RECTIFYING PARTIAL ALGEBRAS OVER OPERADS OF COMPLEXES

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ABSTRACT. In [2] Kriz and May introduced partial algebras over an operad. In this paper we prove that, in the category of chain complexes, partial algebras can be functorially replaced by quasi-isomorphic algebras. In particular, partial algebras contain all of the important homological and homotopical information that genuine algebras do. Applying this result to McClure's partial algebra in [6] shows that the chains of a PL-manifold are quasi-isomorphic to an E_∞-algebra.

1. INTRODUCTION

In [2] Kriz and May introduced partial algebras over an operad and proved that, in the category of simplicial modules, such partial algebras are quasi-isomorphic to genuine algebras. It was left as an open question whether or not such a result also holds in the category of chain complexes. This is important given the recent work of McClure, showing that the chains of a PL-manifold form a partial algebra [6].

In this paper we prove that, in the category of chain complexes, partial algebras can be functorially replaced by quasi-isomorphic algebras. In particular, partial algebras contain all of the important homological and homotopical information that genuine algebras do. Applying this result to McClure's partial algebra in [6] shows that the chains of a PL-manifold are quasi-isomorphic to an $\mathcal{E}_\infty$-algebra. We describe further applications and consequences as well.

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2. PRELIMINARIES

In this section we give some definitions. We work in the category of flat chain complexes over some Dedekind ring R. By a simplicial complex we mean a simplicial object in the category of complexes (so such an object is bi-graded).

Definition 1. *An operad (of complexes over R) is a collection of complexes $\mathcal{O}(j)$ over R, $j \geq 0$, together with a unit map $\eta : R \to \mathcal{O}(1)$, an action of the symmetric group Σ_j on $\mathcal{O}(j)$ for each j, and chain maps*

$$\gamma : \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \to \mathcal{O}(j_1 + \cdots + j_k)$$

for all $k \geq 1$ and $j_i \geq 0$. The maps γ are required to be associative, equivariant with respect to the Σ-actions, and unital with respect to the unit η. See [2].

Intuitively, the component $\mathcal{O}(j)$ encodes operations with j inputs and one output. The maps γ determine the composition of operations. Morphisms of operads are defined naturally.

An algebra over an operad $\mathcal{O}$ is a complex X with chain maps

$$\mathcal{O}(j) \otimes_{R[\Sigma_j]} X^{\otimes j} \to X$$

sending the operad unit to the identity map of X, and satisfying the obvious axiom codifying an action with respect to operad composition. Here Σ_j acts on $\mathcal{O}(j) \otimes_{R[\Sigma_j]} A^{\otimes j}$ by σ on the left and σ^{-1} on the right.

To define a partial algebra we first introduce the notion of a *domain* on which an operad may partially act. This first appeared in [2].

Definition 2. *A domain in a complex X is a collection of subcomplexes*

$$i_j : X_j \to X^{\otimes j}$$

satisfying the following:

(1) $X_1 = X$.
(2) *For all $j = j_1 + \cdots + j_k$, X_j is a Σ_j-invariant subcomplex of $X_{j_1} \otimes \ldots \otimes X_{j_k}$, making the following diagram commute*

$$
\begin{array}{ccc}
X_j & \xrightarrow{i_{j_1,\ldots,j_k}} & X_{j_1} \otimes \ldots \otimes X_{j_k} \\
{\scriptstyle i_j}\downarrow & & \downarrow{\scriptstyle i_{j_1} \otimes \ldots \otimes i_{j_k}} \\
X^{\otimes j} & \xrightarrow{\cong} & X^{\otimes j_1} \otimes \ldots \otimes X^{\otimes j_k}
\end{array}
$$

(3) *The inclusion map $i_j : X_j \hookrightarrow X^{\otimes j}$ is a quasi-isomorphism.*

We remark that our flatness assumption and condition (3) imply that the inclusions $i_{j_1,\ldots,j_k}$ in condition (2) are quasi-isomorphisms.

A morphism f of domains $\{X_j\}$ and $\{Y_j\}$ is a collection of chain maps $f_j : X_j \to Y_j$ such that each map f_j equals the restriction of $f_1^{\otimes j}$ to X_j. We say f is a quasi-isomorphism if each f_j is a quasi-isomorphism. It follows from our flatness assumption, and the diagram

$$
\begin{array}{ccc}
X_j & \xrightarrow{f_j} & Y_j \\
\downarrow & & \downarrow \\
X^{\otimes j} & \xrightarrow{f_1^{\otimes j}} & Y^{\otimes j}
\end{array}
$$

that if f_1 is a quasi-isomorphism, then each f_j is a quasi-isomorphism.

Remark 3. *There is a functor L from domains to complexes taking $\{X_j\}$ to $X_1 = X$. There is also a functor R from complexes to domains taking X to the domain $X_j = X^{\otimes j}$, and $LR = id$.*

Definition 4. *Let $\mathcal{O}$ be an operad. A partial algebra over the operad $\mathcal{O}$ is a domain $\{X_j\}$ in a complex X and a collection of chain maps*

$$\Theta_j : \mathcal{O}(j) \otimes_{R[\Sigma_j]} X_j \to X$$

satisfying the following:

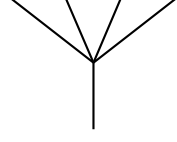(1) *The operad unit acts as the identity: $\Theta_1 \circ (\eta \otimes id_X) = id_X$.*

FIGURE 1. A tree as an operation.

FIGURE 2. The unit element.

(2) *For all $j = j_1 + \cdots + j_k$, the maps*

$$\Theta_{j_1,\ldots,j_k} : \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \otimes_{R[\Sigma_j]} X_j \to X^{\otimes k}$$

given by including X_j into $X_{j_1} \otimes \cdots \otimes X_{j_k}$, applying the shuffle, and then applying $\Theta_{j_1} \otimes \cdots \otimes \Theta_{j_k}$, must factor through X_k.

(3) *The maps Θ_j describe an action with respect to the operad composition. Namely, for all $j = j_1 + \cdots + j_k$,*

$$\Theta_j \circ (\gamma \otimes id_{X_j}) = \Theta_k \circ (id_{\mathcal{O}(k)} \otimes \Theta_{j_1,\ldots,j_k})$$

as maps from $\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \otimes X_j$ to X.

A morphism of partial algebras over an operad is a morphism of domains that commutes with the partial actions. We say a morphism of partial algebras is a quasi-isomorphism if it is a quasi-isomorphism of domains.

Remark 5. *An algebra over an operad is a partial algebra where the domain $\{X_j\}$ is given by $X_j = RX = X^{\otimes j}$.*

We now give a diagrammatic description of operads and their algebras. We represent elements of $\mathcal{O}(k)$ by trees with k inputs, as in Figure 1, and the unit in $\mathcal{O}(1)$ as in Figure 2. Implicit in this are the various structures of an operad over complexes: addition, the differential and the Σ_j-actions.

We represent a generator of $\mathcal{O}(k) \otimes_{R[\Sigma_k]} \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k)$ by a collection of k trees, as in Figure 3, where we have left spaces between trees to indicate this is a tensor product of elements of $\mathcal{O}$. Again various structures are implicit. In particular, the symmetric group acts on the bottom tree, and also by permuting the tensor factor of trees on top.

We represent a generator of

$$\mathcal{O} \boxtimes X = \sum_{k \geq 0} \mathcal{O}(k) \otimes_{R[\Sigma_k]} X^{\otimes k}$$

by a diagram consisting of a tree labeled by elements of X, as in Figure 4, where $x_1 \otimes \cdots \otimes x_k \in X^{\otimes k}$. There are still implicit notions of addition and differential, as well as the symmetric group actions. In particular, this picture is equivalent to the one obtained by acting on the tree by σ and on (the tensor product of) the k labeling elements by σ^{-1}, for all $\sigma \in \Sigma_k$.
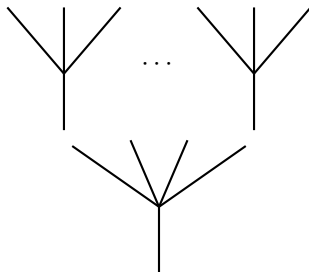
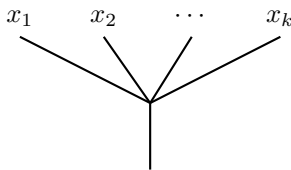FIGURE 3. A generator of $\mathcal{O}(k) \otimes_{R[\Sigma_k]} \mathcal{O}(j_1) \cdots \mathcal{O}(j_k)$.

FIGURE 4. A generator of $\mathcal{O}(k) \otimes_{R[\Sigma_k]} X^{\otimes k}$.

There is a categorical interpretation of operads as monads which allows one to make use of the two sided bar construction. The constructions and proofs in the next section involve minor variations of this bar construction. Rather than appealing to this categorical construction abstractly, we will unravel it explicitly in the case of partial algebras. We do this because it makes our work more transparent, and secondly, because it may be used to give a picture for the bar construction in more general situations.

Finally, some terminology. By the total complex of a simplicial complex $X_{q,k}$, with simplicial grading q and complex grading k, we mean the complex whose degree n is $\sum_{p+k=n} X_{q,p}$ and whose differential is equal to the sum of the simplicial differential $\sum (-1)^i \partial_i$ and $(-1)^q$ times the complex differential. Similarly, maps of simplicial complexes can be added along total degrees to give maps of total complexes.

3. Main Result

In this section we prove the following:

Theorem 6. *Let A be a flat complex and $\mathcal{O} = \bigoplus_{k \geq 0} \mathcal{O}(k)$ be an operad of complexes such that each $\mathcal{O}(k)$ is a projective $R[\Sigma_k]$-module. There is a functor W that assigns to any partial $\mathcal{O}$-algebra A_* an $\mathcal{O}$-algebra WA_* such that A_* and WA_* are quasi-isomorphic as partial $\mathcal{O}$-algebras.*

Let us first give an outline for the proof. We will construct a diagram

$$A_* \underset{\eta}{\overset{\varphi}{\rightleftarrows}} B_* \xrightarrow{\delta} W_*$$

where B_* is a partial $\mathcal{O}$-algebra and W_* is an $\mathcal{O}$-algebra (and therefore also a partial $\mathcal{O}$-algebra, by Remark 5). The maps η and φ are quasi-isomorphisms of domains. Moreover, φ is a morphism of partial $\mathcal{O}$-algebras, and therefore a quasi-isomorphism of partial $\mathcal{O}$-algebras. Finally, δ is a quasi-isomorphism of partial $\mathcal{O}$-algebras. The constructions of B_* and W_* from A_* will be seen to be natural, and the assignment $A_* \mapsto W_*$ will be the desired functor in the statement of the theorem.

The rest of this section is divided into subsections which complete the steps in this outline. Several of the techniques used appear in [2].

3.1. **Definition of the complex B.** First we define a simplicial complex associated to the partial algebra A, whose q-simplicies are denoted by B_q. This first appeared in [2] (Definition 3.2, Example 4.2). The reader may note that this simplicial complex is a minor variation on the two sided bar construction $B(\mathcal{O}, \mathcal{O}, C)$ where C is an $\mathcal{O}$-algebra.

Let A be a partial $\mathcal{O}$-algebra with domain A_* having inclusions $i_j : A_j \to A^{\otimes j}$ and $i_{\alpha_1,\dots,\alpha_k} : A_\alpha \to A_{\alpha_1} \otimes \dots \otimes A_{\alpha_k}$, where $\alpha = \alpha_1 + \dots + \alpha_k$. We let B_0 be the following subcomplex of $\mathcal{O} \boxtimes A$, induced by the domain A_* and the inclusion maps $id \otimes i_j$:

$$B_0 = \bigoplus_{j \geq 0} \mathcal{O}(j) \otimes_{R[\Sigma_j]} A_j$$

Next we define

$$B_1 = \bigoplus_{\substack{k \geq 0 \\ j_1,\dots,j_k \geq 0}} \mathcal{O}(k) \otimes_{R[\Sigma_k]} \mathcal{O}(j_1) \otimes \dots \otimes \mathcal{O}(j_k) \otimes_{R[\Sigma_j]} A_j$$

where $j = j_1 + \dots + j_k$. In other words, B_1 is the natural the subcomplex of $\mathcal{O} \boxtimes \mathcal{O} \boxtimes A$ induced by the domain A_*.

Now, we consider general q. The complex

$$\underbrace{\mathcal{O} \boxtimes \dots \boxtimes \mathcal{O}}_{q+1} \boxtimes A$$

is naturally given as a direct sum of tensor products

$$\mathcal{O}(n_1) \otimes \dots \otimes \mathcal{O}(n_m) \otimes A_{\alpha_1} \otimes \dots \otimes A_{\alpha_k}.$$

By our flatness assumption, each such summand has a subcomplex $\mathcal{O}(n_1) \otimes \dots \otimes \mathcal{O}(n_m) \otimes A_\alpha$, given by tensoring the given inclusions $i_{\alpha_1,\dots,\alpha_k} : A_\alpha \to A_{\alpha_1} \otimes \dots \otimes A_{\alpha_k}$ with the identity map on copies of $\mathcal{O}$. We let B_q be the direct sum of these subcomplexes.

There is a simple diagrammatic description of these complexes B_q. For example a generator of B_0 can be represented as in Figure 5. Also, a generator of B_q can be represented, as in Figure 6, as a stacking of trees of height $q + 1$, with elements $a_1 \otimes \dots \otimes a_\alpha \in A_\alpha$.

The complexes B_q form a simplicial complex with face and degeneracy maps defined on B_q in exactly the same way as they are for the two sided bar construction $B(\mathcal{O}, \mathcal{O}, C)$ where C is an $\mathcal{O}$-algebra. Namely, the zeroth face operator ∂_0 is given by the partial action of $\mathcal{O}$ on A, and for $0 < i \leq q$, the ith face operator is given by the operad composition $\mathcal{O} \boxtimes \mathcal{O} \to \mathcal{O}$ in the ith $\boxtimes$-factor. It follows from the definition of B_q, and condition (2) of the definition of partial algebras, that $\partial_i : B_q \to B_{q-1}$ for all $0 \leq i \leq q$.
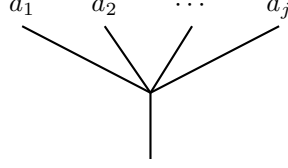
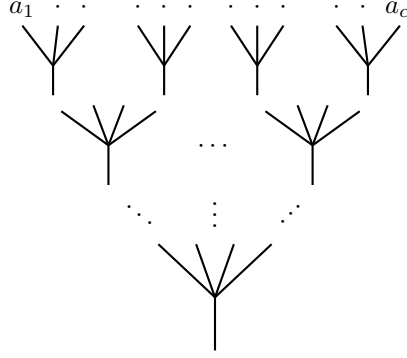FIGURE 5. A generator of B_0 where $a_1 \otimes \ldots \otimes a_j \in A_j$.

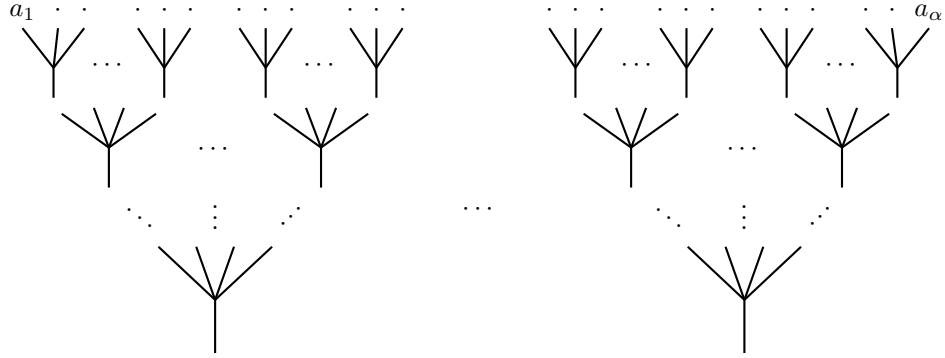FIGURE 6. A generator of B_q, with $a_1 \otimes \ldots \otimes a_\alpha \in A_\alpha$.

Similarly, the ith face operator $s_i : B_q \to B_{q+1}$ is induced by the operadic unit $R \to \mathcal{O}$ in the ith $\boxtimes$-factor. The proof that this forms a simplicial set is exactly the same as the proof for the bar construction, and appears in [2] using the language of monads.

It is easy to visualize B_q and its simplicial structure in terms of our diagrams. As in Figure 6, let us refer to top row of trees at the 1st, the next below the 2nd, etc. The 0th face operator of this simplicial object is given by evaluating the elements of A on the 1st row of trees using the partial algebra structure of A_* over $\mathcal{O}$. For $1 \leq i \leq q$, the ith face operator is given by composing the ith and $(i+1)^{st}$ rows of this diagram using the operad structure. The 0th degeneracy operator of this simplicial object is given by inserting a row of units of $\mathcal{O}$ between the elements of A and the first row of trees. For $1 \leq i \leq q$, the ith degeneracy operator is given by inserting a row of units of $\mathcal{O}$ between the ith and $(i+1)^{st}$ rows of this diagram.

This completes our definition and description of the simplicial complex B. We will use the same notation, B, to denote total complex associated to the simplicial complex B.

3.2. **Domain in B.** We now define a domain in the total complex B, and denote the jth subcomplex of $B^{\otimes j}$ by B_j, or the entire domain simply by B_*. For $j = 1$ we let $B_1 = B$.

For $j > 1$ we note that, again as before, $B^{\otimes j}$ can be written as a sum over terms which are given by a tensor product of copies of the operad components $\mathcal{O}(n_i)$ and of the subcomplexes A_{α_i}. Again by the definition of the domain A_* and our flatness assumption, each summand has a subspace induced by the given inclusions

FIGURE 7. A generator in the domain B_j, with $a_1 \otimes \ldots \otimes a_\alpha \in A_\alpha$.

FIGURE 8. A generator of B_j. Here with $a_1 \otimes \ldots \otimes a_\alpha \in A_\alpha$.

$i_{\alpha_1,\ldots,\alpha_k} : A_\alpha \to A_{\alpha_1} \otimes \ldots \otimes A_{\alpha_k}$, where $\alpha = \alpha_1 + \cdots + \alpha_k$. We let B_j be the sum of these subspaces of $B^{\otimes j}$, and we denote the induced inclusion by $I_j : B_j \to B^{\otimes j}$. It follows from conditions (1) and (2) in the definition of partial algebra that B_j is a subcomplex of $B^{\otimes j}$.

There is a diagrammatic description of B_j given as follows. For simplicial degree $q = 0$ and arbitrary complex degree, we can view a generator as j trees labeled by $a_1 \otimes \ldots \otimes a_\alpha \in A_\alpha$, as in Figure 7.

More generally, we can picture a generator of B_j that is contained in $B_q^{\otimes j}$, for some fixed q, as in Figure 8, by a stacking of trees each of height $q + 1$ labeled on top by elements of A such that $a_1 \otimes \cdots \otimes a_\alpha \in A_\alpha$. In the most general case, we can picture a generator as a collection of j stackings of trees, all of various heights, labeled on top by elements of A such that $a_1 \otimes \cdots \otimes a_\alpha \in A_\alpha$ (not shown).

It remains to show that B_j is a domain in B. First, $B_1 = B$ by definition, and the Σ_j-equivariance of $I_j : B_j \to B^{\otimes j}$ follows from the Σ-equivariance in the domain A_* and the definition of B_j. Secondly, the inclusion map $I_j : B_j \to B^{\otimes j}$ satisfies the factoring condition (2) in Definition 2 since it is induced by the inclusions i_* of the domain A_* which satisfy this condition. Lastly, we claim the inclusions $I_j : B_j \to B^{\otimes j}$ are quasi-isomorphism since they are induced by the inclusions in the domain A_*, that are quasi-isomorphisms.

To prove this, note for each $j > 1$ there is a spectral sequence for each of the bi-complexes B_j and $B^{\otimes j}$ and a morphism between them given by I_j. The induced map on the first page is an isomorphism since these pages are the homology with

respect to the differentials on $\mathcal{O}$ and A_*, the inclusions $i_{\alpha_1,\ldots,\alpha_k}$ are all quasi-isomorphisms, and tensoring with the projective $R[\Sigma_{n_i}]$-module $\mathcal{O}(n_i)$ preserves quasi-isomorphisms. These spectral sequences converge to the homology of their total complexes, since they are bounded, and therefore the induced map on the homology is an isomorphism since it was an isomorphism on the first page.

3.3. **Quasi-isomorphism of domains** B_* **and** A_*. We first construct quasi-isomorphisms $\eta : A \to B$ and $\varphi : B \to A$. This was first done in [2] (Example 4.2), and is word for word the same as the proof that the usual bar construction is a resolution, so we will be brief.

Let $\underline{A}$ denote the constant simplicial object with A in each simplicial degree and all face and degeneracy maps given by the identity. There are canonical maps $\gamma : A \to \underline{A}$ and $\epsilon : \underline{A} \to A$ which are quasi-isomorphisms.

Next, we construct a chain equivalence of $\underline{A}$ and B. There is an inclusion $\psi : \underline{A}_q \to B_q$ of simplicial complexes given by

$$\psi(a) = \underbrace{u \otimes \ldots \otimes u}_{q+1} \otimes a$$

where $u \in \mathcal{O}(1)$ is the operad unit. Next, we have a map $\tau : B \to \underline{A}$ of simplicial complexes given in each simplicial degree by evaluation using the partial action of $\mathcal{O}$ on A. By conditions (2) and (3) of the definition of partial algebra, this is well defined. It is immediate to check that $\tau \circ \psi = id$. Also, there is an explicit simplicial homotopy h such that $\psi \circ \tau$ is homotopic to the identity. It is induced by the simplicial face operators defined in the previous section, as in the usual proof that the bar construction is a resolution, see [2]. Therefore, we have quasi-isomorphisms on the total complexes

$$A \xleftarrow{\;\;\epsilon\;\;}_{\gamma} \underline{A} \xleftarrow{\;\;\tau\;\;}_{\psi} B$$

and it follows that $\varphi = \epsilon \circ \tau : B \to A$ and $\eta = \psi \circ \gamma$ are quasi-isomorphisms. We will refer to the map φ as the evaluation map since, for $x \in B$ of simplicial degree zero, $\varphi(x) \in A$ is given by the partial action of $\mathcal{O}$ on A (while for higher simplicial degrees the map is zero).

Now our goal is to show φ and η each induce a quasi-isomorphism of the domains B_* and A_*. Note that by condition (2) of Definition 4 and our definition of B_j, the restriction of the evaluation map $\varphi^{\otimes j} : B^{\otimes j} \to A^{\otimes j}$ to B_j factors through A_j. Similarly, the restriction of the inclusion map $\eta^{\otimes j} : A^{\otimes j} \to B^{\otimes j}$ to A_j factors through B_j. Thus we have a diagram

$$
\begin{array}{ccc}
B_j & \xrightarrow{\;\;I_j\;\;} & B^{\otimes j} \\
\varphi_j \Big(\Big\uparrow{\eta_j} & \eta^{\otimes j}\Big\uparrow & \Big) \varphi^{\otimes j} \\
A_j & \xrightarrow[\;\;i_j\;\;]{} & A^{\otimes j}
\end{array}
$$

where the square commutes starting from A_j or B_j. By the remark after definition 2, φ_j and η_j are quasi-isomorphisms. Namely the top, bottom and right vertical maps are quasi-isomorphisms, so $\varphi_j : B_j \to A_j$ is also a quasi-isomorphism. For shorthand, we denote this quasi-isomorphism of domains $\varphi_j : B_j \to A_j$ by φ,

as in the outline above. Similarly, η_j is a quasi-isomorphism and gives a quasi-isomorphism η of domains. After defining the partial $\mathcal{O}$-algebra on B_* below, we will see that φ is a map of partial $\mathcal{O}$-algebras.

3.4. B_* as a partial $\mathcal{O}$-algebra. In [2] the authors define, from a partial $\mathcal{O}$-algebra A_*, a simplicial partial algebra. We will recall it's definition here, as it will be used to define a partial $\mathcal{O}$-algebra on the domain B_*.

For each q, j there as a natural subcomplex

$$C_{q,j} \to \underbrace{B_q \otimes \ldots \otimes B_q}_{j}$$

given as before by the subspaces induced by the domain A_* in each summand of the right hand side. In particular, $C_{q,1} = B_q$. Moreover, there are chain maps

$$\mathcal{O}(j) \otimes_{R[\Sigma_j]} C_{q,j} \to C_{q,1}$$

given by the left action of $\mathcal{O}(j)$ on itself, and it is immediate to check this defines a simplicial partial $\mathcal{O}$-algebra. For each j we let C_j be induced total complex and note that $C_1 = B$.

Algebraic structures defined in terms of simplicial maps can be used to define algebraic structure on the chain level, by using the shuffle map ([5], Appendix, or [2]). Recall, that for simplicial complexes $X_{q,r}, Y_{p,s}$ the shuffle map

$$g : T(X)_{q+r} \otimes T(Y)_{p+s} \to T(X \otimes Y)_{q+p+r+s}$$

is defined on these total complexes by

$$g(a \otimes b) = \sum_{(u,v)} \pm (s_{\nu_q} \cdots s_{\nu_1} a \otimes s_{\mu_p} \cdots s_{\mu_1} b)$$

where s_* are the degeneracy operators, the sum is over all shuffles $\nu_1 < \cdots < \nu_q$ and $\mu_1 < \cdots < \mu_p$ of $\{0, 1, \cdots, p+q+1\}$, and the sign is determined by the signature of the corresponding permutation of $\{0, 1, \cdots, p+q+1\}$. It is important to note that g is commutative, associative and unital, and we denote the iterates of the shuffle map also by g.

In our case of interest, we have for each $q = q_1 + \cdots + q_k$, the shuffle map

$$g : B_{q_1} \otimes \ldots \otimes B_{q_j} \to B_q^{\otimes j}$$

We let G_j, with domain $B^{\otimes j}$, denote the sum of these shuffle maps. It is immediate from the definitions of B_j and C_j that the restriction of G_j to B_j factors through C_j, so that there is a well define map Σ_j-equivariant map $B_j \to B^{\otimes j} \to C_j$ given by the inclusion I_j followed by the shuffle map. Thus we can define Θ_j to be the composition

$$\mathcal{O}(j) \otimes_{R[\Sigma_j]} B_j \to \mathcal{O}(j) \otimes_{R[\Sigma_j]} C_j \to C_1 = B$$

In words, this map is given by applying the shuffle product (to obtain an element of correct total degree) followed by the left action of $\mathcal{O}$ on itself. This indeed defines a partial $\mathcal{O}$-algebra on the domain B_*. Property (2) of Definition 4 follows from (and in fact motived) the definition of B_j, while property (3) of Definition 4 follows from the properties of g mentioned above.

Diagrammatically, this action corresponds to inserting operadic units, according to the shuffle map, into a collection of j stackings of trees, followed by composing the trees at the bottom of the diagram with a generator of $\mathcal{O}$.

We close this subsection by noting that the evaluation map φ from subsection 3.3 is a map of partial $\mathcal{O}$-algebras, by properties (1) and (2) of partial $\mathcal{O}$-algebras. By the previous subsection, it follows that φ is a quasi-isomorphism of partial $\mathcal{O}$-algebras.

3.5. **The $\mathcal{O}$-algebra W_*.** In this section we define an $\mathcal{O}$-algebra W_*. First, we define a simplicial complex W. We let $W_0 = \mathcal{O} \boxtimes A$ and for $q \geq 1$ we let

$$W_q = \bigoplus_{k \geq 0} \mathcal{O}(k) \otimes_{R[\Sigma_k]} B_{q-1}^{\otimes k}$$

The face and degeneracy operators of W are defined in the same way as for B, using the the operad partial action for the zeroth face, the operad composition for the other face operations, and operad unit for the degeneracies. We will use the same notation W for the induced total complex.

We note that there is a canonical map $\delta : B \to W$ again induced by the quasi-isomorphisms i_j and $i_{\alpha_1,\dots,\alpha_k}$ of the domain A_* and the identity on all $\mathcal{O}$ tensor factors.

There is a simple diagrammatic description of W_q. As in Figure 9, a generator of W_q may be represented by a stacking of trees $q + 1$ high, labeled on top by elements of A satisfying the following property: for each tree in the qth (second to bottom) row, the elements $a_{i,1}, \cdots, a_{i,\alpha_i}$ of A "lying above" this tree satisfy $a_{i,1} \otimes \cdots \otimes a_{i,\alpha_i} \in A_{\alpha_i}$. Diagrammatically, the simplicial structure can be view as it is for B.

By pre-composition with the shuffle map, followed by the left action of $\mathcal{O}$ on itself, we can define a map

$$\mathcal{O}(j) \otimes_{R[\Sigma_j]} W^{\otimes j} \to W$$

similar to subsection 3.4. The new point to check is that the image of this composition does in fact land in W, but this follows from (and in fact motivates) the definition of W. The other properties follow as in the previous subsection 3.4, so that W is an $\mathcal{O}$-algebra. Said another way, by Remark 5, W_* is a partial $\mathcal{O}$-algebra with domain $W_j = W^{\otimes j}$ for all $j \geq 1$.

3.6. **A quasi-isomorphism of partial $\mathcal{O}$-algebras B_* and W_*.** As noted in the previous subsection 3.5, that there is a canonical map of complexes $\delta : B \to W$ induced by the quasi-isomorphisms of the domain A. It follows from the definition of B_j, and property (2) of the domain A_*, that the restriction of $\delta^{\otimes j}$ to B_j factors through $W^{\otimes j}$, giving a map of domains $\delta_j : B_j \to W_j = W^{\otimes j}$.

This map is a quasi-isomorphism for $j = 1$ by the same spectral sequence argument as in subsection 3.2. By the remark after Definition 2, it is therefore a quasi-isomorphism for all j. Thus we have a quasi-isomorphism of domains $\delta : B_* \to W_*$.

Finally, it is immediate that δ is a map of partial $\mathcal{O}$-algebras since the $\mathcal{O}$-structures are defined in the same way by the shuffle map and the left action of $\mathcal{O}$ on itself. It follows that δ is a quasi-isomorphism of partial $\mathcal{O}$-algebras. It is routine to check, using the techniques already described, that all of our constructions were functorial, so this completes the proof of the theorem.

4. APPLICATIONS

In [6] McClure showed that the PL-chains C of a PL-manifold have a domain $\{C_j\}$ described by "chains in general position". In particular, McClure showed that
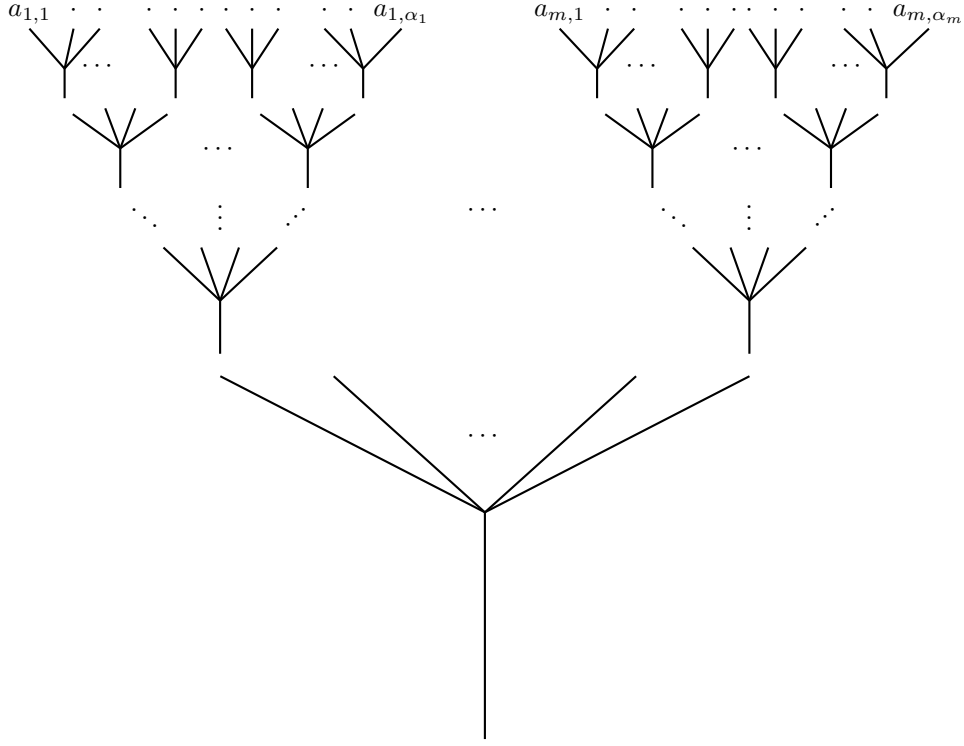
FIGURE 9. A generator of W_q. Here $a_{i,1} \otimes \cdots \otimes a_{i,\alpha_i} \in A_{\alpha_i}$, for all $1 \leq i \leq m$.

C_j is quasi-isomorphic to $C^{\otimes j}$. Moreover, McClure showed this domain is part of a "partial Leinster algebra", defined using the intersection of chains. In the language of this paper, this means they form a partial algebra over the operad $\mathscr{C}$ describing commutative associative algebras.

A technical assumption of Theorem 6 is that the jth component of the operad must be, for each $j > 0$, a projective $R[\Sigma_j]$-module. Over $R = \mathbb{Q}$ every module is $R[\Sigma_j]$-projective, so by Theorem 6 we obtain from any partial $\mathscr{C}$-algebra A (over $\mathbb{Q}$) a commutative associative differential graded algebra on a complex quasi-isomorphic to A. In particular, we have the example $A = C$ above.

On the other hand, over $R = \mathbb{Z}$, $\mathscr{C}$ does not satisfy this property (since the Σ_j actions are trivial on $\mathscr{C}(j) = \mathbb{Z}$). The following operad does satisfy the projective assumption:

Definition 7. *An $\mathcal{E}_\infty$-operad is a unital operad $\mathcal{E}$, i.e. $\mathcal{E}(0) \approx \mathbb{Z}$, such that the maps*

$$\mathcal{E}(j) \otimes \mathcal{E}(0)^j \to \mathcal{E}(0) \approx \mathbb{Z}$$

are quasi-isomorphisms, and each $\mathcal{E}(j)$ is a free $\mathbb{Z}[\Sigma_j]$-module. An algebra over an $\mathcal{E}_\infty$-operad is called an $\mathcal{E}_\infty$-algebra.

Following Kriz and May in [2], we use the given quasi-isomorphisms

$$\mathcal{E}(j) \approx \mathbb{Z} = \mathscr{C}(j),$$

to pull back the partial $\mathscr{C}$-algebra on the PL-chains C of a PL-manifold, [6], to obtain a partial algebra over $\mathcal{E}$. Then by Theorem 6 we obtain the following:

Theorem 8 (McClure [6] using Theorem 6 above)**.** *There is a functor assigning to any closed PL-manifold an $\mathcal{E}_\infty$-algebra on a complex quasi-isomorphic to its PL-chains. This structure induces the intersection product in homology.*

Remark 9. *It is natural to ask how this $\mathcal{E}_\infty$ chain-algebra relates to the known $\mathcal{E}_\infty$ cochain-algebra that, by a theorem of Mandell [3], determines the weak homotopy type of a finite type nilpotent space[1].*

We can take this example a bit further: the (PL) chains of a (PL) manifold embed quasi-isomorphically into the space of bounded linear functionals on differential forms with compact support, i.e. currents, see de Rham [1]. The same domain $\{C_j\}$ of chains as before also gives a domain for currents. By the same argument as above, over $\mathbb{Q}$ or $\mathbb{R}$, Theorem 6 gives a commutative associative differential graded algebra on a complex quasi-isomorphic to the currents. This gives an algebraic resolution to the long standing issue in functional analysis of not being able to multiply distributions, which are precisely zero currents.

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[1]It has recently been announced by D. Chataur that these two $\mathcal{E}_\infty$-algebras are in fact quasi-isomorphic.