A LOOP GROUP EXTENSION OF THE ODD CHERN CHARACTER

SCOTT O. WILSON

ABSTRACT. We show that the universal odd Chern form, defined on the stable unitary group U, extends to the loop group LU as an equivariantly closed differential form. This provides an odd analogue to the Bismut-Chern form that appears in supersymmetric field theories. We also describe the associated transgression form, the so-called Bismut-Chern-Simons form, and explicate some properties it inherits as a differential form on the space of maps of a cylinder into the stable unitary group.

As one corollary, we show that in a precise sense the spectral flow of a loop of self adjoint Fredholm operators equals the lowest degree component of the Bismut-Chern-Simons form, and the latter, when restricted to cylinders which are tori, is an equivariantly closed extension of spectral flow. As another corollary, we construct the Chern character homomorphism from odd K-theory to the periodic cohomology of the free loop space, represented geometrically on the level of differential forms.

CONTENTS

1. INTRODUCTION

The Chern character homomorphism plays a fundamental role in topology and geometry as it relates K-theory to ordinary cohomology. Several recent works have emphasized differential refinements of such cohomology theories [HS], [BS], where one works with geometric representatives, and the data of differential forms. In [TWZ3], the authors show that a careful study of the odd Chern Character, represented geometrically by an odd differential form on the stable unitary group U, leads to a rather explicit and elegant differential refinement of odd K-theory.

In the setting of even K-theory, an analogous role is played by the even degree Chern form, which is associated to a connection ∇ on a bundle over a manifold M.

2010 *Mathematics Subject Classification.* 19L10, 58J28, 22E67.
Key words and phrases. Chern Character, K-theory, loop groups.

The even Chern form on M has been shown by Bismut to extend to a differential form on the free loop space LM, which is closed with respect to an equivariant-type differential [B]. Bismut introduced this differential form as a contribution to the integrand of a path integral over the free loop space which calculates (non-rigorously) the index of a certain twisted Dirac operator. Subsequently, in [Ha] and [ST], the Bismut-Chern form has been given a rigorous field theoretic interpretation, appearing naturally in the dimensional reduction from a $1|1$ supersymmetric Euclidean field theory on M, to a $0|1$ field theory on LM.

This leads to the natural question of whether the odd Chern form on U extends to a differential form on the loop group LU in a way that is closed with respect to the equivariant-type differential. We answer this question affirmatively by constructing this *odd Bismut-Chern* form using new *equivariant* transgression techniques for free loopspaces that we establish below. Interestingly, this differential form on LU is given by an explicit iterated integral formula involving only the left invariant 1-form on U and its directional derivatives.

There are several reasons for studying this differential form on the loop group LU. One reasons is that it provides the universal form required to produced a differential refinement of odd K-theory by differential forms on the free loopspace. This idea, or rather its even K-theoretic analogue, was initiated in [TWZ2] and more recently was completed in [BNV] using sheaves of spectra to obtain a full-fledged differential cohomology theory which is refined by differential forms on the free loopspace of M, rather than M itself.

Another reason for interest in this differential form, and its transgression form, is for its connections with index theory and the spectral flow along families of Fredholm operators. This result is explained below. Finally, it is expected that this differential form on the loop groups LU will have a field theoretic interpretation as well.

The contents of this paper are as follows. In the next section we provide preliminary definitions and results concerning the complex of differential forms on free loopspaces that is used in the paper. These are somewhat scattered throughout the literature and it seems useful to have one comprehensive reference. In section 3 we provide background on the Bismut-Chern form, as well as a re-interpretation as a universal form on the free loopspace LBU. The next section provides a general method for constructing transgression forms on free loopspaces, and defines the transgression form associated to the Bismut-Chern form, first introduced in [TWZ2]. We call this the Bismut-Chern-Simons form since it restricts along constant loops to the Chern-Simons form.

In section 5 we provide a brief review of the odd Chern form on U and in the subsequent sections define the odd Bismut-Chern form and prove several fundamental properties. In particular, we give an explicit formula for the transgression of this form, which is an even differential form on the space of maps of cylinder into U, whose value in degree zero is a lift of the "winding number" function. As we explain, it also agrees in lowest degree with the spectral flow of one parameter family of self adjoint Fredholm operators [P], and as a corollary we obtain an equivariantly-closed extension of the spectral flow to the torus mapping space of self adjoint Fredholm operators . We close with an application which produces a lift of the the odd Chern Character homomorphism $Ch : K^{-1} \to H^{odd}$ to a natural transformation with image in the the completed periodic cohomology of the free loopspace.

Acknowledgments. I would like to thank Thomas Tradler and Mahmoud Zeinalian for helpful discussions concerning the topics of this paper. The author was supported in part by The City University of New York PSC-CUNY Research Award Program.

2. Preliminaries

For a smooth manifold M, let LM denote the space of smooth loops in M, considered as a diffeological (Chen) space [C1], [C2], or as a Fréchet space, [H]. In fact, these structures agree, [L], [Wa] Lemma A.1.7.

The free loop space LM has a natural vector field, given by the circle action, whose induced contraction operator on differential forms is denoted by ι. We use the notation $\Omega^j_{d+\iota}$ to denote the vector space of invariant complex valued differential forms of degree j, i.e. those that are in the kernel of $d\iota + \iota d$.

Let

$$\Omega^{even}_{d+\iota}(LM) = \prod_{k\geq 0} \Omega^{2\mathrm{k}}_{d+\iota}(LM), \quad \Omega^{odd}_{d+\iota}(LM) = \prod_{k\geq 0} \Omega^{2\mathrm{k}+1}_{d+\iota}(LM).$$

The operator $d+\iota$ defines a $\mathbb{Z}_2$-graded complex and we denote the cohomology by

$$H^{\mathrm{even}}_{d+\iota}(LM), \quad H^{\mathrm{odd}}_{d+\iota}(LM).$$

The earliest reference for these groups are [W], [A], and [B], with more recent related work in [KM].

In [JP] the authors show these cohomology groups are isomorphic to the *completed periodic equivariant cohomology* $h^*_{S^1}(LM)$ in even and odd degrees, respectively. Recall this is the cohomology of the *completed periodic equivariant complex* which is defined to be the differential graded ring $(\Omega(LM)[[u,u^{-1}]], d+u\iota)$ consisting of formal power series in u, u^{-1} with coefficients in $\Omega_{d+\iota}(LM)$, where u is an indeterminant of degree 2. Note this allows for elements with arbitrary high powers of u^{-1}, but not u, since forms are concentrated in positive degrees. All of the results in this paper can be restated in terms in the completed periodic equivariant complex by introduction of this formal variables u^{-1}, and the complex with differential $d+\iota$ is recovered by setting $u=1$.

Finally, it follows from Theorem 2.1 of [JP] that the inclusion of constant loops $\rho : M \to LM$ induces an isomorphism

$$\rho^* : h^*_{S^1}(LM) \to u^{-1}H^*_{S^1}(M) = H^*(M) \otimes \mathbb{C}[u,u^{-1}]$$

where $u^{-1}H^*_{S^1}(M)$ is the localization with respect to u of the usual equivariant cohomology of M (with the trivial circle action). In summary, we have (for each $n \geq 0$) that the restriction to constant loops induces an isomorphism

$$H^{even}_{d+\iota}(LM) \cong \left(H^*(M) \otimes \mathbb{C}[u,u^{-1}]\right)_{2n} \cong H^{even}(M)$$
$$H^{odd}_{d+\iota}(LM) \cong \left(H^*(M) \otimes \mathbb{C}[u,u^{-1}]\right)_{2n+1} \cong H^{odd}(M).$$

So we obtain an expression of this equivariant-type cohomology of the free loopspace in terms of the ordinary cohomology of M.

3. The even Bismut-Chern form

It was first shown in [B] that the even degree Chern form of a connection on a bundle extends to an $(d+\iota)$-closed even differential form on the free loopspace of the base. This works was revisited in [GJP], in the context of cyclic homology. In [TWZ], and [TWZ2], the authors re-interpreted the now-called Bismut-Chern as a $(d+\iota)$-closed extension of the trace of holonomy function on the loopspace, and developed several properties with respect to sum and tensor products of connections.

We review this material in this section.

Proposition 3.1. *([B]) For any connection ∇ on a complex vector bundle $E \to M$, there is a differential form $BCh(\nabla) \in \Omega^{even}(LM)$ satisfying*

$$(d+\iota)BCh(\nabla) = 0,$$

which therefore determines a class $[BCh(\nabla)] \in \Omega^{even}_{S^1}(LM)$. Moreover, we have

$$\rho^* BCh(\nabla) = Ch(\nabla)$$

where $\rho^* : \Omega^{even}_{S^1}(LM) \to \Omega^{even}(M)$ is the restriction to constant loops and $Ch(\nabla) = Tr\left(e^R\right) \in \Omega^{even}(M)$ is the ordinary even Chern character.

Let us explain how the differential form $BCh(\nabla)$ is defined using local formulas. On any single local trivialization V of M, we can write the connection locally as a matrix A of 1-forms, with curvature R. There is a degree $2k$ form $Tr(hol^V_{2k})$ on LV given by

$$Tr(hol^V_{2k}) = Tr\left(\sum_{m\geq k}\ \sum_{1\leq j_1<\cdots<j_k\leq m} \int_{\Delta^m} X_1(t_1)\cdots X_m(t_m)dt_1\cdots dt_m\right), \tag{3.1}$$

where

$$X_j(t_j) = \begin{cases} R(t_j) & \text{if } j \in \{j_1,\dots,j_k\} \\ \iota A(t_j) & \text{otherwise} \end{cases}$$

Here $R(t_j)$ is a 2-form taking in two vectors at $\gamma(t_j)$ on a loop $\gamma \in V$, and $\iota A(t_j) = A(\gamma'(t_j))$. Note that $Tr(hol_0)$ is simply the trace of the usual holonomy, and heustically $Tr(hol^V_{2k})$ is given by the same formula for the trace of holonomy, except with exactly k copies of the function ιA replaced by the 2-form R, summed over all possible places.

More generally, a global form is defined on LM is defined using a open covering of LM induced by an open covering of M over which the bundle is locally trivialized. The formula above, essentially glued together by transition data, is shown to define an even differential form on LM, independent of choice of local data, which we denote by $Tr(hol_{2k})$, [TWZ]. Then, by definition,

$$BCh(\nabla) = \sum_{k\geq 0} Tr(hol_{2k}).$$

We have omitted the scalar factor $1/2\pi i$ that is sometimes used in the definition of the even Chern character for its relation to integral cohomology. By replacing R by $R/2\pi i$ in the definition of the Bismut-Chern form, we obtain a form that restricts to $Tr\left(e^{R/2\pi i}\right)$, and is closed with respect to $d + (2\pi i)\iota$. This operator produces a complex that is isomorphic to the one used here. In [TWZ2] the following properties are proved.

Theorem 3.2. *([TWZ2], Theorem 3.3) Let $(E,\nabla) \to M$ and $(\bar{E},\bar{\nabla}) \to M$ be complex vector bundles with connections. Let $\nabla \oplus \bar{\nabla}$ be the induced connection on $E \oplus \bar{E} \to M$, and $\nabla \otimes \bar{\nabla} := \nabla \otimes Id + Id \otimes \bar{\nabla}$ be the induced connection on $E \otimes \bar{E} \to M$. Then*

$$BCh(\nabla \oplus \bar{\nabla}) = BCh(\nabla) + BCh(\bar{\nabla})$$

and

$$BCh(\nabla \otimes \bar{\nabla}) = BCh(\nabla) \wedge BCh(\bar{\nabla}).$$

Remark 3.3. A complex vector bundle $E \to M$ of rank k with hermitian connection ∇ can always be given by the pullback along a map $M \to BU(k)$ of the universal bundle with universal connection over $BU(k)$, [NR], [NR2]. Here our model for $BU(k)$ is the limit of the Grassmann manifolds of k-planes in $\mathbb{C}^n$, with tautological bundles over them given by vectors in the k-planes. These bundles sit inside the trivial bundles with trivial connection given by exterior d, and the tautological sub-bundle has an induced connection given by the composition of projection P onto the k-plane composed with d, i.e. $\nabla = P \circ d$. The curvature of this connection is given by $R = P(dP)^2$. In this way we can describe the universal Bismut-Chern form on $LBU(k)$. An explicit formula is given by

$$Tr\left(\sum_{\substack{2k\geq 0\\ m\geq k}} \sum_{1\leq j_1<\cdots<j_{2k}\leq m} \int_{\Delta^m} X_1(t_1)\cdots X_m(t_m)dt_1\cdots dt_m\right), \tag{3.2}$$

where

$$X_j(t_j) = \begin{cases} P(dP)^2(t_j) & \text{if } j \in \{j_1,\dots,j_{2k}\} \\ \iota P(dP)(t_j) & \text{otherwise} \end{cases}$$

4. Transgression forms on free loop spaces

Assigning a differential form to some geometric data on a manifold often applies as well to a compact one-parameter family of the data. By integration along the family, we obtain a form of degree one lower whose exterior derivative is the difference between the differential forms assigned to the endpoint data. We explain now that this can also be accomplished for the free loopspace LM and the operator $d+\iota$.

Let $I=[0,1]$. There is a map $j: LM \times I \to L(M\times I) = LM \times LI$ which is induced by the inclusion $I \to LI$ of constant loops. Note that $j^*: \Omega^*(L(M\times I)) \to \Omega^*(LM\times I)$ commutes with the contraction operators ι and exterior derivative d so there is a well defined map $j^*: \Omega^*_{d+\iota}(L(M\times I)) \to \Omega^*_{d+\iota}(LM)$.

Consider the composition

$$\begin{array}{ccc} \Omega^*(LM\times I) & \xleftarrow{j^*} & \Omega^*(L(M\times I)) \\ {\scriptstyle\int_I}\big\downarrow & & \\ \Omega^*(LM) & & \end{array}$$

where $\int_I$ is integration along the fiber I. By Stokes' Theorem for integration along fiber, and the fact that for all forms ω we have

$$\iota\int_I j^*\omega = \int_I \iota j^*\omega = \int_I j^*\iota\omega,$$

it follows that we have the following *equivariant Stokes' formula* for integration along the fiber

$$(d+\iota)\int_I j^*\omega = \int_{\partial I} j^*\omega + \int_I j^*(d+\iota)\omega.$$

To our knowledge, this formula is new and of independent interest.

Definition 4.1. Let ∇_s be a path of connections on a bundle $E \to M$, regarded as a bundle with connection over $M \times I$. We define the Bismut-Chern-Simons form $BCS(\nabla_s) \in \Omega^{odd}_{d+\iota}(LM)$ by

$$BCS(\nabla_s) = \int_I j^* BCh(\nabla_s).$$

From the discussion above, and the fact that BCh is $(d+\iota)$-closed, we have

$$(d+\iota)BCS(\nabla_s) = BCh(\nabla_1) - BCh(\nabla_0), \tag{4.1}$$

and this implies $(d\iota + \iota d)BCS(\nabla_s) = 0$.

An explicit formula for $BCS(\nabla_s)$ was previously given (and essentially guessed) in [TWZ2]. That formula agrees with the definition above which now uses the equivariant fiber integration. For completeness, we recall that formula here, again using local formulas. On any local trivialization $V \subset M$, the degree $2k+1$ component of this odd differential form on $LV \subset LM$ is given by

$$BCS^V_{2k+1}(A_s) = Tr\Bigg(\sum_{n \geq k+1} \sum_{\substack{r, j_1, \dots, j_k = 1 \\ \text{pairwise distinct}}}^{n} \int_0^1 \int_{\Delta^n} \iota A_s(t_1) \dots R_s(t_{j_1} \dots A'_s(t_r) \dots R_s(t_{j_k}) \dots \iota A_s(t_n) \quad dt_1 \dots dt_n ds \Bigg) \tag{4.2}$$

Here A_s is the local expression of the connection with curvature R_s and time derivative $A'_s = \frac{\partial A_s}{\partial s}$. A similar formula can be obtained, using transition data, for expressing the entire odd form $BCS(\nabla_s)$ on LM, see [TWZ2] where the following is also proved.

Proposition 4.2. *For any path of connections ∇_s on $E \to M$, the differential form $BCS(\nabla_s)$ is an element of $\Omega^{odd}_{d+\iota}(LM)$ and satisfies*

$$\rho^* BCS(\nabla_s) = CS(\nabla_s),$$

where $\rho^* : \Omega^*_{d+\iota}(LM) \to \Omega^*(M)$ is the restriction to constant loops and

$$CS(\nabla_s) = \mathrm{Tr}\left(\int_0^1 \sum_{n \geq 0} \frac{1}{(n+1)!} \sum_{i=1}^{n+1} \overbrace{R_s \wedge \dots \wedge R_s \wedge \underbrace{(\nabla_s)'}_{i^{th}} \wedge R_s \wedge \dots \wedge R_s}^{n+1 \text{ factors}} \right) ds$$

is the Chern-Simons form associated to ∇_s.

Moreover the following multiplicative properties hold, see [TWZ3].

Theorem 4.3. *Let $E \to M$ and $\bar{E} \to M$ be two complex vector bundles, each with a path of connections (E, ∇_s) and $(\bar{E}, \bar{\nabla}_s)$ for $s \in [0,1]$, respectively. Let $\nabla_s \oplus \bar{\nabla}_s$ be the induced path of connections on $E \oplus \bar{E}$, and let $\nabla_r \otimes \bar{\nabla}_s := \nabla_r \otimes Id + Id \otimes \bar{\nabla}_s$ be the induced connections on $E \otimes \bar{E}$ for any $r, s \in [0,1]$. Then*

$$BCS(\nabla_s \oplus \bar{\nabla}_s) = BCS(\nabla_s) + BCS(\bar{\nabla}_s)$$

and

$$BCS(\nabla_0 \otimes \bar{\nabla}_s) = BCh(\nabla_0) \wedge BCS(\bar{\nabla}_s) \qquad BCS(\nabla_s \otimes \bar{\nabla}_1) = BCS(\nabla_s) \wedge BCh(\bar{\nabla}_1).$$

Finallly, for two composable paths of connections ∇_s and $\bar{\nabla}_s$, where $\nabla_1 = \bar{\nabla}_0$, we have

$$BCS(\nabla_s * \bar{\nabla}_s) = BCS(\nabla_s) + BCS(\bar{\nabla}_s).$$

Remark 4.4. As in Remark 3.3, we may regard this Bismut-Chern-Simons form $BCS(\nabla_s)$ as induced by a map $\phi : M \times I \to BU(k)$, and by the adjoint property there is a universal form on the space of cylinders in $BU(k)$. Namely, $BCS \in \Omega^{\mathrm{odd}}_{d+\iota}(PLBU(k))$ is defined by

$$BCS = \int_I ev_t^* BCh$$

where $ev_t : PLBU \times I \to LBU$ is evaluation at time $t \in I$. To see that this agrees with Definition 4.1, let $\tilde{\phi} : M \to PBU$ denote the adjoint mapping and note that the following diagram commutes

$$\begin{tikzcd}
\Omega^*(LBU) \arrow[r, "ev_t^*"] \arrow[d, "(L(\phi))^*"] & \Omega^*(LPBU \times I) \arrow[r, "\int_I"] \arrow[d, "(L(\tilde{\phi}) \times id)^*"] & \Omega^*(LPBU) \arrow[d, "(L(\tilde{\phi}))^*"] \\
\Omega^*(L(M \times I)) \arrow[r, "j^*"] & \Omega^*(LM \times I) \arrow[r, "\int_I"] & \Omega^*(LM)
\end{tikzcd}$$

Remark 4.5. In [TWZ3] the authors show the Bismut-Chern-Simons form may be used to define a refinement of differential K-theory. We note that the construction applies to the category of S^1-spaces mapping into $L(BU \times \mathbb{Z})$ with universal Bismut Chern form BCh, c.f. Remark 3.3. Then the loop differential K-theory of [TWZ3] is the special case of taking the free loop of a bundle with connection $M \to BU \times \mathbb{Z}$. A sheafified version of this construction was also given in [BNV], producing a differential cohomology theory which refines K-theory of M by differential forms on the free loopspace of M.

5. The odd Chern form

Let U denote the stable unitary group defined as the limit of the finite unitary groups $U(n)$. This is filtered by finite dimensional manifolds and so defines a diffeological space where a finite dimensional plot is a smooth map with image in some $U(n)$. The group U has a left invariant 1-form with values in the Lie algebra $\mathfrak{u}$, which we denote by ω. This generates the canonical bi-invariant closed odd differential form on U given by

$$Ch = \mathrm{Tr} \sum_{n \geq 0} \frac{(-1)^n n!}{(2n+1)!} \omega^{2n+1} \in \Omega^{odd}_{cl}(U; \mathbb{C}).$$

For a smooth map $g : M \to U$, the odd Chern form of g is defined by pullback to be

$$Ch(g) = g^*(Ch) = \mathrm{Tr} \sum_{n \geq 0} \frac{(-1)^n n!}{(2n+1)!} (g^{-1}dg)^{2n+1}.$$

We remark that the scalar factor of $1/(2\pi i)^{n+1}$ is omitted to be consistent with our convention for the even Chern form. These factors may be introduced as mentioned above, where for LM below we modifying the complex of forms to have operator $d + (2\pi i)\iota$. The following lemma explains how the odd Chern form is related to the Chern-Simons form of Proposition 4.2.

Lemma 5.1. *For any $g \in Map(M, U)$ and any n such that $g(M) \subset U(n)$ we have*

$$Ch(g) = CS(d + sg^{-1}dg),$$

where $d + sg^{-1}dg$ is the straight path of connections on the trivial $\mathbb{C}^n$-bundle over M, from the trivial connection d to the gauge equivalent flat connection $g^*(d) = d + g^{-1}dg$.

The universal version of the statement is as follows: the trivial $\mathbb{C}^\infty$-bundle over the stable unitary group comes with a gauge transformation given by left action by g on the fiber over g. The universal Chern form Ch on U equals the Chern Simons form of the straightline path of connections from the trivial connection d to the transform of d by the gauge transformation.

Proof. Let $A = g^{-1}dg$ and $A^s = sg^{-1}dg$ so that $(A^s)' = A$ and $R^s = -s(1-s)A \cdot A$. Then

$$\begin{aligned}
CS(d + sg^{-1}dg) &= \mathrm{Tr} \sum_{n \geq 0} \frac{1}{(n+1)!} \sum_{i=1}^{n+1} \int_0^1 R^s \dots \underbrace{(A^s)'}_{i\text{-th}} \dots R^s ds \\
&= \sum_{n \geq 0} \frac{1}{n!} \mathrm{Tr} \int_0^1 (-s(1-s))^n \underbrace{A \dots A}_{2n+1 \text{ factors}} \; ds \\
&= \sum_{n \geq 0} \frac{(-1)^n}{n!} \frac{n!n!}{(2n+1)!} \mathrm{Tr}(A^{2n+1}) \\
&= \sum_{n \geq 0} \frac{(-1)^n n!}{(2n+1)!} \mathrm{Tr}(A^{2n+1}) \\
&= Ch(g)
\end{aligned}$$

where the identity $\int_0^1 s^k (1-s)^\ell ds = \frac{k!\ell!}{(k+\ell+1)!}$ was used. $\square$

It is straightforward the check that the odd Chern map

$$Ch : Map(M,\ U) \to \Omega_{cl}^{\mathrm{odd}}(M)$$

is a monoid homomorphism, i.e.

$$Ch(g \oplus h) = Ch(g) + Ch(h),$$

and furthermore satisfies $Ch(g^{-1}) = -Ch(g)$.

The odd Chern form $Ch(g)$ depends on g, but for a smooth family $g_t : M \times I \to U(n) \subset U$, there is a smooth even differential form that interpolates between $Ch(g_1)$

and $Ch(g_0)$, in the following way. We define $CS(g_t) \in \Omega^{\text{even}}(M)$ by

$$CS(g_t) = \int_I Ch(g_t), \tag{5.1}$$

where $Ch(g_t) \in \Omega^{\text{odd}}_{cl}(M \times I)$ and $\int_I$ is the integration along the fiber

$$\begin{tikzcd} M \times I \arrow[r, "g_t"] \arrow[d, "\int_I"'] & U \\ M & \end{tikzcd}$$

By Stokes' theorem we have

$$dCS(g_t) = d\int_I Ch(g_t) = \int_{\partial I} Ch(g_t) - \int_I dCh(g_t) = Ch(g_1) - Ch(g_0)$$

since $dCh(g_t) = 0$. By the usual adjunction, our definitions above give a universal form $CS \in \Omega^{even}(PU)$ where PU is the smooth pathspace of U. It follows that the restriction of this form along the inclusion $LU \to PU$ of the free loopspace gives a closed even degree form $CS \in \Omega^{even}(LU)$.

The following explicit formula will be useful. For a proof, see for example [TWZ3].

Lemma 5.2. *For any $g_t \in Map(M \times I, U)$ the* odd Chern Simons form $CS(g_t) \in \Omega^{even}(M)$ *associated to g_t is*

$$CS(g_t) = \mathrm{Tr} \sum_{n \geq 0} \frac{(-1)^n n!}{(2n)!} \int_0^1 (g_t^{-1} g_t') \cdot (g_t^{-1} dg_t)^{2n} dt.$$

We restate the fundamental property for $CS(g_t)$ here, along with several others, whose proofs are immediate from the definitions and Lemma 5.2.

Proposition 5.3. *For any paths $g_t, h_t \in Map(M \times I, U)$ we have*

$$\begin{aligned} dCS(g_t) &= Ch(g_1) - Ch(g_0) \\ CS(g_t \oplus h_t) &= CS(g_t) + CS(h_t) \\ CS(g_t^{-1}) &= -CS(g_t). \end{aligned}$$

*If g_t and h_t can be composed (*i.e. *if $g_1 = h_0$), then the composition $g_t * h_t$ satisfies*

$$CS(g_t * h_t) = CS(g_t) + CS(h_t).$$

Finally, if $g_t : M \to \Omega_* U$, then the degree zero component of $CS(g_t)$ is the integer equal to the winding number.

Defining odd K-theory by $K^{-1}(M) = [M, U]$, it follows that there is an induced homomorphism $Ch : K^{-1}(M) \to H^{\text{odd}}(M)$, which is a geometric representation of the odd Chern Character, which is an isomorphism after tensoring with $\mathbb{C}$.

6. The odd Bismut-Chern form

Let LU be the free loop space of U. Motivated by Lemma 5.1, we make the following definition.

Definition 6.1. Let $BCh \in \Omega^{odd}(LU)$ be defined by

$$BCh = BCS(d + s\omega),$$

where $d + s\omega$ is the path of connections on the trivial $\mathbb{C}^n$-bundle over $U(n)$. We refer to this as the universal odd Bismut-Chern form.

We note that this is well defined since it passes to the limit with respect to n.

Theorem 6.2. *The universal odd Bismut-Chern form is closed, i.e.*

$$(d + \iota)BCh = 0,$$

so that $BCh \in \Omega^{odd}_{d+\iota}(LU)$.

Also, BCh *is an extension to* LU *of the universal odd Chern form* $Ch \in \Omega^{odd}_{cl}(U)$, *in the sense that*

$$\rho^*(BCh) = Ch,$$

where $\rho : U \to LU$ *is the inclusion of constant loops.*

Proof. From Equation 4.1 we have

$$(d + \iota)BCh = (d + \iota)\left(BCS(d + s\omega)\right) = BCh(d + \omega) - BCh(d) = 0$$

where the last equality holds since $d = d + 0$ and $d + \omega$ are gauge equivalent via the left action. (This statement becomes more familiar in the notation $\omega = g^{-1}dg$, where $g^{-1}dg = g^{-1}0g + g^{-1}dg$.) For the second statement we have

$$\rho^* BCS(d + s\omega) = CS(d + s\omega) = Ch$$

where the first equality is Proposition 4.2 and the second equality is Lemma 5.1. □

Letting BCh_γ denote the value of BCh at $\gamma \in LU$, we see from Equation 4.2 where $A_s = s\omega$, $R_s = -s(1-s)\omega^2$ and $A'_s = \omega$, that an explicit formula for the degree $2k+1$ part of BCh is given by

$$Tr\left(\sum_{n \geq k+1} \frac{(-1)^k (n-1)! k!}{(n+k)!} \sum_{\substack{r, j_1, \ldots, j_k = 1 \\ \text{pairwise distinct}}}^{n} \int_{\Delta^n} \iota\omega(t_1) \ldots \omega^2(t_{j_1}) \ldots \omega(t_r) \ldots \omega^2(t_{j_k}) \ldots \iota\omega(t_n) \quad dt_1 \ldots dt_n \right). \tag{6.1}$$

Lemma 6.3. *Let* BCh_γ *denote the value of* BCh *at* $\gamma \in LU$. *Then the following properties hold*

$$BCh_{\gamma_1 \oplus \gamma_2} = BCh_{\gamma_1} + BCh_{\gamma_2}$$

and

$$BCh_{\gamma_1 * \gamma_2} = BCh_{\gamma_1} + BCh_{\gamma_2}$$

Proof. These are immediate from Definition 6.1 and Theorem 4.3. □

7. Transgression, spectral flow, and odd K-theory

The form $BCh \in \Omega^{odd}_{d+\iota}(LU)$ has a transgression form $BCS \in \Omega^{even}_{d+\iota}(PLU)$ defined by

$$BCS = \int_I ev_t^* BCh,$$

where $ev_t : PLU \times I \to LU$ is evaluation at time t. By the equivariant Stokes formula of section 3 we have

$$(d+\iota)BCS = ev_1^* BCh - ev_0^* BCh.$$

Notice that this equation implies that $BCS \in \Omega^{even}_{d+\iota}(PLU)$. It follows that the restriction to $L^2U = LLU \subset PLU$ produces a $(d+\iota)$-closed form of even degree, which we also denote by $BCS \in \Omega^{even}_{d+\iota}(L^2U)$.

By a commutatitive diagram argument similar to Remark 4.4 we see that for a map $g_t : M \times I \to U$ we have $BCS(g_t) \equiv g_t^* BCS$ is given by

$$\int_I j^* BCh(g_t),$$

where $j : LU \times I \to L(U \times I)$ is inclusion of constant loops on I.

An explicit formula for the degree $2k$ component of BCS_γ is similar to Equation 6.1, with an additional sum over possible $\partial/\partial s$ derivatives of any $\omega^2(t_i, s)$ or $\omega(t, s)$:

$$\begin{aligned}(7.1)\quad Tr\Bigg(&\sum_{n \geq k+1} \frac{(-1)^k (n-1)! k!}{(n+k)!} \sum_{\substack{r, j_1, \ldots, j_k = 1 \\ \text{pairwise distinct}}}^{n} \\ &\int_I \int_{\Delta^n} \iota\omega(t_1) \ldots \omega^2(t_{j_1}) \ldots \frac{\partial}{\partial s}\omega(t_r) \ldots \omega^2(t_{j_k}) \ldots \iota\omega(t_n) \\ + \sum_{1 \leq i \leq k} &\int_I \int_{\Delta^n} \pm \iota\omega(t_1) \ldots \omega^2(t_{j_1}) \ldots \omega(t_r) \ldots \frac{\partial}{\partial s}\omega^2(t_{j_i}) \ldots \omega^2(t_{j_k}) \ldots \iota\omega(t_n) \quad ds dt_1 \ldots dt_n \Bigg)\end{aligned}$$

where the sign $\pm$ is positive if $r < j_i$ and negative if $r > j_i$.

Proposition 7.1. *Let $CS \in \Omega^{even}(PU)$ denote the universal (even) Chern-Simons form and let $\rho : Map(I, U) \to Map(S^1 \times I, U)$ be the inclusion of paths in U which are constant in the S^1-direction. Then*

$$\rho^*(BCS) = CS.$$

Consequently, if $\gamma : S^1 \times S^1 \to U$ is independent of the first S^1-factor, then the degree 0 component of $BCS(\gamma)$ equals the winding number of the induced map $S^1 \to U$ obtained by collapsing the first factor of the torus, i.e. the following diagram commutes

$$\begin{tikzcd} L^2U \arrow[r, "BCS_0"] & \mathbb{R} \\ LU \arrow[u, hook] \arrow[r, "W"] & \mathbb{Z} \arrow[u, hook] \end{tikzcd}$$

where W is the winding number of the determinant.

Proof. The first statement follows from the fact that restriction is compatible with integration along the fiber, as well as Theorem 6.2 and Equation 5.1. The second statement follows from the first, in light of the last result in Proposition 5.3. $\square$

One knows from [AS] and [P] that there is a well defined "spectral-flow" function

$$\mathrm{sf} : L\left(\mathcal{F}^*_{sa}\right) \to \mathbb{Z}$$

where $\mathcal{F}^*_{sa}$ is the space of self-adjoint Fredholm operators on a Hilbert space whose essential spectrum contain positive and negative parts, and L is the free loopspace (functor). Moreover, there is a homotopy equivalence $\phi : \mathcal{F}^*_{sa} \to U$ which makes the diagram

$$\begin{tikzcd} L\left(\mathcal{F}^*_{sa}\right) \arrow[dr, "\mathrm{sf}"] \arrow[dd, "L(\phi)"'] & \\ & \mathbb{Z} \\ LU \arrow[ur, "W"'] & \end{tikzcd}$$

commute, where W is the winding number of the determinant. The map ϕ, whose details we needn't reproduce here, is essentially the exponential map composed with a nice weak deformation retraction.

Corollary 7.2. *The spectral flow sf* $: L\left(\mathcal{F}^*_{sa}\right) \to \mathbb{Z}$*, regarded as a function*

$$sf \in \Omega^0\left(L\left(\mathcal{F}^*_{sa}\right)\right)$$

extends to a $(d+\iota)$*-closed differential form on* $LL\left(\mathcal{F}^*_{sa}\right) \supset L\left(\mathcal{F}^*_{sa}\right)$*.*

Proof. Consider the diagram

$$\begin{tikzcd} L^2\left(\mathcal{F}^*_{sa}\right) \arrow[d, "L^2(\phi)"'] & L\left(\mathcal{F}^*_{sa}\right) \arrow[l, hook'] \arrow[d, "L(\phi)"] \\ L^2U & LU \arrow[l, hook'] \end{tikzcd}$$

where the horizontal maps are the inclusions of maps which are constant on the first S^1-factor. The differential form $BCS \in \Omega^{even}_{d+\iota}(L^2U)$ is closed and its restriction to LU equals the winding number in degree zero, which pulls back to the spectral flow on $L\left(\mathcal{F}^*_{sa}\right)$. Since the diagram commutes, the pullback of $BCS \in \Omega^{even}_{d+\iota}(L^2U)$ to $L^2\left(\mathcal{F}^*_{sa}\right)$ is a differential form which satisfies the claim. □

It would be interesting to have explicit calculations of these higher degree forms for such families of self adjoint Fredholm operators.

As an additional corollary to the construction of the even degree Bismut-Chern-Simons form, we have the following

Corollary 7.3. *There is a well defined group homomorphism*

$$[BCh] : K^{-1}(M) \to H^{odd}_{d+\iota}(LM)$$

defined by $[g] \mapsto BCh(g)$*, for* $g : M \to U$*, making the following diagram commute*

$$\begin{tikzcd} & H^{odd}_{d+\iota}(LM) \arrow[d, "\rho^*"] \\ K^{-1}(M) \arrow[ur, "{[BCh]}"] \arrow[r, "{[Ch]}"] & H^{odd}(M) \end{tikzcd}$$

where $[Ch] : K^{-1}(M) \to H^{odd}(M)$ *is the odd Chern character, and* ρ *is the restriction to constant loops.*

There is an even K-theoretic analogue of this statement, first proved in [Z], by completely different methods.

Proof. If g_t is a homtopy from g_0 to g_1 then

$$(d+\iota)BCS(g_t) = BCh(g_1) - BCh(g_0),$$

so $[BCh([g])] \in \Omega^{odd}_{d+\iota}(LM)$ is well defined. The map is a group homomorphism by Lemma 6.3 and the diagram commutes by Theorem 6.2. □

References

[A] M. F. Atiyah, "Circular symmetry and stationary phase approximation," Colloqueen l'honneur de Laurent Schwartz, Astérisque 131 (1985), 311 -323.

[AS] M. F. Atiyah, I.M. Singer, "Index theory for skew-adjoint Fredholm operators," Publ. Math. Inst. Hautes Etudes Sci. (Paris) 37 (1969) 5-26.

[B] J.-M. Bismut, "Index theorem and equivariant cohomology on the loop space," Commun. Math. Phys. 98 (1985), 213-237.

[BNV] U. Bunke, T. Nikolaus, M. Völkl, "Differential cohomology theories as sheaves of spectra," J. Homotopy Related Structures, Oct. 2014.

[BS] U. Bunke, T Schick, "Uniqueness of smooth extensions of generalized cohomology theories," J. Topol. 3 (2010), no. 1, 110-156.

[C1] K.-T. Chen, "Iterated path integrals," Bull. AMS 83, no. 5, (1977), 831-879.

[C2] K.-T. Chen, "Iterated integrals of differential forms and loop space homology," Ann. of Math. (2) 97 (1973), 217-246.

[CS] Chern, S.S. and Simons, James. "Characteristic Forms and Geometric Invariants," Annals of Mathematics. Vol. 99. No. 1. January 1974. pp. 48-69.

[GJP] E. Getzler, J. D. S. Jones, S. Petrack, "Differential forms on loop spaces and the cyclic bar complex," Topology 30 (1991), no. 3, 339-371.

[H] R. Hamilton, "The inverse function theorem of Nash and Moser," Bull. of AMS, Vol. 7, No. 1, 1982.

[Ha] F. Han, "Supersymmetric QFT, super loop spaces and Bismut-Chern Character," Thesis (Ph.D.) University of California, Berkeley. 2005. 61 pp. see also preprint:arXiv:0711.3862.

[HS] M.J. Hopkins, I.M. Singer, "Quadratic functions in Geometry, Topology, and M -theory," J. Diff. Geom 70. 2005. pp. 329-452.

[JP] J.D. Jones, S. Petrack, "The fixed point theorem in equivariant cohomology," Transactions of the American Mathematical Society, Vol. 322, No. 1 (Nov., 1990), pp. 35-49.

[KM] N. Kitchloo, J. Morava, "Thom prospectra for loopgroup representations," Elliptic cohomology, 214-238, London Math. Soc. Lecture Note Ser., 342, Cambridge Univ. Press, Cambridge, 2007.

[L] M. V. Losik, "Fréchet manifolds as diffeological spaces," Izv. Vyssh. Uchebn. Zaved. Mat., 1992, no. 5, 36-42.

[NR] M. Narasimhan and S. Ramanan, "Existence of Universal Connections," Amer. J. Math., 83(3):563–572, 1961.

[NR2] M. Narasimhan and S. Ramanan, "Existence of Universal Connections II," Amer. J. Math., 85(2):223–231, 1963.

[P] Phillips, J. "Self-adjoint fredholm operators and spectral flow," Canad. Math. Bull. Vol. 39 (4), 1996 pp. 460-467.

[ST] S. Stolz and P. Teichner, "Super symmetric Euclidean field theories and generalized cohomology," Mathematical foundations of quantum field theory and perturbative string theory, 279-340, Proc. Sympos. Pure Math., 83, Amer. Math. Soc., Providence, RI, 2011.

[TWZ] T. Tradler, S.O. Wilson, M. Zeinalian, "Equivariant holonomy for bundles and abelian gerbes," Comm. in Math. Physics., Vol. 315, Issue 1, 2012, pg 38-108.

[TWZ2] T. Tradler, S.O. Wilson, M. Zeinalian, "Loop differential K-theory," arxiv:1201.4593.

[TWZ3] T. Tradler, S.O. Wilson, M. Zeinalian, "An elementary differential extension of odd K-theory," J. of K-theory, K-theory and its Applications to Algebra, Geometry, Analysis and Topology, FirstView 04 April 2013, 1-31.

[Wa] K. Waldorf, "Transgression to Loop Spaces and its Inverse, I: Diffeological Bundles and Fusion Maps," Cah. Topol. Géom. Différ. Catég., 2012, Vol. LIII, 162-210.

[W] E. Witten, "Supersymmetry and Morse theory," J. Differential Geom. 17 (1982), 661-692.

[Z] Zamboni, "A Chern character in cyclic homology," Trans. Amer. Math. Soc. 331 (1992), no. 1, 157-163.

Scott O. Wilson, Department of Mathematics, Queens College, City University of New York, 65-30 Kissena Blvd., Flushing, NY 11367

E-mail address: scott.wilson@qc.cuny.edu