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Automorphisms of complexes of curves on punctured spheres and on punctured tori

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Abstract

Let S be either a sphere with $\geqslant 5$ punctures or a torus with $\geqslant 3$ punctures. We prove that the automorphism group of the complex of curves of S is isomorphic to the extended mapping class group $\mathcal{M}_S^*$. As applications we prove that surfaces of genus $\leqslant 1$ are determined by their complexes of curves, and any isomorphism between two subgroups of $\mathcal{M}_S^*$ of finite index is the restriction of an inner automorphism of $\mathcal{M}_S^*$. We conclude that the outer automorphism group of a finite index subgroup of $\mathcal{M}_S^*$ is finite, extending the fact that the outer automorphism group of $\mathcal{M}_S^*$ is finite. For surfaces of genus $\geqslant 2$, corresponding results were proved by Ivanov (IHES/M/89/60, Preprint).

Keywords: Mapping class group; Complex of curves; Surface; Teichmüller space

AMS classification: 57M99; 20F38; 30F60; 32G15

0. Introduction and statement of results

Let S be a connected orientable surface of genus g with b boundary components and with n punctures. The *complex of curves* $C(S)$, first introduced by Harvey [5], is an abstract simplicial complex whose vertices are the isotopy classes of unoriented nontrivial simple closed curves. By definition, a simple closed curve is nontrivial if it bounds neither a disc nor an annulus together with a boundary component, nor a disc with one puncture on S. A set of vertices $\{\alpha_0, \alpha_1, \dots, \alpha_q\}$ forms a q-simplex if and only if $\alpha_0, \alpha_1, \dots, \alpha_q$ have pairwise disjoint representatives. Clearly, the complex of curves of a surface of genus g with b boundary components and with n punctures, and that of a surface of genus g with

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$n + b$ punctures are isomorphic. Therefore, we consider only punctured surfaces. We will usually think of the punctures as distinguished points on a closed surface of genus g.

If $F : S \to S$ is a diffeomorphism of S and a is a nontrivial simple closed curve, then $F(a)$ is also a nontrivial simple closed curve. If F is isotopic to G and a is isotopic to b, then $F(a)$ is isotopic to $G(b)$. It follows that the *extended mapping class group* $\mathcal{M}_S^*$ of S acts on $C(S)$ as automorphisms. That is, there is a natural group homomorphism $\mathcal{M}_S^* \to \operatorname{Aut} C(S)$. By definition, $\mathcal{M}_S^*$ is the group of the isotopy classes of diffeomorphisms $S \to S$. Note that we include the isotopy classes of orientation-reversing diffeomorphisms into $\mathcal{M}_S^*$. The *mapping class group* $\mathcal{M}_S$ is the subgroup of $\mathcal{M}_S^*$ consisting of isotopy classes of orientation-preserving diffeomorphisms of S. Notice that the index of $\mathcal{M}_S$ in $\mathcal{M}_S^*$ is two.

In [9], Ivanov proves that if the genus of the surface S is at least two, then the natural homomorphism $\mathcal{M}_S^* \to \operatorname{Aut} C(S)$ is onto. It follows that if S is not a closed surface of genus two, then, in fact, it is an isomorphism. We prove that this is also true for the surfaces of genus zero and one. More precisely, we prove the following:

Theorem 1. *Let S be a sphere with at least five punctures, or a torus with at least three punctures. Then the group of automorphisms of the complex of curves $C(S)$ is isomorphic to the extended mapping class group $\mathcal{M}_S^*$.*

If S is a sphere with at most three punctures, then there are no nontrivial circles on S. Hence $C(S)$ is empty and the conclusion of Theorem 1 is vacuous.

If S is a sphere with four punctures, or a torus with at most one puncture, then $C(S)$ is infinite discrete and the group of automorphisms $\operatorname{Aut} C(S)$ of $C(S)$ is the infinite symmetric group. In this case, $\operatorname{Aut} C(S)$ is not isomorphic to $\mathcal{M}_S^*$, because, for instance, for any two distinct pairs of vertices of $C(S)$, there is an element in $\operatorname{Aut} C(S)$ interchanging them while this is not possible in $\mathcal{M}_S^*$.

The case that S is a torus with two punctures is still open.

Another result we prove is the following.

Theorem 2. *Let S be a sphere with at least five punctures, or a torus with at least three punctures. Let S' be connected orientable surface of genus at most one. In the case that S is a sphere with five punctures, suppose, in addition, that S' is not a torus with two punctures. If $C(S)$ and $C(S')$ are isomorphic, then S and S' are diffeomorphic.*

If S is a sphere with at most three punctures, then as mentioned above $C(S)$ is empty.

The complex of curves of a sphere with four punctures, that of a closed torus, and that of a torus with one puncture are isomorphic, but certainly these surfaces are not diffeomorphic.

It is not known whether the complexes of curves of a sphere with five punctures and of a torus with two punctures are isomorphic.

In [9], Ivanov sketches the proof of the fact that every automorphism of the complex of curves $C(S)$ is induced by some diffeomorphism of S if S is a surface of genus at least two.

As an application, he states that any isomorphism between two finite index subgroups of the group $\mathcal{M}_S^*$ is the restriction of an inner automorphism $\mathcal{M}_S^*$ if S is not a closed surface of genus two. This gives, in particular, that the group of outer automorphisms of a finite index subgroup is finite, extending the fact that the outer automorphisms of $\mathcal{M}_S^*$ is finite [8,12]. We prove the corresponding theorem for the surfaces of genus zero and one.

Theorem 3. *Let S be a sphere with at least five punctures or a torus with at least three punctures. Let G_1 and G_2 be two subgroups of $\mathcal{M}_S^*$ of finite index. Then any isomorphism $G_1 \to G_2$ is induced by some inner automorphism of $\mathcal{M}_S^*$. In particular, two subgroups of $\mathcal{M}_S^*$ of finite index are isomorphic if and only if they are conjugate. Also, if G is a subgroup of $\mathcal{M}_S^*$ of finite index, then the outer automorphism group* $\operatorname{Out} G$ *of G is finite.*

Another application of Theorem 1 is the following. Theorem 1 allows us to extend Ivanov's geometric proof of Royden–Earl–Kra theorem [9] to spheres with at least five punctures and to tori with at least three punctures. Royden–Earl–Kra theorem asserts that all isometries of the Teichmüller space belongs to $\mathcal{M}_S^*$ [13,3]. Recall that Teichmüller space of a surface S is the space of all hyperbolic metrics divided out by the action of the diffeomorphisms of S which are isotopic to the identity, together with the Teichmüller metric.

Here is an outline of the paper. Section 1 discusses the necessary definitions used in the paper and fixes the notations we use. Some facts on the complexes of curves are also reviewed. In Section 2, we prove that the natural homomorphism $\mathcal{M}_S^* \to \operatorname{Aut} C(S)$ is one-to-one except for a few exceptions. Section 3 proves Theorem 1 for the surfaces of genus zero. Section 4 is devoted to the proofs of Theorem 1 for surfaces of genus one and of Theorem 2. In Section 5, we give a proof of Theorem 3. This proof is outlined in [9].

1. Preliminaries and notations

Let S be a connected orientable surface of genus g with n punctures. A *circle* is a simple closed curve. Circles will be denoted by lower case letters a, b, c, and their isotopy classes by α, β, γ. An embedded arc connecting punctures will be denoted by a', b', c', and their isotopy classes by α', β', γ'.

The *geometric intersection number* $i(\alpha, \beta)$ of two isotopy classes α and β is defined to be the infimum of the cardinality of $a \cap b$ with $a \in \alpha,\ b \in \beta$. From this it follows easily that two distinct vertices $\alpha, \beta \in C(S)$ are joined by an edge if and only if their geometric intersection number is zero. Therefore a set of vertices $\{\alpha_0, \alpha_1, \ldots, \alpha_q\}$ of $C(S)$ form a simplex if and only if $i(\alpha_i, \alpha_j) = 0$ for all $0 \leqslant i, j \leqslant q$.

The geometric intersection numbers $i(\alpha, \beta')$ and $i(\alpha', \beta')$ are defined similarly.

If the Euler characteristic $\chi(S)$ of S is negative, then the surface can be equipped with a complete hyperbolic metric. When S is given with such a metric, the isotopy class of any nontrivial circle contains a unique simple closed geodesic, and the geometric intersection number of two distinct classes is realized by the geodesics in their classes. In this case we have an alternative definition of $C(S)$; the vertices are simple closed geodesics on S and a set of vertices form a simplex if and only if they are pairwise disjoint. This definition is independent of the choice of the metric in the sense that different metrics give rise to isomorphic complexes.

We recall that the maximum number of disjoint pairwise nonisotopic nontrivial circles on S is $3g-3+n$ with some exceptions. The exceptions are spheres with at most three punctures, in which case there is no nontrivial circle, and closed tori, on which this number is one. Thus the dimension of $C(S)$ is $3g-4+n$ if S is not one of the exceptional surfaces above. If S is a sphere with at most three punctures, then $C(S)=\phi$, and if S is a closed torus, then $C(S)$ is infinite discrete, i.e., the dimension of $C(S)$ is zero.

It is known that $C(S)$ is connected whenever its dimension is greater than zero. In fact more is true about the connectedness of $C(S)$ [7,6].

The representatives of the vertices will always be circles and if there are more than one representative under consideration, all representatives will be assumed intersecting each other minimally.

For a circle a, we denote by S_a the surface obtained from S by cutting along a.

Let α be a vertex of $C(S)$ and let $a \in \alpha$. If the surface S_a is connected, then we say that the circle a and the vertex α are *nonseparating*. Similarly, if the surface S_a is not connected, then we say that the circle a and the vertex α are *separating*. If a is separating and if one of the components of S_a is a disc with k punctures, then we call a and α k-*separating*. Therefore, on a sphere with n punctures, a k-separating circle is also $(n-k)$-separating.

Two circles a and b are said to be *topologically equivalent* if there exists a homeomorphism F of S such that $F(a)=b$. If a is topologically equivalent to b and if α and β are their isotopy classes, then we will also say that α and β are *topologically equivalent*.

Let K be an abstract simplicial complex and L a subcomplex of it. L is said to be a *full subcomplex* if, whenever a set of vertices of L is a simplex in K, it is also a simplex in L.

For a vertex α of $C(S)$, we define the link $L(\alpha)$ of α to be the full subcomplex of $C(S)$ whose vertices are those of $C(S)$ which are joined to α by an edge in $C(S)$, i.e., β is a vertex of $L(\alpha)$ if and only if $i(\alpha,\beta)=0$ and $\alpha \neq \beta$. The 'dual' link $L^d(\alpha)$ of α is the graph whose vertices are those of $L(\alpha)$; two vertices β and γ of $L^d(\alpha)$ are joined by an (unoriented) edge if and only if they are not joined by an edge in $C(S)$ (or in $L(\alpha)$), i.e., $i(\beta,\gamma)$ is nonzero.

Let α_1, α_2, α_3, α_4 and α_5 be five distinct vertices of $C(S)$. We will say that $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$ is a *pentagon* in $C(S)$ if $i(\alpha_j,\alpha_{j+1})=0$ for $j=1,2,3,4,5$ and if $i(\alpha_j,\alpha_k)\neq 0$, otherwise $(\alpha_6=\alpha_1)$. This representation is well-defined up to cyclic permutations.

Suppose now that $n \geqslant 1$. We define an abstract simplicial complex $B(S)$ as follows. The vertices of $B(S)$ are the isotopy classes of nontrivial embedded arcs on S joining punctures. By definition, an arc is nontrivial if it cannot be deformed to a puncture. A set of vertices of $B(S)$ forms a simplex if and only if the vertices in the set have representatives which are pairwise disjoint. A simple Euler characteristic argument shows that $\dim B(S)$

is $-3\chi(S) = 6g - 6 + 3n$, and all maximal simplices have the same dimension. As in the case of $C(S)$, a set of vertices $\{\alpha'_0, \alpha'_1, \ldots, \alpha'_q\}$ form a simplex in $B(S)$ if and only if $i(\alpha'_i, \alpha'_j) = 0$ for all $0 \leqslant i, j \leqslant q$.

The following lemma is proved in [4, Expose 2, III], and it will be used throughout this work.

Lemma 1.1. *Let S be a sphere with three punctures. Then*
- (i) *up to isotopy, there exists a unique nontrivial embedded arc joining a puncture P to itself, or P to another puncture Q,*
- (ii) *all circles on S are trivial.*

Suppose that S is either a sphere with at least five punctures, or a surface of positive genus with at least two punctures. If α is a 2-separating vertex of $C(S)$ and $a \in \alpha$, then exactly one of two components of the surface S_a is a disc with two punctures. Any two embedded arcs connecting two punctures on this disc are isotopic. In this way, for each such α we have a vertex α' in the complex $B(S)$. Conversely, any vertex α' of $B(S)$ connecting a puncture to another determines uniquely a 2-separating vertex α of $C(S)$, namely the isotopy class of the boundary component of a regular neighborhood of any embedded representative of α'. Consequently, there is a one-to-one correspondence between 2-separating vertices of $C(S)$ and the isotopy classes of embedded arcs joining different punctures.

Let α and β be two 2-separating vertices of $C(S)$ and let α' and β' be the corresponding vertices in $B(S)$. Suppose that there exist $a' \in \alpha'$ and $b' \in \beta'$ such that a' and b' are disjoint, and they have exactly one common endpoint. Then we say that α and β constitute a *simple pair*, and we denote it by $\langle\alpha; \beta\rangle$. Similarly, we call $\langle a; b\rangle$, $\langle\alpha'; \beta'\rangle$ and $\langle a'; b'\rangle$ *simple pairs*, where $a \in \alpha$, $b \in \beta$.

Let $P_0, P_1, \ldots, P_k$ be distinct punctures on S and let a'_i be an embedded arc connecting P_{i-1} to P_i such that $\langle a'_{i-1}; a'_i\rangle$ is a simple pair and a'_i does not intersect a'_j for $i \neq j$. Then we say that $\langle a'_0; a'_1; \ldots; a'_k\rangle$ is a *chain*. If a_i is a circle corresponding to a'_i and if α'_i and α_i are the isotopy classes of a'_i and a_i, then we also call $\langle a_0; a_1; \ldots; a_k\rangle$, $\langle\alpha'_0; \alpha'_1; \ldots; \alpha'_k\rangle$ and $\langle\alpha_0; \alpha_1; \ldots; \alpha_k\rangle$ *chains*.

Next theorem is a special case of the corollary in [6].

Theorem 1.2. *Let S be a sphere with at least five punctures, or a torus with at least two punctures, or a surface of genus at least two. Then given any two codimension-*

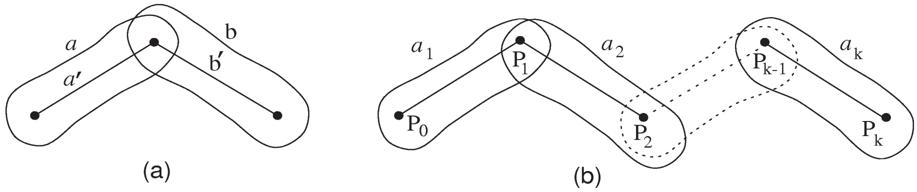

Fig. 1.

zero simplices σ and σ' in the complex of curves $C(S)$, there exists a sequence $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_k = \sigma'$ of codimension-zero simplices such that $\sigma_i \cap \sigma_{i+1}$ is a codimension-one simplex for each i.

2. Injectivity of $\mathcal{M}_S^* \to \operatorname{Aut} C(S)$

The purpose of this section is to prove that the natural homomorphism $\mathcal{M}_S^* \to \operatorname{Aut} C(S)$ is injective but for a few exceptions. The exceptional cases are a sphere with at most four punctures, a torus with at most two punctures and a closed surface of genus two. This result is well known, but it seems that a written proof does not exist in the literature. The proof basically relies on the fact that the mapping class groups are centerless. In fact, if S is a sphere with at most three punctures, there is no such homomorphism since $C(S) = \emptyset$. The kernel of the homomorphism $\mathcal{M}_S \to \operatorname{Aut} C(S)$ is $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ if S is a sphere with four punctures and is $\mathbb{Z}_2$ in the other exceptional cases [10]. In particular, the homomorphism $\mathcal{M}_S^* \to \operatorname{Aut} C(S)$ is not injective.

Let us orient S arbitrarily. Let a be a 2-separating circle on S with the isotopy class α. Denote by D the twice-punctured disc component of S_a. By interchanging two punctures on D leaving a neighborhood of the boundary component pointwise fixed, we get a diffeomorphism of D. Extension of this diffeomorphism to $S \backslash D$ by the identity gives a diffeomorphism of S. Let us denote by $t_\alpha^{1/2}$ the isotopy class of this diffeomorphism. Note that we can choose the twisting of the punctures on D so that $(t_\alpha^{1/2})^2 = t_\alpha$, where t_α is the right Dehn twist about α. We call $t_\alpha^{1/2}$ a *half-twist* about α. It is clear from the definition that, for a mapping class f, if f is orientation-preserving then $f\, t_\alpha^{1/2}\, f^{-1} = t_{f(\alpha)}^{1/2}$, and if f is orientation-reversing then $f\, t_\alpha^{1/2}\, f^{-1} = (t_{f(\alpha)}^{1/2})^{-1}$.

Let S be a connected oriented surface. Suppose that S is not a sphere with at most four punctures, or a torus with at most two punctures, or a closed surface of genus two. First notice that if f is in the kernel of the map $\mathcal{M}_S^* \to \operatorname{Aut} C(S)$ and if f is orientation-preserving, then it commutes with all half-twists and twists. Therefore, f is in $C(\mathcal{M}_S) = \{1\}$ (by [10]), the center of $\mathcal{M}_S$, since the group $\mathcal{M}_S$ is generated by half-twists and twists. Hence, it remains to show that if f is in the kernel, it is orientation-preserving. Blow up the punctures on S into boundary components and denote the resulting surface by R. Consider a pair of pants decomposition $\{c_1, \ldots, c_k\}$ of R as in [4] such that each c_i connects two different pants. Then it follows that f has to send each pant to itself and fix the boundaries of all pants. But then it is simple to conclude that f must be the identity on each pant P (if it is not orientation-preserving, it has to induce the map $x \to x^{-1}$, $y \to y^{-1}$ on $\pi_1(P) = \langle x, y\rangle$, but this is not a homomorphism). Therefore f is orientation-preserving, and we are done.

3. Punctured spheres

In this section we prove Theorem 1 for punctured spheres. Throughout this section unless otherwise stated, S will denote a sphere with $n \geqslant 5$ punctures.

As the first step, we prove that certain pairs of vertices of $C(S)$ can be recognized in the complex of curves. Second, we show that automorphisms of $C(S)$ preserve the topological type of the vertices of $C(S)$. This allows us to conclude that automorphisms of $C(S)$ preserve these pairs of vertices. Next, we show that every automorphism of $C(S)$ induces an automorphism of the complex $B(S)$ in a natural way. The automorphisms of the complex $B(S)$ are completely determined by their action on a single codimension-zero simplex. Finally, using the close relation between codimension-zero simplices of $B(S)$ and isotopy classes of ideal triangulations of S, it is shown that an automorphism of $B(S)$ induced by some automorphism of $C(S)$ agrees with a mapping class.

3.1. Characterization of simple pairs

Since any diffeomorphism of S takes a simple pair of circles to a simple pair of circles, any automorphism of $C(S)$ must take a simple pair of vertices to a simple pair in order to expect that it is induced by some diffeomorphism of S. In this section, we show that simple pairs can be recognized in the complex $C(S)$ and automorphisms of $C(S)$ preserve simple pairs.

Lemma 3.1. *Let σ be a codimension-zero simplex of $C(S)$. Then at least two vertices of σ are* 2*-separating.*

Proof. If S is a sphere with five punctures, then since $\dim C(S) = 1$ (hence card $\sigma = 2$) and since every nontrivial circle on S is 2-separating, we are done.

Let $n \geqslant 6$. Note that every codimension-zero simplex contains at least one 2-separating vertex. Pick a 2-separating vertex $\alpha \in \sigma$ and choose $a \in \alpha$. Then the complex $C(S_a)$ is isomorphic to the complex of curves of a sphere with $n-1$ punctures and $\sigma - \{\alpha\}$ is a codimension-zero simplex in $C(S_a)$. By induction, $\sigma - \{\alpha\}$ contains at least two 2-separating vertices on S_a and at least one of them is 2-separating on S.

The proof is complete. □

Theorem 3.2. *Let α and β be two* 2*-separating vertices of $C(S)$. Then $\langle\alpha;\beta\rangle$ is a simple pair if and only if there exist vertices $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{n-2}$ of $C(S)$ satisfying the following conditions.*

(i) *$(\gamma_1, \gamma_2, \alpha, \gamma_3, \beta)$ is a pentagon in $C(S)$,*

(ii) *γ_1 and γ_{n-2} are* 2*-separating, γ_2 is* 3*-separating, and γ_k and γ_{n-k} are k-separating for $3 \leqslant k \leqslant n/2$,*

(iii) *$\{\alpha,\gamma_3\} \cup \sigma$, $\{\alpha,\gamma_2\} \cup \sigma$, $\{\beta,\gamma_3\} \cup \sigma$ and $\{\gamma_1,\gamma_2\} \cup \sigma$ are codimension-zero simplices, where $\sigma = \{\gamma_4, \gamma_5, \ldots, \gamma_{n-2}\}$.*

Proof. The ‘only if’ part of the proof is very easy. Let $a \in \alpha$ and $b \in \beta$. Then $\langle a;b\rangle$ is a simple pair. It is clear that any two simple pairs of circles are topologically equivalent, i.e., if $\langle c;d\rangle$ is any other simple pair, then there exists a diffeomorphism $F : S \to S$ such that $\langle F(c); F(d)\rangle = \langle a;b\rangle$. Hence we can assume that a and b are the circles illustrated

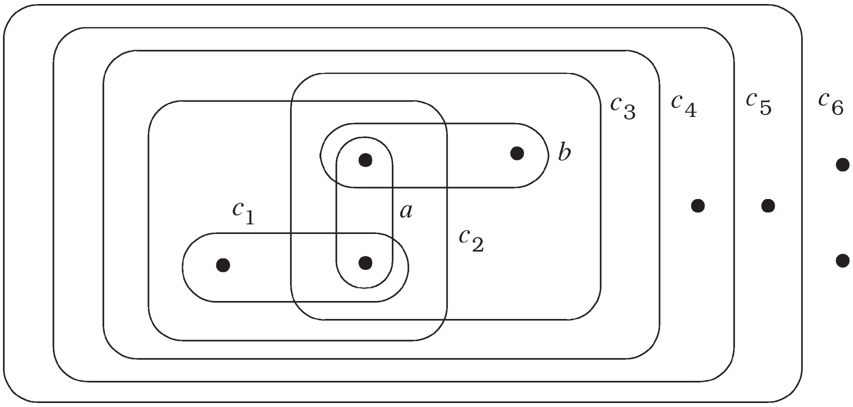

Fig. 2.

in Fig. 2. The figure represents the case $n = 8$. In the figure, we think of the sphere as the one point compactification of the plane. Then the isotopy classes γ_i of the circles c_i satisfy (i)–(iii).

We now prove the converse. Assume that conditions (i)–(iii) above hold. For a k-separating circle c on S with $2 \leqslant k \leqslant n/2$, let us denote by S'_c and S''_c the connected components of S_c having k and $n-k$ punctures, respectively. In the case of $k = n/2$, either of them may represent either component. Let a, b and c_i be the representatives of α, β and γ_i, respectively, intersecting each other minimally.

We claim that $a \cup b$ lies on a thrice-punctured disc bounded by c_3. By (i), a and b intersect transversally at least once, because the number of points in the intersection $a \cap b$ is the geometric intersection number $i(\alpha, \beta)$ of α and β, which is nonzero. Also $a \cup b$ does not intersect c_3. Hence they lie on the same component of S_{c_3}. For $n = 5$ or $n = 6$, since c_3 is 3-separating, each component of S_{c_3} is either a disc with two punctures or a disc with three punctures. Since there is no nontrivial circle on a disc with two punctures, the claim is obvious. Hence we assume that $n \geqslant 7$. Let

$$C = c_3 \cup c_4 \cup \cdots \cup c_{n-3}$$

and consider the surface S_C.

We first prove that S_C is a union of two discs with three punctures, whose boundaries are c_3 and c_{n-3}, and a number of annuli with one puncture. Since every circle on S is separating, the number of components of S_C is $n-4$. Also, for any circle d on S, $\chi(S) = \chi(S'_d) + \chi(S''_d)$. From this it follows that

$$2 - n = \chi(S) = \chi(S_C) = \sum \chi(R),$$

where R runs over the components of S_C. Since all c_i are nonisotopic, $\chi(R)$ is negative for all R. Hence either there is only one R with $\chi(R) = -3$, or there are precisely two components with $\chi(R) = -2$, and the rest of the components have Euler characteristic -1. If there exists a component R with $\chi(R) = -3$, then R is a disc with a number of holes and punctures. The total number of holes and punctures on the disc R is four. We then can find two nonisotopic circles d_1 and d_2 on R such that at most one of d_1 and d_2 is 2-separating on S. Then $\{\delta_1, \delta_2, \gamma_3, \gamma_4, \ldots, \gamma_{n-3}\}$ is a maximal simplex of $C(S)$ containing at most one 2-separating vertex, where δ_i is the class of d_i for $i = 1, 2$. This is a contradiction to Lemma 3.1. Thus S_C has two components with Euler characteristic -2, namely S'_{c_3} and $S'_{c_{n-3}}$.

Since $a \cap b$ is nonempty and since the sets $\{\alpha, \gamma_3, \ldots, \gamma_{n-3}\}$ and $\{\beta, \gamma_3, \ldots, \gamma_{n-3}\}$ are two simplices of $C(S)$, $a \cup b$ lies either on S'_{c_3} or on $S'_{c_{n-3}}$, both of which are discs with three punctures. Now let ∂ be the boundary of the thrice-punctured disc on which a and b lie. Since c_2 is a 3-separating circle intersecting b and is not isotopic to ∂, it follows that c_2 and ∂ intersect nontrivially. As $i(\gamma_2, \gamma_{n-3}) = 0$, we must have $\partial = c_3$. This proves the claim.

Condition (iii) implies that circles a, b, c_1, c_2 and c_3 all lie on $S_{C'}$, where $C' = c_4 \cup c_5 \cup \cdots \cup c_{n-2}$. (If $n = 5$ then c_2 and c_3 are 2-separating circles disjoint from a. Hence the corresponding arcs c'_2 and c'_3 have a common endpoint. In this case, we take C' to be a trivial simple closed curve which can be deformed to this puncture.) By arguing as above, one can see that the surface $S_{C'}$ is the disjoint union of a number of surfaces of Euler characteristic -1 and a disc D with four punctures, with boundary c_4, after changing the roles of c_4 and c_{n-4}, if necessary.

Keeping the correspondence between 2-separating vertices and the arcs in mind, suppose that the endpoints of a' are P and Q. Then c_3 separates P, Q and another puncture, say, R from the forth puncture T on D. Up to a diffeomorphism of S, the picture of a', c_3 and c_4 are as illustrated Fig. 3(a).

Since $i(\alpha, \gamma_2) = 0$ and $i(\gamma_2, \gamma_3) \neq 0$, each component of $c_2 \cap S'_{c_3}$ is an arc connecting two points on c_3, and isotopic to each other by an isotopy of S'_{c_3} leaving the endpoints of the arc on c_3 by Lemma 1.1. (Here S'_{c_3} denotes the thrice-punctured disc component of S which does not contain the circle c_4.) Let λ be one of these arcs and let $\lambda \cap c_3 = \{X, Y\}$.

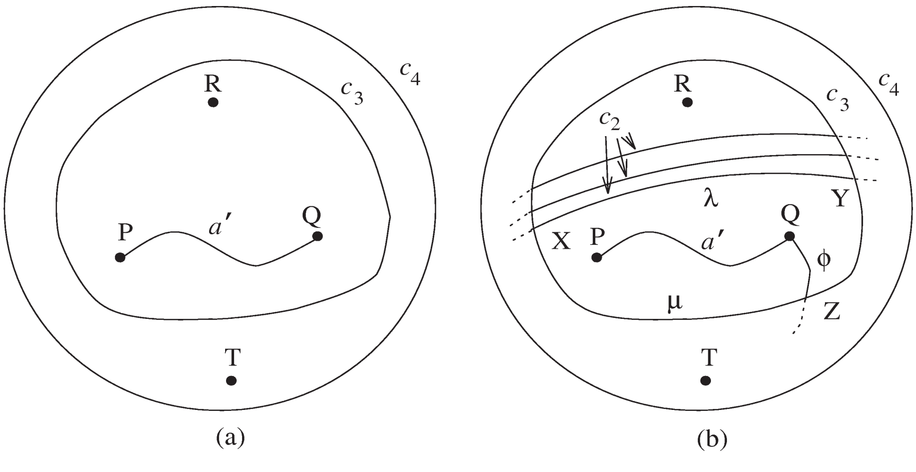

Fig. 3.

Then a component μ of $c_3 - \{X, Y\}$ and λ bound a disc D' with two punctures P and Q. Note that P and Q are on S'_{c_2}, which is a thrice-punctured disc.

Since b' is on S'_{c_3}, its endpoints are among P, Q and R. So one of them must be P, by changing the roles of P and Q if necessary. If the other endpoint of b' is Q, then the endpoints of c'_1 must be T and R since c_1 is disjoint from b. This and $c_1 \cap c_2 = \emptyset$ imply that R and T are on S'_{c_2}, too, since at least one of R and T is on S'_{c_2}. This is a contradiction because the four punctures P, Q, R, T cannot be on S'_{c_2} all together. By this contradiction the endpoints of b' are P and R, and that of c'_1 are Q and T.

Since $c_1 \cap c_2 = \emptyset$, c'_1 does not intersect λ and hence it intersects μ. Let Z be the first point where c'_1 meets μ starting from Q. Denote the segment of c'_1 between Q and Z by ϕ. Up to an isotopy of D' leaving the endpoint at Q fixed and keeping the other endpoint on $\lambda \cup \mu$, such an arc is unique by Lemma 1.1. So ϕ can be chosen so that it does not intersect a'.

Finally, by cutting S'_{c_3} along a regular neighborhood of $\phi \cup \{Q\}$ we get a disc with two punctures P and R. The arc b' must lie on this disc since it meets neither c'_1 nor c_3. Again, up to an isotopy of this disc there is only one arc joining the punctures P and R, which can be chosen disjoint from a'.

The proof of the theorem is now complete. □

Theorem 3.3. *The group* $\operatorname{Aut} C(S)$ *preserves the topological type of the vertices of* $C(S)$.

Proof. Notice that all vertices of $C(S)$ are separating. For a vertex α of $C(S)$, the dual link $L^d(\alpha)$ is connected if and only if α is 2-separating. Recall that $L^d(\alpha)$ is a graph whose vertices are those β such that $\beta \neq \alpha$ and $i(\alpha, \beta) = 0$. Two vertices are connected by an edge in $L^d(\alpha)$ if and only if they are not connected in $C(S)$. Therefore, each automorphism of $C(S)$ permutes the set of 2-separating vertices.

If α is a k-separating vertex for some $2 < k \leqslant n/2$, then the dual link $L^d(\alpha)$ has exactly two connected components. Let us denote these components by $L^d_0(\alpha)$ and $L^d_1(\alpha)$. Then $L^d_0(\alpha)$ and $L^d_1(\alpha)$ are isomorphic to the dual complexes of the complexes of curves of spheres with $k+1$ and $n-k+1$ punctures. These complexes of curves have dimensions $k-3$ and $n-k-3$ and every automorphism of $C(S)$ must preserve these numbers, completing the proof. □

Corollary 3.4. *Let f be an automorphism of $C(S)$. If $\langle \alpha; \beta \rangle$ (and hence $\langle \alpha'; \beta' \rangle$) is a simple pair, then so is $\langle f(\alpha); f(\beta) \rangle$ (and hence $\langle f(\alpha'); f(\beta') \rangle$). Similarly, the image of a chain in $C(S)$ under f is also a chain.*

Proof. It is clear that the conditions (i) and (iii) of Theorem 3.2 are invariant under the automorphisms of $C(S)$. The fact that the condition (ii) is invariant under $\operatorname{Aut} C(S)$ is proved in Theorem 3.3 above.

The second part of the corollary follows easily from the first part. □

3.2. Automorphisms of $C(S)$ as automorphisms of $B(S)$

We now show that every automorphism of the complex $C(S)$ gives rise to an automorphism of $B(S)$ in a natural way. In fact, there is a monomorphism $\operatorname{Aut} C(S) \to \operatorname{Aut} B(S)$. For a punctured surface R, let us denote by $\mathcal{P}(R)$ the set of punctures of R.

Action of $\operatorname{Aut} C(S)$ *on* $\mathcal{P}(S)$

The action of $\operatorname{Aut} C(S)$ on the punctures of S is defined as follows. For $f \in \operatorname{Aut} C(S)$ and for a puncture P of S, take any simple pair $\langle \alpha'; \beta' \rangle$ with center (common endpoint) P and define $f(P)$ to be the center of the simple pair $\langle f(\alpha'); f(\beta') \rangle$. Note that by the one-to-one correspondence between the set of 2-separating vertices of $C(S)$ and the set of those vertices of $B(S)$ which join different punctures, $\operatorname{Aut} C(S)$ has a well-defined action on the latter set.

Lemma 3.5. *The definition of the action of* $\operatorname{Aut} C(S)$ *on the punctures of S is independent of the choice of the simple pair.*

Proof. Let f be an automorphism of $C(S)$, $\langle \alpha'; \beta' \rangle$ a simple pair with center P and $a' \in \alpha'$ and $b' \in \beta'$. Denote by $\overline{P}$ the center of the simple pair $\langle f(\alpha'); f(\beta') \rangle$. Let $f(a')$ and $f(b')$ be the disjoint representatives of $f(\alpha')$ and $f(\beta')$, respectively.

We show first that if c' is an arc joining P to some other puncture, with the class γ', and if $f(c')$ is a representative of $f(\gamma')$ intersecting $f(a')$ and $f(b')$ minimally, then one of the endpoints of $f(c')$ is $\overline{P}$. The proof of this is by induction on $i = i(\alpha', \gamma') + i(\beta', \gamma')$. Let us denote by P_1, P_2 and P_3 the other endpoints of a', b' and c', respectively.

As the first step of the induction, suppose that $i = 0$. The proof splits into two cases.

Case 1. If $\langle \alpha'; \gamma' \rangle$ and $\langle \beta'; \gamma' \rangle$ are simple pairs also (i.e., if P_3 is different from P_1 and P_2), then there is a fourth arc d' such that any two arcs in the set $\{a', b', c', d'\}$ constitute a simple pair with center P (cf. Fig. 4(a)). This is because there are at least five punctures on S. Then any two arcs in $\{f(a'), f(b'), f(c'), f(d')\}$ constitute a simple pair, where $f(d')$ is a representative of the image of the class of d' under f. An easy argument shows that all four arcs $f(a')$, $f(b')$, $f(c')$ and $f(d')$ must have a common endpoint, which must be $\overline{P}$.

Case 2. If the endpoints of a' and c' are the same (i.e., $P_1 = P_3$), then there exist a puncture Q different from P, P_1 and P_2, and an arc d' joining Q and P not intersecting any of the three arcs a', b', c' (cf. Fig. 4(b)). By an application of case 1 to $\{a', b', d'\}$ and then to $\{b', d', c'\}$, we see that one of the endpoints of $f(c')$ is $\overline{P}$. Notice that the center of $\langle f(\beta'); f(\gamma') \rangle$ must be $\overline{P}$.

Suppose now that $i > 0$. Let us orient all three arcs from P to P_j, for $j = 1, 2, 3$. Let X be the first point where c' meets $a' \cup b'$. Without loss of generality we can assume that X is on a'. Let d' be the arc consisting of the segment of c' from P to X and that of a' from X to P_1 (cf. Fig. 5). Then $i(\alpha', \delta') + i(\beta', \delta') = 0$, $\langle \delta'; \beta' \rangle$ is a simple pair, and $i(\delta', \gamma') + i(\beta', \gamma') < i(\alpha', \gamma') + i(\beta', \gamma')$, where δ' is the class of d'. By induction, one of the endpoints of $f(c')$ is the center of the simple pair $\langle f(\delta'); f(\beta') \rangle$. But this center is $\overline{P}$ by case 2.

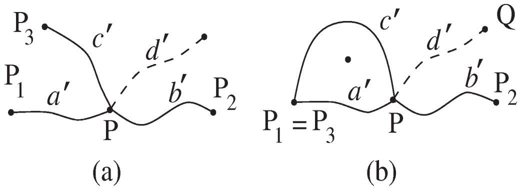

Fig. 4.

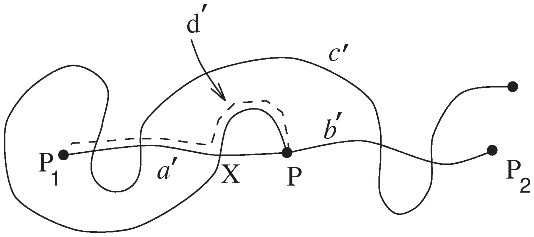

Fig. 5.

If $\langle\gamma'; \delta'\rangle$ is another simple pair with center P, and $c' \in \gamma'$ and $d' \in \delta'$ intersect each other as well as a' and b' minimally, then by applying the argument above first to $\{a', b', c'\}$ and then to $\{a', b', d'\}$ we see that $\overline{P}$ is an endpoint of both $f(c')$ and $f(d')$, which must be the center of the simple pair $\langle f(\gamma'); f(\delta')\rangle$. Hence the action of $\operatorname{Aut} C(S)$ on the set of punctures of S is well-defined. □

Lemma 3.6. *Let $f \in \operatorname{Aut} C(S)$, α a k-separating vertex of $C(S)$, and $a \in \alpha$. If S'_a and S''_a denote the k-punctured and $(n-k)$-punctured disc components of S_a, then $f(\mathcal{P}(S'_a)) = \mathcal{P}(S'_{f(a)})$, and hence $f(\mathcal{P}(S''_a)) = \mathcal{P}(S''_{f(a)})$. In the case of $k = n/2$, we may change the roles of S'_a and S''_a if necessary.*

Proof. Two punctures P and Q are on the same connected component of S_a if and only if P and Q can be joined by an arc disjoint from a, and P and Q can be joined by an arc disjoint from a if and only if $f(P)$ and $f(Q)$ can be joined by an arc disjoint from $f(a)$. The proof now follows. □

Action of $\operatorname{Aut} C(S)$ *on arcs*

We can now define an action of $\operatorname{Aut} C(S)$ on the vertices of $B(S)$. Let $f \in \operatorname{Aut} C(S)$, α' a vertex of $B(S)$ and let $a' \in \alpha'$. If a' is joining two different punctures, then $f(\alpha')$ is already defined by the correspondence with the 2-separating vertices of $C(S)$ and the action of $\operatorname{Aut} C(S)$ on $C(S)$. That is, $f(\alpha')$ is the isotopy class of the arc, which is unique up to isotopy, joining two punctures on the twice-punctured disc component of $S_{f(a)}$ for $f(a) \in f(\alpha)$.

Suppose now that the arc a' is joining a puncture P to itself. Let a_1 and a_2 be the boundary components of a regular neighborhood of $a' \cup \{P\}$ and α_1 and α_2 be their classes. Since a' cannot be deformed to P, at most one of a_1 and a_2 is trivial. If a_1 is trivial then

a_2 is 2-separating, and by Lemma 3.3, $f(\alpha_2)$ is 2-separating. Hence for a representative $f(a_2)$ of $f(\alpha_2)$, one of the components, say $S'_{f(a_2)}$, of $S_{f(a_2)}$ is a twice-punctured disc, and one of the punctures on $S'_{f(a_2)}$ is $f(P)$ by Lemma 3.6. Define $f(\alpha')$ to be the isotopy class of a nontrivial simple arc on $S'_{f(a_2)}$ joining $f(P)$ to itself. Such an arc is unique up to isotopy by Lemma 1.1.

In the case that neither a_1 nor a_2 is trivial, we claim that $f(a_1)$ and $f(a_2)$ bound a once-punctured annulus with only one puncture $f(P)$. Here $f(a_i)$ is a representative of $f(\alpha_i)$. For the proof of this, suppose that the set of punctures on S'_{a_1} and S''_{a_2} are $\mathcal{P}(S'_{a_1}) = \{P_1, \ldots, P_k\}$ and $\mathcal{P}(S''_{a_2}) = \{Q_1, \ldots, Q_{n-k-1}\}$, respectively. Then $P_i \neq Q_j$ for all i, j. By Lemma 3.5,

$$\mathcal{P}(S'_{f(a_1)}) = \{f(P_1), \ldots, f(P_k)\} \quad \text{and} \quad \mathcal{P}(S''_{f(a_2)}) = \{f(Q_1), \ldots, f(Q_{n-k-1})\}.$$

It follows that, since $f(a_1)$ and $f(a_2)$ are disjoint and nonisotopic, they must bound an annulus with only one puncture $f(P)$. Then $f(\alpha')$ is defined to be the isotopy class of the unique arc (up to isotopy) on this annulus joining $f(P)$ to itself.

Lemma 3.7. *Let f be an automorphism of $C(S)$ and α' and β' be two distinct vertices of $B(S)$ such that $i(\alpha', \beta') = 0$. Then $i(f(\alpha'), f(\beta')) = 0$. Hence every automorphism of $C(S)$ induces an automorphism of $B(S)$.*

Proof. Let a' and b' be two disjoint representatives of α' and β', respectively. There are seven cases to examine as illustrated in Fig. 6. In the figure, we assume that the arc on the left is a' and the one on the right is b'.

If a' (respectively b') is joining two different punctures, let us denote by α (respectively β) the 2-separating vertex of $C(S)$ corresponding to α' (respectively β'), and by a (respectively b), a representative of α (respectively β).

If a' (respectively b') is connecting a puncture P to itself, let us denote by a_1 and a_2 (respectively b_1 and b_2) the boundary components of a regular neighborhood of $a' \cup \{P\}$ (respectively $b' \cup \{P\}$). Note that we use the classes of a_i (respectively b_i) to define $f(\alpha')$ (respectively $f(\beta')$). We also denote representatives of $f(\alpha)$, $f(\alpha')$ by $f(a)$, $f(a')$ etc.

We examine each of the seven cases illustrated above.

(i) Since the circles a and b are disjoint, $\{\alpha, \beta\}$ is a 1-simplex. Hence $\{f(\alpha), f(\beta)\}$ is a 1-simplex. It follows that $i(f(\alpha'), f(\beta')) = 0$.

(ii) On the annulus determined by $f(b_1)$ and $f(b_2)$ there is only one puncture. Since f is an automorphism of $C(S)$, the circles $f(a)$, $f(b_1)$ and $f(b_2)$ are all distinct and pairwise nonisotopic. Therefore $f(a)$ cannot lie on this annulus. Since $f(b')$ is on the annulus, we are done.

(i) (ii) (iii) (iv) (v) (vi) (vii)

Fig. 6.

(iii) The once-punctured annuli determined by a_1 and a_2, and b_1 and b_2 are disjoint. Since $a_1,\ a_2,\ b_1$ and b_2 are pairwise disjoint, so are $f(a_1),\ f(a_2),\ f(b_1)$ and $f(b_2)$. So the annuli determined by $f(a_1)$ and $f(a_2)$, and $f(b_1)$ and $f(b_2)$ are disjoint.

(iv) Follows from Corollary 3.4.

(v) Let P and Q denote the endpoints of a' and b'. Let R be any puncture other than P and Q. Connect the punctures P and R via some embedded arc c' disjoint from $a' \cup b'$. Let d' be an embedded arc connecting Q and R disjoint from all these there arcs and isotopic to $b' \cup c' \cup \{P\}$. Consider a regular neighborhood of $b' \cup c' \cup \{P, Q, R\}$ which contains d'. Let e be the boundary of this neighborhood. Notice that any two circles in the set $\{b', c', d'\}$ constitute a simple pair and any arc disjoint from $c' \cup d'$ can be isotoped to an arc disjoint from b'. Then any two circles in $\{f(b'), f(c'), f(d')\}$ constitute a simple pair. Since $f(b') \cup f(c') \cup f(d')$ lie on a thrice punctured disc bounded by $f(e)$, it follows from the first step of the induction in the proof of Lemma 3.5 that any arc disjoint from $f(c') \cup f(d')$ is isotoped to an arc disjoint from $f(b')$. Since $f(a')$ is such an arc, we are done.

(vi) Suppose that a' is connecting P to Q and that b' is connecting P to itself. We can assume that b_2 is disjoint from a'. Clearly, there exists an arc c' disjoint from $a' \cup b' \cup b_1$ joining P to some other puncture, say, R (cf. Fig. 7).
Then $f(c')$ connects $f(P)$ to $f(R)$ and meets $f(b_2)$, but not $f(b_1)$ or $f(a')$. Let λ be the segment of $f(c')$ lying on the once-punctured annulus determined by $f(b_1)$ and $f(b_2)$, and connecting $f(P)$ to $f(b_1)$. Note that the intersection of $f(a')$ with the this annulus is a collection of arcs joining a point on $f(b_1)$ either with another point on $f(b_1)$ or with $f(P)$. Since $f(a')$ is disjoint from λ, the intersection of $f(a')$ and this annulus consists of only one arc connecting $f(P)$ to some point on $f(b_1)$. Then $f(b')$, which is an arc on this annulus joining $f(P)$ to itself, can be chosen so that it does not intersect $f(a')$.

(vii) Let P be the common endpoints of a' and b'. We can assume that a_1 does not intersect $b_1 \cup b_2$. Let $P_1, \ldots, P_k$ be the punctures on the component of S_{a_1} which does not contain $a' \cup b'$. Choose k arcs $c'_1, c'_2, \ldots, c'_k$ such that c'_i joins P_{i-1} to P_i for $1 \leqslant i \leqslant k$, where $P_0 = P$, and $\{\alpha', \beta', \gamma'_1, \ldots, \gamma'_k\}$ is a simplex of $B(S)$. Then the arc $f(c_i)$ joins $f(P_{i-1})$ to $f(P_i)$, and $\{f(\alpha'), f(\gamma'_1), \ldots, f(\gamma'_k)\}$ and $\{f(\beta'), f(\gamma'_1), \ldots, f(\gamma'_k)\}$ are simplices of $B(S)$ by (i), (ii), (iv) and (vi).

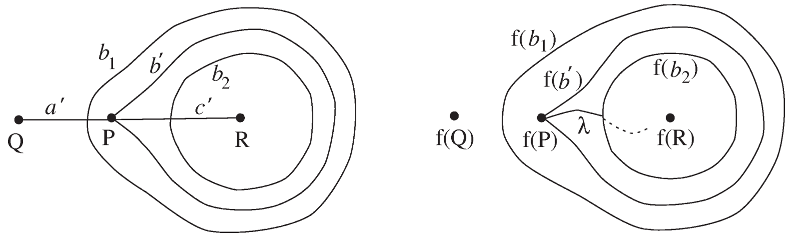

Fig. 7.

Since any arc joining P to itself which is disjoint from $c'_1, c'_2, \ldots, c'_k$ is isotopic to an arc disjoint from a', any arc joining $f(P)$ to itself disjoint from $f(c'_1), f(c'_2), \ldots, f(c'_k)$ is isotopic to an arc disjoint from $f(a')$. Therefore, $f(b')$ can be chosen disjoint from $f(a')$.

This completes the proof. $\square$

Proposition 3.8. *The group* $\operatorname{Aut} C(S)$ *is naturally isomorphic to a subgroup of the group* $\operatorname{Aut} B(S)$.

Proof. By Lemma 3.7 every element of $\operatorname{Aut} C(S)$ induces an element of $\operatorname{Aut} B(S)$. Clearly, the map taking an element of $\operatorname{Aut} C(S)$ to the induced element of $\operatorname{Aut} B(S)$ is a homomorphism.

It remains to show that if an automorphism of $C(S)$ induces the identity automorphism of $B(S)$, then it is, in fact, the identity. Assume that $f \in \operatorname{Aut} C(S)$ induces the identity automorphism of $B(S)$. By the one-to-one correspondence between 2-separating vertices of $C(S)$ and the vertices of $B(S)$ connecting different punctures, f is the identity on the 2-separating vertices of $C(S)$. Now let α be a k-separating vertex of $C(S)$ with $3 \leqslant k \leqslant n/2$ and let $a \in \alpha$. Let us denote by $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_{n-k}$ the punctures on the two connected components of S_a. Choose two arbitrary chains $\langle b'_1; \ldots; b'_{k-1}\rangle$ and $\langle c'_1; \ldots; c'_{n-k-1}\rangle$ disjoint from a such that b'_i connects P_i to P_{i+1} and c'_j connects Q_j to Q_{j+1}. Let $C' = b'_1 \cup \cdots \cup b'_{k-1} \cup c'_1 \cup \cdots \cup c'_{n-k-1}$ and let β'_i and γ'_j be the isotopy classes of b'_i and c'_j, respectively. Notice that $S_{C'}$ is an annulus. Since $i(\beta_i, \alpha) = i(\gamma_j, \alpha) = 0$, $i(\beta_i, f(\alpha)) = i(\gamma_j, f(\alpha)) = 0$ and hence $i(\beta'_i, f(\alpha)) = i(\gamma'_j, f(\alpha)) = 0$. Since the surface $S_{C'}$ obtained from S by cutting along C' is an annulus and since a and a representative of $f(\alpha)$ lie on this annulus, we conclude that $f(\alpha) = \alpha$. $\square$

Ideal triangulations of S and codimension-zero simplices of $B(S)$

Since all maximal simplices in $B(S)$ have the same dimension, there is a well-defined action of the group $\operatorname{Aut} B(S)$ on codimension-zero simplices. Any realization of a codimension-zero simplex is an ideal triangulation of S. An *ideal triangulation* of S is a triangulation of S whose vertex set is the set of punctures on S in the sense that vertices of a triangle can coincide as can a pair of edges. It is clear that when different representatives

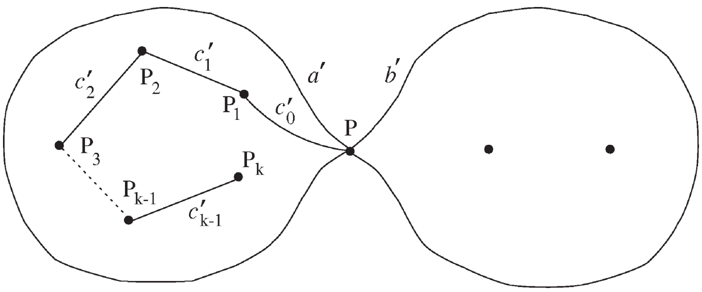

Fig. 8.

are chosen, the resulting triangulations are isotopic. Similarly, isotopy class of any ideal triangulation determines uniquely a codimension-zero simplex in $B(S)$. That is, there is a one-to-one correspondence between the codimension-zero simplices of $B(S)$ and the isotopy classes of ideal triangulations of S. Therefore, $\operatorname{Aut} B(S)$ acts on the isotopy classes of ideal triangulations. It is not immediately clear that the isotopy classes of ideal triangles are mapped to the isotopy classes of ideal triangles.

Let us call an ideal triangle $\{a', b', c'\}$ a *good ideal triangle* if the vertices of $\{a', b', c'\}$ are all distinct. Notice that if the ideal triangle $\{a', b', c'\}$ is good, then $\langle a'; b'\rangle$, $\langle b'; c'\rangle$ and $\langle a'; c'\rangle$ are simple pairs.

Lemma 3.9. *Let $f \in \operatorname{Aut} C(S)$, $\Delta = \{a', b', c'\}$ be a good ideal triangle and let α', β', γ' be the isotopy classes of a', b', c', respectively. Then $\{\alpha', \beta', \gamma'\}$ and, hence, $\{f(\alpha'), f(\beta'), f(\gamma')\}$ is a 2-simplex in $B(S)$. If $f(\Delta) = \{f(a'), f(b'), f(c')\}$ is a realization of the latter simplex, then it is a good ideal triangle on S.*

Proof. By the action of $\operatorname{Aut} C(S)$, distinct punctures go to distinct punctures. It follows that $f(\Delta)$ has three distinct vertices and that $f(a') \cup f(b') \cup f(c')$ separates S into two connected components. In order to show that $f(\Delta)$ is an ideal triangle, we must show that one of these components does not contain any puncture in the interior. Let P and Q be any two punctures different from the vertices of $f(\Delta)$. Since $f^{-1}(P)$ and $f^{-1}(Q)$ can be joined by an arc d' not intersecting any of the edges of Δ, P and Q can be joined by an arc $f(d')$ which has the geometric intersection number zero with each arc in $f(\Delta)$. It is clear that $f(d')$ can be chosen so that it does not intersect any of them. It follows that $f(\Delta)$ is a good ideal triangle. $\square$

Lemma 3.10. *Let f and g be two automorphisms of $B(S)$. If they agree on a codimension-zero simplex, then they agree on all of $B(S)$.*

Proof. Let σ be a codimension-zero simplex of $B(S)$. Suppose that f is equal to g on σ. If σ' is another codimension-zero simplex, then by Theorem 1.2 there exist codimension-zero simplices $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$ such that $\sigma_{i-1} \cap \sigma_i$ is a codimension-one simplex for each i. Since any codimension-one simplex is a face of either one or two codimension-zero simplices, if two automorphisms $B(S)$ agree on σ_{i-1} then they agree on σ_i. Clearly, this implies that f must be equal to g on σ'. Since every simplex of $B(S)$ is a face of a codimension-zero simplex, we are done. $\square$

3.3. Proof of Theorem 1 for punctured spheres

The fact that the map $\mathcal{M}_S^* \to \operatorname{Aut} C(S)$ is injective is proved in Section 2. Let us now show that it is surjective. Let $f \in \operatorname{Aut} C(S)$ and let C' be an arbitrary ideal triangulation of S such that each triangle has three different vertices, i.e., a 'good' triangulation. Existence of such a triangulation is clear. Let σ be the isotopy class of C'. Then σ is a codimension-zero simplex of $B(S)$. By Lemma 3.9, 'good' ideal triangles are mapped to 'good' ideal triangles by f, and it is a well known fact that f can be realized by a homeomorphism on each such triangle. Since each edge of C' is an edge of exactly two 'good' ideal triangles, the homeomorphisms of these triangles give rise to a homeomorphism F of S. By replacing F with a diffeomorphism isotopic to F, we can assume that F itself is a diffeomorphism. If $[F]$ denotes the isotopy class of F, then f agrees with $[F]$ on the codimension-zero simplex σ of $B(S)$. By Lemma 3.10, they agree on $B(S)$. We have shown in the proof of Proposition 3.8 that the natural map $\operatorname{Aut} C(S) \to \operatorname{Aut} B(S)$ is injective. It follows now that $[F]$ is equal to f as automorphisms of $C(S)$.

4. Punctured tori

This section is devoted to the proofs of Theorem 1 for tori with at least three punctures and of Theorem 2. Unless otherwise stated, S denotes a torus with n punctures.

Instead of simple pairs, we consider the pairs of circles having geometric intersection number 1. The main problem in showing that these pairs are preserved by automorphisms of $C(S)$ is to prove that topological types of the vertices of $C(S)$ are invariant under the action of $\operatorname{Aut} C(S)$. After this problem is solved for certain type of vertices, we use a theorem of Ivanov (see Theorem 4.4). This theorem is used in [9], but a written proof of it has not been published yet. We will give our own proof of this theorem for punctured tori at the end of this section using the methods of Section 3.

4.1. Automorphisms of $C(S)$ and $\mathcal{M}_S^*$

If f is a mapping class and if $\alpha,\ \beta$ are two vertices of $C(S)$ such that their geometric intersection number $i(\alpha,\beta)$ is one, then clearly $i(f(\alpha), f(\beta))$ is one. Hence if an automorphism of $C(S)$ is induced by a diffeomorphism, then it must preserve the same relation between the nonseparating vertices of $C(S)$. This is the starting point of this section.

The following theorem enables us to recognize whether or not two vertices of $C(S)$ have geometric intersection number one, by looking at the complex $C(S)$.

Theorem 4.1. *Let S be a torus with at least two punctures, and let α and β be two vertices of $C(S)$. Then $i(\alpha,\beta)=1$ if and only if there exist three vertices $\gamma_1,\ \gamma_2$ and γ_3 of $C(S)$ such that*

(i) *$(\gamma_1,\alpha,\gamma_2,\beta,\gamma_3)$ is a pentagon in $C(S)$, and*
(ii) *$\alpha,\ \beta$ and γ_3 are nonseparating, and γ_1 and γ_2 are n-separating.*

Proof. Let us first prove the 'only if' clause of the theorem. Clearly, $i(\alpha,\beta)=1$ implies that α and β are both nonseparating. Thus there exist $a\in\alpha$ and $b\in\beta$ such that a and b intersect transversally at only one point. It is well known that if c and d are any other pair of circles intersecting transversally at only one point, then there exists a diffeomorphism $F\colon S\to S$ such that $F(c)=a$ and $F(d)=b$. Hence we can assume that a and b are the standard circles in Fig. 9(a). The existence of the other circles whose isotopy classes satisfy (i) and (ii) is now obvious from Fig. 9(b).

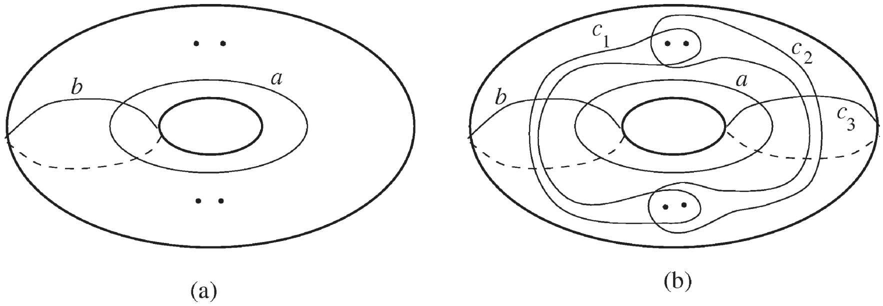

Fig. 9.

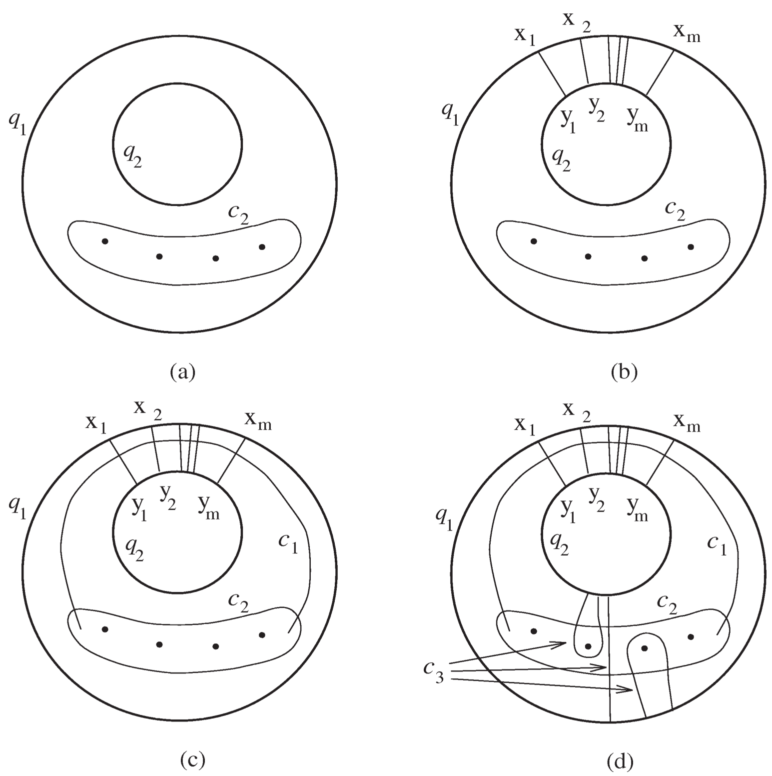

Fig. 10.

For the converse, let $a \in \alpha$, $b \in \beta$ and $c_i \in \gamma_i$ intersect each other minimally. Since a is nonseparating, the surface S_a is an annulus with n punctures. Let us denote by q_1 and q_2 the boundary components of S_a. Then S is a quotient space of S_a. Let $p: S_a \to S$ be the quotient map. So $p(q_1) = p(q_2) = a$. Up to a diffeomorphism of S_a preserving q_1 and q_2, we can assume that the picture of $p^{-1}(c_2)$ in S_a is as in Fig. 10(a). In the figures, c_i represents $p^{-1}(c_i)$.

Let $i(\alpha, \beta) = m$, which is the cardinality of $a \cap b$. Since α and β are not connected by an edge in the pentagon (and hence in $C(S)$), this geometric intersection number m must be positive.

We now consider the components of the preimage $p^{-1}(b)$ of b, which is a collection of arcs. Since $i(\beta, \gamma_2) = 0$, the components of $p^{-1}(b)$ lie on a disc with two holes whose boundary components are q_1, q_2 and c_2, and they do not intersect c_2. Thus it follows from Lemma 1.1 that each arc in $p^{-1}(b)$ joins either a point on q_1 to a point on q_2, or, two points on q_1, or, two points on q_2. Let m_{11}, m_{22} m_{12} be the number of these components joining q_1 to q_1, q_2 to q_2 and q_1 to q_2, respectively. Then $m = m_{12} + 2m_{11} = m_{12} + 2m_{22}$ and hence $m_{11} = m_{22}$. On the other hand, if λ is an embedded arc on S_a connecting two points on q_1 such that λ is not isotopic to a segment of q_1 and does not intersect c_2, then every embedded arc connecting two points on q_2 which is disjoint from λ must be trivial, i.e., isotopic to a segment of q_2. Therefore m_{11} and m_{22} must be zero, so each component of $p^{-1}(b)$ connects a point on q_1 to a point on q_2. Therefore the picture of these arcs on S_a is as in Fig. 10(b).

Let us now orient q_1 and q_2 so that the induced orientations of $p(q_1)$ and $p(q_2)$ agree in S, and let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_m$ be the consecutive intersection points of $p^{-1}(b)$ with q_1 and q_2, respectively, such that X_i is joined with Y_i by some arc in $p^{-1}(b)$ for each $i = 1, 2, \ldots, m$. It is clear that there exists a k, $0 \leqslant k < m$, such that $p(X_i) = p(Y_{i+k})$ for each i. By convention we set $Y_{m+i} = Y_i$.

Since $i(\beta, \gamma_1) \neq 0$ and $i(\alpha, \gamma_1) = 0$, the preimage of c_1, also denoted by c_1 in the figure, intersects every component of $p^{-1}(b)$ (cf. Fig. 10(c)). As $i(\gamma_3, \gamma_1) = i(\gamma_3, \beta) = 0$, the components of $p^{-1}(c_3)$ intersect q_1 only in the open interval $]X_m, X_1[$ and q_2 only in $]Y_m, Y_1[$. Therefore $p(]X_m, X_1[) = p(]Y_m, Y_1[)$ in S. In particular, $p(X_1) = p(Y_1)$, and hence $k = 0$. Finally, $p(b)$ is a connected curve only if m is equal to 1.

This finishes the proof of the theorem. □

Lemma 4.2. *Let $n \geqslant 3$. Let α, β and γ be three distinct vertices of $C(S)$. If α is nonseparating, β is n-separating and γ is separating, and $i(\alpha, \beta) = i(\beta, \gamma) = 0$, then $i(\alpha, \gamma) = 0$.*

Proof. Let a, b and c be representatives of α, β and γ in minimal position. The nonseparating circle a and the separating circle c are, respectively, nonseparating and separating on the surface S_b, the surface obtained from S by cutting along b. But nonseparating and separating circles on S_b lie on different components (cf. Fig. 11). □

Lemma 4.3. *Let $n \geqslant 2$ and let S and S' denote a torus with n punctures and a sphere with $n + 3$ punctures, respectively. If every automorphism of the complex $C(S)$ is induced by some diffeomorphism of S, then $C(S)$ and $C(S')$ are not isomorphic.*

Proof. Suppose that φ is an isomorphism from $C(S)$ to $C(S')$. Then φ induces a group isomorphism $\varphi_* : \operatorname{Aut} C(S) \to \operatorname{Aut} C(S')$, defined by $\varphi_*(f) = \varphi f \varphi^{-1}$. This implies that

$$\operatorname{Aut} C(S') = \left\{\varphi f \varphi^{-1} \colon\ f \in \operatorname{Aut} C(S)\right\}.$$

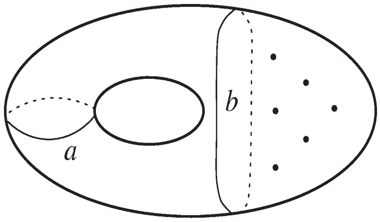

Fig. 11.

We now show that this is impossible. Note that for a vertex α of $C(S)$, the dual link $L^d(\alpha)$ of α is connected if and only if α is either nonseparating or 2-separating, and for a vertex β of $C(S')$, the dual link $L^d(\beta)$ of β is connected if and only if β is 2-separating. From this it follows that the image of the union of the set of nonseparating vertices and the set of 2-separating vertices of $C(S)$ is precisely the set of 2-separating vertices of $C(S')$.

Let α be a nonseparating vertex of $C(S)$, and choose a 2-separating vertex β of $C(S)$ such that $i(\alpha,\beta)=0$, i.e., α and β are joined by an edge in $C(S)$. Then $\varphi(\alpha)$ and $\varphi(\beta)$ are two 2-separating vertices of $C(S')$ and are joined by an edge in $C(S')$. Let c and d be representatives of $\varphi(\alpha)$ and $\varphi(\beta)$, respectively. Then c and d are two disjoint 2-separating circles on S'. By the classification of surfaces there exists a diffeomorphism G of S' such that $G(c)=d$. Hence $g(\varphi(\alpha))=\varphi(\beta)$, where g is the isotopy class of G. Then the automorphism $\varphi_*^{-1}(g)=\varphi^{-1}g\varphi$ of $C(S)$ takes the nonseparating vertex α to the separating vertex β. This is impossible since every automorphism of $C(S)$ is induced by a diffeomorphism of S, by hypothesis. Hence we have the lemma. □

Theorem 4.4. *Let S be an orientable surface of genus at least one. Suppose that $f \in \operatorname{Aut} C(S)$ and α and β two vertices of $C(S)$ with $i(\alpha,\beta)=1$ imply that $i(f(\alpha), f(\beta))=1$. Then every element of $\operatorname{Aut} C(S)$ is induced by some diffeomorphism of S.*

Remark. Theorem 4.4 was proved by Ivanov [9]. The proof of it has not been published yet. Since it will be used in the proof of Theorem 1 for punctured tori below, for the sake of completeness we give a proof of it for tori with at least three punctures in Section 4.4. The proof we give is basically the same as the one we gave for punctured spheres in Section 3.

We will prove the following lemma in the next subsection.

Lemma 4.5. *Let $n \geqslant 3$. If S is a torus with n punctures and S' is a sphere with $n+3$ punctures, then $C(S)$ and $C(S')$ are not isomorphic.*

4.2. Proofs of Theorem 1 for punctured tori and of Lemma 4.5

We have already proved in Section 2 that the homomorphism $\mathcal{M}_S^* \to \operatorname{Aut} C(S)$ is one-to-one. Hence we need to show that the automorphisms of the complex $C(S)$ are induced by diffeomorphisms of S.

As we have mentioned in the proof of Lemma 4.3, for a vertex α of $C(S)$, $L^d(\alpha)$ is connected if and only if α is either nonseparating or 2-separating. Hence an element of the group $\operatorname{Aut} C(S)$ maps a nonseparating vertex either to a nonseparating vertex or to a 2-separating one, and a k-separating vertex to a k'-separating one for $k, k' \geqslant 3$.

Let α be a k-separating vertex of $C(S)$ with $k \geqslant 3$ and let $a \in \alpha$. Let us denote by $S_a^{(0)}$ and $S_a^{(1)}$ the components of S_a of genus zero and of genus one, respectively. The graph $L^d(\alpha)$ has exactly two connected components, say, $L_0^d(\alpha)$ and $L_1^d(\alpha)$. The vertices of these components correspond to the isotopy classes of circles on the connected components of S_a. We can choose $L_0^d(\alpha)$ and $L_1^d(\alpha)$ so that the vertices of $L_i^d(\alpha)$ are the isotopy classes of circles on $S_a^{(i)}$. We then define two full subcomplexes $L_0(\alpha)$ and $L_1(\alpha)$ of $C(S)$ as follows. The set of vertices of $L_i(\alpha)$ are those of $L_i^d(\alpha)$. Recall that $L_i(\alpha)$ is a full subcomplex of $C(S)$ means that a set of vertices of $L_i(\alpha)$ is a simplex in $L_i(\alpha)$ if and only if it is a simplex in $C(S)$. Clearly, $L_i(\alpha)$ is isomorphic to $C(S_a^{(i)})$.

If f is an automorphism of $C(S)$ and α is a k-separating vertex with $k \geqslant 3$, then f induces an isomorphism from $L^d(\alpha) = L_0^d(\alpha) \cup L_1^d(\alpha)$ to $L^d(f(\alpha)) = L_0^d(f(\alpha)) \cup L_1^d(f(\alpha))$. Since $L_j^d(\alpha)$ and $L_j^d(f(\alpha))$ are connected components, we get $f(L_0^d(\alpha)) = L_r^d(f(\alpha))$ and $f(L_1^d(\alpha)) = L_{1-r}^d(f(\alpha))$ for some $r = 0$ or $r = 1$. Then $f(L_0(\alpha)) = L_r(f(\alpha))$ and $f(L_1(\alpha)) = L_{1-r}(f(\alpha))$. Since $\dim L_0(\alpha) = \dim C(S_a^{(0)}) = k - 3$ and $\dim L_1(\alpha) = \dim C(S_a^{(1)}) = n - k$, $\dim L_r(\alpha) = k - 3$ and $\dim L_{1-r}(\alpha) = n - k$. From this it is easy to conclude that if $r = 0$ (respectively $r = 1$), then $f(\alpha)$ is k-separating (respectively $(n-k+3)$-separating). In particular, if α is n-separating on S then $f(\alpha)$ is either n-separating or 3-separating. The proofs now proceed simultaneously by induction on n, the number of punctures on S.

Suppose that $n = 3$. Let $f \in \operatorname{Aut} C(S)$ and let α and β be two vertices of $C(S)$ with $i(\alpha, \beta) = 1$. By Theorem 4.1, there exist vertices γ_1, γ_2 and γ_3 of $C(S)$ such that $(\gamma_1, \alpha, \gamma_2, \beta, \gamma_3)$ is a pentagon, γ_1 and γ_2 are 3-separating, and γ_3 is nonseparating. Hence $(f(\gamma_1), f(\alpha), f(\gamma_2), f(\beta), f(\gamma_3))$ is a pentagon in $C(S)$. From the discussion given in the preceding paragraph, it follows that $f(\gamma_1)$ and $f(\gamma_2)$ are 3-separating. Note that any two distinct nonisotopic 3-separating circles on S, a torus with three punctures, must intersect. Therefore none of the vertices $f(\alpha)$, $f(\beta)$ and $f(\gamma_3)$ can be 3-separating. Similarly, any two distinct nonisotopic 2-separating circles on S must intersect. We conclude that one of the vertices $f(\beta)$ and $f(\gamma_3)$, say $f(\beta)$, is nonseparating. By applying Lemma 4.2 twice to the pentagon $(f(\gamma_1), f(\alpha), f(\gamma_2), f(\beta), f(\gamma_3))$, we first see that $f(\alpha)$, and then $f(\gamma_3)$, is nonseparating, i.e., $f(\alpha)$, $f(\beta)$, $f(\gamma_1)$, $f(\gamma_2)$ and $f(\gamma_3)$ satisfy the conditions (i) and (ii) of Theorem 4.1. Hence $i(f(\alpha), f(\beta)) = 1$. This is true for any automorphism of $C(S)$. Therefore, by Theorem 4.4 every automorphism of $C(S)$ is induced by some diffeomorphism of S. Using this and Lemma 4.3, we see that $C(S)$ and $C(S')$ are not isomorphic if S' is a sphere with six punctures.

Now suppose that $n \geqslant 4$. Let α be a nonseparating and β be a 2-separating vertex of $C(S)$, and let $a \in \alpha$ and $b \in \beta$. Then $L(\alpha)$ is isomorphic to the complex of curves on S_a, a sphere with $n+2$ punctures, and $L(\beta)$ is isomorphic to the complex of curves on $S_b^{(1)}$, a torus with $n-1$ punctures. Here, $S_b^{(1)}$ is the component of S_b which has genus one. By the induction hypothesis, every automorphism of $C(S_b^{(1)})$ is induced by a diffeomorphism of $S_b^{(1)}$. Then by Lemma 4.3, $C(S_a)$ is not isomorphic to $C(S_b^{(1)}) = C(S_b)$, i.e., $L(\alpha)$ is not isomorphic to $L(\beta)$. It follows that if $f \in \operatorname{Aut} C(S)$ then $f(\alpha)$ cannot be 2-separating. Therefore nonseparating vertices are preserved under the action of $\operatorname{Aut} C(S)$.

To show that n-separating vertices are also preserved under the action of $\operatorname{Aut} C(S)$, we assume the converse. Suppose that there exist an $f \in \operatorname{Aut} C(S)$ and an n-separating vertex α of $C(S)$ such that $f(\alpha) = \beta$ is not n-separating. Then β is 3-separating by the discussion given above. The automorphism f restricts to an isomorphism from the disjoint union $L_0(\alpha) \cup L_1(\alpha)$ to the disjoint union $L_0(\beta) \cup L_1(\beta)$. Since $L_1(\alpha)$ and $L_0(\beta)$ are discrete, and since $L_0(\alpha)$ and $L_1(\beta)$ are, for instance, connected, we must have $f(L_1(\alpha)) = L_0(\beta)$. But this means that f takes the nonseparating vertices in the link of α to separating vertices, a contradiction. Thus the automorphisms of $C(S)$ preserve the set of n-separating vertices as well.

Now it follows from Theorem 4.1 that if $f \in \operatorname{Aut} C(S)$ and if $i(\alpha, \beta) = 1$, then $i(f(\alpha), f(\beta)) = 1$. Then Theorem 4.4 implies that automorphisms of $C(S)$ are induced by diffeomorphisms of S. This completes the proof of Theorem 1.

Again, that $C(S)$ and $C(S')$ are not isomorphic if S' is a sphere with $n + 3$ punctures follows from Lemma 4.3. So the proof of Lemma 4.5 is complete, too.

4.3. *Proof of Theorem 2*

If S and S' are either both punctured spheres or both punctured tori, and if $C(S)$ and $C(S')$ are isomorphic, then their dimensions are equal. Hence the number of punctures on S and S' are equal. By the classification of surfaces, these two surfaces are diffeomorphic.

Let S be a sphere with at least five punctures and let S' be torus with at least three punctures. Certainly, if the dimensions of $C(S)$ and $C(S')$ are not equal, then S and S' are not diffeomorphic. But if their dimensions are the same, then S has three more punctures than S'. In this case, that $C(S)$ and $C(S')$ are not isomorphic is proved in Lemma 4.5.

4.4. *Proof of Theorem 4.4 for punctured tori*

Let S be a torus with at least three punctures. By the hypothesis, if f is an automorphism of $C(S)$, and if α and β are two vertices of $C(S)$ with $i(\alpha, \beta) = 1$, then $i(f(\alpha), f(\beta)) = 1$. The idea of the proof is the same as the idea of the proof of the Theorem 1 for punctured spheres given in Section 3.

Note that for a nonseparating vertex α of $C(S)$, there exists a vertex β such that $i(\alpha, \beta) = 1$. By assumption, $i(f(\alpha), f(\beta)) = 1$ for $f \in \operatorname{Aut} C(S)$. Hence $f(\alpha)$ is nonseparating. Therefore $\operatorname{Aut} C(S)$ preserves nonseparating vertices. The dual link of a vertex is connected if and only if the vertex is either nonseparating or 2-separating. Therefore, 2-separating vertices are preserved under the action of $\operatorname{Aut} C(S)$, as well.

Recall the one-to-one correspondence between 2-separating vertices of $C(S)$ and the vertices of $B(S)$ connecting different punctures. For a 2-separating vertex α of $C(S)$, the corresponding vertex of $B(S)$ is denoted by α'. This gives rise to a well-defined action of Aut $C(S)$ on the set of isotopy classes of arcs joining different punctures defined by $f(\alpha') = f(\alpha)'$.

Lemma 4.6.

(i) *Let f be an automorphism of $C(S)$ and let α_1 and α_2 be two nonseparating vertices of $C(S)$ such that a_1 and a_2 bound an annulus with one puncture for $a_i \in \alpha_i$. Then $f(a_1)$ and $f(a_2)$ bound an annulus with one puncture, where $f(a_i) \in f(\alpha_i)$.*

(ii) *Let f be an automorphism of $C(S)$ and let α be a nonseparating and β be a 2-separating vertex of $C(S)$. If $i(\alpha, \beta') = 1$, then $i(f(\alpha), f(\beta')) = 1$.*

Proof. (i) Let α and β be two nonseparating vertices of $C(S)$ such that $\{\alpha, \beta\}$ is a 1-simplex of $C(S)$. We define the link $L(\alpha, \beta)$ of $\{\alpha, \beta\}$ to be the full subcomplex of $C(S)$ with the vertex set

$$\left\{\gamma \in C(S)\colon \gamma \neq \alpha,\ \gamma \neq \beta,\ i(\gamma, \alpha) = i(\gamma, \beta) = 0\right\}.$$

In fact, $L(\alpha, \beta) = L(\alpha) \cap L(\beta)$ and $L^d(\alpha, \beta) = L^d(\alpha) \cap L^d(\beta)$.

Let $a \in \alpha$ and $b \in \beta$ be disjoint representatives. Then the surface $S_{a\cup b}$ has two connected components. The vertices of $L(\alpha, \beta)$ are the isotopy classes of nontrivial circles on these two components. That is, $L(\alpha, \beta)$ is isomorphic to the complex of curves $C(S_{a\cup b})$.

Since the circles on different components do not intersect, if two vertices of $L^d(\alpha, \beta)$ form an edge, then their representatives can be isotoped to circles which lie on the same connected component of $S_{a\cup b}$. It follows that $L^d(\alpha, \beta)$ is connected if and only if the complex of curves of one of the connected components of $S_{a\cup b}$ is empty. That is, $L^d(\alpha, \beta)$ is connected if and only if one of the components of $S_{a\cup b}$ is a once-punctured annulus.

Since $g(L^d(\alpha, \beta)) = L^d(g(\alpha), g(\beta))$ for any automorphism g of $C(S)$, the proof of (i) follows.

(ii) Let $i(\alpha, \beta') = 1$. We can find nonseparating vertices $\alpha_0, \alpha_1, \ldots, \alpha_n$ of $C(S)$ such that $\alpha_1 = \alpha$, α_i and α_{i+1} bounds an annulus with one puncture for $1 \leqslant i \leqslant n$ $(\alpha_{n+1} = \alpha_1)$, $i(\alpha_0, \alpha_i) = 1$ for $1 \leqslant i \leqslant n$, and all of the unmentioned intersection numbers are zero. Then by using the hypothesis of Theorem 4.4 and part (i), we see that the configuration formed by minimally intersecting representatives $f(a_j)$ of $f(\alpha_j)$ is diffeomorphic to the one formed by a_j. Any diffeomorphism between these two configurations can extended to a diffeomorphism of S. (Such a diffeomorphism is constructed in [8] and [12].) Hence, we can assume that $f(\alpha_j) = \alpha_j$ for all j. But, up to isotopy there exists a unique 2-separating circle, which must be b, disjoint from every a_j for $j \neq 1$. Hence the conclusion follows. □

Lemma 4.7. *If $\langle \alpha'; \beta' \rangle$ is a simple pair, then so is $\langle f(\alpha'); f(\beta') \rangle$ for any automorphism f of $C(S)$.*

Proof. Notice that any simple pair $\langle \alpha'; \beta' \rangle$ is determined uniquely by the existence of vertices $\alpha_0, \alpha_1, \ldots, \alpha_n$ such that

(a) each α_i is nonseparating, $0 \leqslant i \leqslant n$;

(b) α_i and α_{i+1} bounds an annulus with one puncture, $1 \leqslant i \leqslant n$ $(\alpha_{n+1} = \alpha_1)$;

(c) $i(\alpha_0, \alpha_i) = 1$, $1 \leqslant i \leqslant n$;
(d) $i(\alpha', \alpha_1) = i(\beta', \alpha_2) = 1$; and
(e) all the other unmentioned geometric intersection numbers are zero.

Since $\operatorname{Aut} C(S)$ preserves the conditions (a)–(e) and 2-separating vertices, the simple pairs are preserved by $\operatorname{Aut} C(S)$. □

Now one can define an action of $\operatorname{Aut} C(S)$ on the punctures and then on the isotopy classes of arcs. One can easily see that a monomorphism $\operatorname{Aut} C(S) \to \operatorname{Aut} B(S)$ can be defined this way. The proofs of these are completely similar to the case of punctured spheres with a few more cases. Then the proof of the theorem will follow as in Section 3.3.

5. Subgroups of $\mathcal{M}_S^*$

The purpose of this section is to give a proof of Theorem 3, as an application of Theorem 1. The proof we give is basically the same as the one sketched in [9] for the surfaces of genus at least two.

Let S be a connected oriented surface. For a vertex α of $C(S)$, we denote by t_α the right Dehn twist about α. It is well known that for $f \in \mathcal{M}_S^*$, $f t_\alpha f^{-1} = t_{f(\alpha)}$ if f is orientation-preserving and $f t_\alpha f^{-1} = t_{f(\alpha)}^{-1}$ if f is orientation-reversing. An immediate consequence of the definition of Dehn twists is the following theorem.

Theorem 5.1. *Let α and β be two vertices of $C(S)$ and let N, M be two nonzero integers. Then $t_\alpha^N = t_\beta^M$ if and only if $\alpha = \beta$ and $N = M$.*

The following relations between Dehn twists are well known. A proof of the theorem may be found in [8] or [12].

Theorem 5.2. *Let α and β be two vertices of $C(S)$ and let N, M be two nonzero integers. Then*
(i) $i(\alpha, \beta) = 0$ *if and only if* $t_\alpha^N t_\beta^M = t_\beta^M t_\alpha^N$,
(ii) (braid relations) $i(\alpha, \beta) = 1$ *if and only if* $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$.

For a group G and for $f \in G$, we denote by $C_G(f)$ the centralizer of f in G, i.e.,

$$C_G(f) = \{g \in G\colon\ gf = fg\}.$$

We denote the center of G by $C(G)$.

Let $m \geqslant 3$ be an integer. Let Γ be a subgroup of finite index of the kernel of the natural homomorphism

$$\mathcal{M}_S^* \to \operatorname{Aut} H_1(S, \mathbb{Z}_m).$$

Then clearly Γ is of finite index in $\mathcal{M}_S^*$.

The following theorem is proved in [8] for the case of punctured spheres (see Theorem 2.3 there). It can be proved similarly for the case of punctured tori.

Theorem 5.3. *Let S be a sphere with at least five punctures or a torus with at least three punctures. An element $f \in \Gamma$ is a power of a Dehn twist if and only if*
 (i) *$C(C_\Gamma(f))$ is isomorphic to $\mathbb{Z}$, and*
(ii) *$C(C_\Gamma(f))$ is not isomorphic to $C_\Gamma(f)$.*

5.1. Proof of Theorem 3

Let $\Phi : G_1 \to G_2$ be an isomorphism. Consider the subgroup $\Phi^{-1}(G_2 \cap \Gamma) \cap \Gamma$ of finite index in G_1. Let Γ_1 be a finite index subgroup of $\Phi^{-1}(G_2 \cap \Gamma) \cap \Gamma$ and let $\Gamma_2 = \Phi(\Gamma_1)$. Then, Γ_1 and Γ_2 are of finite index in $\mathcal{M}_S^*$ and Φ restricts to an isomorphism from Γ_1 to Γ_2. Clearly, $\Phi(C(C_{\Gamma_1}(f))) = C(C_{\Gamma_2}(\Phi(f)))$ and $\Phi(C_{\Gamma_1}(f)) = C_{\Gamma_2}(\Phi(f))$. It follows from Theorem 5.3 that Φ takes sufficiently high powers of Dehn twists to powers of Dehn twists. More precisely, for a vertex α of $C(S)$, since the index of Γ_1 is finite, there exists a nonzero integer N such that $t_\alpha^N \in \Gamma_1$. If $t_\alpha^N \in \Gamma_1$ then $\Phi(t_\alpha^N) = t_\beta^M$ for some vertex β of $C(S)$ and an integer M. If $t_\alpha^{N_1}, t_\alpha^{N_2} \in \Gamma_1$ and $\Phi(t_\alpha^{N_i}) = t_{\beta_i}^{M_i}$, then

$$t_{\beta_1}^{M_1 N_2} = \Phi(t_\alpha^{N_1})^{N_2} = \Phi(t_\alpha^{N_2})^{N_1} = t_{\beta_2}^{M_2 N_1}.$$

Hence by Theorem 5.1, $\beta_1 = \beta_2$. That is, we have a well-defined map φ from the vertex set of $C(S)$ to itself, defined by the equation $\Phi(t_\alpha^N) = t_{\varphi(\alpha)}^M$, which is independent of the choice of the powers involved.

Next, we show that φ is an automorphism $C(S) \to C(S)$. Obviously, we also have a map ψ from the vertex set of $C(S)$ to itself induced by Φ^{-1}. Now for any α of $C(S)$

$$t_\alpha^N = \Phi^{-1}\big(\Phi(t_\alpha^N)\big) = \Phi^{-1}(t_{\varphi(\alpha)}^M) = t_{\psi(\varphi(\alpha))}^T$$

and

$$t_\alpha^N = \Phi\big(\Phi^{-1}(t_\alpha^N)\big) = \Phi(t_{\psi(\alpha)}^K) = t_{\varphi(\psi(\alpha))}^L$$

for some appropriate integers N, M, T, K, L. Then again, by Theorem 5.1, we have $\psi(\varphi(\alpha)) = \alpha$ and $\varphi(\psi(\alpha)) = \alpha$, so φ is a bijection.

Let α, β be two vertices of $C(S)$ with $i(\alpha, \beta) = 0$, then $t_\alpha^{N_1}$ commutes with $t_\beta^{N_2}$, and hence $t_{\varphi(\alpha)}^{M_1}$ commutes with $t_{\varphi(\beta)}^{M_2}$. It follows from Theorem 5.2 that $i(\varphi(\alpha), \varphi(\beta)) = 0$, i.e., φ is an automorphism of the complex of curves $C(S)$.

Now, the automorphism $\varphi : C(S) \to C(S)$ is induced by a mapping class f of S, ($\varphi(\alpha) = f(\alpha)$), by Theorem 1 if S is a sphere with at least five puncture or a torus with at least there punctures. Then

$$\Phi(t_\alpha^N) = t_{\varphi(\alpha)}^M = t_{f(\alpha)}^M = f\, t_\alpha^{\pm M}\, f^{-1}.$$

Let $g \in G_1$. If α is a vertex of $C(S)$, then for appropriate integers N and M,

$$\Phi(g\, t_\alpha^N\, g^{-1}) = \Phi(g)\, t_{\varphi(\alpha)}^M\, \Phi(g)^{-1} = \Phi(g)\, t_{f(\alpha)}^M\, \Phi(g)^{-1} = t_{\Phi(g)(f(\alpha))}^{\pm M}.$$

On the other hand,

$$\Phi(g\,t_\alpha^N\,g^{-1}) = \Phi(t_{g(\alpha)}^{\pm N}) = t_{\varphi(g(\alpha))}^K = t_{f(g(\alpha))}^K$$

for some K. Therefore, we have $\Phi(g)(f(\alpha)) = f(g(\alpha))$, or equivalently, $(\Phi(g)f)(\alpha) = (fg)(\alpha)$, and thus $(fg)^{-1}\Phi(g)f$ is in the kernel of the map $\mathcal{M}_S^* \to \operatorname{Aut} C(S)$, which is trivial by Section 2. Hence $\Phi(g)f = fg$, or, $\Phi(g) = fgf^{-1}$.

The second conclusion follows easily from the first.

Let G be a subgroup of $\mathcal{M}_S^*$ of finite index. Let us denote by $N_{\mathcal{M}_S^*}(G)$ the normalizer of G in $\mathcal{M}_S^*$. That is,

$$N_{\mathcal{M}_S^*}(G) = \{f \in \mathcal{M}_S^*\colon\ f\,G\,f^{-1} \subset G\}.$$

There is a homomorphism $\psi : N_{\mathcal{M}_S^*}(G) \to \operatorname{Aut} G$ defined by $\psi(f)$ to be the conjugation with f. The homomorphism ψ is surjective by the first part of the theorem. Hence we have a surjective homomorphism $\overline{\psi} : N_{\mathcal{M}_S^*}(G) \to \operatorname{Out} G$. Clearly, G is in the kernel of $\overline{\psi}$. Consequently, the order of $\operatorname{Out} G$ is

$$[\operatorname{Out} G : 1] = [N_{\mathcal{M}_S^*}(G) : \ker\overline{\psi}] \leqslant [N_{\mathcal{M}_S^*}(G) : G] \leqslant [\mathcal{M}_S^* : G],$$

and $[\mathcal{M}_S^* : G]$ is finite by the hypothesis.

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