Name: Nathan Schwerdfeger

Math 3012-L
Spring 2018
Exam 1
1 Feb
Time Limit: 70 Minutes

This exam contains 8 pages (including this cover page) and 6 questions. There are 0 points in total. Justify all answers. Any expression for a number is acceptable; there is no need to find a decimal representation. Write explanations of proofs clearly and in complete thoughts. Points are reserved for clarity. Use the blank side of paper for scratch work. No calculators or notes may be used.

On my honor, I pledge that I will not give or receive aid in examinations; I will not use unapproved materials in examinations; I will not misrepresent my work or represent the work of another as my own; and I will avoid any activity which will encourage others to violate their own pledge of honor.

Signature: Nathan Schwerdfeger

Print Name: Nathan Schwerdfeger

Formal Symbols Crib Sheet

$\neg$	not	$\wedge$	and	$\vee$		or
$\Rightarrow$	implies	$\lightning$	contradiction	$\in$		element of
$\forall$	for all	$\exists$	there exists	$\Leftrightarrow$		equivalence
$\varnothing$	empty set	$\mathbb{N}$	natural numbers	$\mathbb{Z}$		integers
$\mathbb{Z}_+$	positive integers	$\mathbb{Z}_{\geq 0}$	non-negative integers	$\equiv$	$\pmod{n}$	congruence mod n
$\mathbb{Q}$	rationals	$\mathbb{R}$	reals	$\mathbb{C}$		complex numbers
$\times$	Cartesian product	$\subset$	subset	$\setminus$		set minus
$\cap$	intersection	$\cup$	union	$\mathcal{O}$		big-O asymptotic order
2^A	power set of set A	$\lvert A\rvert$	cardinality of set A	A^B		set of functions $B \to A$

The Twelvefold Way:

$|\{f : k \to n\}|$

How many ways to sort k balls into n boxes?

	Arbitrary any sorting	Injective max 1 ball per box	Surjective each box gets ball
Distinct Balls Distinct Boxes	n^k	$\frac{n!}{(n-k)!}$	$n!\left\{ {k \atop n} \right\}$
Identical Balls Distinct Boxes	$\binom{n+k-1}{k}$	$\binom{n}{k}$	$\binom{k-1}{n-1}$
Distinct Balls Identical Boxes	$\sum_{j=0}^{n} \left\{ {k \atop j} \right\}$	1 if $k \leq n$	$\left\{ {k \atop n} \right\}$
Identical Balls Identical Boxes	$p_{\leq n}(k)$	1 if $k \leq n$	$p_n(k)$

1. (a) (3 points) What makes a decision problem **P**? What makes a decision problem **NP**?

P decision problems can have an answer found in polynomial time

NP decision problems can be checked if given a proposed solution in polynomial time, but a solution can not be found in polynomial time.

(b) Consider the following decision problem:

Given a list of n positive integers less than $50n$, decide if two distinct numbers in the list multiply to $4n + 8$.

Describe an algorithm that can answer the decision problem and estimate the $\mathcal{O}$ complexity of your algorithm. You must state what basic operations you are counting.

Basic operations: multiplication, addition, comparison, indexed list lookups

1. find target value $(4n+8)$, this is constant time
2. select an item from the list (n_i), over the course of the problem requires n looku
3. select a different item than n_i, this is $n-1$ lookups per every step 2
4. multiply the two values and compare to the value found in step 1, this is constant time per basic ops listed above

This algorithm is $\boxed{O(n^3)}$

2. (a) (3 points) Circle True or False.

A. For a graph $G = (V, E)$ we have $|E| = \mathcal{O}(|V|^2)$.

TRUE **FALSE** $|E| = O(|V| \cdot |E|)$

B. If S is a set and w is the width of the poset of subsets of S, then $w = \mathcal{O}(|S|^2)$.

$= O(n < \frac{n}{2})$

$\{a, b\}$ $\{a\}$ $\{b\}$ $\{\}$ 2

$w = \binom{n}{\frac{n}{2}}$ n elements

$\frac{n!}{n!(n-\frac{n}{2})!} = \frac{n!}{n!(\frac{n}{2})!}$

TRUE **FALSE**

C. If $H = (V', E')$ is a subgraph of $G = (V, E)$, then $|E'| = \mathcal{O}(|E|)$.

TRUE FALSE

(b) (3 points) How many subgraphs of the complete graph K_{11} with vertex set $\{0, \ldots, 10\}$ are trees?

n^{n-2} = # of trees possible w/ n verticies, comes from Prüfer code (string w/ len n-2) where each character has n options.

$(11)^{(11)-2} = \boxed{11^9}$

(c) (3 points) Suppose a graph G' has 11 vertices. Recall that the *symmetries* of G' are the graph isomorphisms from G' to itself. What is the maximum number of symmetries G' might have? What is the least number of symmetries G' might have?

if G' is complete (K_{11}), Any of the distinct verticies could be in any of the "vertex spots", thus allowing for a maximum of 11! Symmetries.

The least number of symmetries is zero

3. (a) (4 points) Circle True or False.

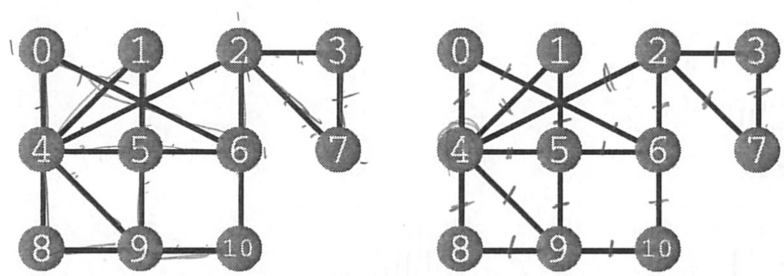
A. If a graph G is planar, then G is also Hamiltonian.
TRUE FALSE

B. If a graph G is 4-colorable, then G is also planar.
TRUE FALSE

C. Deciding if G is planar is an **NP**-problem.
TRUE FALSE

connected BFS — $|V||E|$
check for homeomorphic K_5, $K_{3,3}$ subgraph

D. Deciding if G is planar is not a **P**-problem.
TRUE FALSE



(b) (3 points) Consider the graph shown above. (Two copies are provided for your convenience.) Is the graph Eulerian? Justify your claim.

Yes, the following vertex sequence is a valid Eulerian path:
~~0, 6, 5, 4, 1, 5, 9, 4, 8, 9, 10, 6, 2, 7, 3, 2, 4~~

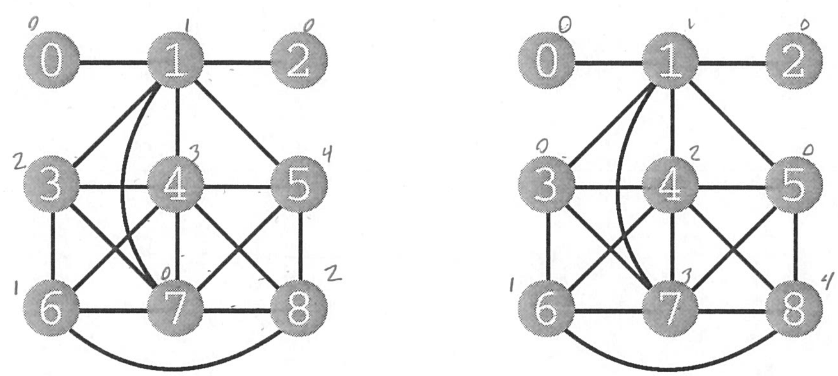
4, 0, 6, 5, 4, 1, 5, 9, 4, 8, 9, 10, 6, 2, 7, 3, 2, 4

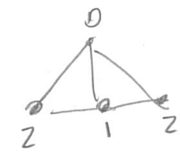
(c) (3 points) Consider the graph shown above. Is the graph Hamiltonian? Justify your claim.

No, when walking / creating the path, you would become "trapped" in the 237 subgraph. If you started outside and entered via the 2, you would be unable to leave. If you started inside, you would be unable to return to complete the walk.

4. (a) (3 points) What is a k-coloring of a graph?

A coloring in which every vertex is assigned a color and no vertices directly connected by a single edge share a color





(b) (3 points) Consider the graph above. What is the chromatic number of this graph? Explain.

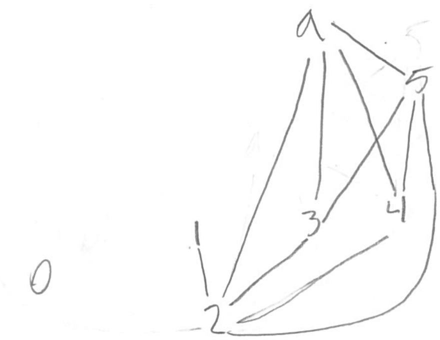
Since there is a clique containing 1347, the chromatic number must be at least 4, This overlaps with another clique containing 4678, which also must contain 4 distinct numbers. Thus the chromatic number is 5 (5-coloring shown above).

(c) BONUS: Suppose G is known to have chromatic number 3 and has vertex set $\{0, \ldots, 9\}$. Both $\{0, 1\}$ and $\{1, 2\}$ are edges in G, but the other edges of G are not known. How many possible 3-colorings of G are consistent with this information, up to relabeling of the colors?

5. (a) (3 points) Draw the Hasse diagram for the poset

$$\{(a,a),(0,0),(1,1),\overset{(2,2)}{}(3,3),(4,4),(5,5),(2,1),(2,3),$$

$$(2,4),(2,5),(2,a),(3,a),(3,5),(4,a),\underline{(4,5)},\underline{(5,a)}\}$$



(b) (3 points) A graph has degree sequence $(4,4,4,4,3,2,2,2,1,1,1)$. Must it be planar, must it be nonplanar, or might it be either? Explain.

$11 = V$

$E \overset{?}{\leq} 3V - 6$

$(24) \overset{?}{\leq} 3(11) - 6$

$24 \overset{?}{\leq} 33 - 6$

$24 \overset{?}{\leq} 27$

$\therefore$ a graph with the given degree sequence cannot be planar.

6. The 2018 Winter Olympics were held in PyeongChang, Soth Korean.

(a) (2 points) Competing were 2,922 athletes representing exactly 92 National Olympic Committees. How many ways might the 2,922 different athletes have come from the 92 different National Olympic Committees if we track which athlete competes for which nation?

Distinct : "balls" athletes
Distinct: "boxes" nations

$\boxed{92! \left\{ {2{,}922 \atop 92} \right\}}$

(b) (3 points) Athletes competed in 102 events in 15 sports, with a gold, silver, and bronze medal awarded in each event. How many ways might the medals have been awarded to the 92 National Olympic Committees if we track the number of each type of medal?

$= (\text{ways to assign gold})(\text{silver})(\text{bronze})$

symmetric balls: medals of same type
Distinct boxes: NOC

$= (\text{ways to assign a medal})^3$

$= \left[\binom{92+102-1}{92}\right]^3$

(c) (3 points) In fact Norway had the highest total medal count with 39, and only 30 National Olympic Committees won any medals. How many ways may the remaining 267 medals have been distributed among the other 29 nations? Note no nation but Norway won more than 38. All ≤ 37

$= \sum_{j=0}^{29} (-1)^j \binom{267-37j+29}{29}$

(d) BONUS: What nation had the second highest medal count?

USA ? I know cross Country skiing went well...