NAME: Key **SECTION:**

Quiz 4: Consider the matrix $A = \begin{pmatrix} -3 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 2 & -3 \end{pmatrix}$ for the differential equation $\frac{d}{dt}\mathbf{x} = A\mathbf{x}$. The matrix A has eigenvalue -3 with algebraic multiplicity 3, but geometric multiplicity 1.

1. (3 pts) Find an eigenvector v for A and give a corresponding solution to $\frac{d}{dt}\mathbf{x} = A\mathbf{x}$.

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} V = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad V = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \bar{X}_1 = e^{-3t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

2. (3 pts) Find a rank 2 generalized eigenvector $\mathbf{w}_2$ for A (so that $(A - \lambda I)\mathbf{w}_2 = \mathbf{v}$) and give a corresponding solution to $\frac{d}{dt}\mathbf{x} = A\mathbf{x}$.

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \mathbf{w}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow w_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \bar{X}_2 = e^{-3t}\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = e^{-3t}\begin{pmatrix} t \\ 0 \\ 1 \end{pmatrix}$$

3. (3 pts) Find a rank 3 generalized eigenvector $\mathbf{w}_3$ for A (so that $(A - \lambda I)\mathbf{w}_3 = \mathbf{w}_2$) and give a corresponding solution to $\frac{d}{dt}\mathbf{x} = A\mathbf{x}$.

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow w_3 = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} \qquad \bar{X}_3 = e^{-3t}\left(\begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{t^2}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = e^{-3t}\begin{pmatrix} t^2/2 \\ 1/2 \\ t \end{pmatrix}$$

4. (3 pts) Compute the matrix exponential e^{At}.

$$\mathcal{X}(t) = e^{-3t}\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 0 & 1/2 \\ 0 & 1 & t \end{pmatrix} \qquad \mathcal{X}(0) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & 1 & 0 \end{vmatrix} \qquad (\mathcal{X}(0))^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\text{rref}\left|\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right| = \left|\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{array}\right|$$

$$e^{At} = e^{-3t}\begin{vmatrix} 1 & t & t^2/2 \\ 0 & 0 & 1/2 \\ 0 & 1 & t \end{vmatrix}\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix}$$

$$e^{At} = e^{-3t}\begin{pmatrix} 1 & t^2 & t \\ 0 & 1 & 0 \\ 0 & 2t & 1 \end{pmatrix}$$