Section 4.5 : Indeterminate Forms and L'Hôpital's Rule

Chapter 4 : Applications of Derivatives

Math 1551, Differential Calculus

"I owe much to the insights of the Messrs. Bernoulli, especially to those of the young (John), currently a professor in Groningen. I did unceremoniously use their discoveries, as well as those of Mr. Leibniz."
- Guillaume de L'Hôpital

L'Hôpital's rule was actually was discovered by John Bernoulli.

Section 4.5 Indeterminate Forms and L'Hôpital's Rule

Topics

1. Indeterminate forms.
2. L'Hôpital's rule.

Learning Objectives

For the topics in this section, students are expected to be able to:

1. Determine whether a limit yields an indeterminate form.
2. Evaluate limits using L'Hôpital's rule.

Students are not required to understand the Cauchy Mean Value Theorem.

Limits and Indeterminant Forms

- Earlier in the course we used algebraic manipulation to evaluate limits of the form

$$\lim_{x \to \infty} \frac{2x^2 + 1}{x^2 - x + 1}$$

- Now consider:

$$\lim_{x \to \infty} \frac{x^2 + 1}{e^x + x}$$

- We need other techniques to evaluate such limits.

Indeterminant Forms

- Quantities such as
$$\frac{\infty}{\infty} \quad \text{and} \quad \frac{0}{0}$$
are known as **indeterminant forms**.
- Five other indeterminant forms that we explore in this section:
    - $0 \cdot \infty$
    - $\infty - \infty$
    - 1^{∞}
    - 0^{0}
    - ∞^{0}
- These expressions **are not numbers**.

L'Hôpital's Rule (LHR)

Theorem

Suppose f and g are differentiable, $g'(a) \neq 0$ on an interval containing a (except possibly at a), and either:

- $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, or
- $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$.

then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

- LHR not valid for the $\frac{0}{\infty}$, $\frac{\infty}{0}$ cases.
- LHR is valid for one-sided limits, and for limits at infinity.
- A proof of the $\frac{0}{0}$ case is in the textbook, it's about 2 pages long.

Example

Evaluate

$$\lim_{x \to 0} \frac{\cos(2x) - \cos(3x)}{x^2 + 1}$$

Indeterminate Products

Evaluate

$$\lim_{x\to 0^+} x \ln x$$

Indeterminate Differences

Evaluate

$$\lim_{x \to \frac{\pi}{2}^-} (\tan x - \sec x)$$

In-Class Participation Activity: Worksheet

(if time permits)

The remainder of the examples in this lecture are incorporated into a worksheet.

- Please solve worksheet problems in groups of **1 to 3 students**
- Each group submits **one** completed worksheet
- Clearly print full names at the top of your sheet
- Every student in a group gets the same grade
- Grading scheme per question:
  - 0 marks for no work, or for working in a group of 4 or more
  - 1 mark for starting the problem or for a final answer with insufficient justification
  - 2 marks for a complete solution

Indeterminate Powers

Several indeterminate forms arise from $\lim_{x\to a}[f(x)]^{g(x)}$. They are:

a) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ type 0^0

b) $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ type ∞^0

c) $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ type 1^∞

For all of these cases, we use natural logarithm:

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln\left(f(x)^{g(x)}\right)}$$

$$=$$

Examples

Evaluate the following limits.

1. $\lim_{x \to 0^+} x^x$
2. $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$
3. $\lim_{x \to 0^+} \sin(x) \ln(x)$

Summary

forms	strategy
$\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$	differentiate: $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$
$0\cdot\infty$	$\displaystyle\lim_{x\to a} f(x)g(x) = \lim_{x\to a}\frac{f(x)}{1/g(x)}$
$\infty-\infty$	find common denominator
1^{∞} or 0^0 or ∞^0	use natural logarithm to convert to $0\cdot\infty$