

# Time series analysis

EDS 222

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Tamma Carleton

Fall 2023

# Announcements/check-in

- Assignment 04 posted, due 12/08

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- A note on depth in coming lectures

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- Assignment 04 posted, due 12/08
- A note on depth in coming lectures
- **No class** 11/23
- Final projects: due in 3.5 weeks!
  - Presentations: 12/12 4:00-7:00pm (Bren Hall 1424)
  - Blog posts: 12/15

# Today

What are time series data?

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Decomposition

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Autocorrelation



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Autocorrelation

Time series and OLS

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Up to this point, we focused on **cross-sectional data**.

- Sampled *across* a population (e.g., people, counties, countries).
- Sampled at *one moment* in time (e.g., Jan. 1, 2015).
- We had  $n$  individuals, each indexed  $i$  in  $\{1, \dots, n\}$ .

# What are time series data?

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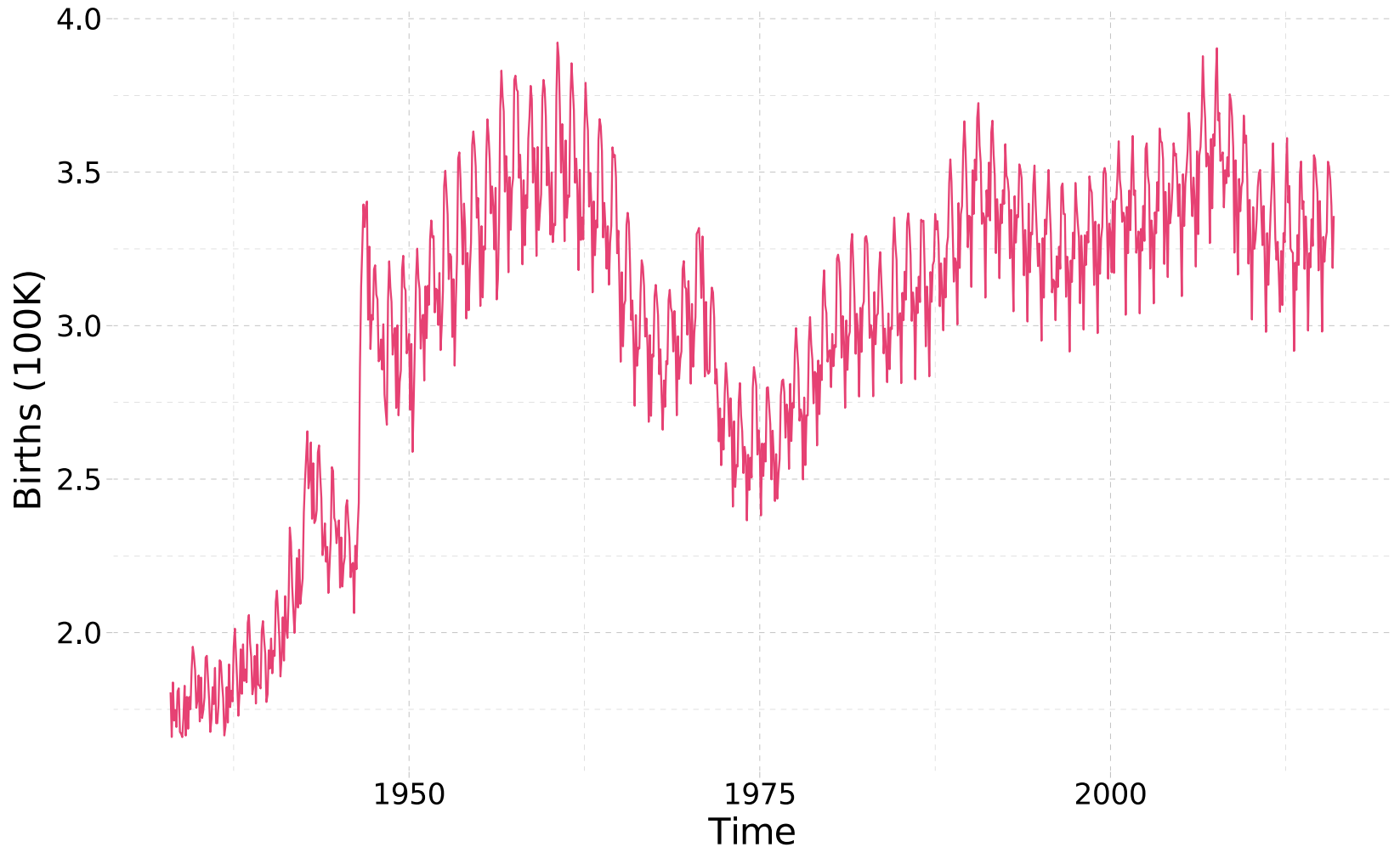
- Sampled *across* a population (e.g., people, counties, countries).
- Sampled at *one moment* in time (e.g., Jan. 1, 2015).
- We had  $n$  individuals, each indexed  $i$  in  $\{1, \dots, n\}$ .

Today, we focus on a different type of data: **time-series data**.

- Sampled within **one unit/individual** (e.g., Oregon).
- Observe **multiple times** for the same unit (e.g., Oregon: 1990–2020).
- We have  **$T$  time periods**, each indexed  $t$  in  $\{1, \dots, T\}$ .

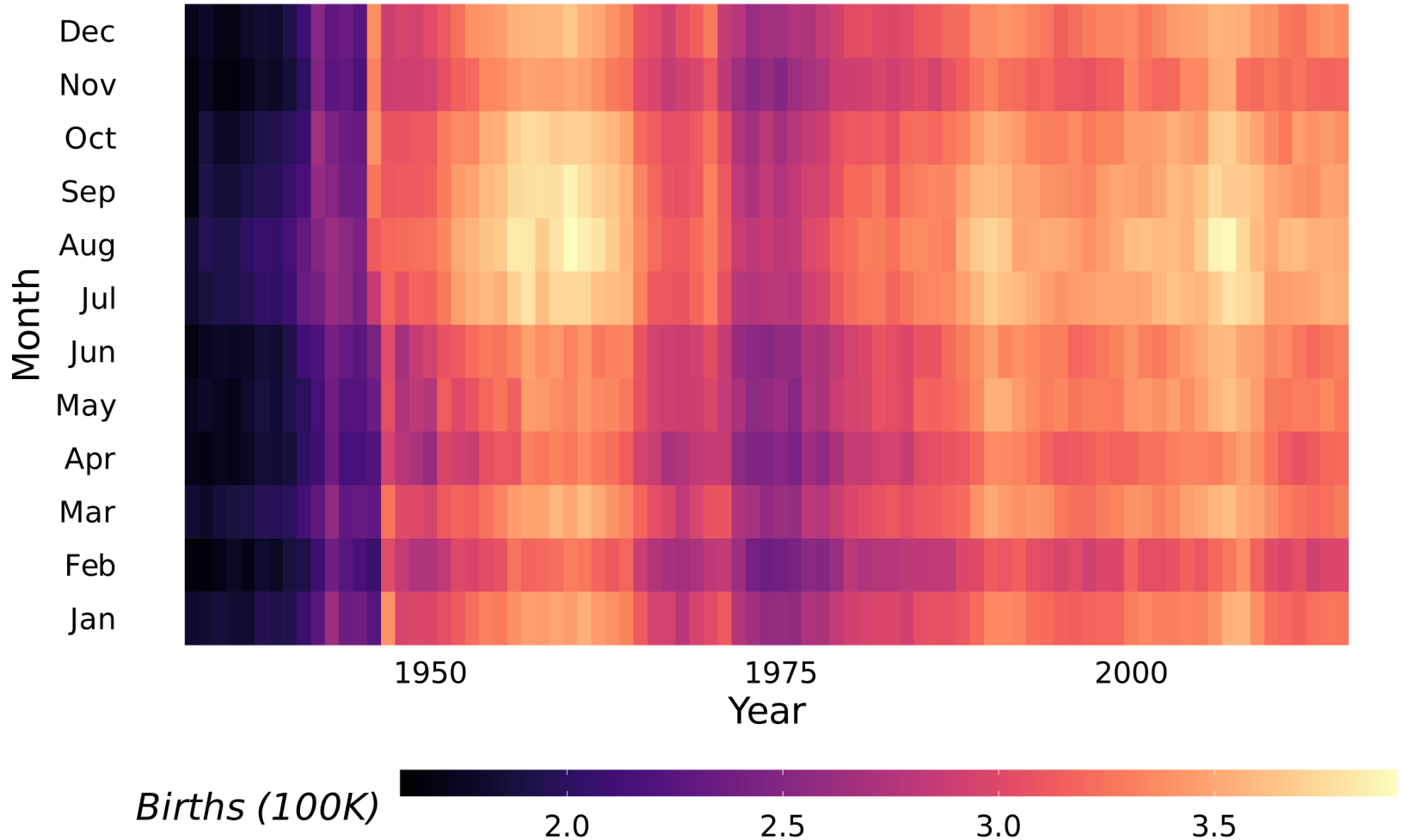
# Time series data: Example

**US monthly births, 1933–2015:** Classic time-series graph



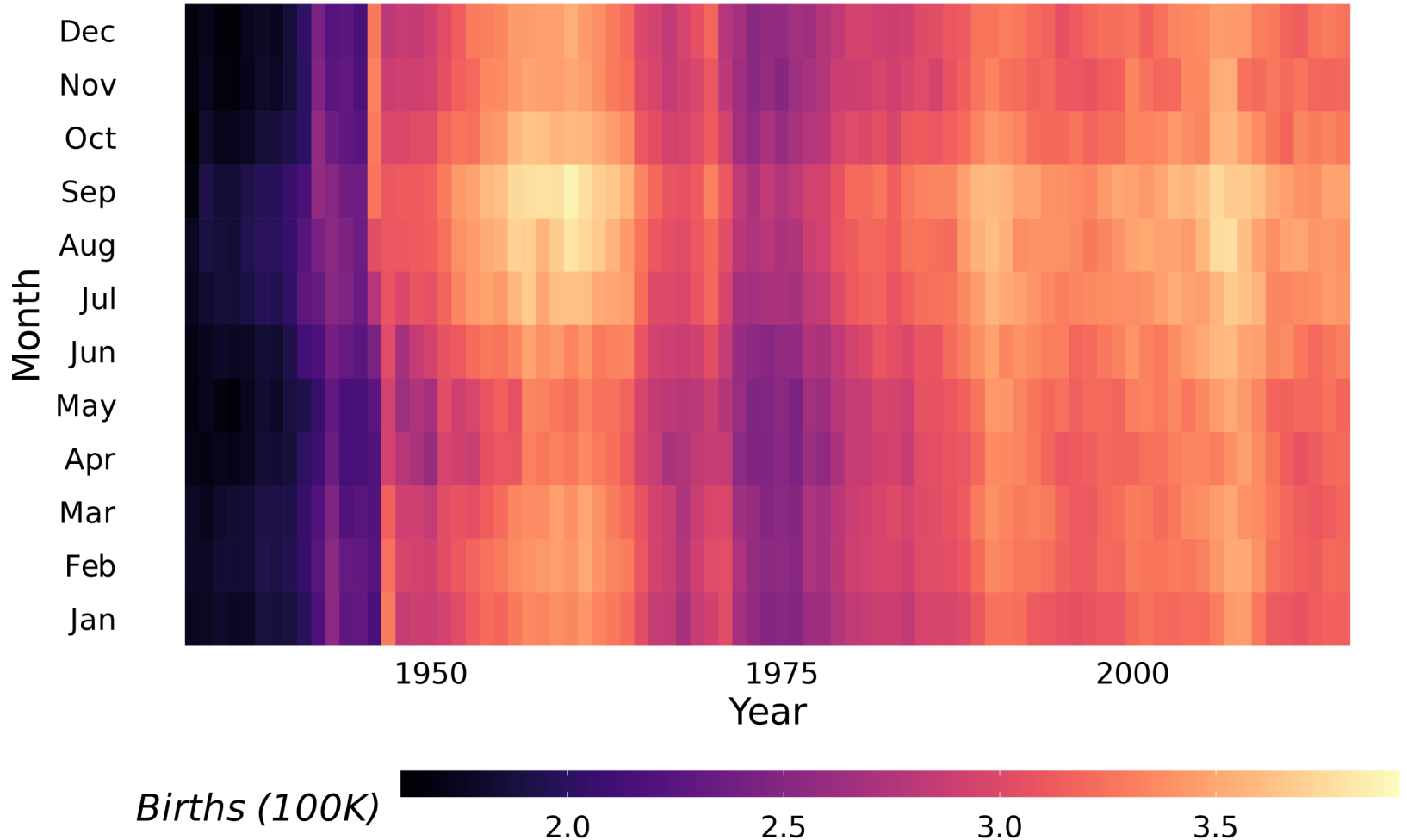
# Time series data: Example

**US monthly births, 1933–2015:** Newfangled time-series graph



# Time series data: Example

**US monthly births per 30 days, 1933–2015:** Newfangled time-series graph



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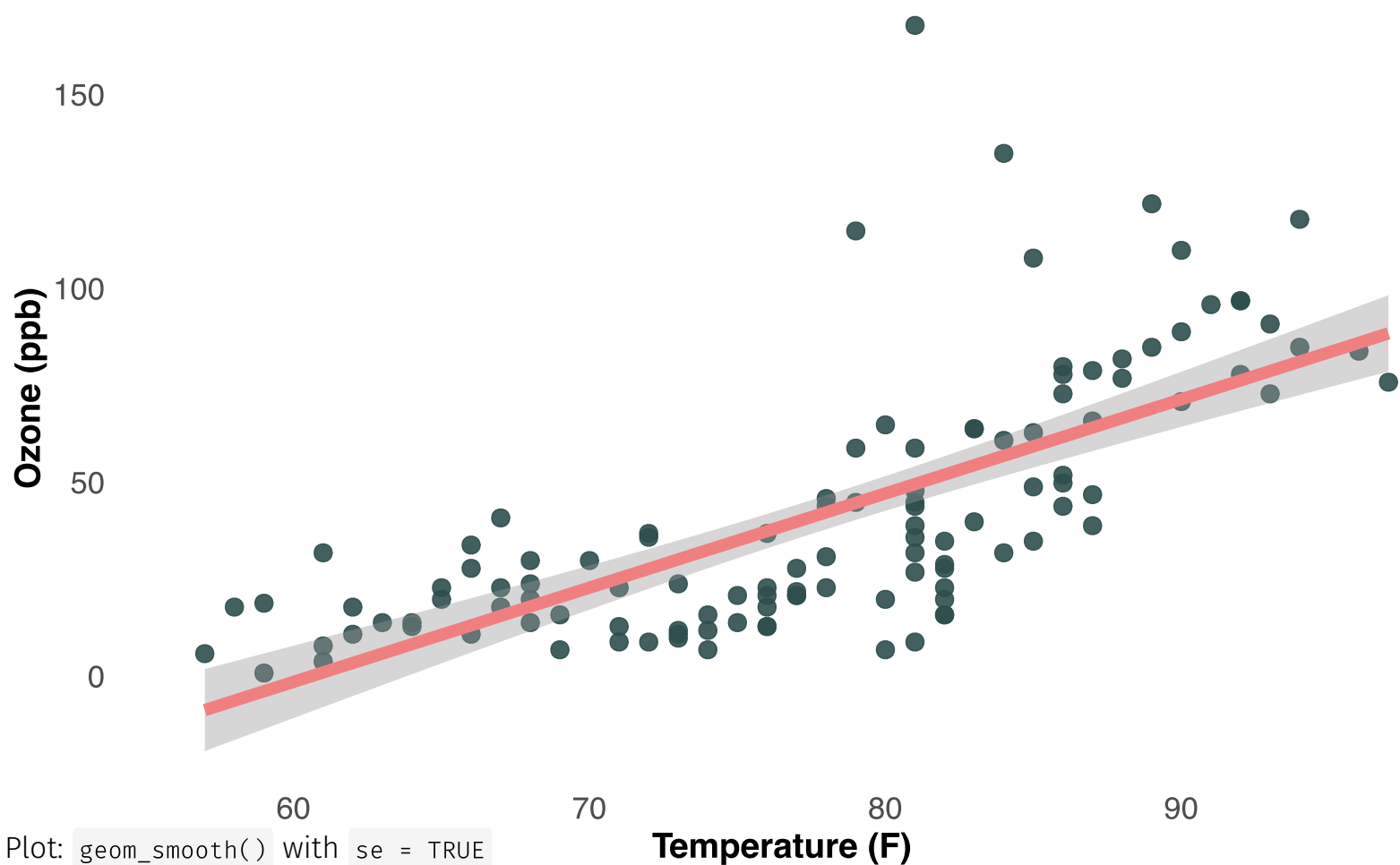
- Description of `airquality` data:

Daily air quality measurements in New York, May to September 1973.

- These are **time series data** and we already ran an OLS regression with them!

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airqts = airquality %>% mutate(date = make_datetime(1973, Month, Day))  
head(airqts)
```

```
#>   Ozone Solar.R Wind Temp Month Day      date  
#> 1    41     190  7.4   67     5   1 1973-05-01  
#> 2    36     118  8.0   72     5   2 1973-05-02  
#> 3    12     149 12.6   74     5   3 1973-05-03  
#> 4    18     313 11.5   62     5   4 1973-05-04  
#> 5    NA      NA 14.3   56     5   5 1973-05-05  
#> 6    28      NA 14.9   66     5   6 1973-05-06
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- Regression of *Ozone* on *date* estimates a **linear trend** in ozone
- Tip: `make_datetime()` from the `lubridate` package (handy for dates and times)

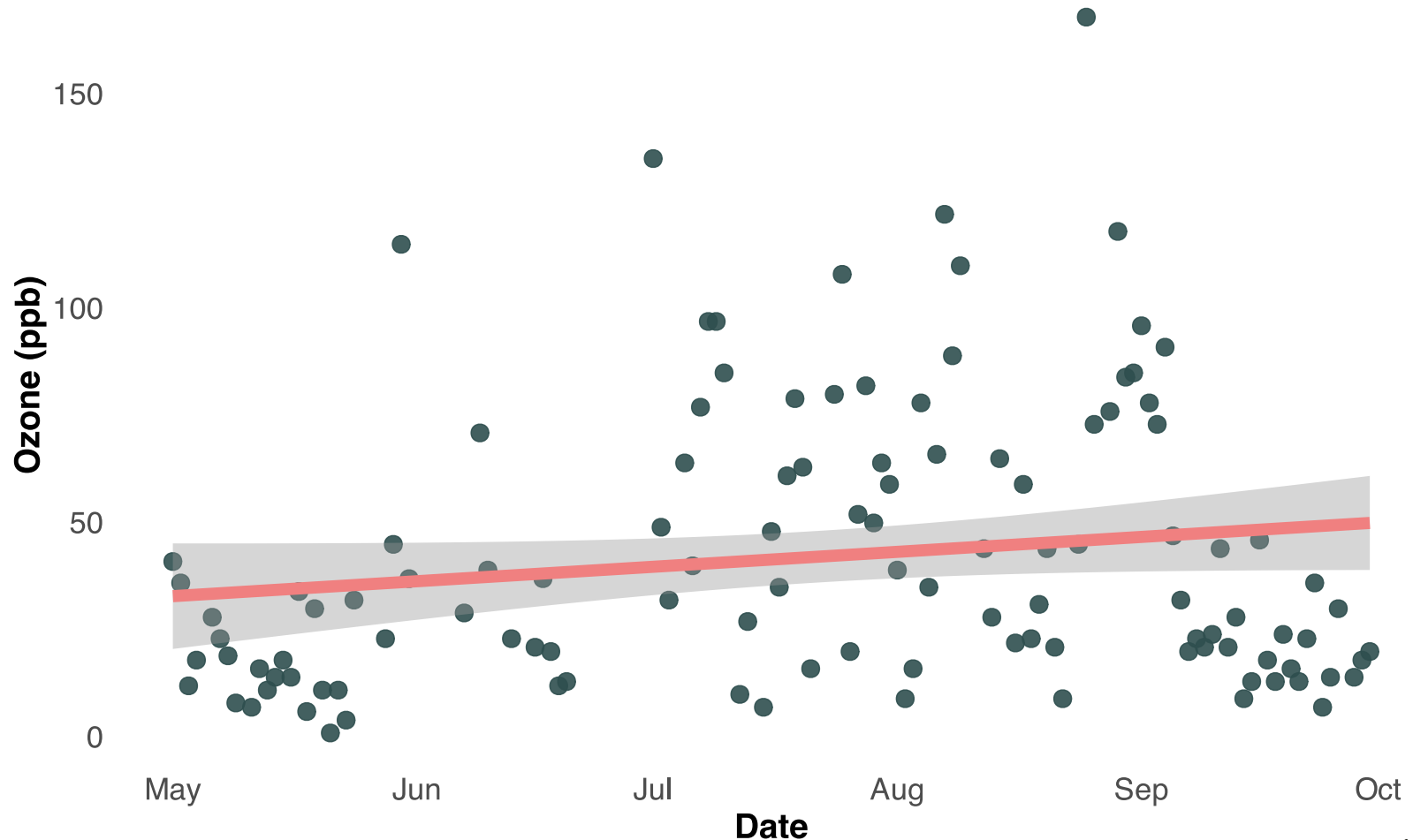
# You already have (many of) the tools

$$\text{Ozone}_t = \beta_0 + \beta_1 \text{date}_t + \varepsilon_t$$

```
summary(lm(Ozone ~ date, data = airqts))
#>
#> Call:
#> lm(formula = Ozone ~ date, data = airqts)
#>
#> Residuals:
#>      Min       1Q   Median       3Q      Max
#> -42.32 -24.58  -8.39   20.46 122.05
#>
#> Coefficients:
#>              Estimate Std. Error t value Pr(>|t|)
#> (Intercept) -1.04e+02   8.59e+01  -1.21    0.230
#> date         1.30e-06   7.65e-07   1.70    0.092 .
#> ---
#> Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
#>
#> Residual standard error: 32.7 on 114 degrees of freedom
#> (37 observations deleted due to missingness)
#> Multiple R-squared:  0.0247,    Adjusted R-squared:  0.0162
#> F-statistic: 2.89 on 1 and 114 DF,  p-value: 0.092
```

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- But there are some new **features** we want to explore:
  - Does my data have exhibit **trending behavior**?
  - Is there **seasonality**?
  - Is my data **cyclical**?

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- Many of the summary statistics, regression, and hypothesis testing tools apply to time series data without much adjustment
- But there are some new **features** we want to explore:
  - Does my data have exhibit **trending behavior**?
  - Is there **seasonality**?
  - Is my data **cyclical**?
- And some new **challenges** to overcome:
  - Additional **assumptions** needed in OLS
  - Threat to existing assumptions: Are our error terms **independent**? Is **exogeneity** harder now?

# Decomposition

# Time series components

## Seasonality

A repeated pattern over known and equal periods (e.g., month; quarter, decade)



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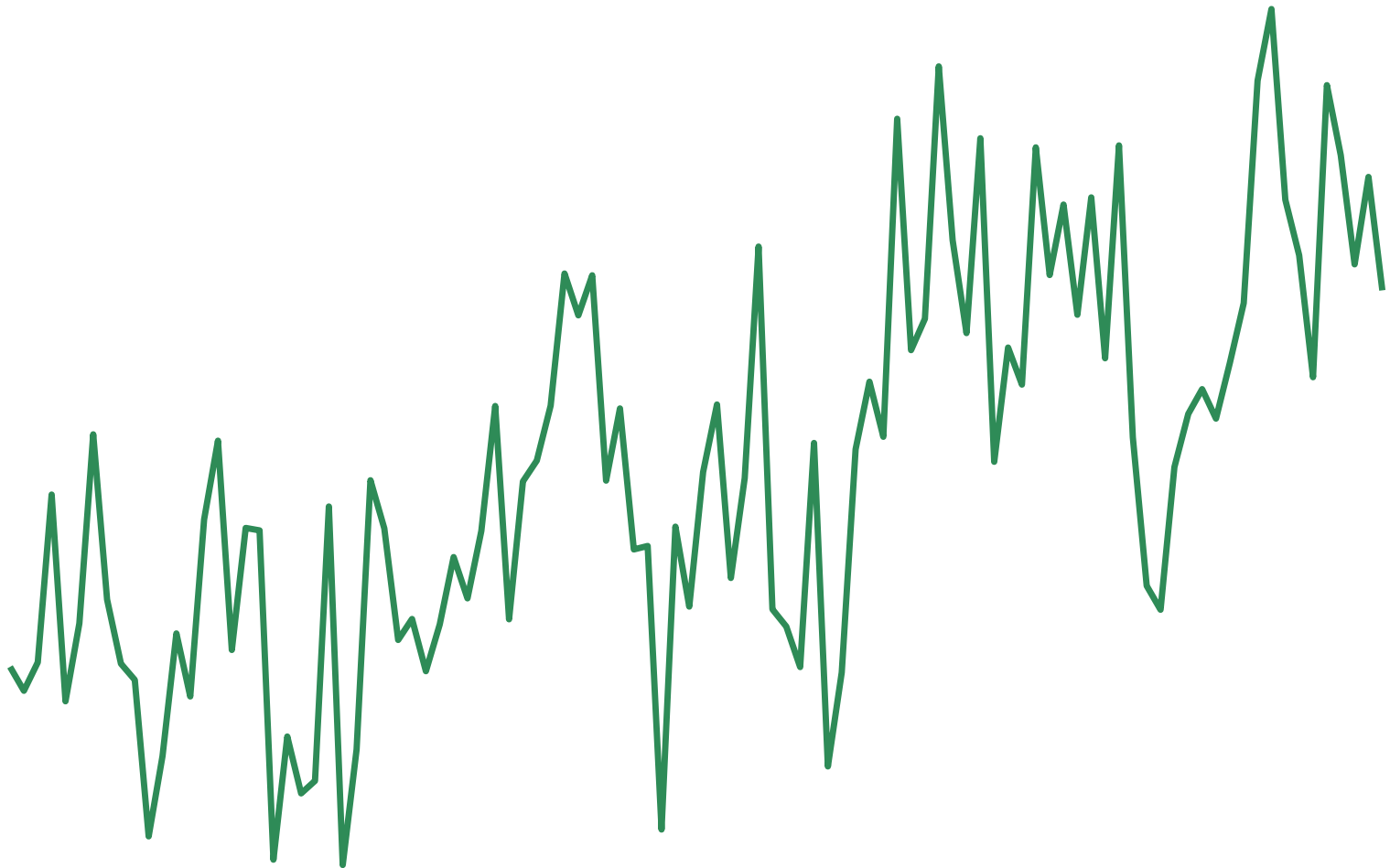
A broader cyclical trend with unknown and/or unequal periods (e.g., business cycle, ENSO)

## Trends

Long-term increase or decrease in the data (not necessarily linear!)

# Time series components

Often, seasonality, cyclicality and trends occur all at the same time:



# Time series components

For many time series,<sup>\*</sup> we can decompose the data into:

$$y_t = S_t + T_t + R_t$$

where  $S_t$  is a **seasonal** component,  $T_t$  is the cycle *and* trend components, and  $R_t$  is the remainder.

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**Decomposition** allows us to isolate each component of the time series visually and quantitatively.

[\*]: This decomposition is "additive", which works for many time series. See [Hyndman](#) for details on more complex "multiplicative" decomposition.

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where  $m = 2k + 1$ .

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The moving average gives you an estimate of the irregular trend-cycle component  $T$  at time  $t$  by averaging values of the time series within  $k$  periods of  $t$



# Moving average example

Computing an  $m = 5$  moving average over the data plotted on the last slide:

```
df = as.data.frame(cbind(x, y)) # these are the data we plotted above
df = df %>% mutate(ma = slider::slide_dbl(y, mean,
                                           .before = 2, .after = 2, .complete = TRUE))
```

# Moving average example

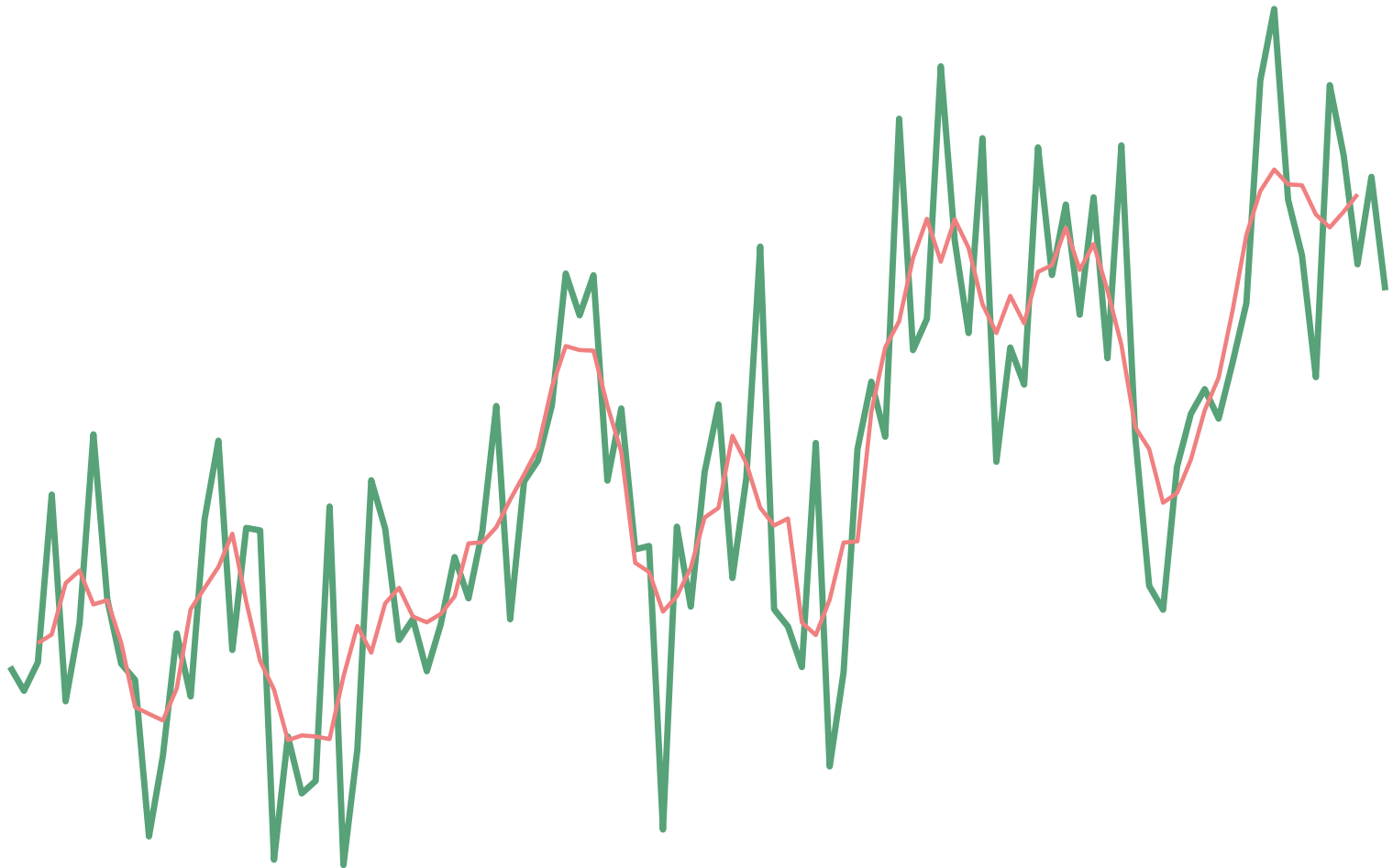
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```

- Helpful package: `slider` (there are others too!)
- Option `.complete=TRUE` ensures only moving windows with complete data are computed

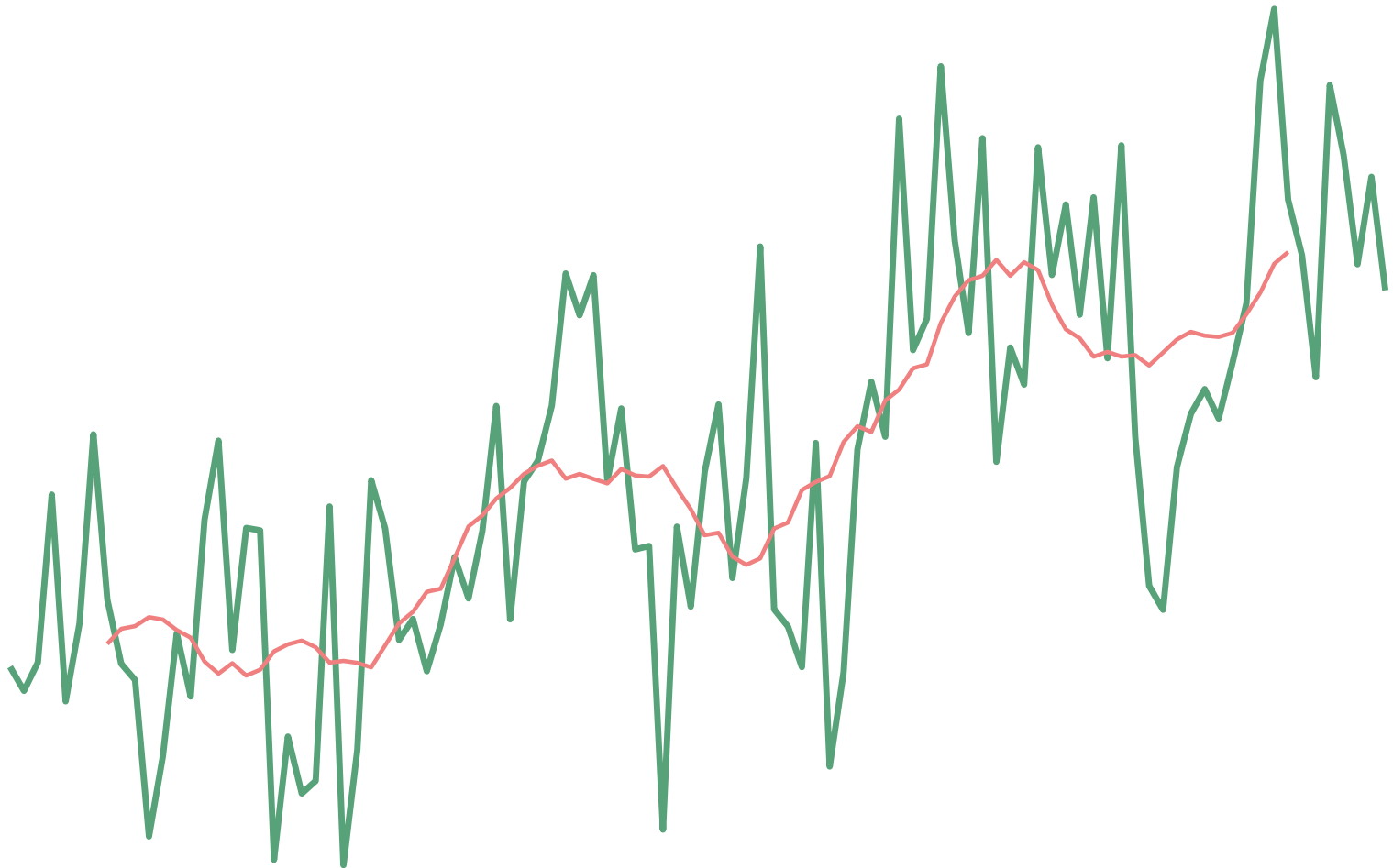
# Moving average example

Computing an  $m = 5$  moving average:



# Moving average example

Computing an  $m = 15$  moving average:



# Classical decomposition

Step 1: estimate a moving average

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Simple average over de-trended series for each season  $s$

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## Step 4: remainder

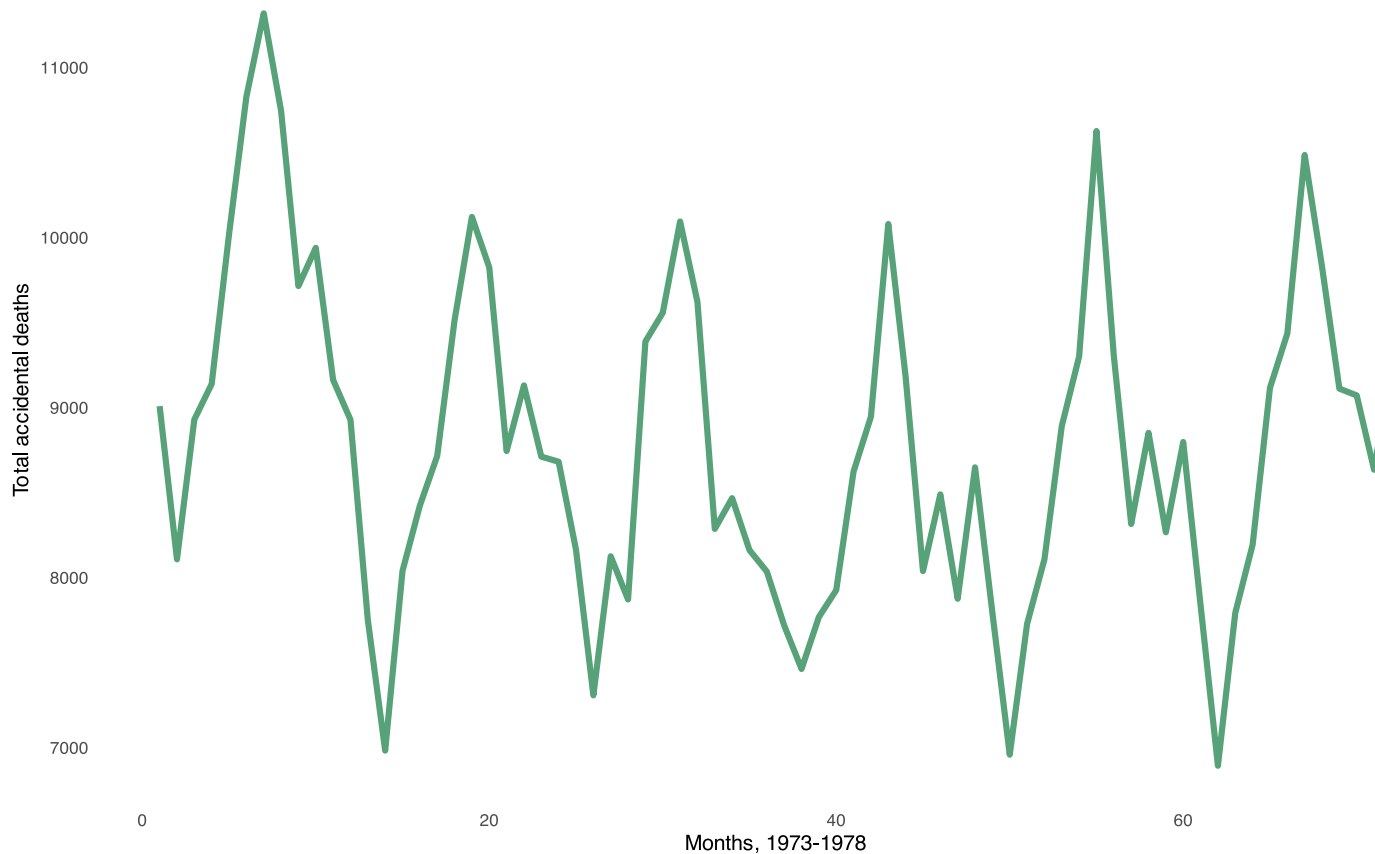
Whatever is left over



# Classical decomposition

Consider a time series of monthly totals of accidental deaths in the USA:

```
df = USAccDeaths
```



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Let's decompose the accidental deaths time series.

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```
decomp = as_tsibble(USAccDeaths) %>%  
  model(  
    classical_decomposition(value, type = "additive")  
  ) %>%  
  components()  
head(decomp)
```

```
#> # A dtable: 6 x 7 [1M]  
#> # Key:      .model [1]  
#> # :      value = trend + seasonal + random  
#>   .model      index value trend seasonal random season_adj  
#>   <chr>      <mth> <dbl> <dbl>    <dbl>    <dbl>    <dbl>  
#> 1 "classical_decomposition(v... 1973 Jan   9007     NA    -806.     NA     98  
#> 2 "classical_decomposition(v... 1973 Feb   8106     NA   -1523.     NA     96  
#> 3 "classical_decomposition(v... 1973 Mar   8928     NA    -741.     NA     96  
#> 4 "classical_decomposition(v... 1973 Apr   9137     NA   -515.     NA     96  
#> 5 "classical_decomposition(v... 1973 May  10017     NA    340.     NA     96  
#> 6 "classical_decomposition(v... 1973 Jun  10826     NA    745.     NA    106
```

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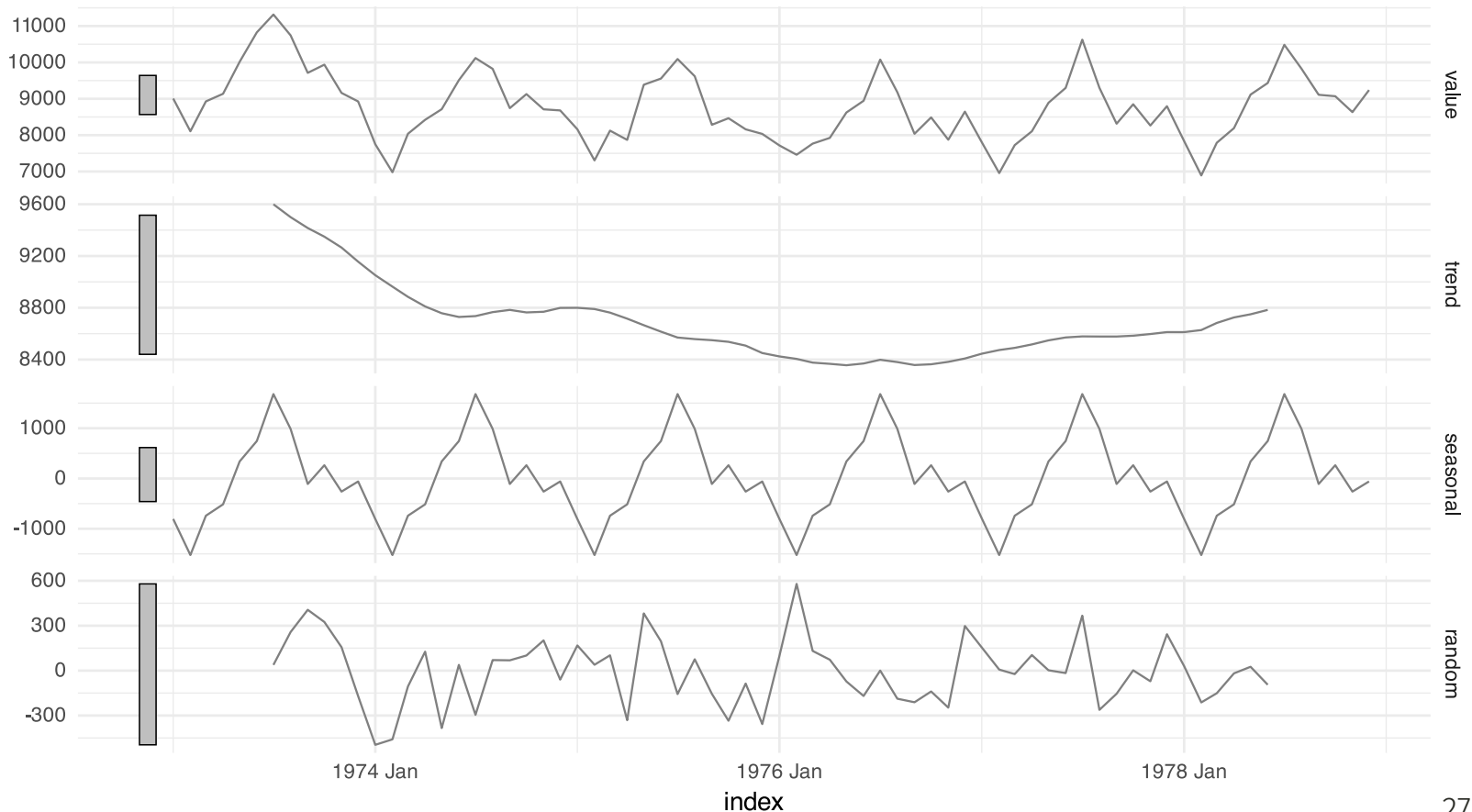
```
as_tsibble(USAccDeaths) %>%  
  model(  
    classical_decomposition(value, type = "additive")  
  ) %>%  
  components() %>%  
  autoplot() +  
  labs(title = "Classical additive decomposition of accidental deaths in the USA")
```

# Classical decomposition

You can do this by hand, or...

Classical additive decomposition of accidental deaths in the USA

value = trend + seasonal + random



# Decomposition

- As outlined in Hyndman & Athanasopoulos, **classical decomposition has some drawbacks:**
  - Assumes the seasonal component is fixed over time
  - Loses data at the start and end (due to moving average)
  - Can be sensitive to outliers/short-run anomalous behavior

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  - Can be sensitive to outliers/short-run anomalous behavior
- **Seasonal and Trend Decomposition using Loess (STL)**
  - Flexible and versatile method
  - Seasonal component can change over time
  - Robust to outliers
  - use `STL()` in place of `classical_decomposition()`

# Decomposition

## Why decompose a time series?

1. To **better understand** your data
  - Do summers tend to have higher crime?
  - Is there an positive trend in ocean temperatures?
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- Do summers tend to have higher crime?
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- Does deforestation follow business cycles?

2. To aid in **forecasting**

- You can forecast using estimated seasonality and trend-cycles
- Details are not covered in this class, see Hyndman & Athanasopoulos for an overview and implementation in `R`

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This matters both for interpreting OLS output (in a few slides), and for understanding our data (helpful for identifying any seasonality).

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- Today's temperature is **positively** correlated with yesterday's temperature:  $cor(y_t, y_{t-1}) > 0$
- Today's temperature is **negatively** correlated with temperatures 6 months ago:  $cor(y_t, y_{t-182}) < 0$
- Today's temperature may have **no correlation** with temperatures 7 days ago:  $cor(y_t, y_{t-7}) = 0$



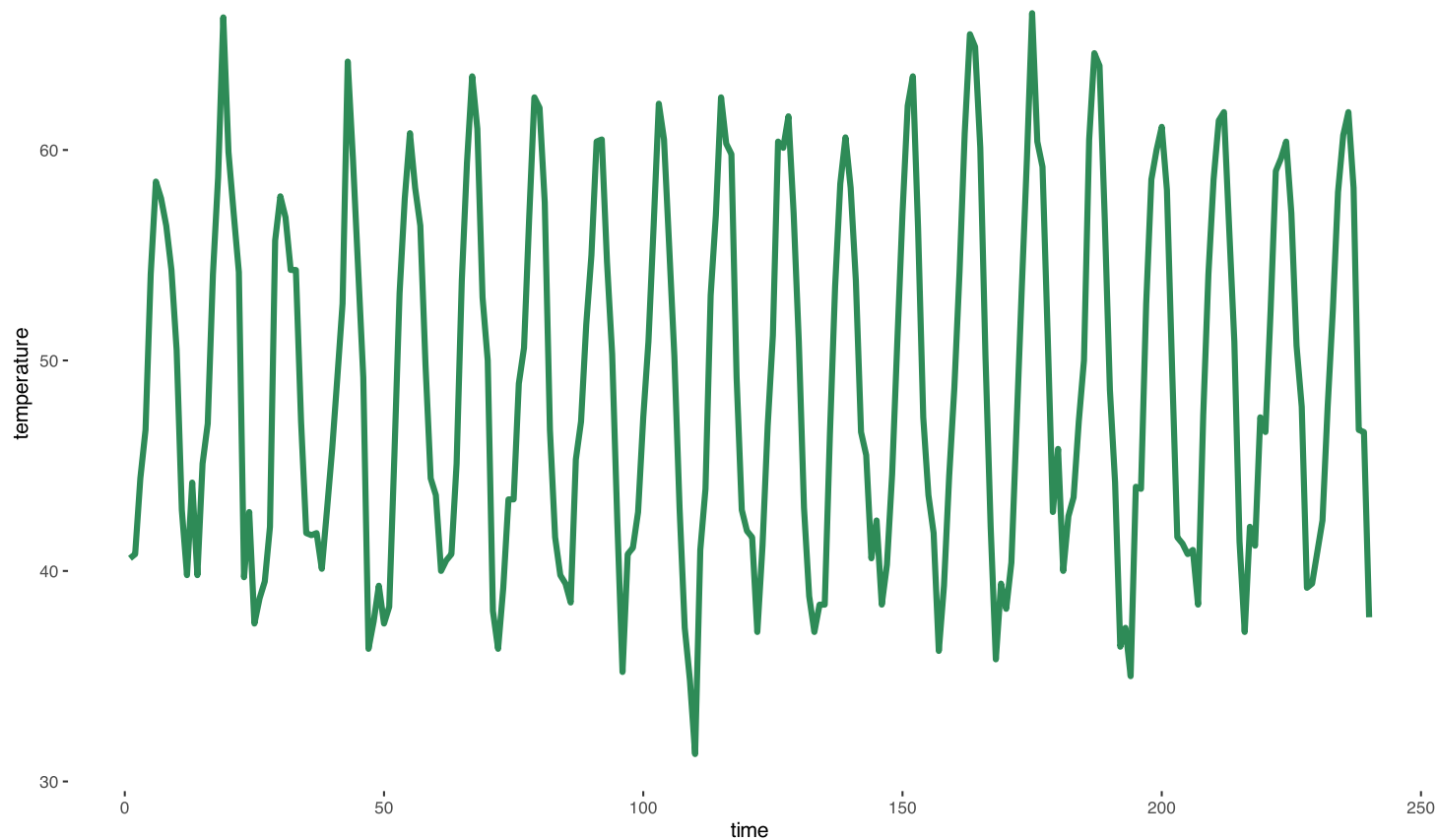
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# Autocorrelation

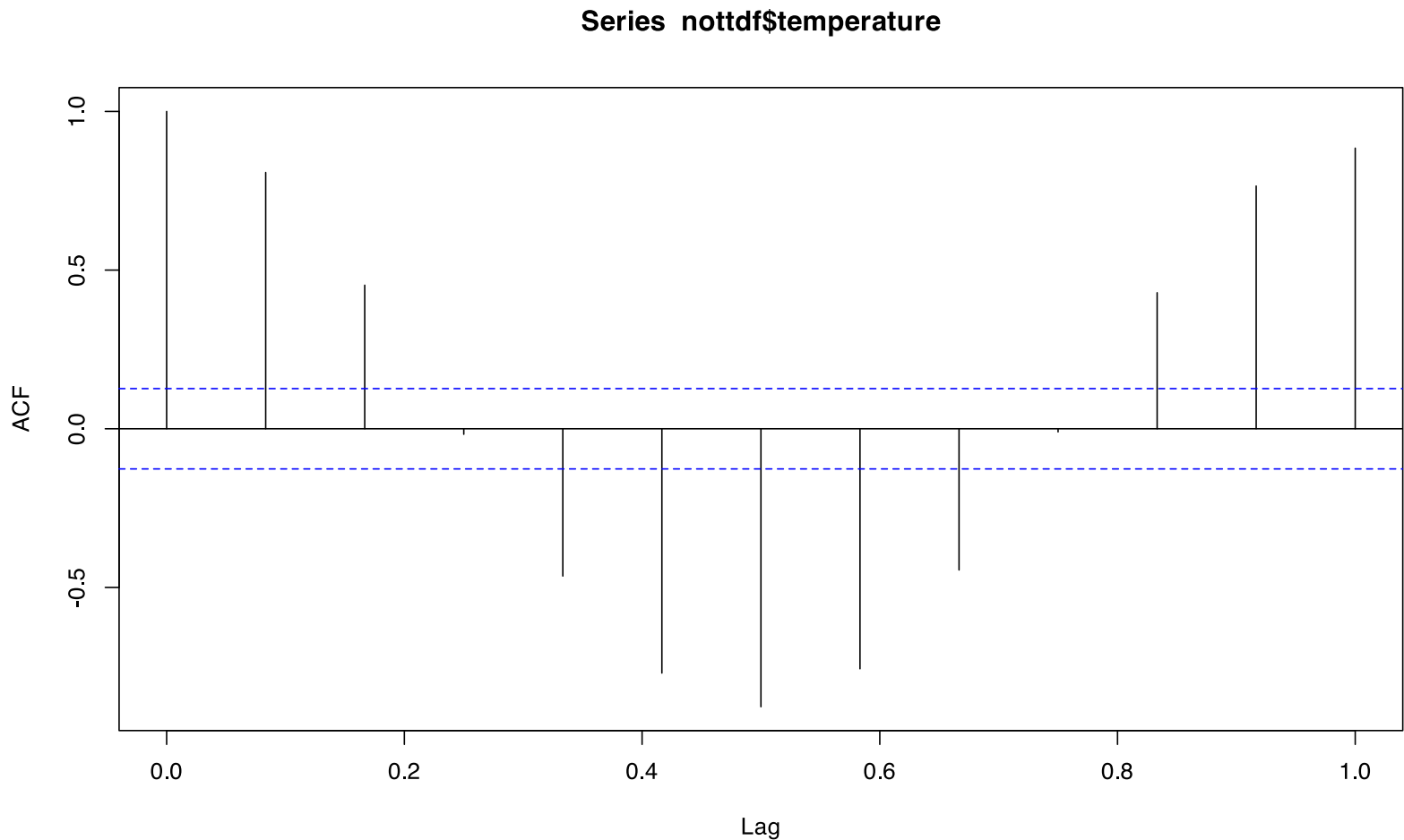
We can describe autocorrelation using an **autocorrelation function** or ACF.

Consider a **monthly** temperature time series for Nottingham Castle



# Autocorrelation Function (ACF)

```
acf(nottdf$temperature, lag.max=12)
```



# Autocorrelation Function (ACF)

| `acf()` plots an ACF for you!

- The height of each line indicates the correlation between temperature today and temperature  $l$  days ago
- Confidence intervals are shown in blue by default -- indicate if  $cor(y_t, y_{t-l})$  is statistically distinguishable from zero (or not)
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Definition: **white noise** is a random time series in which there is no correlation across time periods (rare in the real world!). Here, the ACF would look noisy and correlations would largely fall within the blue confidence interval.

# Time series and OLS

# Intro to time series and OLS

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where  $t - 1$  denotes the time period prior to  $t$  (*lagged* income or births).

# Time-series models

## Assumptions

1. **New: Weakly persistent outcomes**—essentially,  $x_{t+k}$  in the distant period  $t + k$  is weakly correlated with period  $x_t$  (when  $k$  is "big").
2.  $y_t$  is a **linear function** of its parameters and disturbance.
3. There is **some variation** in our explanatory variables
4. **Harder to satisfy:** The  $u_t$  have conditional mean of zero (**exogeneity**),  $E[u_t|X] = 0$ .
5. **Harder to satisfy:** The  $u_t$  are **normally distributed** and **homoskedastic** with **zero correlation** between  $u_t$  and  $u_s$ , i.e.,  $u_t \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$ ,  $\text{Var}(u_t|X) = \text{Var}(u_t) = \sigma^2$ , and  $\text{Cor}(u_t, u_s|X) = 0$ .

# Time-series models

## Model options

Time-series modeling boils down to two classes of models.

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2. **Dynamic models:** Allow for persistent effects.

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1. **Static models:** Do not allow for persistent effect.
2. **Dynamic models:** Allow for persistent effects.
  - Models with **lagged explanatory** variables
  - **Autoregressive, distributed-lag** (ADL) models

# Model options

## Option 1: Static models

**Static models** assume the outcome depends upon **only the current period**.

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We also need to believe current births do not depend upon previous births.

Can be a very restrictive way to consider time-series data.

# Model options

## Option 2: **Dynamic models**

**Dynamic models** allow the outcome to depend upon **other periods**.

# Model options

**Option 2a: Dynamic models** with lagged explanatory variables

These models allow the outcome to depend upon the explanatory variable(s) in other periods.

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Income}_{t-1} + \beta_3 \text{Income}_{t-2} + \beta_4 \text{Income}_{t-3} + u_t$$

# Model options

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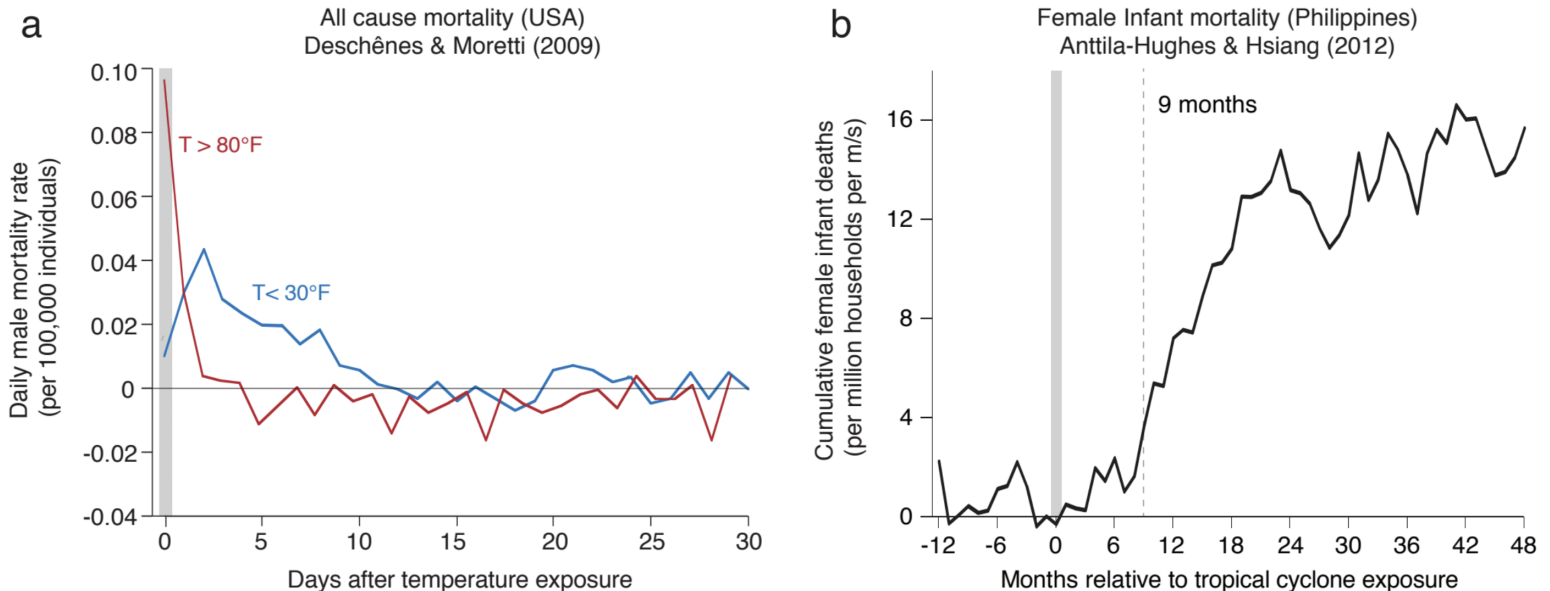
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*Note:* We still assume current births don't affect future births.

# Model options

Lagged explanatory variables in empirical research:

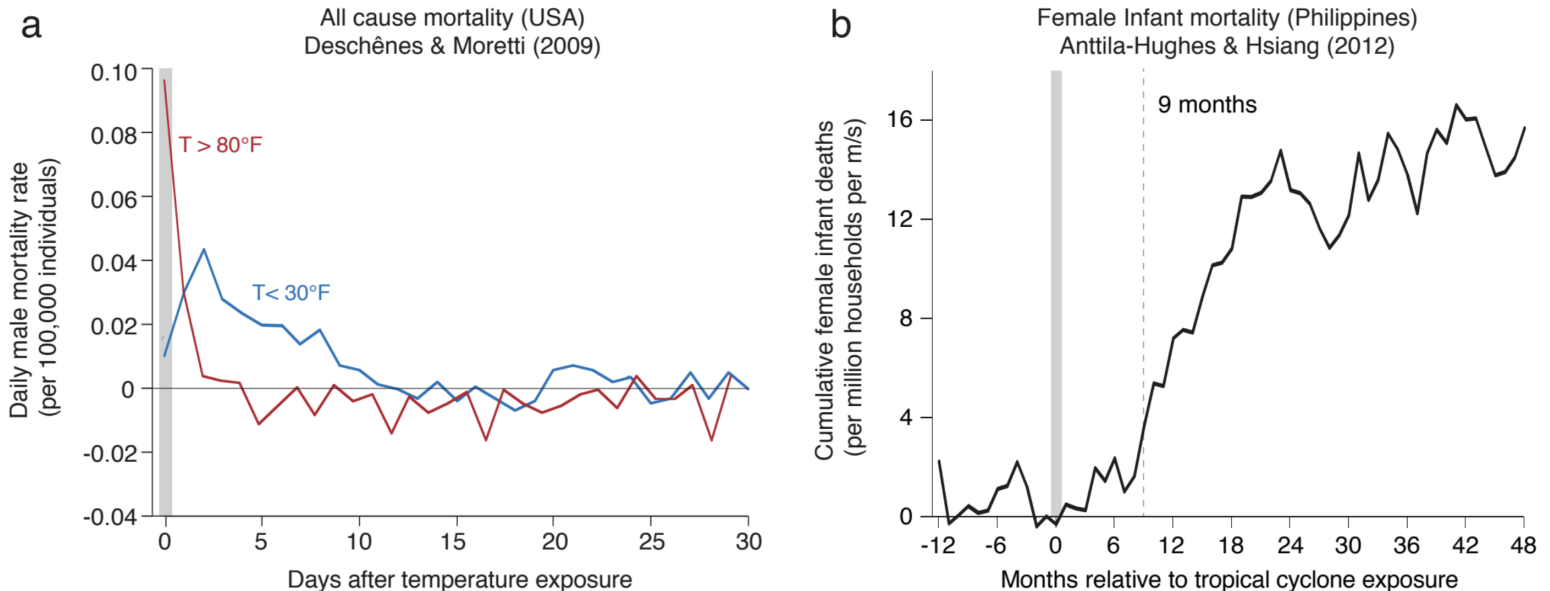


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# Model options

Lagged explanatory variables in empirical research:



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Q: Can you think of other examples of lagged effects?

# Model options

## Option 2b: Autoregressive distributed-lag (ADL) models

These models allow the outcome to depend upon the explanatory variable(s) and/or the outcome variable in prior periods.

$$\text{Births}_{t_{\text{red}}} = \beta_0 + \beta_1 \text{Income}_{t_{\text{red}}} + \beta_2 \text{Income}_{t_{\text{blue}}-1} + \beta_3 \text{Births}_{t_{\text{blue}}-1} + u_{t_{\text{red}}}$$

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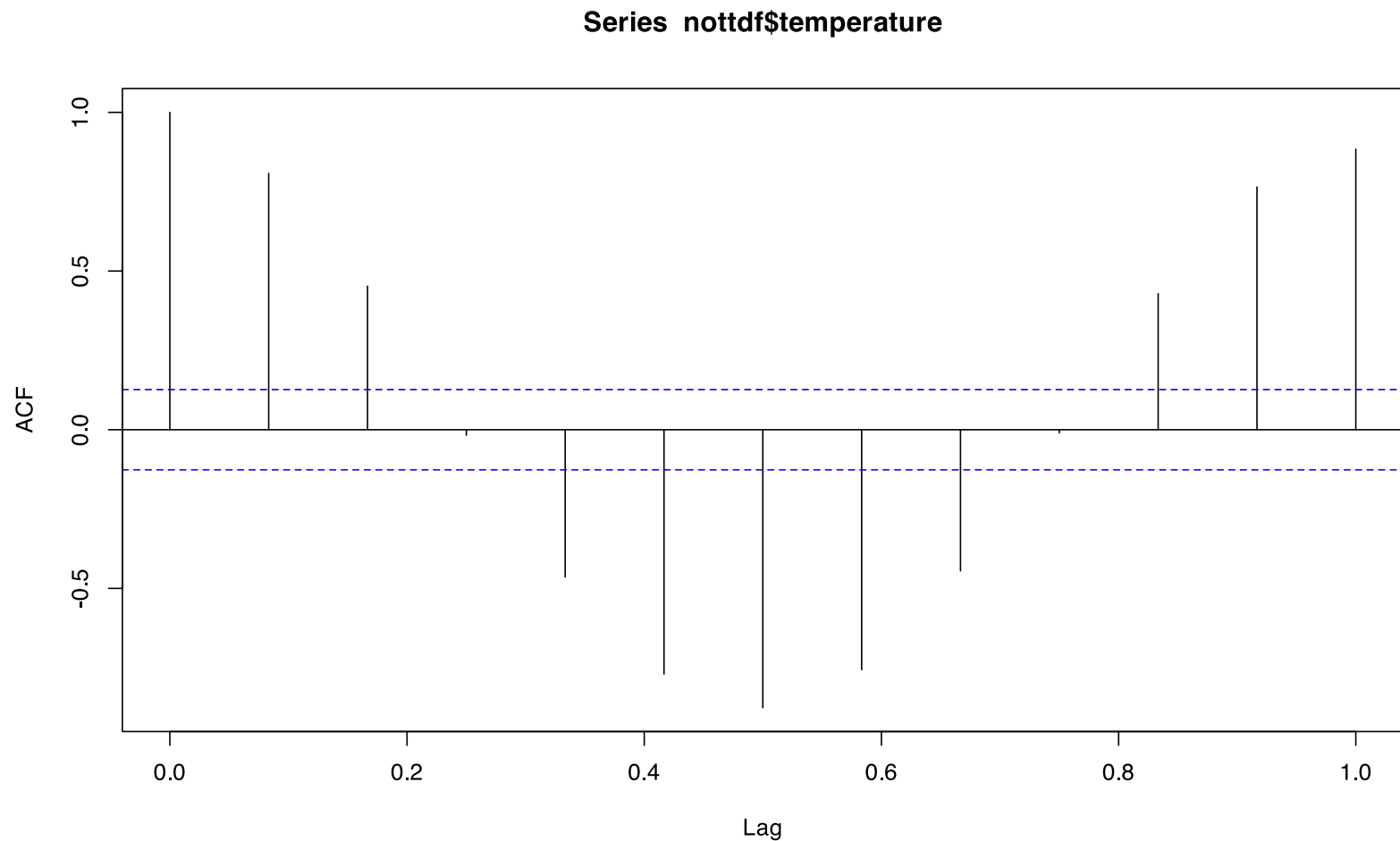
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Here, current income affects affects **current** births and **future** births.

In addition, **current births affect future births**—we're allowing lags of the outcome variable.

# Do you need an ADL?

Hint: Autocorrelation Function (ACF)



# Autoregressive distributed-lag models

## Numbers of lags

ADL models are often specified as  $\text{ADL}(p, q)$ , where

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which we can substitute in for  $\text{Births}_{t-1}$  in the first equation, *i.e.*,

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + \underbrace{\beta_2(\beta_0 + \beta_1 \text{Income}_{t-1} + \beta_2 \text{Births}_{t-2} + u_{t-1})}_{\text{Births}_{t-1}} + u_t$$

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Continuing...

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We could then substitute in the equation for  $\text{Births}_{t-2}$ ,  $\text{Births}_{t-3}$ , ...



# Complexity

Eventually we arrive at

$$\begin{aligned}\text{Births}_t = & \beta_0 (1 + \beta_2 + \beta_2^2 + \beta_2^3 + \dots) + \\ & \beta_1 (\text{Income}_t + \beta_2 \text{Income}_{t-1} + \beta_2^2 \text{Income}_{t-2} + \dots) + \\ & u_t + \beta_2 u_{t-1} + \beta_2^2 u_{t-2} + \dots\end{aligned}$$

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## The point?

By including just **one lag of the dependent variable**—as in a ADL(1, 0)—we implicitly include for *many lags* of the explanatory variables and disturbances.<sup>†</sup>

<sup>†</sup> These lags enter into the equation in a very specific way—not the most flexible specification.

# Time-series models

## Assumptions

1. **New: Weakly persistent outcomes**—essentially,  $x_{t+k}$  in the distant period  $t + k$  is weakly correlated with period  $x_t$  (when  $k$  is "big").
2.  $y_t$  is a **linear function** of its parameters and disturbance.
3. There is **some variation** in our explanatory variables
4. **Harder to satisfy:** The  $u_t$  have conditional mean of zero (**exogeneity**),  $E[u_t|X] = 0$ .
5. **Harder to satisfy:** The  $u_t$  are **normally distributed** and **homoskedastic** with **zero correlation** between  $u_t$  and  $u_s$ , i.e.,  $u_t \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$ ,  $\text{Var}(u_t|X) = \text{Var}(u_t) = \sigma^2$ , and  $\text{Cor}(u_t, u_s|X) = 0$ .

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As before, the unbiased-ness of OLS is going to depend upon our exogeneity assumption, *i.e.*,  $\mathbf{E}[u_t|X] = 0$ .

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Thus, **OLS is biased for dynamic models with lagged outcome variables.**

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To see why dynamic models with lagged outcome variables violate our exogeneity assumption, consider two periods of our simple ADL(1, 0) model.

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This correlation violates the second part of our exogeneity requirement.

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With contemporaneous exogeneity, OLS estimates for the coefficients in a time series model are **consistent** (whew)

# Autocorrelation in the error term

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Are we worried? In a static model with lagged explanatory variables:

- OLS is **inefficient**, i.e., no longer the lowest variance unbiased estimator
- That is, your standard errors are no longer correct
- However, violating this assumption does not introduce bias (whew!)

# Autocorrelation

## OLS and lagged outcome variables

Consider a model with one lag of the outcome variable—ADL(1, 0)—model with AR(1) disturbances

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**A:** It violates **contemporaneous exogeneity**, *i.e.*,  $\text{Cov}(x_t, u_t) \neq 0$ .

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  - Test whether  $\hat{\theta}$  is statistically distinguishable from zero in

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- Implement in R with: `dwtest()`, `bgtest()`

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- Test whether  $\hat{\theta}$  is statistically distinguishable from zero in

$$e_t = \theta_1 e_{t-1} + \theta_2 e_{t-2} + \dots$$

- Implement in R with: `dwtest()`, `bgtest()`
- Autocorrelation may arise because your model is **misspecified**. Consider adding additional lags and/or explanatory variables if errors are correlated



# Summary: Time series and OLS

- Our model now has  $t$  subscripts for **time periods**.
- **Dynamic models** allow **lags** of explanatory and/or outcome variables.
- We changed our **exogeneity** assumption to **contemporaneous exogeneity**, i.e.,  $E[u_t|X_t] = 0$
- Including **lags of outcome variables** can lead to **biased coefficient estimates** from OLS (but fortunately they are still **consistent**)
- **Lagged explanatory variables** make **OLS inefficient** (i.e., mess up our standard errors)
- **Autocorrelation in the error + lagged dependent variables** make **OLS biased**. Watch out! Test for serial/autocorrelation, check for misspecification of your model.

Slides created via the R package **xaringan**.

Some slide components were borrowed from **Ed Rubin** and Allison Horst.