

# Origins of Matrix/Vector and Matrix/Matrix Multiplication

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## 1 Algebraic properties of Matrices

Why is matrix/vector and matrix/matrix multiplication defined the way it is? One motivation is to try to extend the (albeit trivial) solution of linear equations in 1-dimension to  $n$  dimensions. To do this, we go through the solution of the 1-dimensional case in careful detail.

The scalar problem is:

$$a x = b \tag{1}$$

Here is a very explicit solution keeping an eye towards generalization.

$$a x = b \tag{2}$$

$$a^{-1}(a x) = a^{-1}b \quad (\text{Multiply by Inverse}) \tag{3}$$

$$(a^{-1}a)x = a^{-1}b \quad (\text{Use associativity of multiplication}) \tag{4}$$

$$1 x = a^{-1}b \quad (\text{What Inverse multiplication does}) \tag{5}$$

$$x = a^{-1}b \quad (\text{What identity multiplication does}) \tag{6}$$

The multi-dimensional problem has a lot more variables and coefficients. To start, we need to be more systematic about the naming of these coefficients. With this in mind, the multi-dimensional linear problem can be written:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

Here,  $a_{ij}$  is the coefficient in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column.

To make this look like the 1-dimensional case, we need to think of the  $b$ 's as a single unit. Our single unit will be the *vector* of the  $b$ 's.

The multi-dimensional case can be rewritten in vector terms as:

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad (7)$$

We can replace the tedious list of sums on the left hand side of the above with standard mathematical summing notation:

$$\begin{bmatrix} \sum_{j=1}^n a_{1j}x_j \\ \sum_{j=1}^n a_{2j}x_j \\ \vdots \\ \sum_{j=1}^n a_{nj}x_j \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad (8)$$

Two things need to be done: treat the  $x$ 's as a unit – as we did with the  $b$ 's – and separate the  $a$  coefficients from the  $x$ 's. This must be done formally and yet have the same meaning as the original problem formulation. This is done in the most natural of ways:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad (9)$$

Taking the rectangular collection (matrix) of  $a$ 's to be represented by  $A$ ; the (vector) collection of  $x$ 's to be  $\mathbf{x}$ ; and the collection of the  $b$ 's to be  $\mathbf{b}$ ; we may write (9) as:<sup>1</sup>

$$A\mathbf{x} = \mathbf{b} \quad (10)$$

This now looks like the scalar problem, (1). Just as one writes  $ax = b$ , interpreting the proximity of  $a$  and  $x$  to mean multiplication, we want the proximity of  $A$  and  $\mathbf{x}$  to mean matrix/vector multiplication – something we need to work out.

We proceed to try solving (10) using the solution procedure used above for the scalar case. In the process, we need to:

- Vectorize the input variable  $x$  and the outputs  $b$ . – **Done.**
- Define an, “ $a$ ”, matrix. – **Done.**
- Define matrix-vector multiplication.
- Define matrix-matrix multiplication.
- Define the identity matrix.

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<sup>1</sup>In general, we use uppercase italic letters for matrices; lowercase bold italic letters for vectors; and lowercase italic letters for scalars.

- Define the inverse matrix.

Here, in our new notation, is what the solution outline looks like for the multi-dimensional problem:

$$A\mathbf{x} = \mathbf{b} \quad (11)$$

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} \quad (\text{Matrix/vector multiply the vector equation by } A \text{ Inverse}) \quad (12)$$

$$(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b} \quad (\text{Use associativity of matrix/matrix multiplication}) \quad (13)$$

$$I\mathbf{x} = A^{-1}\mathbf{b} \quad (\text{What Inverse matrix/matrix multiplication does}) \quad (14)$$

$$\mathbf{x} = A^{-1}\mathbf{b} \quad (\text{What identity matrix/vector multiplication does}) \quad (15)$$

How do we make sense of this matrix/vector syntax on the left hand side of (11)? It should have the proper meaning; that is, (9) should have the same meaning as (8).

Examining (9) and the left hand side of (8), gives us our definition of matrix/vector multiplication:

$$[A\mathbf{x}]_i \equiv \sum_{j=1}^n A_{ij}x_j \quad (16)$$

That is,  $A$  acts on a vector,  $\mathbf{x}$ , to create a new vector,  $A\mathbf{x}$ , whose  $i^{\text{th}}$  entry is the  $i^{\text{th}}$  entry of the left hand side of (8).

Another way to write this is:

$$A\mathbf{x} = \sum_{j=1}^n x_j \mathbf{A}^j \quad (17)$$

Here,  $\mathbf{A}^j$  is the  $j^{\text{th}}$  column vector of  $A$ .

This can be seen by writing (7) as:<sup>2</sup>

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad (18)$$

From this we can see that matrix/vector multiplication of  $A$  on  $\mathbf{x}$  is a “weighted” combination of the columns of  $A$  using the elements of  $\mathbf{x}$  as the weights. Notice that if we pick an “ $\mathbf{x}$ ” with all of the weight concentrated on the  $j^{\text{th}}$  entry, then matrix/vector multiplication on this specific vector yields the  $j^{\text{th}}$  column of  $A$ ,  $\mathbf{A}^j$ . Specifically, applying  $A$  to the special vectors,  $\mathbf{e}_j$  – which are 0 everywhere except at  $j$  where they are 1 – we see that  $A\mathbf{e}_j = \mathbf{A}^j$ . Consequently, matrix/vector multiplication on this set of vectors uniquely determines  $A$  – if there is another matrix that gives the same result for these vectors with respect to matrix/vector multiplication, its columns would have to match  $A$ .

We note the following for future reference.

$$[\mathbf{A}^j]_i = A_{i,j} \quad (19)$$

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<sup>2</sup>Here, we assume the reader has a passing knowledge of how vectors are added and multiplied by scalars.

That is, the  $i^{\text{th}}$  entry of the column vector,  $\mathbf{A}^j$ , is the  $i^{\text{th}}$  row,  $j^{\text{th}}$  column of the matrix  $A$ .

Before moving on with our outline, we note one very important property of matrix/vector multiplication: It is *linear*. By that we mean the following:<sup>3</sup>

$$A(a\mathbf{x} + b\mathbf{y}) = aA\mathbf{x} + bA\mathbf{y} \quad (20)$$

One can show this from our definition of matrix/vector multiplication which we leave to the reader. What this means in practice is that for a given vector,  $\mathbf{z} = a\mathbf{x} + b\mathbf{y}$ , we can compute  $A\mathbf{z}$  by computing  $A$  on the “components” of  $\mathbf{z}$  and then multiplying by scalars and adding the resulting vectors. This can be an easier way to compute the *action* of a matrix on a vector in some cases. The linearity result above can be extended to arbitrary sums as:

$$A\left(\sum_{j=1}^n c_j \mathbf{x}_j\right) = \sum_{j=1}^n c_j A\mathbf{x}_j \quad (21)$$

Continuing with our solution outline, matrix/matrix multiplication is needed to go from equation (12) to (13). How must this be defined? Well, we need to make sense of:

$$(A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x}) \quad (22)$$

This involves the inverse of the matrix  $A$ , which we also need to define. What we are imposing on matrix multiplication is that it be associative in the specific case of a matrix and its inverse. But this is not what happens in the scalar case; associativity works not just for a special case of multiplication, but for *all* numbers. We will forgo what the inverse of a matrix might mean and focus now on imposing the condition that matrix/matrix multiplication be associative; meaning for *all* input vectors. Specifically, the condition we impose is:

$$(BA)\mathbf{x} = B(A\mathbf{x}) \quad \forall \mathbf{x} \in R^n \quad (23)$$

Notice that we are imposing what matrix/matrix multiplication is by saying how the new matrix formed,  $BA$ , acts on an *arbitrary* vector,  $\mathbf{x}$ . And we specify that action by the right hand side of (23), which involves only matrix/vector multiplication – something we have defined.

We want to emphasize that this is a strong condition to impose; meaning, that the requirement that this is true for *all*  $\mathbf{x}$  gives us a lot to work with. For instance, if one had two  $n \times n$  matrices,  $C$  and  $D$ , what could you conclude if someone told you that  $C\mathbf{x} = D\mathbf{x}$  for *some*  $n$  vector,  $\mathbf{x}$ ? Well, the answer is: not much. However, if I told you that  $C\mathbf{x} = D\mathbf{x}$  for *all*  $\mathbf{x}$ , what could you say? I claim that tells us that  $C \equiv D$ ; meaning that every entry in  $C$  is the same as the corresponding entry in  $D$ . We can see this by applying  $C$  and  $D$  to each of the  $\mathbf{e}_j$ , ( $j \in [1, n]$ ) vectors. For each  $j$ ,  $C\mathbf{e}_j$  and  $D\mathbf{e}_j$  pick off the  $j^{\text{th}}$  column of  $C$  and  $D$  respectively. So we see that each column of  $C$  and  $D$  must match; consequently,  $C \equiv D$ .

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<sup>3</sup>You might ask: “How did someone think up such a strange property?” But it is not strange, it is what one might hope for, as all we are saying is that matrix/vector multiplication works the same way that ordinary *scalar* multiplication does – it *distributes* over addition and scalar multiplication.

We apply this idea to (23) to find a formula for matrix/matrix multiplication. Since (23) must be true *for all*  $\mathbf{x}$ ; in particular, it must be true for the vectors  $\{\mathbf{e}_j\}_{j=1}^n$ . This gives us the following equations:

$$(BA)\mathbf{e}_j = B(A\mathbf{e}_j) \quad j \in [1, n] \quad (24)$$

Or,

$$(\mathbf{B}\mathbf{A})^j = \mathbf{B}\mathbf{A}^j \quad j \in [1, n] \quad (25)$$

For any given,  $i$ , the  $i^{\text{th}}$  entry of the left and right hand side is (after using the definition of matrix/vector multiplication, (16), on the right hand side)

$$[(\mathbf{B}\mathbf{A})^j]_i = \sum_{k=1}^n B_{i,k} [\mathbf{A}^j]_k \quad \forall i, j \quad i \in [1, n] \wedge j \in [1, n] \quad (26)$$

Using (19) on the left and right hand sides of the previous equation we have

$$(BA)_{i,j} = \sum_{k=1}^n B_{i,k} A_{k,j} \quad \forall i, j \quad i \in [1, n] \wedge j \in [1, n] \quad (27)$$

And we are done, we have shown what the new matrix,  $BA$ , is by showing what every entry,  $(BA)_{i,j}$ , of the new matrix is. So using the set of vectors,  $\{\mathbf{e}_j\}_{j=1}^n$ , we have completely determined what the matrix multiplication of  $BA$  is in order that matrix/matrix multiplication be associative when using these vectors. Does this mean that associativity works for *arbitrary* vectors? The answer is yes and this is because of the linearity of matrix/vector multiplication. Any given  $\mathbf{x}$  can be written as a *linear combination*<sup>4</sup> of the vectors  $\{\mathbf{e}_j\}_{j=1}^n$ :  $\mathbf{x} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_n\mathbf{e}_n$ . Matrix multiply the left and right hand sides by the matrix,  $BA$ . This gives

$$\begin{aligned} (BA)\mathbf{x} &= (BA) \left( \sum_{j=1}^n a_j \mathbf{e}_j \right) && \text{Replace } \mathbf{x} \text{ with its components.} \\ &= \sum_{j=1}^n a_j (BA)\mathbf{e}_j && \text{Linearity of matrix/vector multiplication by } (BA). \\ &= \sum_{j=1}^n a_j B(A\mathbf{e}_j) && \text{Associativity of matrix/matrix multiplication on } \mathbf{e}_j. \\ &= B \left( \sum_{j=1}^n a_j A\mathbf{e}_j \right) && \text{Linearity of matrix/vector multiplication by } B. \\ &= B \left( A \left( \sum_{j=1}^n a_j \mathbf{e}_j \right) \right) && \text{Linearity of matrix/vector multiplication by } A. \\ &= B(A\mathbf{x}) && \text{Replace components with } \mathbf{x}. \end{aligned} \quad (28)$$

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<sup>4</sup>Again, we assume the reader is familiar with expanding a vector in terms of the standard basis vectors.

In the above calculations, we applied our new matrix,  $(BA)$  to an arbitrary  $\mathbf{x}$ , and used the *linearity* of matrix/vector multiplication and the *associativity* of matrix/matrix multiplication on the components,  $\{\mathbf{e}_j\}_{j=1}^n$ , to show that matrix/matrix multiplication, as we have defined it, is associative – *irrespective* of the input vector.

The next order of business is to identify an  $n \times n$  matrix which serves as an identity (in terms of matrix/vector multiplication). Let us suppose that we have such a matrix and let's call it  $I$ . Then we must have  $I\mathbf{x} = \mathbf{x} \quad \forall \mathbf{x} \in R^n$ . Then, in particular,  $I\mathbf{e}_j = \mathbf{e}_j \quad \forall j \in [1, n]$ . However, we know that  $I\mathbf{e}_j = \mathbf{I}^j$ . Therefore,  $I$  must have the property that  $\mathbf{I}^j = \mathbf{e}_j$ . Consequently,  $I$  must have the form:

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (29)$$

We have shown that the only candidate for the identity matrix with respect to matrix/vector multiplication is the matrix  $I$ . That is, if there is an identity matrix, it must be the matrix of equation (29). Does it satisfy the property of being an identity matrix (again, in the matrix/vector multiplication world)? That is, do we have  $I\mathbf{x} = \mathbf{x}$  for all  $\mathbf{x} \in R^n$ ? As before, we can express  $\mathbf{x}$  as:  $\mathbf{x} = \sum_{j=1}^n c_j \mathbf{e}_j$ . By the *linearity* of matrix/vector multiplication, we have

$$\begin{aligned} I\mathbf{x} &= I \left( \sum_{j=1}^n c_j \mathbf{e}_j \right) && \text{Replace } \mathbf{x} \text{ with components.} \\ &= \sum_{j=1}^n c_j I\mathbf{e}_j && \text{Linearity of matrix/vector multiplication.} \\ &= \sum_{j=1}^n c_j \mathbf{e}_j && \text{Identity of } I \text{ on components.} \\ &= \mathbf{x} && \text{Replace components with } \mathbf{x}. \end{aligned} \quad (30)$$

One can show that this identity matrix,  $I$ , is also the identity operator for matrix/matrix multiplication.

The only thing left is to know when a matrix inverse exists and how to compute it. We do not attempt to do this in this paper. In the next section we continue with a qualitative comparison of the solution to the scalar problem,  $ax = b$ , and its vectorized cousin.

## 2 Qualitative Features of Solutions

We can view the multiplication of two numbers,  $a$  and  $x$ , as just that. Or, we can think of  $a$  being fixed and letting  $x$  “run-through” all numbers. Here we see two cases: if  $a \neq 0$ , then letting  $x$  run through all of the numbers in  $R$  will produce all of the numbers in  $R$ . We could think of  $a$  as an “operator” and call the set of

all possible outputs, the range of  $a$  and denote it:  $\mathcal{R}(a)$ . There is another case,  $a$  could be zero. In this case it's range is the set  $\{0\}$ . In the first case, with non-zero  $a$  it is clear that we can find an  $x$  to “hit” a given value  $b$ . That is, we can solve  $ax = b$ .

One can define the same concept for a matrix,  $A$ . Using this language of ranges, here is what we can say about the scalar problem:  $ax = b$ .

**Unique Solution:** If  $a^{-1}$  exists ( $a \neq 0$ ),  $b$  is any number (that is:  $b \in \mathcal{R}(a)$ ) then there is a **unique** solution.

**No Solution:** If  $a^{-1}$  does not exist (i.e.,  $a = 0$ ) *AND*  $b$  is **not** in the range of  $a$  (that is:  $b \neq 0$ ), then there is **no** solution.

**Infinite Solutions:** If  $a^{-1}$  does not exist (i.e.,  $a = 0$ ) *BUT*  $b$  is in the range of  $a$  (that is:  $b = 0$ ), then there are an **infinite** number of solutions.

Here is the analog of this solution categorization for the multi-dimensional case:  $A\mathbf{x} = \mathbf{b}$ .

**Unique Solution:** If  $A^{-1}$  exists,  $\mathbf{b}$  is in the range of  $A$  ( $\mathbf{b} \in \mathcal{R}(A)$ ) then there is a **unique** solution.

**No Solution:** If  $A^{-1}$  does not exist *AND*  $\mathbf{b}$  is not in the range of  $A$  (that is  $\mathbf{b} \notin \mathcal{R}(A)$ ), then there is **no** solution.

**Infinite Solutions:** If  $A^{-1}$  does not exist *BUT*  $\mathbf{b}$  is in the range of  $A$  (that is:  $\mathbf{b} \in \mathcal{R}(A)$ ), then there are an **infinite** number of solutions.