Differential Equations: Basic Definitions and Classifications

- What is a differential equation (DE)?
- ODE vs. PDE
- Order of a DE / Solution of a DE
- Initial-Value Problems (IVPs)
- General Solution vs. Particular Solution

Differential Equation (Definition)

A differential equation is an equation involving a function $y = f(x)$ that interests us, along with terms that may have derivatives of y, where the solution to this equation is $y = f(x)$ itself.

Simply put, differential equations are equations that have some function y (some of its derivatives) act as variables and the solution of the equation is a function (namely $y = f(x)$).

Examples of DEs

- $\dfrac{dy}{dx} = kx$
- $3x \cdot \dfrac{dy}{dx} + \dfrac{2}{x} = x^3$
- $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} = 5 - 6\sin(x)$

Examples of non-DEs

- $x^2 - x + 2 = 0$
- $2x + 4 = 8$
- $e^x = \ln(x-3)$
- $\dfrac{x+3}{x-4} = 6$

Ordinary vs. Partial DEs

- Ordinary DE (i.e. ODE): A DE that involves some function "y" where y is a function of only 1 variable.

- Partial DE (i.e. PDE): A DE that involves some function "y" where y is a function of more than 1 variable.

For example, $3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$ is a PDE because the function "u" is assumed to be a function of both x and y. (i.e. $u = f(x,y)$).

Order of a DE

The order of a DE is just the order of the highest derivative explicitly appearing the equation.

Ex. of 1st order equations: $\frac{dy}{dx} = 3x^5$, $\frac{dy}{dx} - x^2 = e^x y$, & $\frac{dy}{dx} = k(T - T_0)$

Ex. of 2nd order equations: $\frac{d^2y}{dx^2} + 5y = 2\frac{dy}{dx} + 8\tan(3x)$ & $3x^3\frac{d^2y}{dx^2} = (8x-1)y$

Ex. of 3rd order equations: $\frac{d^3y}{dx^3} = 5e^{7x}$ and $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} = 2x^2 + y$

NOTE: Consider the ODE $\frac{d^6y}{dx^6} = \left(\frac{d^2y}{dx^2}\right)^{11}$. The order of this ODE is "6" (i.e. 6th-order), not "11". The term $\left(\frac{d^2y}{dx^2}\right)^{11}$ is just the 2nd-derivative of y raised to the 11^{th} power!

Solution of a DE

If you can take some function "y" (if you know what it is explicitly) and substitute it into a DE AND both sides of the equation come out to be the same thing for the interval of values (e.g. x-values) then we say that the (explicit) function "y" is a solution to our DE. Another way we can state this is to say that "y" satisfies the equation.

Ex: Consider the equation: $\frac{d^2y}{dx^2} = \frac{dy}{dx} + 2y$. (Note that we can also express this in prime notation as $y'' = y' + 2y$).

It follows that the function $y = 2e^{-x}$ is a solution to our aforementioned ODE because...

$$y = 2e^{-x} \Rightarrow y' = -2e^{-x} \Rightarrow y'' = 2e^{-x}$$

$$\therefore\ y'' = y' + 2y \Rightarrow 2e^{-x} = -2e^{-x} + 2\left[2e^{-x}\right]$$

$$\Rightarrow 2e^{-x} = -2e^{-x} + 4e^{-x}$$

$$\Rightarrow 2e^{-x} = 2e^{-x}\ \checkmark$$

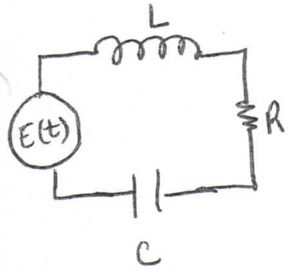
NOTE: Our function $y = 2e^{-x}$ satisfies our ODE for $x = (-\infty, \infty) = \mathbb{R}$!!

<u>ATTENTION</u>: The last note brings up a very important point that might get overlooked when solving ODEs and PDEs. <u>KNOWIN THE INTERVAL OF VALUES FOR WHICH YOUR SOLUTIONS FOR ODEs PDEs ARE VALID IS VERY IMPORTANT !!</u>

One of the reasons why learning how to solve ODEs / PDEs is important is because we use them to model system and/or phenomena on our environment. Some examples of "real-world" systems of phenomena include...

- <u>Population Growth</u>: $\frac{dP}{dt} = kP$, where $P = P(t) = P_0 e^{kt}$, $k > 0$, $P_0 \in \mathbb{R}$
- <u>Radioactive Decay</u>: $\frac{dD}{dt} = kD$; where $D = D(t) = D_0 e^{kt}$, $k < 0$, $D_0 \in \mathbb{R}$
- <u>Spread of Disease</u>: $\frac{dx}{dt} = kxy = kx(n+1-x)$, where $x = x(t) = $ # people infected by disease, $y = y(t) = $ # people not yet infected (by disease), $n = $ # of people in given population, $x + y = n+1$ (i.e. relationship between $x, y,$ and n if 1 extra person/host introduced to population), and $k > 0$.
- <u>Chemical Reactions</u>: $\frac{dX}{dt} = k(\alpha - X)(\beta - X)$, where $X(t) = $ amount of substance A remaining at time t (that has not decomposed into smaller molecules) of substance B, $\alpha = $ given (initial) amount of substance A, $\beta = $ given (initial) amount of substance B, + $k < 0$.

- **Series Circuits** : * $L\frac{di}{dt} = L\frac{d^2q}{dt^2}$ (inductor)



* $iR = R\frac{dq}{dt}$ (resistor)

* $\frac{1}{C}q$ (capacitor)

* $i(t)$ = current of circuit at time t

* $q(t)$ = charge across capacitor at time t

* $i(t) = \frac{dq}{dt}$

Kirchhoff's Voltage Law (KVL) states that voltage $E(t)$ on a closed (circuit) loop must equal the sum of the voltage drops in the loop.

$$\therefore \underbrace{L\frac{d^2q}{dt^2}}_{\text{volt drop across inductor}} + \underbrace{R\frac{dq}{dt}}_{\text{volt drop across resistor}} + \underbrace{\frac{1}{C}q}_{\text{volt drop across capacitor}} = \underbrace{E(t)}_{\text{volt source or electromotive force of circuit}}$$

- **Draining a Tank** : $\frac{dh}{dt} = -\frac{A_h}{A_w}\sqrt{2gh}$

In hydrodynamics, Torricelli's law states that the speed $v = \sqrt{2gh}$ of efflux of water through a sharp-edged hole at the bottom of a tank filled to a depth of h is the same as the speed that a body (i.e. drop of water in this case) would acquire in falling freely from height h. (Note that g = acceleration due to gravity and A_h = area of hole in bottom of tank and A_w = constant area of the upper surface of the water in the tank).

So, the reason why knowing what interval of values your DE (and the solution (function) for it) is important is that in the context of what your DE is modeling, some values (e.g. of "x") may not make sense to use! (Find these interval of values the same way you find the domain of a function).

Initial - Value Problems / General vs. Particular Solutions

Note that a DE whose solution is a function (e.g. say $y = f(x)$), this function may be a general solution (i.e. the solution is really a family of functions vs. being a specific function. For example, $y(x) = 2e^{-x} + 3x + C$ could be a general solution to a DE whereas $y(x) = 2e^{-x} + 3x + \frac{3}{4}$ is a specific (function) solution within the family of functions $y(x) = 2e^{-x} + 3x + C$.

NOTE: A specific function (that is a solution to a DE) is better known as a particular solution to the DE. In Initial-Value Problems (IVPs), the particular solution is usually found by specifying certain characteristics about the solution $y = f(x)$. We may be given information about a point on the graph of the function or information about the (1^{st}, 2^{nd}, and/or N^{th} derivative) of the function.

In a 1st year Calculus course, you normally learn about how to solve IVP problems when learning (initially) how to find the antiderivative of a function. Here is an example of what a problem like this might look like.

Ex: A particle is moving with the given data (below). Find the position function, $s(t)$, for the data.

$\boxed{\text{NOTE: } a(t) = v'(t) = s''(t)}$

$$\cancel{a(t)}\,\frac{d^2s}{dt^2} = 10\sin(t) + 3\cos(t) \; ; \; \underbrace{s(0) = 0 \; ; \; s'(2\pi) = 12}_{\text{initial conditions}} \; ;$$

s in meters
t in seconds

$$\therefore \cancel{v(t)}\,\frac{ds}{dt} = \int a(t)\,dt = \int \left(10\sin(t) + 3\cos(t)\right) dt = -10\cos(t) + 3\sin(t) +$$

$$\therefore s(t) = \int v(t)\,dt = \int \left(-10\cos(t) + 3\sin(t) + C\right) dt$$

$$\therefore s(t) = -10\sin(t) - 3\cos(t) + Ct + D$$

general solution to the ODE $\frac{d^2s}{dt^2} = 10\sin(t) + 3\cos$

Using initial conditions to find $C + D$

$$\therefore \underbrace{s(0) = 0}_{\text{s(t) has point (0,0) on its graph}} \Rightarrow -10\sin(0) - 3\cos(0) + C(0) + D = 0 \Rightarrow -3 + D = 0 \Rightarrow \boxed{D = 3}$$

$$\therefore \underbrace{s'(2\pi) = 12}_{\text{velocity of particle @ time } t = 2\pi \text{ (secs) is 12 m/s}} \Rightarrow -10\cos(2\pi) + 3\sin(2\pi) + C = 12 \Rightarrow -10 + C = 12 \Rightarrow \boxed{C = 22}$$

$$\therefore \boxed{s(t) = -10\sin(t) - 3\cos(t) + 22t + 3}$$

particular solution of the ODE $\frac{d^2s}{dt^2} = 10\sin(t) + 3\cos(t)$ and where $s(0) = 0$ and $s'(2\pi) = 12$

In general, for an N^{th}-order ODE/PDE with initial conditions will have exactly N values being assigned (i.e. assigned values $y(x_0)$, $y'(x_0)$, $y''(x_0), \ldots, y^{N-1}(x_0)$) where the highest derivative in this set is of order $N-1$.

Also, note that our solution to our (IVP) ODE/PDE satisfies the ODE/PDE as well as the initial conditions of the ODE/PDE. If a given function $y=f(x)$ satisfies either the ODE/PDE or initial conditions (BUT NOT BOTH), then $y=f(x)$ IS NOT A SOLUTION TO THE IVP ODE/PDE !!

Now we will do a few examples to put into practice what we have just (conceptually) learned

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Ex. 1 : Find $f(t)$ for the ODE $\frac{d^2f}{dt^2} = 2e^t + 3\sin(t)$ where $f(0) = 0$ and $f(\pi) = 0$.

Let $\frac{d^2f}{dt^2} = f''(t) = 2e^t + 3\sin(t)$.

$\therefore \int f''(t)\,dt = \int \left(2e^t + 3\sin(t)\right) dt$

$\Rightarrow f'(t) = 2e^t - 3\cos(t) + C$

$\Rightarrow \int f'(t)\,dt = \int \left(2e^t - 3\cos(t) + C\right) dt$

$\Rightarrow \boxed{f(t) = 2e^t - 3\sin(t) + Ct + D}$ → general solution

Apply initial cond'n $f(0) = 0$

$f(0) = 0 \Rightarrow 2e^0 - 3\cancel{\sin(0)} + \cancel{C(0)} + D = 0$

$\Rightarrow 2 + D = 0 \Rightarrow \boxed{D = -2}$

Apply initial cond'n $f(\pi) = 0$

$f(\pi) = 0 \Rightarrow 2e^\pi - 3\cancel{\sin(\pi)} + C(\pi) + \cancel{D} = 0$

$\Rightarrow 2e^\pi + C\pi - 2 = 0$

$\Rightarrow C\pi = 2 - 2e^\pi \Rightarrow \boxed{C = \frac{2 - 2e^\pi}{\pi}}$

Particular Sol'n : $\boxed{f(t) = 2e^t - 3\sin(t) + \frac{2 - 2e^\pi}{\pi} t - 2}$

Ex. 2 : Determine which, if any, of the given functions $y = y(x)$ are a solution to the given ODE. Also, identify the interval of interest (i.e. the interval of values that are valid for our solution).

Given ODE : $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0 \Rightarrow y'' - 6y' + 9y = 0$

Is $y(x) = e^{3x}$ a solution? : $y'(x) = 3e^{3x}$; $y''(x) = 9e^{3x}$

$\therefore y'' - 6y' + 9y = 9e^{3x} - 6[3e^{3x}] + 9[e^{3x}] = 9e^{3x} - 18e^{3x} + 9e^{3x} \Rightarrow$

$18e^{3x} - 18e^{3x} = 0$ ✓

yes!

Is $y(x) = xe^{3x}$ a solution? : $y'(x) = 1\cdot e^{3x} + x[3e^{3x}] = e^{3x} + 3xe^{3x}$;

$y''(x) = 3e^{3x} + 3[e^{3x} + 3xe^{3x}] = 3e^{3x} + 3e^{3x} + 9xe^{3x} = 6e^{3x} + 9xe^{3x}$

$\therefore y'' - 6y' + 9y = 6e^{3x} + 9xe^{3x} - 6[e^{3x} + 3xe^{3x}] + 9[xe^{3x}] =$

$\cancel{6e^{3x}} + 9xe^{3x} - \cancel{6e^{3x}} - 18xe^{3x} + 9xe^{3x} = 18xe^{3x} - 18xe^{3x} = 0$

yes!

Ex. 2 (cont'd)

Recall that $\frac{d}{dx}\left[xe^{3x}\right] = e^{3x} + 3xe^{3x}$ and

$\frac{d^2}{dx^2}\left[xe^{3x}\right] = 6e^{3x} + 9xe^{3x}$

Is $y(x) = 7e^{3x} - 4xe^{3x}$ a solution?

$$\therefore y'(x) = 7\left[3\cdot e^{3x}\right] - 4\left[e^{3x} + 3xe^{3x}\right] = 21e^{3x} - 4e^{3x} - 12xe^{3x}$$
$$= 17e^{3x} - 12xe^{3x}$$

$$\therefore y''(x) = 17\left[3e^{3x}\right] - 12\left[e^{3x} + 3xe^{3x}\right] = 51e^{3x} - 12e^{3x} - 36xe^{3x}$$

So, $y'' - 6y' + 9y = \left[51e^{3x} - 12e^{3x} - 36xe^{3x}\right] - 6\left[17e^{3x} - 12xe^{3x}\right] + 9\left[7e^{3x} - 4xe^{3x}\right]$

$$= 51e^{3x} - 12e^{3x} - 36xe^{3x} - 102e^{3x} + 72xe^{3x} + 63e^{3x} - 36xe^{3x}$$

$$= e^{3x}\left[51 - 12 - 102 + 63\right] + xe^{3x}\left[\cancelto{0}{-36 + 72 - 36}\right]$$

$$= e^{3x}\left[\cancelto{0}{-114 + 114}\right] + 0$$

$$= 0$$

$\therefore y'' - 6y' + 9y = 0 \Rightarrow y(x) = 7e^{3x} - 4xe^{3x}$ is a sol'n to this ODE!

Ex. 3 : Follow the same directions as Ex. 2. (1)

Given ODE ! $\frac{d^2y}{dx^2} + 4y = 12x \Rightarrow y'' + 4y = 12x$

Is $y(x) = \sin(2x)$ a solution? : $y'(x) = 2\cos(2x)$; $y''(x) = -4\sin(2x)$

$\therefore\ y'' + 4y = -4\sin(2x) + 4[\sin(2x)] = 0 \neq 12x$

NO!

Is $y(x) = \sin(2x) + 3x$ a solution?

$y'(x) = 2\cos(2x) + 3$; $y''(x) = -4\sin(2x)$

$\therefore\ y'' + 4y = -4\sin(2x) + 4[\sin(2x) + 3x]$

$\quad " \quad = \cancel{-4\sin(2x)} + \cancel{4\sin(2x)} + 12x$

$\quad " \quad = 12x$

$\therefore\ y'' + 4y = 12x \Rightarrow$ YES!

Ex. 3 (cont'd)

Is $y(x) = 3x$ a solution? : $y'(x) = 3$; $y''(x) = 0$

$\therefore \quad y'' + 4y \stackrel{?}{=} 12x$

$\Rightarrow \quad 0 + 4[3x] \stackrel{?}{=} 12x$

$\Rightarrow \quad 12x = 12x \checkmark$ yes!

Ex. 4: Consider the 1st-order ODE $\frac{dy}{dx} \overset{y'}{=} 6x + 2x^2$. Do the followi[ng]

a) Find a general solution y(x) to this equation. (Show all your work)

$\frac{dy}{dx} = 6x + 2x^2 \Rightarrow \frac{dy}{dx} \cdot dx = (6x + 2x^2) \cdot dx \Rightarrow dy = (6x + 2x^2)\, dx$

$\therefore \int dy = \int (6x + 2x^2)\, dx \Rightarrow y = \frac{6x^2}{2} + \frac{2x^3}{3} + C \Rightarrow y(x) = 3x^2 + \frac{2}{3}x^3 + C$

$\boxed{\text{General sol'n: } y(x) = 3x^2 + \tfrac{2}{3}x^3 + C}$

b) Find the value of C if $y(0) = 1$, where C = arbitrary constant of integration.

$y(0) = 1 \Rightarrow \cancel{3(0)^2} + \cancel{\tfrac{2}{3}(0)^3} + C = 1 \Rightarrow \boxed{C = 1}$

c) Find the value of C if $y(1) = 0$, where C = arbitrary constant of integration

$y(1) = 0 \Rightarrow 3(1)^2 + \frac{2}{3}(1)^3 + C = 0 \Rightarrow 3 + \frac{2}{3} + C = 0 \Rightarrow C = -(3 + \tfrac{2}{3})$

$\Rightarrow \boxed{C = -\tfrac{11}{3}}$

Ex. 5: Verify that $y(x) = Ax + Bx\ln|x|$ is a solution to the ODE:

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0 \Rightarrow x^2y'' - xy' + y = 0$$

on the interval $x = (-\infty, 0) \cup (0, \infty)$ regardless of what A and B are assign to be. Afterwards, find a particular solution to this ODE (i.e. find specific values of A and B) if $y(1) = 3$ and $y'(1) = 8$.

$$y'(x) = A + B\left[1\cdot\ln|x| + x\cdot\frac{1}{x}\right] = A + B\left[\ln(x) + 1\right] \; ; \; x = (-\infty, 0) \cup (0, \infty)$$

$$y''(x) = 0 + B\left[\frac{1}{x} + 0\right] = \frac{B}{x} \; ; \; x = (-\infty, 0) \cup (0, \infty)$$

$$\therefore x^2y'' - xy' + y \stackrel{?}{=} 0 \Rightarrow x^2\left[\frac{B}{x}\right] - x\left[A + B\ln(x) + B\right] + \left[Ax + Bx\ln(x)\right] \stackrel{?}{=} 0$$

$$\therefore \cancel{Bx} - \cancel{Ax} - \cancel{Bx\ln(x)} - \cancel{Bx} + \cancel{Ax} + \cancel{Bx\cdot\ln(x)} \stackrel{?}{=} 0$$

$\Rightarrow 0 = 0$ ✓ So, $y(x)$ is a sol'n for any $A, B \in \mathbb{C}$.

Apply initial cond'n $y(1) = 3$

$$y(1) = 3 \Rightarrow A(1) + B(1)\cdot\cancel{\ln|1|}^{0} = 3 \Rightarrow \boxed{A = 3}$$

Apply initial cond'n $y'(1) = 8$

$$y'(1) = 8 \Rightarrow A + B\left[\cancel{\ln(1)}^{0} + 1\right] = 8 \Rightarrow \cancel{A}^{3} + B = 8 \Rightarrow B = 8 - 3 = 5 \Rightarrow \boxed{B = 5}$$

$$\therefore \boxed{\text{Particular sol'n}: \; y(x) = 3x + 5x\cdot\ln|x| \; ; \; x = (-\infty, 0) \cup (0, \infty)}$$