# Intro to Higher-Order ODEs and Use of 1st-order ODE Techniques



- Definitions and Examples of Higher-Order ODEs
- Substitution Techniques for Basic 2nd order ODEs with...
  - ↳ Terms involving $x$, $dy/dx$, and $\frac{d^2y}{dx^2}$ only (i.e. no "$y$" terms)
  - ↳ Terms involving $y$, $dy/dx$, and $\frac{d^2y}{dx^2}$ only (i.e. no "$x$" terms)
- Subsititution Techniques for Higher-Order ODEs using 2nd-order ODE substitution techniques

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## Definitions

- Second-order ODE: An ODE where the highest derivative is of order 2
- Third-order ODE: An ODE where the highest derivative is of order 3
- $N^{th}$-order ODE: An ODE where the highest derivative is of order $N$

## Examples of ODEs of 2nd-order or more

**2nd-order**

* $y'' - y' + y = 0$
* $2y' + y = e^{2x}$
* $(y+2)y'' = (3y')^2$

**3rd-order (or higher)**

* $7y''' + 3y'' - 8y' + 37y = -9e^{3x}$
* $y''' - y'' + y' = 0$
* $x^2 y^{(vi)} + 4xy''' - 42\cos(x)y = -5e^{3x}$
* $y^{(41)} + 2y^2 y^{(21)} - xy''' = 22$

## Substitution Techniques for $2^{nd}$-order ODEs

As we study techniques for solving higher-order ODEs, we will see that the techniques we learn for solving $2^{nd}$-order ODEs will be most important to us. We will also see that most of the techniques that we learned to deal with $1^{st}$-order ODEs won't directly apply to the ODEs that we will deal with of the $2^{nd}$-order type.

However, there are 2 specific types of $2^{nd}$-order ODEs that lend themselves to (possibly) being solved via substitution techniques that was introduced when we were working on solving $1^{st}$-order ODEs. These $2^{nd}$-order ODEs are of 1 or 2 types:

* ODEs with terms possessing $x$, $\frac{dy}{dx}$, and/or $\frac{d^2y}{dx^2}$ only
* ODEs with terms possessing $y$, $\frac{dy}{dx}$, and/or $\frac{d^2y}{dx^2}$ only (i.e. autonomous $2^{nd}$-order ODEs)

Next, we will state the suggested substitutions along with other tips and "FYI" info that will aid in solving these type of equations

# Solving ODEs w/ $x$, $\frac{dy}{dx}$, and/or $\frac{d^2y}{dx^2}$ terms only

1. Let $v = \frac{dy}{dx}$
2. If $v = \frac{dy}{dx}$, then $\frac{dv}{dx} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2}$
3. Make the substitutions of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ to $v$ and $\frac{dv}{dx}$, respectively in the original ODE.
4. Resulting equation (for the types we will study in this section) should turn out to be a 1st-order ODE is either separable, directly integrable, 1st-order Linear, 1st-order homogeneous, 1st-order Bernoulli, or 1st-order Linear sub. eqn.
5. Use appropriate 1st-order solving techniques to solve ODE.

$\boxed{*}$ NOTE: In solving for general solutions of 2nd-order ODEs, please be mindful that we will have 2 constants of integration, not just one! Also, note that for IVP problems of 2nd-order ODEs will have to have 2 initial conditions in order to solve for a particular solution to your given ODE.

# Solving ODEs w/ $y$, $\frac{dy}{dx}$, and/or $\frac{d^2y}{dx^2}$ terms only

1. Assume that $v = v(y)$ (i.e. $v$ is a function of $y$) instead of a function of $x$. Also, $\boxed{\text{let } v = \frac{dy}{dx}}$.

   <u>NOTE</u>! Remember that $y = y(x)$ (i.e. $y$ is a function of $x$) which means that $v = v(y) = v(f(x))$ (i.e. $v$ is a composition function (of $x$)). Therefore, find $\frac{dv}{dx}$ would consist of using the chain rule!

2. If $v = v(y) = \frac{dy}{dx}$, then find $\frac{dv}{dx}$ via the chain rule

   $$\therefore \frac{dv}{dx} = \frac{dv}{dx} \cdot \frac{dy}{dy} = \frac{dv}{dy} \cdot \cancel{\frac{dy}{dx}}^{\,v} = \frac{dv}{dy} \cdot v \Rightarrow \boxed{\frac{dv}{dx} = \frac{dv}{dy} \cdot v}$$

3. Make the substitutions of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ to $v$ and $\frac{dv}{dy} \cdot v$, respectively in the original ODE.

4. The resulting equation (for the types we will study in this section) should turn out to be either separable, directly integrable, $1^{st}$-order Linear, $1^{st}$-order homogeneous, $1^{st}$-order Bernoulli, or $1^{st}$-order Linear substitution equation!

5. Use appropriate $1^{st}$-order ODE solving techniques and Notes ⊛ apply.

# Examples of solving $2^{nd}$-order ODEs w/ $x$, $\frac{dy}{dx}$, + $\frac{d^2y}{dx^2}$ terms (only)

NOTE: All the examples in this (sub)-section are ODE's that do not contain a term with "y" stated explicitly!!

Ex. 1 Solve $y'y'' = 1$ for a general solution for $y = y(x)$.

Let $v = \frac{dy}{dx} = y'$. Then, $y'' = \frac{d}{dx}[v] = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2}$.

$\therefore\ y' = \frac{dy}{dx} = v$ and $\frac{d^2y}{dx^2} = \frac{dv}{dx} = v'$. Making these substitutions into our ODE yields...

$$y'y'' = 1 \Rightarrow v \cdot v' = 1 \Rightarrow v \cdot \frac{dv}{dx} = 1 \Rightarrow v \cdot dv = dx$$

(separable equation)

$$\therefore \int v\, dv = \int dx \Rightarrow \frac{v^2}{2} = x + C \Rightarrow v^2 = 2x + 2C$$

$\therefore\ v = \pm\sqrt{2x+2C}$. But $v = \frac{dy}{dx}$. Thus, $\frac{dy}{dx} = \pm\sqrt{2x+2C}$

$$\Rightarrow dy = \pm\sqrt{2x+2C} \cdot dx \Rightarrow \cancel{\int dy}^{\,y} = \pm\int \underbrace{\sqrt{2x+2C}\ dx}$$

let $w = 2x + 2C$

$\therefore\ dw = 2 \cdot dx$

$\Rightarrow \frac{1}{2}\, dw = dx$

$$\therefore y = \pm\int \overset{w^{1/2}}{\cancel{\sqrt{w}}} \cdot \tfrac{1}{2}\, dw = \pm\tfrac{1}{2}\left[\tfrac{2}{3} w^{3/2}\right] + C_2$$

$$\therefore y = \pm\tfrac{1}{3}(2x + C_1)^{3/2} + C_2, \text{ where } C_1 = 2C \Rightarrow \boxed{y(x) = C_2 \pm \tfrac{1}{3}(2x + C_1)^{3/2}}$$

Ex 2: Solve $(x^2+1)y''+2xy'=0$ for $y=y(x)$.

Let $v=y'=\frac{dy}{dx}$. Then, $v'=\frac{dv}{dx}=\frac{d^2y}{dx^2}$.

$\therefore (x^2+1)y''+2xy'=0 \Rightarrow (x^2+1)\cdot\frac{dv}{dx}+2xv=0$

$\therefore (x^2+1)\cdot\frac{dv}{dx}=-2xv \Rightarrow \frac{dv}{v}=\frac{-2x}{x^2+1}\,dx$ (separable eqn.)

$\therefore \underbrace{\cancel{\int\frac{dv}{v}}}_{\ln|v|} = -\int\frac{2x}{x^2+1}\,dx$. Let $w=x^2+1 \Rightarrow dw=2x\cdot dx$

$\therefore \ln|v| = -\int\frac{dw}{w} = -\ln|w|+C \Rightarrow \underbrace{\cancel{\ln|v|+\ln|w|}}_{\ln|vw|}=C$

$\therefore e^{\ln|vw|}=e^C \Rightarrow |vw|=e^C \Rightarrow |v(x^2+1)|=e^C \Rightarrow |v|(x^2+1)=e^C$

Let $C_1=e^C$. Then, $|v|=\frac{C_1}{x^2+1} \Rightarrow \left|\frac{dy}{dx}\right|=\frac{C_1}{x^2+1}$

$\therefore \frac{dy}{dx}=\frac{C_1}{x^2+1}$; noting that $y'(x)\geq 0$ for all $x$ in the domain of $y(x)$ (i.e. $y(x)$ must be either increasing or constant for all $x$ in the domain of $y(x)$).

$\therefore \int dy=\int\frac{C_1}{x^2+1}\,dx \Rightarrow y=C_1\arctan(x)+C_2$. Since $C_1=e^C>0$ always and $\arctan(x)$ is always increasing on $x=(-\infty,\infty)$, we see that our result for $y=y(x)$ is suitable!

$$\therefore \boxed{y=y(x)=C_1\arctan(x)+C_2}$$

Ex. 3: Solve $y'' + 4y' = 9e^{-3x}$ for $y = y(x)$.

Let $v = y' = \frac{dy}{dx}$. Then, $v' = \frac{dv}{dx} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d^2y}{dx^2}$

$$\therefore \quad v' + 4v = 9e^{-3x} \Rightarrow \frac{dv}{dx} + \underbrace{4}_{p}v = \underbrace{9e^{-3x}}_{r} \quad \left(\begin{array}{c}1^{st}\text{-order}\\ \text{Linear ODE}\end{array}\right)$$

NOTE: $v$ is assumed to be a function of $x$ here, hence this is why we can (ultimately) say that our ODE in terms of $v$ and $x$ is a $1^{st}$-order Linear ODE!

$$\therefore \quad v = e^{-h}\left[\int e^{h} \cdot r \cdot dx + C\right], \text{ where } h = \int p \cdot dx$$

So, $h = \int 4\,dx = 4x \Rightarrow e^{h} = e^{4x}$ and $e^{-h} = e^{-4x}$

$$\therefore \quad v = e^{-4x}\left[\int\left(e^{4x} \cdot 9e^{-3x}\right)dx + C\right] = e^{-4x}\left[\left(\int 9e^{x}\,dx\right) + C\right]$$

$9e^{x}$

$$\Rightarrow v = e^{-4x} \cdot 9e^{x} + Ce^{-4x} \Rightarrow \boxed{v = 9e^{-3x} + Ce^{-4x}}$$

But $v = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 9e^{-3x} + Ce^{-4x} \Rightarrow \int dy = \int\left(9e^{-3x} + Ce^{-4x}\right)dx$

$$\therefore \quad y = y(x) = 9\frac{e^{-3x}}{-3} + C\frac{e^{-4x}}{-4} + D \Rightarrow \boxed{y(x) = -3e^{-x} - \frac{C}{4}e^{-4x} + D}$$

So, $\boxed{y(x) = -9e^{-x} + Ae^{-4x} + D \text{ is our final answer}}$, where $A = -\frac{C}{4}$ and $A, D \in \mathbb{R}$.

Ex. 4: Solve $xy'' + 4y' = 18x^2$ with $y(1) = 8$ and $y'(1) = -3$ for a particular solution of $y = y(x)$.

let $v = y' = \frac{dy}{dx}$. Then, $v' = y'' = \frac{d^2y}{dx^2}$.

$$\therefore xy'' + 4y' = 18x^2 \Rightarrow xv' + 4v = 18x^2 \Rightarrow v' + \overset{p}{\left(\frac{4}{x}\right)} v = \overset{r}{18x}$$

(1st order linear ODE)

$$\therefore v = e^{-h}\left[\int e^{h} \cdot r \cdot dx + C\right], \text{ where } h = \int p \cdot dx = \int \frac{4}{x} dx = 4\ln|x| = \ln|x^4| = \ln(x^4)$$

$$\therefore e^{h} = e^{\ln(x^4)} = x^4 \text{ and } e^{-h} = e^{-\ln(x^4)} = e^{\ln\left(\frac{1}{x^4}\right)} = \frac{1}{x^4} = x^{-4}$$

$$\therefore v = x^{-4}\left[\int \left(x^4 \cdot 18x\right) dx + C\right] = x^{-4}\left[18\int x^5 \cdot dx + C\right] = x^{-4}\left[18\frac{x^6}{6} + C\right]$$

$$\therefore v = x^{-4}\left[3x^6 + C\right] = 3x^2 + Cx^{-4} = y'(x) \text{ since } v = y'.$$

Using our initial condition $y'(1) = -3$, we can find $C$.

$$\therefore y'(1) = -3 \Rightarrow -3 = 3(1)^2 + C(1)^{-4} \Rightarrow -3 = 3 + C \Rightarrow \boxed{C = -6}$$

$$\therefore y'(x) = \frac{dy}{dx} = 3x^2 - 6x^{-4} \Rightarrow dy = \left(3x^2 - 6x^{-4}\right) dx$$

$$\therefore \int dy = \int \left(3x^2 - 6x^{-4}\right) dx \Rightarrow y = 3\frac{x^3}{3} - 6\frac{x^{-3}}{-3} + D = x^3 + \frac{2}{x^3} + D$$

Ex. 4 : (cont'd)

$\therefore\ y(x) = x^3 + \frac{2}{x^3} + D$. Apply our other initial condition $y(1) = 8$, we can find our constant $D$.

$$\therefore\ y(1) = 8 \Rightarrow 8 = \cancel{(1)^3}^{\,1} + \cancel{\frac{2}{(1)^3}}^{\,2} + D \Rightarrow 8 = 3 + D \Rightarrow \boxed{D = 5}$$

$\therefore$ Final answer : $\boxed{y(x) = x^3 + \frac{2}{x^3} + 5}$

# Examples of Solving Autonomous $2^{nd}$-order ODEs



__NOTE:__ All the examples in this (sub)-section are ODEs that do not contain a term with "x" stated explicitly !!

__Ex. 1:__ Solve $y'y'' = 1$ for a general solution of $y = y(x)$

__NOTE 1:__ We solved this ODE in Ex. 1 for the $2^{nd}$-order ODEs that do not have terms that have "y" stated explicitly. This ODE happens to not have "x" stated explicitly in any terms of it as well!! Therefore, this ODE can be solved either by letting $v = \frac{dy}{dx} \Rightarrow \frac{dv}{dx} = \frac{d^2y}{dx^2}$ (as in Ex. 1 of the previous subsection) __or__ by letting $v = v(y)$ and $v = \frac{dy}{dx}$

$$\Rightarrow \frac{dv}{dx} = \frac{d}{dx}\left[v(y)\right] = \frac{dv}{dy} \cdot \overset{v}{\cancel{\frac{dy}{dx}}} = \frac{dv}{dy} \cdot v \text{ !!}$$

__NOTE 2:__ Recall that our solution to this ODE was ...
$y(x) = C_2 \pm \frac{1}{3}(2x + C_1)^{3/2}$. We should end up with the same answer when we solve this ODE this time!

Let $v = v(y)$ and $v = \frac{dy}{dx} = y'$. Then, $\frac{dv}{dx} = y'' = \frac{d}{dx}\left[v(y)\right] = \frac{dv}{dy} \cdot \overset{v}{\cancel{\frac{dy}{dx}}}$.

Thus, $\frac{dv}{dx} = y'' = \frac{dv}{dy} \cdot v$. We will substitute $y' = v$ and $y'' = \frac{dv}{dy} \cdot v$ into our original ODE.

Ex. 1 : cont'd



$\therefore\ y'y'' = 1 \Rightarrow v \cdot \frac{dv}{dy} \cdot v = 1 \Rightarrow v^2 \cdot \frac{dv}{dy} = 1 \Rightarrow \frac{dv}{dy} = v^{-2}$

(separable ODE)

$\therefore\ \frac{dv}{dy} = v^{-2} \Rightarrow \frac{dv}{v^{-2}} = dy \Rightarrow v^2 dv = dy$

$\therefore \int v^2 \cdot dy = \int dy \Rightarrow \frac{v^3}{3} = y + C \Rightarrow v^3 = 3y + 3C$

$\Rightarrow v = (3y + C_1)^{1/3}$

where $C_1 = 3C$

But $v = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = (3y + C_2)^{1/3} \Rightarrow \frac{dy}{(3y+C_2)^{1/3}} = dx$

$\therefore \int (3y + C)^{-1/3} \cdot dy = \int dx$. let $w = 3y + C \Rightarrow dw = 3\,dy \Rightarrow \frac{dw}{3} = dy$

$\therefore \int w^{-1/3} \cdot \frac{1}{3} dw = \int dx \Rightarrow \frac{1}{3}\left[\frac{3}{2} w^{2/3}\right] = x + D \Rightarrow \frac{1}{2}(3y + C)^{2/3} = x + D$

$\therefore \left[(3y + C)^2\right]^{1/3} = 2x + 2E \Rightarrow (3y + C)^2 = (2x + 2E)^3$

$\Rightarrow 3y + C = \pm\sqrt{(2x + 2E)^3}$

$\Rightarrow 3y + C = \pm\sqrt{(2x + C_1)^3}$ ; $C_1 = 2E$

$3y = -C \pm \sqrt{2x + C_1} \Rightarrow y = -\frac{C}{3} \pm \frac{1}{3}\sqrt{(2x + C_1)^3}$. let $C_2 = -\frac{C}{3}$.

$\therefore$ Final answer: $\boxed{y = y(x) = C_2 \pm \frac{1}{3}\sqrt{(2x + C_1)^3}}$

**Ex. 2:** Solve the ODE $y'' = y'(y'-2)$ for $y=y(x)$.

Let $v = v(y)$ and $v = \frac{dy}{dx} = y'$. Then, $v' = y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}[v(y)] = \frac{dv}{dy}\cdot\frac{dy}{dx} = \frac{dv}{dy}\cdot v$. So, $y' = v$ and $y'' = \frac{dv}{dy}\cdot v$.

$\therefore\ y'' = y'(y'-2) \Rightarrow \frac{dv}{dy}\cdot v = v(v-2) \Rightarrow \underbrace{\frac{dv\cdot\cancel{v}}{\cancel{v}(v-2)} = dy}_{\text{separable ODE!}}$

$\therefore \int\frac{dv}{v-2} = \int dy \Rightarrow \ln|v-2| = y + C \Rightarrow e^{\ln|v-2|} = e^{y+C}$

$\therefore\ |v-2| = C_1 e^y$, where $\boxed{C_1 = e^C > 0}$.

So $|v-2| = C_1 e^y \Rightarrow v-2 = C_1 e^y$ or $v-2 = -C_1 e^y$

$\therefore\ v = C_1 e^y + 2$ or $v = -C_1 e^y + 2 \Rightarrow v = \pm C_1 e^y + 2$

But $v = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \pm C_1 e^y + 2 \Rightarrow \frac{dy}{\pm C_1 e^y + 2} = dx$

NOTE: $\frac{dy}{\pm C_1 e^y + 2} = \frac{dy}{\pm C_1 e^y + 2}\cdot\frac{e^{-y}}{e^{-y}} = \frac{e^{-y}\cdot dy}{\pm C_1 + 2e^{-y}}$. Now, if we let $u = \pm C_1 + 2e^{-y}$, then $du = -2e^{-y}\cdot dy \Rightarrow -\frac{1}{2}du = e^{-y}$.

$\therefore\ \frac{e^{-y}\cdot dy}{\pm C_1 + 2e^{-y}} = \frac{-\frac{1}{2}du}{u}$. So, we can use the u-sub. technique!!

Ex. 2 : cont'd

$$\therefore \frac{dy}{\pm C_1 e^{y} + 2} = dx \Rightarrow \frac{e^{-y} \cdot dy}{\pm C_1 + 2e^{-y}} = dx \Rightarrow \int \frac{-\frac{1}{2}\, du}{u} = \int dx$$

$$\therefore -\tfrac{1}{2} \ln|u| = x + D \Rightarrow \ln|u| = -2x - 2D \Rightarrow e^{\ln|u|} = e^{-2x} \cdot e^{-2D}$$

$$\therefore |u| = e^{-2x} \cdot e^{-2D} \Rightarrow \left| \pm C_1 + 2e^{-y} \right| = e^{-2x} \cdot C_2 \text{, where } \boxed{C_2 = e^{-2D} > 0}$$

$$\therefore \left| \pm C_1 + 2e^{-y} \right| = e^{-2x} \Rightarrow \pm C_1 + 2e^{-y} = C_2 e^{-2x} \text{ or } \pm C_1 + 2e^{-y} = -C_2 e^{-2x}$$

$$\therefore \pm C_1 + 2e^{-y} = C_2 e^{-2x} \text{ or } \pm C_1 + 2e^{-y} = -C_2 e^{-2x}$$

$$\Rightarrow 2e^{-y} = C_2 e^{-2x} \mp C_1 \text{ or } 2e^{-y} = -C_2 e^{-2x} \mp C_1$$

$$\Rightarrow \boxed{e^{-y} = \frac{C_2 e^{-2x}}{2} \mp \frac{C_1}{2} \text{ or } e^{-y} = \frac{-C_2 e^{-2x}}{2} \mp \frac{C_1}{2}}$$

$$\Rightarrow \ln\left|e^{-y}\right| = \ln\left|C_3 e^{-2x} \mp C_4\right| \text{ or } \ln\left|e^{-y}\right| = \ln\left|-C_3 e^{-2x} \mp C_4\right| \text{, where}$$

$$\boxed{C_3 = \frac{C_2}{2} \text{ and } C_4 = \frac{C_1}{2}}.$$

$$\therefore -y = \ln\left|C_3 e^{-2x} \mp C_4\right| \text{ or } -y = \ln\left|-C_3 e^{-2x} \mp C_4\right|$$

$$\Rightarrow y = -\ln\left|C_3 e^{-2x} \mp C_4\right| \text{ or } y = -\ln\left|-C_3 e^{-2x} \mp C_4\right|$$

(!) NOTE: On the previous 2 pages, we noted that $C_1 = e^{c} > 0$ and $C_2 = e^{-2D} > 0$. Thus, $C_3 = \frac{C_2}{2} > 0$ and $C_4 = \frac{C_1}{2} > 0$. Also, note that $e^{-y} > 0$ and that $e^{-y} = \frac{C_2}{2} e^{-2x} \mp \frac{C_1}{2}$ or $e^{-y} = -\frac{C_2}{2} e^{-2x} \mp \frac{C_1}{2}$

$$\Rightarrow e^{-y} = C_3 e^{-2x} \mp C_4 \quad \text{or} \quad e^{-y} = -C_3 e^{-2x} \mp C_4$$

ATTENTION: We need to find out in each case of what $e^{-y}$ equals whether we should have "$+C_4$" or "$-C_4$" in the expression.

Case 1 ($e^{-y} = C_3 e^{-2x} \mp C_4$)

Since $e^{-y} > 0 \Rightarrow C_3 e^{-2x} \mp C_4 > 0 \Rightarrow e^{-2x} > \frac{\pm C_4}{C_3}$. Since $C_3$ and $C_4 > 0$, the only way this statement can be true is if $\boxed{\pm C_4 = C_4}$.

$\therefore e^{-y} = C_3 e^{-2x} \mp C_4 \Rightarrow e^{-y} = C_3 e^{-2x} + C_4$ since $e^{-2x} > 0$ as well!

Case 2 ($e^{-y} = -C_3 e^{-2x} \mp C_4$)

Since $e^{-y} > 0 \Rightarrow -C_3 e^{-2x} \mp C_4 > 0 \Rightarrow e^{-2x} < \frac{\pm C_4}{-C_3}$. Since $C_3$ and $C_4 > 0$, the only way this statement can be true is if $\boxed{\pm C_4 = -C_4}$ since $e^{-2x} > 0$ as well!

$\therefore$ From Cases 1 + 2, we see that ...

$$y = -\ln\left|C_3 e^{-2x} \mp C_4\right| \quad \text{or} \quad y = -\ln\left|-C_3 e^{-2x} \mp C_4\right|$$

$$\Rightarrow y = -\ln\left|C_3 e^{-2x} + C_4\right| \quad \text{or} \quad y = -\ln\left|-C_3 e^{-2x} - C_4\right|$$

But $\left|-C_3 e^{-2x} - C_4\right| = \left|(-1)(C_3 e^{-2x} + C_4)\right| = \cancel{|-1|}\left|C_3 e^{-2x} + C_4\right| = \left|C_3 e^{-2x} + C_4\right|$

$$\therefore \quad y = -\ln\left|C_3 e^{-2x} + C_4\right| \quad \text{or} \quad y = -\ln\left|-C_3 e^{-2x} - C_4\right|$$

$\Rightarrow \quad y = -\ln\left|C_3 e^{-2x} + C_4\right| \Rightarrow y = -\ln\left(C_3 e^{-2x} + C_4\right)$ (i.e. we can drop the absolute value signs on the argument for our natural logarithm expression since $C_3 e^{-2x} + C_4 > 0$)! Finally, let $A = C_3 > 0$ and $B = C_4 > 0$.

$$\therefore \boxed{y = y(x) = -\ln\left(A e^{-2x} + B\right)}$$

Ex. 3 : Solve the IVP : $3yy'' = 2(y')^2$ with $y(1) = 1$ and $y'(1) = 9$.

Let $v = v(y)$ and $v = \frac{dy}{dx} = y'$. Then, $\frac{dv}{dx} = y'' = \frac{d}{dx}[v(y)] = \frac{dv}{dy} \cdot \frac{dy}{dx}$

$= \frac{dv}{dy} \cdot v$. So $y' = v$ and $y'' = \frac{dv}{dy} \cdot v$

$\therefore 3yy'' = 2(y')^2 \Rightarrow 3y\left(\frac{dv}{dy} \cdot v\right) = 2v^2 \Rightarrow \frac{dv}{dy} = \frac{2}{3}\frac{v}{y}$

$\therefore \frac{dv}{dy} = \frac{2}{3} \cdot \frac{v}{y} \Rightarrow \frac{dv}{v} = \frac{2}{3} \cdot \frac{dy}{y} \Rightarrow \int \frac{dv}{v} = \frac{2}{3}\int \frac{dy}{y}$

$\therefore \ln|v| = \frac{2}{3}\ln|y| + C \Rightarrow e^{\ln|v|} = e^{\ln|y^{2/3}| + C} \Rightarrow |v| = Ay^{2/3}$

where $A = e^C$.

But $v = \frac{dy}{dx}$. So, $\left|\frac{dy}{dx}\right| = Ay^{2/3} \Rightarrow \frac{dy}{dx} = Ay^{2/3}$, where $\frac{dy}{dx} \geq 0$

NOTE: $\frac{dy}{dx} \geq 0$ means that $y(x)$ must be a function that is either increasing or constant for $x$ in the domain of $y(x)$.

$\therefore \frac{dy}{dx} = Ay^{2/3} \Rightarrow y^{-2/3}\,dy = A\,dx \Rightarrow \int y^{-2/3}\,dy = \int A\,dx$

$\therefore 3y^{1/3} = Ax + B \Rightarrow 27y = (Ax+B)^3 \Rightarrow y = \frac{(Ax+B)^3}{27}$

Ex. 3 : (cont'd)

Applying initial condition $y(1) = 1$

$$\therefore\ y(1) = 1 \Rightarrow 1 = \frac{(A(1)+B)^3}{27} \Rightarrow 27 = (A+B)^3 \Rightarrow \sqrt[3]{27} = \boxed{3 = A+B}$$

Applying initial condition $y'(1) = 9$

$$y(x) = \frac{(Ax+B)^3}{27} \Rightarrow y'(x) = \frac{3(Ax+B)^2 \cdot A}{27} = \frac{A(Ax+B)^2}{9}$$

$$\therefore\ y'(1) = 9 \Rightarrow 9 = \frac{A(A(1)+B)^2}{9} = \frac{A(\overset{3}{\cancel{A+B}})^2}{9} = \frac{A(3)^2}{9} = \frac{A(\overset{1}{\cancel{9}})}{\underset{1}{\cancel{9}}} = A$$

$$\therefore \boxed{A = 9}.\ \text{So, } \overset{9}{A} + B = 3 \Rightarrow B = 3 - 9 = -6 \Rightarrow \boxed{B = -6}$$

$$\therefore\ y = y(x) = \frac{(Ax+B)^3}{27} = \frac{(9x-6)^3}{27} = \frac{[3(3x-2)]^3}{27} = \frac{\overset{1}{\cancel{3^3}} \cdot (3x-2)^3}{\underset{1}{\cancel{27}}}$$

$$\therefore \boxed{y(x) = (3x-2)^3}$$

Ex. 4: Solve the IVP: $y'' = -y'e^{-y}$ with $y(0)=0$ and $y'(0)=2$.

Let $v = v(y)$ and $v = \frac{dy}{dx}$, where $y = y(x)$. Then, if $v = \frac{dy}{dx} = y' \Rightarrow$

$$\frac{dv}{dx} = \frac{d}{dx}\left[v(y)\right] = \frac{dv}{dy} \cdot \frac{dy}{dx} = \frac{dv}{dy} \cdot v.$$

$$\therefore\ y'' = -y'e^{-y} \Rightarrow \frac{dv}{dy} \cdot v = -(v) \cdot e^{-y} \Rightarrow \frac{v \cdot dv}{-v} = e^{-y} \cdot dy$$

$$\Rightarrow \int -dv = \int e^{-y} \cdot dy \Rightarrow -v + c = -e^{-y} \Rightarrow \boxed{v = e^{-y} + c}$$

But $v = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = e^{-y} + c \Rightarrow \frac{dy}{e^{-y}+c} = dx$

NOTE: Since there is not an easy way to know what the anti-derivative of $\frac{1}{e^{-y}+c}$ is "straight up", we consider using the u-substitution and rewriting our integrand in another way to possibly help us out.

$$\therefore\ \frac{dy}{e^{-y}+c} = \frac{dy}{e^{-y}+c} \cdot \frac{e^{y}}{e^{y}} = \frac{e^{y} \cdot dy}{1 + ce^{y}}.$$ Let $u = 1 + ce^{y}$. Then,

$du = ce^{y}\,dy \Rightarrow \frac{1}{c}\,du = e^{y} \cdot dy$. Therefore, $\frac{e^{y} \cdot dy}{1+ce^{y}} = \frac{\frac{1}{c}\,du}{u}$

$$\therefore\ \frac{dy}{e^{-y}+c} = dx \Rightarrow \frac{\frac{1}{c}\,du}{u} = dx \Rightarrow \int \frac{du}{u} = c\int dx$$

$\therefore \ln|u| = Cx + D \Rightarrow \ln|1+Ce^{y}| = Cx + D$

$\Rightarrow \ln(1+Ce^{y}) = Cx + D$ since $1+Ce^{y} > 0$ for all $y \in \mathbb{R}$

$\therefore e^{\ln(1+Ce^{y})} = e^{Cx+D} \Rightarrow 1 + Ce^{y} = Ae^{Cx}$ ; $A = e^{D}$

$\therefore Ce^{y} = Ae^{Cx} - 1 \Rightarrow e^{y} = \frac{A}{C}e^{Cx} - \frac{1}{C} \Rightarrow \ln(e^{y}) = \ln\left(\frac{A}{C}e^{Cx} - \frac{1}{C}\right)$

$\therefore y = \ln\left(\frac{A}{C}e^{Cx} - \frac{1}{C}\right)$

Applying initial condition $y(0) = 0$ to find $A$

$y(0) = 0 \Rightarrow 0 = \ln\left(\frac{A}{C}\cdot 1 - \frac{1}{C}\right) = \ln\left(\frac{A-1}{C}\right) = \ln(A-1) - \ln(C)$

$\therefore 0 = \ln(A-1) - \ln(C) \Rightarrow \ln(C) = \ln(A-1) \Rightarrow C = A - 1 \Rightarrow \boxed{A = C+1}$

Applying initial condition $y'(0) = 2$ to find $C$

$y = y(x) = \ln\left(\frac{A}{C}e^{Cx} - \frac{1}{C}\right) = \ln\left(\frac{Ae^{Cx} - 1}{C}\right) = \ln(Ae^{Cx} - 1) - \ln(C)$

$\therefore y'(x) = \frac{1}{Ae^{Cx} - 1}\cdot(Ae^{Cx} - 1)' = \frac{ACe^{Cx}}{Ae^{Cx} - 1} = \frac{(C+1)(C)e^{Cx}}{(C+1)e^{Cx} - 1}$

$\therefore y'(0) = 2 \Rightarrow \frac{(C+1)(C)\cancel{e^{C(0)}}}{(C+1)\cancel{e^{C(0)}} - 1} = 2 = \frac{(C+1)(C)}{C+\cancel{1}-\cancel{1}} = 2 \Rightarrow \frac{(C+1)\cancel{(C)}}{\cancel{C}} = 2$

$\therefore C + 1 = 2 \Rightarrow \boxed{C = 1} \Rightarrow A = C + 1 \Rightarrow \boxed{A = 2}$

$\therefore \boxed{y = \ln\left(\frac{2}{1}e^{1x} - \frac{1}{1}\right) = \ln(2e^{x} - 1)}$

## Higher-Order ODEs

For the ODEs of order 3 or more, we will use similar strategies to achieve our ultimate tactic/goal of finding ways to solve equations of this ilk. Before we get into examples of solving these type of equations, we will need to note the following points.

- Make sure to allow $v = \frac{d^M y}{dx}$, where $M$ = lowest order of derivative present in the ODE.

- Remember that an $N^{th}$-order IVP will have an $N^{th}$-order ODE with $N$ initial conditions which will look like $y(x_0) = y_0$, $y'(x_0) = y_1$, $y''(x_0) = y_2$, $\ldots$, <u>and/or</u> $y^{(N-1)}(x_0) = y_{N-1}$.

- For IVP type of problems, if you have an $N^{th}$-order ODE to solve, you should expect to have "$N$" undetermined coefficients (from various constants of integration). These undetermined coefficients along with all the initial condition equations $y(x_0) = y_0, \ldots,$ and/or $y^{(N-1)}(x_0) = y_{N-1}$ will form a system of algebraic equations that you can choose to solve for algebraically or by using matrix algebra techniques.

# Examples of solving higher-order ODEs

Ex. 1: Solve $x y''' + 2y'' = 6x$ for $y = y(x)$.

Let $v = y''$. Then, $\frac{dv}{dx} = v' = y'''$

$\therefore\ x y''' + 2y'' = 6x \Rightarrow x v' + 2v = 6x \Rightarrow v' + \frac{2}{x} v = 6$

$\therefore\ \frac{dv}{dx} + \frac{2}{x} v = 6 \Rightarrow$ 1st-order Linear ODE where $p = \frac{2}{x}$ and $r = 6$

$\therefore\ v = e^{-h}\left[\int e^{h} \cdot r \cdot dx + C\right]$, where $h = \int p \cdot dx = \int \frac{2}{x}\,dx = 2\ln|x| = \ln|x^2| = \ln(x^2)$

So, $e^{h} = e^{\ln(x^2)} = x^2$ and $e^{-h} = e^{-\ln(x^2)} = e^{\ln(x^{-2})} = x^{-2}$

$\therefore\ v = x^{-2}\left[\int x^2 \cdot 6 \cdot dx + C\right] = x^{-2}\left[\frac{6x^3}{3} + C\right] = 2x + Cx^{-2}$

$\Rightarrow v = 2x + Cx^{-2} \Rightarrow \frac{d^2y}{dx^2} = 2x + Cx^{-2} \Rightarrow \underbrace{\frac{d}{dx}\left[\frac{dy}{dx}\right] = 2x + Cx^{-2}}_{\text{directly integrable!}}$

$\therefore\ \frac{dy}{dx} = \frac{2x^2}{2} + \frac{Cx^{-1}}{-1} + D = x^2 - \frac{C}{x} + D$

$\therefore\ y = y(x) = \frac{x^3}{3} - C\ln|x| + Dx + E$

Ex. 2: Solve $y^{(4)} = -2y'''$ for $y = y(x)$.

Let $v = y''' = \frac{d^3y}{dx^3}$. Then, $v' = y^{(4)} = \frac{d^{(4)}y}{dx^4} = \frac{dv}{dx}$.

$\therefore\ y^{(4)} = -2y''' \Rightarrow \frac{dv}{dx} = -2v \Rightarrow \frac{dv}{v} = -2\,dx$

$\therefore \int \frac{dv}{v} = \int -2\,dx \Rightarrow \ln|v| = -2x + C_1 \Rightarrow e^{\ln|v|} = e^{-2x+C_1}$

$\therefore\ |v| = e^{-2x} \cdot A$, where $A = e^{C_1} > 0$.

$\Rightarrow \left|\frac{d^3y}{dx^3}\right| = Ae^{-2x} \Rightarrow \frac{d^3y}{dx^3} = \pm Ae^{-2x}$

$\therefore\ \frac{d^3y}{dx^3} = \pm Ae^{-2x} \Rightarrow \frac{d^2y}{dx^2} = \frac{\pm Ae^{-2x}}{-2} + B \Rightarrow \frac{dy}{dx} = \frac{\pm Ae^{-2x}}{4} + Bx + C_2$

$\Rightarrow y(x) = \frac{\pm Ae^{-2x}}{-8} + \frac{Bx^2}{2} + C_2 x + D$. Let $\alpha_1 = \frac{\pm A}{-8}$, $\alpha_2 = \frac{B}{2}$,

$\alpha_3 = C_2$, and $\alpha_4 = D$

$$\therefore \boxed{y(x) = \alpha_1 e^{-2x} + \alpha_2 x^2 + \alpha_3 x + \alpha_4}$$

Ex. 3: Solve the IVP for $y = y(x)$.

$$x y''' + 2y'' = 6x \quad \text{with} \quad y(1) = 2,\ y'(1) = 1, \text{ and } y''(1) = 4.$$

Note 1: In Ex. 1 for the set of examples for higher-order ODEs, we found that the general solution to the ODE was...

$$y(x) = \frac{x^3}{3} - C\ln|x| + Dx + E$$

Therefore, we only need to apply the initial conditions in order to find C, D, and E

Note 2: From the work we did in Ex. 1, we also note that...

$$y'(x) = \frac{dy}{dx} = x^2 - \frac{C}{x} + D$$

$$y''(x) = \frac{d^2y}{dx^2} = 2x + \frac{C}{x^2}$$

$$\therefore\ y''(1) = 4 \Rightarrow 4 = \cancelto{2}{2(1)} + \frac{C}{(1)^2} \Rightarrow \boxed{2 = C}$$

$$\therefore\ y'(1) = 1 \Rightarrow 1 = \cancelto{1}{(1)^2} - \frac{\cancelto{2}{C}}{1} + D \Rightarrow 1 = -1 + D \Rightarrow \boxed{D = 2}$$

$$\therefore\ y(1) = 2 \Rightarrow 2 = \frac{(1)^3}{3} - \cancelto{2}{C}\,\cancelto{0}{\ln|1|} + \cancelto{2}{D}(1) + E$$

$$\Rightarrow 2 = \tfrac{1}{3} - 0 + 2 + E \Rightarrow 2 - \tfrac{1}{3} - 2 = E \Rightarrow \boxed{E = -\tfrac{1}{3}}$$

$$\therefore\ \boxed{y(x) = \frac{x^3}{3} - 2\ln|x| + 2x - \tfrac{1}{3}}$$