Automorphisms of Some Complexes

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Todo:

1. signposting
2. section on Hatcher's normal form for spheres
3. rewrite $\Gamma_{n,s}$ as relative $\mathrm{Out}(F, B)$
4. understand which -¿ that
5. rewrite Putnam a la metaconjecture discussion
6. more detail: $\pi_2(\mathbb{R}^3 - finite)$
7. printable colors?
8. https://arxiv.org/abs/math/0611241 Kent Leininger Schleimer discussion of fiber of point forget map
9. bad base of sepsphere dimension proof
10. add putnams sym(m)
11. sphsere diagram spacing
12. pentagons vs handlepairs
13. Some comments on the beginning of Sec 5. Sec 5 line 4. Say what you mean by "object", perhaps by giving some examples.

    Page 8 line -2. "maximal" appears too many times

    This is cool! Can you include a proposition with proof about the different dimensions of maximal simplices?

    Page 10 line 1. Putnam −¿ Putman

    If n=2k+1 are there $M_{k+1,1}$-bounding spheres? Is the k+1 allowed to be on the larger side?

    What is the difference between a k-separating sphere and an $M_{k,1}$-separating sphere?

    Is t a transposition or a transvection?

When you talk about inversions, is it clear that you are referring to a standard inversion and not some conjugate? Same for transvections. I could maybe refer to them as being standard.

In the third paragraph of the proof it would help to have "we must have" in a couple of places.

The hyphen in one of the "k-separating"s is a minus sign.

Have you defined push maps? Maybe you should be consistent about referring to group elements either algebraically or topologically.

a $M_{k,1} - - > anM_{k,1}$ (as we discussed)

Don't need parentheses for Aut X. Typesets better.

Awesome pictures!

Is it weird that pentagons give you sharing pairs in the Out(Fn) case? Maybe you can make a comment about the relationship between your pentagons, Ivanov's pentagons, and the horrible mess that Tara and I use to characterize sharing pairs. (I know we've discussed this before.)

The way that Tara and I describe sharing pairs is topological. Then to show that they are preserved by automorphisms we show that they are characterized by certain shapes in the complex.

In Lemma 11, maybe change the wording of "uniquely engulf"?

Did you define y^{in}?

14. Alan Metaconj

15. The Bass-Serre trees and graph of groups

16. The metaconjecture is a kind of generalized noncommutative Poincare duality? Here are two functors from Top to Group: $\mathrm{Out}\,\pi_1$ and $\mathrm{Aut}\,\mathcal{C}$ where the first gives a description of geometric permutations of dimension 1 submanifolds, while $\mathrm{Aut}\,\mathcal{C}$ C should be some complex of codimension 1 manifolds, so that geometric permutations of dimension 1 and dimension n-1 submanifolds are dual. The connection is through the Van Kampen and Bass-Serre splitting of π_1 according to the codimension 1 splitting of the manifold

17. Shouldn't this whole thing just be some sort of rigorous description of separating doubling?

18. def trinion pants

19. https://mathoverflow.net/questions/142699/analogues-of-the-curve-complex-for-outf?rq=1

20.

1 intro

the metaconjecture positive results in the metaconjecture the metatheorem Dans metametaconjecture interpretations of Out: graphs disks spheres change of coordinates for spheres MOd analog complex of surfaces Toward Dan's metatheorem must be closed handlebody, mod with boundary

2 Index of Notation

3 Bass-Serre Theory

Work of Bass and Serre, see [47].

A *graph of groups* Γ is a connected graph (V, E) together with a collection of vertex groups $\{G_v\}_{v\in V}$ and a collection of edge groups $\{G_e\}_{e\in E}$ together with inclusions of the edge groups into their incident vertex groups, that is for each edge $e = uv$ there are injections

$$G_u \xleftarrow[i_{uv}]{} G_e \xrightarrow{i_{vu}} G_v$$

Then for any spanning tree T of Γ the fundamental group $\pi_1(\Gamma, T)$ is the group generated by E and the vertex groups G_v for $v \in V$, together with the relations $i_{uv}(g) = e i_{vu}(g) e^{-1}$ for all $e = uv$ and $g \in G_e$, and $e = 1$ for all $e \in T$.

The universal cover $\tilde{\Gamma}$ of the graph of groups Γ (with respect to $\pi_1(\Gamma, T)$) is the tree with vertices given by left cosets vertex groups in $\pi_1(\Gamma, T)$ and edges given by the left cosets of edge groups in $\pi_1(\Gamma, T)$. So if gG_e is a left coset with $g \in \pi_1(\Gamma, T)$ and $e = uv$ is an edge, then gG_e is an edge between the $\tilde{\Gamma}$ vertices gG_u and geG_v. The quotient $p : \tilde{\Gamma} \to \Gamma$ is given by $gG_x \mapsto x$ for any vertex or edge. In fact $\tilde{\Gamma}$ is a tree, and is equipped with action of $\pi_1(\Gamma, T)$ by left multiplication

$$h \cdot (gG_x) = (hg)G_x$$

for any $g, h \in \pi_1(\Gamma, T)$. The action of $\pi_1(\Gamma, T)$ on the tree $\tilde{\Gamma}$ thus has

$$\operatorname{stab}_{\pi_1(\Gamma,T)}(gG_x) = gG_xg^{-1}$$

and respects the projection p and acts without inverting any edges of the tree.

Theorem 1. *Let T be a tree with group G acting without inversions. If Γ is the quotient graph of groups with T any spanning tree, then G is isomorphic to $\pi(\Gamma, T)$, and there is an G-equivariant isomorphism betweeen T and the universal cover $\tilde{\Gamma}$ of Γ.*

4 background

free group

Theorem 2 (Nielson [42])**.** *$\mathrm{Aut}\, F_n$ is generated by permutations, inversions, and transvections.*

x^{in} x^{out} Van Kampen cut apart change of coordinates

$M_{n,s}$ is the compact 3-manifold obtained from n copies of $S^1 \times S^2$ with the interiors of s disjoint balls. $Diff(M_{n,s})$ is the group of orientation-preserving diffeomorphisms of $M_{n,s}$. $\Gamma_{n,s}$ is the group $\pi_0 Diff(M_{n,s}, \partial)$ but its normal subgroup of Dehn twists about embedded 2-spheres.

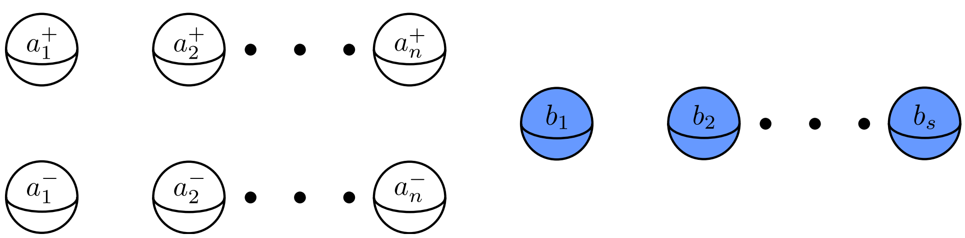


Figure 1

Theorem 3 (Laudenbach [33])**.** $\Gamma_{n,0} \cong \mathrm{Out}\, F_n$ *and* $\Gamma_{n,1} \cong \mathrm{Aut}\, F_n$

$\mathcal{G}_n$ is the graph of free splittings of F_n

$\mathcal{S}_{n,s}$ is the complex of spheres in $M_{n,s}$ whose k simplices are $k+1$ disjointely essential embedded 2-spheres in $M_{n,s}$ $\mathcal{S}_\infty$ is the subcomplex of $\mathcal{S}_{n,0}$ consisting of sphere systems having at least one non-simply connected complimentary component

Theorem 4 (Hatcher [20])**.** *$\mathcal{S}_{n,0} - \mathcal{S}_\infty$ is homeomorphic to Culler-Vogtmann outer space*

Theorem 5 (Aramayona, Souto [4])**.** *The natural map* $\operatorname{Out} F_n \to \operatorname{Aut} S_n \cong \operatorname{Aut} \mathcal{G}_n$ *is an isomorphism for* $n \geq 3$*.*

Theorem 6 (Pandit [43])**.** *For* $n \geq 3$ *we have* $\operatorname{Aut}(\mathcal{S}_n^{nonsep}) \cong \operatorname{Out}(F_n)$*.*

Theorem 7 (Putman's Lemma [45])**.** *Let group* G *with generators* H *act on simplicial set* X*. Fix a basepoint* $v \in X^{(0)}$*. If*

1. *for all* $v' \in X^{(0)}$ *the orbit* $G \cdot v$ *intersects the connected component of* v' *in* X *and*
2. *for all* $h \in H^{\pm}$ *there is some path from* v *to* $h \cdot v$

then X *is connected.*

Lemma 8. *An essential separating sphere of* $M_{n,s}$ *is uniquely determined by any of the following data:*

1. *A bipartition of any* n *disjoint nonseparating spheres and the boundary spheres.*
2. *A collection of* $\alpha_1, \dots, \alpha_k$ *independent loops and disjoint nonseparating spheres* $a_1, \dots, a_k$ *such that every* α_i *intersects a* a_j*, and a subset of the boundary spheres.*
3. *A free splitting of each* $\pi_1(M_{n,s}, x_i)$ *for a point* x_i *on each boundary component, or else the conjugacy class of a free splitting if* $s = 0$*.*

5 The Birman Exact Sequence in the Curve Complex

Definitions:!!!! surface $S_{g,p}$ with marked points P

The purpose of this section is to sketch the surface analog of the proof we will follow for the complex of spheres.

The works of Ivanov [27], Korkmaz [30], and Luo [36], describe the automorphisms of complexes of curves. Their theorem may be summarized

Theorem 9. *The natural map*

$$\operatorname{MCG}^{\pm} S_{g,p} \to \operatorname{Aut} \mathcal{C}S_{g,p}$$

is an isomorphism whenever the curve complex $\mathcal{C}S_{g,p}$ *has positive dimension* $3g + p - 4$ *and* $(g,p) \neq (1,2)$*.*

Although their methods of proof are general and do not require separate consideration of the closed and punctured cases, we will demonstrate that additional punctures of the surface leave the isomorphism

$$\mathrm{MCG}^{\pm} S_{g,p} \to \mathrm{Aut}\, \mathcal{C}S_{g,p}$$

intact, if we consider the celebrated Birman exact sequence. The Birman exact sequence describes the mapping class group of a punctured surface as a fibration over the mapping class group of the unpunctured surface, where the fundamental group is the fiber [8].

Theorem 10. *Let $q \in S_{g,p}$. The surface inclusion $S_{g,p+1} = S_{g,p} - \{q\} \hookrightarrow S_{g,p}$ induces the following short exact sequence*

$$1 \longrightarrow \pi_1(S_{g,p}, q) \longrightarrow \mathrm{MCG}^{\pm}(S_{g,p+1}, q) \longrightarrow \mathrm{MCG}^{\pm} S_{g,p} \longrightarrow 1$$

Our goal in this subsection will be an independent proof of the following weaker version of Theorem 9, in preparation for the free group analog.

Theorem 11. *If the natural map*

$$\mathrm{MCG}^{\pm} S_{g,p} \to \mathrm{Aut}\, \mathcal{C}S_{g,p}$$

is an isomorphism, then so is

$$\mathrm{MCG}^{\pm} S_{g,p+1} \to \mathrm{Aut}\, \mathcal{C}S_{g,p+1}.$$

The proofs of Theorems 11 and 10

Every simplex Δ of $\mathcal{C}S_{g,p}$ is a collection of disjoint curves cuts up the surface $S_{g,p}$ a number of connected components. This gives a Bass-Serre graph of groups [**?**] for $\pi_1 S_{g,p}$ induced by the Δ specified splitting. The underlying simple graph is the adjacency graph studied in Margalit, Behrstock [7] and Shackleton [48]. Also appear as graphs associated to pants decompositions/markings in [19].

Definition 12. Let $\Delta \subset \mathcal{C}S_{g,p}$ be a simplex. The *region adjacency graph* $\mathcal{G}_{\Delta}$ of Δ is the graph whose vertices are the connected components of the cut surface

$$S_{g,p} - \bigcup_{c \in \Delta} c$$

with an edge for every curve c incident to the regions it bounds.

We will also consider the graph simplification $\mathcal{G}_\Delta^{simp}$ obtained from the (possibly looped, multi-edged) graph $\mathcal{G}_\Delta$ by removing any self-loops and identifying multi-edges.

Automorphisms of the curve complex act naturally on the set of adjacency graphs by isomorphism, as we show in the following lemma, due to Margalit and Behrstock [7]. Or Irmak?!!!! For completeness, we argue similarly here.

Lemma 13. *Curve complex automorphisms preserve the edge incidence of region adjacency graphs.*

Let $\phi \in \operatorname{Aut} \mathcal{C}S_{g,p}$ and let Δ be a simplex of $\mathcal{C}S_{g,p}$ with adjacency graph $\mathcal{G}_\Delta$. Then $e_c, e_{c'}$ are incident edges of $\mathcal{G}_\Delta$ if and only if $e_{\phi(c)}, e_{\phi(c')}$ are incident edges of $\mathcal{G}_{\phi(\Delta)}$.

Proof. We will argue that ϕ induces a map $\phi_* : E_{\mathcal{G}_\Delta} \to E_{\mathcal{G}_{\phi(\Delta)}}$ on the set of edges that preserves the incidence and non-incidence of edges.

Let e_c be an edge of $\mathcal{G}_\Delta$ given by curve c. Then $\phi_*(e_c) = e_{\phi(c)}$ defines a bijection between the edges of $\mathcal{G}_\Delta$ and $\mathcal{G}_{\phi(\Delta)}$. We will show e_c is incident to $e_{c'}$ if and only if there is a curve of $\mathcal{C}S_{g,n}$ intersecting c and c', but no other curve of Δ. Then $e_{\phi(c)}$ is incident to $e_{\phi(c')}$ if and only if there is a curve of $\mathcal{C}S_{g,n}$ intersecting $\phi(c)$ and $\phi(c')$, but no other curve of $\phi(\Delta)$.

Suppose e_c is incident to $e_{c'}$. Observe every region of $S_{g,p} - \bigcup_{c\in\Delta} c$ contains an embedded pair of pants $S_{0,3}$. So if we consider regluing regions along c and c', we obtain the component R of $S_{g,p} - \bigcup_{b\neq c,c'} b$ with $c, c' \subset R$. Then R must contain an embedded $S_{0,5}$ or a $S_{1,1}$. So R contains a curve c'' intersecting c and c', and since $c'' \subset R$, it does not intersect any other curve of Δ.

Suppose e_c is not incident to $e_{c'}$ in $\mathcal{G}_\Delta$. Then there is a multicurve $\Delta' \subset \Delta$ which separates c from c' in $S_{g,p}$. But then every curve that intersects c and c' must intersect a curve of Δ'. □

Example 14. Edge incidence isn't always enough for an isomorph: Explain $S_1, 2$, $K_3 \cong K_{1,3}$, loop-multiedge swap with 2 vertices

We recall the Whitney Graph Isomorphism Theorem [53] states that the edge-incidence relation determines a simple graph, with a single exceptional pair.

Theorem 15. *An edge-incidence preserving bijection between two simple, connected graphs is a isomorphism, provided neither is the complete graph K_3.*

There is an edge-incidence preserving edge bijection between the complete graph K_3 and the complete bipartite graph $K_{1,3}$.

Edge incidence can be preserve and swap a loop with a multiedge, as in $\operatorname{Aut}\mathcal{C}S_{1,2}$, Show example !!!!

Corollary 16. *Curve complex automorphisms induce isomorphisms of region adjacency graphs of maximal simplices, except in the case of $S_{1,2}$.*

Let $\phi \in \operatorname{Aut}\mathcal{C}S_{g,p}$ for $(g,p) \neq (1,2)$, and let Δ be a maximal simplex of $\mathcal{C}S_{g,p}$. Then $\mathcal{G}_\Delta$ and $\mathcal{G}_{\phi(\Delta)}$ are isomorphic graphs.

Proof. Any maximum simplex Δ gives a pants decomposition!!!! of the surface $S_{g,p}$ with $3g+p-3$ curves and $2g+p-2$ pairs of pants. So $\mathcal{G}_\Delta^{simp}$ and $\mathcal{G}_{\phi(\Delta)}^{simp}$ are simple, connected graphs with the same number of vertices and the same edge-incidence relations. Then by Whitney's Theorem 15, $\mathcal{G}_\Delta^{simp}$ is isomorphic to $\mathcal{G}_{\phi(\Delta)}^{simp}$.

To see that self-loops are preserved, observe that as Δ cuts $S_{g,p}$ into pairs of pants, every vertex of G_Δ has degree at most 3. Then if e_c is a self-loop at vertex v_R, it is incident to exactly one other edge e_x which cannot be a self-loop, so e_x is uniquely represented in $\mathcal{G}_\Delta^{simp}$. If $(g,p) \neq (1,2)$, then e_x is incident to another edge e_y. Then $e_\phi(x)$ is also uniquely represented in the isomorphic graph $\mathcal{G}_\Delta^{simp}$ and has a degree 1 vertex. Then in G_Δ^{simp}, $e_\phi(x)$ is incident to $e_\phi(y)$, and $e_\phi(c)$ is incident to $e_\phi(x)$, but not $e_\phi(y)$ or any other edge, so $e_\phi(c)$ must be a loop at the vertex which is degree 1 in $G_{\phi(\Delta)}^{simp}$. $\square$

Lemma 17. *For positive dimension and $(g,p) \neq (1,2)$, any automorphism of $\operatorname{Aut}\mathcal{C}S_{g,p}$ preserves the topological type of curves sides of separating curves.*

Proof. We will characterize each topological type of curve by a combinatorial property of a corresponding region adjacency graph, and apply Lemmas 13 and 16.

- Nonseparating curves: Observe that a curve c is nonseparating if and only if there is a maximal simplex Δ for which e_c is a self-loop in the region adjacency graph $\mathcal{G}_\Delta$.

- Separating curves: Observe that if curve x separates $S_{g,p}$

$$S_{g,p} = S_{g',p'} \sqcup_c S_{g-g',p-p'+2}$$

then the corresponding edge e_x of the region adjacency graph $\mathcal{G}_\Delta$ is a cut edge. More specifically, if

$$\Delta = \Delta_+ \cup \{x\} \cup \Delta_-$$

with Δ_+ and Δ_- the curves on each side of the separating curve c, then e_x separates $\mathcal{G}_\Delta$

$$\mathcal{G}_\Delta - \{e_x\} = G_{\Delta_+} \sqcup G_{\Delta_-}$$

into the components G_{Δ_+}, with $3g' + p' - 3$ edges and genus g', and G_{Δ_-} with $3(g - g') + p - p' - 1$ edges and genus $g - g'$.

$\square$

Remark 18. For the closed surface S_g the inclusion $S_g - \{q\} \hookrightarrow S_g$ has a well defined adjoint forgetful map

$$\mathcal{C}S_{g,1} \to \mathcal{C}S_g$$

since we do not allow peripheral curves in $\mathcal{C}S_{g,1}$. However, in the case of multiple punctures P, the surface $S_{g,p}$ has curves bounding twice-punctured disks, which may become peripheral if a puncture is forgotten. However, excluding these curves gives a subcomplex $\mathcal{C}(S_{g,p}, q) \subset \mathcal{C}S_{g,p}$ where the puncture-forgetting map is well-defined.

$$\rho_q : \mathcal{C}(S_{g,p}, q) \to \mathcal{C}S_{g,p-1}$$

Kent, Leininger, and Schleimer [28] show that this forgetful projection has fibers described by Bass-Serre trees of the surface fundamental group so that there is a fibration of the form

$$\mathcal{T} \to \mathcal{C}(S_{g,p}, q) \to \mathcal{C}S_{g,p-1}.$$

More rigorously,

Theorem 19. *Let $\Delta \subset \mathcal{C}S_{g,p}$ be a simplex with interior point $x \in \Delta$. Then the fiber $\rho_q^{-1}(x)$ is $\pi_1(S_{g,p}, q)$-equivariantly homeomorphic to the tree $\mathcal{T}_\Delta$, the Bass-Serre tree for the splitting of $\pi_1(S_{g,p}, q)$ determined by the multicurve Δ.*

Observe that $\mathcal{C}(S_{g,p}, q)$ is not characteristic in $\mathcal{C}S_{g,p}$, since in general automorphisms of $\mathcal{C}S_{g,p}$ will permute the punctures. Let $\operatorname{Aut}(\mathcal{C}S_{g,p}, q) < \operatorname{Aut}\mathcal{C}S_{g,p}$ be the subgroup of $\operatorname{Aut}\mathcal{C}S_{g,p}$ which preserves the fibration $\mathcal{C}(S_{g,p}, q) \to \mathcal{C}(S_{g,p-1})$, i.e. $\phi\left(\rho_q^{-1}\rho_q(x)\right) = \rho_q^{-1}\rho_q(\phi(x))$. These automorphisms display the structure of the Birman exact sequence.

Lemma 20. *This diagram commutes*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(S_{g,p-1}, q) \arrow[r] \arrow[d] & \mathrm{MCG}^{\pm}(S_{g,p}, q) \arrow[r, "f_q"] \arrow[d] & \mathrm{MCG}^{\pm} S_{g,p-1} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(S_{g,p-1}, q) \arrow[r, "\alpha"] & \mathrm{Aut}\,\mathcal{C}(S_{g,p}, q) \arrow[r, "\rho_q"] & \mathrm{Aut}\,\mathcal{C}S_{g,p-1} \arrow[r] & 1
\end{tikzcd}$$

and has exact rows when ρ_q is surjective.

Proof. The map α is defined by the first square, so it certainly commutes. And α gives a well defined injection, since for any nontrivial loop γ there is a nonseparating curve c which intersects γ so that the push map $\alpha(\gamma)$ moves c, and $c \in \mathcal{C}(S_{g,p}, q)$.

The second square must commute, since if $[\psi] \in \mathrm{MCG}^{\pm}(S_{g,p}, q)$ is a mapping class and c a curve of $S_{g,p}$, the homotopy class of $\psi(c)$ is the same if we first forget that ψ fixes q or if we first allow ψ with q fixed, then homotope $\psi(c)$ forgetting q.

According to Theorem 19, a fiber $\rho_q^{-1}(x)$ of the projection $\rho_q : \mathcal{C}(S_{g,p}, q) \to \mathcal{C}S_{g,p}$ for x in interior point of simplex $x \in \Delta$ is homeomorphic to the Bass-Serre tree $\mathcal{T}_\Delta$. Then the kernel $\ker \rho_q$ is a group acting on the tree $\mathcal{T}_\Delta$, so by the Fundamental Theorem of BassSerre Theory (how to reference?!!!!), $\ker \rho_q$ is isomorphic to the fundamental group π_1 of the quotient graph of groups, but the corresponding graph of groups is exactly the Van Kampen splitting of π_1 induced by Δ. Thus

$$\ker \rho_q = \text{image } \alpha \cong \pi_1(S_{g,p}, q)$$

and the second row is exact. $\square$

We will show that, though curve complex automorphisms might not preserve the fibers of ρ_q for any particular puncture q, they do permute the fibers of the puncture-forgetting projections $(\rho_q)_{q \in P}$. Korkmaz's proof of Theorem 9 utilizes a slightly more general arc complex allowing peripheral arcs [30]

Definition 21. Define the arc complex complex $\mathcal{A}S_{g,p}$ to be the complex of homotopy classes of embedded non-peripheral arcs in $S_{g,p}$ with endpoints in P, where an arc is peripheral if it is a separating loop based at a single puncture, and one of its sides is a punctured monogon of $S_{g,p}$. Two arcs are adjacent in $\mathcal{A}S_{g,p}$ if their homotopy classes have disjoint representatives and share no punctures as endpoints.

Remark 22. The arc complex has as vertices both arcs with two distinct endpoints and loops based at a single puncture. Since loops that are disjoint are always based at distinct punctures, there is an obvious way to color (in the graph-theoretic sense) the loop-vertices of $\mathcal{A}S_{g,p}$: assign a color to each puncture and all the loops based at that puncture. The arcs with distinct endpoints require two colors, however. We make a slight generalization of k-colorings to allow a privileged set of vertices that which require two colors.

Definition 23. A k, η-painting of a graph $G = (V, E)$ is an assignment to each vertex v of a number of colors $\eta(v)$ and a choice $f(v) \subset \{0, \ldots, k-1\}$ of $\eta(v)$ colors such that two adjacent vertices have no common colors. I.e. functions $\eta : V \to \mathbb{Z}_+$ and $f : V \to 2^{\{0,\ldots,k-1\}}$ so that $|f(v)| = \eta(v)$ and

$$f(v) \cap f(u) = \varnothing$$

if v is adjacent to u in G. Call G k-paintable if it admits a k, η-painting.

MOVE THIS TO INTROMATERIAL? Margalit in [38] discusses the trinion pants complex. Pants decompositions of $S_{g,p}$ are maximal simplices of $\mathcal{C}S_{g,p}$, with two such pants decompositions giving sharing an edge in the pants complex if they differ by a single pair of minimally intersecting curves. Hatcher and Thurston demonstrated the pants complex (which they call *markings*) of a surface is connected and simply connected in [19]. We recall their result as the following helpful lemma.

Lemma 24. *Let Δ, Δ' be maximal k-simplicies of $\mathcal{C}S_{g,p}$. Then there is a sequence $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta'$ of maximal simplices such that $\Delta_i \cap \Delta_{i+1}$ is a $k-1$ simplex and the curves $c_i \in \Delta_i - \Delta_{i+1}$ and $c_i' \in \Delta_{i+1} - \Delta_i$ are contained in a single component R of $S_{g,p} - \bigcup_{x \in \Delta_i \cap \Delta_{i+1}} x$. Further, c_i, c_i' can be chosen to intersect once if $R \cong S_{1,1}$ and twice if $R \cong S_{0,4}$.*

Lemma 25. *The arc complex is uniquely paintable.*

Let $3g + p \geq 6$. The arc complex $\mathcal{A}S_{g,p}$, it admits a unique p-painting, up to permutation of the colors.

Proof. We will argue with a modification of the Putman trick 7. Observe that (since we exclude peripheral arcs) every maximal simplex of $\mathcal{A}S_{g,p}$ has an arc with an end at every puncture. Observe that if $p \leq 2$ the result is trivial, so we assume $p \geq 3$. Let f be any p-coloring of the arc complex $\mathcal{A}S_{g,p}$.

We first fix a collection X of curves. If $g \geq 1$, then $S_{g,p}$ has a nonseparating curve, and we let X contain p parallel nonseparating loops based at the punctures as in Figure 2.

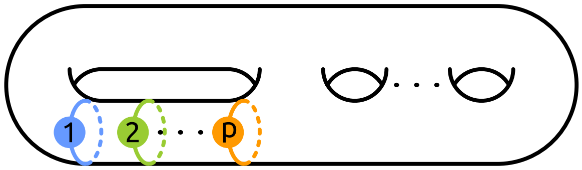


Figure 2: A base collection of parallel nonseparating loops.

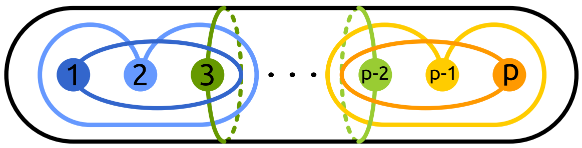


Figure 3: A base collection of mostly parallel loops in the punctured sphere.

If $g = 0$ we take as $X = \{x_i\}_{i=1}^{p}$ to be as in Figure 3 with $p-4$ parallel loops based at $p_2, \ldots, p_{p-2}$ and 4 additional curves x_1, x_2, x_{p-1}, x_p so that x_1, x_2 and x_3 pairwise intersect twice and are disjoint from all other curves of X, and similarly x_p, x_{p-1}, x_{p-2} pairwise intersect twice and are disjoint from all other curves of X. We will argue that each loop of X must be colored differently. Observe that there is an arc α_{12} from p_1 to p_2 which is disjoint from all loops of X except x_1 and x_2. Similarly there is $\alpha_{p-1,p}$ disjoint from all loops of X except x_p and x_{p-1}. So the collection $\alpha_{12}, x_3, \ldots, x_{p-2}, \alpha_{p-1,p}$ require all p-colors to paint, and it must be that $f(x_1) \cup f(x_2) = f(\alpha_{12})$ and $f(x_{p-1}) \cup f(x_p) = f(\alpha_{p,p-1})$. So X requires p-colors to paint.

We may then assume, possibly after relabeling the colors, that f colors the arcs of X by their punctures. We will show that the painting on X forces the painting on all of $\mathcal{A}S_{g,p}$. Our technique will be to contruct paths with the group action of $\operatorname{MCG} S_{g,p}$ on $\mathcal{A}S_{g,p}$ and show that f is determined along these paths. Let $\alpha_{i,i+1}$ be an arc which is disjoint from all loops of X except x_i and x_{i+1} and contained in the annulus they bound if they are disjoint. Take as a generating set of $\operatorname{MCG} S_{g,p}$ the Dehn half-twists along the arcs $\alpha_{i,i+1}$ and the usual Humphrey's generators of Dehn twists about nonseparating curves which are disjoint from X, except for one curve z which intersects each loop of X exactly once.

We now claim that for $h \in H$ the painting f is determined on the loops $h \cdot X$ by the painting on X. We must consider several cases depending on whether h is a twist or half-twist, and how $\alpha_{i,i+1}$ intersects X. These cases are considered in Figures 4-7.

Observe that if $g \in \operatorname{MCG} S_{g,p}$ we can write $g = h_n \cdots h_1$ with $h_i \in H$.

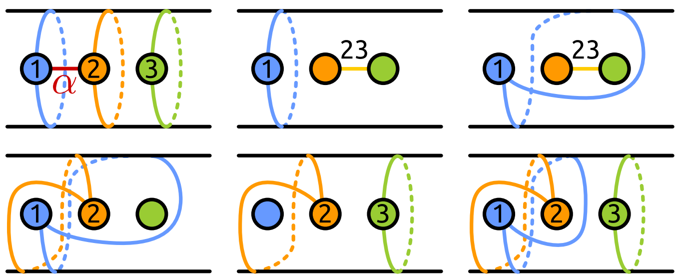


Figure 4: Case 1: The half-twist h is about an arc α in an annulus between x_i and x_{i+1} and disjoint from the other loops of X, as in the top left. The lower right shows $h(x_1)$, $h(x_2)$, and $h(x_3) = x_3$. Then if $f(x_i) = p_i$ the sequence of curve replacements shows that $f(h(x_i)) = p_i$.

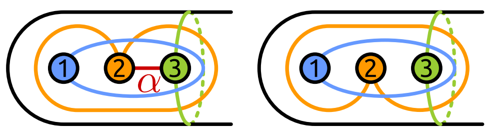


Figure 5: Case 2: The half-twist h is about an arc α x_i and x_{i+1} and disjoint from the other loops of X, where x_i and x_{i+1} intersect twice. We may assume the configuration on the left and note that $h(x_2) = x_3$, so that $f(h(x_3))$ must be p_2.

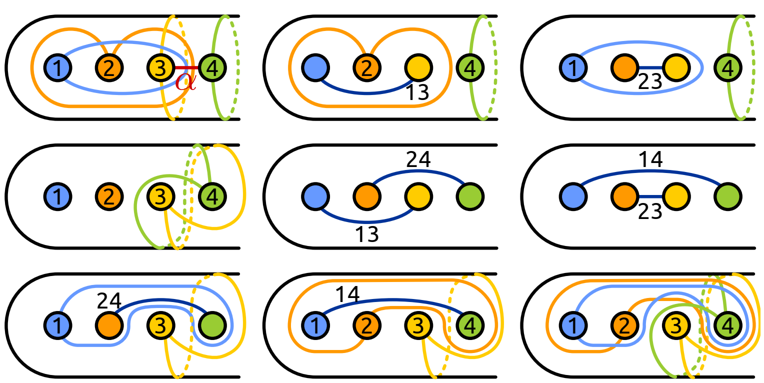


Figure 6: Case 3: The half-twist h is about arc α between disjoint curves x_i and x_{i+1}, and α intersects other curves of X. We may assume the configuration on the top left. Then the image of h is shown in the bottom right. Note that by Case 1, $f(h(x_3)) = p_3$ and $f(h(x_4)) = p_4$ so that we need only determine $f(h(x_1))$ and $f(h(x_2))$.

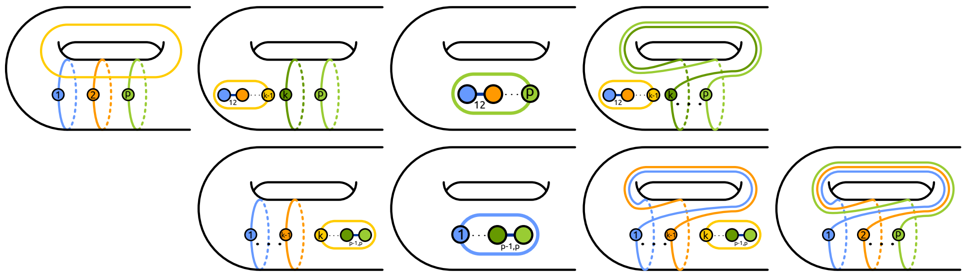


Figure 7: Case 4: The Dehn twist $h = T_z$ about a nonseparating curve which links with the loops of X. We may assume the configuration of X and z as in the top left. The image $T_z(X)$ is shown in the bottom right. In the top row: Replace x_1 and x_2 with $\alpha_{1,2}$ so that $f(\alpha_{12}) = \{p_1, p_2\}$. Then iteratively replace x_k with the loop y_k separating $\alpha_{1,2}, y_3, \ldots, y_{k-1}$ from the other punctures, so that $f(y_k) = p_k$ for $k = 3, \ldots, p$. Then replace y_k with $h(x_k)$ so that $f(T_z(x_k)) = p_k$ for $k = p, \ldots, 3$. A similar process reversing the order of the punctures as in the bottom row shows $f(T_z(x_k)) = p_k$ for $k = 1, \ldots, p-2$.

And if the loops of $h_k \cdots h_1 \cdot X$ are painted by their punctures, then so are the loops of $h_{k+1} \cdots h_1 \cdot X$ by the above argument.

Observe in the case of $g = 0$ every curve is in $\operatorname{MCG} S_{g,p} \cdot X$. If $g \geq 1$ then $\operatorname{MCG} S_{g,p} \cdot X$ includes all nonseparating loops, and any separating loop x is disjoint from $p-1$ mutually disjoint loops $x_1, \ldots, x_{p-1}$ so that the color $f(x)$ is determined by the colors $f(x_i)$ and so $f(x)$ must be colored by its puncture. $\square$

Lemma 26. *Curve complex automorphisms induce arc complex $\mathcal{A}S_{g,p}$ automorphisms.*

Proof. By Lemma 17 automorphisms of $\mathcal{S}_{g,p}$ preserve the class of two curves. Observe that curves c, c' of $S_{g,p}$ bound a punctured annulus if and only if there is a maximal simplex $\Delta \subset \mathcal{C}S_{g,p}$ containing c, c' so that in the region adjacency graph $\mathcal{G}_\Delta$, the edges e_c and $e_{c'}$ are incident at a degree 2 vertex v_a. Then applying Lemma 16 $\phi(c)$ and $\phi(c')$ must cobound a punctured annulus.

Let $\phi \in \operatorname{Aut} \mathcal{C}S_{g,p}$ and let a be an punctured annulus $\mathcal{A}S_{g,p}$ with bounding curves c, c'. If $(g,p) \neq (1,2)$ then c, c' uniquely specify the annulus. Then $\phi(c), \phi(c')$ specify cobound a punctured annulus $\phi_*(a)$.

Finally a, a' are disjoint arcs if and only if there is a maximal simplex $\Delta \subset \mathcal{C}S_{g,p}$ such that the bounding curves of regular neighborhoods of a and

a' are in Δ. Then a and a' are represented by distinct vertices v_a and $v_{a'}$ of $\mathcal{G}_\Delta$. So a and a' are disjoint if and only if $\phi_*(a)$ and $\phi_*(a')$ are disjoint.

Thus $\phi_* : \mathcal{A}S_{g,p} \to \mathcal{A}S_{g,p}$ is an isomorphism. $\square$

Lemma 27. *Curve complex automorphism permute puncture-projection fibers.*

Let $\phi \in \operatorname{Aut} \mathcal{C}S_{g,p}$ for $3g + p \geq 6$. Then if x, y are curves of $S_{g,p}$ with $\rho_q(x) = \rho_q(y)$, then there is a puncture $q' \in P$ such that $\rho_{q'}(\phi(x)) = \rho_{q'}(\phi(y))$.

Proof. Consider the structure of $\rho_q^{-1}(\rho_q(x))$. We have it is a subtree of $\mathcal{C}S_{g,p}$ with $x, x' \in \rho_q^{-1}(\rho_q(x))$ adjacent if and only if they bound an annulus punctured by q. Then we have a path $x = x_0 \to \ldots \to x_n = y$ such that x_i, x_{i+1} bound an annulus punctured by q. Using we have $\phi(x_i), \phi(x_{i+1})$ bound an annulus punctured by some q_i. Then since x_i, x_{i+1}, x_{i+2} bound annuli punctured by q, which are the regular neighborhoods of loops $a_i, a_{i+1} \in \mathcal{A}S_{g,p}$ based at q. Then $\phi(x_i), \phi(x_{i+1}), \phi(x_{i+2})$ bound annuli which are the regular neighborhoods of loops $\phi_*(a_i), \phi_*(a_{i+1})$. By Lemma 25 $\phi_*(a_i)$ and $\phi_*(a_{i+1})$ are based at a common point q'. $\square$

Proof of Theorem 11. Assume that the natural map

$$\mathrm{MCG}^{\pm} S_{g,p-1} \xrightarrow{\gamma} \operatorname{Aut} \mathcal{C}S_{g,p-1}$$

is an isomorphism. Then by Lemma 40 the following diagram commutes

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(S_{g,p-1}, q) \arrow[r] \arrow[d, "\alpha"] & \mathrm{MCG}^{\pm}(S_{g,p}, q) \arrow[r, "f_q"] \arrow[d, "\beta"] & \mathrm{MCG}^{\pm} S_{g,p-1} \arrow[r] \arrow[d, "\gamma"] & 1 \\
1 \arrow[r] & \pi_1(S_{g,p-1}, q) \arrow[r] & \operatorname{Aut} \mathcal{C}(S_{g,p}, q) \arrow[r, "\rho_q"] & \operatorname{Aut} \mathcal{C}S_{g,p-1} \arrow[r] & 1.
\end{tikzcd}$$

and since $\gamma f_q = \rho_q \beta$ is a surjection we have that ρ_q is a surjection and the rows are exact. By the Five Lemma β is an isomorphism.

Let $\phi \in \operatorname{Aut} \mathcal{C}S_{g,p}$. We have that by Lemma 27 that ϕ permutes the fibers $\{\rho_q^{-1}\}$, so there is $\psi \in \mathrm{MCG}^{\pm} S_{g,p}$ so that $\psi_* \in \operatorname{Aut} \mathcal{C}S_{g,p}$ is such that $\psi_* \phi$ maintains the fibers ρ_q^{-1}. So $\psi_* \phi \in \operatorname{Aut} \mathcal{C}(S_{g,p}, q)$. But then there is $\psi' \in \mathrm{MCG}^{\pm} S_{g,p}$ so that $\psi_* \phi = \psi'_*$. But then $\phi = \left(\psi^{-1} \psi'\right)_*$ is also induced by a mapping class we have the natural map

$$\mathrm{MCG}^{\pm} S_{g,p} \longrightarrow \operatorname{Aut} \mathcal{C}S_{g,p}$$

is an isomorphism. $\square$

6 All Mapping Class Group Extensions are Products

See the primer [16]

Theorem 28. *The center of* $\operatorname{MCG} S_{g,p}$ *is trivial, unless*

$$(g,p) \in \{(0,2),(1,0),(1,1),(1,2),(2,0)\}$$

and in all these cases $Z(\operatorname{MCG} S_{g,p}) \cong \mathbb{Z}/2$.

This is also obviously true for $\mathrm{MCG}^{\pm}$, but perhaps I must argue it. Mirror reflection doesn't commute with Dehn twisting.

McCarthy [39], Ivanov [27], Korkmaz [30]

Theorem 29. *Let* (g,p) *have* $g \geq 2$ *and* $g+p \geq 3$*, or* $g=1$ *and* $p \geq 3$*, or* $g=0$ *and* $p \geq 5$*. Let* G *and* G' *be finite index subgroups of* $\mathrm{MCG}^{\pm} S_{g,p}$*. Then any isomorphism* $G \to G'$ *is induced by an inner automorphism of* $\mathrm{MCG}^{\pm} S_{g,p}$*.*

In general the isomorphism class of a fibration of groups

$$1 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 1$$

is classified by an $\operatorname{Out} G$ and the cohomology $H^*(B; Z(A))$ see e.g. Brown [14]. But mapping class groups are centerless and out-less!

Theorem 30. *A centerless, out-less group always fibers trivially.*

Suppose that

$$1 \longrightarrow A \hookrightarrow E \xrightarrow{\pi} B \longrightarrow 1$$

is a short exact sequence of groups. If A *has trivial center and outer automorphism group, then* $E \cong A \times B$*.*

Proof. Let $B = \langle S | R \rangle$ be a presentation for B. For each $s \in S$ choose $e_s \in E$ so that $\pi(e_s) = s$. Note that $A = \ker \pi$ is normal in E. So the conjugation $x \mapsto e_s x e_s^{-1}$ restricts to an automorphism of A. By hypothesis $\operatorname{Aut} A = \operatorname{Inn} A$, so there must be $a_s \in A$ such that $e_s x e_s^{-1} = a_s x a_s^{-1}$ for all $x \in A$. Since

$$\pi(e_s) = \pi(e_s a_s^{-1}) = s$$

we may replace e_s with $e_s a_s^{-1}$ so that $e_s x e_s^{-1} = x$ for all $x \in A$. So $\langle e_s \rangle_{s \in S}$ commutes with A in E. But then if $\prod_i s_i \in R$ is a relation of B, we have $\prod_i e_{s_i} \in A$ by the exact sequence, so $\prod_i e_{s_i} \in Z(A)$ is in the center of A. But by hypothesis $Z(A) \cong 1$, so $\prod_i e_{s_i} = 1$ in E. So $s \mapsto e_s$ is a homomorphism $B \to E$ which splits the exact sequence, and we have $E \cong A \times B$. □

Corollary 31. *The mapping class group has only trivial extensions. (Or gives only trivial fibers?)*

Let (g, p) have $g \geq 2$ and $g + p \geq 3$, or $g = 1$ and $p \geq 3$, or $g = 0$ and $p \geq 5$. Suppose that

$$1 \longrightarrow \mathrm{MCG}^{\pm} S_{g,p} \hookrightarrow E \twoheadrightarrow B \longrightarrow 1$$

is an exact sequence of groups. Then $E \cong B \times \mathrm{MCG}^{\pm} S_{g,p}$.

7 The Punctured Sphere Complex

$\Gamma'_{n,p} = \pi_0 Diff(M_{n,p})$ i.e. don't fix boundary components. There is an exact sequence

$$1 \longrightarrow \Gamma_{n,p} \longrightarrow \Gamma'_{n,p} \longrightarrow \mathrm{Sym}(p) \longrightarrow 1$$

We can describe a generating set for $\Gamma'_{n,p}$. By capping every boundary component of $M_{n,p}$ with a copy of $M_{1,1}$ we can include $\Gamma'_{n,p} \hookrightarrow \mathrm{Out}\, F_{n+p}$ and consider diffeomorphisms of $M_{n+p,0}$ which preserve (setwise with orietation) the set of separating spheres where we glued the capping $M_{1,1}$ and the nonseparating spheres contained inside. Consider a free basis F_{n+p} given by $a_1, \ldots, a_n, b_1, \ldots, b_p$. Then $\Gamma_{n,p}$ is generated by

1. permutations of $\{a_1, \ldots, a_n\}$
2. inversion: $a_1 \mapsto a_1^{-1}$
3. transvection: $a_1 \mapsto a_1 a_2$
4. a-conjugation of b: $b_i \mapsto a_1 b_i a_1^{-1}$

and $\Gamma'_{n,s}$ by the addition of permutations of $\{b_1, \ldots, b_p\}$.

The usual transvection $a_1 \mapsto a_1 a_2$ corresponds to cutting the nonseparating sphere representing a_1, then pushing a_1^- through a_2^- before regluing a_1^+ and a_1^-. The map $b_1 \mapsto a_1 b_1 a_1^{-1}$ corresponds to pushing the bounding sphere through a_1^+.

This belongs in an intro about OUT and relative OUT Out is like the projectivization of Aut, b is like the point at infinty

The main theorem of this section is the following

Theorem 32. *The natural map $\mathrm{Out}(F_n, F_s) \to \mathrm{Aut}\, \mathcal{S}_{n,s}$ is an isomorphism for $n \geq 3$ and $s \geq 0$.*

The proof is directly analogous to our proof of Theorem 11. Induct on the number of punctures. A base, unpunctured case is considered by the Theorem of Aramayona and Souto [4]. We will show that automorphisms of the punctured sphere complex respect the fibration induced by forgetting punctures, so that adding additional punctures expands the automorphism group of the complex of spheres according to a Birmann exact sequence.

Theorem 33. *If the natural map*

$$\mathrm{MCG}^{\pm} S_{g,p} \to \mathrm{Aut}\, \mathcal{C}S_{g,p}$$

is an isomorphism, then so is

$$\mathrm{MCG}^{\pm} S_{g,p+1} \to \mathrm{Aut}\, \mathcal{C}S_{g,p+1}.$$

Definition 34. As in Definition 12 for surfaces, let $\Delta \subset \mathcal{S}_{g,p}$ be a simplex. The *region adjacency graph* $\mathcal{G}_\Delta$ of Δ is the graph whose vertices are the connected components of the cut manifold

$$M_{n,p} - \bigcup_{x \in \Delta} x$$

with an edge e_x for every sphere x and with the edge e_x incident to the connected components it bounds.

We will also consider the graph simplification $\mathcal{G}_\Delta^{simp}$.

Lemma 35. *Sphere complex automorphisms preserve edge incidence of region adjacency graphs.*

Let $\phi \in \mathrm{Aut}\, \mathcal{S}_{n,p}$ and let Δ be a simplex of $\mathcal{S}_{n,p}$ with adjacency graph $\mathcal{G}_\Delta$. Then $e_x, e_{x'}$ are incident edges of $\mathcal{G}_\Delta$ if and only if $e_{\phi(x)}, e_{\phi(x')}$ are incident edges of $\mathcal{G}_\phi(\Delta)$.

Proof. We will argue that the induced bijection ϕ_* between the edges of $\mathcal{G}_\Delta$ and the edges of $\mathcal{G}_{\phi(\Delta)}$ preserves incidence.

Let $x, x' \in \Delta$ be distinct spheres of $M_{n,p}$. Suppose that the associated edges e_x and e'_x of $\mathcal{G}_\Delta$. It suffices to show that e_x and $e_{x'}$ are incident if and only if there is a third $y \in \Delta$ with y intersecting x and x' but no other sphere of Δ. In that case $e_{\phi(x)}$ and $e_{\phi(x')}$ are incident if and only if there is no third $\phi(y) \in \phi(\Delta)$ with $\phi(y)$ intersecting $\phi(x)$ and $\phi(x')$ but no other sphere of $\phi(\Delta)$. So ϕ induces an incidence-preserving edge bijection between $\mathcal{G}_\Delta$ and $\mathcal{G}_{\phi(\Delta)}$.

Suppose that e_x and $e_{x'}$ are incident. Then there is a region R of $M_{n,p} - \bigcup_{z \neq x,x'} z$ containing x and x', and since every region of $M_{n,p} - \bigcup_{z \in \Delta} z$ contains at least an $M_{0,3}$, it must be that R contains a $M_{1,2}$ or $M_{0,5}$.

If R contains a $M_{1,2}$ the subcomplex of spheres in R contains a copy of $\mathcal{S}_{1,2}$ and so must have infinite diameter. So there must be a sphere y in R that intersects both x and x'. Since y is in R, it intersects no other sphere of Δ.

If R is simply connected then it must have a copy of $M_{0,5}$ and x and x' are essential and separating in R. Then there are two boundary spheres y', y'' of R with a path α between them that passes through both x and x'. Let y be the boundary of a regular neighborhood of $\alpha \cup y' \cup y''$ in R. Then y intersects both x and x', but since y is in R, y does not intersect any other sphere of Δ.

Suppose that e_x and $e_{x'}$ are not incident. Then there is a collection of spheres $\Delta' \subset \Delta$ that separate x from x' in $M_{n,p}$. So any sphere intersecting x and x' must also intersect a sphere of Δ. $\square$

Corollary 36. *Sphere complex automorphisms preserve the adjacency graphs of maximal simplices.*

Let $3n+p \geq 6$. Let $\phi \in \operatorname{Aut} \mathcal{S}_{n,p}$ and let Δ be a maximal simplex. Then $\mathcal{G}_\Delta$ and $\mathcal{G}_{\phi(\Delta)}$

Proof. Any maximum simplex Δ contains $3n+p-3$ spheres and cuts $M_{n,p}$ $2n+p-2$ copies of $M_{0,3}$. So $\mathcal{G}_\Delta^{simp}$ and $\mathcal{G}_{\phi(\Delta)}^{simp}$ are simple, connected graphs with the same number of vertices and the same edge incidence relations. So by Whitney's Theorem 15, $\mathcal{G}_\Delta^{simp}$ and $\mathcal{G}_{\phi(\Delta)}^{simp}$ are isomorphic.

To see that self-loops are preserved, observe that as Δ cuts $M_{n,p}$ into copies of $M_{0,3}$, every vertex of $\mathcal{G}_\Delta$ has degree at most 3. Then if e_x is a self-loop at vertex v_R it is incident to exactly one other edge $e_{x'}$ which cannot be a self-loop or have a parallel edge since v_R is degree 3. So $e_{\phi(x')}$ has a degree one vertex in $\mathcal{G}_{\phi(\Delta)}^{simp}$. Since $3n+p-3 \geq 3$ we have $\mathcal{G}_{\phi(\Delta)}$ has at least 3 edges. So if both vertices of $e_{\phi(x')}$ are degree one in $\mathcal{G}_{\phi(Delta)}^{simp}$, then $\mathcal{G}_{\phi(Delta)}$ has two vertices with a self loop at each. So $e_{\phi(x)}$ is a self loop. If $e_{\phi(x')}$ has only one degree one vertex in $\mathcal{G}_{\phi(Delta)}^{simp}$, then $e_{\phi(x)}$ is only incident to $e_{\phi(x')}$. So $e_{\phi(x)}$ must be a self loop in $\mathcal{G}_{\phi(\Delta)}$.

If ϕ preserves both the graph simplification and the self loops of $\mathcal{G}_\Delta$, it must be that ϕ also preserves the multi-edges, and so ϕ induces a graph isomorphism. $\square$

Lemma 37. *Sphere complex automorphisms preserve the topological type of spheres, and the sides of the spheres.*

Let $3n+p \geq 6$. Let $\phi \in \operatorname{Aut} \mathcal{S}_{n,p}$. Let x be a sphere of $M_{n,p}$. Then x and $\phi(x)$ have the same topological type. Further if x is separating and y, y' are spheres in the same connected component of $M_{n,p} - x$, then $\phi(x)$ is separating and $\phi(y), \phi(y')$ are in the same connected component of $M_{n,p} - \phi(x)$.

Proof. By Lemma 36 it suffices to characterise the topological type and sides of a sphere in terms of the region adjacency graph of a maximal simplex.

· Nonseparating spheres: Observe that x is a nonseparating sphere if and only if there is a maximal simplex Δ in which the corresponding edge e_x is a self-loop in the region adjacency graph $\mathcal{G}_\Delta$.

· Separating spheres: Observe that if x separates $M_{n,p}$

$$M_{n,p} = M_{n',p'} \sqcup_x M_{n-n',p-p'+2}$$

if and only if the corresponding edge e_x of the region adjacency graph $\mathcal{G}_\Delta$ is a cut edge. More specifically, if

$$\Delta = \Delta_+ \cup \{x\} \cup \Delta_-$$

with Δ_+ and Δ_- the spheres on each side of x, then e_x separates $\mathcal{G}_\Delta$

$$\mathcal{G}_\Delta - e_x = \mathcal{G}_{\Delta_+} \sqcup \mathcal{G}_{\Delta_-}$$

into the components $\mathcal{G}_{\Delta_+}$ with $3n' + p' - 3$ edges and rank n', and $\mathcal{G}_{\Delta_-}$ with with $3(n-n') + p - p' - 1$ edges and rank $n - n'$.

□

Remark 38. Consider the inclusion which ignores the puncture q

$$i_q : M_{n,p} \hookrightarrow M_{n,p-1}.$$

Then for the homotopy class $[x]$ of a sphere in $M_{n,p}$ we have the class $[i_q(x)]$ in $M_{n,p-1}$ by forgetting the puncture q. Separating spheres of $M_{n,p}$ bounding a copy of S^3 containing only q and one other puncture will become non-essential in this inclusion, but other homotopy classes of spheres have well defined essential representatives up to homotopy forgetting q.

Let $\mathcal{S}^{(q)}_{n,p} \subset \mathcal{S}_{n,p}$ be the subcomplex for which the puncture forgetful map $\rho_q : [x] \mapsto [i_q(x)]$ is well defined. So we have a surjective projection map

$$\rho_q : \mathcal{S}^{(q)}_{n,p} \to \mathcal{S}_{n,p-1}.$$

As in the case for surfaces, the fibers of this map are Bass-Serre trees. Let x be a sphere of $\mathcal{S}_{n,p-1}$. Homotope x in $M_{n,p-1}$ so that it is pointed at q. Then the two boundary spheres of a regular neighborhood of x in $M_{n,p-1}$ gives a well edge of $\rho_q^{-1}(x) \subset \mathcal{S}_{n,p}$. Let Γ be the corresponding be the graph of groups given by the splitting of $F_n = \pi_1(M_{n,p-1}, q)$. So the vertices of Γ are the components of $M_{n,p-1} - x$ with vertex groups given by the π_1 of the component, and an edge with trivial edge group between two components if x is separating, or a self-loop if x is nonseparating. Then there is a an isomorphism between $\rho_q^{-1}(x)$ and the Bass-Serre tree $\tilde{\Gamma}$ given by associating every edge xx' of $\rho_q^{-1}(x)$ with the edge of Γ given by uv if

$$\mathrm{stab}_{\pi_1(M_{n,p-1},q)}(u) * \mathrm{stab}_{\pi_1(M_{n,p-1},q)}(v)$$

is the splitting specified by the pointed sphere at q whose regular neighborhood in $M_{n,p}$ has boundary spheres x and x'.

Definition 39. Let $\mathrm{Aut}(\mathcal{S}_{n,p}, q) < \mathrm{Aut}\,\mathcal{S}_{n,p}$ be the subgroup which preserves the fibration of the forgetful map ρ_p. That is $\phi \in \mathrm{Aut}(\mathcal{S}_{n,p}, q)$ if

$$\phi\left(\rho_p^{-1}\rho_p(x)\right) = \rho_p^{-1}\rho_p(\phi(x))$$

$\pi_1(M_{n,p-1}, q)$

Lemma 40. *This diagram commutes*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(M_{n,p-1}, q) \arrow[r] \arrow[d] & \Gamma_{n,p}^{(q)} \arrow[r, "f_q"] \arrow[d] & \Gamma_{n,p-1} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(M_{n,p-1}, q) \arrow[r, "\alpha"] & \mathrm{Aut}(\mathcal{S}_{n,p}, q) \arrow[r, "\rho_q"] & \mathrm{Aut}\,\mathcal{S}_{n,p-1} \arrow[r] & 1
\end{tikzcd}$$

and has exact rows when ρ_q is surjective.

Proof. The map α is defined by the first square, so it commutes. The map α is injective, since for any loop γ based at q, there is a nonseparating sphere x intersecting γ so that the push map $\alpha(\gamma)$ acts non-trivially on x and so cannot be the identity on $\mathcal{S}_{n,p}^{(q)}$.

The second square must commute, since if $[\psi] \in \Gamma_{n,p}^{(q)}$ is a mapping class of $M_{n,p}$ and x a sphere of $M_{n,p}$, the homotopy class of $\psi(x)$ is the same if we first forget that ψ fixes q or if we first allow ψ with q fixed, then homotope $\psi(x)$ forgetting q.

A fiber $\rho_q^{-1}(x)$ of the forgetful map $\rho_q : \mathcal{S}_{n,p}^{(q)} \to \mathcal{S}_{n,p-1}$ is isomorphic to the Bass-Serre tree associated to the splitting. Then the kernel $\ker \rho_q$ is a group acting on the tree $\mathcal{T}_\Delta$, so by the Fundamental Theorem of BassSerre Theory 1, $\ker \rho_q$ is isomorphic to the fundamental group π_1 of the quotient graph of groups, but the corresponding graph of groups is exactly the Van Kampen splitting of π_1 induced by x. Thus

$$\ker \rho_q = \text{image } \alpha \cong \pi_1(S_{g,p}, q)$$

and the second row is exact. □

Remark 41. The edges of the fibers of the forgetful map are between two spheres which cobound a $S^2 \times I$ punctured by q. Such spheres are specified by the boundaries of regular neighborhoods of pointed spheres of $M_{n,p}$, so we consider the associated complex.

Definition 42. Let $\mathcal{PS}_{n,p}$ be the *pointed sphere complex* defined as follows. Let the vertices of $\mathcal{PS}_{n,p}$ be pointed spheres in $M_{n,p}$, i.e. homotopy classes of maps $(S^2, s_0) \to (M_{n,p}, P)$ where s_0 is a basepoint of the 2-sphere S^2, and unpointed spheres which bound a twice punctured ball. A collection of pointed spheres in $M_{n,p}$ span a simplex of $\mathcal{PS}_{n,p}$ if they have disjoint representatives.

Lemma 43. *The pointed sphere complex is uniquely paintable.*

Let $\eta(x)$ be 1 if x is a pointed sphere and 2 if x is an unpointed sphere which bounds a twice punctured ball of $M_{n,p}$. Then there is k, η-painting of $\mathcal{S}_{n,p}$ given by the puncture labels. Further it is the only k, η-painting up to relabeling of the k colors.

Proof. The argument is by a modified Putman trick 7. Observe that if $p \leq 2$ the result is trivial, so we assume $p \geq 3$. Let f be any p, η-painting of the pointed sphere complex $\mathcal{PS}_{n,p}$.

Choose a nonseparating sphere x of $M_{n,p}$. Let X be a collection of p disjoint, pointed spheres all parallel to x, as in Figure !!!! Since they are all disjoint they form a p-clique which requires so must all be distinctly colored. We may assume, possibly after relabeling, that f colors each pointed sphere of X by the label of its puncture; so $X = \{x_i\}_{i \in P}$ and $f(x_i) = \{i\}$.

FINISH ALL THE DIAGRAMS !!!! □

Lemma 44. *Sphere complex automorphisms induce automorphisms of the based sphere complex.*

Proof. □

Lemma 45. *Sphere complex automorphisms permute the fibers of the puncture forgetful map.*

Proof. $\square$

Proof of Theorem 33. Induct on the number boundary spheres of s. The base case with $s = 0$ is the Theorem of Aramayona and Souto [4]. In the proof of Proposition 1 of [22], Hatcher and Vogtmann construct the following short exact sequence. Let P be $s-1$ marked points in M_n and $p \in M_n$ an s^{th} marked point. There is a fibration

$$\begin{tikzcd} Diff(M_n, P \cup \{p\})) \arrow[r] & Diff(M_n, P) \arrow[d, "q"] \\ & M - P \end{tikzcd}$$

where the projection q is given by evaluation at p. The long exact sequence of homotopy groups associated to the fibration yields a Birmann-like short exact sequence

$$1 \longrightarrow F_n \longrightarrow \pi_0 Diff(M_n, P \cup \{p\})) \longrightarrow \pi_0 Diff(M_n, P)) \longrightarrow 1$$

which after a quotient by the finite normal Dehn-twist subgroup yields an exact sequence

$$1 \longrightarrow F_n \longrightarrow \Gamma_{n,s} \longrightarrow \Gamma_{n,s-1} \longrightarrow 1$$

Assume that the natural map $\Gamma'_{n,s-1} \to \operatorname{Aut} \mathcal{S}_{n,s-1}$ is an isomorphism. We have subgroups $N_k \leq \operatorname{Aut} \mathcal{S}_{n,k}$ which are the image of $\Gamma_{n,s}$ under the natural map.

Need a combinatorial characterization of spheres that are equal if you delete a boundary sphere in $S_{n,s}$ to show the forgetful map fits into the exact sequence. $\square$

8 Complex of Separating Spheres

Let $\mathcal{S}^{sep}_{n,s} \subset \mathcal{S}_{n,s}$ be the complex of embedded homotopy classes of separating spheres in $M_{n,s}$.

Lemma 46. $\mathcal{S}^{sep}_{n,s}$ *is a flag complex of dimension* $2n + s - 4$..

Proof. $\mathcal{S}_{n,s}^{sep}$ is the induced subcomplex of $\mathcal{S}_{n,s}$, which is known to be flag [4]. We show by induction that any collection Σ of disjoint spheres in $M_{n,s}$ is a subset of a maximal collection of $\max(2n+s-3,0)$ disjoint spheres.

Suppose for a base case that $\Sigma = \varnothing$. Certainly $M_{1,1}$ contains a unique class of nonseparating sphere, and no separating spheres. Every sphere of $M_{0,3}$ is homotopic to a boundary component. Observe that $M_{n,s}$ contains n disjoint 1-genus spheres. Cutting along these spheres yields n copies of $M_{1,1}$ and the sphere with $n+s$ balls removed, $M_{0,n+s}$, which contains a collection of $n+s-3$ disjoint spheres. Cutting along these spheres yields n copies of $M_{1,1}$ and $n+s-2$ copies of $M_{0,3}$, so the collection of $2n+s-3$ spheres in $M_{n,s}$ is maximal.

Assume that any collection of k or fewer disjoint spheres in $M_{n,s}$ is a subset of a maximal collection of $2n+s-3$ disjoint spheres. Let Σ be a collection of k disjoint spheres and let x be a sphere disjoint from all spheres of Σ. Then cutting $M_{n,s}$ along x yields two components homeomorphic to M_{n_1,s_1} and M_{n_2,s_2} where $n_1+n_2=n$ and $s_1+s_2=s+2$. By inductive hypothesis, the set spheres of Σ in each component can extended to maximal sets Σ_1 and Σ_2 of size $2n_1+s_1-3$ and $2n_2+s_2-3$ respectively. Then $\Sigma \cup \{x\}$ is contained in the maximal set $\Sigma_1 \cup \Sigma_2 \cup \{x\}$ of size

$$(2n_1+s_1-3)+(2n_2+s_2-3)+1 = 2n+s-3.$$

□

Lemma 47. *$\mathcal{S}_{n,s}^{sep}$ is connected whenever it has positive dimension, except if $(n,s)=(2,1)$.*

Proof. Consider first the case where $n=0$. Observe there are only finitely many spheres in $M_{0,s}$, with each sphere totally determined by its bipartition of the boundary spheres. Let Σ be the set of boundary spheres. So $\mathcal{S}_{0,s}^{sep}$ is the isomorphic to the complex of subsets of Σ with size $2,\dots,\left\lfloor \frac{s}{2} \right\rfloor$, with two sets adjacent if they are disjoint or if one is a subset of another. Then if $s \geq 5$ there is a path in $\mathcal{S}_{0,s}^{sep}$ between any two spheres made by dropping or adding one element of Σ at a time.

Consider the case where $n \geq 1$. We make use of Putman's Lemma 7 where $G = \Gamma_{n,s}$ and $X = \mathcal{S}_{n,s}^{sep}$. Fix a sphere v which bounds an embedded copy of $M_{1,1}$ in $M_{n,s}$. Note that $\Gamma_{n,s}$ acts transitively on such spheres, and every separating sphere which does not bound an embedded copy of $M_{1,1}$ is disjoint from such a sphere. So the orbit $\Gamma_{n,s} \cdot v$ intersects the connected component of every separating, considered as a vertex of $\mathcal{S}_{n,s}^{sep}$.

Consider a free basis $a_1, \ldots, a_n, b_1, \ldots, b_s$ of the free group F_{n+s} with v disjoint from the spheres representing the basis and v separating a_1 from $a_2, \ldots, a_n$. We use the generating set described above **??** !!!! for $\Gamma'_{n,s}$

Observe that transvections of $\{b_1, \ldots, b_s\}$ leave v fixed. Further, permutations of $\{a_1, \ldots, a_n\}$ either fix a_1 and thus v, or move v to a disjoint separating sphere at distance 1 from v in $\mathcal{S}^{sep}_{n,s}$.

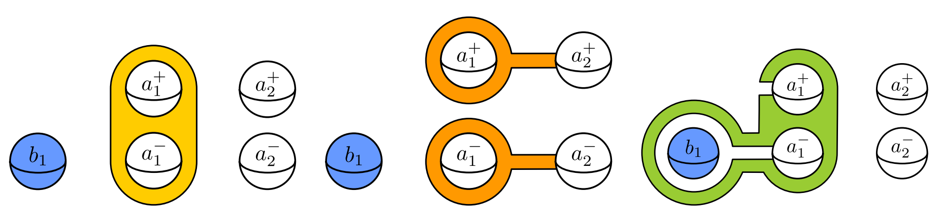
Consider the image v' of v under the diffeomorphism corresponding to conjugation $b_1 \mapsto a_1 b_1 a_1^{-1}$, as shown in green in Figure 8.

Then v' and v are contained in a copy of $M_{1,2}$ bounded by b_1 and the sphere u separating a_1 and b_1 from $a_2, \ldots, a_n$ and $b_2, \ldots, b_s$. If $n \geq 2$ or $s \geq 3$ we have that u is essential and this gives a length 2 path v to u to v' in $\mathcal{S}^{sep}_{n,s}$. The inverse conjugation $b_1 \mapsto a_1^{-1} b_1 a_1$ similarly moves v distance 2 in $\mathcal{S}^{sep}_{n,s}$. Appealing to Putman's Lemma 7, we conclude that $\mathcal{S}^{sep}_{n,s}$ is connected for $n = 1$ and $s \geq 3$.

Suppose $n \geq 2$. Then generation of $\Gamma_{n,s}$ also requires transvection. Consider the image v' of v under the diffeomorphism corresponding to the transvection $a_1 \mapsto a_1 a_2^{-1}$, as shown in orange in Figure 8. Then v' and v are contained in a copy of $M_{2,1}$ bounded by the sphere u separating a_1 and a_2 from $a_3, \ldots, a_n$ and $b_1, \ldots, b_s$. If $n \geq 3$ or $s \geq 2$ we have that u is essential and this gives a length 2 path v to u to v' in $\mathcal{S}^{sep}_{n,s}$. The inverse transvection $b_1 \mapsto a_1 a_2$ similarly moves v distance 2 in $\mathcal{S}^{sep}_{n,s}$. Appealing to Putman's Lemma 7, we conclude that $\mathcal{S}^{sep}_{n,s}$ is connected for $n = 2$ and $s \geq 2$ or $n \geq 3$.

Finally, we show that $\mathcal{S}^{sep}_{2,1}$ is disconnected. By capping the boundary component with a sphere we obtain a map

$$\phi : \left(\mathcal{S}^{sep}_{2,1}\right)^{(0)} \to \left(\mathcal{S}^{sep}_{2,0}\right)^{(0)}$$



(a) The basepoint sphere v separates a_1 from $a_2, \ldots, a_n$ and $b_1, \ldots, b_s$.

(b) The transvection $a_1 \mapsto a_1 a_2^{-1}$ corresponds to pushing a_1^+ through a_2^-.

(c) The conjugation $b_1 \mapsto a_1 b_1 a_1^{-1}$ corresponds to pushing b_1 through a_1^+.

Figure 8: Nontrivial $\Gamma'_{n,s}$ generator actions on the base sphere v move v at most distance 2 in $\mathcal{S}^{sep}_{n,s}$.

Observe that if u and v are disjoint spheres of $\mathcal{S}^{sep}_{2,1}$ then $\phi(u) = \phi(v)$. So ϕ gives a surjective simplicial map

$$\mathcal{S}^{sep}_{2,1} \to \mathcal{S}^{sep}_{2,0}.$$

But as $S^{sep}_{2,0}$ is totally disconnected it must be that $S^{sep}_{2,1}$ is disconnected. $\square$

We say that a sphere is M'_n, s'-bounding if it bounds an embedded copy of $M_{n',s'} \subset M_{n,s}$.

Lemma 48. *For $k \leq n/2$, $M_{k,1}$-bounding spheres are characteristic in $\mathcal{S}^{sep}_n$ for $n \geq 3$.*

Proof. Suppose that $x \in \mathcal{S}^{sep}_n$ bounds an $M_{k,1}$. Observe the link of x is isomorphic to a join $\mathcal{S}^{sep}_{k,1} * \mathcal{S}^{sep}_{n-k,1}$. By Lemma 46 the dimensions of the sides of the join are $2k-3$ and $2n-2k-3$, so any automorphism of $\mathcal{S}^{sep}_n$ must send x to a genus k-bounding sphere. $\square$

Observe that $M_{1,0} = S_1 \times S_2$ so that $\pi_2(M_{1,0}, p) \cong \mathbb{Z}$. Using the long exact sequence of the pair $(M_{1,1}, \partial M_{1,1})$ we compute $\pi_2(M_{1,0}) \cong \pi_2(M_{1,1}, S^2)$, so $M_{1,1}$ contains a unique homotopy class of nonseparating sphere generating the second homotopy group.

Then for any automorphism $\phi \in \mathrm{Aut}\,(\mathcal{S}^{sep}_{n,s})$ we can extend ϕ to a map $\hat{\phi} : \mathcal{S}_{n,s} \to \mathcal{S}_{n,s}$. If x is a separating sphere we assign $\hat{\phi}(x) = \phi(x)$. If a is a nonseparating sphere, then there is there is an $M_{1,1}$-bounding sphere x bounding an $M_{1,1}$ which contains a. Then $\phi(x)$ bounds an $M_{1,1}$ by Lemma 48. We define $\hat{\phi}(a)$ to be the nonseparating sphere in the $M_{1,1}$ bounded by $\phi(x)$. We must first demonstrate that $\hat{\phi}$ is well defined.

Fix a nonseparating sphere a. Define a *sharing pair* $\{x, x'\}$ (sharing a) to be $M_{1,1}$-bounding spheres x and x' such that x and x' each bound an $M_{1,1}$ containing a and are contained in a common $M_{2,1}$ bounded by separating sphere y.

Lemma 49. *If $\{x, x'\}$ is a sharing pair, then $\{\phi(x), \phi(x')\}$ is a sharing pair for any automorphism $\phi \in \mathrm{Aut}\,(\mathcal{S}^{sep}_n)$.*

Proof. Let a_1 be a nonseparating sphere of $M_{n,s}$ and let $\{x, x'\}$ be a sharing pair for a_1. Then x and x' are adjacent to an $M_{2,1}$-bounding sphere y, but not each other in $\mathcal{S}^{sep}_n$. Let a_2 be a nonseparating sphere disjoint from $a1$ in the $M_{2,1}$ bounded by y. Observe further we may find an $M_{1,1}$-bounding sphere z which intersects y, but not x or x'. Then, appealing to Lemma 49, $\phi(x)$ and $\phi(x')$ must be intersecting $M_{1,1}$-bounding spheres. Let A be the

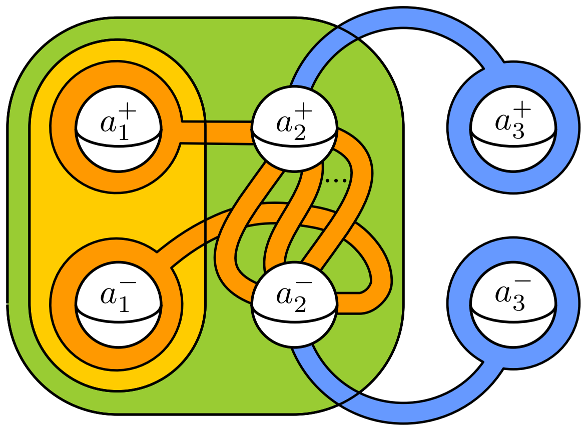


Figure 9: The sharing pair x and x' bound $M_{1,1}$ shown in yellow and orange. They are contained in the green $M_{2,1}$ bounded by y. The blue $M_{1,1}$ is bounded by z. Observe an $M_{1,1}$-bounding sphere containing a_1 can be represented by drawing two parallel copies a_1^+ and a_1^- and then connecting them by attaching a handle given by the regular neighborhood of an arc from a_1^- to a_1^+ disjoint from a_1. Fixing a_1, the spheres x and x' are determined by their respective intersection numbers with a_2.

$M_{2,1}$ bounded by $\phi(y)$. $\phi(z)$ is disjoint from $\phi(x)$ and $\phi(x')$, but not $\phi(y)$. Consider the image of $\phi(z)$ in the A. If $\phi(z)$ bound a region containing a nonseparating sphere in A, there would only be one class of separating sphere in A disjoint from $\phi(z)$. Then $\phi(z)$ must bound in A a handle given by the boundary of a regular neighborhood of an arc of $\pi_1(A, \partial A)$ which must pass through a nonseparating sphere a of A. But then $\phi(x)$ and $\phi(x')$ must both bound the nonseparating sphere of A disjoint from a. So $\{\phi(x), \phi(x')\}$ is a sharing pair. □

Let a be a nonseparating sphere of M_n. We will show that any two $M_{1,1}$-bounding spheres which contain a on their $M_{1,1}$-side are connected by a sequence of sharing pairs. Let $\mathcal{P}_a$ be the *sharing pair* graph defined as follows. The vertices of $\mathcal{P}_a$ are genus 1-bounding separating spheres of M_n which bound an $M_{1,1}$ containing a. Two vertices of $\mathcal{P}_a$ are adjacent if they form a sharing pair for a.

Lemma 50. *The sharing pair graph $\mathcal{P}_a$ is connected.*

Proof. We appeal to Putman's Lemma 7 using the graph $X = \mathcal{P}_a$ and the group $G \leq \Gamma_n$ fixing a setwise. Let $a_1, \ldots, a_n$ be a basis for F_n.

Then G is generated by diffeomorphisms corresponding to permutations of $\{a_2, \ldots, a_n\}$, inversions, and the transvections $a_1 \mapsto a_1 a_2^{-1}$ and $a_2 \mapsto a_2 a_3^{-1}$.

Observe that G acts transitively on $M_{1,1}$-bounding spheres which contain a on their genus 1-side. Let v be the sphere separating a_1 from $a_2, \ldots, a_n$. Observe that of the chosen generators only the transvection $\phi : a_1 \mapsto a_1 a_2^{-1}$ has nontrivial action on v. But, as can be seen in figure 8, v and $\phi(v)$ are contained in an $M_{2,1}$ so that $\{v, \phi(v)\}$ is a sharing pair.

It follow by Putman's Lemma 7 that $\mathcal{P}_a$ is connected. □

The previous Lemma shows that $\hat{\phi}$ is well defined. If a is a nonseparating sphere of M_n and x and x' $M_{1,1}$-bounding sphere bounding an $M_{1,1}$ containing a, then as P_a is connected there is a sequence of sharing pairs from x to x'. By Lemma 49 this gives a sequence of sharing pairs from $\phi(x)$ to $\phi(x')$. But then $\phi(x)$ and $\phi(x')$ share the same nonseparating sphere so that $\hat{\phi}(a)$ is well defined.

Certainly $\hat{\phi}$ is simplicial. If a and a' are disjoint nonseparating spheres then there are disjoint $M_{1,1}$-bounding spheres x and x' bounding disjoint copies of $M_{1,1}$ separating a and a', respectively. Since $\phi(x)$ and $\phi(x')$ are disjoint $M_{1,1}$-bounding spheres, $\hat{\phi}(a)$ and $\hat{\phi}(a')$ are also disjoint. If y is a separating sphere disjoint from a, then either there is an $M_{1,1}$-bounding sphere separating a from y or y is an $M_{1,1}$-bounding sphere, so that $\hat{\phi}(a)$ is disjoint from $\hat{\phi}(y) = \phi(y)$.

Theorem 51 (Proposition)**.** *The natural map* $\mathrm{Out}(F_n) \to \mathrm{Aut}\,(\mathcal{S}_n^{sep})$ *is an isomorphism for* $n \geq 3$*.*

Proof. The map constructed above

$$\Phi : \mathrm{Aut}\,(\mathcal{S}_n^{sep}) \to \mathrm{Aut}\,(\mathcal{S}_n)$$

with $\phi \mapsto \hat{\phi}$ is an isomorphism with the map simply restricting automorphisms

$$\mathrm{Aut}\,(\mathcal{S}_n) \to \mathrm{Aut}\,(\mathcal{S}_n^{sep})$$

giving an inverse to Φ. Then the result follows from Theorem 5. □

9 Complexes of High Genus Separating Spheres

For $k \leq n/2$, we call a sphere $x \in [S^2, M_{n,s}]$ k-separating if both components of $M_{n,s} - x$ contain either a boundary component or at least k disjoint separating spheres. If x bounds a copy of $M_{j,1}$ with $j < n/2$, we refer to it as to as x^{in}, or the *inside* of x. We will also describe objects disjoint from and inside of x as *engulfed* by x, and disjoint objects on the outside as *exgulfed* by x.

Let $\mathcal{S}_{n,s}^{sep,k} \subset \mathcal{S}_{n,s}^{sep}$ be the subcomplex spanned by homotopy classes of essential k-separating spheres.

Observe that for $k > 1$, $\mathcal{S}_{n,s}^{sep,k}$ does not have a uniform dimension. For example, in the case with no boundary components, $s = 0$, we can construct a maximal (with respect to inclusion) simplex of maximal dimension $n - 2k$ as in Figure 10. If we write $n = qk + r$ by Euclidean division, then we can construct maximal (with respect to inclusion) simplices of smaller dimension $2q + r - 4$ as in Figure 11.

If there are boundary spheres we can construct a maximal dimension simplex similar to Figure 10 by replacing the $M_{k,1}$-bounding spheres with

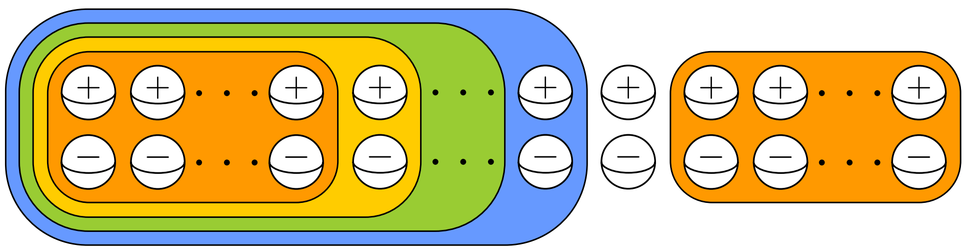


Figure 10: A maximal dimension maximal simplex of $\mathcal{S}_n^{sep,k}$ for $k > 1$ is spanned by $n - 2k + 1$ spheres and cuts M_n into 2 copies of $M_{k,1}$ and $n - 2k$ copies of $M_{1,2}$. The corresponding graph of M_n components is an unbranched tree with 2 leaves of weight k and $n - 2k$ internal vertices of weight 1.

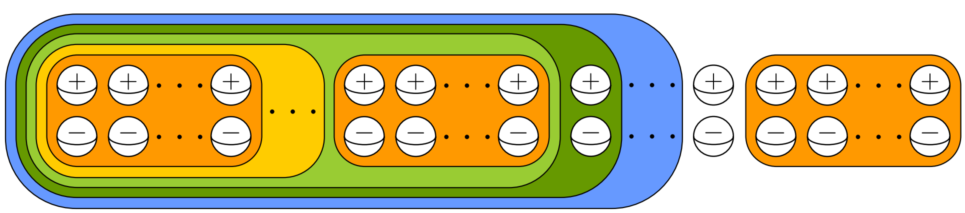


Figure 11: A minimal dimension maximal simplex of $\mathcal{S}_n^{sep,k}$ for $k > 1$ is spanned by $2q + r - 3$ spheres and cuts M_n into q copies of $M_{k,1}$ and r copies of $M_{1,2}$ and $q - 2$ copies of $M_{0,3}$. The corresponding graph of M_n components is a tree with q leaves of weight k and $q - 2$ internal vertices weight 0 and r internal vertices of weight 1.

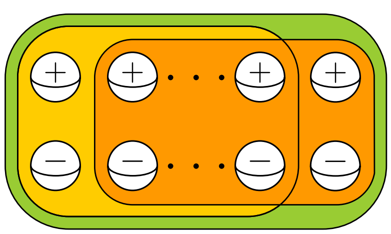
boundary spheres. Similar linear nesting shows any k-separating sphere can be contained in a maximum dimension simplex of dimension for $k > 1$

$$\max_n \left\{ \Delta^n \hookrightarrow \mathcal{S}^{sep,k}_{n,s} \right\} = \begin{cases} n - 2k & \text{if } s = 0 \\ n - k & \text{if } s = 1 \\ n + s - 3 & \text{if } s \geq 2 \end{cases}.$$

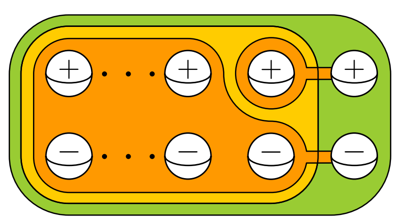
Lemma 52. *For $1 < k < n/2$, the complex of k-separating spheres $\mathcal{S}^{sep,k}_{n,s}$ is connected whenever it has positive dimensional simplices if $s = 0$, and whenever it has 2 dimensional simplices if $s > 0$.*

Proof. The proof is similar to the proof of Lemma **??**!!!!, utilizing Putnam's Lemma with the group $\Gamma'_{n,s}$.

Consider first the case with $s = 0$ and suppose that $\mathcal{S}^{sep,k}_{n,s}$ has positive dimensional simplices. So $n > 2k$, and in particular there are $M_{k,1}$ and $M_{k+1,1}$-bounding spheres in $\mathcal{S}^{sep,k}_{n,s}$. Choose a sphere v to be an $M_{k,1}$-bounding sphere. Observe that every k-separating sphere is disjoint from a $M_{k,1}$-bounding sphere, and the $M_{k,1}$-bounding spheres are exactly the Γ_n orbit of v. Let $a_1, \ldots, a_n$ be a maximal collection of disjoint non-separating spheres of M_n with $a_1, \ldots, a_k$ engulfed by v. Consider as a generating set for Γ_n the transpositions and inversions of $a_1, \ldots, a_n$ and the transposition diffeomorphism t corresponding to $a_1 \mapsto a_1 a_{k+1}^{-1}$. Observe that every inversion fixes v. Observe that a transposition ϕ either fixes v, in the case it swaps spheres on the same side of $M_n - v$, or $\phi(v)$ and v are contained in a common $M_{k+1,1}$-bounding sphere which is k-separating, as in Figure 12a. Finally, v and $t(v)$ are contained in a common $M_{k+1,1}$-bounding sphere as in Figure 12b. The connectivity then follows by Putnam's Lemma.



(a) Transpositions move $M_{k,1}$-bounding spheres distance 2.



(b) Transvections move $M_{k,1}$-bounding spheres distance 2.

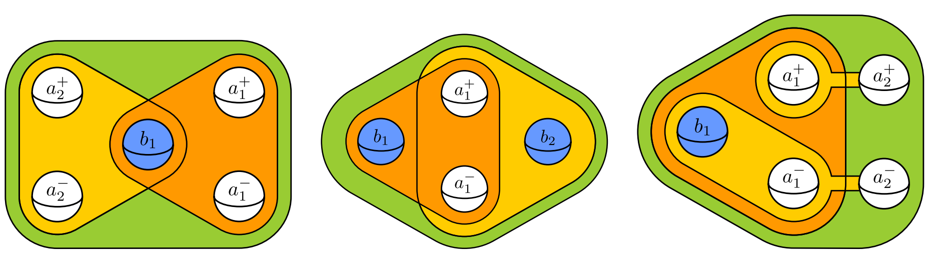
Figure 12: Nontrivial Γ_n generator actions on the base sphere v move v at most distance 2 in $\mathcal{S}^{sep,k}_{n,s}$.

Consider the case with $s > 0$. If $s = 1$ then to have dimension 2 simplices $n \geq k + 2$, and $M_{2,2}$-bounding spheres are k-separating. If $s > 1$ then to have dimension 2 simplices $n + s \geq 5$. If $s = 1$ then $n \geq 4$ so that $M_{2,2}$-bounding spheres are disjoint from a $M_{k,1}$-bounding sphere and must be $k-separating$. If $s > 1$ then $n \geq 4$ so that $M_{1,3}$ and $M_{2,2}$-bounding spheres are disjoint from a $M_{1,2}$ or $M_{k,1}$-bounding sphere and must be $k-separating$.

Choose a sphere v to be an $M_{1,2}$-bounding sphere. Observe that every k-separating sphere is disjoint from a $M_{1,2}$-bounding sphere, and the $M_{1,2}$-bounding spheres are exactly the $\Gamma'_{n,s}$ orbit of v. Let $b_1, \ldots, b_s$ be the bounding spheres and let a_1 be a nonseparating sphere engulfed by v and $a_2, \ldots, a_n$ disjoint nonseparating spheres disjoint from v and a_1. Consider as a generating set for $\Gamma'_{n,s}$ diffeomorphisms corresponding to transpositions of $a_1, \ldots, a_n$, transpositions of $b_1, \ldots, b_s$, t the transvection $a_1 \mapsto a_1 a_2^{-1}$, and u the b_1 push corresponding to conjugation $b_1 \mapsto a_1 b_1 a_1^{-1}$. Observe first that u leaves v fixed. Observe that ϕ a transposition of $a_1, \ldots, a_n$ either leaves v if it fixes a_1, or swaps a_1, and then $\phi(v)$ and v are engulfed by an $M_{2,2}$-bounding sphere as in Figure 13a. Observe that ψ a transposition of $b_1, \ldots, b_s$ either leaves v if it fixes b_1, or swaps b_1, and then $\psi(v)$ and v are engulfed by an $M_{1,3}$-bounding sphere as in Figure 13b. Finally, v and $t(v)$ are engulfed by an $M_{2,2}$-bounding sphere as in Figure 13c. The connectivity then follows by Putnam's Lemma. ☐

Lemma 53. *Let $n \geq 3$ and $\phi \in \mathrm{Aut}\left(\mathcal{S}_n^{sep,k}\right)$. For $n/2 > j \geq k$, if x is a $M_{j,1}$-bounding spheres engulfing a sphere y, then $\phi(x)$ is a $M_{j,1}$-bounding spheres engulfing the sphere $\phi(y)$.*

Proof. Suppose that x bounds an $M_{j,1}$ in $\mathcal{S}_n^{sep,k}$. Consider the subcomplex $\mathcal{E}_x$ spanned by spheres engulfed by x and the subcomplex $\mathcal{F}_x$ spanned by



(a) a-Transposition (b) b-Transposition (c) Transvection

Figure 13: Nontrivial $\Gamma'_{n,s}$ generator actions on the base sphere v move v at most distance 2 in $\mathcal{S}_{n,s}^{sep,k}$.

spheres disjoint but not engulfed by x. The link of x is a join $\mathcal{E}_x * \mathcal{F}_x$ and $\mathcal{E}_x \cong \mathcal{S}_{j,1}^{sep,k}$ and $\mathcal{F}_x \cong \mathcal{S}_{n-j,1}^{sep,k}$. Then according to Lemma 52, $\mathcal{E}_x$ and $\mathcal{F}_x$ have simplices with maximal dimension $j-k$ and $n-j-k$, respectively. So the link of $\phi(x)$ must have the same structure and $\phi(x)$ must be $M_{j,1}$-bounding. Note that since

$$\phi\left(\mathcal{E}_x\right) = \mathcal{E}_{\phi(x)}$$

any sphere y engulfed by x has $\phi(y)$ engulfed by $\phi(x)$. $\square$

We hope to extend automorphisms of $\mathcal{S}_n^{sep,k}$ to automorphisms of $\mathcal{S}_n^{sep,k-1}$ by a combinatorial characterization of $M_{k-1,1}$-bounding spheres in $\mathcal{S}_n^{sep,k}$.

This is the direct analog of handle pairs of **??**METACONJECTURE!!!!

Definition 54. If x is a $M_{k,1}$-bounding sphere engulfed by $M_{k+1,1}$-bounding sphere y we say that a pair v, w of $M_{k,1}$-bounding spheres *carve* x from y if

(1) Each pair of v, w, y intersects, but v, w, y are all disjoint from x.

$$\begin{tikzcd} v \arrow[r, dash] & x \arrow[r, dash] & y \\ w \arrow[ur, dash] & & \end{tikzcd}$$

(2) The $M_{k,1}$-bounding sphere x is the unique sphere engulfed by y and disjoint from both v and w.

(3) There is more than one $M_{k,1}$-bounding sphere engulfed by y and disjoint from v but not w.

(4) There is more than one $M_{k,1}$-bounding sphere engulfed by y and disjoint from w but not v.

It follows immediately from this combinatorial definition and Lemma 53 that carving is characteristic. The following Lemma shows additionally that there is a unique non-separating sphere which was "carved away" from y.

Lemma 55. *Let $\phi \in \operatorname{Aut} \mathcal{S}_n^{sep,k}$. If v, w carve x from y, then*

(1) $\phi(v), \phi(w)$ carve $\phi(x)$ from $\phi(y)$

(2) One of the spheres v or w contains a disk or annulus s with $\partial s \subset y$ whose image in y^{in}/y is homotopic to a nonseparating sphere.

(3) There is an arc α with endpoints on y such that v and w separate α from x.

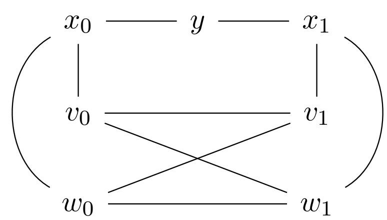
(4) x is the unique $M_{k,1}$-bounding sphere engulfed by y and disjoint from s and α.

Proof. (1) follows from Lemma 53 and the combinatorial definition of carving.
(2) Fix representatives for v, w, x, and y which intersect minimally and transversely. Then $w \cap y^{in}$ is a collection of disks and annuli with boundary on y. No component disk or annulus of $w \cap y^{in}$ can be separating, or else there would be at most $M_{k,1}$ in y^{in} disjoint from w. Similarly $v \cap y^{in}$ is a collection of disks and annuli, no one of which separates y^{in}. Let β be any nontrivial loop in $y^{in} - x^{in}$ and based at a point on x. Then β must intersect either v or w, or else the pushes of any nonseparating sphere of x about α would yield infinitely many $M_{k,1}$-bounding spheres engulfed by y and disjoint from v and w, contrary to the hypothesis. Since no such β exists, there must a component s of $v \cap y^{in}$ or $w \cap y^{in}$ whose image in the quotient y^{in}/y is homotopic to a nonseparating sphere.
(3) Let a be a nonseparating sphere engulfed by y and exgulfed by x and disjoint from the nonseparating component s as above. If $v \cap y^{in}$ or $w \cap y^{in}$ have a component which intersects a, then as v and w are separating there must be an arc intersecting a with endpoints on y which they separate from x. Suppose that v and w are disjoint from a. If there is a loop γ based at a which winds through a noseparating sphere engulfed by x and is disjoint from v and w, then the pushes of a along γ leave v and w unchanged, but the images of x give infintely many $M_{k,1}$-bounding spheres engulfed by y and disjoint from v and w, in contradiction with the definiton of carving. Then v and w must separate a from x in y, and there is an arc α with end points on y which intersects a once and so must be separated from x by v and w.
(4) Assume to the contrary there is some x' engulfed by y, distinct from x, and disjoint from s and α. Then there must be some nonseparating sphere a engulfed x' but not x. Note that $y^{in} - x^{in} \cong M_{1,2}$ and consider the components of $a \cap (y^{in} - x^{in})$. If $a \cap (y^{in} - x^{in})$ has a nonseparating disk, it must intersect α. If $a \cap (y^{in} - x^{in})$ contains a nontrivial arc, it must intersect s. So a must be engulfed by x. □

Definition 56. Define a $M_{k-1,1}$-sharing pair $\{x_0, x_1\}$ to be a pair of $M_{k,1}$-bounding spheres $x_0, x_1 \in \mathcal{S}_n^{sep,k}$ such that:

(1) There are $M_{k,1}$-bounding spheres x_2, x_3 and a $M_{k+1,1}$-bounding sphere y such that the induced subgraph of $\mathcal{S}_n^{sep,k}$ on $y, x_0, x_1, v_0, v_1, w_0, w_1$ is

exactly



and

(2) For $i = 0, 1$ the spheres v_i, w_i carve x_i from y_i.

(3) For $z_0 \in \{v_0, w_0\}$ and $z_1 \in \{v_1, w_1\}$, there is no $M_{k,1}$-bounding sphere engulfed by y and disjoint from both z_0 and z_1.

Lemma 57. *The spheres of an $M_{k-1,1}$-sharing pair uniquely engulf a $M_{k-1,1}$-bounding sphere in M_n.*

Proof. Let $\{x_0, x_1\}$ be a sharing pair with y, v_0, w_0, v_1, w_1 as above. Let s_i and α_i be as specified in Lemma 55, so that, without loss of generality, s_i is a component of $v_i \cap y^{in}$ which is nonseparating in y. And α_i is a loop with endpoints on y which v_i and w_i separate from x_i. But as v_0 and v_1 are disjoint, so are s_0 and s_1.

Since there is no $M_{k,1}$-bounding sphere disjoint from both v_0 and v_1, it must be no $M_{k-1,1}$-bounding sphere is disjoint from both s_0 and s_1 or from both α_0 and α_1.

Then α_1 must intersect x_0, and there are $k-1$ disjoint nonseparting spheres $a_1, \dots, a_{k-1}$ engulfed by x and disjoint from α_1.

Consider the images in $y^{in}/y \cong M_{k+1,1}$. Then the images of s_0 and s_1 are distinct nonseparating spheres in y^{in}/y. Further since there is no $M_{k,1}$-bounding sphere disjoint from both, forgetting the basepoint y/y in y^{in}/y gives distinct, disjoint spheres $\overline{s_0}$ and $\overline{s_1}$ of y^{in}. Let Σ be a system of $n-k-1$ disjoint spheres exgulfed by y.

The let z be the unique sphere separating $a_1, \dots, a_k$ from $\overline{s_0}$, $\overline{s_1}$, and Σ. Then z is $M_{k-1,1}$-bounding and uniquely engulfed by both x_0 and x_1. $\square$

Lemma 58. *Sharing pairs are characteristic.*
If $\{x_0, x_1\}$ is an $M_{k-1,1}$-sharing pair with $x_0, x_1 \in \mathcal{S}_n^{sep,k}$ and $\phi \in \operatorname{Aut} \mathcal{S}_n^{sep,k}$, then $\{\phi(x_0), \phi(x_1)\}$ is an $M_{k-1,1}$-sharing pair.

Proof. Let $y, x_0, v_0, w_0, x_1, v_1, w_1$ be as in Definition 56. By Lemma 53 $\phi(x_0), \dots, \phi(x_3)$ are $M_{k,1}$-bounding and $\phi(y)$ is $M_{k+1,1}$-bounding. The ϕ

image of the induced subgraph on $y, x_0, v_0, w_0, x_1, v_1, w_1$ is an isomorphic graph. Property (2) is preserved by Lemma 55. Property (3) of Definition 56 is preserved by its combinatorial definition. □

Although the representation of $M_{k-1,1}$-bounding spheres are

Lemma 59. *Every $M_{k-1,1}$-bounding sphere x admits an $M_{k-1,1}$-sharing pair in $\mathcal{S}_n^{sep,k}$ engulfing x, provided $n \geq 3k$.*

Proof. By a change of coordinates the, arrangement in Figure 14 shows a possible pair sharing x.

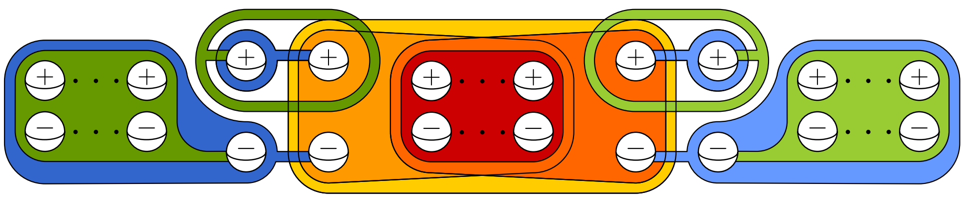


Figure 14: A sharing pair and the requisite carvings. The pair $\{x_0, x_1\}$ is shown bounding dark and light orange, respectively. The pair shares the $M_{k-1,1}$-bounding sphere x shown bounding red. The pair is engulfed by $M_{k+1,1}$-bounding sphere y shown in yellow. Dark orange x_0 is carved by v_0 and w_0 shown in dark green and blue. Light orange x_1 is carved by v_1 and w_1 shown in light green and blue.

□

We call three spheres an $M_{k-1,1}$-sharing triple if the spheres pairwise form sharing pairs and all engulf a common $M_{k-1,1}$-bounding sphere.

Lemma 60. *Sharing triples are characteristic.*
If $\{x_0, x_1, x_2\}$ is an $M_{k-1,1}$-sharing triple with $x_0, x_1, x_2 \in \mathcal{S}_n^{sep,k}$ and $\phi \in \operatorname{Aut} \mathcal{S}_n^{sep,k}$, then $\{\phi(x_0), \phi(x_1), \phi(x_2)\}$ is an $M_{k-1,1}$-sharing triple.

Proof. According to Lemma 58 if x_0, x_1, x_2 pairwise form sharing pairs, then so do $\phi(x_0), \phi(x_1), \phi(x_2)$. It remains only to see that $\phi(x_0), \phi(x_1), \phi(x_2)$ all engulf a common $M_{k-1,1}$-bounding sphere, rather that a distinct $M_{k-1,1}$-bounding sphere for each pair. We reduce the proof to showing $\phi(x_0), \phi(x_1), \phi(x_2)$ all engulf a common $M_{k-1,1}$-bounding sphere if and only if there is no $M_{k+1,1}$-bounding sphere y engulfing $\phi(x_0), \phi(x_1)$, and $\phi(x_2)$. Then if there were a y engulfing $\phi(x_0), \phi(x_1)$, and $\phi(x_2)$, $\phi^{-1}(y)$ engulfs x_0, x_1, x_2, which would contradict that $\{x_0, x_1, x_2\}$ is a sharing triple.

Observe that, as in the proof of Lemma 58, since $\phi(x_0), \phi(x_1), \phi(x_2)$ are pairwise sharing pairs, there are three pairwise-disjoint $M_{1,1}$-bounding spheres z_0, z_1, z_2 such that z_i is uniquely engulfed by $\phi(x_i)$ and disjoint but not engulfed by $\phi(x_{i+1})$, for $i \in \mathbb{Z}/3$. The sphere shared by $\{\phi(x_i), \phi(x_{i+1})\}$ is in $\phi(x_i)^{in} - z_i^{in}$. If $\phi(x_0), \phi(x_1), \phi(x_2)$ all engulf a common $M_{k-1,1}$-bounding sphere x, then any sphere engulfing $\phi(x_0), \phi(x_1), \phi(x_2)$ contains x, z_0, z_1, and z_2 so must be $M_{j,1}$-bounding for $j \geq k+2$. If $\phi(x_0), \phi(x_1), \phi(x_2)$ do not engulf a common $M_{k-1,1}$-bounding sphere, then for $i \in \mathbb{Z}/3$ we have a distinct $M_{k-1,1}$-bounding sphere shared by $\{\phi(x_i), \phi(x_{i+1})\}$ and which engulfs z_{i-1}. But then $\phi(x_0), \phi(x_1), \phi(x_2)$ do all engulf a common $M_{k-2,1}$-bounding sphere x such that $\phi(x_i)$ engulfs x and z_i and z_{i+1}. Then the same $M_{k+1,1}$-bounding sphere y fits into the defining pentagon of Definition 56 for all three sharing pairs. $\square$

Let x be an $M_{k-1,1}$-bounding sphere. We will show that any two sharing pairs engulfing x are connected by a sequence of sharing triples. Let $\mathcal{P}_x$ be the *sharing pair* graph defined as follows. The vertices of $\mathcal{P}_x$ are $M_{k-1,1}$-sharing pairs in $\mathcal{S}_n^{sep,k}$ engulfing x, where $n \geq 3k$. Two vertices $\{x_0, x_1\}$ and $\{x_1, x_2\}$ of $\mathcal{P}_x$ are adjacent if the pairs have a common member and the three spheres form an $M_{k-1,1}$-sharing triple engulfing x.

Lemma 61. *The sharing pair graph $\mathcal{P}_x$ is connected for $n \geq 3k$ and $k \geq 2$.*

Proof. We appeal to Putman's Lemma 7. Fix an $M_{k-1,1}$-bounding sphere x and an $M_{k-1,1}$-sharing pair $v = \{x_0, x_1\}$ engulfing x with x_0 and x_1 having geometric intersection 1. Let $G \leq \Gamma_n$ be the subgroup fixing x^{in}, so that $G \cong \Gamma_{n-k+1,1}$.

Observe that if two x-sharing pairs $\{x_0, x_1\}$ and $\{x_1, x_2\}$ contain a common member x_1 and with all three spheres engulfed by a common $M_{k+1,1}$-bounding sphere y, then we can find a length 2 path in $\mathcal{P}_x$ by choosing x_3 with intersection 1 with y and engulfing an $M_{1,1}$-bounding sphere which is disjoint from y. Then $\{x_0, x_1, x_3\}$ and $\{x_1, x_2, x_3\}$ are sharing triples, since they are pairwise sharing pairs and all engulf x. So if y_0 is the $M_{k+1,1}$-bounding sphere engulfing $v = \{x_0, x_1\}$, and $\{x_0', x_1'\}$ is any sharing pair engulfed by y_0', then there is $g \in G$ such that $g(x_1) = x_1'$ and $g(y_0) = y_0'$. So the orbit $G \cdot v$ is at most distance 2 from any sharing pair vertex of $\mathcal{P}_x$. This completes the first criterion of Putnam's Lemma 7.

Fix a system of nonseparating spheres $a_0, \ldots, a_{n-k}$ disjoint and not engulfed by x with a_0 but not a_{n-k} engulfed by x_0 and a_{n-k} but not a_0 engulfed by x_1. Then $G \cong \mathrm{Aut}\langle a_0, \ldots, a_{n-k}, \rangle$ is generated by diffeomorphism classes corresponding to inversions of $a_0, \dots, a_{n-k}$ (though these always fix v), transpositions of $a_0, \dots, a_{n-k}$ the transvection $t' : a_0 \mapsto a_0 a_1^{-1}$.

Consider first the action of transpositions t on the sharing pair $v \in \mathcal{P}_x$. If neither a_0 nor a_{n-k} are swapped by t, then the sharing pair v is fixed. If both a_0 and a_{n-k} are swapped by t, then x_0 and x_1 are swapped, so that the sharing pair $v = \{x_0, x_1\}$ is still fixed. If exactly one of a_0 or a_{n-k} is swapped by t transposition then exactly one of x_0, x_1 are exchanged from the sharing pair, so that the transposition action moves the sharing pair v distance 1 in $\mathcal{P}_x$, as shown in Figure 15.

Finally consider the transvection action $t' : a_0 \mapsto a_0 a_1^{-1}$ on $\mathcal{P}_x$. Then $t'(x_0)$ (shown in yellow in Figure 16) intersects x_0 (shown in orange) twice and $t'(x_1) = x_1$ (shown in light green). Since $n - k \geq 4$ there is a nonseparating sphere a_2 disjoint and not engulfed from $x_0, x_1, t'(x_0)$. So there is an $M_{k,1}$ bounding sphere x_2 (shown in dark green) engulfing a_2 and such that $\{t'(x_0), x_1, x_2\}$ and $\{x_0, x_1, x_2\}$ are sharing triple— let x_2 be the image of x_1 under the transposition $(a_2 a_{n-k})$. Then we have a length 2 path of in $\mathcal{P}_x$ from $t'(v)$ to v:

$$\{t'(x_0), x_1\} \to \{x_1, x_2\} \to \{x_0, x_1\}.$$

It follows from Putman's Lemma 7 that $\mathcal{P}_x$ is connected. $\square$

We define a map $\operatorname{Aut} \mathcal{S}_n^{sep,k} \to \operatorname{Aut} \mathcal{S}_n^{sep,k-1}$ as $\phi \mapsto \hat{\phi}$ by extending $\phi \in \operatorname{Aut} \mathcal{S}_n^{sep,k}$ to $M_{k-1,1}$-bounding spheres via $M_{k-1,1}$-sharing pairs. More explicitly, if $x \in S_n^{sep,k-1}$ is an $M_{k-1,1}$-bounding sphere then by Lemma 59

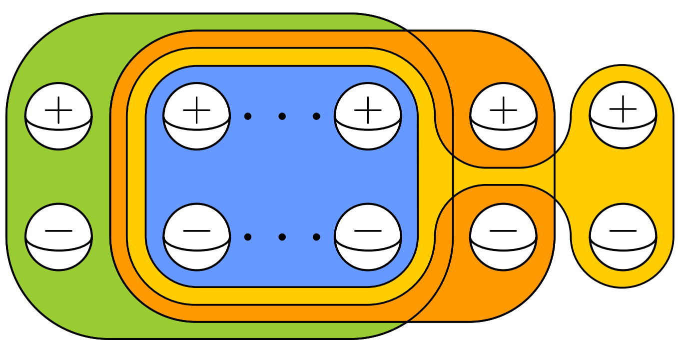


Figure 15: The sharing pair v is formed by the green x_1 and orange x_0 spheres. Transpositions move the sharing pair v either distance 0 in $\mathcal{P}_x$, by swapping orange and green, or distance 1, by, for example, swapping orange and yellow. Observe that the orange, yellow, and green $M_{k,1}$-bounding spheres form a sharing triple for the blue sphere x.

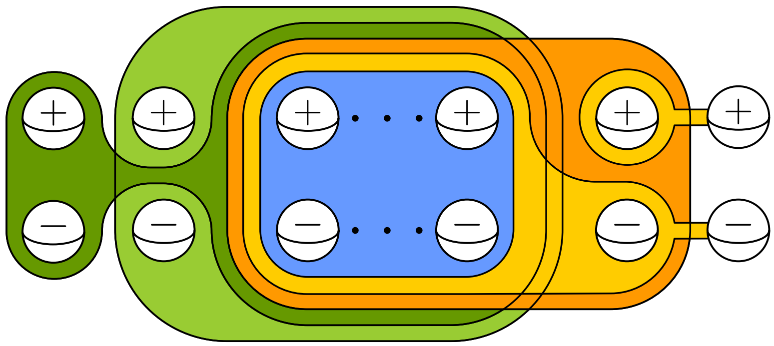


Figure 16: The chosen transvection moves v distance 2 in $\mathcal{P}_x$. Observe that x_0 orange, x_1 light green, x_2 dark green, and $t'(x_0)$ yellow can be organized into two sharing triples: orange with the greens and yellow with the greens.

there is an $M_{k-1,1}$-sharing pair $\{x_0, x_1\}$ that engulfs x uniquely. Then by Lemma 58, $\{\phi(x_0), \phi(x_1)\}$ is a sharing pair. We define $\hat{\phi}(x)$ as the $M_{k-1,1}$-bounding sphere engulfed by $\{\phi(x_0), \phi(x_1)\}$. By Lemma 61 any other choice $\{x'_0, x'_1\}$ of x-sharing pair is connected by a sequence of sharing triples, which by Lemma 60 gives a sequence of sharing triples from $\{\phi(x_0), \phi(x_1)\}$ to $\{\phi(x'_0), \phi(x'_1)\}$, so that both share the same $M_{k-1,1}$-bounding sphere $\hat{\phi}(x)$, which is thus well defined.

Certainly $\hat{\phi}$ is simplicial. To see that observe that if x and x' are disjoint $M_{k-1,1}$-bounding spheres, then $n \geq 3k$ so there are disjoint $M_{k-1,1}$-sharing pairs which ϕ takes to disjoint sharing pairs. Then $\hat{\phi}(x)$ is disjoint from $\hat{\phi}(x')$. If $y \in \mathcal{S}_n^{sep,k}$ is disjoint from x, then y is $M_{j,1}$-bounding with $j \leq \frac{n}{2}$ so there is an x-sharing pair disjoint from y, with its ϕ-image disjoint from $\phi(y)$.

Lemma 62. *For $n \geq 3k$ and $k \geq 2$, the natural restriction map* $\operatorname{Aut} \mathcal{S}_n^{sep,k-1} \to \operatorname{Aut} \mathcal{S}_n^{sep,k}$ *is an isomorphism.*

Proof. We claim that the constructed map extension $\operatorname{Aut} \mathcal{S}_n^{sep,k} \to \operatorname{Aut} \mathcal{S}_n^{sep,k-1}$ given by $\phi \mapsto \hat{\phi}$ is the inverse homomorphism to the restriction $\operatorname{Aut} \mathcal{S}_n^{sep,k-1} \to \operatorname{Aut} \mathcal{S}_n^{sep,k}$ with $\psi \mapsto \psi|_k$. By definition the restriction of $\hat{\phi}$ to $\mathcal{S}_n^{sep,k}$ is ϕ. So the extension is injective and restriction is surjective. But restriction must also be injective, since if $\psi \in \mathcal{S}_n^{sep,k-1}$ restricts to the identity, then for any $M_{k-1,1}$-bounding sphere x there is a x-sharing pair $\{x_0, x_1\}$ which ψ fixes. But then $\psi(x) = x$ is the unique $M_{k-1,1}$-bounding sphere engulfed by $\{x_0, x_1\}$. $\square$

Theorem 63. *For $n \geq 3k$, the natural map* $\operatorname{Out} F_n \to \operatorname{Aut} \mathcal{S}_n^{sep,k}$ *is an isomorphism.*

Proof. The proof is by induction on k. Theorem 51 □

10 This Was Bad

This is supposed to get you the free factor complex, but there is a complicated fibration and not FF itself

Let F_n be the rank n free group. If F_n can be expressed as the internal free product of subgroups $A, B \leqslant F_n$, then A and B are *free factors* of F_n. The free factor complex $\mathcal{FF}_n$ is the simplicial complex with a k-simplex given by conjugacy classes of length $k+1$ chains of proper free factors. The purpose of this note is to give a new proof of the following theorem of Bestvina and Bridson [11].

Theorem 1. (Bestvina–Bridson) For $n \geq 3$ we have $\operatorname{Aut}(\mathcal{FF}_n) \cong \operatorname{Out}(F_n)$.

Let M_n be the connect sum of n copies of $S^1 \times S^2$, with the convention that $M_0 = S^3$. We will consider a series of simplicial complexes where the simplices correspond to collections of spheres in M_n.

Hatcher [22] characterized the free factor complex as a complex of spheres in M_n. When discussing spheres or submanifolds of M_n below, we will always mean their homotopy classes. We define the following three simplicial complexes related to the free factor complex:

- Let $\mathcal{S}_n^{\text{nonsep}}$ be the simplicial complex with k-simplices specified by $k+1$ disjoint nonseparating spheres in M_n.
- Let $\mathcal{S}_n^{\text{coc}}$ be the subcomplex of $\mathcal{S}_n^{\text{nonsep}}$ with simplices given by collections of spheres which are coconnected (i.e. have connected complement) in M_n.
- Let $\mathcal{SF}_n$ be the barycentric subdivision of the $(n-2)$-skeleton of $\mathcal{S}_n^{\text{coc}}$. Thus vertices of $\mathcal{SF}_n$ are coconnected sets of at most $n-1$ spheres, and simplices are given by chains of proper subsets.

For a simplex $\Sigma_0 \subset \cdots \subset \Sigma_k$ of $\mathcal{SF}_n$, we obtain a corresponding simplex of $\mathcal{FF}_n$ by the (conjugacy class of) free factors

$$\pi_1(M_n - \Sigma_k, x_0) \leqslant \cdots \leqslant \pi_1(M_n - \Sigma_0, x_0)$$

so that as posets $\mathcal{SF}_n \cong (\mathcal{FF}_n)^{op}$, and as simplicial complexes they are isomorphic. We thus have the following theorem of Hatcher.

Our contribution is the following pair of isomorphisms.

Theorem 3. For $n \geq 3$ we have $\mathrm{Aut}\,(\mathcal{SF}_n) \cong \mathrm{Aut}\,(\mathcal{S}_n^{\mathrm{coc}}) \cong \mathrm{Aut}\,(\mathcal{S}_n^{\mathrm{nonsep}})$.

Theorem 1 then follows from Theorems 2, 3, and the following result of Pandit [43].

Theorem 4. (Pandit) For $n \geq 3$ we have $\mathrm{Aut}\,(\mathcal{S}_n^{\mathrm{nonsep}}) \cong \mathrm{Out}(F_n)$.

Our first goal is to show that $\mathrm{Aut}\,(\mathcal{SF}_n) \cong \mathrm{Aut}\,(\mathcal{S}_n^{\mathrm{coc}})$.

Let $M_{n,s}$ be the manifold M_n with interiors of s disjoint closed balls removed. We call n the *genus* of $M_{n,s}$. If Σ is a set of disjoint embedded spheres of $M_{n,s}$, we will denote by $M_{n,s}|\Sigma$ the manifold $M_{n,s}$ cut along Σ.

Lemma 5. Automorphisms of $\mathcal{SF}_n$ preserve the cardinality of sets of spheres.

Proof. We induct downward on the cardinality of sets of spheres. We claim as a base case that a set of spheres $\Sigma \in \mathcal{SF}_n^{(0)}$ has $n-1$ spheres if and only if it is adjacent to finitely many sets of spheres in $\mathcal{SF}_n$, namely, the proper subsets of Σ. If $\Sigma \in \mathcal{SF}_n^{(0)}$ has fewer than $n-1$ spheres, then $M_n|\Sigma$ has genus $k \geq 2$. The complex of coconnected nonseparating spheres of $M_n|\Sigma$ is isomorphic to $\mathcal{S}_k^{\mathrm{coc}}$, which is infinite. Choose any nonseparating sphere a of $M_n|\Sigma$. Then $\Sigma \cup \{a\}$ is coconnected in M_n and adjacent to Σ in $\mathcal{SF}_n$.

Assume that automorphisms of $\mathcal{SF}_n$ preserve the size of sets of spheres with at least $k+1$ spheres. Let $A_k \subset \mathcal{SF}_n^{(0)}$ be the sets of spheres of $\mathcal{SF}_n$ with k or fewer spheres. A set of spheres $\Sigma \in A_k$ has k spheres if and only if $\mathrm{link}(\Sigma) \cap A_k$ is finite. By hypothesis automorphisms of $\mathcal{SF}_n$ preserve A_k and its complement, so must preserve the class of sets of k spheres. $\square$

We now prove the first isomorphism of Theorem 3.

Proposition 6. For $n \geq 3$ we have $\mathrm{Aut}\,(\mathcal{SF}_n) \cong \mathrm{Aut}\,(\mathcal{S}_n^{\mathrm{coc}})$.

Proof. As $\mathcal{SF}_n$ is the barycentric subdivision of the $n-2$ skeleton $\mathcal{S}_n^{\mathrm{coc}\,(n-2)}$, there is a natural map

$$\Phi : \mathrm{Aut}\,(\mathcal{S}_n^{\mathrm{coc}}) \to \mathrm{Aut}\,(\mathcal{SF}_n)\,.$$

We will construct the inverse. Let $\phi \in \mathrm{Aut}\,(\mathcal{SF}_n)$. The vertices of $\mathcal{SF}_n$ are the simplices of $\mathcal{S}_n^{\mathrm{coc}}$ with dimension $n-2$ or less. Then ϕ induces a bijection ϕ_* of simplices of $\mathcal{S}_n^{\mathrm{coc}(n-2)}$. By Lemma 5 we have ϕ_* preserves the dimension of simplices, so ϕ_* is an automorphism of $\mathcal{S}_n^{\mathrm{coc}(n-2)}$.

It remains to see that ϕ_* also preserves $n-1$ simplices. To see this we will show that a collection of n disjoint separating spheres Σ form a simplex in $\mathcal{S}_n^{\mathrm{coc}}$ if and only if

$$\mathcal{S}^{\mathrm{coc}(0)}_{n} \cap \left(\bigcap_{x\in\Sigma} \mathrm{link}(x) \right)$$

is finite. Note that if Σ is a coconnected set of n spheres, then $M_n|\Sigma$ is homeomorphic to $M_{0,2n}$. Then $\pi_2(M_n|\Sigma)$ is the free abelian group generated by any $2n-1$ of the balls, and an embedded sphere must be degree at most 1 over any generator. There are thus finitely many embedded spheres of $M_n|\Sigma$. Then $\bigcap_{x\in\Sigma} \mathrm{link}(x)$ contains finitely many vertices of $\mathcal{S}_n^{\mathrm{coc}}$. Conversely suppose Σ is a non-coconnected set of n disjoint spheres. Then $M_n|\Sigma$ has a component M' with genus at least one and at least two boundary spheres. Choose a non-separating sphere x of M', a boundary sphere y, and a loop α based at y intersecting x once. The push map of x along α produces a collection A of infinitely many spheres of M_n. Each $a \in A$ is nonseparating in $M' \subset M|\Sigma$, so $\{a, x\}$ is coconnected for any $x \in \Sigma$. Then $A \subset \bigcap_{x\in\Sigma} \mathrm{link}(x)$. Thus ϕ_* must also preserve $n-1$ simplices and gives a simplicial automorphism of $\mathcal{S}_n^{\mathrm{coc}}$. Then $\phi \mapsto \phi_*$ gives the inverse homomorphism to Φ. □

Call a collection of m disjoint spheres $\Sigma \subset \mathcal{S}_n^{\mathrm{coc}(0)}$ a *bounding m-tuple* (pair, triple, etc.) if Σ is not coconnected but every proper subset of Σ is. The genus of the bounding tuple is the smaller of the genera of the two components of $M_n|\Sigma$. The following lemma shows we can detect the genus combinatorially.

Lemma 7. The link of a genus k bounding m-tuple of $\mathcal{S}_n^{\mathrm{coc}}$ is isomorphic to the join $\mathcal{S}_k^{\mathrm{coc}} * \mathcal{S}_{n-k-m+1}^{\mathrm{coc}}$.

Proof. Consider $\Sigma \subset \mathcal{S}_n^{\mathrm{coc}(0)}$ a bounding m-tuple with genus k. Then $M_n|\Sigma$ has two components, $R_1 \cong M_{k,m}$ and $R_2 \cong M_{n-k-m+1,m}$. Let V_i be the complex of coconnected nonseparating spheres in R_i. So $V_1 \cong \mathcal{S}_k^{\mathrm{coc}}$ and $V_2 \cong \mathcal{S}_{n-k-m+1}^{\mathrm{coc}}$. We claim that $\mathrm{link}(\Sigma)$ is the join $V_1 * V_2$. Certainly $\mathrm{link}(\Sigma) \subset V_1 * V_2$. Consider sets of spheres Σ_i giving simplices of V_i. The $R_i|\Sigma_i$ are connected. $M_n|(\Sigma_1 \cup \Sigma_2)$ is $R_1|\Sigma_1$ and $R_2|\Sigma_2$ glued along Σ, and hence connected. So $\Sigma_1 \cup \Sigma_2$ must be coconnected in M_n and the join $\Sigma_1 * \Sigma_2$ lies in $\mathrm{link}(\Sigma)$. □

We now prove the second isomorphism of Theorem 3.

Proposition 8. For $n \geq 3$ we have $\mathrm{Aut}\,(\mathcal{S}_n^{\mathrm{coc}}) \cong \mathrm{Aut}\,(\mathcal{S}_n^{\mathrm{nonsep}})$.

Proof. Restriction gives a natural map

$$\Phi : \mathrm{Aut}\,(\mathcal{S}_n^{\mathrm{nonsep}}) \to \mathrm{Aut}\,(\mathcal{S}_n^{\mathrm{coc}})\,.$$

We will construct the inverse. Observe that since ${\mathcal{S}^{\mathrm{coc}}}_n^{(0)} = \mathcal{S}_n^{\mathrm{nonsep}(0)}$ any $\phi \in \mathrm{Aut}\,(\mathcal{S}_n^{\mathrm{coc}})$ induces a set map ϕ_* of $\mathcal{S}_n^{\mathrm{nonsep}(0)}$. If ϕ_* is a simplicial automorphism, then $\phi \mapsto \phi_*$ is the inverse homomorphism to Φ. As $\mathcal{S}_n^{\mathrm{nonsep}}$ is a flag complex (Lemma 3 of [4]), it will suffice to show that ϕ_* sends pairs of disjoint spheres to pairs of disjoint spheres. Disjoint nonseparating spheres form a bounding pair if and only if they are not adjacent in $\mathcal{S}_n^{\mathrm{coc}}$. So it suffices to show that ϕ preserves bounding pairs of $\mathcal{S}_n^{\mathrm{coc}}$. We will demonstrate this through the stronger result that ϕ preserves the set of genus k bounding m-tuples.

Case 1. Suppose Σ is a genus k bounding m-tuple with $m > 2$. Any $\Sigma' \subset \mathcal{S}_n^{\mathrm{coc}(0)}$ is a bounding m-tuple if and only if Σ' does not span a simplex in $\mathcal{S}_n^{\mathrm{coc}}$, but every proper subset of Σ' does. Hence if $\phi \in \mathrm{Aut}\,(\mathcal{S}_n^{\mathrm{coc}})$, then $\phi(\Sigma)$ is a bounding m-tuple. By Lemma 7, $\mathrm{link}(\Sigma)$ is isomorphic to $\mathcal{S}_k^{\mathrm{coc}} * \mathcal{S}_{n-k-m+1}^{\mathrm{coc}}$. We can determine k by the maximal simplex dimension on the sides of the join. Then $\phi(\Sigma)$ is also genus k.

Case 2. Suppose $\Sigma = \{x, y\}$ has $m = 2$ spheres. Choose a collection Σ' of disjoint nonseparating spheres such that there are two separate components

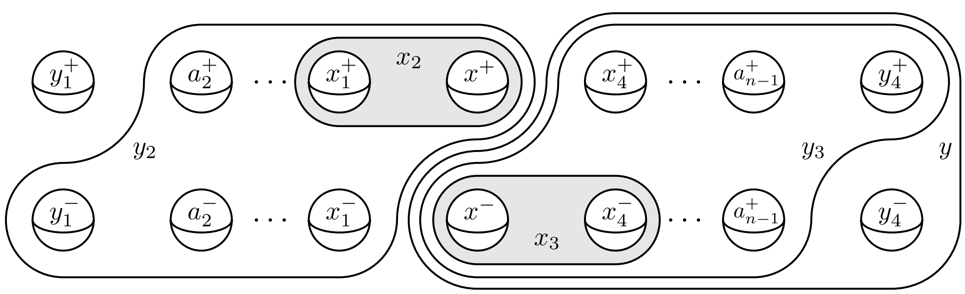


Figure 17: The manifold $M_n|\{a_i\}_{i=1}^n$ is S^3 with $2n$ balls removed. We obtain M_n by again identifying the spheres with $+$ and $-$ labels via a vertical reflection. The spheres $\Sigma' = \{x_i, y_i\}_{i=1}^4$ are such that $M_n|\Sigma'$ contains x and y in disjoint copies of $M_{0,4}$. The $M_{0,4}$ containing x (identify x^+ and x^-) is shaded. The $M_{0,4}$ containing y is the exterior of y_2 and y_3.

of $M_n|\Sigma'$ homeomorphic to $M_{0,4}$ and containing x and y respectively. We can construct Σ' as follows. $M_n|\Sigma$ has two components, homeomorphic to $M_{k,2}$ and $M_{n-k-1,2}$. So we have a set of spheres $\{a_i\}_{i=1}^n$ coconnected in M_n disjoint from y with $a_{k+1} = x$. Choose x_2, x_3, y_2, y_3 as shown in figure 17 and relabel $a_1 = y_1$, $x_1 = a_k$, $x_4 = a_{k+2}$, $y_4 = a_n$. Then $\{x_1, \ldots, x_4\}$ (resp. $\{y_1, \ldots, y_4\}$ are the boundary spheres of a component of $M|\Sigma'$ homeomorphic to $M_{0,4}$ and containing x (resp. y). Further $\{x, x_1, x_2\}$ and $\{x, x_3, x_4\}$ are genus 0 bounding triples. Let $\Sigma' = \{x_i, y_i\}_{i=1}^4$.

By Case 1 we have that $\{\phi(x_1), \ldots, \phi(x_4)\}$ is a genus 0 bounding 4-tuple and $\{\phi(x), \phi(x_1), \phi(x_2)\}$ and $\{\phi(x), \phi(x_3), \phi(x_4)\}$ are genus 0 bounding triples. So $\{\phi(x_1), \ldots, \phi(x_4)\}$ define a component of $M|\Sigma'$ homeomorphic to $M_{0,4}$ and containing $\phi(x)$.

If $\{x_1, \ldots, x_4\} \neq \{y_1, \ldots, y_4\}$ then $\phi(x)$ and $\phi(y)$ lie in disjoint $M_{0,4}$ homeomorphic components of $M|\phi(\Sigma')$. Then $\phi(x)$ and $\phi(y)$ are are disjoint. They are also not adjacent in $\mathcal{S}_n^{\text{coc}}$, so they are bounding a pair.

Suppose $\{x_1, \ldots, x_4\} = \{y_1, \ldots, y_4\}$. Then $n = 3$ and $M_3|\{x_i\}_{i=1}^4$ is homeomorphic to two copies of $M_{0,4}$. As x, y form a bounding pair, the bounding triples must be $\{x, x_1, x_2\}$, $\{x, x_3, x_4\}$, $\{y, x_1, x_2\}$, and $\{y, x_3, x_4\}$. Then the ϕ image of these triples are bounding triples giving $\phi(x)$ and $\phi(y)$ contained in disjoint $M_{0,4}$. Then $\phi(x)$ and $\phi(y)$ are disjoint and must form a bounding pair. $\square$

11 Complex of Strongly Separating Curves

Joint with Alan McLeay

The complex of strongly separating spheres is the induced subcomplex of the curve complex of $C^{ss}(S_{g,p})$ whose vertices are homotopy classes of separating curves in the genus g, n-punctured surface $S_{g,p}$ such that both components of the complement have complexity $3g' + p' \geq 3$.

Bowditch motivation "Weil-Petersson metric on Teichmuller space. There it was shown that the rigidity of Gss() implies the quasi-isometric rigidity of the Weil-Petersson metric associated to ." Bowditch proved this [9]

Theorem 64. *If $g + p \geq 7$, then the natural map*

$$\mathrm{MCG}^{\pm}(S_{g,p}) \to \mathrm{Aut}\left(C^{ss}(S_{g,p})\right)$$

is an isomorphism.

Bowditch asked which of the We extend the results of Bowditch

Theorem 65. *The natural map*

$$\mathrm{MCG}^{\pm}(S_{g,p}) \to \mathrm{Aut}\left(C^{ss}(S_{g,p})\right)$$

is an isomorphism for the green entries in the table

$g \setminus p$	0	1	2	3	4	5	6
0	×	×	×	×	×	×	×
1	×	×	×	×	?	✓	[9]
2	×	×	×	✓	✓	[9]	[9]
3	[10]	[29]	✓	✓	[9]	[9]	[9]
4	[10]	[29]	✓	[9]	[9]	[9]	[9]
5	[10]	[29]	[9]	[9]	[9]	[9]	[9]
6	[10]	[29]	[9]	[9]	[9]	[9]	[9]

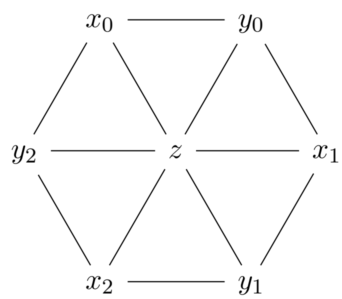
11.1 Pentagonal Sharing Pairs

11.2 Hexagonal Sharing Pairs

11.2.1 $S_{1,5}$

Lemma 66. *??Curve types are preserved There's hexagons inside a 5 curve. There's no hexagons outside a 3 curve since there's no hexagons in $S_1, 4$?*

Definition 67. 3-disk sharing pair is x_0, x_1 such that there is



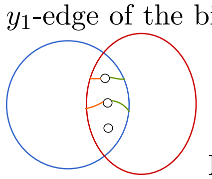
so that z is a 5-disk, y_i are 4-disks, x_i is a 3-disk

Lemma 68. *The complex of 3 and 4-disks in an annulus has no cycles smaller than an octagon.*

Lemma 69. *Suppose hexagon $(x_i, y_i)_{i\in\mathbb{Z}/3}$ as above. If y_0 and y_1 have geometric intersection 2, and p_0, p_1 are in $y_0 \cap y_1 \cap y_2$, then there is a unique 2-curve shared by y_0, y_1, y_2.*

Proof. Observe if there is a 2-disk, it has to be unique as if there are a pair of 2-disk shared by all y_i, then that would give a 3-disk shared by all y_i.

Assume to the contrary there's no path p_0 and p_1 all three y_i. There there are arcs α and β from p_0 to p_1 with $\alpha \subset x_0 \subset y_0$ and $\beta \subset x_2 \subset y_1$. Then $\alpha\beta^{-1}$ is a nontrivial loop that must be contained in the disk y_2. So Consider the projections of α and β in the bigon $y_0 \cap y_1$. α has arcs from p_0, p_1 to the y_0-edge of the bigon, similarly β has arcs from p_0, p_1 to the y_1-edge of the bigon, so that these four arcs form a band across the bigon.



No arc of α, β can disconnect the band, so the band there must contain a path p_0 to p_1 which is in y_1.

□

Lemma 70. *A 3-disk sharing pair uniquely determines a 3-disk.*

Proof. We are working in the 3 & 4-disk complex of a 5-disk. Consider the possible distribution of the marked points $p_0, \ldots, p_4$ in the 5-disk. If there is a point that is in none of the 4-disks y_i, then we would have a hexagon in the 3 & 4-disk complex of an annulus. So up to relabeling $p_4 \notin y_0$ and $p_3 \notin y_1$ and there are three cases for 4-disk point configuration

Case 0: $p_2 \notin y_2$

Case 1: $p_4 \notin y_2$ and $p_2 \notin x_0$

Claim that ∂y_0 and ∂y_1 have geometric intersection 2. The 3-disk x_1 is contained in both y_0 and y_1. So there is only one possible projection for y_1 into y_0 bands. The idea is supposed to be find two arcs in y_2 that you assume are disjoint but the bands get in the way Now $p_0, p_3 \in x_0 \subset y_0 \cap y_2$ and $p_1, p_4 \in x_2 \subset y_1 \cap y_2$.

Case 2: $p_4 \notin y_2$ and $p_3 \notin x_0$

Claim that ∂y_0 and ∂y_1 have geometric intersection 2. The 3-disk x_1 is contained in both y_0 and y_1. So there is only one possible projection for y_1 into y_0 bands, which separate p_3 from x_1 in y_0. Now $p_0, p_1, p_2 \in x_2 \subset y_1 \cap y_2$ so there must be an arc α from p_0 to p_1 contained in x_2 but not in x_1 so it must pass through the bands. Similarly $p_0, p_1, p_2 \in x_0 \subset y_2 \cap y_0$ so there must be an arc β from p_0 to p_1 contained in x_0 but not in x_1, so it must cross the bands. But then β and α form a bigon around p_3 so that p_3 must be in x_1, a contradiction.

Now p_0 and p_1 arc α from p_0 to p_1 arc β from p_0 to p_1

The idea is supposed to be find two arcs in y_2 that you assume are disjoint but the bands get in the way Now $p_0, p_3 \in x_0 \subset y_0 \cap y_2$ and $p_1, p_4 \in x_2 \subset y_1 \cap y_2$.

Then you're supposed to assume that the $p_0 \to p_1$ arcs in x_2 and x_0 are not in x_1 so they have to be distinct and you force them to bound a disk that has to be in y_2 and show that defines the 4-curve and then its not the one you want?

□

11.3 Octagonal Sharing Pairs?

Observe that $C^{ss}(S_1, 4)$ is a bipartite graph consisting of 3-disks and 4-disks.

Consider a $2n$-cycle $(x_i, y_i)_{i \in 2n}$ in $C^{ss}(S_{1,4})$ with x_i a 3-disk and y_i a 4-disk.

Claim Up to relabeling the point configurations of is one of the following three types:

1. $p(x_i) = \{p_i\}_{i \neq j}$
2. $p(x_0) = p(x_2) = \{p_1, p_2, p_3\}$ and $p(x_1) = p(x_3) = \{p_0, p_1, p_2\}$
3. $p(x_0) = p(x_1) = \{p_1, p_2, p_3\}$ and $p(x_2) = p(x_3) = \{p_0, p_1, p_2\}$

Fix minimally intersecting representatives of the homotopy classes of octagon $(x_i, y_i)_{i\in 4}$.

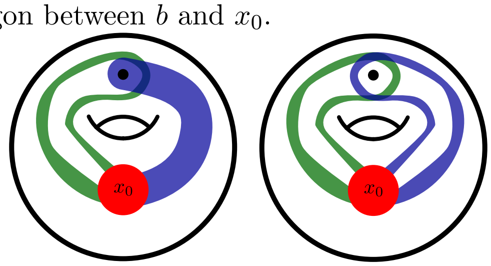
Case 1

If no two 3-disks contain the same 3 points, then up to label permutation the labels are as above. This configuration is obtained by the following octagon

FIG!!!!

Case 2

Suppose that x_i and x_{i+1} contain the same 3 points. We may assume that x_0 and x_1 both contain p_1, p_2, p_3. Consider the image of x_1 in $S_{1,4} - x_0$. Since x_0 and x_1 are both contained in the 4-disk y_0, there is a unique nontrivial simple closed curve based at the image of x_0 in $S_{1,4}/x_0$, so x_1 must contain at least one band b so that the component of $y_0 - x_0 - b$ containing p_0 is a bigon between b and x_0.



The 3-disk x_2 has two marked points in common with x_0, suppose without loss of generality that $p_1, p_2 \in x_0 \cap x_2$. Consider an arc a_1 from p_1 to p_2 in x_2. Assume to the contrary that a_1 is not contained in y_0. The 4-disk y_0 cannot contain x_2. Then there is an arc p_2 to p_1 in $x_1 \subset y_0$. But then $a_1 a_2$ is nontrivial curve in the torus, contained in the 4-disk $y_1 \supset x_1 \cup x_2$, a contradiction. Then it must be that $a_1 \subset y_0$.

If $p_0 \notin x_2$, there must be an arc $a_1 \subset x_2$ from p_1 to p_3 not contained in y_0. Then there is an arc p_3 to p_1 in $x_1 \subset y_0$. But then $a_1 a_2$ is nontrivial curve in the torus, contained in the 4-disk $y_1 \supset x_1 \cup x_2$, a contradiction.

We have that x_2 contains the points p_0, p_1, p_2. By a similar argument x_3 contains the same points.

This configuration is obtained by the following octagon.

FIG!!!!

Case 3

Suppose that x_i and x_{i+2} contain the same points. We may assume that x_0 and x_2 contain p_1, p_2, p_3. As x_0 and x_2 are not contained in the same

4-disk there must be two points, say p_1 and p_3, and arcs $a_0 \subset x_0$ from p_1 to p_3 and $a_2 \subset x_2$ from p_3 to p_1 such that $a_0 a_2$ is a nontrivial curve in the unpunctured torus. Assume to the contrary that p_1 and p_3 are both in x_1, then there is an arc $a_1 \subset x_1$ from p_3 to p_1. Since $a_0 a_1 \subset y_1$, we have $a_0 a_1$ is nullhomotopic in the unpunctured torus. But then $a_0 a_2 = rev(a_1) a_2$ is nontrivial in the unpunctured torus so not contained in y_2, a contradiction. It must be that either p_1 or p_3 not in x_1. We may assume that $p_3 \notin x_1$.

A similar argument forces p_1 or $p_3 \notin x_3$. Suppose that $p_3 \in x_3$. Consider an arc $b_3 \subset x_3$ from p_2 to p_3. All arcs from p_1 to p_2 in x_0, x_1, and x_2 must lie in a common 4-curve by the above argument. But then the arcs from p_2 to p_3 in x_0 and x_2 cannot lie in a 4-disk, but this contradicts that p_2 to p_3 in x_3 lies in both. It must be that $p_3 \notin x_3$.

This configuration can be realized as follows FIG!!!! $\square$

Def Let $A_{p,q}$ be the set of classes of arcs from point p to point q considered up to homotopy relative to P. By forgetting all the points but p and q we may also consider the arcs $A_{p,q}$ up to homotopy relative to p, q. Observe that if $a, a' \in A_{p,q}$ are arcs contained in a common 4-disk then a and a' are homotopic relative to $\{p, q\}$.

Claim Suppose that x_0 and x_2 are distance four 3-disks in an octagon. Then x_0 and x_2 contain the same points if and only if there more than 2 length 4 paths from x_0 to x_2.

Suppose that there are at least 3 paths from x_0 to x_2, each of which pass through a distinct 3-disk, say x_1, x_2, x_3. If x_1, x_2, x_3 all contain distinct sets of marked points, then by the possible point configurations described by Lemma !!!!, there are not possible choice for the marked points contained in x_0 and x_2. If two of x_1, x_2, x_3 contain the same points. But then by the possible point configurations described by Lemma !!!!, the x_0 and x_2 must also contain the same points.

Suppose x_0 and x_2 contain the same points, say p_1, p_2, p_3. Then by the Lemma !!!! it must be that x_1 and x_3 contain the same points, say p_0, p_1, p_2. Suppose that there is a nonseparating curve α based at p_0 and disjoint from x_0 and x_2. Let

$$x_0, y, x, y', x_2$$

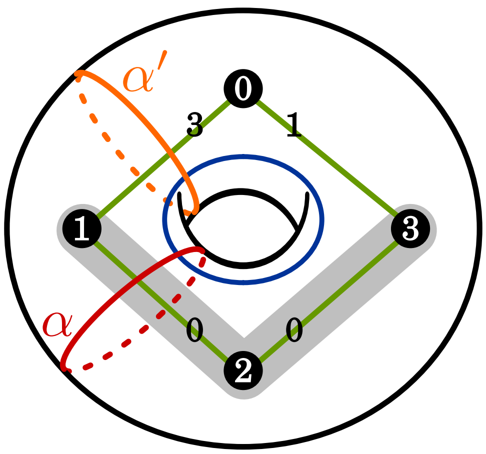
be a length 4 path. Since $p_0 \in y, x, y'$ we have that

$$x_0 \to T_\alpha^k y \to T_\alpha^k x \to T_\alpha^k y' \to x_2$$

gives infinitely many paths of length 4.

If there is no nonseparating curve then p_0 is in 2n-gon. Connect p0 to all the corners!

Claim Up to homeomorphism there is only one type of octagon with $p(x_i) = \{p_j\}_{j \neq i}$.



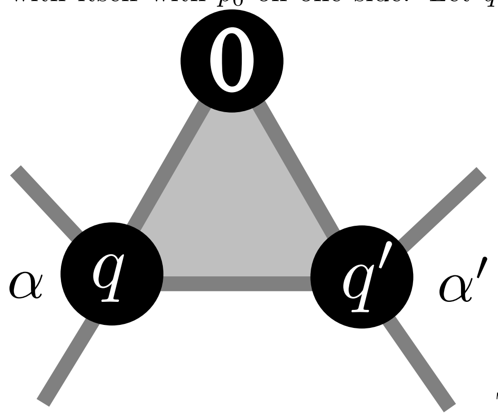
Consider the two 3-disks x_0 and x_2. Observe that x_0 and x_2 cannot be contained in a common 4-disk, or else x_0, x_1, x_2 would all be contained in a common 4-disk which would force $y_0 = y_1$ by Lemma !!!!.

Assume to the contrary that x_0 and x_2 contain a common arc $a_{13} \in A_{p_1,p_3}$. Consider arcs $a_{12}^{(0)} \in A_{p_1,p_2}$ in x_0 and $a_{10}^{(2)} \in A_{p_1,p_0}$ in x_2 which cannot be contained in a common 4-disk, since otherwise a regular neighborhood of $a_{12} \cup a_{13}^{(0)} \cup a_{01}^{(2)}$ is a 4-disk containing x_0 and x_2. Then x_3 contains arcs $a_{12}^{(3)} \in A_{p_1,p_2}$, which must be homotopic to $a_{12}^{(0)}$ relative to p_1, p_2, and $a_{10}^{(3)} \in A_{p_1,p_0}$, which must be homotopic to $a_{10}^{(2)}$ relative to p_1, p_0. But then $a_{12}^{(3)}$ and $a_{10}^{(3)}$ cannot be contained in a common 4-disk, and yet are contained in x_3, a contradiction.

Let $a_{i+1,i+2}^{(i)} \in A_{p_{i+1},p_{i+2}}$ for $i \in 4$ be an arc in x_i. Then we have a loop $a = a_{12}^{(0)} a_{23}^{(1)} a_{30}^{(2)} a_{01}^{(3)}$ of S_1. We may assume any self-intersections of a occur transversely at points not in P.

Assume to the contrary that a is nullhomotopic in S_1. So a must be a non-simple separating curve in the torus. There must be a point $p \in$

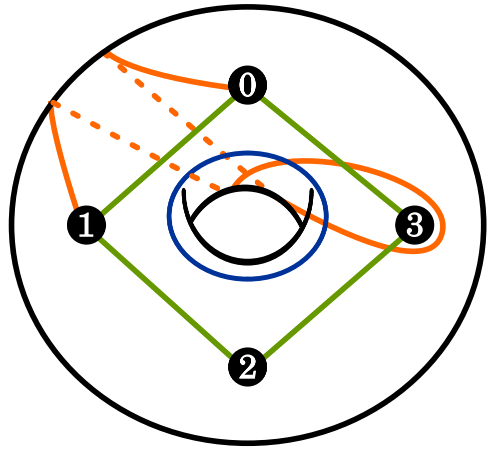
P such that a is not nullhomotopic relative to $\{p\}$. Relabel P so that a is not nullhomotopic relative to $\{p_0\}$. So a forms an innermost bigon b with itself with p_0 on one side. Let $q, q' \in a$ be the vertices of the bigon.



The image of $a-(b-q)$ must contain a simple closed curve α based at q, and $a-(b-q')$ contains a simple closed curve α' based at q'. Since $a^{(i)}$ and $a^{(i+1)}$ must be contained in a common 4-disk, their union cannot contain loops of the torus. So both α and $\alpha' \subset a$ must have at least 2 points of $P-\{0\}$, but they do not intersect at P, a contradiction.

Assume to the contrary that a is not homotopic in S_1 to a simple closed curve of S_1. Let q be a self-intersection of a and let α and $\alpha' \subset a$ be two nontrivial loops of S_1 sharing no common subarcs. Then α and α' must each contain two points of P. Say α contains p_0 and p_1 while α' contains p_2 and p_3. So $a_{12}^{(0)}$ and $a_{30}^{(2)}$ must intersect at q. Observe x_1 must contain an arc $a_{30}^{(1)} \in A_{p_3p_0}$ homotopic to $a_{30}^{(2)}$ relative $\{p_3p_0\}$ and disjoint from $a_{23}^{(1)}-\{p_3\}$. Then $a_{30}^{(2)}$ and $a_{30}^{(1)}$ cannot be homotopic relative to P, as if $a_{30}^{(1)}$ intersects $a_{12}^{(0)}$ the 4-disk y_0 would contain α'. Then $a_{23}^{(1)}$ must link with $a_{01}^{(3)}$. But then $a_{01}^{(3)}$ is homotopic relative $\{p_0, p_1\}$ to a curve $a_{01}^{(2)} \in A_{p_0p_1}$ in x_2 which must link with $a_{23}^{(1)}$. But then $y_1 \supset x_1 \cup x_2$ contains a nontrivial loop of the torus. It must be that a is a homotopic in S_1 relative $\varnothing$ to a simple closed curve of S_1.

Let b be the simple closed curve obtained from a by homotoping the arcs



$a^{(i-1)}_{i,i+1}$ relative p_i, p_{i+1}. We may assume that $a^{(3)}_{01} = b^{(3)}_{01}$.

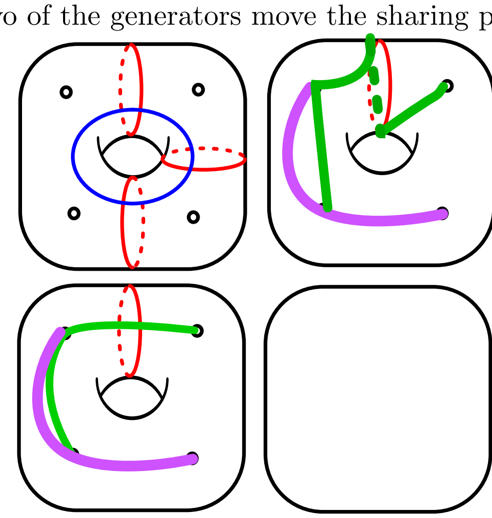
Assume to the contrary that $a^(0)_{12}$ is not supported on the annulus $N(b)$. Then let D be an innermost bigon formed by $a^{(0)}_{12}$ and $b^{(3)}_{12}$, which must contain a point of $P - \{p_1, p_2\}$.

If $a^{(0)}_{12}$ and $a^{(3)}_{01}$ intersect at a point $q \in \partial D$, then $a^{(3)}_{01}$ has an arc that cros $a^{(0)}_{12}$ contains a loop

Remark Fix a 2-disk z A pair x, x' of 3-disks containing z are a sharing pair for z if they fit into a common octogon. Call a triple x, x', x'' of 3-disks containing z a sharing triple if every two form a sharing pair.

Make a graph P_z whose vertices are sharing pairs x, x' of z and with two sharing pairs adjacent if their union is a sharing triple.

Well-definedness is shown by Putman's Lemma on P_z. Observe that only two of the generators move the sharing pair at all and only distance 1.



12 Computational Evidence

Fix a triangulation of the surface with all points at the marked points.

Data is a triangulation T

1. A triangulated polygon embedded in the plane $\mathbb{R}^2$
2. List of oriented triangles given by triples of directed edges

Normal coordinates Laminations are functions $f : T \to \mathbb{R}$ Multicurves are integer laminations $f : T \to \mathbb{Z}_{\geq 0}$

$S_{1,4}$ has triangulation

$$
\begin{tikzcd}
p_0 \arrow[r, "e_0"] & p_1 \arrow[r, "e_3"] & p_2 \arrow[r, "e_6"] & p_3 \arrow[r, "e_9"] & p_0 \\
p_0 \arrow[u, "e_1"] \arrow[r, "e_0"'] & p_1 \arrow[u, "e_4"'] \arrow[ul, "e_2"] \arrow[r, "e_3"'] & p_2 \arrow[u, "e_7"'] \arrow[ul, "e_5"] \arrow[r, "e_6"'] & p_3 \arrow[u, "e_{10}"'] \arrow[ul, "e_8"] \arrow[r, "e_9"'] & p_0 \arrow[u, "e_1"'] \arrow[ul, "e_{11}"]
\end{tikzcd}
$$

Normal coordinates uniquely determine a number of line segements at each angle of each triangle

Also write as crossing sequence, cyclic reduced word in $T^{\pm}$

Computing dehn twists

possible in polynomial time

simplify $T_\alpha(\beta)$ computation by a choice of end for every edge so that α always passes through the end and β always passes through the middle

Hyperbolicity constant of the curve graph Hyperbolicity constant of the strongly separating curve complex

Github

13 Complex of Surfaces

Def A genus g *rosebud* is the wedge of a sphere and a genus g rose. We consider spheres genus 0 rosebuds, and roses degenerate rosebuds.

Claim Every embedded compact closed orientable surface in M_g is homotopy equivalent to a sphere, rose, or rosebud.

Proof. Consider the image in S^3 minus spheres. Every disk is compressible. $\square$

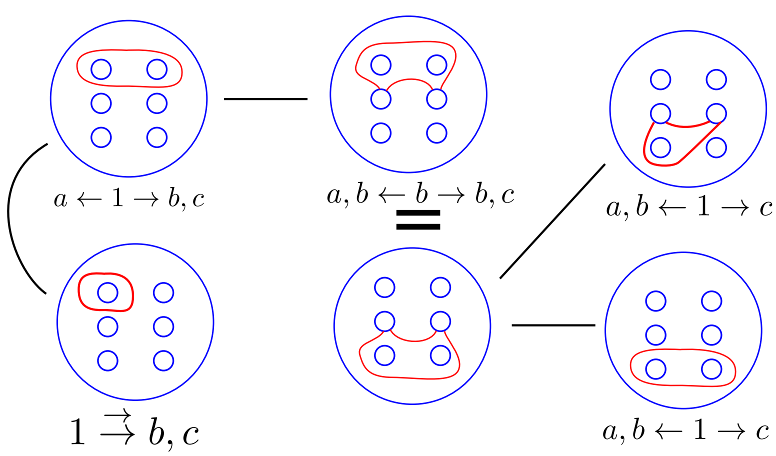
Claim An embedded surface S_g has $\pi_1(S_g)$ contains a free factor at most rank g.

Proof. If $\pi_1(S \hookrightarrow M)$ contains a free factor rank k, then S intersects at least k disjoint spheres in M. Then there are at least k compressible disks with $\partial D \subset S$. Each one kills a generator of $\pi_1(S)$

Claim There is a correspondence between homotopy classes of embedded rosebuds in M_n and one edge Bass Serre graph-of-groups decompositions of F_n.

Proof. By Van Kampen every rosebud gives a graph of groups. Conversely if $F_n = \pi_1(A \leftrightarrow C \to B)$ then there is a basis $\{a_i\}_{i \in n}$ of F_n with the first k in A and the last $n-k$ in B. This corresponds to a system of nonseparating spheres. There is a unique separating sphere S which separates the A spheres from the B spheres. From a basepoint on S there is a rose whose petals give the arcs forming a basis of C. $\square$

Def. The graph of embedded rosebuds $\mathcal{R}_n$ has embeddings of rosebuds (and roses) of genus at most n as vertices. Two embedded rosebuds are adjacent if the embeddings of their homotopy classes differ by the wedge of a (homologically nontrivial?) 1- or 2-sphere.



$\mathcal{R}_n$ is equivalent to the graph of one-edge Bass Serre graph-of-group decompositions of F_n with adjacency given by the following moves:

1. tube tunneling at $a \in A - B$

$$A \leftarrow C \rightarrow B \;\Leftrightarrow\; A \leftarrow C * a \rightarrow B * a$$

2. nonseparating sphere scooping (primitive $b \notin A$)

$$C \overset{\rightarrow}{\rightarrow} A * A'b \;\Leftrightarrow\; C \overset{\rightarrow}{\rightarrow} A * A' \;\Leftrightarrow\; b * C \leftarrow C \rightarrow A * A'$$

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