COMS 4721: Machine Learning for Data Science
Lecture 13, 3/10/2015

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BOOSTING

For more, see the textbook: Robert E. Schapire and Yoav Freund, *Boosting: Foundations and Algorithms*, MIT Press, 2012. (I borrow some figures from that book.)

Algorithm: Bagging binary classifiers

Given $(x_1, y_1), \ldots, (x_n, y_n)$, $x \in \mathcal{X}$, $y \in \{-1, +1\}$

- For $b = 1, \ldots, B$
  - Sample a bootstrap dataset $\mathcal{B}_b$ of size n from $D_0 = \sum_{i=1}^{n} \frac{1}{n} \delta_{x_i}$.
  - Learn a classifier f_b using data in $\mathcal{B}_b$.
- Set the classification rule to be

$$f_{bag}(x_0) = \text{sign}\left(\sum_{b=1}^{B} f_b(x_0)\right).$$

- With bagging, we saw how a *committee* of classifiers votes on a label.
- Each classifier is learned on a *bootstrap sample* from the data set.
- Learning a collection of classifiers is referred to as an *ensemble method*.

BOOSTING

How is it that a committee of blockheads can somehow arrive at highly reasoned decisions, despite the weak judgment of the individual members?

- Schapire & Freund, "Boosting: Foundations and Algorithms"

Boosting is another powerful method for ensemble learning. It is similar to bagging in that a set of classifiers are combined to make a better one.

Free to choose a classifier, but a "weak" one is usually picked – i.e., one with accuracy a little better than random guessing, but very fast to learn.

Short history

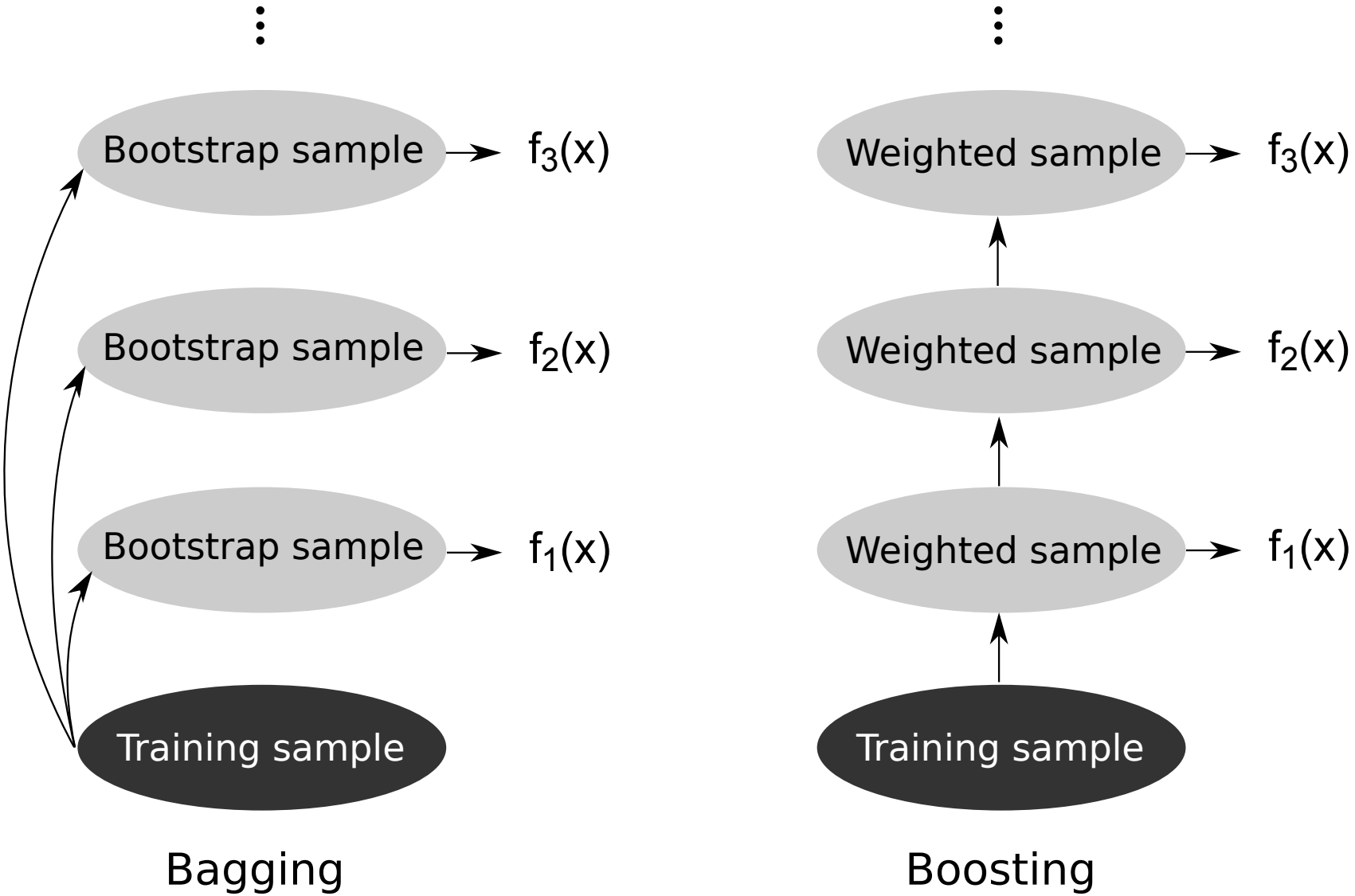
1984 : Leslie Valiant and Michael Kearns ask if "boosting" is possible.

1989 : Robert Schapire creates first boosting algorithm.

1990 : Yoav Freund creates an optimal boosting algorithm.

1995 : Freund and Schapire create AdaBoost (Adaptive Boosting), the major boosting algorithm.

Bagging vs Boosting (schematic)



The AdaBoost Algorithm (a sampling version)

Algorithm: Boosting a binary classifier

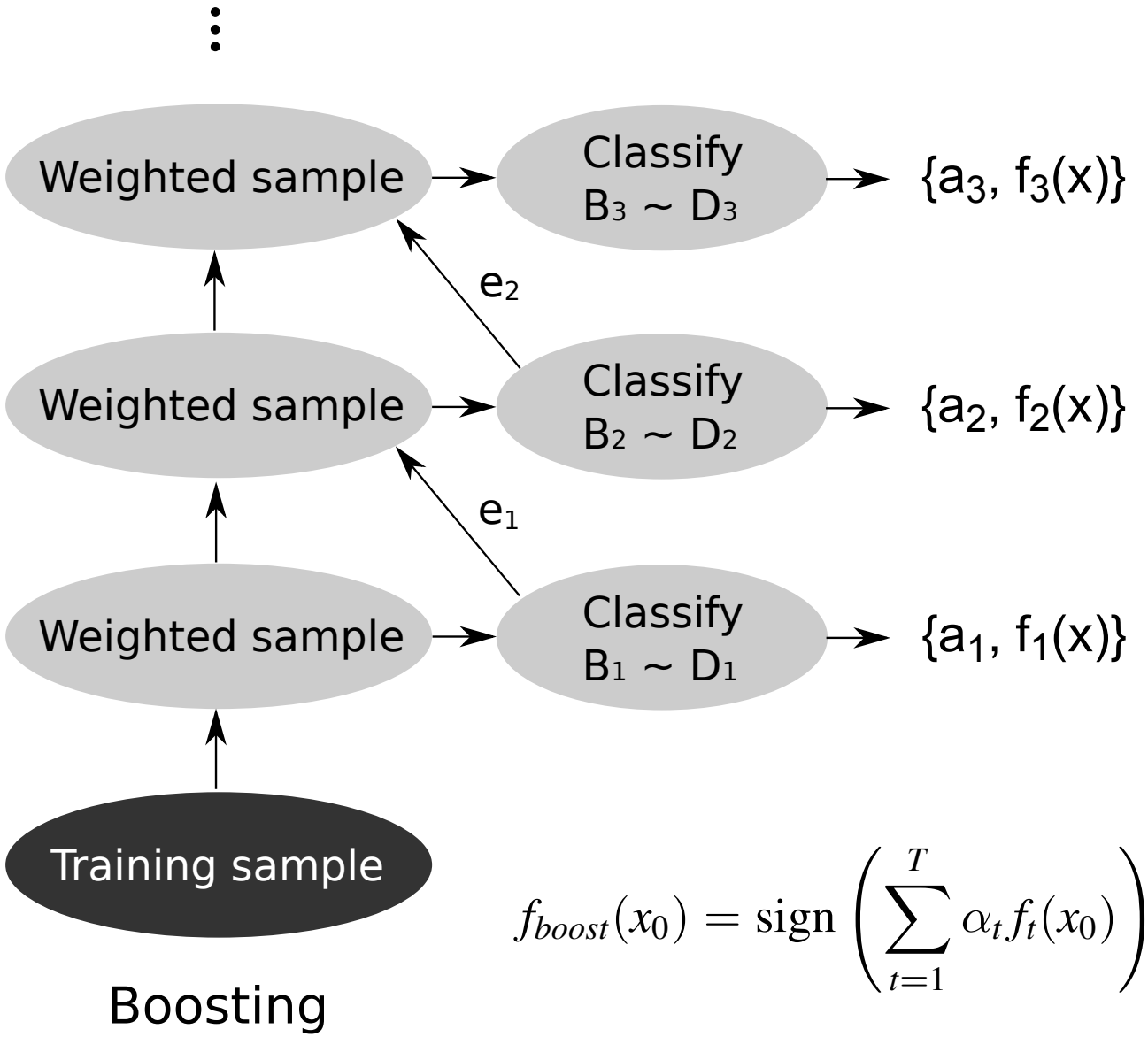
Given $(x_1, y_1), \ldots, (x_n, y_n)$, $x \in \mathcal{X}$, $y \in \{-1, +1\}$, set $w_1(i) = \frac{1}{n}$

- For $t = 1, \ldots, T$
  1. Sample a bootstrap dataset $\mathcal{B}_t$ of size n from $D_t = \sum_{i=1}^n w_t(i)\delta_{x_i}$.
  2. Learn a classifier f_t using data in $\mathcal{B}_t$.
  3. Set $\epsilon_t = \sum_{i=1}^n w_t(i)\mathbb{1}\{y_i \neq f_t(x_i)\}$ and $\alpha_t = \frac{1}{2}\ln\left(\frac{1-\epsilon_t}{\epsilon_t}\right)$.
  4. Update $\tilde{w}_{t+1}(i) = w_t(i)\exp\{-\alpha_t y_i f_t(x_i)\}$.
  5. Normalize $w_{t+1}(i) = \tilde{w}_{t+1}(i)/\sum_j \tilde{w}_{t+1}(j)$.
- Set the classification rule to be

$$f_{boost}(x_0) = \text{sign}\left(\sum_{t=1}^T \alpha_t f_t(x_0)\right).$$

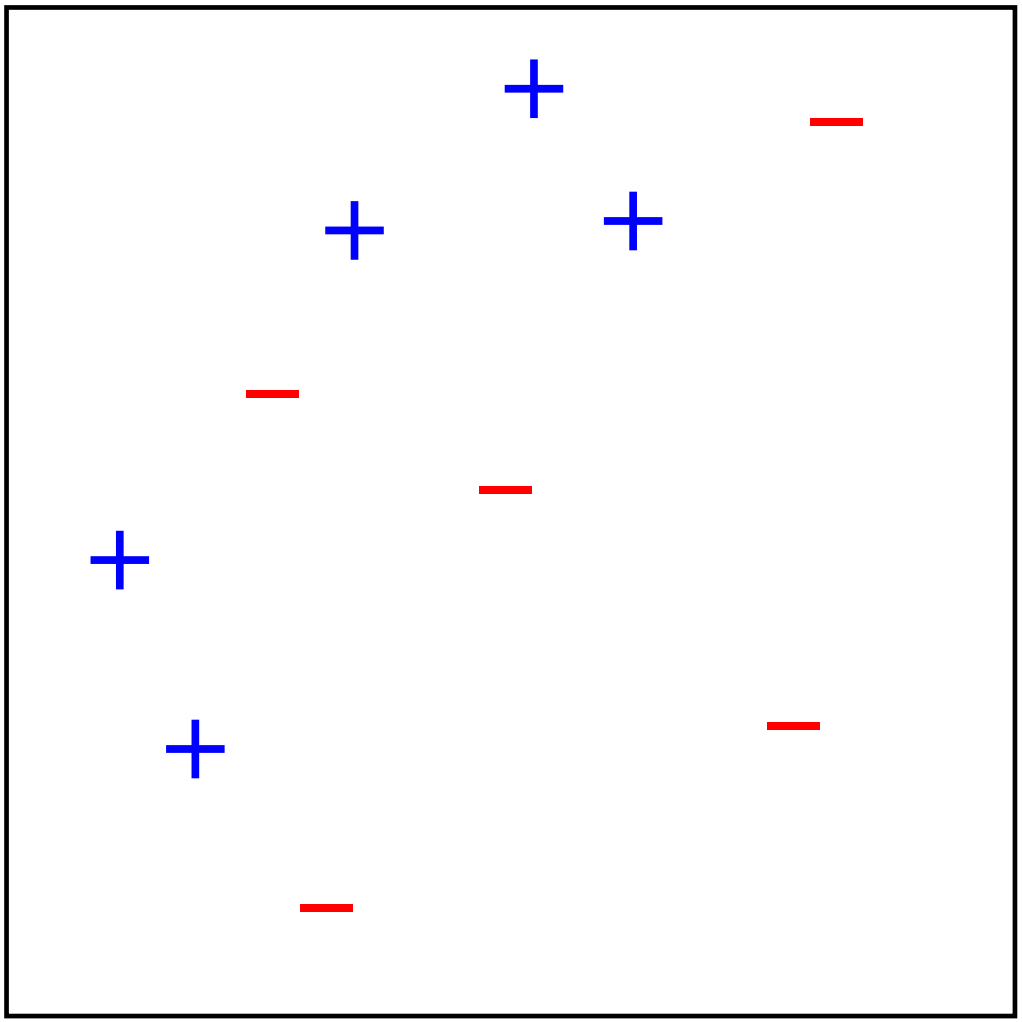
Comment: Step 1 can often be skipped and w_t used directly in Step 2, which is the way AdaBoost is normally presented.

The AdaBoost Algorithm (schematic)



$$f_{boost}(x_0) = \text{sign}\left(\sum_{t=1}^{T} \alpha_t f_t(x_0)\right)$$

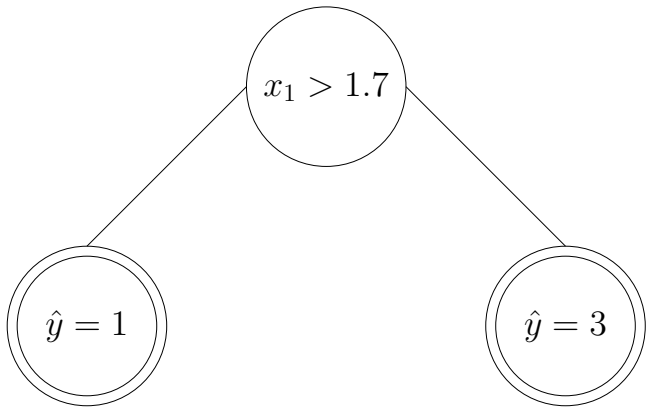
BOOSTING A DECISION STUMP (EXAMPLE 1)



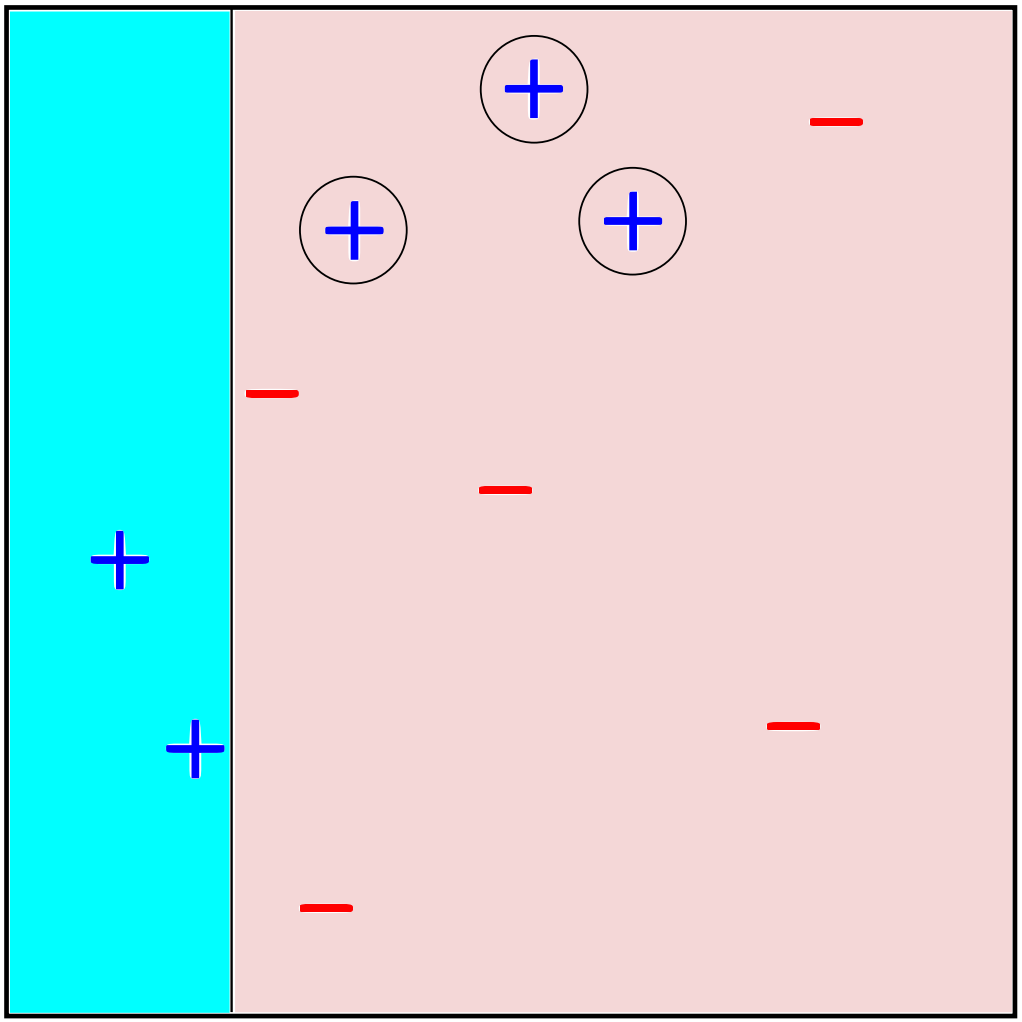
Original data

Uniform data distribution
Learn *weak classifier*

Here: Use a decision stump



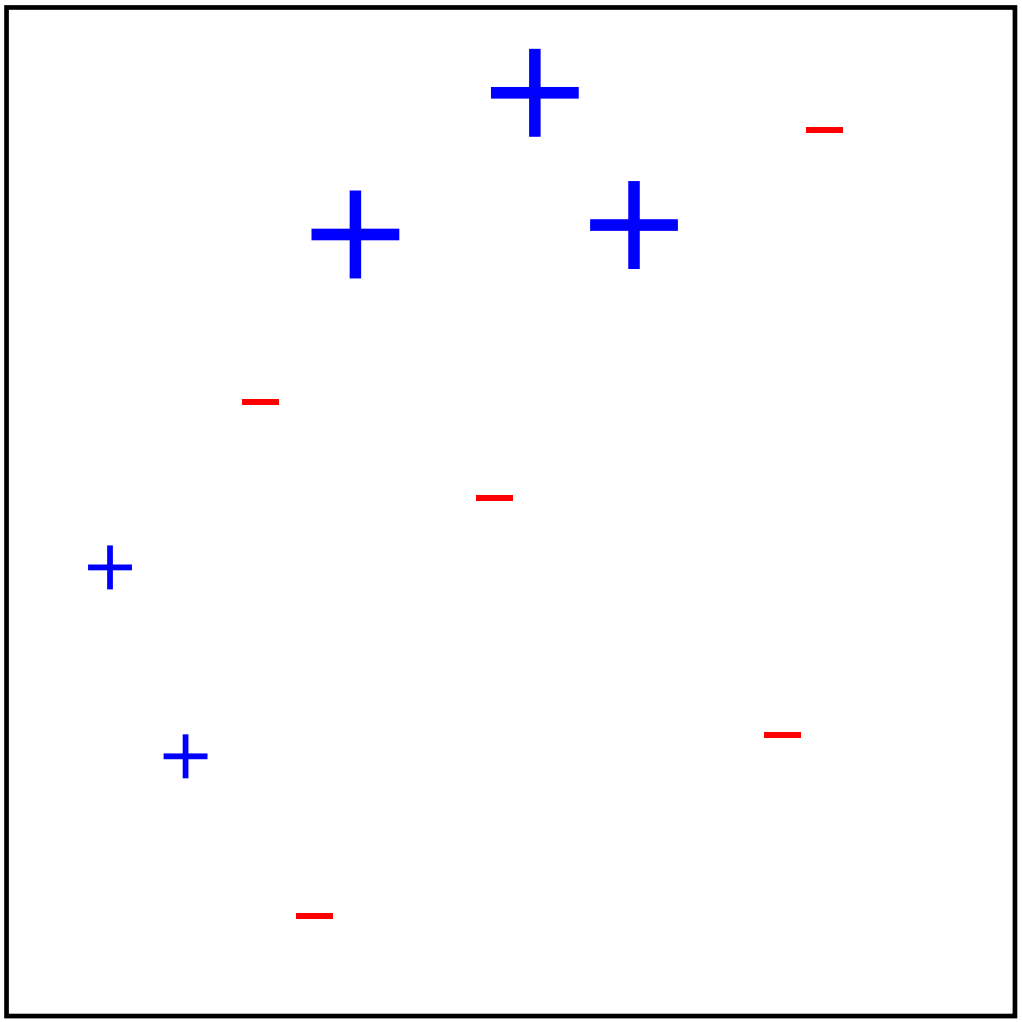
BOOSTING A DECISION STUMP (EXAMPLE 1)



Round 1 classifier

Weighted error: $\epsilon_1 = 0.3$
Weight update: $\alpha_1 = 0.42$

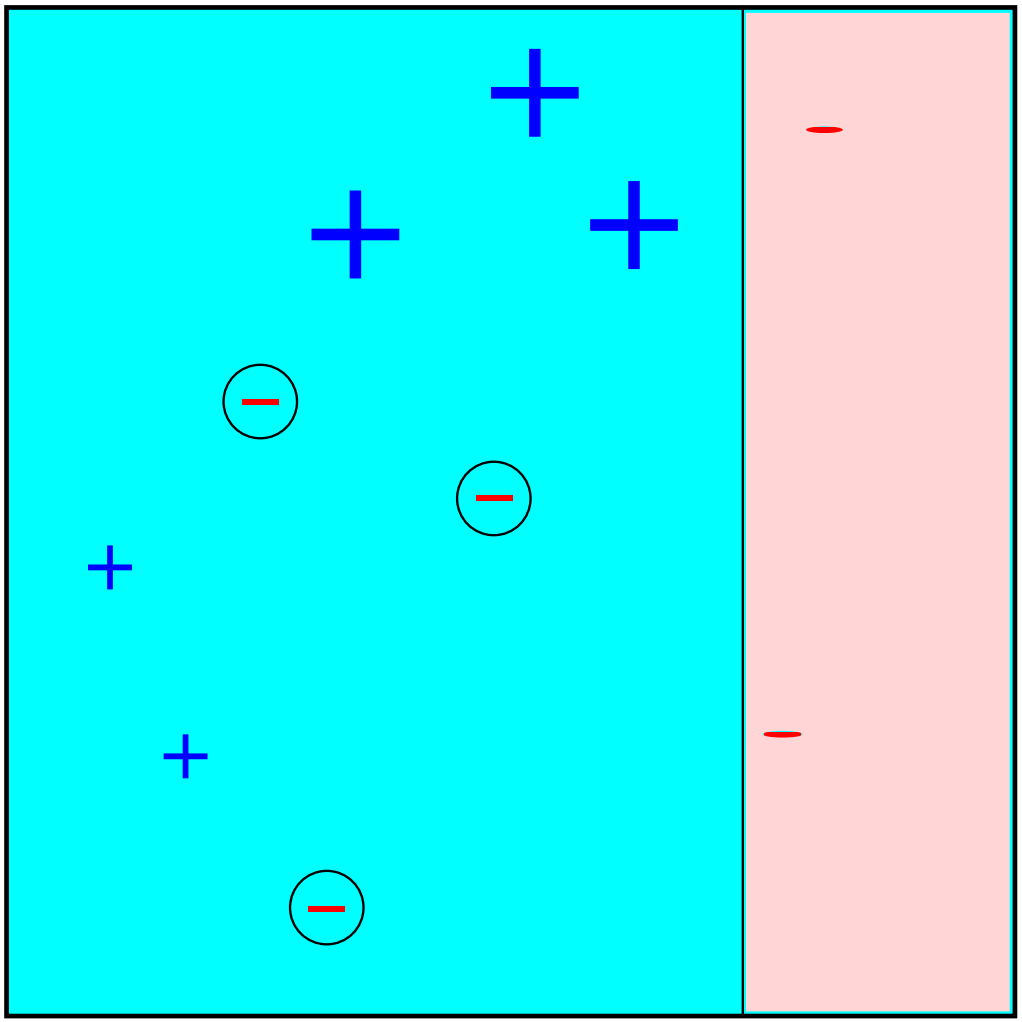
BOOSTING A DECISION STUMP (EXAMPLE 1)



Weighted data

After round 1

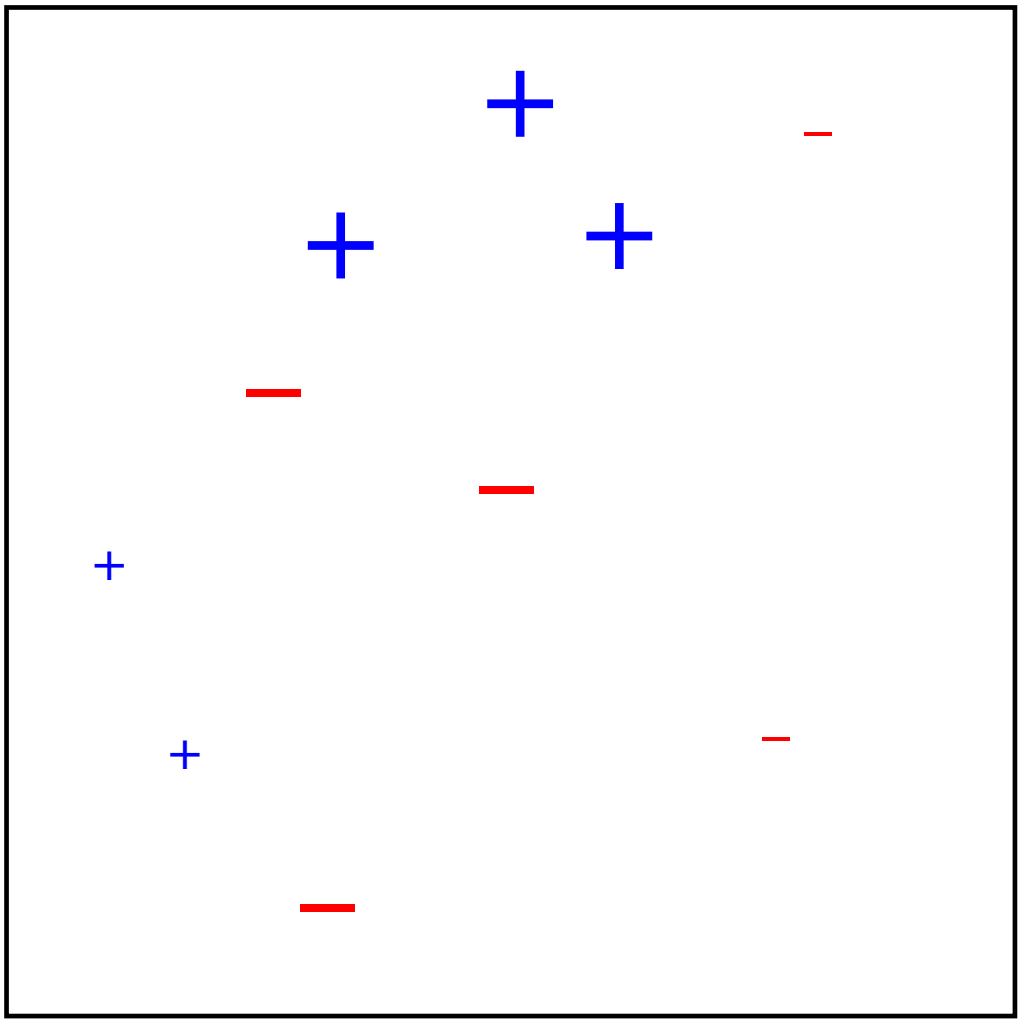
BOOSTING A DECISION STUMP (EXAMPLE 1)



Round 2 classifier

Weighted error: $\epsilon_2 = 0.21$
Weight update: $\alpha_2 = 0.65$

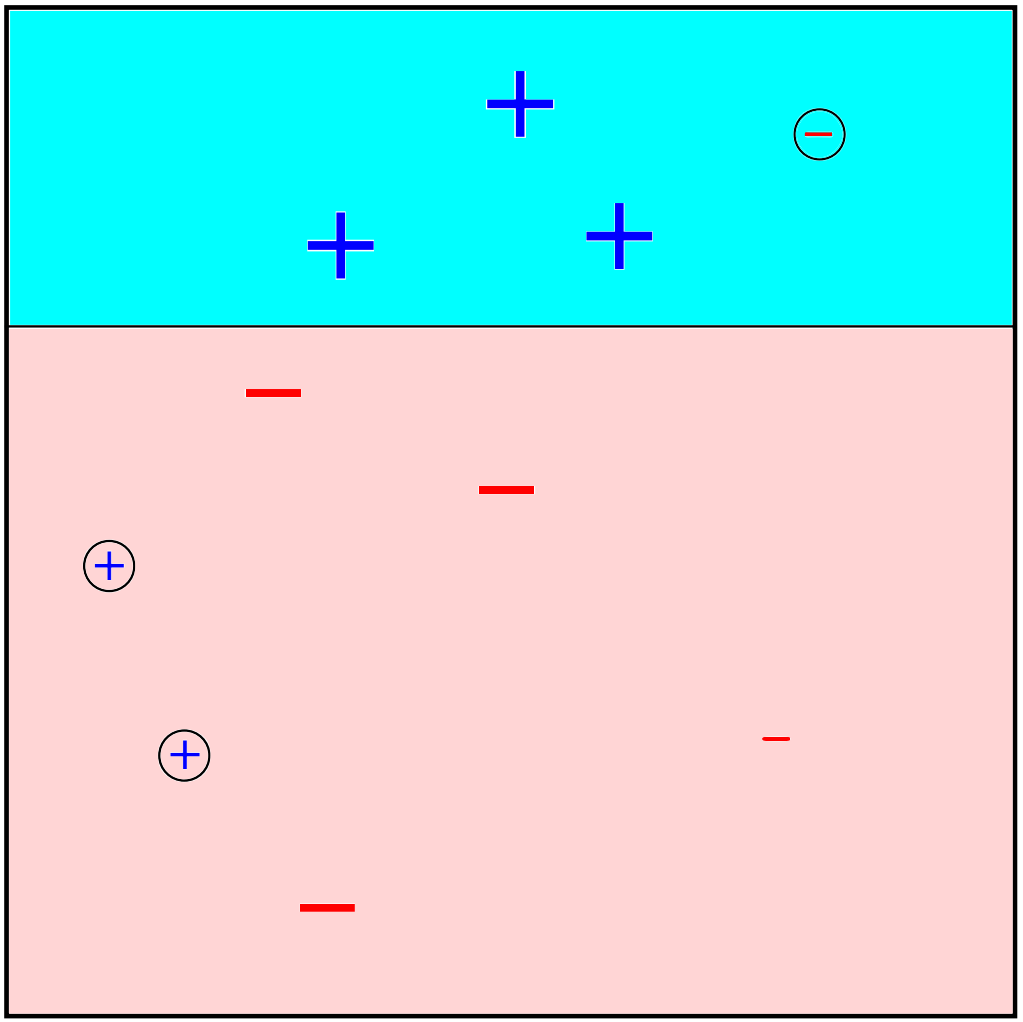
BOOSTING A DECISION STUMP (EXAMPLE 1)



Weighted data

After round 2

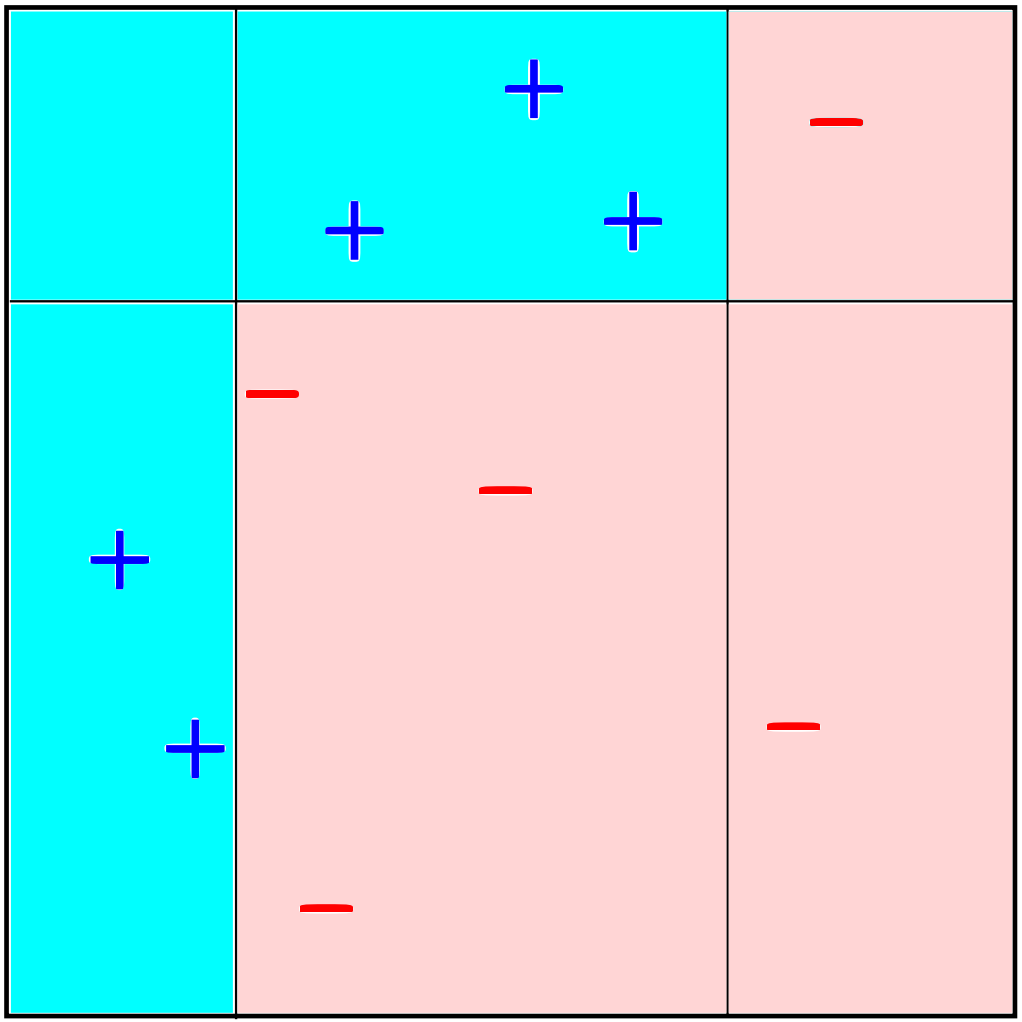
BOOSTING A DECISION STUMP (EXAMPLE 1)



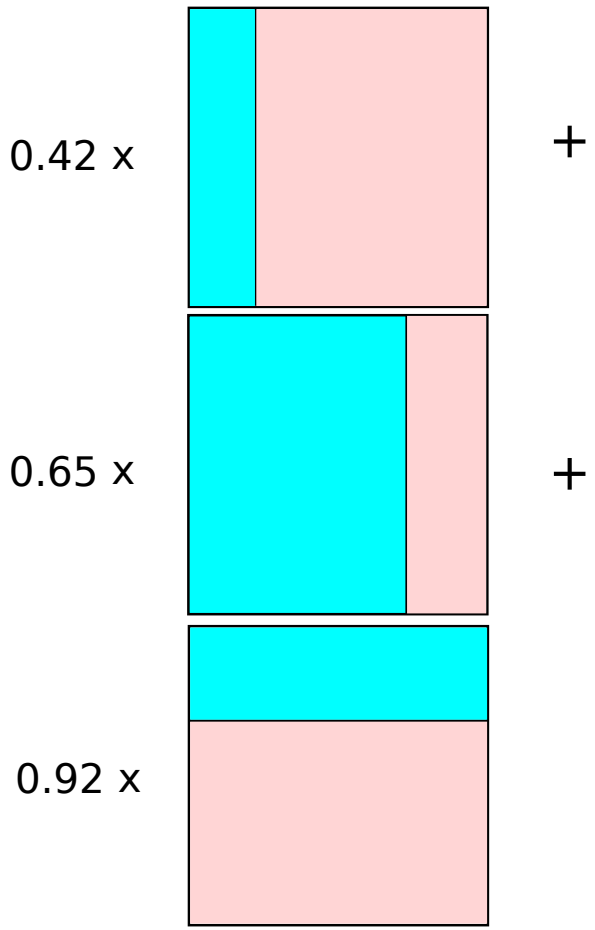
Round 2 classifier

Weighted error: $\epsilon_3 = 0.14$
Weight update: $\alpha_3 = 0.92$

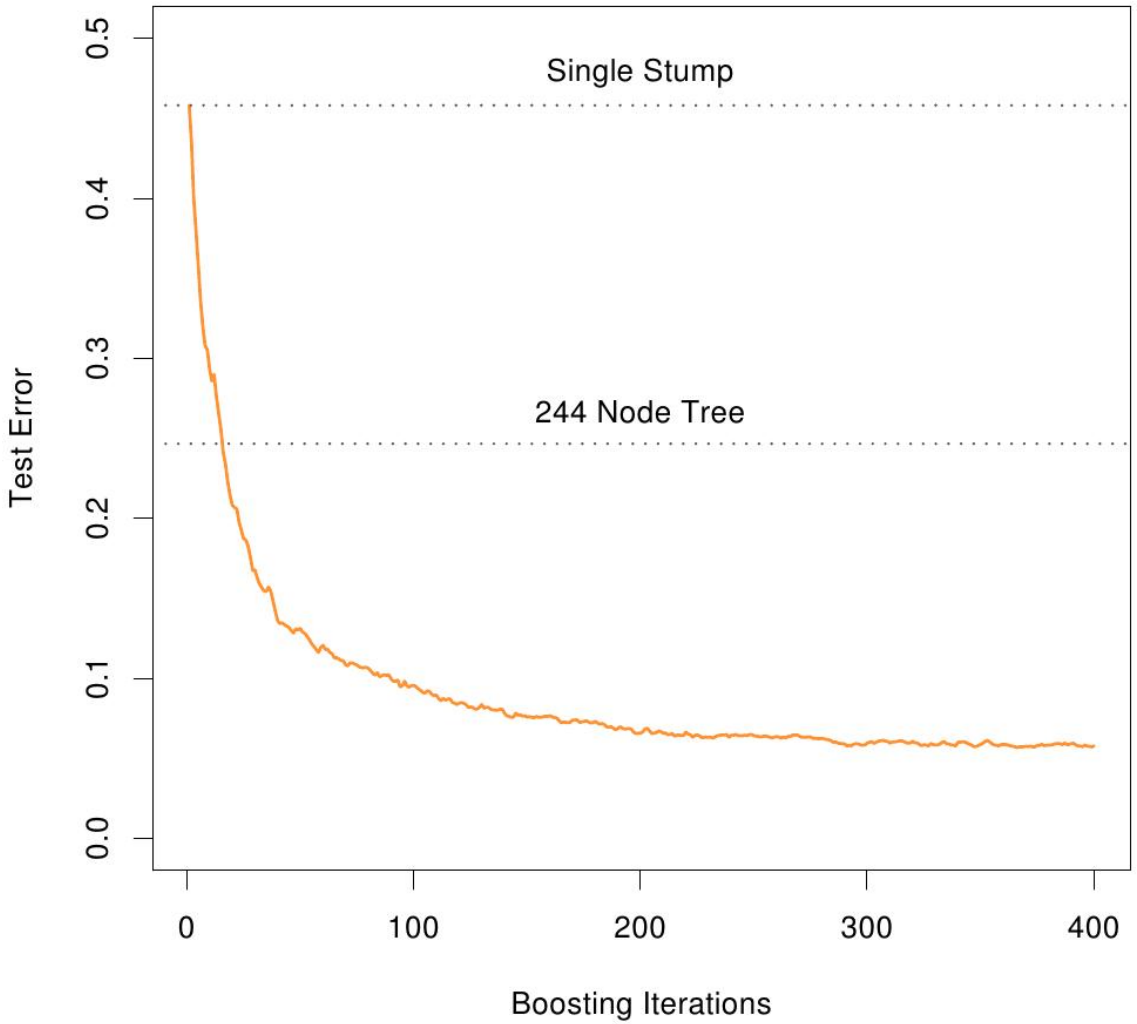
BOOSTING A DECISION STUMP (EXAMPLE 1)



Classifier after three rounds



BOOSTING A DECISION STUMP (EXAMPLE 2)



A Toy Problem

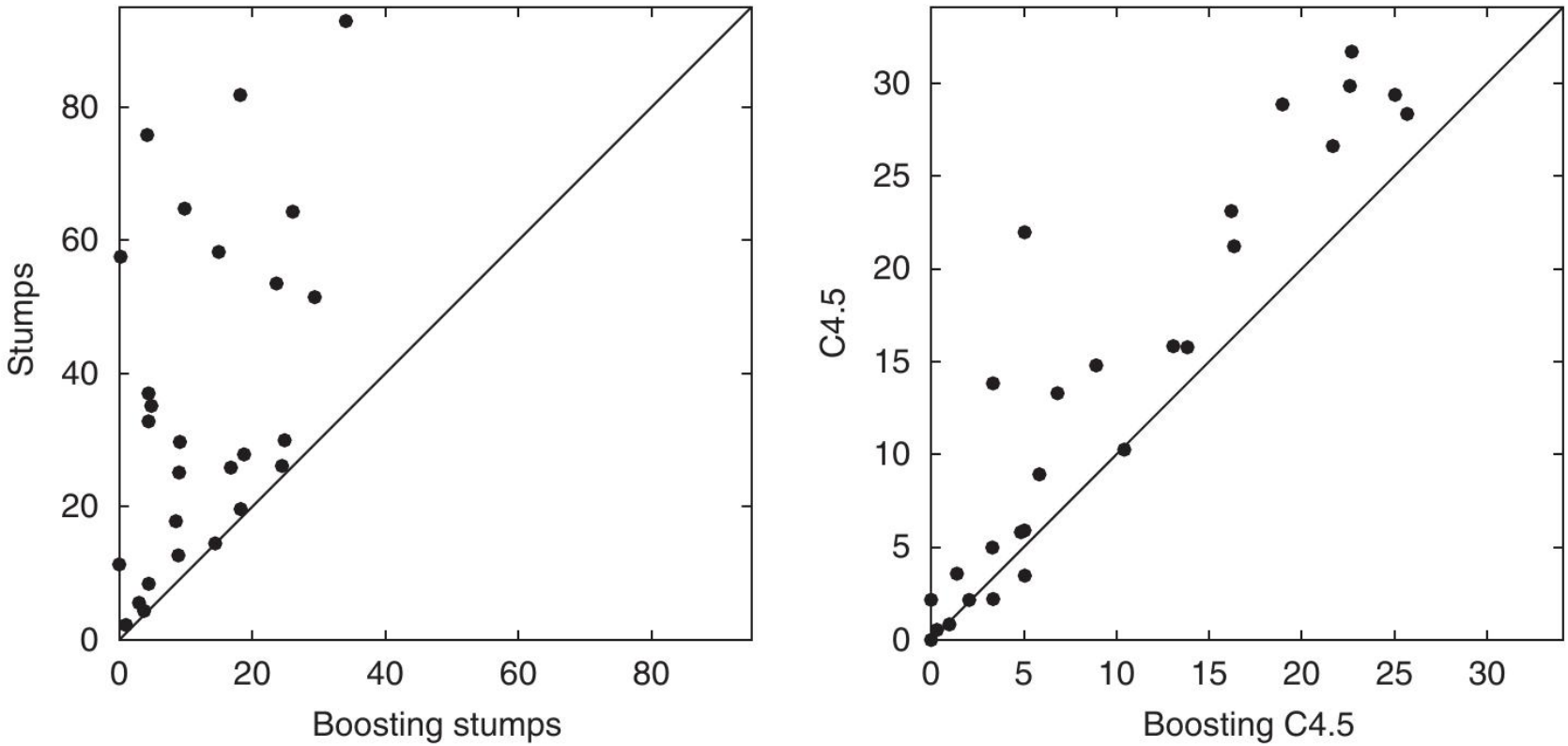
Random guessing
50% error

Decision stump
45.8% error

Full decision tree
24.7% error

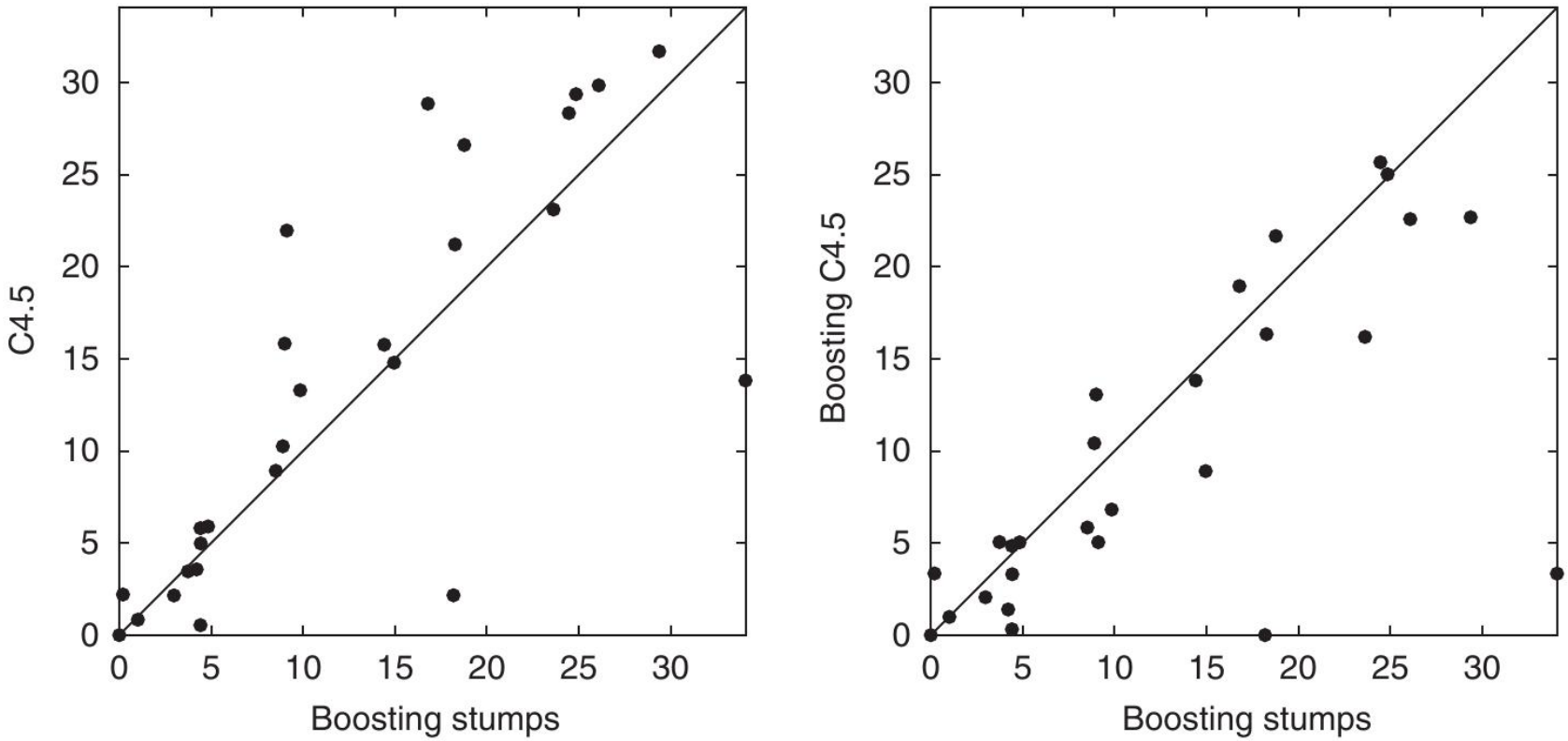
Boosted stump
5.8% error

BOOSTING



Point = one dataset. Location = error rate w/ and w/o boosting. The boosted version of the same classifier almost always produces better results.

BOOSTING



(left) Boosting a bad classifier is often better than not boosting a good one.
(right) Boosting a good classifier is often better (but can take more time).

BOOSTING AND FEATURE MAPS

Q: What makes boosting work so well?

A: This is a very well studied question. We will present one analysis later, but we can also give intuition by tying it in with what we've discussed.

The classification for a new x_0 from boosting is

$$f_{boost}(x_0) = \text{sign}\left(\sum_{t=1}^{T} \alpha_t f_t(x_0)\right).$$

Define $\phi(x) = [f_1(x), \ldots, f_T(x)]^T$, where each $f_t(x) \in \{-1, +1\}$.

- We can think of $\phi(x)$ as a high dimensional feature map of x.
- The vector $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_T]^T$ correspond to hyperplane.
- So the classifier can be written $f_{boost}(x_0) = \text{sign}(\phi(x_0)^T \boldsymbol{\alpha})$.
- Boosting learns the feature mapping and hyperplane simultaneously.

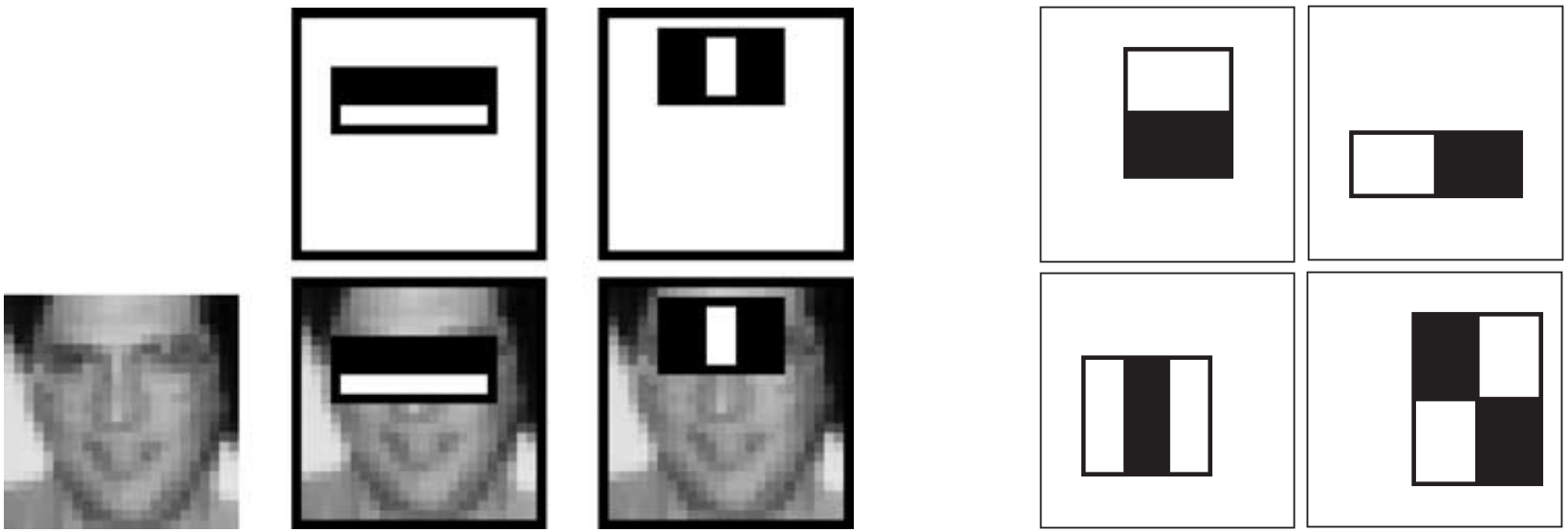
Application: Face detection

FACE DETECTION (VIOLA & JONES, 2001)

Problem: Locate the faces in an image or video.

Processing: Divide image into patches of different scales, e.g., 24×24, 48×48, etc. Extract *features* from each patch.

Classify each patch as face or no face using a *boosted decision stump*. This can be done in real-time, for example by your digital camara (at 15 fps).



- One patch from a larger image. Mask it with many "feature extractors."
- Each pattern gives one number, which is the sum of all pixels in black region minus sum of pixels in white region (total of 45,000+ features).

FACE DETECTION (EXAMPLE RESULTS)

Analysis of boosting

Training error theorem

We can use *analysis* to make a statement about the predictive accuracy of boosting *on the training data*.

Theorem: Under the AdaBoost framework, if ϵ_t is the weighted error of classifier f_t, then for the classifier $f_{boost}(x_0) = \text{sign}(\sum_{t=1}^{T} \alpha_t f_t(x_0))$,

$$\text{training error } = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{y_i \neq f_{boost}(x_i)\} \leq \exp\Big(-2\sum_{t=1}^{T}(\tfrac{1}{2}-\epsilon_t)^2\Big).$$

Even if each ϵ_t is only a little better than random guessing, the accumulation of them over T classifiers can lead to a substantial value in the exponent.

For example:

$$\epsilon_t = 0.45,\ T = 1000 \ \rightarrow\ \text{training error } \leq\ 0.0067.$$

Setup

We break the proof into three steps. It is an application of the fact that

$$\text{if} \quad \underbrace{a < b}_{\text{Step 2}} \quad \text{and} \quad \underbrace{b < c}_{\text{Step 3}} \quad \text{then} \quad \underbrace{a < c}_{\text{conclusion}}$$

- Step 1 allows us to know what b is above.
- Steps 2 and 3 correspond to the two inequalities.

Also recall the following step from AdaBoost:

- Update $\tilde{w}_{t+1}(i) = w_t(i)e^{-\alpha_t y_i f_t(x_i)}$ and normalize $w_{t+1}(i) = \frac{\tilde{w}_{t+1}(i)}{\sum_j \tilde{w}_{t+1}(j)}$.

We define $Z_t = \sum_j \tilde{w}_{t+1}(j)$ for use in the proof.

Step 1

We first want to expand the equation of the weights to show that

$$w_{T+1}(i) = \frac{1}{n} \frac{\exp\{-y_i \overbrace{\sum_{t=1}^{T} \alpha_t f_t(x_i)}^{= f_{boost}(x_i)}\}}{\prod_{t=1}^{T} Z_t}.$$

Derivation of Step 1:
To do so, first notice the recurrence: $w_{t+1}(i) = w_t(i) e^{-\alpha_t y_i x_i} / Z_t$.

We can break down $w_t(i)$ in exactly the same way, and continue until $w_1(i)$,

$$\begin{aligned} w_{T+1}(i) &= w_1(i) \frac{\exp\{-\alpha_1 y_i x_i\}}{Z_1} \times \cdots \times \frac{\exp\{-\alpha_T y_i x_i\}}{Z_T} \\ &= \frac{1}{n} \frac{\exp\{-y_i \sum_{t=1}^{T} \alpha_t f_t(x_i)\}}{\prod_{t=1}^{T} Z_t} \end{aligned}$$

Step 2

We next need to show that the training error of the classifier after $T+1$ steps is not greater than $\prod_{t=1}^{T} Z_t$.

Derivation of Step 2:

From Step 1: $w_{T+1}(i) = \frac{1}{n} \frac{\exp\{-y_i f_{boost}(x_i)\}}{\prod_{t=1}^{T} Z_t} \rightarrow w_{T+1}(i) \prod_{t=1}^{T} Z_t = \frac{1}{n} e^{-y_i f_{boost}(x_i)}$

$$\begin{aligned}
\frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{y_i \neq f_{boost}(x_i)\} &\leq \frac{1}{n} \sum_{i=1}^{n} \exp\{-y_i f_{boost}(x_i)\} \\
&= \sum_{i=1}^{n} w_{T+1}(i) \prod_{t=1}^{T} Z_t \\
&= \prod_{t=1}^{T} Z_t
\end{aligned}$$

Step 3

The final step is to calculate an upper bound on Z_t, and therefore of $\prod_{t=1}^{T} Z_t$.

Since $\prod_{t=1}^{T} Z_t$ is an upper bound on the training error, the upper bound from Step 3 is *also* of the training error.

Derivation of Step 3:

This step is slightly more involved. It also shows why $\alpha_t = \frac{1}{2} \ln\left(\frac{1-\epsilon_t}{\epsilon_t}\right)$.

$$\begin{aligned} Z_t &= \sum_{i=1}^{n} w_t(i) \exp\{-\alpha_t y_i f_t(x_i)\} \\ &= \sum_{i\,:\,y_i = f_t(x_i)} e^{-\alpha_t} w_t(i) + \sum_{i\,:\,y_i \neq f_t(x_i)} e^{\alpha_t} w_t(i) \\ &= e^{-\alpha_t}(1-\epsilon_t) + e^{\alpha_t}\epsilon_t \end{aligned}$$

PROOF OF THEOREM

Derivation of Step 3 (continued):

We're currently at $Z_t = e^{-\alpha_t}(1-\epsilon_t) + e^{\alpha_t}\epsilon_t$. Remember from Step 2 that

$$\text{training error} \;=\; \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{y_i \neq f_{boost}(x_i)\} \;\leq\; \prod_{t=1}^{T} Z_t \,.$$

We want the training error to be small. We therefore pick α_t to *minimize* Z_t. This minimum is independent for each t and occurs at

$$\alpha_t = \frac{1}{2}\ln\left(\frac{1-\epsilon_t}{\epsilon_t}\right).$$

Plugging this value in gives $Z_t = 2\sqrt{\epsilon_t(1-\epsilon_t)}$.

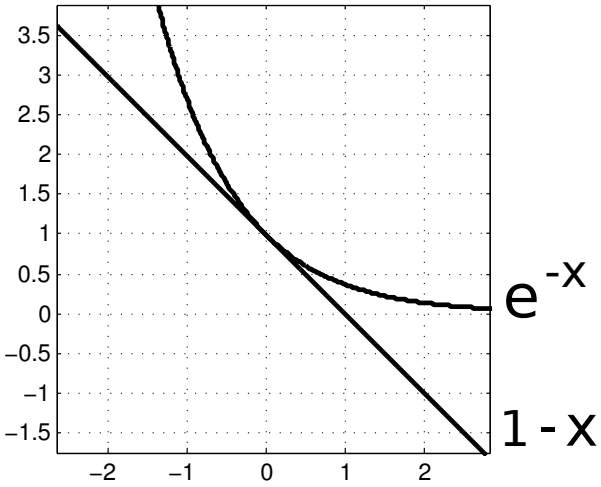
PROOF OF THEOREM

Derivation of Step 3 (conclusion):

Thus $Z_t = 2\sqrt{\epsilon_t(1-\epsilon_t)}$

We re-write this as

$$Z_t = \sqrt{1 - 4(\frac{1}{2} - \epsilon_t)^2}.$$



We use the general inequality $1 - x \leq e^{-x}$ to conclude that

$$Z_t = \left(1 - 4(\tfrac{1}{2} - \epsilon_t)^2\right)^{\frac{1}{2}} \leq \left(e^{-4(\frac{1}{2}-\epsilon_t)^2}\right)^{\frac{1}{2}} = e^{-2(\frac{1}{2}-\epsilon_t)^2}.$$

Putting it all together

Step 3 showed that $Z_t \leq e^{-2(\frac{1}{2}-\epsilon_t)^2}$. Because both sides are positive, the product over t doesn't change this inequality,

$$\prod_{t=1}^{T} Z_t \;\leq\; \prod_{t=1}^{T} e^{-2(\frac{1}{2}-\epsilon_t)^2} \;= e^{-2\sum_{t=1}^{T}(\frac{1}{2}-\epsilon_t)^2}.$$

From the earlier steps we showed $\prod_{t=1}^{T} Z_t$ was also an *upper bound*,

$$\text{training error} \;=\; \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{y_i \neq f_{boost}(x_i)\} \;\leq\; \prod_{t=1}^{T} Z_t \;\leq e^{-2\sum_{t=1}^{T}(\frac{1}{2}-\epsilon_t)^2}.$$

The two ends of this chain is what we set out to prove.

TRAINING VS TESTING ERROR

Q: Driving the training error to zero leads one to ask, does boosting overfit?

A: Sometimes, but very often it doesn't!