Helmholtz Forward Solver

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December 10, 2020

1 Introduction

In this program, we simulate the propagation of acoustic waves through a medium of inhomogeneous sound speed. The setup of the problem is meant to imitate the setup in Ultrasound Computed Tomography (USCT), in which ultrasonic sound waves are emitted by a single transducer on the boundary of the domain to be investigated and measured by every other transducer on the boundary. Mathematically, the propagation of sound waves through a medium is modeled by the wave equation. However, if we assume that the sound waves are time harmonic then the wave equation reduces to the Helmholtz equation:

$$-\Delta p^k(\boldsymbol{x}) - \gamma(\boldsymbol{x})(\omega^k)^2 p^k(\boldsymbol{x}) = g^k(\boldsymbol{x}) \quad \text{in } \Omega \subset \mathbb{R}^2 \tag{1}$$

$$p^k(\boldsymbol{x}) = h^k(\boldsymbol{x}) \quad \text{on} \quad \partial\Omega \tag{2}$$

where

$$\begin{aligned} \gamma(\boldsymbol{x}) &= \frac{1}{c^2(\boldsymbol{x})}, \\ \omega^k &= 2\pi f^k, \\ c(\boldsymbol{x}) &= \text{ inhomogeneous sound speed of background,} \\ f^k &= \text{ temporal frequency of incident wave at kth experiment,} \end{aligned}$$

and where the index k runs over the total number of experiments at each iteration. We will have $1 \leq k \leq N$ where $N = K * M$, with K the number of transducers and M the number of frequencies.

The functions $g^k(\boldsymbol{x})$ represent any acoustic sinks or sources in the problem at experiment k. Our domain will not include any sinks or sources, so $g^k(\boldsymbol{x}) = 0$ for all k. However, we leave the term in the following derivations in case the user wants to model a different domain. The functions $h^k(\boldsymbol{x})$ represent acoustic pressure sources on the boundary of the domain at experiment k. For us, $h^k(\boldsymbol{x})$ will represent an acoustic pulse at one of transducers that transmits at temporal frequency f^k.

We proceed by multiplying (1) by a test function $\phi^k \in V_0 = \{\phi \in H^1(\Omega) : \phi|_{\partial\Omega} = 0\}$ and integrating over Ω to get

$$-\int_\Omega \phi^k \Delta p^k d\boldsymbol{x} - \int_\Omega \phi^k \gamma(\omega^k)^2 p^k d\boldsymbol{x} = \int_\Omega \phi^k g^k d\boldsymbol{x}. \tag{3}$$

We will focus on the term $\int_\Omega \phi^k \Delta p^k d\boldsymbol{x}$. Integrating this by parts gives

$$\int_\Omega \phi^k \Delta p^k d\boldsymbol{x} = \int_{\partial\Omega} \phi^k \nabla p^k d\boldsymbol{x} - \int_\Omega \nabla \phi^k \nabla p^k d\boldsymbol{x} \tag{4}$$

$$= -\int_\Omega \nabla \phi^k \nabla p^k d\boldsymbol{x} \tag{5}$$

where the last equality holds since $\phi^k \in V_0$. Now plugging this into (3) gives

$$\int_\Omega \nabla \phi^k \nabla p^k d\boldsymbol{x} - \int_\Omega \phi^k \gamma(\omega^k)^2 p^k d\boldsymbol{x} = \int_\Omega \phi^k g^k d\boldsymbol{x} \tag{6}$$

$$\Rightarrow (\nabla \phi^k, \nabla p^k)_\Omega - (\phi^k, \gamma(\omega^k)^2 p^k)_\Omega = (\phi^k, g^k)_\Omega \tag{7}$$

where the notation $(\cdot,\cdot) \mapsto \mathbb{R}$ in (6) is the bilinear notation for the inner products in (4).

Now we proceed in the usual manner and seek an approximation $p_h(\boldsymbol{x}) = \sum_j P_j \phi_j(\boldsymbol{x})$ where P_j are the unknown expansion coefficients we need to determine and $\phi_i(\boldsymbol{x})$ are the finite element shape functions we will use. Plugging these into (7) gives

$$(\nabla \phi_i^k, \nabla p_h^k) - (\phi_i^k, \gamma(\omega^k)^2 p_h^k) = (\phi_i^k, g^k). \tag{8}$$

Then plugging in the representation $p_h(\boldsymbol{x}) = \sum_j P_j \phi_j(\boldsymbol{x})$ gives

$$(\nabla \phi_i^k, \nabla p_h^k) - (\phi_i^k, \gamma(\omega^k)^2 p_h^k) = \left(\nabla \phi_i^k, \nabla \left[\sum_j P_j^k \phi_j^k\right]\right) - \left(\phi_i^k, \gamma(\omega^k)^2 \left[\sum_j P_j^k \phi_j^k\right]\right) \tag{9}$$

$$= \sum_j \left(\left(\nabla \phi_i^k, \nabla \left[P_j^k \phi_j^k\right]\right) - \left(\left(\phi_i^k, \gamma(\omega^k)^2 \left[P_j^k \phi_j^k\right]\right)\right)\right) \tag{10}$$

$$= \sum_j \left(\left(\nabla \phi_i^k, \phi_j^k\right) - \left(\phi_i^k, \gamma(\omega^k)^2 \phi_j^k\right)\right) P_j^k. \tag{11}$$

The problem is now to find N vectors P_j^k such that

$$A^k P^k = G^k \tag{12}$$

where $A_{ij}^k = (\nabla \phi_i^k, \nabla \phi_j^k)$, and $G_i^k = (\phi_i^k, g^k)$. This gives N different linear systems we must solve in order to acquire the pressure throughout the domain given the "true" value of the parameter γ (and thus the true value of the inhomogeneous sound speed $c(\boldsymbol{x})$). In practice, we can treat each of the K experiments as independent from each of the other experiments. Thus at each experiment we have a linear system defined by (12) (for a fixed k) that we must solve in order to determine the pressure throughout the domain when a certain transducer transmits a pressure wave at a fixed amplitude and frequency.

We must take care in this program to ensure that the correct transducer is transmitting a wave at the correct frequency during each experiment. A description of how we set the transducer location and frequency is given in the commented program below. While determining the pressure on the entire domain is useful, in a real-life application we could only determine the pressure at locations on the boundary in which we have a measuring transducer. As such, in this program we determine the pressure throughout the domain and also return the pressure measurements on every transducer except the transmitting one at each experiment. This data will be used in the accompanying program Helmholtz Inverse Solver (name subject to change). A description of the steps to acquire the pressure on the transducers is given in the commented program below.

2 Commented Program

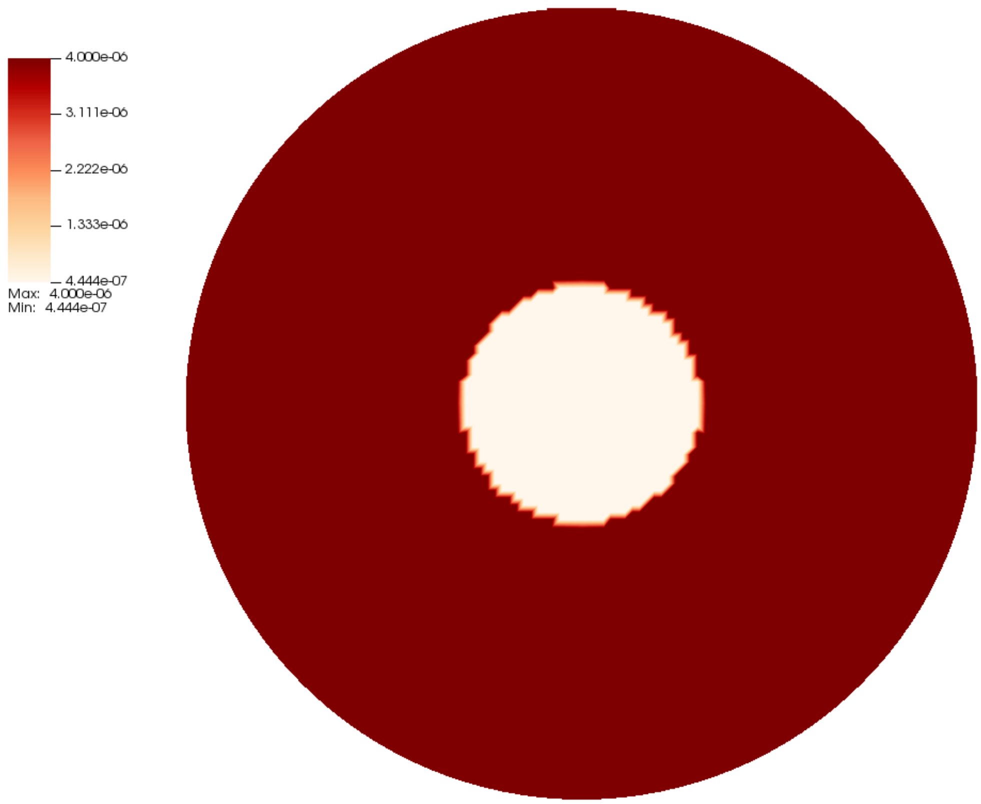
See attached file.

3 Results

The output of the program will change a bit depending on the number of transducers and transmitted frequencies set by the user, but will look something as follows. The following output was generated on a mesh that was refined 5 times with 8 transducers and 3 transmitted frequencies on each transducer (for a total of 24 experiments). The output to the console looks like the following:

Number of active cells: 5120
Number of degrees of freedom: 5185
1181 CG iterations needed to obtain convergence.
7879 CG iterations needed to obtain convergence.
9085 CG iterations needed to obtain convergence.
1362 CG iterations needed to obtain convergence.
8014 CG iterations needed to obtain convergence.
8209 CG iterations needed to obtain convergence.
1180 CG iterations needed to obtain convergence.
9792 CG iterations needed to obtain convergence.
8059 CG iterations needed to obtain convergence.
1205 CG iterations needed to obtain convergence.
7610 CG iterations needed to obtain convergence.
8312 CG iterations needed to obtain convergence.
1099 CG iterations needed to obtain convergence.
8469 CG iterations needed to obtain convergence.
9638 CG iterations needed to obtain convergence.
1116 CG iterations needed to obtain convergence.
7530 CG iterations needed to obtain convergence.
8130 CG iterations needed to obtain convergence.
1319 CG iterations needed to obtain convergence.
7429 CG iterations needed to obtain convergence.
8468 CG iterations needed to obtain convergence.
1375 CG iterations needed to obtain convergence.
8087 CG iterations needed to obtain convergence.
7603 CG iterations needed to obtain convergence.

This output displays the number of active cells and number of degrees of freedom after refining our mesh as well as the number of conjugate gradient iterations needed to solve the linear system at each experiment. In addition to this console output, the program generates 25 .vtk files. The first of those .vtk files is a plot of the parameter γ. The plot of the parameter used for this run is given below, as generated by the visualization program VisIt.

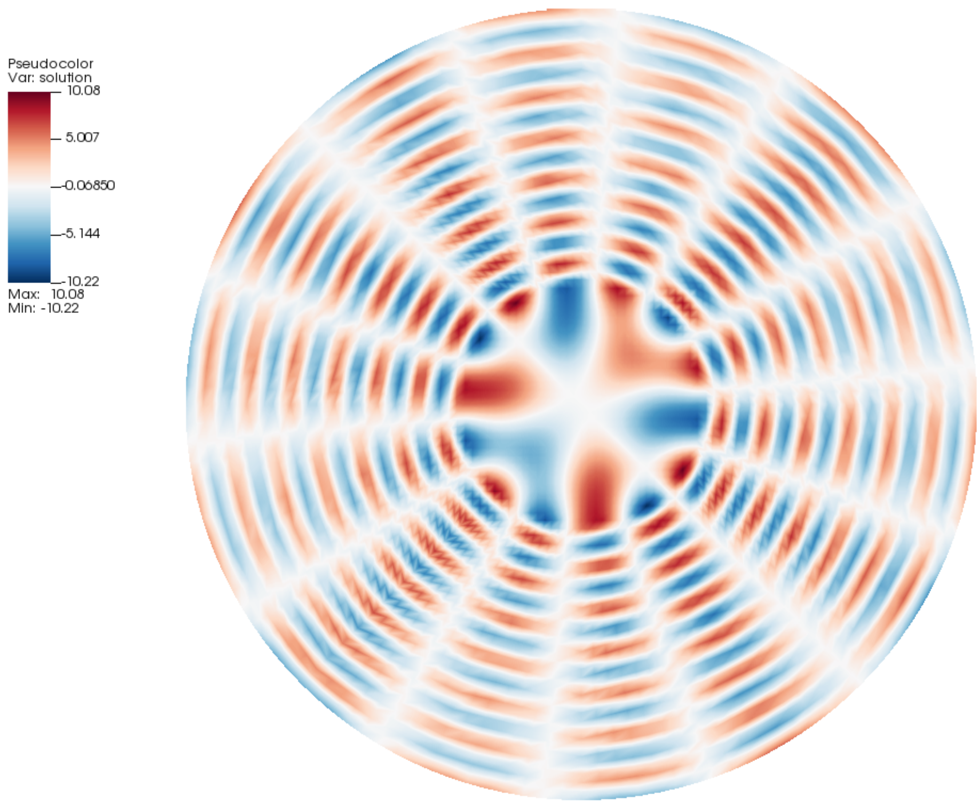


There is not a lot of interest in this plot except to note the inhomogeneity in the center of the domain. The values shown in the legend are somewhat meaningless, since what is actually of interest is the the values of the sound speed $c(\boldsymbol{x})$. However, the values of the sound speed throughout the domain can be recovered by noting that $\gamma(\boldsymbol{x}) = 1/c^2(\boldsymbol{x})$.

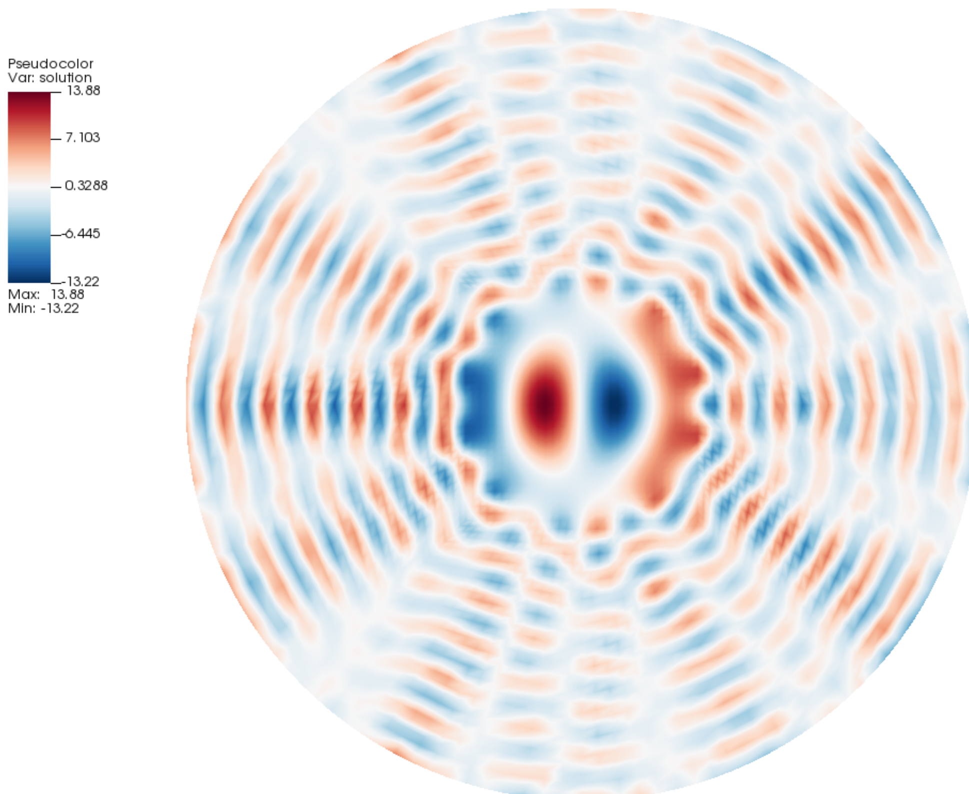
The final 24 .vtk the pressure throughout the domain at each experiment and have a file name of the form solution$_{\mathrm{k,j.vtk}}$ where k is an index for the transmitting transducer and j is an index for the active frequency. These indices are somewhat arbitrary but the position of transducer k is given by

$(\cos(2\pi k/K), \sin(2\pi k/K))$ and the frequency can be determined by taking $5000Hz \pm 100j$ where we add 100Hz if j is odd and subtract 100Hz if j is even. A few of these plots are shown below.

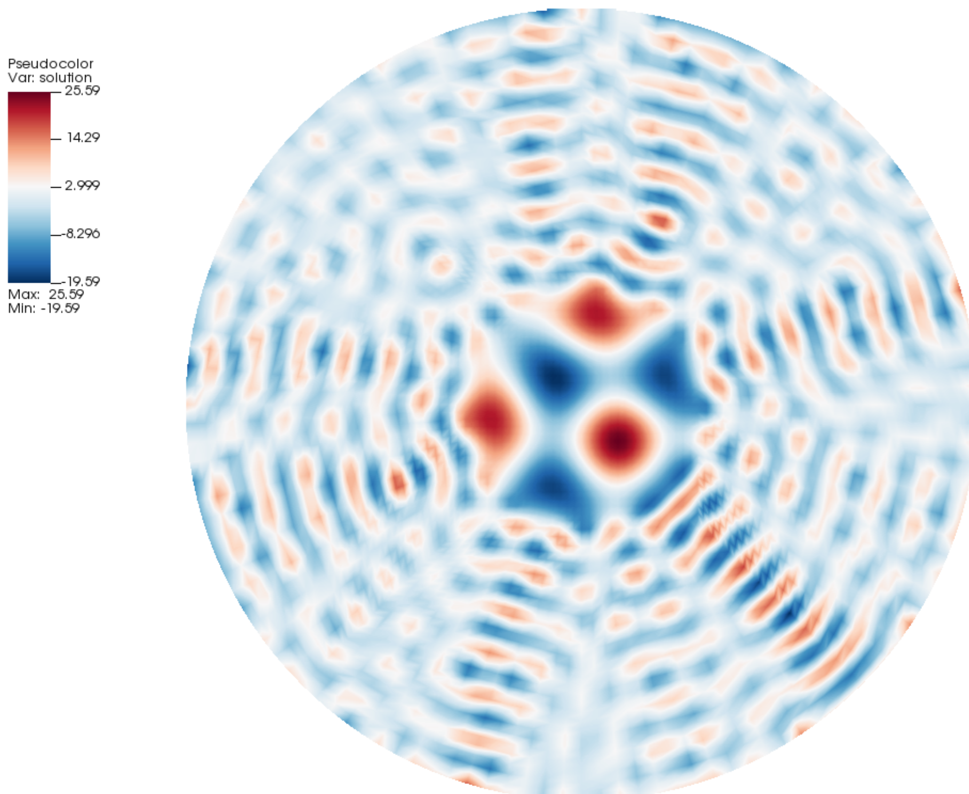
Transducer 1 emits at a frequency of 5100Hz:



Transducer 4 emits at a frequency of 4900Hz:



Transducer 7 emits at a frequency of 4900Hz:



These plots clearly show a pressure wave interacting with the inhomogeneity in the center of our domain. Note that although we transmit a pressure wave with amplitude 5 Pa we see pressures of much higher amplitudes throughout the domain. This is due to the fact that there are no sinks in the domain, so the pressure waves will interfere constructively (and destructively) in the interior of the domain to produce larger and smaller amplitudes.