Variation of Parameters and Use w/ Non-homogeneous ODEs of Order 2+

In our introduction to learning about how to solve 2^{nd}-order Linear Non-homogeneous ODEs, we learned how to solve such equations as long as our "force function" (aka external force function) $g(x) \neq 0$ was either a polynomial, exponential, sine, or cosine function (or a linear combination or product of any # of these types of functions). The Method of Undetermined Coefficients was used to make an educated guess about what y_p could be in the (non-homogeneous) solution $y = y_h + y_p$. However, our Method of Undetermined Coefficients does not work at all (or at best is very cumbersome to work with) when $g(x)$ is outside of the list of the type of functions aforementioned that work well with the Method of Undetermined Coefficients.

It turns out that we have a more general way of solving non-homogeneous ODEs that utilize the Reduction of Order (ROO), but with a few "twists and turns" that we normally don't see during performing ROO on an ODE. These "twists and turns" are actually choices where we get to randomly select some conditions about our ODE (and some of its byproducts) in order to make our non-homogeneous ODE follow the principles of a pseudo-superposition principle similar to that of homogeneous ODEs of order 2 and higher.

Our primary goal will be to get you familiar with the steps that are involved in doing this process which we call Variation of Parameters.

<u>NOTE</u>: The purpose of the Variation of Parameters method of solving non-homogeneous ODEs (of order 2+) is to have a systematic way of solving non-homogeneous, linear ODEs when $g(x) \neq 0$ is not a polynomial, exponential, sine, or cosine function (as well as any linear combination or product of any # of these types of functions).

Afterwards, we will do several examples to get you familiar with the process of working out these types of problems. Lastly, we will take some time to discuss how and why this process works (as well as why the process is called "Variation of Parameters" in the 1st place) by deriving this process from scratch.

<u>FYI (Why Method of Undetermined Coefficients is Limited to Function Type for $g(x)$)</u>

Note that all polynomial, exponential, sine, and cosine functions all have a finite # of (basis or fundamental) functions that can represent the derivative of these functions for an order N. Thus, $y^{(N)}$ can be expressed in a finite # of terms! This makes y_p easy to characterize w/ functions like these! Most of other functions we deal w/ don't have this property!

Steps of Variation of Parameters (2^{nd}-order Linear ODEs) - Traditional way

Step 1: Consider the ODE $ay'' + by' + cy = g$ $\rightarrow$ eqn($\star$), where a, b, c can be constants and/or functions (of x) and g is a function (of x). Also, assume that $y_p = u(x) \cdot y_1 + v(x) \cdot y_2$ is a particular solution to ($\star$) where $u = u(x)$ + $v = v(x)$ are functions (of x) that are not constants. Also recall that $y = y_h + y_p$ for ($\star$)!

Step 2: Find homogeneous solution $y_h = c_1 y_1 + c_2 y_2$ corresponding to eqn. ($\star$) and note that $\{y_1, y_2\}$ forms a basis of solutions (i.e. a fundamental set of solutions) for y_h.

Step 3: Given that y_1 + y_2 are known and that $u = u(x)$ and $v = v(x)$, find $u' = u'(x)$ and $v' = v(x)$ from the system of equations

$$\begin{cases} u'y_1 + v'y_2 = 0 \\ u'y_1' + v'y_2' = \frac{g}{a} \end{cases}$$

NOTE: In order to solve this system of equations, we have to find y_1' + y_2'.

by your prefered method. (The most common methods are Substitution, Elimination/Addition Method, Cramer's Rule, or Gauss-Jordan Elimination (aka RREF method). (Note that some methods may be more efficient than others).

Step 4: Integrate u' + v' to find general function for u + v.

Step 5: Take root function for u + v (i.e. drop all constant). Sub in $y_p = uy_1 + vy_2$ for final solution $y = y_h + y_p$!

Alternative Method for Variation of Parameters (Formula Way) - 2^{nd}-order ODEs

Consider $ay''+by'+cy=g$, where $g\neq 0$ and a, b, c can be constants and/or functions (of x). The function $g=g(x)$. Noting that y_1 and y_2 are function (of x) that create a basis for the corresponding homogeneous eqn $ay''+by'+cy=0$, the solution $y(x)=y_h+y_p=(c_1y_1+c_2y_2)+(uy_1+vy_2)$, where $u+v$ are functions (of x), can be found by...

→ Do not forget your integration constants!!

$$(\Delta)\quad \boxed{y(x) = -y_1(x)\int \frac{y_2(x)\cdot g/a}{w(x)}\,dx + y_2(x)\int \frac{y_1(x)\cdot g/a}{w(x)}\cdot dx}$$

where $W(x)=$ Wronskian of $\{y_1, y_2\} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2$

$$\therefore\ y = y(x) = y_h(x) + y_p(x) = c_1y_1 + c_2y_2 - y_1\int\frac{y_2\cdot g/a}{w(x)}\,dx + y_2\int\frac{y_1\cdot g/a}{w(x)}\,dx$$

↳ this is equation (Δ) with integration constants explicitly written out.

Formula for Variation of Parameters for N^{th}-order ODEs

For the N^{th}-order ODE $a_0y^{(N)}+a_1y^{(N-1)}+\dots+a_{N-1}y'+a_Ny=g$, where $g=g(x)$ and $g\neq 0$ and $a_0,\dots,a_N$ are constants and/or functions (of x), the function $y=y_h+y_p$ is a solution to our N^{th}-order ODE where $y_h=c_1y_1+c_2y_2+\dots+c_ny_n$, where $c_1,\dots,c_n$ are constants and $\boxed{y=y(x)=\sum_{k=1}^{N}\left[y_k(x)\cdot\int\frac{W_k(x)}{w(x)}\,dx\right]}$, where

$$w(x) = \begin{array}{cccc} k=1 & k=2 & & k=N \\ \end{array}$$
$$w(x)=\begin{vmatrix} y_1 & y_2 & \cdots & y_N \\ y_1' & y_2' & \cdots & y_N' \\ \vdots & \vdots & \ddots & \\ y_1^{(N-1)} & y_2^{(N-1)} & & y_N^{(N-1)} \end{vmatrix}$$

and $W_k = W(x)$ with the k^{th} column of $W(x)$ replaced with $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ g/a \end{bmatrix}$.

Ex. 1: Solve $y''-2y'+y=\frac{e^x}{x}$ via Traditional Way of Variation of Parameters. Use Cramer's Rule to solve for u' and v'.

Steps 1+2: $g=\frac{g}{a}=\frac{e^x}{x}$ where $a=1$. Let $y=y_h+y_p$, where $y_p=uy_1+vy_2$, where y_1 + y_2 are basis solutions of y_h

$\therefore\ y''-2y'+y=0 \Rightarrow$ Char. eqn.: $r^2-2r+1=0 \Rightarrow (r-1)^2=0 \Rightarrow \boxed{r=1}$

$\therefore\ y_h=c_1y_1+c_2y_2=c_1e^x+c_2xe^x \Rightarrow \{e^x, xe^x\}=\{y_1,y_2\}=$ basis for y_h.

Steps 3+4: Solve the system of equations below for u' + v'. (Choosing Cramer's Rule).

$$\begin{cases} u'y_1+v'y_2=0\\ u'y_1'+v'y_2'=\frac{g}{a}\end{cases} \Rightarrow u'=\frac{\begin{vmatrix}0 & y_2\\ \frac{g}{a} & y_2'\end{vmatrix}}{\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}}=\frac{-\frac{g}{a}y_2}{y_1y_2'-y_1'y_2} \text{ and } v'=\frac{\cancel{\begin{vmatrix}y_1 & 0\\ y_1' & \frac{g}{a}\end{vmatrix}} \rightarrow y_1\cdot\frac{g}{a}}{y_1y_2'-y_1'y_2}$$

$\therefore\ u'=\frac{-\frac{g}{a}y_2}{y_1y_2'-y_1'y_2}$ and $v'=\frac{y_1\cdot\frac{g}{a}}{y_1y_2'-y_1'y_2}$. Note... $y_1'=(e^x)'=e^x$, $y_2'=(xe^x)=e^x+xe^x=e^x(1+x)$

$$\therefore\ u'=\frac{-\left(\frac{e^x}{x}\right)(xe^x)}{e^x(e^x(1+x))-e^x(xe^x)}=\frac{-\cancel{e^{2x}}^{\,1}}{\cancel{e^{2x}}_{\,1}[1+\cancel{x}-\cancel{x}]}=-1 \Rightarrow u=\int \cancel{u'}^{\,-1}\cdot dx=-x+K_1$$

$$\therefore\ v'=\frac{(e^x)\left(\frac{e^x}{x}\right)}{e^{2x}}=\frac{\frac{1}{x}\cdot\cancel{e^{2x}}^{\,1}}{\cancel{e^{2x}}_{\,1}}=\frac{1}{x} \Rightarrow v=\int \cancel{v'}^{\,1/x}\cdot dx=\ln|x|+K_2$$

Step 5: For simplicity, let $u=-x$ and $v=\ln|x| \Rightarrow y_p=-xe^x+\ln|x|\cdot xe^x$

$\therefore\ y=y_h+y_p=c_1e^x+c_2xe^x+(-xe^x+xe^x\ln|x|) \Rightarrow$ $\boxed{y=c_1e^x+(c_3+\ln|x|)xe^x,\text{ where } c_3=c_2-1.}$

Ex. 2 : Solve $y''+y=\cot(x)$ via Formula Way of Variation of Parameters.

For 2^{nd}-order ODEs, $y=y_h+y_p$ where y_h is the homogeneous solution corresponding our ODE and y_p is...

$$y_p(x)=-y_1(x)\int\frac{y_2(x)\cdot g/a}{W(x)}\,dx+y_2(x)\int\frac{y_1(x)\cdot g/a}{W(x)}\,dx$$

In this example, I'm waiting to write integration constant results, y_h terms, until end of problem for convenience!

Find y_h

$$y''+y=0\Rightarrow r^2+1=0\Rightarrow r=\pm i\Rightarrow y_h=e^0\left[c_1\cos(x)+c_2\sin(x)\right]$$

$$\Rightarrow y_h=c_1\cos(x)+c_2\sin(x)$$

$\therefore \{y_1,y_2\}=\{\cos(x),\sin(x)\}$ = basis of solutions for y_h

Find y_p

Note that $y_1(x)=\cos(x)$ and $y_2(x)=\sin(x)$. Also, $g/a=g=\cot(x)$.

$$\therefore W(x)=\begin{vmatrix}y_1 & y_2\\ y_1' & y_2'\end{vmatrix}=y_1y_2'-y_1'y_2=(\cos(x))(\sin(x))'-(\cos(x))'(\sin(x))=$$

$$(\cos(x))(\cos(x))-(-\sin(x))(\sin(x))=\cos^2(x)+\sin^2(x)=1$$

$$\therefore y_p(x)=-\cos(x)\int\frac{\cancel{\sin(x)\cdot\cot(x)}^{\cos(x)}}{1}\,dx+\sin(x)\int\frac{\cos(x)\cdot\cancel{\cot(x)}^{\frac{\cos(x)}{\sin(x)}}}{1}\,dx=$$

$$-\cos(x)\int\cos(x)\,dx+\sin(x)\int\frac{\cancel{\cos^2(x)}^{1-\sin^2(x)}}{\sin(x)}\,dx=-\cos(x)\left[-\sin(x)\right]+\sin(x)\int(\csc(x)-\sin(x))\,dx=$$

$$\sin(x)\cos(x)+\sin(x)\left[\ln|\csc(x)-\cot(x)|-\cos(x)\right]=\cancel{\sin(x)\cos(x)}+\sin(x)\cdot\ln|\csc(x)-\cot(x)|\cancel{-\sin(x)\cos(x)}$$

$$\Rightarrow y_p=\sin(x)\cdot\ln|\csc(x)-\cot(x)|$$

$$\therefore y=y_h+y_p$$

$$\Rightarrow \boxed{y=c_1\cos(x)+c_2\sin(x)+\sin(x)\cdot\ln|\csc(x)-\cot(x)|}$$

Ex. 3: Solve $x^2y''+xy'-y=\sqrt{x}$ via Traditional Way of Variation of Parameters. Use substitution method to solve for u' and v'.

Let $y=y_h+y_p$, where y_h = homogeneous solution corresponding to our ODE and $y_p=u(x)\cdot y_1+v(x)\cdot y_2=uy_1+vy_2$. We need to find y_h and y_p.

Find y_h: Since $x^2y''+xy'-y=0$ is an E-C equation, we know that x^r is a general solution for this equation. We need to find the roots r_1 & r_2 so that $y_h=c_1x^{r_1}+c_2x^{r_2}$ can be found. The characteristic eqn. for an E-C ODE is $\alpha r^2+(\beta-\alpha)r+\gamma=0$, where (in this case) $\alpha=\beta=1$ and $\gamma=-1$.

$\therefore \alpha r^2+(\beta-\alpha)r+\gamma=0 \Rightarrow r^2+0r-1=0 \Rightarrow r^2-1=0 \Rightarrow r=\pm1 \Rightarrow \begin{matrix} r_1=-1 \\ r_2=1 \end{matrix}$

$\therefore \boxed{y_h=c_1x^{-1}+c_2x^1} \Rightarrow \{\underbrace{x^{-1}}_{y_1},\underbrace{x}_{y_2}\}$ form a basis of solutions for y_h

Find y_p: Since $y_p=uy_1+vy_2 \Rightarrow y_p=ux^{-1}+vx$. We need to find u & v!
To find u & v, we will 1st need to find u' & v' from the system of equations

$$\begin{cases} u'y_1+v'y_2=0 \\ u'y_1'+v'y_2'=g/a \end{cases}$$, where $y_1=x^{-1}$, $y_1'=-x^{-2}$, $y_2=x$, $y_2'=1$, and $g/a=\frac{\sqrt{x}}{x^2}=x^{-3/2}$.

$\therefore u'y_1+v'y_2=0 \Rightarrow v'=\frac{-u'y_1}{y_2}=\frac{-u'x^{-1}}{x}=-\frac{u'}{x^2}=-u'x^{-2} \Rightarrow \boxed{v'=-u'x^{-2}}$

$\therefore u'y_1'+v'y_2'=g/a \Rightarrow u'(-x^{-2})-u'x^{-2}\cdot(1)=x^{-3/2} \Rightarrow -2x^{-2}\cdot u'=x^{-3/2} \Rightarrow \boxed{u'=\frac{x^{-3/2}}{-2x^{-2}}=\frac{x^{1/2}}{-2}}$

$\therefore v'=-\left(\frac{x^{1/2}}{-2}\right)(x^{-2})=\frac{1}{2}x^{-3/2}$ and $u'=-\frac{1}{2}x^{1/2}$

$\Rightarrow v=\int v'dx=\int \frac{1}{2}x^{-3/2}dx=\frac{1}{2}\frac{x^{-1/2}}{-1/2}+K_1=-x^{-1/2}+K_1$, where K_1 = constant.

and $u=\int u'dx=\int -\frac{1}{2}x^{1/2}dx=-\frac{1}{2}\frac{x^{3/2}}{3/2}+K_2=-\frac{1}{3}x^{3/2}+K_2$, where K_2 = constant.

Ex. 3 (cont'd)

For simplicity, let $u = -\frac{1}{3}x^{3/2}$ and $v = -x^{-1/2}$

$$\therefore y_p = u y_1 + v y_2 = \left(-\tfrac{1}{3}x^{3/2}\right)\left(x^{-1}\right) + \left(-x^{-1/2}\right)\left(x\right) = -\tfrac{1}{3}x^{1/2} - 1x^{1/2} = -\tfrac{4}{3}x^{1/2}$$

$$\therefore y_p = -\tfrac{4}{3}x^{1/2} = -\tfrac{4}{3}\sqrt{x}$$

$\therefore$ Our solution to our non-homogeneous (E-C) equation is ...

$$\boxed{y = y_h + y_p = c_1 x^{-1} + c_2 x - \tfrac{4}{3}\sqrt{x}}$$

Ex. 4: Solve $t^2 y'' - 2ty' + 2y = t \cdot \ln(t)$ via Formula Way of Variation of Parameters.

For 2^{nd}-order (E-C) ODEs, our formula for Variation of Parameters is...

$$y(t) = -y_1(t) \int \frac{y_2(t) \cdot g/a}{W(t)} \, dt + y_2(t) \int \frac{y_1(t) \cdot g/a}{W(t)} \, dt,$$ where $W(x) =$ Wronskian of $\{y_1, y_2\}$ $= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$.

The solution for our (Non-homogeneous) ODE is $y = y_h + y_p$.

Find y_h: The corresponding homogeneous ODE for our given ODE is $t^2 y'' - 2ty' + 2y = 0$. The corresponding characteristic eqn will be $\alpha r^2 + (\beta - \alpha) r + \gamma = 0$ where $\alpha = 1$, $\beta = -2$, & $\gamma = 2$.

$\therefore r^2 + (-2-1) r + 2 = 0 \Rightarrow r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0 \Rightarrow r = 1, 2$

$\therefore y_h = c_1 y_1 + c_2 y_2 = c_1 t^1 + c_2 t^2 \Rightarrow \{t, t^2\} = \{y_1, y_2\} =$ basis for y_h

Find $y(t)$*: Note that $g/a = \dfrac{t \ln(t)}{t^2} = \dfrac{\ln(t)}{t}$, $y_1(t) = t$, $y_2(t) = t^2$, $y_1'(t) = 1$, and $y_2'(t) = 2t$. Also, $W(t) = y_1 y_2' - y_1' y_2 = t(2t) - 1(t^2) = t^2$.

$$\therefore y(t) = -t \int \frac{t^2 \cdot \frac{\ln(t)}{t}}{t^2} \, dt + t^2 \int \frac{t \cdot \frac{\ln(t)}{t}}{t^2} \, dt = -t \underbrace{\int \frac{\ln(t)}{t} \, dt}_{\text{Use substitution rule to evaluate}} + t^2 \underbrace{\int \frac{\ln(t)}{t^2} \, dt}_{\text{Use integration by parts to evaluate}}$$

Normally we would just look for $y_p(t)$ here. However, I am trying to showcase how we can use Formula Way of Variation of Parameters to recover y_h terms through listing integration const. terms!

Ex. 4: (cont'd)

$\therefore -t\int \frac{\ln(t)}{t}\,dt$. Let $z=\ln(t)$. Then, $dz=\frac{1}{t}\,dt$

$$\therefore -t\int z\cdot dz = -t\left[\frac{z^2}{2}+K_1\right] = -t\left[\frac{\ln^2(t)}{2}+K_1\right] = -\frac{t\ln^2(t)}{2} - K_1 t$$

Also, let $m=\ln(t)$ and $dn=\frac{1}{t^2}\,dt = t^{-2}\cdot dt$. Then, using integration by parts on $\int \frac{\ln(t)}{t^2}\,dt$, we see that $dm=\frac{1}{t}\cdot dt$ and $n=-t^{-1}=-\frac{1}{t}$

$$\therefore \int \frac{\ln(t)}{t^2}\,dt = \int m\,dn = mn - \int n\,dm = \ln(t)\cdot\frac{-1}{t} - \int -\frac{1}{t}\cdot\frac{1}{t}\,dt =$$

$$\frac{\ln(t)}{-t} + \int t^{-2}\,dt = -\frac{\ln(t)}{t} - t^{-1} + K_2 = -\frac{\ln(t)}{t} - \frac{1}{t} + K_2$$

$$\therefore t^2\int \frac{\ln(t)}{t^2}\,dt = \frac{-t^2\ln(t)}{t} - \frac{t^2}{t} + K_2 t^2 = -t\ln(t) - t + K_2 t^2$$

$$\therefore y(t) = -t\int\frac{\ln(t)}{t}\,dt + t^2\int\frac{\ln(t)}{t^2}\,dt = -\frac{t\ln^2(t)}{2} - K_1 t - t\ln(t) - t + K_2 t^2$$

$$\therefore \boxed{y(t) = c_1 t + c_2 t^2 - \frac{t\ln^2(t)}{2} - t\ln(t), \text{ where } c_1 = -K_1 - 1 \text{ and } c_2 = K_2}$$

NOTE: We went ahead and found all of $y(t)$ instead of just $y_p(t)$ first, and, then $y(t)=y_h+y_p$, because I wanted to showcase how you use Formula Way of Variation of Parameters and list out integration constant terms as well.

So, explicitly stated, is $y_p = -\frac{1}{2}t\ln^2(t) - t\ln(t)$ and $y_h = c_1 t + c_2 t^2$!

Ex. 5: Solve $y''' + y' = \sec(x)$ via Formula Way of Variation of Parameters.

NOTE: This is a 3^{rd}-order Non-homogeneous ODE. We need to find $y = y_h + y_p$.

Find y_h: The characteristic eqn for $y''' + y' = 0$ is $r^3 + r = 0$

$\therefore r(r^2+1) = 0 \Rightarrow r = 0, \pm i \Rightarrow \{r_1, r_2, r_3\} = \{1, \cos(x), \sin(x)\}$ is the basis of solutions for $y_h \Rightarrow \boxed{y_h = c_1 + c_2\cos(x) + c_3\sin(x)}$

Find y_p: Note that $y_1 = 1$, $y_2 = \cos(x)$, and $y_3 = \sin(x) \Rightarrow y_1' = 0$, $y_2' = -\sin(x)$, and $y_3' = \cos(x)$. Also, $g/a = g = \sec(x)$.

So, $y(x) = \sum_{k=1}^{N}\left[(-1)^{N+k} \cdot y_k(x) \cdot \int \frac{W_k(x)}{W(x)}\,dx\right]$, where $W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} =$

$$\begin{vmatrix} 1 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \\ 0 & -\cos(x) & -\sin(x) \end{vmatrix} = 1 \cdot \begin{vmatrix} -\sin(x) & \cos(x) \\ -\cos(x) & -\sin(x) \end{vmatrix} = (-\sin(x))^2 - (-\cos(x))(\cos(x)) = \sin^2(x) + \cos^2(x) = 1,$$

$N = 3$, $k = 1, 2, 3$; $W_1 = \begin{vmatrix} 0 & \cos(x) & \sin(x) \\ 0 & -\sin(x) & \cos(x) \\ \sec(x) & -\cos(x) & -\sin(x) \end{vmatrix} = \sec(x) \cdot \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} =$

$$\sec(x)\left[\cos^2(x) - (-\sin(x))(\sin(x))\right] = \sec(x)\left[\xcancel{\cos^2(x) + \sin^2(x)}^{\,1}\right] = \sec(x),$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin(x) \\ 0 & 0 & \cos(x) \\ 0 & \sec(x) & -\sin(x) \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & \cos(x) \\ \sec(x) & -\sin(x) \end{vmatrix} = 1\left[0 - \xcancel{\sec(x)\cdot\cos(x)}^{\,1}\right] = -1, \text{ and}$$

$$W_3 = \begin{vmatrix} 1 & \cos(x) & 0 \\ 0 & -\sin(x) & 0 \\ 0 & -\cos(x) & \sec(x) \end{vmatrix} = 1 \begin{vmatrix} -\sin(x) & 0 \\ -\cos(x) & \sec(x) \end{vmatrix} = 1\left[\xcancel{-\sin(x)\cdot\sec(x)}^{\,\tan(x)} + 0\right] = -\tan(x).$$

Ex. 5: (cont'd)

$$\therefore\ y(x) = \overset{1}{\cancel{(-1)^{3+1}}}\, y_1(x)\int \frac{W_1(x)}{W(x)}\,dx + \overset{-1}{\cancel{(-1)^{3+2}}}\, y_2(x)\int \frac{W_2(x)}{W(x)}\,dx + \overset{1}{\cancel{(-1)^{3+3}}}\, y_3(x)\int \frac{W_3(x)}{W(x)}\,dx$$

$$y(x) = 1\cdot\int \frac{\sec(x)}{1}\,dx - \cos(x)\int \frac{-1}{1}\,dx + 1\cdot\sin(x)\int \frac{-\tan(x)}{1}\,dx$$

$$y(x) = \Big(\ln\left|\sec(x)+\tan(x)\right| + K_1\Big) + \Big(x\cos(x) + K_2\cos(x)\Big) + \Big(-\sin(x)\cdot\left[-\ln\left|\cos(x)\right| + K_3\right]\Big)$$

$$y(x) = \ln\left|\sec(x)+\tan(x)\right| + K_1 + x\cos(x) + K_2\cos(x) + \sin(x)\cdot\ln\left|\cos(x)\right| - K_3\sin(x)$$

$$\therefore\ y(x) = K_1 + K_2\cos(x) - K_3\sin(x) + \ln\left|\sec(x)+\tan(x)\right| + x\cos(x) + \sin(x)\cdot\ln\left|\cos(x)\right|$$

$$\Rightarrow\ \boxed{\begin{array}{l} y(x) = c_1 + c_2\cos(x) + c_3\sin(x) + \ln\left|\sec(x)+\tan(x)\right| + x\cos(x) + \sin(x)\cdot\ln\left|\cos(x)\right| \\ \text{where } c_1 = K_1,\ c_2 = K_2, \text{ and } c_3 = -K_3. \end{array}}$$

Ex. 6: Recall from Ex. 2 in Ch. 19 & 20 notes on solving 2^{nd}-order Non-homogeneous ODEs that $y = c_1 e^{2x} + c_2 e^{3x} - 4xe^{2x}$ is a solution to the ODE $y'' - 5y' + 6y = 4e^{2x}$ using the Method of Undetermined Coefficients. Show that we can arrive at this same solution using Variation of Parameters.

Sol'n: Since our ODE is a 2^{nd}-order Linear Non-homogeneous ODE, we know our solution $y = y_h + y_p$, where $y_h = c_1 y_1 + c_2 y_2$ and $y_p = u y_1 + v y_2$, where u + v are functions (of x).

Find y_h: $y'' - 5y' + 6y = 0 \Rightarrow$ Char. eqn: $r^2 - 5r + 6 = 0$

$\therefore (r-2)(r-3) = 0 \Rightarrow r = 2, 3 \Rightarrow \{y_1, y_2\} = \{e^{2x}, e^{3x}\}$ = basis of solutions for $y_h \Rightarrow \boxed{y_h = c_1 e^{2x} + c_2 e^{3x}}$

Find y_p: Using the Traditional Way of Variation of Parameters, we must solve the system of equations $\begin{cases} u'y_1 + v'y_2 = 0 \\ u'y_1' + v'y_2' = g/a \end{cases}$, where

$y_1 = e^{2x}$, $y_1' = 2e^{2x}$, $y_2 = e^{3x}$, $y_2' = 3e^{3x}$, $g/a = g = 4e^{2x}$, $u = \int u' \, dx$, and $v = \int v' \, dx$. We will solve this system of equations for u' + v' via Cramer's Rule.

$$\therefore u' = \frac{W_1(x)}{W(x)} = \frac{\begin{vmatrix} 0 & y_2 \\ g/a & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-g/a \, y_2}{y_1 y_2' - y_1' y_2} = \frac{-4e^{2x} \cdot e^{3x}}{e^{2x}(3e^{3x}) - (2e^{2x})(e^{3x})} = \frac{-4e^{5x}}{e^{5x}} = -4 \quad \therefore \boxed{u = -4x + K_1}$$

$$v' = \frac{W_2(x)}{W(x)} = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g/a \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 \cdot g/a}{e^{5x}} = \frac{e^{2x} \cdot 4e^{2x}}{e^{5x}} = \frac{4e^{4x}}{e^{5x}} = 4e^{-x} \Rightarrow \boxed{v = -4e^{-x} + K_2}$$

$\therefore y_p = uy_1 + vy_2 = -4xe^{2x} + K_1 e^{2x} + (-4e^{2x}) + K_2 e^{3x}$

$\therefore y = y_h + y_p$. Let $c_1 = K_1 - 4$ + $c_2 = K_2$.

$\Rightarrow \boxed{y = c_1 e^{2x} + c_2 e^{3x} - 4xe^{2x}}$

How & Why Does Variation of Parameters Method Work? (optional)

Without Loss of Generality (WLOG), we will consider deriving the method for Variation of Parameters for a 2^{nd}-order Linear Non-homogeneous ODE. Afterwards we will state the extension of this method for an N^{th} order and hopefully you will see the pattern that ensues.

Consider the ODE $\underbrace{ay'' + by' + cy = g}_{(\star)}$, where $g \neq 0$. Then it follows that the solution of this ODE is in the form $y = y_h + y_p$, where y_h is the corresponding homogeneous solution to equation $(\star)$ and y_p is a particular solution to equation $(\star)$. We observe the following facts about our situation.

(i) $y_h = c_1 y_1 + c_2 y_2$, where c_1, c_2 are constants & $\{y_1, y_2\}$ create a basis for y_h.
Thus $(\star)$ when $g = 0$ follows the principle of Superposition (of solutions).

(ii) Since $(\star)$ when $g \neq 0$ is Non-homogeneous, the solution to $(\star)$ does not follow the principle of Superposition (of solutions).

(iii) Let y_a and y_b be 2 particular solutions of $(\star)$. Then it is true that y_h must be the difference of these 2 solutions (i.e. $y_h = y_a - y_b$ or $y_h = y_b - y_a$).

(iv) If y_w and y_z are 2 known (or assumed) solutions to an ODE (regardless of type), then by the Reduction of Order (ROO) Method, uy_w and vy_z can be solutions of the ODE as well, where $u = u(x)$ and $v = v(x)$ are functions (of x).

We can conclude from statements (i) – (iv) that...

- If y_p is a solution to (★), then y_p will not be able to be expressed as $y_p = K_1 y_1 + K_2 y_2$, where K_1, K_2 are constants such that $K_1 \neq c_1$ and $K_2 \neq c_2$ because (★) has solutions that fail to follow the Principle of Superposition in this way.
- If we assumed 2 solutions of (★) are y_w and y_z, then we could use ROO to state that uy_w and vy_z are solutions of (★) as well. Since u and v are functions (of x), we might have to satisfy some conditions for which this situation could be true if necessary

From these conclusions, we know that $y_p = K_1 y_1 + K_2 y_2$ can't be a solution to (★) since K_1, K_2 are constants. BUT WHAT IF WE LET K_1 and K_2 be functions (of x) instead of constants? If we did, then y_p would take on the form $y_p = uy_1 + vy_2$. Since u and v are constants, we don't know if the Principle of Superposition would or would not apply here. Since it is uncertain if solutions for (★) in the form $y_p = uy_1 + vy_2$ would satisfy the Superposition Principle, let's assume it would! If we made this assumption, then what other assumptions or conditions would have to be satisfied in order to make this situation work? Performing ROO under these conditions on (★) will answer these questions and yield the Var. of Parameters Method.

Therefore, let $y_p = uy_1 + vy_2$ be a solution for (★) via ROO Method. We will need to (attempt to) try to find u and v. We observe the following expectations as we go through the ROO process on (★).

- We must find y_p' + y_p''. These results, along with y_p, must be substituted back into (★).
- Since "uy_1" and "vy_2" are assumed solutions to (★) via ROO, we expect that the sum of all terms of lowest order (i.e. sum of all "u" like terms and "v" like terms) will cancel to zero.
- Note that since y_1 + y_2 are homogeneous solutions of (★) when $g=0$, this means that $ay_k'' + by_k' + cy_k = 0$, where $k = 1, 2$.
- Once all terms of "u" and "v" are cancelled to zero <u>AND</u> since we are assuming $g \neq 0$ (since (★) in Non-homogeneous), we are expecting that our resulting equation for (★) (after substituting in y_p, y_p', + y_p'' into (★)) will be a 1^{st}-order Linear ODE (since (★) is a 2^{nd}-order Linear ODE).

Now let's start the process of finding "u" and "v".

$\therefore\ y_p = uy_1 + vy_2$

$\therefore\ y_p' = u'y_1 + uy_1' + v'y_2 + vy_2' = (u'y_1 + v'y_2) + (uy_1' + vy_2')$

we are expecting these terms to eventually cancel out with others in (★) after y_p, y_p', + y_p'' substituted in (★)

$\therefore\ y_p'' = (u'y_1 + v'y_2)' + (u'y_1' + uy_1'' + v'y_2' + vy_2'')$

Since expanding this expression out will not yield any "u" or "v" terms, we refrain from expanding it out to reduce amount of computation!

$$\therefore \; a y_p'' + b y_p' + c y_p = g$$

$$\Rightarrow \left\{ \begin{array}{l} a\left[(u'y_1 + v'y_2)' + (u'y_1' + uy_1'' + v'y_2' + vy_2'')\right] \\ + b\left[(u'y_1 + v'y_2) + (uy_1' + vy_2')\right] \\ + c\left[uy_1 + vy_2\right] \end{array} \right\} = g$$

$$\Rightarrow \left\{ \begin{array}{l} a(u'y_1 + v'y_2)' + au'y_1' + auy_1'' + av'y_2' + avy_2'' \\ + b(u'y_1 + v'y_2) + buy_1' + bvy_2' \\ + cuy_1 + cvy_2 \end{array} \right\} = g$$

$$\Rightarrow \left\{ \begin{array}{l} a(u'y_1 + v'y_2)' + b(u'y_1 + v'y_2) + au'y_1' + av'y_2' \\ + u\left[\cancelto{0}{ay_1'' + by_1' + cy_1}\right] + v\left[\cancelto{0}{ay_2'' + by_2' + cy_2}\right] \end{array} \right\} = g \quad (\star\star)$$

Recall that $ay_k'' + by_k' + cy_k = 0$, for $k = 1, 2$. Therefore, equation $(\star\star)$ will reduce to...

$$a(u'y_1 + v'y_2)' + b(u'y_1 + v'y_2) + au'y_1' + av'y_2' = g \quad (\star\star\star)$$

<u>NOTE</u>! Since we cancelled out all like terms of "u" and "v" (via $ay_k'' + by_k'' + cy_k = 0$ for $k = 1, 2$), we now concentrate on making equation $(\star\star\star)$ look like a 1st-order Linear ODE. Since the expression "$u'y_1 + v'y_2$" shows up in the 2nd term of $(\star\star\star)$, the derivative "$(u'y_1 + v'y_2)'$" is also present in the 1st term of $(\star\star\star)$,

and "$u'y_1 + v'y_2$" can be assumed to be a function (of x), we let $f = f(x) = u'y_1 + v'y_2$. Therefore, equation (★★★) can be expressed as...

$$af' + bf + a[u'y_1' + v'y_2'] = g$$

$$\Rightarrow f' + \frac{b}{a} f + [u'y_1' + v'y_2'] = g/a \quad (★★★★) \quad (1^{st}\text{-order Linear ODE})$$

where a, b are constants and f, f', u', v', y_1', and y_2' are functions (of x).

BIG NOTE: At this point in our derivation, please realize that we have introduced 6 unknowns (i.e. unknown functions) but only 5 equations (of significance) that contain at least 1 of the unknowns. (See chart below).

Unknown (Functions)	Equations (w/ unknowns acting like variables)
1. u	1. $u = \int u' \, dx$
2. v	2. $v = \int v' \, dx$
3. u'	3. equation (★★★★) above
4. v'	4. $u'y_1 + v'y_2 = f$
5. f	5. $f = \int f' \, dx$
6. f'	

Since # unknowns > # equations, by definition, we have a system of (linear) equations that is dependent $\Rightarrow$ we have at least 1 "free variable" (aka "free parameter"). Therefore, this system of equations technically has an infinite number of solutions. We can allow any 1 of our 6 unknowns to be a "free parameter" by allowing it to equal any (real) # (or function) we want. It turns out that the easiest and most convenient unknown to select as our "free parameter" should be "f"! Why? Because it is the primary function in equation (★★★★) and this equation possesses the most unknowns in it!

$\therefore$ Let $f = z$, where z = any real # (or function)!

So, depending on what we vary f to be, we will find a specific solution for u' and v' (and, thus, u and v) within the family of (infinite) solutions for our system of equations ...

$$\begin{cases} u = \int u'\,dx \\ v = \int v'\,dx \\ f = u'y_1 + v'y_2 \\ f' + \frac{b}{a}f + [u'y_1' + v'y_2'] = \frac{g}{a} \\ f = \int f'\,dx \end{cases}$$

, where y_1 + y_2 are the (known) solutions to the corresponding homogeneous solution, y_h.

Since we only need 1 set of solutions (and not infinitely-many), we will make the selection of f that we presume will be the easiest to work with for this system of equations. THUS, IT IS CONVENIENT TO LET $f = 0$!

NOTE: The double-underlined phrase is a phrase you will see often in most textbooks, but rarely (if ever) will you get an explanation as to why it is so convenient to let $f = u'y_1 + v'y_2 = 0$. Now you KNOW WHY!! IT IS BECAUSE "f" IS CHOSEN TO BE A PARAMETER IN A SYSTEM OF EQUATIONS w/ 6 UNKNOWN FUNCTION (VARIABLES) AND ONLY 5 EQUATIONS! Also, the ability to let "f" be various (real) #s or functions is why this method is called Variation of Parameters!!!

So, if $f = 0$, then equation (★★★★) reduces to...

$$\cancelto{0}{f''} + \frac{b}{a}\cancelto{0}{f'} + [u'y_1' + v'y_2'] = g/a \quad (★★★★)$$

$$\Rightarrow u'y_1' + v'y_2' = g/a \quad (5★) \quad \text{since } f = 0 \Rightarrow f' = 0 \text{ !}$$

So now our system of equations is...

$$\begin{cases} u = \int u'\,dx \\ v = \int v'\,dx \\ f' + b/a\, f + [u'y_1' + v'y_2'] = g/a \\ f = u'y_1 + v'y_2 \\ u'y_1' + v'y_2' = g/a \\ f = 0 \end{cases}$$

, which gives us 6 equations to match the 6 unknowns we have !!

Since u, v, and f could be easily found from u', v', and f', respectively, we can reduce this system of equations down to a system of 3 equations and 3 unknowns where the unknowns are u', v', and f.

$$\therefore \begin{cases} f = u'y_1 + v'y_2 \\ f = 0 \\ u'y_1' + v'y_2' = g/a \end{cases} \Rightarrow \begin{cases} 0 = u'y_1 + v'y_2 \\ u'y_1' + v'y_2' = g/a \end{cases}$$

This is our system of 2 equations / 2 unknowns for the Variation of Parameters for 2^{nd}-order Linear Non-homogeneous ODEs !!

This system of equations can be solved (preferably) via Cramer's Rule or Substitution Method from (Linear) Algebra !

Variation of Parameters - Derived Formula Way for 2^{nd}-order ODEs

Solving $\begin{cases} 0 = u'y_1 + v'y_2 \\ u'y_1' + v'y_2' = g/a \end{cases}$ via substitution method, ...

- $u' = \dfrac{-v'y_2}{y_1} \Rightarrow u = \int u'\,dx = \int -\dfrac{v'y_2}{y_1}\,dx$

- $u'y_1' + v'y_2' = g/a \Rightarrow \left(\dfrac{-v'y_2}{y_1}\right)y_1' + v'y_2' = g/a$

$$\Rightarrow v'\left[y_2' - \frac{y_2y_1'}{y_1}\right] = g/a$$

$$\Rightarrow v'\left[\frac{y_2'y_1 - y_2y_1'}{y_1}\right] = g/a$$

$$\Rightarrow v' = \frac{g/a \cdot y_1}{y_2'y_1 - y_2y_1'} = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g/a \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{W_2(x)}{W(x)},$$

where $W(x)$ = Wronskian of $\{y_1, y_2\} = \det\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$ and $W_2(x) = W(x)$ where $k=2$ column replaced with $\begin{bmatrix} 0 \\ g/a \end{bmatrix}$.

Also, $u' = -\dfrac{v'y_2}{y_1} = \left[\dfrac{-g/a \cdot \cancel{y_1}^{\,1}}{y_2'y_1 - y_2y_1'}\right]\left[\dfrac{y_2}{\cancel{y_1}_{\,1}}\right] = \dfrac{-g/a\, y_2}{y_2'y_1 - y_2y_1'} = \dfrac{\begin{vmatrix} 0 & y_2 \\ g/a & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \dfrac{W_1(x)}{W(x)},$

where $W_1(x) = W(x)$ where $k=1$ column replaced with $\begin{bmatrix} 0 \\ g/a \end{bmatrix}$

$\therefore$ $u = \int u'\,dx = \int \dfrac{W_1(x)}{W(x)}\,dx$ and $v = \int v'\,dx = \int \dfrac{W_2(x)}{W(x)}\,dx$

our integration does have integration constants, but they are recovered by writing y_h terms

Recall that our solution for our ODE is of the form $y = y_h + y_p$, where $y_h = c_1y_1 + c_2y_2$ and $y_p = uy_1 + vy_2$.

$$\boxed{\begin{aligned} &\therefore\ y_p = y_p(x) = y_1(x)\int \frac{W_1(x)}{W(x)}\,dx + y_2(x)\int \frac{W_2(x)}{W(x)}\,dx \\ &\therefore\ y = y_h + y_p = c_1y_1(x) + c_2y_2(x) + y_1(x)\int \frac{W_1(x)}{W(x)}\,dx + y_2(x)\int \frac{W_2(x)}{W(x)}\,dx \end{aligned}}$$

Variation of Parameters for 3rd-order (Linear) ODEs

If we let $a_0 y''' + a_1 y'' + a_2 y' + a_3 y = g$ be our ODE and assume that $y = y_h + y_p$ be a solution to this ODE where ...

- $y_h = c_1 y_1 + c_2 y_2 + c_3 y_3$, where $\{y_1, y_2, y_3\}$ = basis of solutions for y_h, and
- $y_p = u y_1 + v y_2 + p y_3$, where $u, v, + p$ are functions of x,

then it turns out that our (reduced) system of equations is ...

$$\begin{cases} 0 = u' y_1 + v' y_2 \\ 0 = u' y_1' + v' y_2' \\ g/a = u' y_1'' + v' y_2'' \end{cases}$$

$$\therefore y(x) = y_h + y_p = \left(c_1 y_1 + c_2 y_2 + c_3 y_3\right) + \left(y_1 \int \frac{W_1(x)}{W(x)}\,dx + y_2 \int \frac{W_2(x)}{W(x)}\,dx + y_3 \int \frac{W_3(x)}{W(x)}\,dx\right)$$

where $W(x)$ = Wronskian for $\{y_1, y_2, y_3\} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$ and $W_k(x) = W(x)$

where the k^{th} column is replaced with $\begin{bmatrix} 0 \\ 0 \\ g/a \end{bmatrix}$.

For the Variation of Parameters for an N^{th}-order (Linear) ODE, it turns out to be ...

$$y(x) = y_h + y_p = (c_1 y_1 + \dots + c_n y_n) + \left(y_1 \int \frac{W_1(x)}{W(x)}\,dx + \dots + y_N \int \frac{W_N(x)}{W(x)}\,dx\right)$$

$\Rightarrow$ $\boxed{y = y(x) = \sum_{k=1}^{N} \left[y_k(x) \int \frac{W_k(x)}{W(x)}\,dx \right]}$, where the $y_h(x)$ terms are recovered by making sure that the integration constant terms are listed when performing integration for each term.