Outer spaces and Automorphisms of Free Groups

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1 Introduction / Motivation

In 1880 Dyck defined free groups. But the study of automorphism of free groups was started in the 1920-30 by Nielsen, Magnus, Whitehead. They proved basics properties of $\mathrm{Out}(F_n)$. Most of the results used algebraic methods. In 1970's Stallings introduced new topological methods - based on the fact that $\pi_1(\text{graph}) = \text{free}$. "The topology of finite graphs" 1983. Recent work was heavily influenced by Gromov, Thurston - used geometry and dynamics to study groups.

Many recent results: analogies between $\mathrm{Out}(F_n)$ and:

1. lattices in semisimple lie groups
2. mapping class groups of surfaces

How can you see the analogy in 1. ? Consider $F_n \to \mathbb{Z}^n$ (abelianize), this gives a map $\mathrm{Aut}(F_n) \to \mathrm{Aut}(\mathbb{Z}^n) = \mathrm{GL}_n(\mathbb{Z})$. Innerautomorphism $x \mapsto gxg^{-1} \in F_n \to x \in \mathbb{Z}^n$. So the map factors through

$$\begin{tikzcd} \mathrm{Aut}(F_n) \arrow[rr] \arrow[dr] & & \mathrm{Aut}(\mathbb{Z}^n) \\ & \mathrm{Out}(F_n) \arrow[ur] & \end{tikzcd}$$

So we have a map $\mathrm{Out}(F_n) \to \mathrm{GL}_n(\mathbb{Z})$. Nielson (1924) showed this map is surjective. He also showed that for $n = 2$ the kernel is trivial, while for $n > 2$ the kernel is called IA_n ("*Identity on abelianization*"). Magnus (1934) proved that IA_n is finitely generated. He then asked: Is it finitely presented? In 1997, Kirstic-McCool showed that IA_3 is <u>not</u> finitely presented. But for $n > 3$ we don't know.

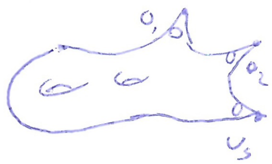
Note. We can show $\mathrm{Out}(F_n)$ is <u>not</u> a lattice in any Lie group.

How can you see the analogy in 2. ? Mapping class groups: Consider $S_{g,s}$, for examples:

Then $\pi_1(S_{g,s}) \cong F_n$ where $n = 2g + s - 1$. If we have a homeomorphism $h : S_{gs} \to S_{g,s}$ this induces $h_* : \pi_1(S_{g,s}) \xrightarrow{\cong} \pi_1(S_{g,s})$. If we choose an isomorphism $F_n \to \pi_1 S_{g,s}$, then we get

$$\begin{tikzcd} \pi_1 S_{g,s} \arrow[r, "\cong"] & \pi_1 S_{g,s} \\ F_n \arrow[u, "\cong"] \arrow[r] & F_n \arrow[u, "\cong"] \end{tikzcd}$$

This gives a map $\mathrm{Homeo}(S_{g,s}) \to \mathrm{Out}(F_n)$ (this is because we do not pick a base point, as if we did we might end up with conjugate elements depending of paths picks). Homotopic homeomorphism have the same image so we can actually consider this map on $\pi_0(\mathrm{Homeo} S_{g,s}) \to \mathrm{Out}(F_n)$.



Note that since punctures needs to be map to each other, the circles are permuted (up to homotopy) by h. They correspond to conjugacy classes in F_n. If h induces ϕ, then note that ϕ stabilises $\{u_1, \ldots, u_s\}$. So the image is in $\mathrm{Stab}\{u_1, \ldots, u_s\} \leq \mathrm{Out}(F_n)$.

Theorem (Zieschang)**.** *The map* $\pi_0(\mathrm{Homeo}S_{g,s}) \to \mathrm{Out}(F_n)$ *is injective, with image* $\mathrm{Stab}\{(u_1, \ldots, u_s\})$.

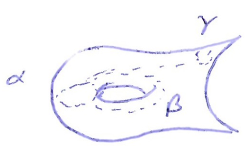
We call the mapping class group $\mathrm{Mod}(S_{g,s}) = \pi_0(\mathrm{Homeo}S_{g,s})$. This is a subgroup of $\mathrm{Out}(F_n)$.

Exercise. Does $\mathrm{Mod}(S_{g,0})$ ever embed in $\mathrm{Out}(F_n)$?

Question: Is <u>every</u> $\phi \in \mathrm{Out}(F_n)$ in the image of some $\mathrm{Mod}(S_{g,s})$? Answer: (Stallings, 1970s): No.

Example. The automorphism $\phi : \begin{cases} x \mapsto y \\ y \mapsto z \\ z \mapsto xy \end{cases}$ is not in the image of any $\mathrm{Mod}(S_{g,s})$.

Proof. The orientable surfaces giving F_n are $S_{0,4}, S_{1,2}$. Let us consider $S_{1,2}$, so suppose that ϕ was induced by a homeomorphism $h : S_{1,2} \to S_{1,2}$. Look at h^2 instead: it must preserve orientation and fixes the punctures. On H_1: $h_* : H_1(S_{1,2}) \to H_1(S_{1,2})$, note that H_1 is generated by $[\alpha], [\beta], [\gamma]$.



Note that $h(\gamma) \cong \gamma$, so $[\gamma]$ is an eigenvector with eigenvalue 1. So in the basis $[\gamma], [\alpha], [\beta]$ is $\begin{pmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$. Other eigenvalues must be λ, λ^{-1}. But let us look at the matrix of ϕ, it is $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. This has eigenvalue (μ_1, μ_2, μ_3) with $\mu_1 > 1$, $\mu_2, \mu_3 < 1$. So the eigenvalue of ϕ^2 have the same properties. So ϕ is not equal to h_* on π_1. $\square$

Exercise. What happens for $S_{0,4}$? And the non-orientable cases?

Stallings: If $S_{g,s} \to S_{g,s}$ fixing orientation and fixing punctures, then the eigenvalues of h_* are $\{\underbrace{1, \ldots, 1}_{s-1}, \{\lambda, \lambda^{-1}\}, \{\mu, \mu^{-1}\}, \ldots\}$

To study lattices you look at their action on the symmetric space $K \backslash G$
To study $\mathrm{Mod}(S)$ we look at the action on Teichmuller space $J(S)$
To study $\mathrm{Out}(F_n)$ we look at the action on Outer space. This was introduced in 1980s by Culler-Votgmann. Outerspace is a space of <u>graphs</u>. Many things are parameterised by graphs.
Publicity: There exists connections with:

- Moduli spaces of punctured surfaces
- Systems of spheres in a 3-manifold
- Tropical curves ("tropical algebraic geometry")
- Feynman diagrams

- Invariants in sympletic modules
- Classical modular forms
- Phylogenetic trees

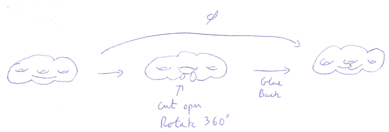
We want to use topology to study free groups and their automorphism. So we need X with $\pi_1(X) \cong F_n$ then we get $\pi_0(\text{homotopy equivalence of } X) \to \mathrm{Out}(F_n)$.

We tried $X = S_{g,s}$ with $n = 2g + s - 1$, but it was not surjective. (Not every automorphism can be realised)

How about X =finite graph. We need homotopy equivalence, as homeomorphism have finite order (up to homotopy). (Exercise). But: using homotopy equivalences, we get $\pi_0 HE(X) \to \mathrm{Out}(F_n)$.

Exercise. This map is an isomorphism if $F_n \cong \pi_1 X$.

What about $X = H_n$ (handle body), then $\pi_1 H_n = F_n$. We do get a map $\pi_0(\mathrm{Homeo} H_n) \to \mathrm{Out}(F_n)$. We claim this map is surjective, but it is not injective. To see it is not injective see the diagram

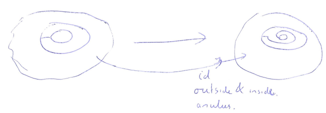


The curve on the boundary can not be straighten up, so it is not isomorphic to the identity. But it is the identity on $\pi_1(H_n)$. So ϕ is in the kernel, and ϕ has ∞ order.

What about $M_n = H_n \cup_{\text{id on } \partial} H_n$. Now $H_n = \#S^1 \times D^2$, so $M_n = \#S^1 \times S^2$. In particular $\pi_1(M_n) = F_n$. So we can look at $\pi_0(\mathrm{Homeo} M_n) \to \mathrm{Out}(F_n)$

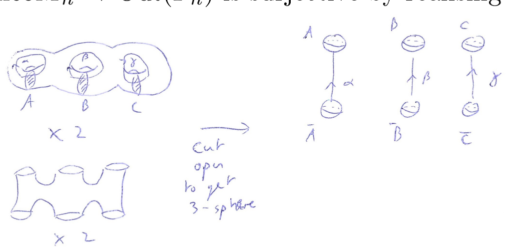
Theorem (Laudenbach)**.** *This map is surjective with kernel a finite* 2*-group* $(\mathbb{Z}/2\mathbb{Z})^n$*.*

What is in the kernel? (Doubled twists) Dehn twist in a surface

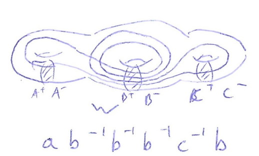


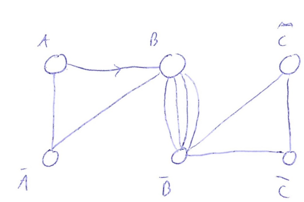
I.e. think of the annulus as a bunch of concentrated sphere, which we rotate by angle from 0 to 2π as we go towards the centre. Dehn twist, D, in 3-manifold M, supported on $S^2 \times I \subseteq M$. It's the same idea, except this time we rotate spheres along a fixed axes by angles $0 \le t \le 2\pi$ as we go in. So this are loop in $SO(3)$. Recall that $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$, so $D^2 \cong \mathrm{id}$. Hence $D^2 \in \ker(\pi_0 \mathrm{Homeo} M_n \to \mathrm{Out}(F_n))$.

Hence $\mathrm{Homeo}(M_n)$ is a good model for $\mathrm{Out}(F_n)$. In fact Whitehead used this to give a combinatorial algorithm to decide whether a map $F_n \to F_n$ is an automorphism. For example, consider $a \mapsto ab$, $b \mapsto ba^{-1}bc^{-1}$, $c \mapsto cb$, is it an automorphism? Nielson has already shown (and we will prove later) that the generators of $\mathrm{Out}(F_n)$ are : $\rho_{i,j} : a_i \mapsto a_i a_j$ and $a_k \mapsto a_k$ for $k \neq i$; $\lambda_{i,j} : a_i \mapsto a_i a_j$ and $a_k \mapsto a_k$ for $k \neq i$; $\epsilon : a_i \mapsto a_i^{-1}$. So we can show that $\mathrm{Homeo} M_n \to \mathrm{Out}(F_n)$ is surjective by realising the $\rho_{i,j} \lambda_{i,j}, \epsilon$. We are going to $\rho_{a,b}$ by a diffeomorphism of M_n:

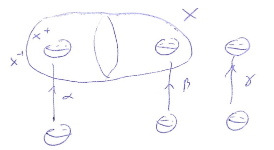


Suppose we have a word, then we can read it off

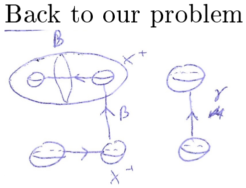




This is called the *stargraph ofw*.

Back to our problem, consider: 

Take a diffeomorphism interchanging X and B.



Then the we have $a \mapsto ab$, $b \mapsto b$ and $c \mapsto c$, so we have modelled $\rho_{a,b}$. Now take $\phi \in \mathrm{Out}(F_n)$, $\phi(a_i) = \omega_i$ and model ϕ by a diffeomorphism of M. The image of a_i is a loop, put all the w_i is the same picture. This is called the *stargraph* of ϕ (denoted $\mathrm{st}(\phi)$.)

Lemma 1.1. *If ϕ is an automorphism, then $\mathrm{st}(\phi)$ has a cut vertex (i.e., a vertex that if you remove it, the graph becomes disconnected.)*

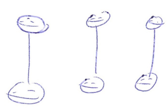
Proof. Consider the graph and also consider $\phi(A)$. We can make it transverse to A, B, C and they intersects in circles. Now $A \cap \{\alpha, \beta, \gamma\}$ =one point, so one disc of $\phi(A)$ does not intersects $\mathrm{st}(\phi)$. This disk starts at $v \in A, \overline{A}, B, \overline{B}, C, \overline{C}$ so v is a cut vertex. $\square$

Now for the algorithm:

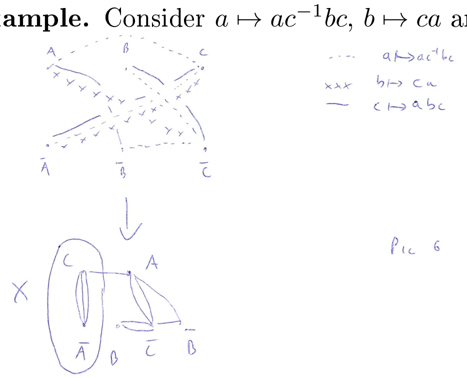
Draw $\mathrm{st}(w_1, \ldots, w_n)$. If there is no cut vertex, then we are done as it is not an automorphism.

Definition 1.2. We define the complexity of $\{w_1, \ldots, w_n\}$ to be $\frac{1}{2}\sum_{v \in \mathrm{St}(w_1,\ldots,w_n)}$ valence of v = total #of letters in cyclic words w_i #of edges in $\mathrm{st}(w_1, \ldots, w_n)$.

If there is a cut vertex, v, we can find a diffeomorphism decreasing complexity. There are two spheres that en-globes the two components of the graph, and suppose we choose the sphere X that separates v from $\overline{v}$. Choose a diffeomorphism replacing v by X. So we get a new stargraph which, since the sphere intersects $\mathrm{st}(w_1, \ldots, w_n)$ in fewer points, has lower complexity. If it has no cut vertex we are done (and it is not an automorphism) or repeat.

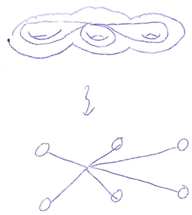
Stop once you get  , as then we have a map that send each a_i to a conjugate to itself (or some other a_j)

Example. Consider $a \mapsto ac^{-1}bc$, $b \mapsto ca$ and $c \mapsto abc$. This has complexity 9:



Interchange c and X this should be $a \mapsto c^{-1}a$, $b \mapsto b$, $c \mapsto c$, and we get $a \mapsto ac^{-1}bc \mapsto c^{-1}ac^{-1}bc \sim ac^{-1}b$, $b \mapsto ca \mapsto cc^{-1}a = a$, $c \mapsto abc \mapsto c^{-1}abc \sim ab$. This has complexity 6. If you keep going we get $a \mapsto c$, $b \mapsto a^{-1}bc^{-1}acb^{-1}a$ and $c \mapsto b$. This is not an automorphism (e.g., can not make a with this basis)

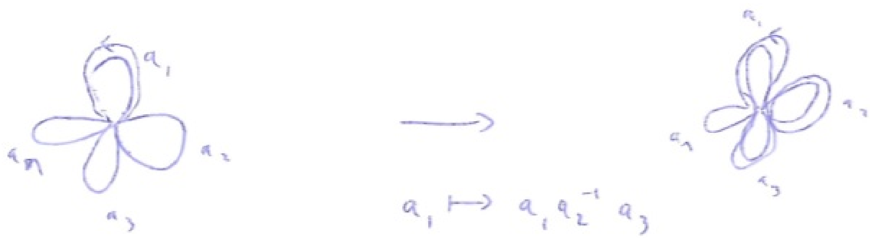
If you want automorphisms, do the same thing, but put in a base point:



Lemma 1.3. *If ϕ is an automorphism, then the pointed star graph has a cut vertex* <u>*not*</u> *at the basepoint.*

1.1 Fast forward 40 years

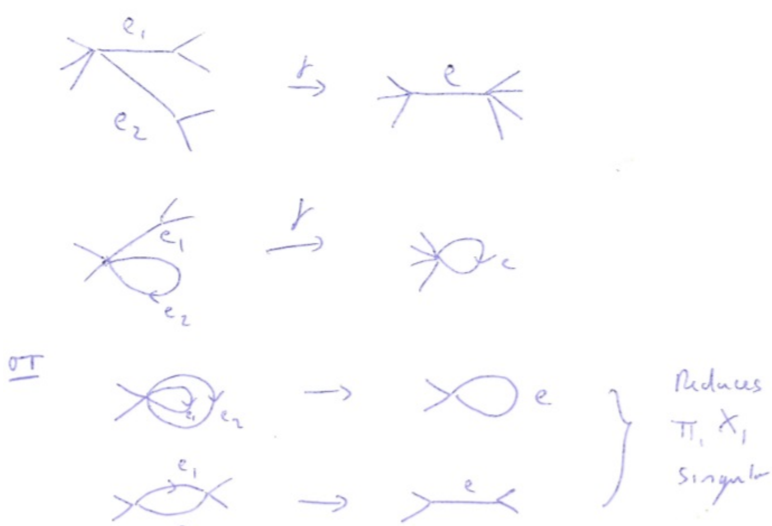
Nielson proved $\rho_{i,j}, \lambda_{i,j}$ and ϵ generate automorphism of F_n in 1924. But we are going to use Stalling's proof which uses <u>graphs</u> to model automorphism: Suppose $\phi(a_i) = w_i$



Definition 1.4. Let X, Y be a graph. We say that $f; X \to Y$ is a <u>graph</u> morphism if:

1. vertices are mapped to vertices
2. can subdivide edges of X so that either edges are collapsed to vertices or edges get mapped linearly to edges.

Definition 1.5. A *stalling fold* is a graph morphism which identifies two edges from the same vertex



e.g.

Lemma 1.6. *If $f : X \to Y$ is a graph morphism which is* <u>not</u> locally injective, then either

- *f collapses on an edge or*
- *two edges coming out of the same vertex have the same image*

Now let $\phi \in \mathrm{Aut}(F_n)$ with $\phi(a_i) = w_i$ and model it as above by a graph morphism. If it's not locally injective, then we can fold and factor the map through the fold. If it is not locally injective, repeat. Keep going until ϕ_k is locally injective (this happens as each fold reduces the number of total edges). Hence we end up with ϕ_k a homotopy equivalence. We claim that ϕ_k is a <u>homeomorphism</u> (Exercise)

So we had a series of graphs

$$\begin{tikzcd} R \arrow[r] \arrow[rrr, bend left, "\phi"] & X_1 \arrow[r] & \dots \arrow[r] & X_k \cong R \end{tikzcd}$$

We want to identify $\pi_1 X_i$ with F_n, and hope to get ρ's and λ's:

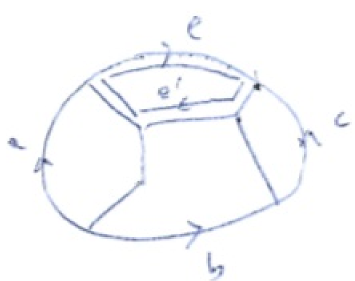
$$\begin{tikzcd} \pi_1 R \arrow[r] \arrow[d, equal] & \pi_1 X_1 \arrow[r] \arrow[d, equal] & \dots \arrow[r] & \pi_1 X_k = \pi_1 R \arrow[d, equal] \\ F_n \arrow[r, bend left, "\rho"] & F_n \arrow[r, bend left, "\rho"] & {} \arrow[r, bend left, "\lambda"] & F_n \end{tikzcd}$$

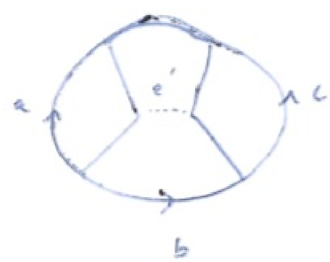
The usual way to identity $\pi_1 X$ with F_n is pick $\Gamma \subset X$ maximal tree oriented, and take the rest of edges by generators of F_n. How to choose the T_i?

Do the fold in two stages, half the edge at a time. What happens to T? In the first part T is extended, labels don't change. Collapsing T

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X \arrow[d] \\ X/\Gamma \arrow[r, "\mathrm{id}"] & X/\Gamma \end{tikzcd}$$

What about the second part: collapse e, if $e \in T$ then nothing happens as we get the identity of $\pi_1 X$. If $e \notin T$





Then choose e' in cycle containing e and change the tree T to be as in the second picture of above diagram. Then upon collapsing, we have $e \mapsto e'$, $b \mapsto be'$, $c \mapsto e'^{-1}ce'$ and $a \mapsto a$. This is a composition of ρ's and λ's. Last stage flips and permutes generators and hence ϕ is a composition of ρ's, λ's and ϵ's

2 Outer Spaces

We now want to define outer spaces,

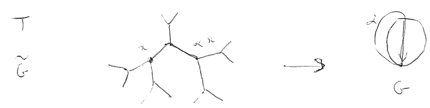
$$\begin{tikzcd} \mathrm{Out}(F_n) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] \arrow[rr, leftrightarrow, bend left] & \mathrm{GL}_n(\mathbb{Z}) \arrow[d, leftrightarrow] & \mathrm{Mod}(S_{g,s}) \arrow[d, leftrightarrow] \\ \text{acts on } \underline{\text{Outer Spaces}} & \text{acts on } SO(n)\backslash \mathrm{SL}_n \mathbb{R} & \text{acts on } \mathcal{J}_{g,s} (\text{Teichmiuller spaces}) \end{tikzcd}$$

Useful properties:

- Contractible
- Finite dimensions
- Action is proper: stabilisers are finite.

We have models for F_n through graphs of doubled handlebodies. But we need another characterisation of free groups. Γ is free if and only if it acts freely on a tree (Where a tree is a one dimensional simplicial complex which is connected and no loops and the action is by simplicial automorphisms, so it is free if every $g \in \Gamma \setminus \{1\}$ moves every $x \in T$)

Example. Let G be a graph, connected, $\Gamma = \pi_1(G)$, $T = \widetilde{G}$ universal cover. Then $\pi_1 G$ acts on T by dick transformation.



To make a space:

1. Just use minimal actions, i.e., there are no invariant subtrees.
2. Think of T as a metric tree: make each edge isometric to $[x, y] \leq \mathbb{R}$. This gives a path metric on T
3. Want actions by isometries $\rho : F_n \to \mathrm{Isom}(T)$ is equivalent to $\rho' : F_n \to \mathrm{Isom}(T')$ if the there's an isometry $T \to T'$ commuting with the actions
$$\begin{tikzcd} T \arrow[r] \arrow[d, "\rho(g)"] & T' \arrow[d, "\rho'(g)"] \\ T \arrow[r] & T' \end{tikzcd}$$

2.1 Definition 1

Definition 2.1. cv_n is the spaces of (equivalence classes of) free isometric actions of F_n on metric simplicial trees.

What is the topology?? It's the Equivariant Gromov-Hausdorff topology. Neighbourhood basis for the topology is: Let $\rho : F_n \to \mathrm{Isom}(T)$, consider the neighbourhood $V_\rho(X, A, \epsilon)$ where $X \subset T$ is finite, $A \subset F_n$ is finite and $\epsilon > 0$ is real. Then we say that $\rho' : F_n \to \mathrm{Isom}(T')$ is in $V_\rho(X, A, \epsilon)$ if there is $X' \subset T'$, a bijection $X \to X'$ defined by $x \mapsto x'$ such that $|d(x, gy) - d(x', gy')| < \epsilon$ for all $x, y \in X$, $g \in A$.

What is the action of $\phi \in \mathrm{Out}(F_n)$: Lift ϕ to $\widehat{\phi} : F_n \to F_n$ an automorphism, $\rho : F_n \to \mathrm{Isom}(T)$, then $\rho \circ \phi = \rho \circ \widehat{\phi}$ by

$$\begin{tikzcd} F_n \arrow[r, "\rho"] & \mathrm{Isom}(T) \\ F_n \arrow[u, "\widehat{\phi}"] \arrow[ur, "\rho\circ\widehat{\phi}"'] & \end{tikzcd}$$

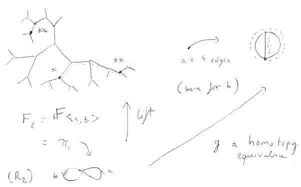
This changes the action, not the tree!

Exercise. Why does this give an action of $\mathrm{Out}(F_n)$? I.e., show that the inner automorphism act trivially.

This definition is (relatively) succinct. It's other nice features is that it generalises easily to other classes of groups which act on trees (e.g., free product). But what does cv_n look like?

2.2 Definition 2

Using trees. F_n acts freely on T implies the quotient $T/(gx \sim x)$ is a graph G.



A metric on T descends to a metric on G. An equivariant isometry $T \to T'$ descends to an isometry $G \to G'$ (Exercise: action minimal implies that G is a finite graph and has no univalent vertices).

Definition 2.2. Fix a rose R_n, identify $\pi_1 R_n \equiv F_n$. Then cv_n is the space of (equivalence classes of) marked metric graphs (G, g), where:

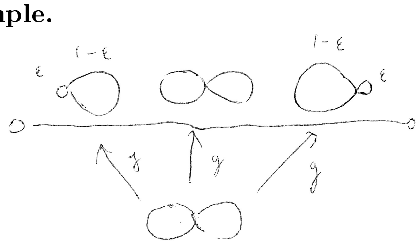
- G is a finite metric graph, no univalent or bivalent vertices
- $g : R_n \to G$ is a homotopy equivalence
- $(G, g) \sim (G', g')$ if there exists an isometry $G \to G'$ making the diagram $\begin{tikzcd} G \arrow[r] & G' \\ R_n \arrow[u, "g"] \arrow[ur, "g'"'] & \end{tikzcd}$ commute up to homotopy.

$\phi \in \mathrm{Out}(F_n)$ acts in the following way. Represent ϕ by $\delta : R_n \to R_n$ (remember Stallings), $\begin{tikzcd} R_n \arrow[r, "g"] & G \\ R_n \arrow[u, "\delta"] \arrow[ur, "g \circ \delta"'] & \end{tikzcd}$, $(G, g) \circ \phi =$ $(G, g \circ \delta)$.

We're ignoring base points because we're talking about $\mathrm{Out}(F_n)$. If I want to think about $\mathrm{Aut}(F_n)$ instead, we can use basepointed graph and basepointed homotopy equivalence in all the definitions and get the Auterspace space.

We often normalises so that $\sum_{e \in G} \mathrm{length}(e) = 1$. Then get CV_n instead of cv_n, and CV_n =union of open simplices, $\sigma(G, g)$ contains (G, g) and $\sigma(G, g) = \{(G', g) | \mathrm{length\,on}\, G' \,\mathrm{varies}\}$

Example.



$\mathrm{CV}_n = \coprod \sigma(G, g)/\mathrm{face\,identification}$

Theorem 2.3 (Paulin). *The quotient topology is equivalent to the equivariant Gromov-Hausdorff topology.*

The "Fins" look unnecessary. <u>Reduced</u> outer spaces $\overline{\mathrm{CV}_n}$ is the spaces of (G, h) such that G has no separating edges. Shrink them - get equivariant deformation retraction $\mathrm{CV}_n \to \overline{\mathrm{CV}_n}$

Exercise. $\dim(\mathrm{CV}_n) = 3n - 4$ (Use $\chi(G) = 1 - n$ and vertices have valence 3)

2.3 Definition 3

Using $M_n = \#S^1 \times S^2$

Definition 2.4. A *sphere system* S is a collection of embedded 2-spheres, $S = \{s_1, \dots, s_k\}$, such that:

- Each s_i are disjoint from each other
- s_i does not bound a ball
- s_i is not homotopic to s_j

S is equivalent to S' if there is an isotopy $M_n \times I \to M$ taking S to S'. (Isotopy means homotopy through homeomorphism).

Definition 2.5. S is *simple* if $M \setminus S$ consists of simply connected components (i.e., punctured balls)

Weights on S: assign real numbers $w_i > 0$ to S_i

Definition 2.6. cv_n is the space of weighted simple spheres systems in M_n.

The set of all sphere systems form the vertices of a simplicial complex $S^1(M_n)$, where we get a k-simplex $= S_0 \subset S_1 \subset \cdots \subset S_k$. In fact $S^1(M_n)$ is a barycentric subdivison of $S(M_n)$. The vertices are isotopy classes of 2-spheres in M (not bounding a ball), k-simplices $\leftrightarrow$ sphere systems on k spheres.

Put barycentric coordinates on each simples

$$\begin{array}{l} \circ\ \epsilon, 1-\epsilon \\ \Big|\ {\scriptstyle 1/2, 1/2} \\ \circ\ 1-\epsilon, \epsilon \end{array}$$

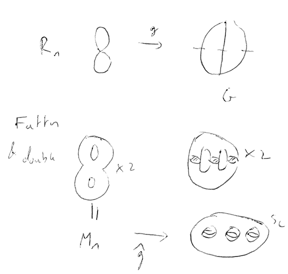
this gives a weighted sphere systems. CV_n(normalised so that the sum of weight is 1) is a subspace, with subspaces topology (same topology)

2.4 Sphere systems and marked graphs

Details for the the identification between graphs and sphere systems, read Hatcher: "Homology stability for $\mathrm{Out}(F_n)$"

Sphere system to marked graph: Start with a sphere systems, this gives a graph G: vertex for each components of $M \setminus S$ and edge for each $s \in S$. Marking $g^{-1} : G \hookrightarrow M_n \to R_n$ induces $\pi_1 G \to \pi_1 M_n \equiv F_n$.

Marked graph to sphere system: Start with a marked graph $g : R_n \to G$ and X_e on each edge of G. Fatten R_n and the graph, to get solid 3d shapes and double. Then R_n gets identified to M_n, and the X_e have becomes balls. This realize g by a diffeomorphism $\widehat{g}$ and $S = (\widehat{g}^{-1}(S_l)\}$.



Why is this picture good: Every point in CV_n is representable by a subobject of M_n. You can use it to compare points.

Action of $\mathrm{Out}(F_n)$? Realise $\phi \in \mathrm{Out}(F_n)$ by $M_n \to M_n$ diffeomorphism. $S \circ \phi = \{\phi_{s_1}, \dots, \phi_{s_k}\}$ gives an action of $\mathrm{Diff}(M_n)$ on Sphere systems. But this is not F_n, Laudeubach said

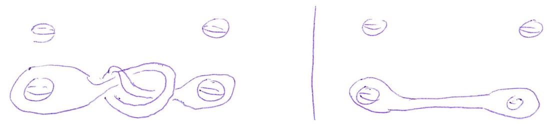
$$\pi_0 \mathrm{Diff}(M_n) \to \mathrm{Out}(F_n) \to 1$$

has kernel spanned by Dehn twists. To get an action of $\mathrm{Out}(F_n)$, we need to show that a Dehn twist acts trivially.

3 CV_n is contractible

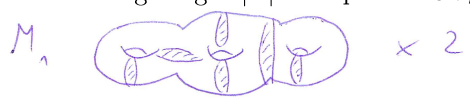
There are proofs using all three models. All uses an idea: fix $x_0 \in CV_n$, retract all of CV_n to x_0 by following paths which reduces some form of complexity.

The sphere system proof: has nice intuitive paths, but it requires some non-trivial facts about 3-manifold theory, based ultimately on Laudeubach: If two sets of disjointly embedded 2-spheres in $M_n = \#S^1 \times S^2$ are homotopic, then they are isotopic.



These are clearly homotopic, and they are isotopic by the "light bulb trick". But it is not so obvious if you have many spheres, all knotted and linked together.

Need a point $x_0 \in CV_n$. x_0 =weighted sphere system $= \sum = \{\sigma_0, \ldots, \sigma_{3n-4}\}$ maximal sphere system. Let us put the following weight $|\Sigma|$ =simplex in CV_n = all possible positive weights on Σ. E.g.

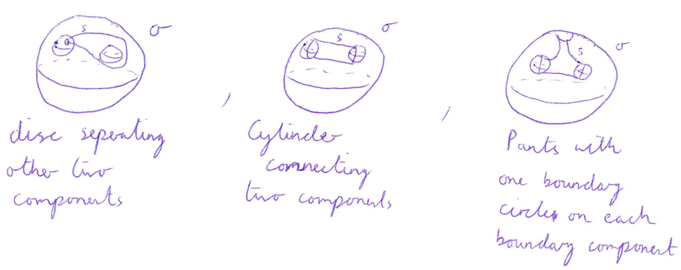


x_0: same weight on each sphere. We will retract CV_nonto $|\Sigma|$.

Retraction: Let $S = \{s_0, \ldots, s_k\}$ be another weighted sphere system.

Definition 3.1. S is in *normal form* with respect to Σ if:

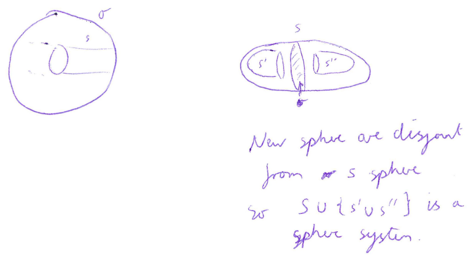
- each $s \in S$ is transverse to all $\sigma \in \Sigma$, so $S \cap \Sigma$ is the union of circles.
- The circles cut each $s \in S$ into pieces, each piece is one (or unions) of the following



Hatcher's normal form theorem: Every sphere system is isotopic to one in normal form. If S and S' are isotopic and both in normal form, then there is an isotopy preserving the intersection patterns of circles on each σ (i.e., the dual trees are isometric)

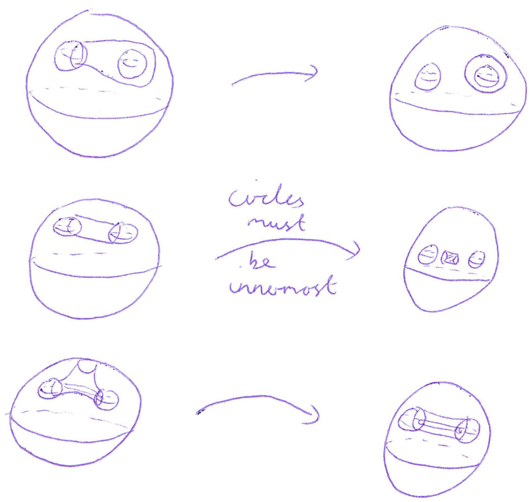
We want a path from S to $|\Sigma|$:

Idea: do (weighted) surgery on spheres in S to eliminate intersection circles. Let us forget weights for now on



Exercise. If S is simple, show that so is $S \cup \{s', s''\}$ and so is $S \cup \{s', s''\} \setminus \{s\}$.

S was in normal form with respect to Σ, so what happens under surgery?



This proves the lemma:

Lemma 3.2. *Surgery on S in normal form produces S' in normal form (after discarding trivial spheres)*

Path from a weighted sphere system. Do surgery on all innermost circles simultaneously to produce a (possibly) larger sphere system. Linearly transfer the weight of the old spheres to the new spheres, when the weight of a sphere becomes zero, throw it away and start over. Note that the complexity (:= number of intersection circles) has decreased, new system still in normal form.

At the end, I have S' with $S' \cap \Sigma = \emptyset$, why am I in $|\Sigma|$?

Fact. *Any sphere in a 3-punctured spheres is parallel to one of the boundary components (so is in Σ)*

Continuity? It is a problem: Hatcher justifies it in 2 pages.

$S(M_n)$ =sphere complex, which we identified with CV_n^*. Simplicies correspond to sphere systems, hence vertices correspond to single spheres. $S'(M_n)$ =barycentric subdivison, so vertices correspond to sphere systems. A k-simplex is a chain of k inclusions $S_0 \subset S_1 \subset \cdots \subset S_k$.

Definition 3.3. Let $K_n \subset S'(M_n)$ be the subcomplex spanned by simple systems. It is called the *spine* of the outer space.

$\mathrm{Out}(F_n)$ acts on K_n (in the guise of $\pi_0 \mathrm{Diff}^+ M_n$). It takes a simple system to a simple system.

Correction: We meant orientation preserving diffeomorphism, i.e., $1 \to (\mathbb{Z}/2\mathbb{Z})^n \to \pi_0 \mathrm{Diff}^+ M_n \to \mathrm{Out}(F_n) \to 1$

Fact. *CV_n deformations retracts to K_n*

$x \in CV_n$ is a weighted simple sphere system. $x = a_0 s_0 + a_1 s_1 + \cdots + a_k s_k$. x is in some piece of $S^1(M)$, say $\{s_0\} \subset \{s_0 s_1\} \subset \cdots \subset \{s_0 \ldots s_k\} = S$, call these $S_0 \subset \cdots \subset S_k = S$ and write $x = b_0 S_0 + \cdots + b_k S_k$. Since S_k is simple, let S_i be the first simple. So push x along the straight line from $b_0 S_0 + \cdots + b_{i-1} S_{i-1}$ (not simple) to $b_i S_i + \cdots + b_k S_k$ (in K_n). Need to normalise so $\sum b_i = 1$.

Exercise. If τ' is a face of τ open simplicies in CV_n, then retraction of τ extends continuously to the retraction of τ'.

Corollary 3.4. *The spine is contractible*

Now $\dim K_n = 2n - 3$ and recall $\dim CV_n = 3n - 4$. Need at least n spheres in any simple system (by Euler characteristic argument). Hence the longest chain of inclusion has length $2n - 3$.

Action of $\mathrm{Out}(F_n)$ is cocompact: because there is only a finite number of ways to build M_n from punctured balls (up to diffeomorphism)

The stabiliser of a simple sphere system is finite. A diffeomorphism of M_n sending S to itself permutes the spheres.

Theorem 3.5. *A diffeomorphism of $S^3 \setminus \cup Ball$ which is identity on the boundary is isotopic to identity.*

We have: A contractible $(2n-3)$-dimensional simplicial complex, an action by $\mathrm{Out}(F_n)$ which is cocompact and has finite stabilizers.

What is this good for:

Theorem 3.6 (Hurewicz). *Let X be a CW complex, $\pi_1(X) = G$. If $\widetilde{X}$ is contractible, then $H^*(X)$ is an invariant of G.*

(This is one way to define $H^*(G)$. There are other purely algebraic ways; see Ken Brown's book, Cohomology of Groups)

To find X: note $\pi_1 X$ acts freely on $\widetilde{X}$ by Dick transformations. So take Y contractible with a free G-action $G \to \pi_1(Y/G)$ defined by $g \mapsto [\gamma_g]$ is an isomorphism. We have K_n contractible, but $\mathrm{Out}(F_n)$ action is not free.

Theorem 3.7. $\mathrm{Out}(F_n)$ *has finite index subgroups* Γ *with no torsion.*

So Γ acts freely on K_n. So $H^*(\Gamma) = H^*(K_n/\Gamma)$

Theorem 3.8. $H^i(\Gamma) = 0$ *if* $i > 2n - 3$.

$K_n/\mathrm{Out}(F_n)$ is finite (i.e., finite number of cells). $[\mathrm{Out}(F_n) : \Gamma] < \infty \Rightarrow K_n/\Gamma$ is also finite.

Theorem 3.9. $H^i(\Gamma)$ *is finitely-generated*

Theorem 3.10. *If Γ, Γ' are both finite torsion free finite index subgroups of G, then they have the same cohomological dimensions.*

Today we will look more at the spine K_n. I.e., the subcomplex spanned by simple systems contained in the barycentric subdivision $S^1(M_n)$. We've showed that it is:

- Contractible
- $\dim 2n - 3$
- $\mathrm{Out}(F_n)$ acts properly
- The quotient is compact

All of this implies that K_n is quasi-isometric to $\mathrm{Out}(F_n)$.

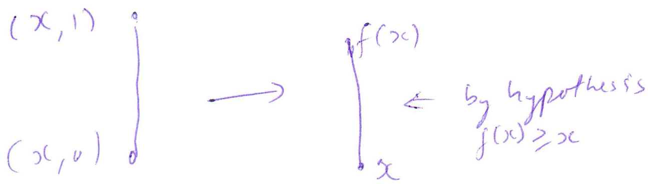
3.1 K_n as a complex of marked graph

So we have a sphere system $S = \{s_0, \dots, s_k\}$. We say that S is simple f $M \setminus S$ components are punctured balls. To each sphere systems we get a dual graph G. S being simple is equivalent to $G \hookrightarrow M_n$ induces $\pi_1 G \overset{\cong}{\to} \pi_1 M_n$, so this serves as a marking of G. Edges of K_n correspond to $S_0 \subseteq S_1$.

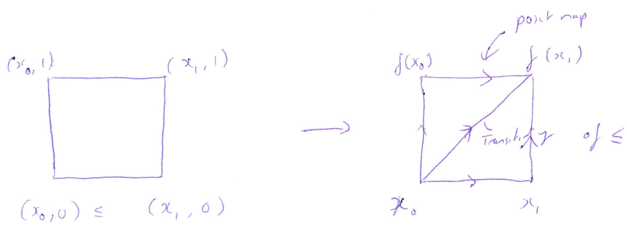
Digression: Partially ordered sets (posets). If X is a simplicial complex, then the vertices of X are a poset under inclusion. Given any poset $(\Omega, \leq)$, there a simplicial complex (the *geometric realisation*, or *order complex* of Ω), denoted by $|\Omega|$. The vertices are $x \in \Omega$ and the k-simplex are $x_0 \leq x_1 \leq \dots \leq x_k$. A poset map $f : \Omega \to Z$ is a map f such that $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$.

Lemma 3.11. *If we have a poset map $f : \Omega \to \Omega$ and $f(x) \geq x$ for all x, then $|\Omega| \cong |f(\Omega)|$.*

Proof. We want a homotopy $h : |\Omega| \times I \to |\Omega|$ with $H(x, 0) = x$ and $H(x, 1) = f(x)$. We define this on 0 simplices:

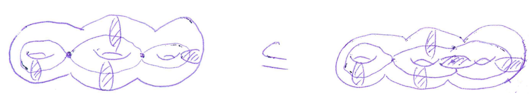


We define this for 1-simplices:



In general: If H is defined on $\delta(\sigma \times I)$, we triangulate using the prism operator. Each simplex maps to a simplex in Ω □

Back to K_n. Suppose we have tow systems $S_0 \subset S_1$



To go ← we collapse an edge (called edge collapse), while to go → we split a vertex into 2 vertices and add an edge ("blowing up" a vertex).

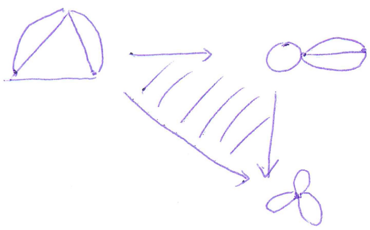
Note. we can't collapse a subgraph with a cycle.

A subgraph with no cycles is a forest ϕ (union of trees). A forest collapse collapses each edge of a forest to a point.

Another description of K_n: the vertices are marked graphs (g, G). The poset relations $(g, G) \geq (g', G')$ is there is a collapse

$$\begin{tikzcd} G \arrow[r, "c_\phi"] & G_\Phi \\ R_n \arrow[u, "g"] \arrow[ur, "c_\phi \circ g"'] & \end{tikzcd}$$

such that $(g, G') \sim (c_\phi \circ g, G_\Phi)$. Then K_n is realisation of this poset.

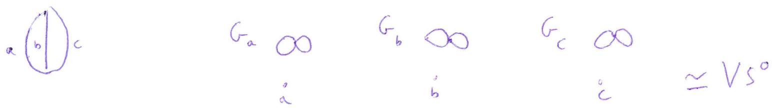


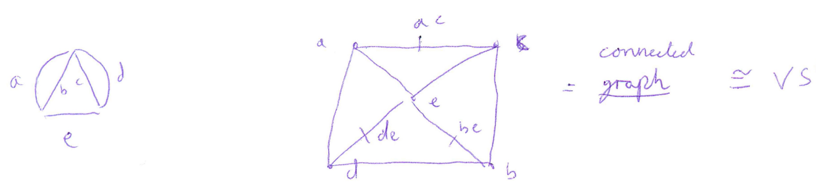
Example.

3.2 Local structure of K_n

Link of a vertex: S edges are two types, $S \subset S'$ and $S'' \subset S$. Notice that every S'' is connected to every S'. So the link of S, $lk(S) = lk_{<S} * lk_{>S}$ (where $*$ is simplicial join). Want to understand the two piece separately.

$lk_{<S}$: Use graph picture. e.g:





Theorem 3.12. $\mathrm{lk}_{<G}$ *is contractable or homotopic equivalent to* $\vee S^{V-2}$ *where* V *is the number of vertices of* G.

What about $lk_{>S}$? Here we use sphere systems, not graphs. $M \setminus S$ is a union of pieces P. The boundary of P are spheres in S. To add spheres to S, need to add inside some P. To add s to S: the isotopy class of s depends only on which boundary components are inside s, which are outside. Can add spheres to P or P' independently. So $\mathrm{lk}_{>S} =_P *\mathrm{lk}_{>P}$. So $\mathrm{lk}_{>P} = S(P)$ =sphere complex of P.

Theorem 3.13. $S(P) \cong \vee S^{B-4}$ *where* B *is the number of boundary components of* P.

Theorem 3.14. *If* $\phi \in \mathrm{Out}(F_n)$ *acts on* K_n *by a simplicial automorphism. So we have a map* $\mathrm{Out}(F_n) \to \mathrm{SAut}(K_n)$. *This map is an isomorphism.*

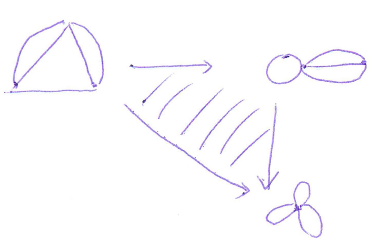
First step in proof: Given $f \in \mathrm{SAut}(K_n)$, to show $f((g,g)) = (G,g')$, show link $(G,g) \neq$ link (G',g) if $G \ncong G'$.

Today: We want to look at K_n as a cube complex structure, and use that to do homology computation and see connections with Kontseurch's graph homology.

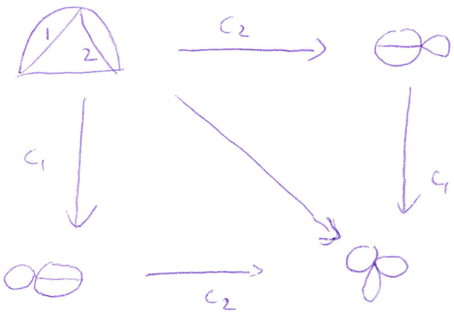
Recall that a simplex in K_n:

$$\begin{tikzcd}
G \arrow[r] & G_{\Phi_1} \arrow[r] & (G_{\Phi_1})_{\Phi_2} \arrow[r] & \dots \arrow[r] & (\dots)_{\Phi_K} \\
R_n \arrow[u, "g"] \arrow[ur] \arrow[urr] \arrow[urrrr] & & & &
\end{tikzcd}$$

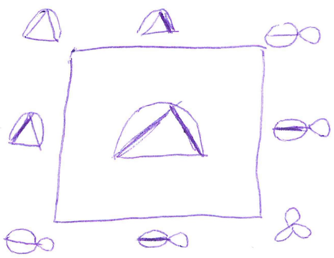
E.g.,



So another way to describe a simplex is by $(g, G, \Phi_1 \leq \cdots \leq \Phi_K)$.
We can collapse the edges in any order, e.g.



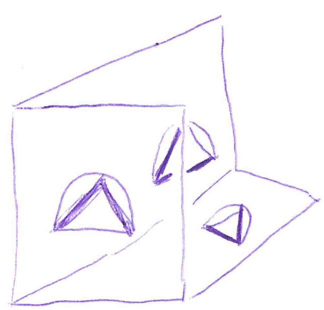
If $\Phi < G$ has n edges, there are 2^n simplices (n-simplices) fitting together into a cube. Hence we get an n-cube for each $\Phi < G$ with n edges. Ordering the edges of Φ gives an orientation on the cube. $c(G, \Phi, g)$.



In this case the boundary $\partial(c(G, \Phi, g)) = \sum_{e \in \Phi} c(G, \Phi - e) - c(G_e, \Phi_e)$

- Aside: A cube complex is CAT(0) if link of vertices are flag complexes, i.e., if δ of a simplex is in link, so is the simplex.

  Remark. K_n is not a CAT(0) cube complex e.g.:

  The cube is not there: it would have to be 

  Bridsen's thesis: Can't fudge angles to make K_n CAT(0).

  Bridsen latter proved: $\mathrm{Out}(F_n)$ can no act property, compactly on any proper CAT(0) metric spaces.

3.2.1 Spectral sequences

Recall: Suppose $\Gamma < \mathrm{Out}(F_n)$ is torsion free and finite index. Then $H^*(\Gamma) = H^*(K_n/\Gamma)$, with any coefficients.

Heuristically: $H^*(\,;\mathbb{Q})$ with trivial coefficients "can't see" finite subgroups, i.e., it thinks the action is free. So in fact $H^*(\mathrm{Out}F_n, \mathbb{Q}) \cong H^*(K_n/\mathrm{Out}F_n, \mathbb{Q})$.

Example. That $H^*(\,;\mathbb{Q})$ with trivial coefficients "can't see" finite subgroups:
Let C_n be the cyclic graph of order n

$$\dots \to \overset{c_4}{\mathbb{Z}} \xrightarrow{\times n} \overset{c_3}{\mathbb{Z}} \xrightarrow{0} \overset{c_2}{\mathbb{Z}} \xrightarrow{\times n} \overset{c_1}{\mathbb{Z}} \xrightarrow{0} \overset{c_0}{\mathbb{Z}} \to 0$$

Tensoring by $\mathbb{Q}$ we get

$$\cdots \to \overset{c_4}{\mathbb{Q}} \xrightarrow{\cong} \overset{c_3}{\mathbb{Q}} \xrightarrow{0} \overset{c_2}{\mathbb{Q}} \xrightarrow{\cong} \overset{c_1}{\mathbb{Q}} \xrightarrow{0} \overset{c_0}{\mathbb{Q}} \to 0$$

giving that $H_0 = \mathbb{Q}$ and $H_i = 0 \forall i$.

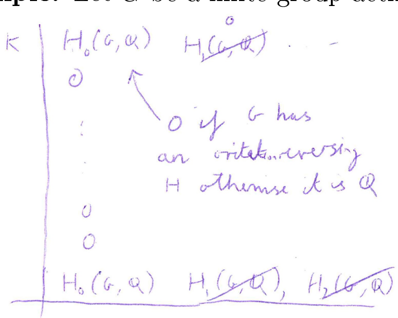
Equivariant homology spectral sequences (details can be found in K. Brown's book): Let G acting on X simplicially. $C_*EG \otimes_{\mathbb{Z}G} C_*X$ is a double complex, so you can filter it both vertically and horizontally, i.e.,

$$\begin{tikzcd}
C_pEG \otimes C_qX \arrow[r, "1\otimes\delta"] \arrow[d, "\delta\otimes 1"'] & C_pEG \otimes C_{q-1}X \arrow[d] \\
C_{p-1}EG \otimes C_qX \arrow[r] & C_{p-1}EG \otimes C_{q-1}
\end{tikzcd}$$

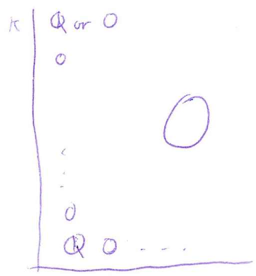
We look at diagonals $p+q$, $p+q-1$, etc chains.

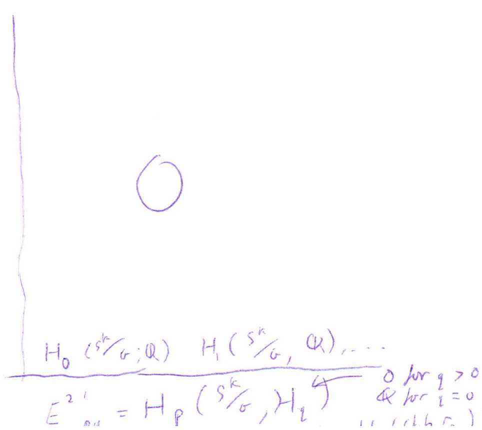
This gives rises to an array $E^2_{pq} = H_p(G; H_q(X))$ going one way and $E'^2_{pq} = H_p(X/G, \mathcal{H}_q)$ where $\mathcal{H}_q = \oplus H_*(\mathrm{stab}\sigma_q)$ the other way. They both converge to the same thing.

Example. Let G be a finite group acting on $X = S^k$.



i.e.



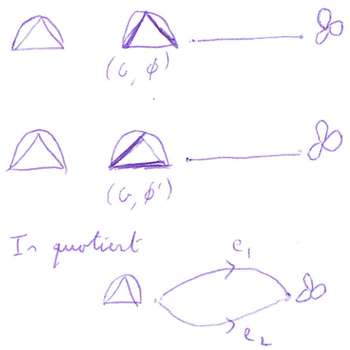


All the rest of differential are 0. So this shows that the quotient of S^k by a finite group action is:

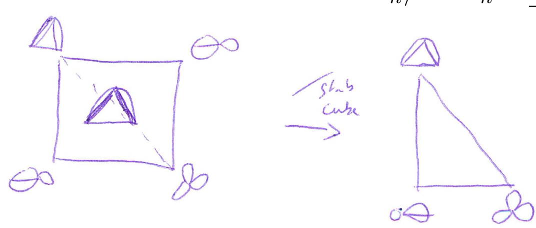
- A rational disk if G has an orientation-reversing automorphism
- A rational H_* sphere if not

Exercise. Use the equivariant H_* spectral sequence to prove that $H_*(K_n/\mathrm{Out}(F_n), \mathbb{Q}) \cong H_*(\mathrm{Out}(F_n), \mathbb{Q})$.

Now we compute $H_*(\mathrm{Out}(F_n), \mathbb{Q})$. So we have to consider $K_n/\mathrm{Out}(F_n)$ =vertices are unmarked graphs and we get a k-simplex for each unmarked pair $(G, \Phi_1 < \Phi_2 < \cdots < \Phi_k)$. E.g.:



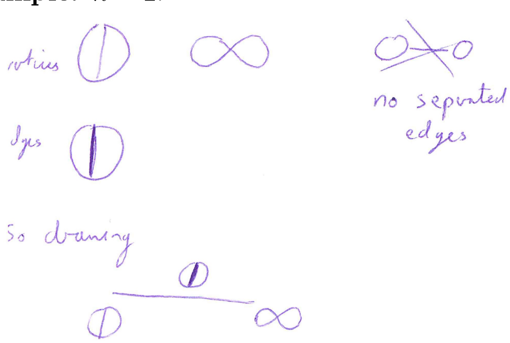
Note that it is two edges because there are no automorphism taking one forest to the other.
So $K_n/\mathrm{Out}(F_n)$ is the union of simplices, not simplicial complex.
So let's use cubes instead: But $K_n/\mathrm{Out}F_n$ is <u>not</u> a union of cubes! e.g.:



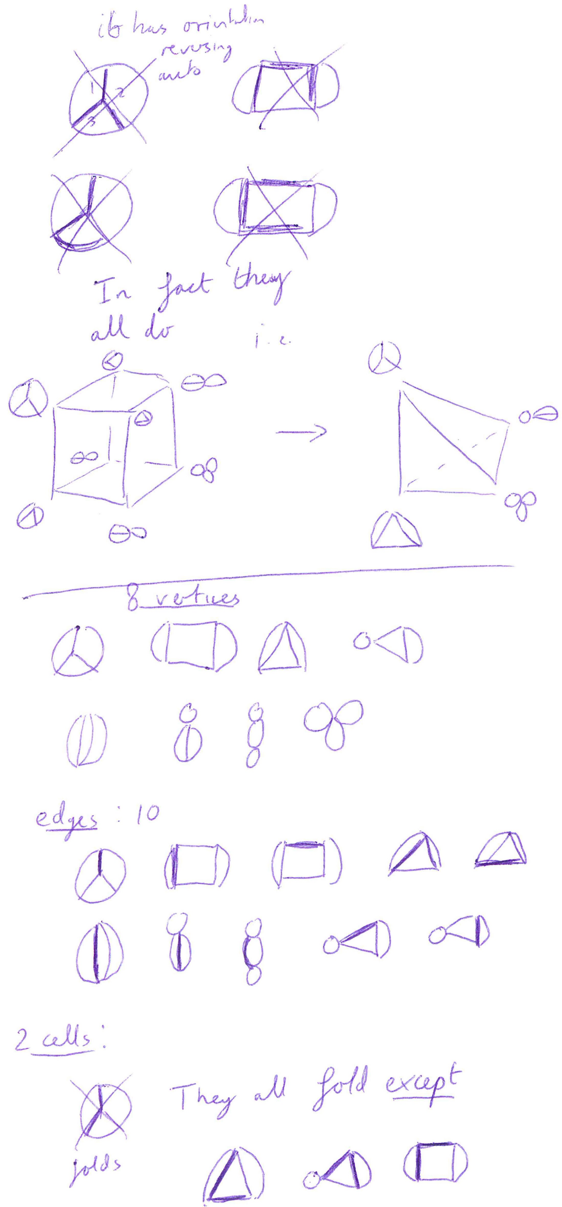
Cells in $K_n/\mathrm{Out}F_n$ are quotients of cubes. These are either cones on a rational homology sphere or disk (because stab(cube) acts simplicially on the boundary of the cube). So to get a chain complex for $K_n/\mathrm{Out}(F_n)$:

$$C_k = \bigoplus_{(G,\Phi)\text{ k edges in }\Phi,\text{ no orientation reversing auto in }(G,\Phi)} \mathbb{Q}$$

Example. $n = 2$:



Hence $H_*(K_n/\mathrm{Out}(F_2)) = \begin{cases} F_2 & * = 0 \\ 0 & * > 0 \end{cases}$
$n = 3$: K_n is 3 dimensional:
3-cubes:

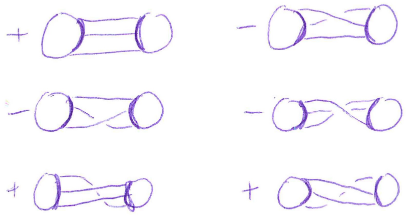


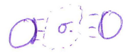
So we have $c_3 \to c_2 \to c_1 \to c_0 \to 0$ is $0 \to \mathbb{Q}^3 \to \mathbb{Q}^{10} \to \mathbb{Q}^8 \to 0$.

Exercise. Show that $H_0 = \mathbb{Q}$, $H_1 = H_2 = H_3 = 0$.

$n = 4$ is also possible and find that $H_0(\mathrm{Out}F_4) = \mathbb{Q}$, $H_1 = H_2 = H_3 = H_5 = 0$ and $H_4 = (\mathrm{Out}F_4, \mathbb{Q}) = \mathbb{Q}$. Let us describe that cycle.

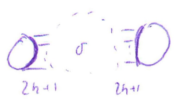
6 cube quotient



So $\mu_1 = \sum_{\sigma \in S_3}$ 

$\cdot\, \operatorname{sgn}(\sigma)$

In general

$\mu_k = \sum_{\sigma \in S_{2k+1}}$ 

$\cdot\, \operatorname{sgn}(\sigma)$

μ_k is always a cycle.

Conjecture. $\mu_k \neq 0$ *in* $H_{4k}(\mathrm{Out}(F_{2k+2}), \mathbb{Q})$.

This is true for $k = 1, 2, 3$.

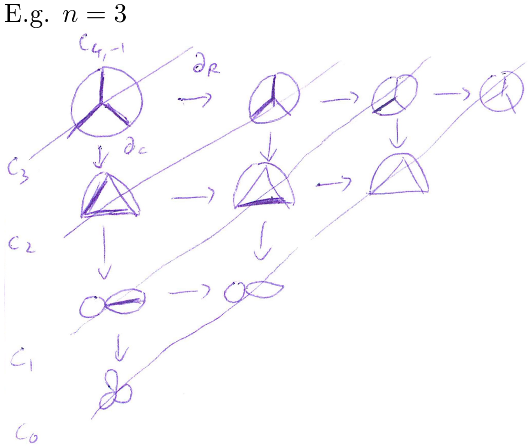
Morita found the μ_k by exploiting connections between $H^*(\mathrm{Out}F_n, \mathbb{Q})$ and Kantsevich's graph homology. From now on all homology will be with trivial $\mathbb{Q}$ (or $\mathbb{R}$, or any k with characteristic 0) coefficients. Kantsevich's graph homology is a way to compute H^* of certain infinite dimensional Lie algebras.

To make the connection, we need to simplify our chain complex. $C_k = \oplus_{(G,\Phi)}\mathbb{Q}$, where G is a connected graph with vertices of valence ≥ 3 and $\pi_1 G = F_n$ and Φ is a forest with k edges. Can further split up C_* into $C_k = \oplus C_{p,q}$ where

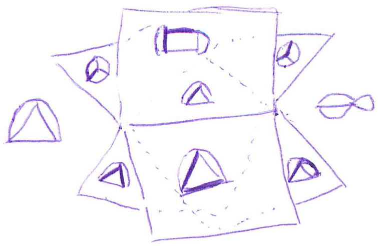
$$C_{p,q} = \bigoplus_{G \text{ has } p \text{ vertices}, \Phi \text{ has } p+q=k \text{ edges}} (G, \Phi)$$

We have boundary map $\partial : C_k \to C_{k-1}$ where $\partial = \partial_R + \partial_C$. Recall that (G, Φ) is oriented by ordering the edges of $\Phi = \{e_1, \ldots, e_k\}$. Then $\partial_R(G,\Phi) = \sum_i (-1)^i (G, \Phi - e_i)$ and $\partial_C(G,\Phi) = \sum (-1)^{i+1}(G_e, \Phi_e)$ (where (G_e, Φ_e) is the collapse of (G, Φ) at e).

E.g. $n = 3$



For the Kontseirh connection, it is convenient to use cohomology.



So $\delta(G,\Phi)=\delta_C+\delta_R$, with $\delta_R=\sum_{\Phi\cup\{e\}\,\text{a forest}}\pm(G,\Phi\cup\{e\})$ and $\delta_C=\sum_{\alpha=\text{way of dividing the edge of vertices into two pieces}}\pm(G^\alpha,\Phi^\alpha)$. This is easier to explain with a sphere system.

$G\leftrightarrow\mathcal{S}_G$ a simple sphere systems in M_n, $\Phi\leftrightarrow\mathcal{S}_\phi$ a subsphere system (not simple). Then $\delta_C(\mathcal{S}_G,\mathcal{S}_\phi)=\sum_{\mathcal{S}_\alpha\,\text{compatible in}\,\mathcal{S}_G}\pm(\mathcal{S}_G\cup\alpha,\mathcal{S}_\phi\cup\alpha)$. Reference: Conant-Vogtmann: "Morita classes vanishes after one stabilisation"

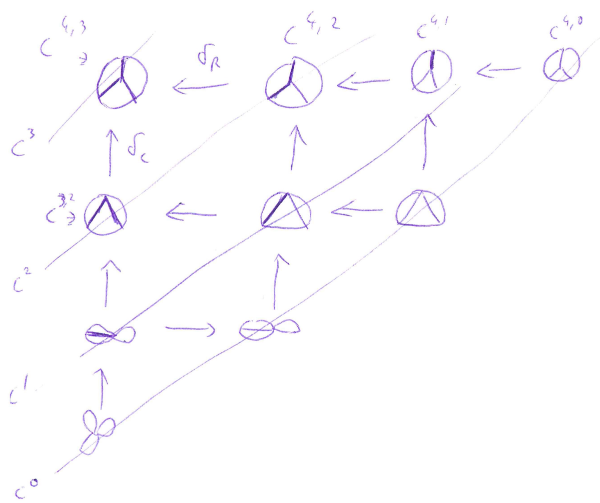
Exercise. After calculating the right signs, check that $\delta^2=0$

$C^k=\oplus_{\Phi\,\text{has}\,k\,\text{edges}}(G,\Phi)=\oplus_{v(G)=v,e(\Phi)=k}C^{v,k}$. Recall that our top dimension was $2n-3$, so $C^{2n-3}\ni$(trivalent graph, maximal tree). So

- $C^{2n-3}=C^{2n-2,2n-3}$
- $C^{2n-4}=C^{2n-2,2n-4}\oplus C^{2n-3,2n-4}$ etc

Arrange the $C^{v,k}$ in a grid. To see what happen, put a sample (G,Φ) in each slot.

E.g. $n=3$



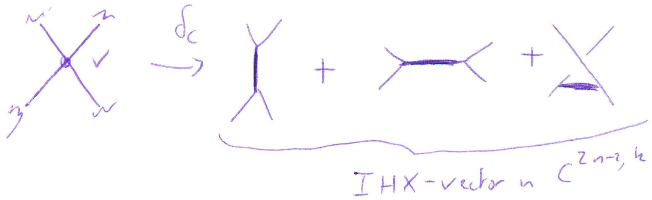
To compute the cohomology, we split it into horizontal chunks (spectral sequences).

We're looking at unmarked graphs in $K_n/\mathrm{Out}(F_n)$. If we had marked graphs each columns would be the chain complex of $lk_{>G}$ for G the sum of the bottom graph. Recall that $lk_{>G}\cong\vee S^{2n-3-v(G)}$. So it has no homology except in the top dimension! So in the vertical columns, taking $\widetilde{H}_*$ kills all but the top term.

This is OK in the quotient too: Using equivariant H_*spectral sequences, $H_*(\mathrm{lk}_{>G}/\mathrm{stab}(\mathrm{lk}_{>G}))$ but $\mathrm{stab}(lk_{>G})$ is a finite group Γ, and let $X=\mathrm{lk}_{>0}$. This gives $E^2_{p,q}=H_p(\Gamma;H_q(X))$ converges to $H_{p+q}(X/\Gamma)$. So you get X/Γ only has H_* in the top dimension. So chain complex is reduced to (all graphs G are trivalent)

$$\bigoplus_{e(T)=2n-3}(G,T)/\mathrm{im}\delta_C\leftarrow\bigoplus_{e(\Phi)=2n-4}(G,\Phi)/\mathrm{im}\delta_C\leftarrow\cdots\leftarrow\bigoplus_{e(\Phi)=1}(G,e)/\mathrm{im}\delta_C\leftarrow\oplus G$$

Now $\mathrm{im}\delta_C$: $\delta_C(G,\Phi)$ is in top row if G has exactly one 4-valent vertex (restrict to trivalent)



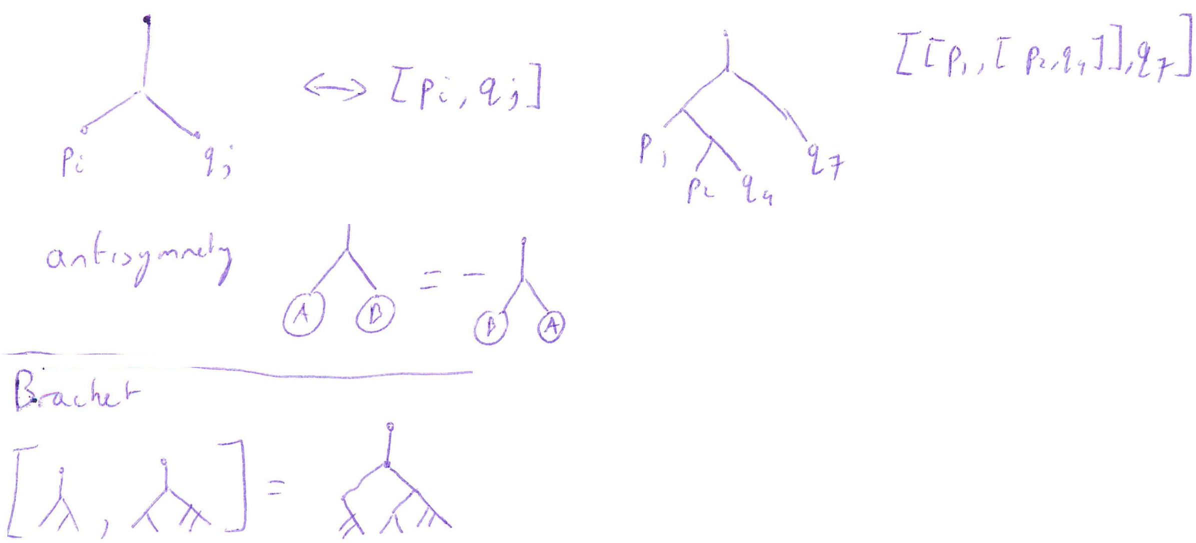
Kantsevich

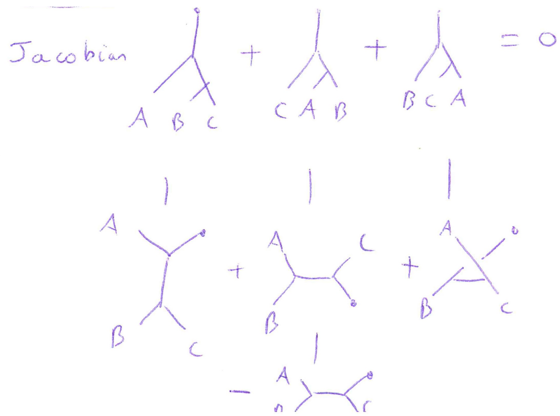
$V_d = \mathbb{R}^{2d}$ with simpletic form $\langle\, , \rangle$, with basis $B = \{p_1, \dots, p_d, q_1, \dots, q_d\}$. We have $\langle x, y\rangle = 0$ for x, y in the basis expect for $\langle p_i, q_i\rangle = 1 = -\langle q_i, p_i\rangle$.

- Let C_d is the free commutative algebra on V, i.e., polynomials in p_i's and q_i's with real coefficients.
- We let A_d be the free associative algebra on the basis B, ie.., the polynomials in non-commutative variables p_i, q_i.
- We have L_d be the free Lie algebra on B, generated by bracket expressions (e.g., $[[p_1, [p_2, q_4]], q_7]$) modulo
  - antisymmetry $[A, B] = -[B, A]$
  - Jacobi $0 = [A, [B, C]] + [C, [A, B]] + [B, [C, A]]$

In terms of graphs:

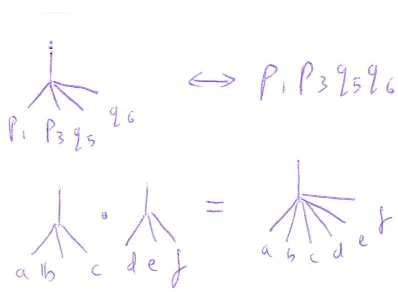
- L_d is a rooted planar binary tree (which correspond to a bracket expression) with leaves labelled by B.





Hence the Jacobi relation is just the IHX relation!

- A_d is generated by planar rooted trees with 1 internal vertex

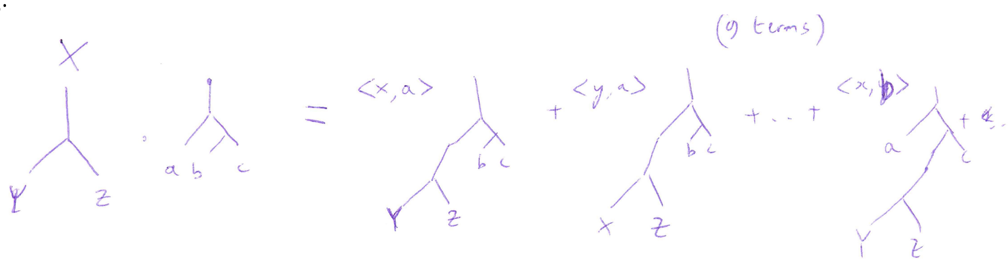


- C_d are rooted tress, with one internal vertex, and labelled leaves. (Not that since we are not asking them to be planar we can interchange leaves)

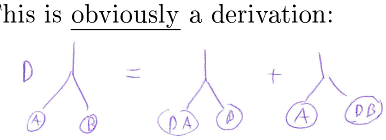
Let $H_d = A_d, C_d$ or L_d and we will define $\mathfrak{h}_d$ to be the Lie algebra based on H_d generated by (appropriate) trees, with all leaves labelled (no root)

Claim: Such an object correspond to a derivation $D : H_d \to H_d$, i.e., $DAB = DA \cdot B + A \cdot DB$, or $D[A,B] = [DA,B] + [A,DB]$.

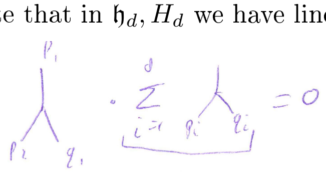
E.g.



This is obviously a derivation:

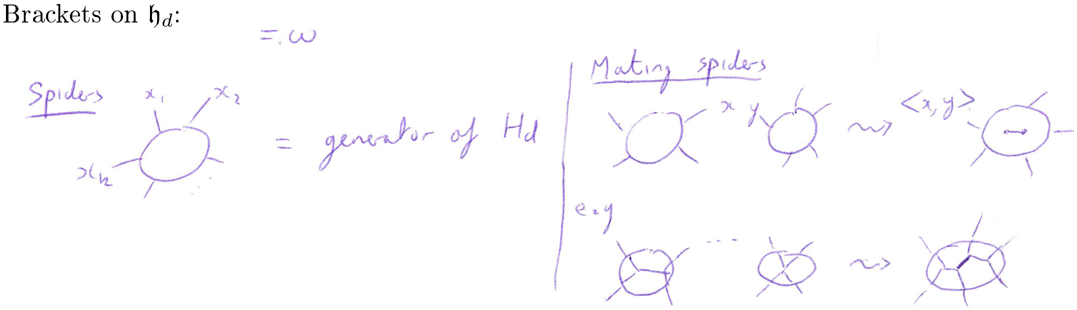


Also note that in $\mathfrak{h}_d, H_d$ we have linearity.

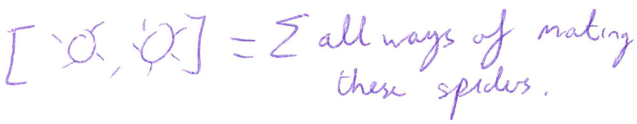
Exercise. 

Fact. *Let $H_d = L_d$. Then $\mathfrak{h}_d$ is the set of derivation that kills ω*

Brackets on $\mathfrak{h}_d$:



Then the brackets is



It is easy to see that $[A, B] = -[B, A]$, check the Jacobi takes a bit more effort, but it also holds.

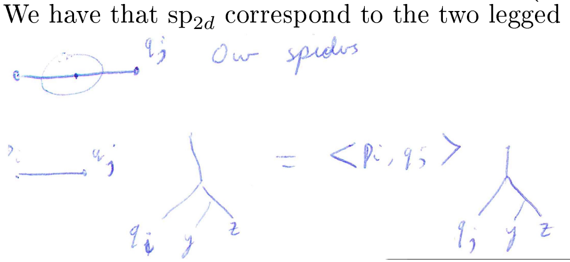
Theorem (Kantsevich)**.** *Let $\mathfrak{h}_\infty = \lim_{d\to\infty} \mathfrak{h}_d$. Let $H_d = L_d$, then $PH_*(l_\infty) = H_*(\mathrm{sp}_\infty) \oplus_{n\geq 2} H^*(\mathrm{Out}(F_n))$ where sp_∞ is the simpletic Lie algebra.*

How to compute $H_*(l_\infty)$, you make a chain complex using $C_k = \wedge^k(l_\infty)$. We're going to cover how to calculate H_* of a Lie algebra. Let G be a compact, simply connected Lie group, then $\mathfrak{g}$ determines G. Lie algebra homology translates $H_*(G)$ to $H_*(\mathfrak{g})$, using de Rham.

For $\mathfrak{a}$ any Lie algebra, look at $\partial : C_k = \wedge^k\mathfrak{a} \to \wedge^{k-1}\mathfrak{a}$ defined by $a_1 \wedge \cdots \wedge a_k \mapsto \sum_{i<j}(-1)^{i+j-1}[a_i, a_j] \wedge a_1 \wedge \cdots \wedge \widehat{a_i} \wedge \cdots \wedge \widehat{a_j} \wedge \cdots \wedge a_k$. This is the Chevalley-Eilenberg complex.

It doesn't look much like our first chain complex. The trick: $\mathfrak{h}_d$ contains a copy of sp_{2d}, where sp_{2d} are $2d \times 2d$ matrices A such that ${}^tAJ + JA = 0$ where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

We have that sp_{2d} correspond to the two legged spiders:

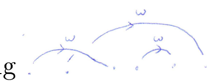


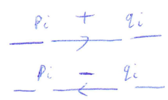
So multiplying the spider sends $q_i \mapsto q_j$ and $p_j \mapsto p_i$. So the matrix associated to our two legged spiders is $\begin{pmatrix} E_{ij} & 0 \\ 0 & E_{ji} \end{pmatrix}$. A two legged spiders of the form  correspond to $\begin{pmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix}$ respectively. Note that these three matrices generate sp_{2d}. So $\mathrm{sp}_{2d} \subset \mathfrak{h}_d$. Note that sp_{2d} acts on $\mathfrak{h}_d$ (by preserving the number of legs).

Exercise. The sp_{2d} action commutes with ∂.

Let us define C_k^{sp} =invariants = killed by every $A \in \mathrm{sp}_{2d}$. So $C_k^0 \to C_k^{\mathrm{sp}} \to C_{k-1}^{\mathrm{sp}} \to \dots$ is a subcomplex

Theorem 3.15. *sp_{2d} is a simple algebra, so $H_*(C_*^{\mathrm{sp}}) \cong H_*(C_*)$*

So now we just need to figure out $(\wedge \mathfrak{h}_d)^{\mathrm{sp}}$. Hermann Weyl already did it for us! Let V be any sympletic vector space. Consider $V \otimes V \ni \sum p_i \wedge q_i = \omega$ is killed by any elements of $\mathrm{sp}(v)$ so it is in $(V \wedge V)^{\mathrm{sp}}$. In fact $(\wedge^2 V)^{\mathrm{sp}} = \langle \omega \rangle$. In $V^{\otimes k}$ there are no invariants if k is odd. If $k = 2l$ is even, $\underbrace{\omega \otimes \cdots \otimes \omega}_{l}$ is invariants. Let $\omega = p_i \otimes q_i - q_i \otimes p_i$ and this invariants correspond to . Insert another invariant is different pairing .
Weyl: These span the spaces of invariants.

$(\wedge^k \mathfrak{h})^{\mathrm{sp}}$, so the space of invariants translates to: Each way of pairings the legs of $S_1 \wedge \cdots \wedge S_k$ gives an invariant $= \sum$ of terms, in each term the labels on matched leaves match. The sign of the terms is the product of the sign of the edge as follows 

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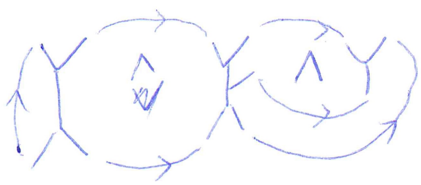
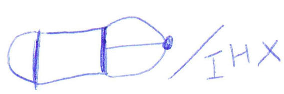
Generators of $C_k = (\wedge^k \mathfrak{h})^{\mathrm{sp}}$ looks like:



This has a lot of data:

- The wedge summands  are planar tree modulo IHX and antisymmetry
- The arrows are orientation on edges
- There is an ordering on the wedge summands

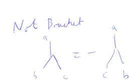
This looks likes 

. We have an ordering on edges of Φ.

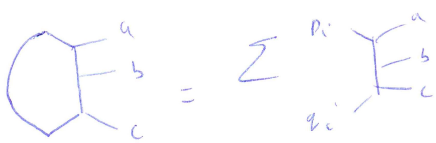
Theorem 3.16. *Orientation data in both combinatorial objects is the same.*

This theorem is not transparent, but it is true. So the boundary maps on the two chain complexes are the same.

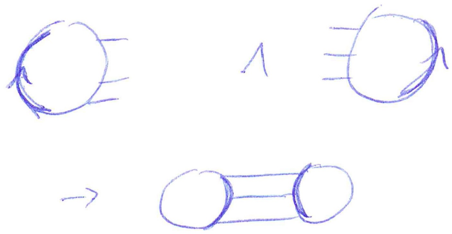
Unfortunately composition $g \overset{\phi}{\mapsto} (\wedge^k \mathfrak{h})^{\mathrm{sp}} \overset{\psi}{\to} g$ is not the identity as we end up with other things that what you started with. But in the limit as $d = \frac{1}{2} \dim V \to \infty$ these accidental terms don't matter, so composition is an isomorphism.

Let $\mathfrak{h} = \mathfrak{h}_\infty$, then $\mathfrak{h} \to \mathfrak{h}_{ab} = \mathfrak{h}/[\,,\,]$. E.g.: 

. So $\wedge^3 V \mathfrak{h}_{ab}$. Also this is not a bracket



Why do we care about the abelianisation? $\mathfrak{h} \to \mathfrak{h}_{\mathrm{ab}}$, $H^* \mathfrak{h}_{\mathrm{ab}} \to H^* \mathfrak{h} = H^*(\mathrm{Out}(F_n))$.
Marita:

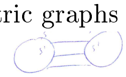


so 4 edges in Φ, giving a cycle in $C_4(\mathrm{Out}F_4)$.

Theorem 3.17. *μ_1, μ_2, μ_3 are non zero classes in $H_{*4k}(\mathrm{Out}F_{2k+2})$.*

Conjecture. *μ_k is always non-trivial.*

Let's see these μ_k in CV_n mod $\mathrm{Out}F_n$.
$\mathrm{CV}_n/\mathrm{Out}F_n$ =marked metric graphs / change of marking = metric graphs Q_n.

Claim: $S^1 \times S^1 \times S^1 \times S^1 = T^4 \to Q_n$ image is non trivial cycle  So $T_4 \to Q_4$.