int_{\Lambda} \mathbb{E}[f/\mathcal{F}] dP = \int_{\Lambda} f dP \quad \forall \Lambda \in \mathcal{F} \quad (1)$$

That is, $\mathbb{E}[f/\mathcal{F}]$ is a measurable function in the probability space (P, X, \mathcal{F}) . Although it seems that f itself satisfies this equation you have to be careful. The function we are looking for must be measurable with respect to \mathcal{F} , and since \mathcal{F} is a sub-sigma algebra of \mathcal{E} , it is quite possible that f is not \mathcal{F} measurable. However, if f is measurable with respect to \mathcal{F} then it is its own conditional expectation. We have the following basic facts about conditional expectation:

- $\mathbb{E}[f/\mathcal{F}] = f$ when f is measurable with respect to \mathcal{F} ;
- $\mathbb{E}[\mathbb{E}[f/\mathcal{F}]/\mathcal{F}] = \mathbb{E}[f/\mathcal{F}]$;
- $\mathbb{E}[(f - \mathbb{E}[f/\mathcal{F}])/\mathcal{F}] = 0$.

Consider the function of the last section, $f(D_i) = i$, which is measurable in the space (P, X, \mathcal{E}) . Let \mathcal{F} be the sub-sigma algebra of the last section. We now compute the conditional expectation of f with respect to \mathcal{F} . Using (1) we choose two Λ 's: $\Lambda_1 = \{1, 3\}$ and $\Lambda_2 = \{2, 4, 6\}$. We have

$$\int_{\Lambda_1} \mathbb{E}[f/\mathcal{F}] dP = \int_{\Lambda_1} f dP \quad (2)$$

and

$$\int_{\Lambda_2} \mathbb{E}[f/\mathcal{F}] dP = \int_{\Lambda_2} f dP \quad (3)$$

We know $\mathbb{E}[f/\mathcal{F}]$ is constant on Λ_1 . We label this value as C_{Λ_1} . From (2) we have

$$\begin{aligned} \int_{\Lambda_1} \mathbb{E}[f/\mathcal{F}] dP &= \int_{\Lambda_1} f dP \\ C_{\Lambda_1} \int_{\Lambda_1} dP &= \int_{\Lambda_1} f dP \\ C_{\Lambda_1} * P(\Lambda_1) &= \int_{\Lambda_1} f dP = f(D_1) * P(D_1) + f(D_3) * P(D_3) \\ C_{\Lambda_1} * P(\Lambda_1) &= 1 * \frac{1}{6} + 3 * \frac{1}{6} = \frac{2}{3} \end{aligned}$$

Since $P(\Lambda_1) = \frac{2}{6} = \frac{1}{3}$, we have that

$$C_{\Lambda_1} = 2$$

Consequently,

$$\mathbb{E}[f/\mathcal{F}](D_1) = \mathbb{E}[f/\mathcal{F}](D_3) = C_{\Lambda_1} = 2$$

Likewise, $\mathbb{E}[f/\mathcal{F}]$ is constant on the set Λ_2 . As we did above, we find the value of $\mathbb{E}[f/\mathcal{F}]$ on the set Λ_2 . Let C_{Λ_2} be the constant value of $\mathbb{E}[f/\mathcal{F}]$ on Λ_2 . We have

$$\begin{aligned} \int_{\Lambda_2} \mathbb{E}[f/\mathcal{F}] dP &= \int_{\Lambda_2} f dP \\ C_{\Lambda_2} \int_{\Lambda_2} dP &= \int_{\Lambda_2} f dP \\ C_{\Lambda_2} * P(\Lambda_2) &= \int_{\Lambda_2} f dP = f(D_2) * P(D_2) + f(D_4) * P(D_4) + f(D_6) * P(6) \\ C_{\Lambda_2} * P(\Lambda_2) &= 2 * \frac{1}{6} + 4 * \frac{1}{6} + 6 * \frac{1}{6} = 2 \end{aligned} \tag{*}$$

Since $P(\Lambda_2) = \frac{3}{6} = \frac{1}{2}$, we have that

$$C_{\Lambda_2} = 4$$

Consequently,

$$\mathbb{E}[f/\mathcal{F}](D_2) = \mathbb{E}[f/\mathcal{F}](D_4) = \mathbb{E}[f/\mathcal{F}](D_6) = C_{\Lambda_2} = 4$$

Since a function is determined once we know what its values are on every $x \in X$, we have found the conditional expectation function, $\mathbb{E}[f/\mathcal{F}]$, as we know its values on every $x \in X$.

In general, from equation (*) we see, that the value of the function (f/\mathcal{F}) evaluated on any 'x' value in a minimal set, Λ , is

$$(f/\mathcal{F})(x) = \frac{\int_{\Lambda} f dP}{\mu(\Lambda)}$$