Math 3012-L
Spring 2018
Exam 1
1 Feb
Time Limit: 70 Minutes

Name: PRABHAV CHAWLA

This exam contains 8 pages (including this cover page) and 6 questions. There are 0 points in total. Justify all answers. Any expression for a number is acceptable; there is no need to find a decimal representation. Write explanations or proofs clearly and in complete thoughts. Points are reserved for clarity. Use the blank side of paper for scratch work. No calculators or notes may be used.

On my honor, I pledge that I will not give or receive aid in examinations; I will not use unapproved materials in examinations; I will not misrepresent my work or represent the work of another as my own; and I will avoid any activity which will encourage others to violate their own pledge of honor.

Signature: Prabhav

Print Name: PRABHAV CHAWLA

Formal Symbols Crib Sheet

$\neg$	not	$\wedge$	and	$\vee$		or
$\Rightarrow$	implies	$\lightning$	contradiction	$\in$		element of
$\forall$	for all	$\exists$	there exists	$\Leftrightarrow$		equivalence
$\varnothing$	empty set	$\mathbb{N}$	natural numbers	$\mathbb{Z}$		integers
$\mathbb{Z}_+$	positive integers	$\mathbb{Z}_{\geq 0}$	non-negative integers	$\equiv$	$\pmod{n}$	congruence mod n
$\mathbb{Q}$	rationals	$\mathbb{R}$	reals	$\mathbb{C}$		complex numbers
$\times$	Cartesian product	$\subset$	subset	$\setminus$		set minus
$\cap$	intersection	$\cup$	union	$\mathcal{O}$		big-O asymptotic order
2^A	power set of set A	$\lvert A\rvert$	cardinality of set A	A^B		set of functions $B \to A$

The Twelvefold Way:

$|\{f : k \to n\}|$

How many ways to sort k balls into n boxes?

	Arbitrary any sorting	Injective max 1 ball per box	Surjective each box gets ball
Distinct Balls Distinct Boxes	n^k	$\frac{n!}{(n-k)!}$	$n!\left\{ {k \atop n} \right\}$
Identical Balls Distinct Boxes	$\binom{n+k-1}{k}$	$\binom{n}{k}$	$\binom{k-1}{n-1}$
Distinct Balls Identical Boxes	$\sum_{j=0}^{n} \left\{ {k \atop j} \right\}$	1 if $k \leq n$	$\left\{ {k \atop n} \right\}$
~~Identical Balls~~ Identical Boxes	$p_{\leq n}(k)$	1 if $k \leq n$	$p_n(k)$

1. (a) (3 points) What makes a decision problem **P**? What makes a decision problem **NP**?

⇒ A decision problem is in P if there exists a polynomial time deterministic algorithm to find a solution for it.

⇒ A decision problem is in NP if there exists a polynomial time deterministic algorithm to verify if a proposed solution to the problem is correct or not.

⇒ $P \subseteq NP$
subset

(b) Consider the following decision problem:

Given a list of n positive integers less than $50n$, decide if two distinct numbers in the list multiply to $4n + 8$.

Describe an algorithm that can answer the decision problem and estimate the $\mathcal{O}$ complexity of your algorithm. You must state what basic operations you are counting.

⇒ We pick two numbers in $\binom{n}{2}$ ways and then multiply them to check if we get $4n+8$. Since each number is $< 50n$, multiplying should not take a long time & won't factor into Big-Oh

⇒ $O\left(\binom{n}{2}\right) = \boxed{O(n^2)}$

2. (a) (3 points) Circle True or False.

A. For a graph $G = (V, E)$ we have $|E| = \mathcal{O}(|V|^2)$.

TRUE FALSE

6

B. If S is a set and w is the width of the poset of subsets of S, then $w = \mathcal{O}(|S|^2)$.

TRUE FALSE

C. If $H = (V', E')$ is a subgraph of $G = (V, E)$, then $|E'| = \mathcal{O}(|E|)$.

TRUE FALSE

(b) (3 points) How many subgraphs of the complete graph K_{11} with vertex set $\{0, \ldots, 10\}$ are trees?

$\Rightarrow 11^9$ since we know that:

# of trees on n $\longleftrightarrow$ # of length $n-2$ strings on n

(c) (3 points) Suppose a graph G' has 11 vertices. Recall that the *symmetries* of G' are the graph isomorphisms from G' to itself. What is the maximum number of symmetries G' might have? What is the least number of symmetries G' might have?

$\Rightarrow$ It must be greater than 1, since the identity would be a valid isomorphism: $f(v) = v, v \in V$

$\Rightarrow$ It must be less than or equal to $\binom{11}{1}$ since we have 11 vertices to choose from. $\binom{11}{1} = 11$

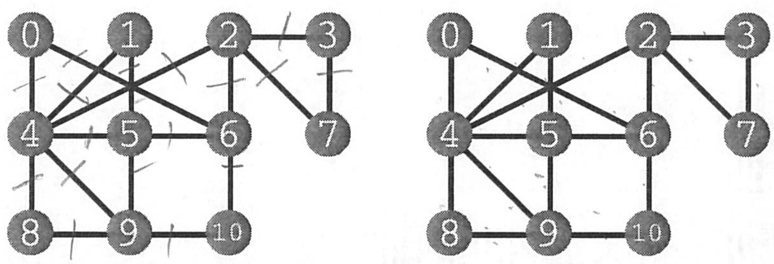
3. (a) (4 points) Circle True or False.

   A. If a graph G is planar, then G is also Hamiltonian.
   TRUE FALSE

   B. If a graph G is 4-colorable, then G is also planar.
   TRUE FALSE

   C. Deciding if G is planar is an **NP**-problem.
   TRUE FALSE

   D. Deciding if G is planar is not a **P**-problem.
   TRUE FALSE

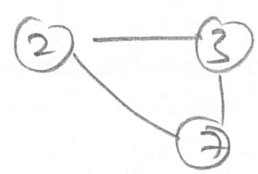


(b) (3 points) Consider the graph shown above. (Two copies are provided for your convenience.) Is the graph Eulerian? Justify your claim.

⇒ Yes it is. The graph is connected and degree of each vertex is even.

(c) (3 points) Consider the graph shown above. Is the graph Hamiltonian? Justify your claim.

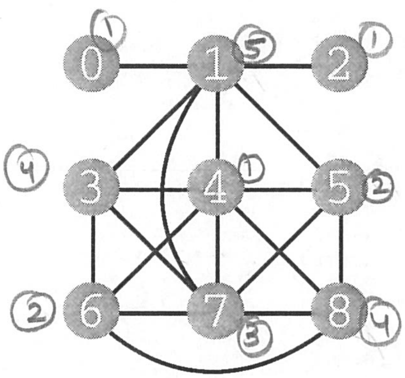
⇒ No. We look at the cycle

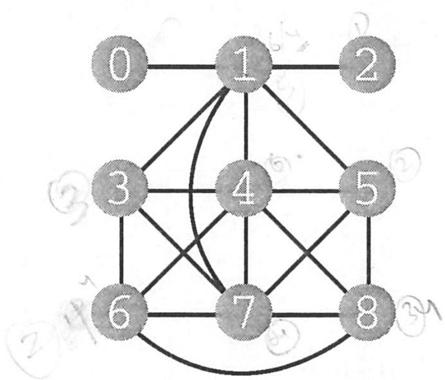


We can't enter and exit this cycle without visiting (2) twice, therefore the graph is not Hamiltonian.

4. (a) (3 points) What is a k-coloring of a graph?

$\Rightarrow$ a k-coloring is a function $f : V \longrightarrow k$ that assigns each vertex $v \in V$ to a 'color' / number in $\{0, 1, 2 \ldots k-1\}$ such that if edge $uv \in E$, then $f(u) \neq f(v)$





(b) (3 points) Consider the graph above. What is the chromatic number of this graph? Explain.

$\Rightarrow$ The graph has a 4-clique given by $\{4, 5, 7, 8\}$

$\therefore \chi(G) \geq 4$.

$\Rightarrow$ As illustrated above on the left, G is 5-colorable. Therefore $\chi(G) \leq 5$. However

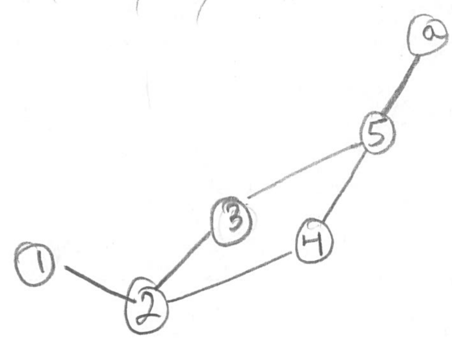
$\Rightarrow$ Since vertex 1 is connected to 3, 7, 4 and 5 and each of them must have a unique color ($\{3, 4, 6, 7\}$ also form a clique and 6 and 8 are connected), ~~the~~ the graph isn't four colorable and $\chi(G) = 5$

(c) BONUS: Suppose G is known to have chromatic number 3 and has vertex set $\{0, \ldots, 9\}$. Both $\{0, 1\}$ and $\{1, 2\}$ are edges in G, but the other edges of G are not known. How many possible 3-colorings of G are consistent with this information, up to relabeling of the colors?

5. (a) (3 points) Draw the Hasse diagram for the poset

(2,2)

$$\{(a,a),(0,0),(1,1),(3,3),(4,4),(5,5),(2,1),(2,3),$$
$$(2,4),(2,5),(2,a),(3,a),(3,5),(4,a),(4,5),(5,a)\}$$



1 2 3 4 5 6 7 8 9 10 11

(b) (3 points) A graph has degree sequence (4, 4, 4, 4, 3, 2, 2, 2, 1, 1, 1). Must it be planar, must it be nonplanar, or might it be either? Explain.

$\Rightarrow |V| = 11$

$\Rightarrow |E| = \dfrac{\sum_{v \in V} \deg(v)}{2} = \dfrac{4+4+4+4+3+2+2+2+1+1+1}{2} = 14$

$\Rightarrow$ since $|V| \geq 3$, check if:

$\Rightarrow \quad 3|V| \geq |E| + 6$

$\Rightarrow \quad 33 \geq 20$, which is true. But this doesn't necessarily imply that G is planar. Therefore it may be either.

6. The 2018 Winter Olympics were held in PyeongChang, Soth Korean.

(a) (2 points) Competing were 2,922 athletes representing exactly 92 National Olympic Committees. How many ways might the 2,922 different athletes have come from the 92 different National Olympic Committees if we track which athlete competes for which nation?

$f : A \longrightarrow N$. $|A| = 2922$, $|N| = 92$. This function is surjective and elements in A and N are unique.

$\Rightarrow \therefore$ # of ways $= \underbrace{S(2922, 92)}_{\text{Surjective \#}} = 92! \left\{ {2922 \atop 92} \right\}$

(b) (3 points) Athletes competed in 102 events in 15 sports, with a gold, silver, and bronze medal awarded in each event. How many ways might the medals have been awarded to the 92 National Olympic Committees if we track the number of each type of medal?

$\Rightarrow$ Let's do for gold medals first:

$\Rightarrow$ Let x_i = # of gold medals won by committee i.

$\Rightarrow$ Then, $x_1 + x_2 + x_3 \cdots + x_{92} = 102$, each $x_i \geq 0$

$\Rightarrow$ # of ways to do this $= \binom{102+92-1}{91} = \binom{193}{91}$

$\Rightarrow$ To account for all three types of medals $= \binom{193}{91}^3$

(c) (3 points) In fact Norway had the highest total medal count with 39, and only 30 National Olympic Committees won any medals. How many ways may the remaining 267 medals have been distributed among the other 29 nations? Note no nation but Norway won more than 38.

$\Rightarrow$ Let x_i = # of medals won by committee i.

$\Rightarrow$ Let $1 \leq i \leq 29$ represent the 29 nations we care about.

$\Rightarrow x_1 + x_2 + x_3 \cdots + x_{29} = 102 \times 3 - 39 = 267$ and $1 \leq x_i \leq 38$ for $1 \leq i \leq 29$

$\Rightarrow$ To change inequality we get:

$\Rightarrow \sum_{1}^{29} x_i = 267 - 29 = 238$, $0 \leq x_i \leq 37$

$\Rightarrow$ By inclusion exclusion: $\binom{238+29-1}{29-1} - \binom{29}{1}\binom{238-38+29-1}{29-1} + \binom{29}{2}\binom{238-2\times38+29-1}{29-1}$

$$= \boxed{\sum_{j=0}^{29} (-1)^j \binom{266-38j}{28}\binom{29}{j}}$$

(d) BONUS: What nation had the second highest medal count?

$\Rightarrow$ USA