Time series analysis

EDS 222

Tamma Carleton

Fall 2023

Announcements/check-in

- Assignment 04 posted, due 12/08

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- A note on depth in coming lectures

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- A note on depth in coming lectures
- **No class** 11/23
- Final projects: due in 3.5 weeks!
    - Presentations: 12/12 4:00-7:00pm (Bren Hall 1424)
    - Blog posts: 12/15

Today

What are time series data?

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Decomposition

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Autocorrelation

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Autocorrelation

Time series and OLS

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Up to this point, we focused on **cross-sectional data**.

- Sampled *across* a population (*e.g.*, people, counties, countries).
- Sampled at *one moment* in time (*e.g.*, Jan. 1, 2015).
- We had n *individuals*, each indexed i in $\{1, \ldots, n\}$.

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Up to this point, we focused on **cross-sectional data**.

- Sampled *across* a population (*e.g.*, people, counties, countries).
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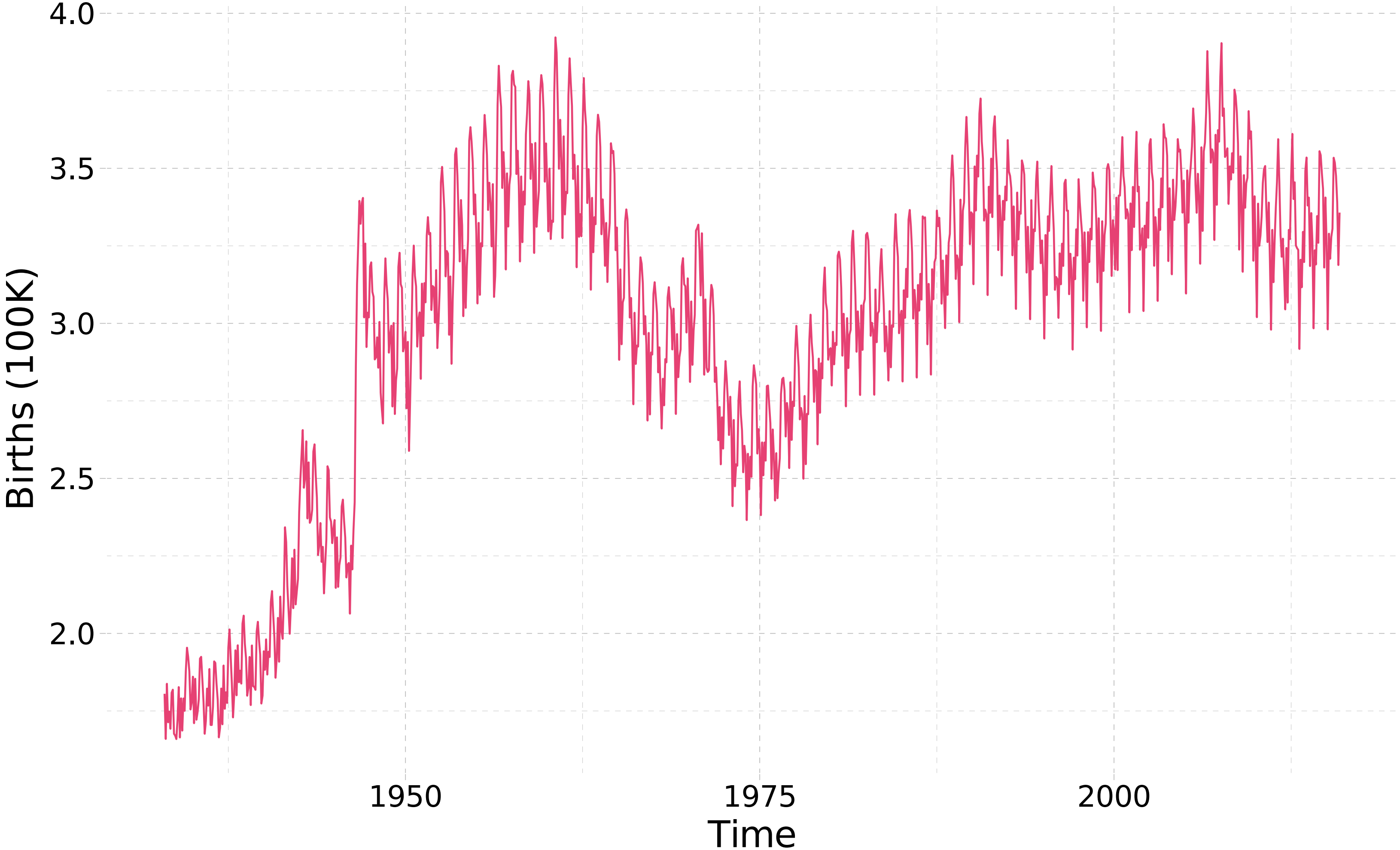
Today, we focus on a different type of data: **time-series data**.

- Sampled within one unit/individual (*e.g.*, Oregon).
- Observe multiple times for the same unit (*e.g.*, Oregon: 1990–2020).
- We have T time periods, each indexed t in $\{1, \ldots, T\}$.

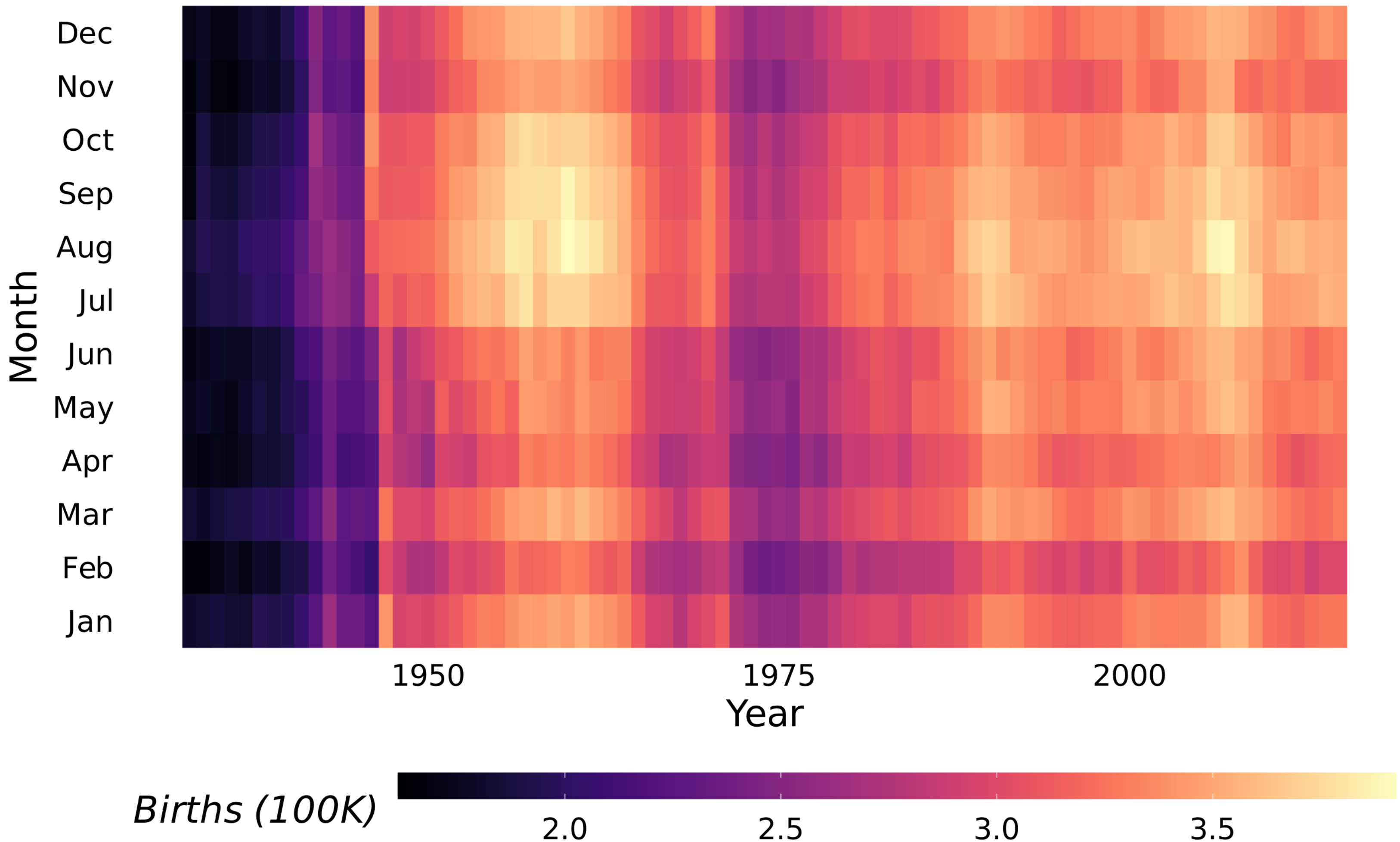
Time series data: Example

US monthly births, 1933–2015: Classic time-series graph



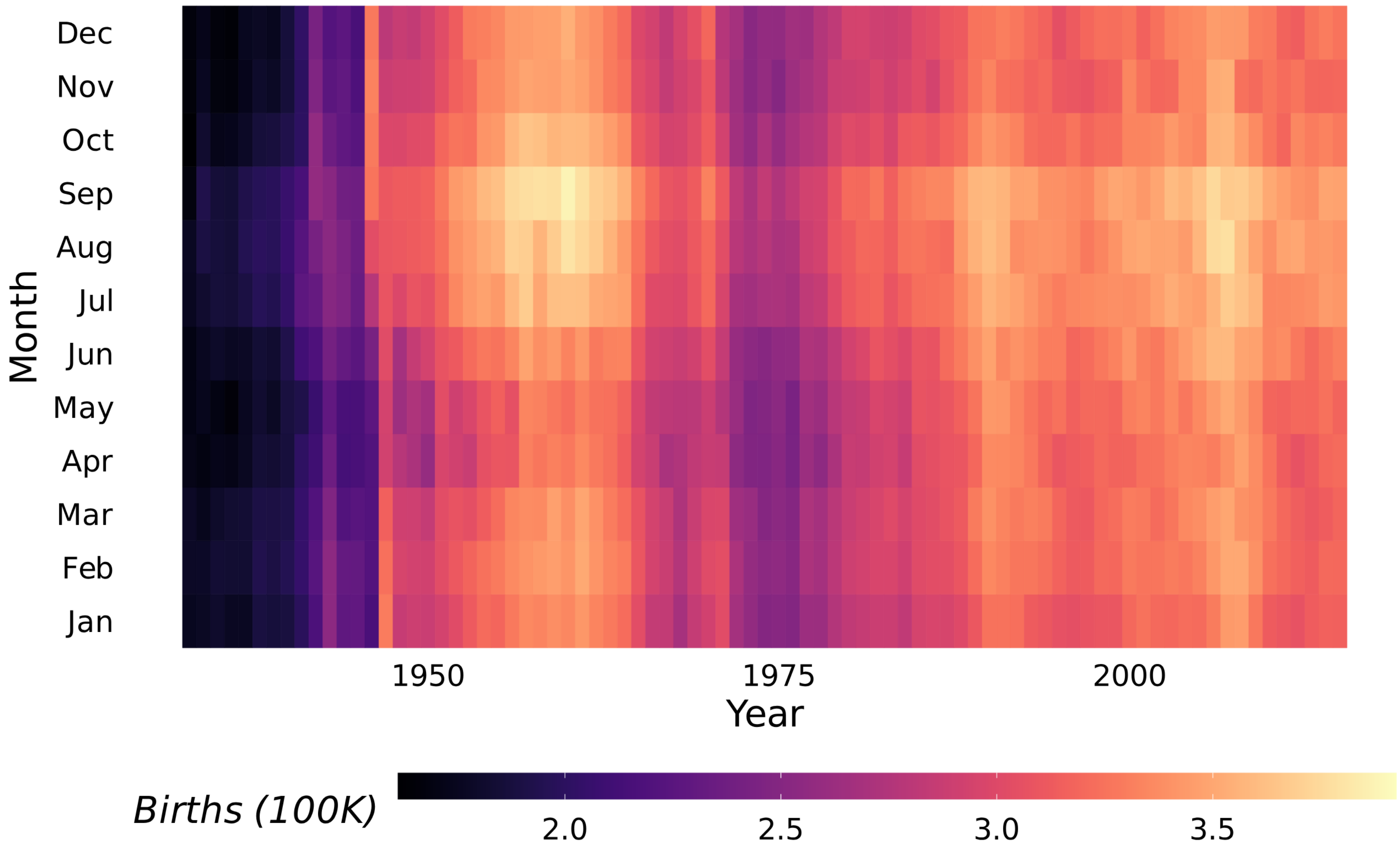
Time series data: Example

US monthly births, 1933–2015: Newfangled time-series graph



Time series data: Example

US monthly births per 30 days, 1933–2015: Newfangled time-series graph



You already have (many of) the tools

- Time series data open some **new questions and new challenges** for statistical analysis

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$$Ozone_t = \beta_0 + \beta_1 Temp_t + \varepsilon_t$$

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- Description of `airquality` data:

> Daily air quality measurements in New York, May to September 1973.

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- Description of `airquality` data:

> Daily air quality measurements in New York, May to September 1973.

- These are **time series data** and we already ran an OLS regression with them!

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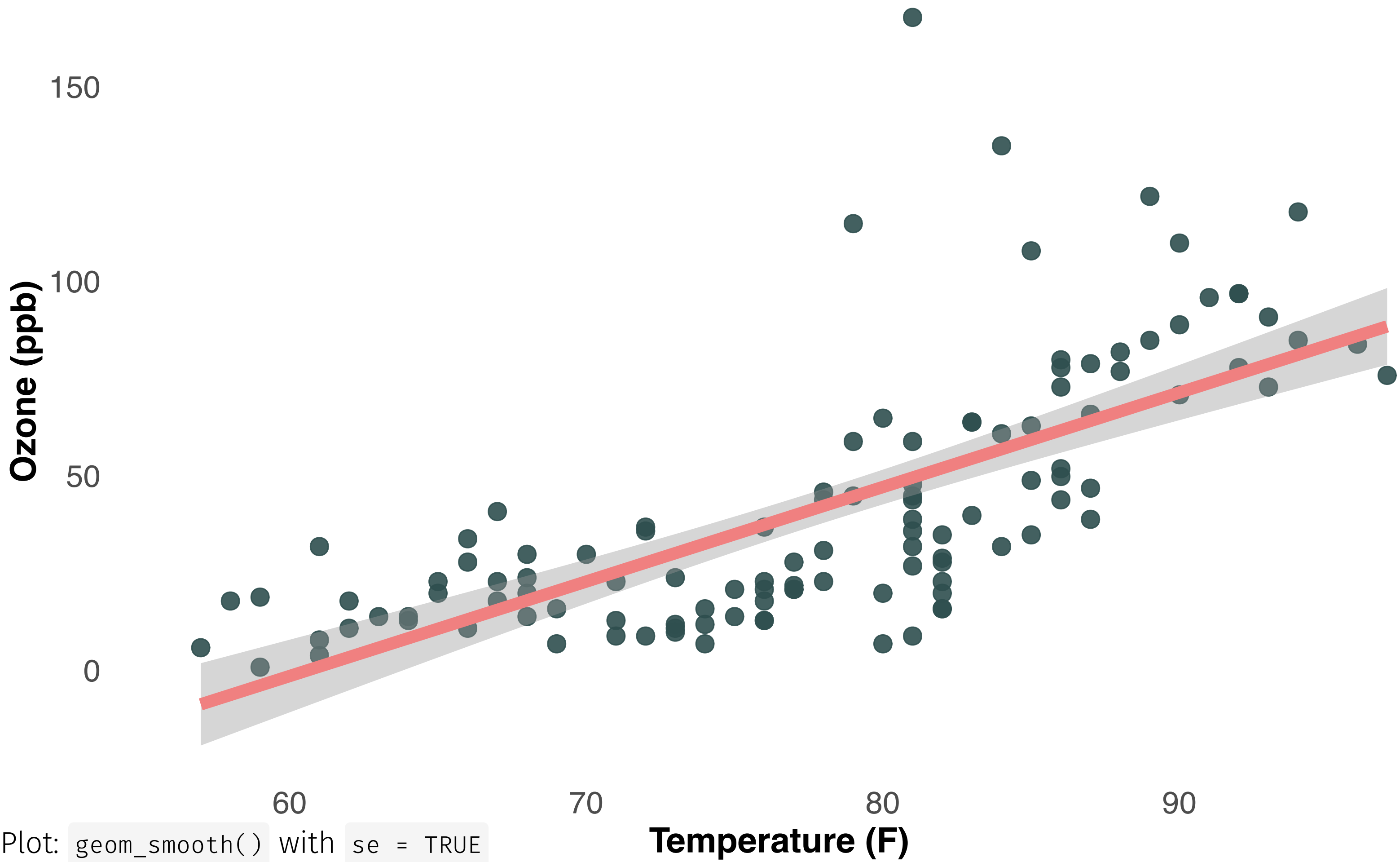

Plot: `geom_smooth()` with `se = TRUE`

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```
airqts = airquality %>% mutate(date = make_datetime(1973, Month,Day))
head(airqts)
#>   Ozone Solar.R Wind Temp Month Day       date
#> 1    41     190  7.4   67     5   1 1973-05-01
#> 2    36     118  8.0   72     5   2 1973-05-02
#> 3    12     149 12.6   74     5   3 1973-05-03
#> 4    18     313 11.5   62     5   4 1973-05-04
#> 5    NA      NA 14.3   56     5   5 1973-05-05
#> 6    28      NA 14.9   66     5   6 1973-05-06
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```

- Regression of *Ozone* on *date* estimates a **linear trend** in ozone
- Tip: `make_datetime()` from the `lubridate` package (handy for dates and times)

You already have (many of) the tools

$$Ozone_t = \beta_0 + \beta_1 date_t + \varepsilon_t$$

```
summary(lm(Ozone ~ date, data = airqts))
#> 
#> Call:
#> lm(formula = Ozone ~ date, data = airqts)
#> 
#> Residuals:
#>    Min      1Q  Median      3Q     Max 
#> -42.32 -24.58  -8.39  20.46 122.05 
#> 
#> Coefficients:
#>              Estimate Std. Error t value Pr(>|t|)
#> (Intercept) -1.04e+02   8.59e+01   -1.21    0.230
#> date         1.30e-06   7.65e-07    1.70    0.092 .
#> ---
#> Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
#> 
#> Residual standard error: 32.7 on 114 degrees of freedom
#>   (37 observations deleted due to missingness)
#> Multiple R-squared:  0.0247,    Adjusted R-squared:  0.0162 
#> F-statistic: 2.89 on 1 and 114 DF,  p-value: 0.092
```

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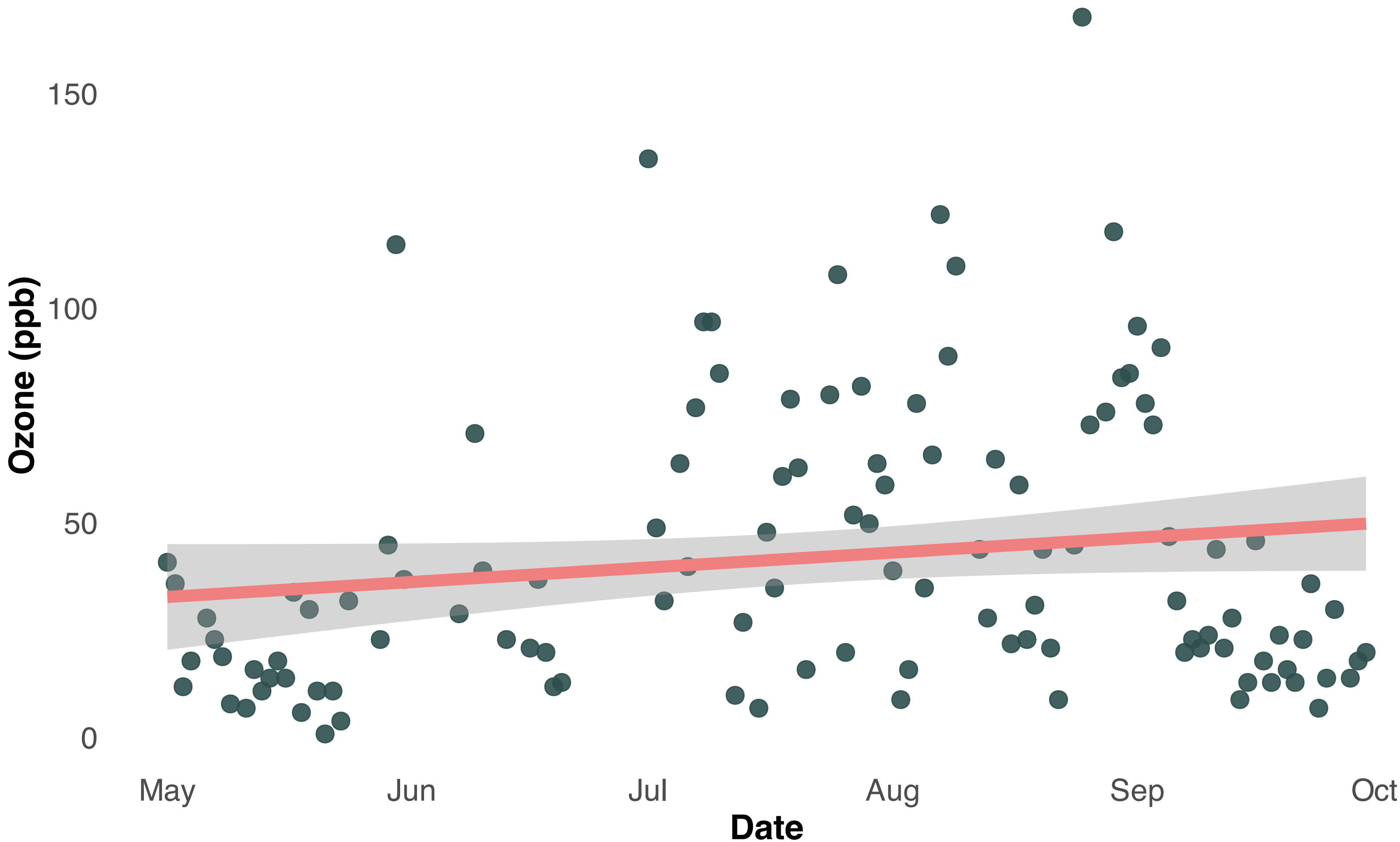
$$Ozone_t = \beta_0 + \beta_1 date_t + \varepsilon_t$$



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- But there are some new **features** we want to explore:
    - Does my data have exhibit **trending behavior**?
    - Is there **seasonality**?
    - Is my data **cyclical**?

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- But there are some new **features** we want to explore:
  - Does my data have exhibit **trending behavior**?
  - Is there **seasonality**?
  - Is my data **cyclical**?
- And some new **challenges** to overcome:
  - Additional **assumptions** needed in OLS
  - Threat to existing assumptions: Are our error terms **independent**? Is **exogeneity** harder now?

Decomposition

Time series components

Seasonality

A repeated pattern over known and equal periods (e.g., month; quarter, decade)

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A broader cyclical trend with unknown and/or unequal periods (e.g., business cycle, ENSO)

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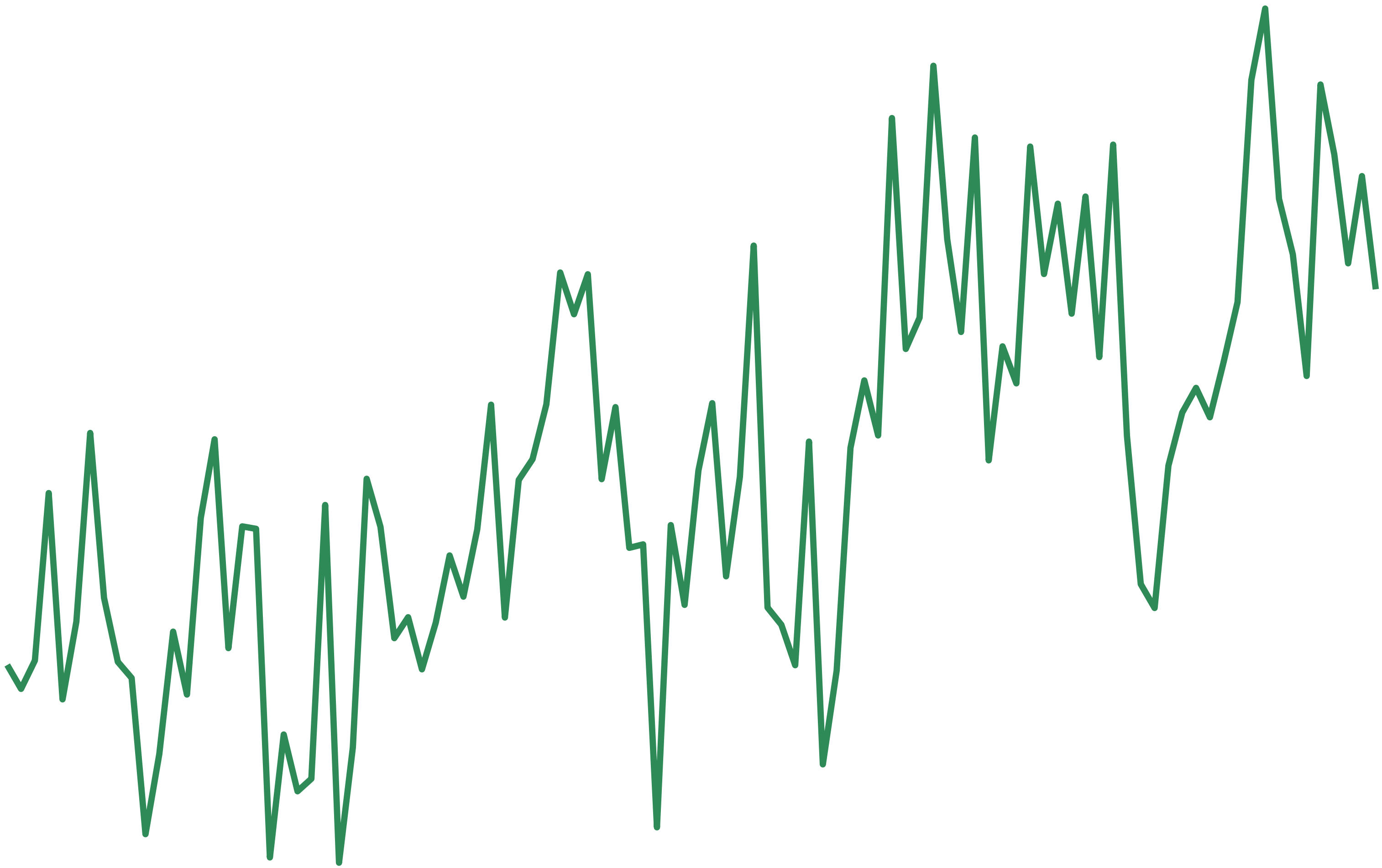
Trends

Long-term increase or decrease in the data (not necessarily linear!)

Time series components

Often, seasonality, cyclicality and trends occur all at the same time:



Time series components

For many time series,* we can decompose the data into:

$$y_t = S_t + T_t + R_t$$

where S_t is a **seasonal** component, T_t is the cycle *and* trend components, and R_t is the remainder.

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where S_t is a **seasonal** component, T_t is the cycle *and* trend components, and R_t is the remainder.

Decomposition allows us to isolate each component of the time series visually and quantitatively.

[*]: This decomposition is "additive", which works for many time series. See Hyndman for details on more complex "multiplicative" decomposition.

Decomposition: Moving averages

A key tool in "decomposing" a time series into its component parts is computing a **moving average**

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A moving average of order *m* is computed as:

$$\hat{T}_t = \frac{1}{m} \sum_{j=-k}^{k} y_{t+j}$$

where $m = 2k + 1$.

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A moving average of order *m* is computed as:

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where $m = 2k + 1$.

The moving average gives you an estimate of the irregular trend-cycle component T at time *t* by averaging values of the time series within *k* periods of *t*

Moving average example

Computing an $m = 5$ moving average over the data plotted on the last slide:

```
df = as.data.frame(cbind(x, y)) # these are the data we plotted above
df = df %>% mutate(ma = slider::slide_dbl(y, mean,
                   .before = 2, .after = 2, .complete = TRUE))
```

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```

- Helpful package: `slider` (there are others too!)
- Option `.complete=TRUE` ensures only moving windows with complete data are computed

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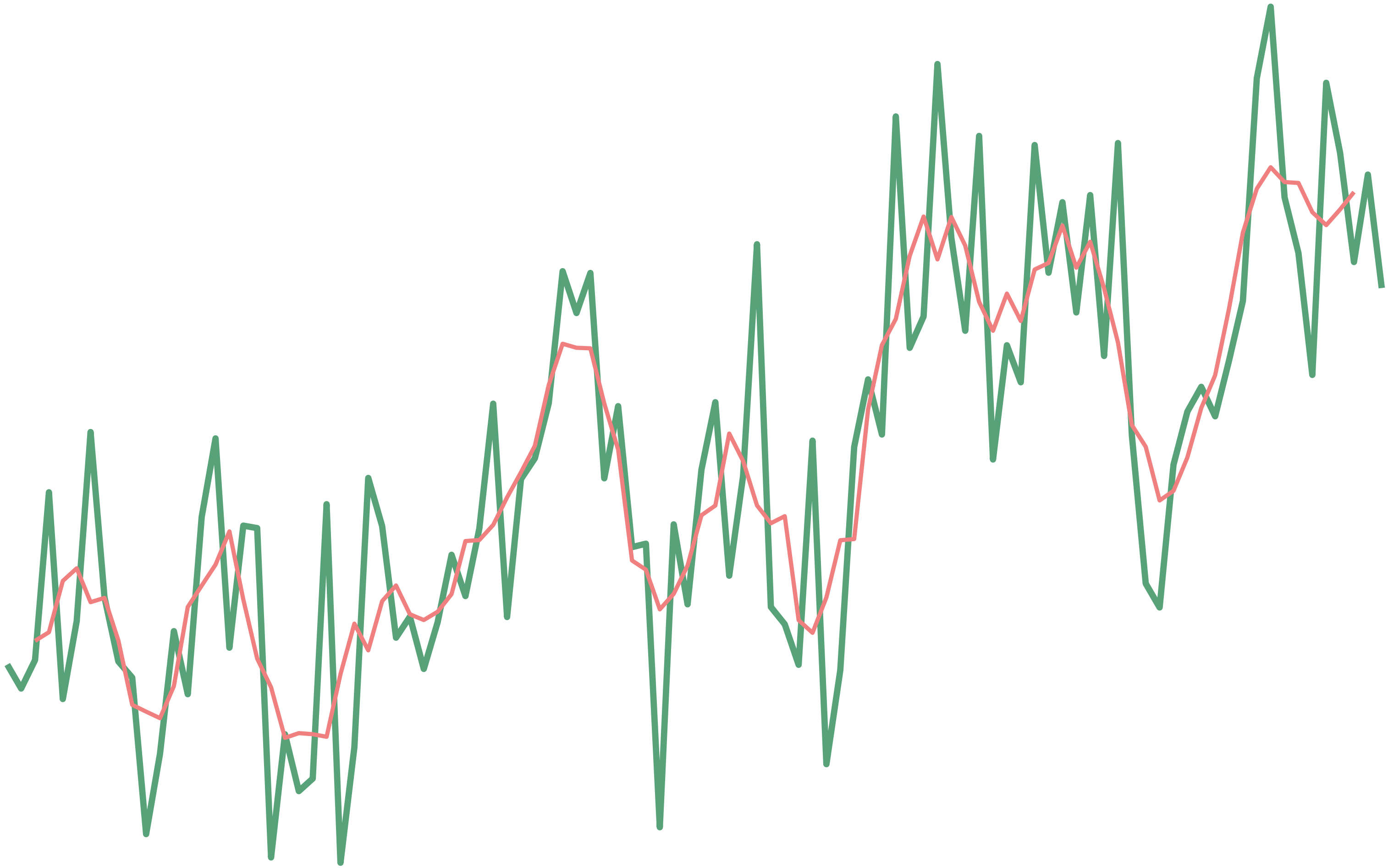
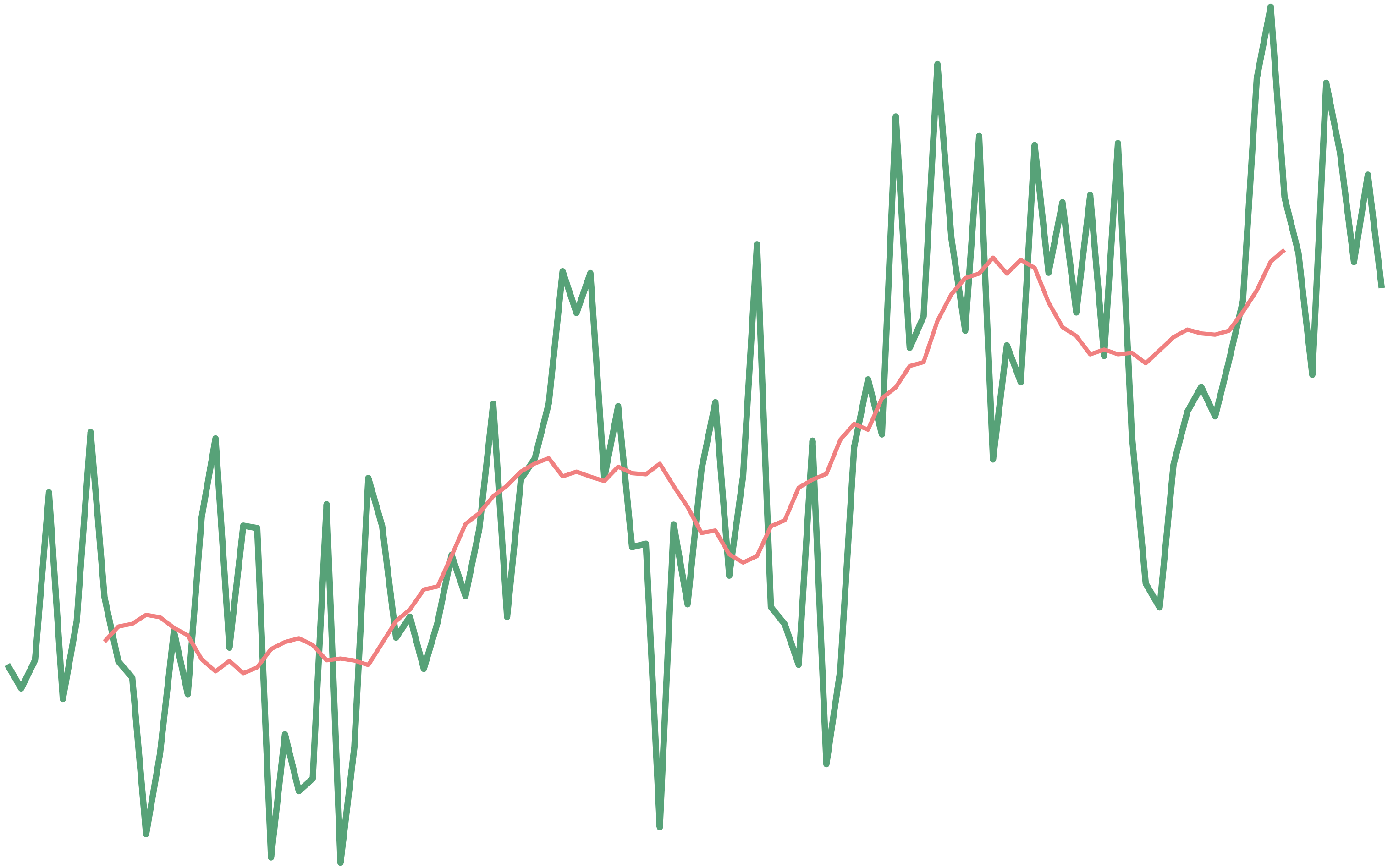


Moving average example

Computing an $m = 15$ moving average:



Classical decomposition

Step 1: estimate a moving average

Estimate an m-moving average to compute $\hat{T}_t$

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De-trended series $= y_t - \hat{T}_t$

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Simple average over de-trended series for each season s

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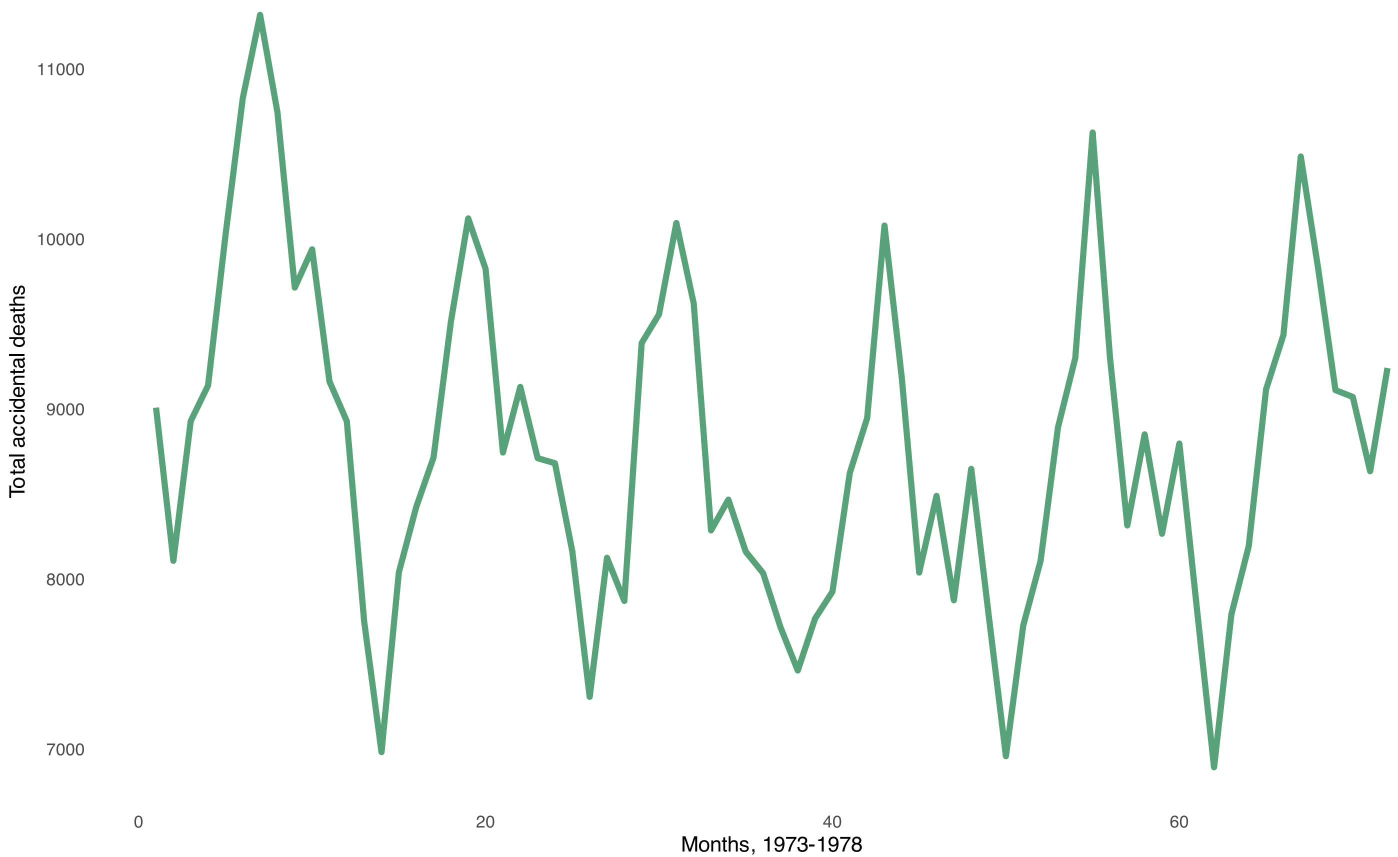
Step 4: remainder

Whatever is left over

Classical decomposition

Consider a time series of monthly totals of accidental deaths in the USA:

```
df = USAccDeaths
```



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Let's decompose the accidental deaths time series.

You can do this by hand, or...

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```
decomp = as_tsibble(USAccDeaths) %>%
  model(
    classical_decomposition(value, type = "additive")
  ) %>%
  components()
head(decomp)
#> # A dable: 6 x 7 [1M]
#> # Key:      .model [1]
#> # :         value = trend + seasonal + random
#>   .model                          index value trend seasonal random season_adj
#>   <chr>                           <mth> <dbl> <dbl>    <dbl>  <dbl>          <d
#> 1 "classical_decomposition(v… 1973 Jan  9007    NA    -806.     NA           98
#> 2 "classical_decomposition(v… 1973 Feb  8106    NA   -1523.     NA           96
#> 3 "classical_decomposition(v… 1973 Mar  8928    NA    -741.     NA           96
#> 4 "classical_decomposition(v… 1973 Apr  9137    NA    -515.     NA           96
#> 5 "classical_decomposition(v… 1973 May 10017    NA     340.     NA           96
#> 6 "classical_decomposition(v… 1973 Jun 10826    NA     745.     NA           100
```

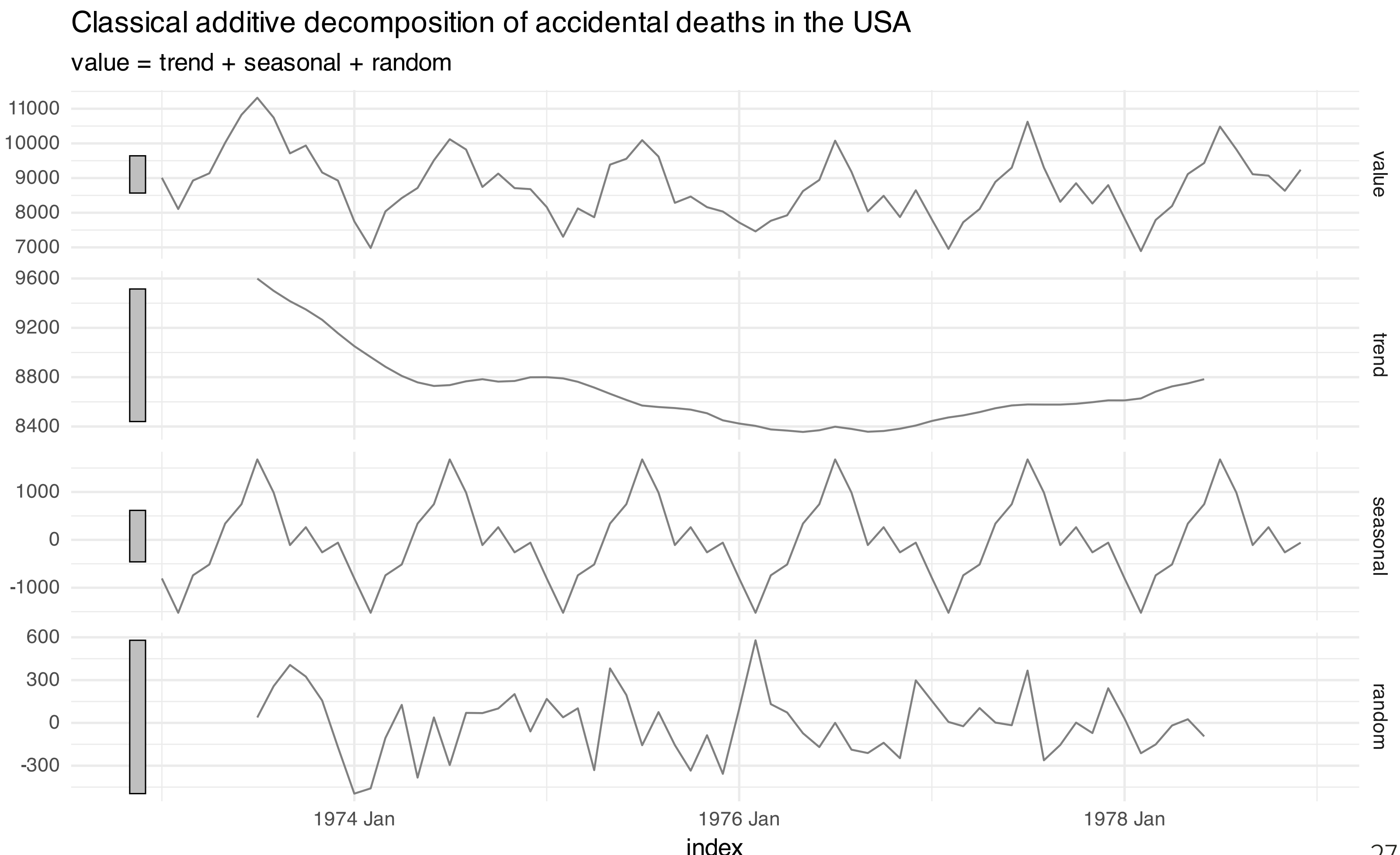
Classical decomposition

You can do this by hand, or...

```
as_tsibble(USAccDeaths) %>%
  model(
    classical_decomposition(value, type = "additive")
  ) %>%
  components() %>%
  autoplot() +
  labs(title = "Classical additive decomposition of accidental deaths in the USA
```

Classical decomposition

You can do this by hand, or...



Decomposition

- As outlined in Hyndman & Athanasopoulos, **classical decomposition has some drawbacks**:
  - Assumes the seasonal component is fixed over time
  - Loses data at the start and end (due to moving average)
  - Can be sensitive to outliers/short-run anomalous behavior

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  - Assumes the seasonal component is fixed over time
  - Loses data at the start and end (due to moving average)
  - Can be sensitive to outliers/short-run anomalous behavior
- **Seasonal and Trend Decomposition using Loess** (STL)
  - Flexible and versatile method
  - Seasonal compoenent can change over time
  - Robust to outliers
  - use `STL()` in place of `classical_decomposition()`

Decomposition

Why decompose a time series?

1. To **better understand** your data
    - Do summers tend to have higher crime?
    - Is there an positive trend in ocean temperatures?
    - Does deforestation follow business cycles?

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1. To **better understand** your data
    - Do summers tend to have higher crime?
    - Is there an positive trend in ocean temperatures?
    - Does deforestation follow business cycles?
2. To aid in **forecasting**
    - You can forecast using estimated seasonality and trend-cycles
    - Details are not covered in this class, see Hyndman & Athanasopoulos for an overview and implementation in `R`

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That is, y_t may be correlated with y_{t-1}, y_{t-2}, y_{t-12}, etc.

This matters both for interpreting OLS output (in a few slides), and for understanding our data (helpful for identifying any seasonality).

Autocorrelation

For example:

- Today's temperature is **positively** correlated with yesterday's temperature: $cor(y_t, y_{t-1}) > 0$

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- Today's temperature is **positively** correlated with yesterday's temperature: $cor(y_t, y_{t-1}) > 0$
- Today's temperature is **negatively** correlated with temperatures 6 months ago: $cor(y_t, y_{t-182}) < 0$
- Today's temperature may have **no correlation** with temperatures 7 days ago: $cor(y_t, y_{t-7}) = 0$

Autocorrelation

We can describe autocorrelation using an **autocorrelation function** or ACF.

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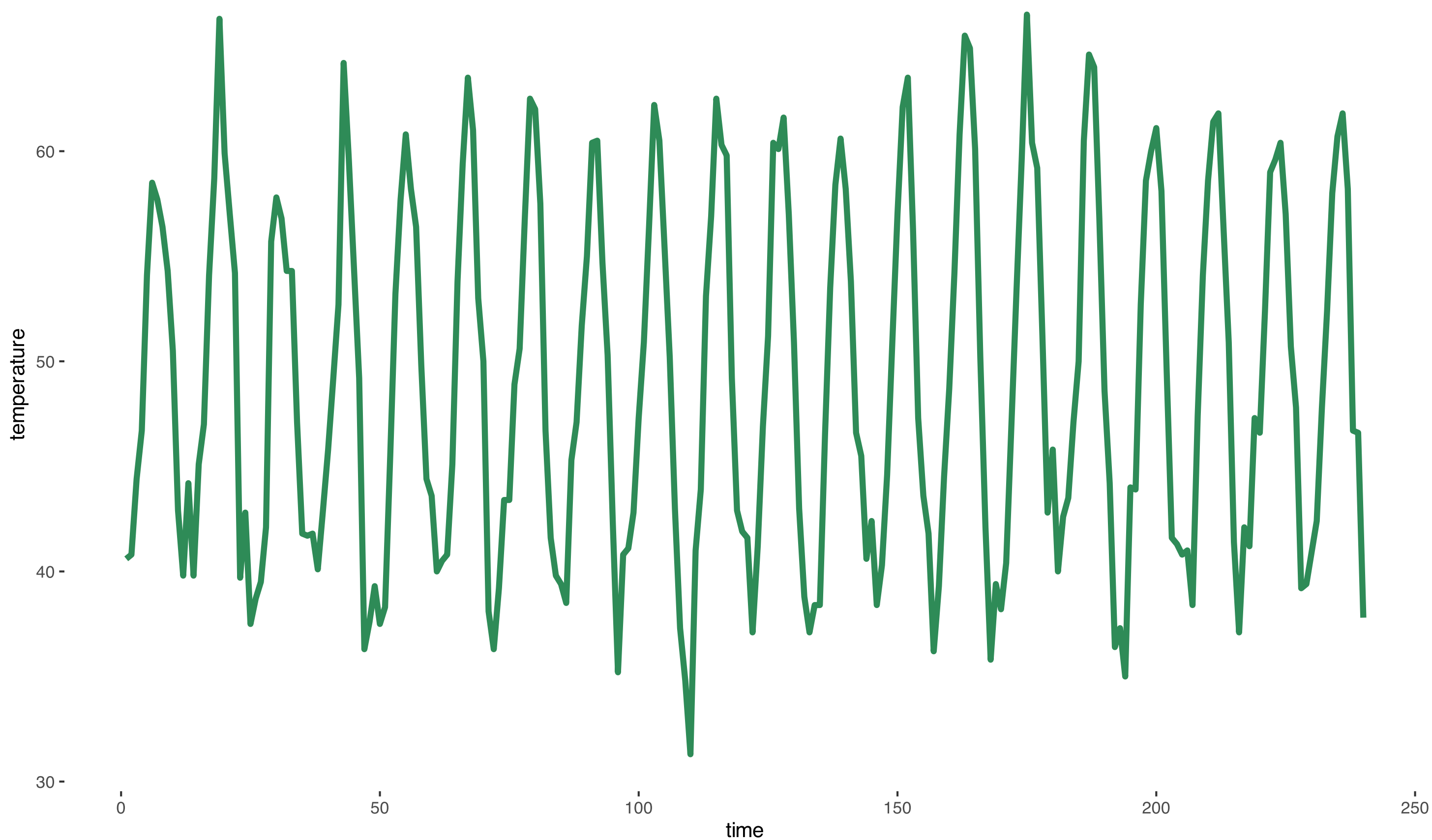
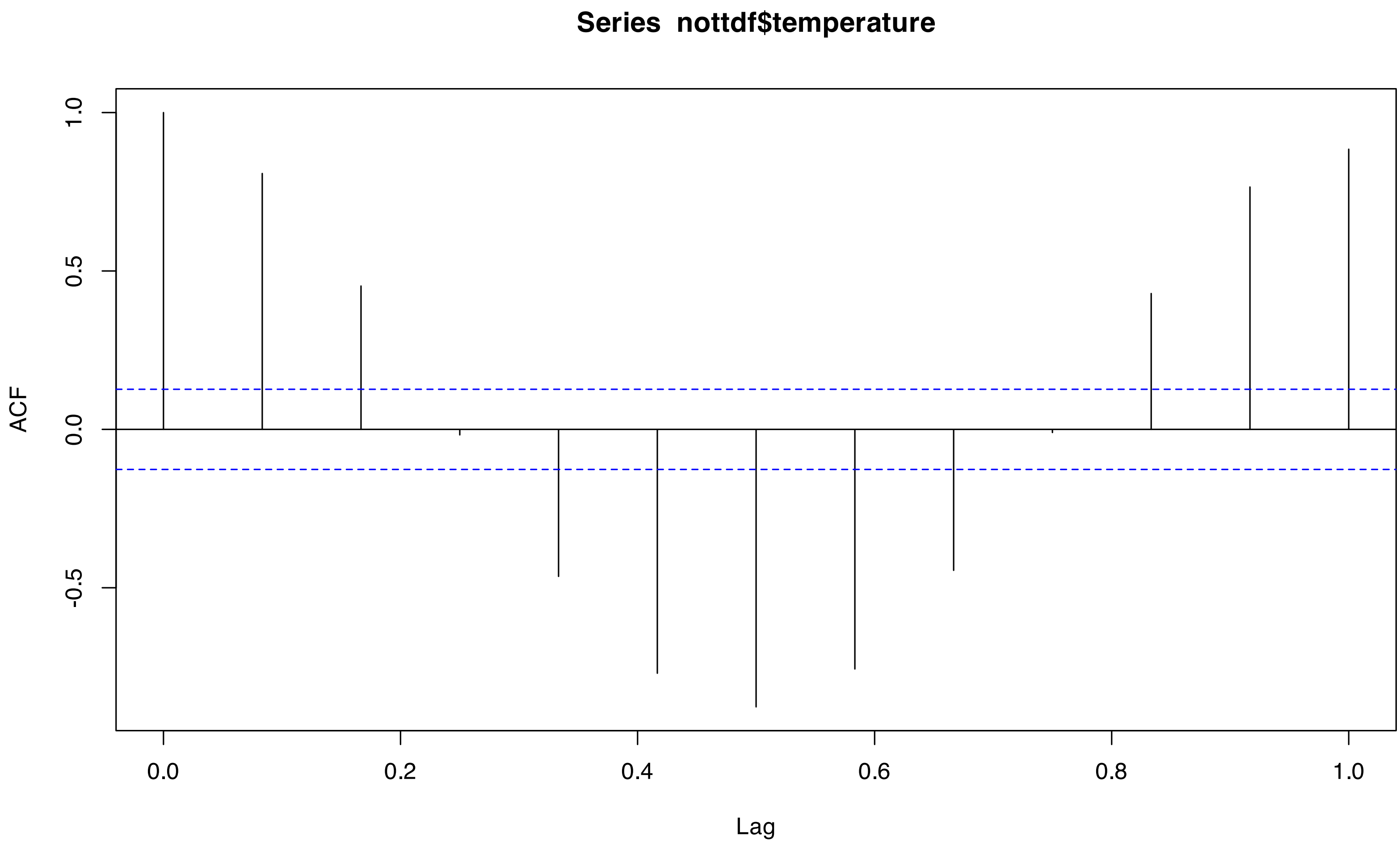
Consider a **monthly** temperature time series for Nottingham Castle



Autocorrelation Function (ACF)

```
acf(nottdf$temperature, lag.max=12)
```



Autocorrelation Function (ACF)

> `acf()` plots an ACF for you!

- The height of each line indicates the correlation between temperature today and temperature l days ago
- Confidence intervals are shown in blue by default -- indicate if $cor(y_t, y_{t-l})$ is statistically distinguishable from zero (or not)
- Helps to identify periodicity of seasonality

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> Definition: **white noise** is a random time series in which there is no correlation across time periods (rare in the real world!). Here, the ACF would look noisy and correlations would largely fall within the blue confidence interval.

Time series and OLS

Intro to time series and OLS

Our model now looks something like

$$\mathbf{Births}_t = \beta_0 + \beta_1 \mathbf{Income}_t + u_t$$

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where $t-1$ denotes the time period prior to t (*lagged* income or births).

Time-series models

Assumptions

1. **New: Weakly persistent outcomes**—essentially, x_{t+k} in the distant period $t+k$ is weakly correlated with period x_t (when k is "big").

2. y_t is a **linear function** of its parameters and disturbance.

3. There is **some variation** in our explanatory variables

4. **Harder to satisfy:** The u_t have conditional mean of zero (**exogeneity**), $\boldsymbol{E}[u_t|X] = 0$.

5. **Harder to satisfy:** The u_t are **normally distributed** and **homoskedastic** with **zero correlation** between u_t and u_s, *i.e.*, $u_t \overset{\text{iid}}{\sim} N\left(0,\, \sigma^2\right)$, $\text{Var}(u_t|X) = \text{Var}(u_t) = \sigma^2$, and $\text{Cor}(u_t,\, u_s|X) = 0$.

Time-series models

Model options

Time-series modeling boils down to two classes of models.

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1. **Static models:** Do not allow for persistent effect.
2. **Dynamic models:** Allow for persistent effects.
   - Models with **lagged explanatory** variables
   - **Autoregressive, distributed-lag** (ADL) models

Model options

Option 1: **Static models**

Static models assume the outcome depends upon only the current period.

$$\text{Births}_t = \beta_0 + \beta_1 \text{Income}_t + u_t$$

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We also need to believe current births do not depend upon previous births.

Can be a very restrictive way to consider time-series data.

Model options

Option 2: Dynamic models

Dynamic models allow the outcome to depend upon other periods.

Model options

Option 2a: Dynamic models with lagged explanatory variables

These models allow the outcome to depend upon the explanatory variable(s) in other periods.

$$\begin{aligned}\text{Births}_t = &\beta_0 + \beta_1 \text{Income}_t + \beta_2 \text{Income}_{t-1} + \\ &\beta_3 \text{Income}_{t-2} + \beta_4 \text{Income}_{t-3} + u_t\end{aligned}$$

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Model options

Option 2a: **Dynamic models** with lagged explanatory variables

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Here, income **immediately** affects the number of births *and* affects **future** numbers of births. In other words: Births today depend today's income and *lags* of income—*e.g.*, last month's income, last year's income, ...

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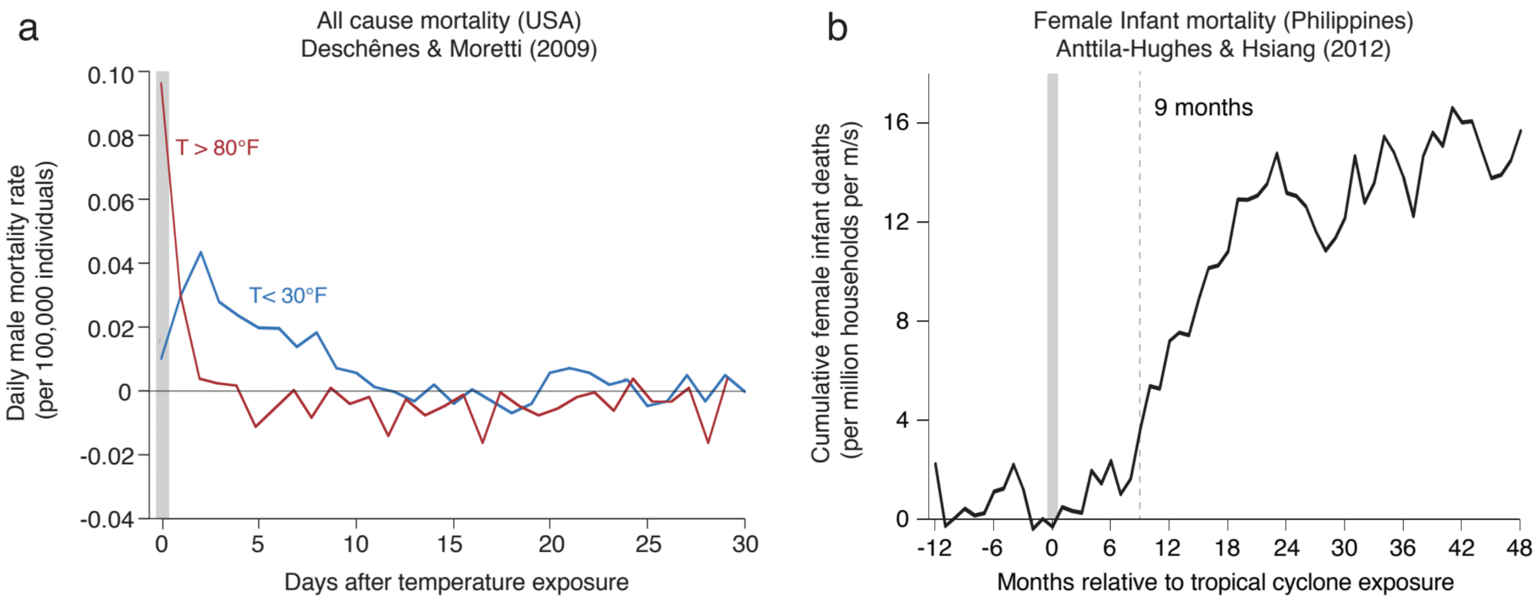
Note: We still assume current births don't affect future births.

Model options

Lagged explanatory variables in empirical research:



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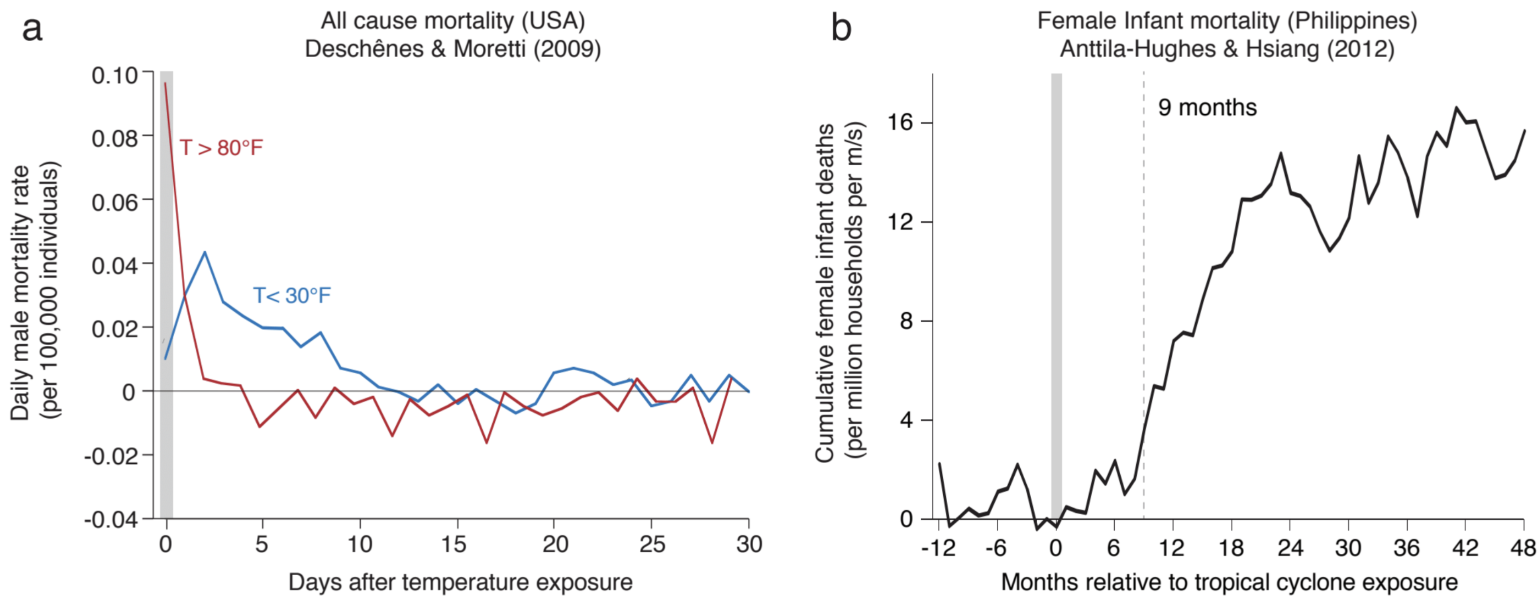

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> Q: Can you think of other examples of lagged effects?

Model options

Option 2b: Autoregressive distributed-lag (ADL) models

These models allow the outcome to depend upon the explanatory variable(s) and/or the outcome variable in prior periods.

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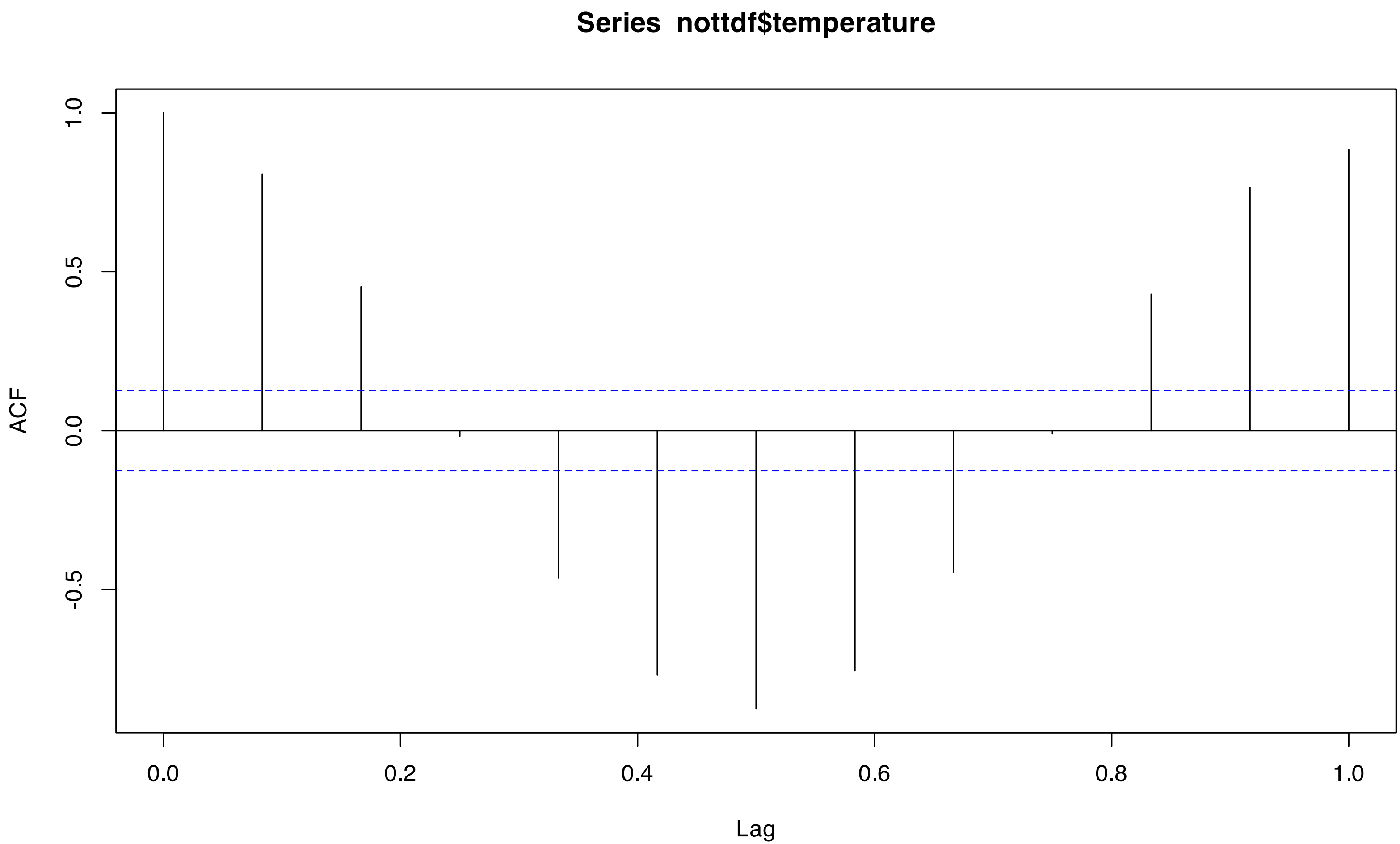
Here, current income affects affects **current** births and **future** births.

In addition, current births affect future births—we're allowing lags of the outcome variable.

Do you need an ADL?

Hint: Autocorrelation Function (ACF)



Autoregressive distributed-lag models

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ADL models are often specified as $\mathrm{ADL}(p, q)$, where

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which we can substitute in for Births_{t-1} in the first equation, *i.e.*,

$$\begin{aligned} \text{Births}_t =& \beta_0 + \beta_1 \text{Income}_t + \\ & \beta_2 \underbrace{\left(\beta_0 + \beta_1 \text{Income}_{t-1} + \beta_2 \text{Births}_{t-2} + u_{t-1} \right)}_{\text{Births}_{t-1}} + u_t \end{aligned}$$

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Continuing...

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We could then substitute in the equation for Births_{t-2}, Births_{t-3}, ...

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Eventually we arrive at

$$\begin{aligned}\text{Births}_t = &\beta_0 \left(1 + \beta_2 + \beta_2^2 + \beta_2^3 + \cdots\right) + \\ &\beta_1 \left(\text{Income}_t + \beta_2 \text{Income}_{t-1} + \beta_2^2 \text{Income}_{t-2} + \cdots\right) + \\ &u_t + \beta_2 u_{t-1} + \beta_2^2 u_{t-2} + \cdots\end{aligned}$$

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The point?

By including just **one lag of the dependent variable**—as in a ADL(1, 0)—we implicitly include for *many lags* of the explanatory variables and disturbances.†

† These lags enter into the equation in a very specific way—not the most flexible specification.

Time-series models

Assumptions

1. **New: Weakly persistent outcomes**—essentially, x_{t+k} in the distant period $t+k$ is weakly correlated with period x_t (when k is "big").

2. y_t is a **linear function** of its parameters and disturbance.

3. There is **some variation** in our explanatory variables

4. **Harder to satisfy:** The u_t have conditional mean of zero (**exogeneity**), $\boldsymbol{E}[u_t|X] = 0$.

5. **Harder to satisfy:** The u_t are **normally distributed** and **homoskedastic** with **zero correlation** between u_t and u_s, *i.e.*, $u_t \overset{\text{iid}}{\sim} N\left(0,\, \sigma^2\right)$, $\text{Var}(u_t|X) = \text{Var}(u_t) = \sigma^2$, and $\text{Cor}(u_t,\, u_s|X) = 0$.

Unbiased coefficients

As before, the unbiased-ness of OLS is going to depend upon our exogeneity assumption, *i.e.*, $\boldsymbol{E}[u_t|X] = \mathbf{0}$.

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I.e., to guarantee the numerator equals zero, we need $\boldsymbol{E}[u_t|X] = \mathbf{0}$—for both $\boldsymbol{E}[u_t|X_t] = 0$ *and* $\boldsymbol{E}[u_t|X_s] = 0$ $(s \neq t)$.

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Thus, **OLS is biased for dynamic models with lagged outcome variables**.

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To see why dynamic models with lagged outcome variables violate our exogeneity assumption, consider two periods of our simple ADL(1, 0) model.

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This correlation violates the second part of our exogeneity requirement.

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With contemporaneous exogeneity, OLS estimates for the coefficients in a time series model are **consistent** (whew)

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Are we worried? In a static model with lagged explanatory variables:

- OLS is **inefficient**, i.e., no longer the lowest variance unbiased estimator
- That is, your standard errors are no longer correct
- However, violating this assumption does not introduce bias (whew!)

Autocorrelation

OLS and lagged outcome variables

Consider a model with one lag of the outcome variable—ADL(1, 0)—model with AR(1) disturbances

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A: It violates contemporaneous exogeneity, *i.e.*, $\text{Cov}(x_t, u_t) \neq 0$.

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- Autocorrelation may arise because your model is **misspecified**. Consider adding additional lags and/or explanatory variables if errors are correlated

Summary: Time series and OLS

- Our model now has t subscripts for **time periods**.
- **Dynamic models** allow **lags** of explanatory and/or outcome variables.
- We changed our **exogeneity** assumption to **contemporaneous exogeneity**, *i.e.*, $\boldsymbol{E}[u_t|X_t] = 0$
- Including **lags of outcome variables** can lead to **biased coefficient estimates** from OLS (but fortunately they are still **consistent**)
- **Lagged explanatory variables** make **OLS inefficient** (i.e., mess up our standard errors)
- **Autocorrelation in the error + lagged dependent variables** make **OLS biased**. Watch out! Test for serial/autocorrelation, check for misspecification of your model.

Slides created via the R package **xaringan**.

Some slide components were borrowed from Ed Rubin and Allison Horst.