

NAME:

SECTION:

Quiz 10: Find the general solution to the inhomogenous differential equation

$$\frac{d}{dt}\mathbf{x} = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \mathbf{x} + \begin{pmatrix} e^t/t + te^{3t} \\ te^{3t} \end{pmatrix}$$

Hint: a fundamental matrix for the homogenous equation is given by

$$X(t) = \begin{pmatrix} e^t & e^{3t} \\ 0 & e^{3t} \end{pmatrix}$$

Solution

$$\begin{aligned} \mathbf{x}(t) &= X(t) \int X(t)^{-1} \begin{pmatrix} e^t/t + te^{3t} \\ te^{3t} \end{pmatrix} dt \\ &= \begin{pmatrix} e^t & e^{3t} \\ 0 & e^{3t} \end{pmatrix} \int e^{-6t} \begin{pmatrix} e^{3t} & -e^{3t} \\ 0 & e^t \end{pmatrix} \begin{pmatrix} e^t/t + te^{3t} \\ te^{3t} \end{pmatrix} dt \\ &= \begin{pmatrix} e^t & e^{3t} \\ 0 & e^{3t} \end{pmatrix} \int e^{-4t} \begin{pmatrix} e^{4t}/t \\ te^{4t} \end{pmatrix} dt \\ &= \begin{pmatrix} e^t & e^{3t} \\ 0 & e^{3t} \end{pmatrix} \int \begin{pmatrix} 1/t \\ t \end{pmatrix} dt \\ &= \begin{pmatrix} e^t & e^{3t} \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} \ln|t| + c_1 \\ \frac{t^2}{2} + c_2 \end{pmatrix} \\ &= \begin{pmatrix} e^t(\ln|t| + c_1) + e^{3t}(\frac{t^2}{2} + c_2) \\ e^{3t}(\frac{t^2}{2} + c_2) \end{pmatrix} \end{aligned}$$

for any constants $c_1, c_2 \in \mathbb{R}$. An alternative form might be

$$\mathbf{x}(t) = \begin{pmatrix} e^t \ln|t| + \frac{t^2}{2} e^{3t} \\ \frac{t^2}{2} e^{3t} \end{pmatrix} + \tilde{c}_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tilde{c}_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for any constants \tilde{c}_1, \tilde{c}_2