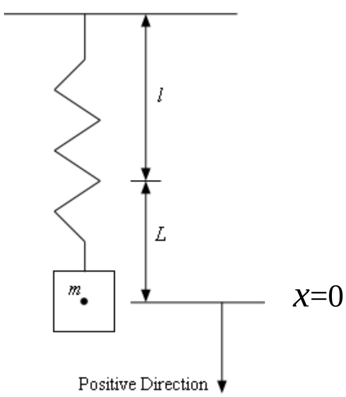
Department of Electrical and Computer Engineering
The University of Alabama in Huntsville

CPE 381: Fundamentals of Signals and Systems for Computer Engineers

Homework #3 Solution

1. (15 points) Write differential equation describing displacement x of suspended weight m on spring with elastic constant k.



At any time, sum of all forces is equal to zero

$$m\ddot{x} + c\dot{x} + kx = 0$$

With initial conditions

$$x(0)[m] \; and \; \dot{x}(0)$$

By using Laplace transform

$$\mathcal{L}(m\ddot{x} + c\dot{x} + kx) = ms^2X(s) - msx(0) - m\dot{x}(0) + csX(s) - cx(0) + kX(s) = 0$$

$$(ms^2 + cs + k)X(s) = msx(0) + cx(0)$$

$$X(s) = \frac{msx(0) + cx(0)}{ms^2 + cs + k} = \frac{sx(0) + \frac{c}{m}x(0)}{s^2 + \frac{c}{m}s + \frac{k}{m}}$$

Example #1:

$$m = 1\ [kg], k = 2\ \left[\frac{kg}{s^2}\right], c = 2\ \left[\frac{kg}{s}\right], x(0) = 1\ [m], and\ \ \dot{x}(0) = 0\ \left[{}^{m}/{}_{s^2}\right]$$

$$X(s) = \frac{s+2}{s^2+2s+17} = \frac{s+2}{(s+1)^2+16}$$

and

$$x(t) = \mathcal{L}^{-1}\big(X(s)\big) = e^{-t}\left(\cos(4t) + \frac{1}{4}\sin(4t)\right)$$

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% CPE381 Example: differential equation
t=0:0.01:4;
x=exp(-t).*(cos(4*t)+0.25*sin(4*t));
plot(t,x),title('Displacement in time'),xlabel('time [s]'),ylabel('y[m]'),grid
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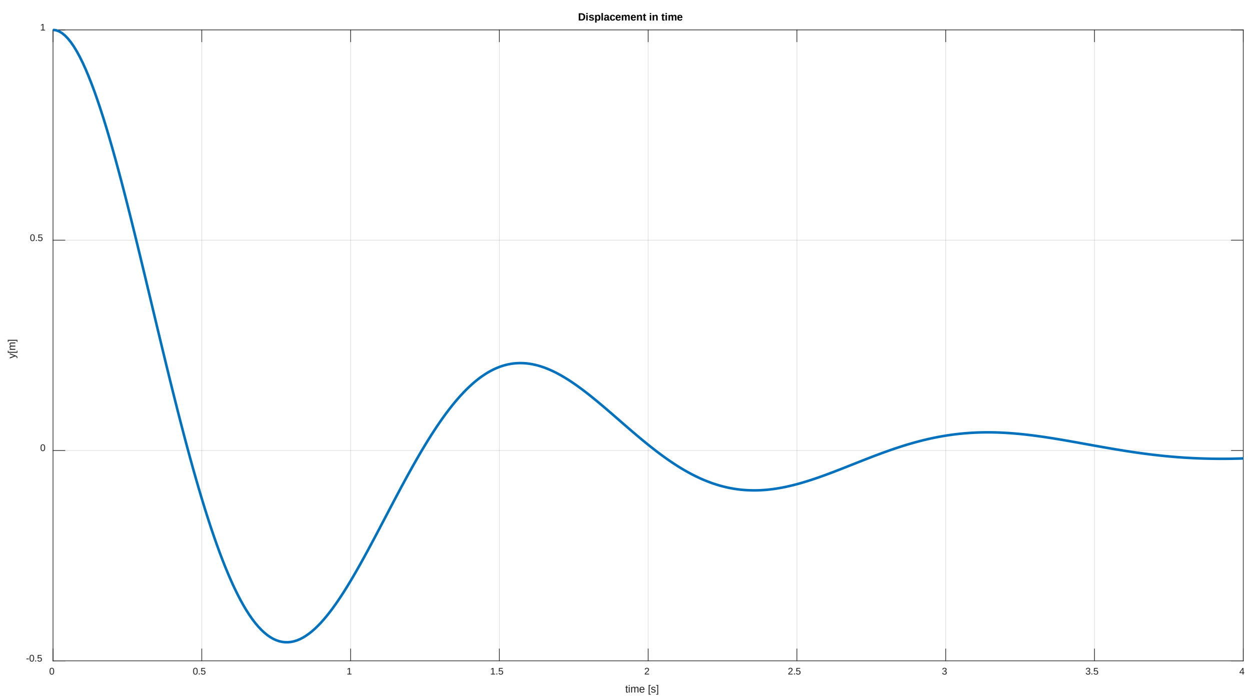


Example #2: A 1 kg weight is hung on the end of a vertically suspended spring, thereby stretching the spring *L*=10 cm. The weight is raised 5 cm above its equilibrium position and released from rest at time *t*=0. Find the displacement x of the weight from its equilibrium position at time *t.* Use g=10m/s^2.

$$F = kL, \qquad k = \frac{F}{L} = \frac{mg}{L} = \frac{1[kg]\ 10\left[\frac{m}{s^2}\right]}{0.1[m]} = 100\left[\frac{kg}{s^2}\right]$$

At any time, sum of all forces is equal to zero

$$m\ddot{x} + kx = 0$$

With initial conditions

$$x(0) = -0.05[m] \ \ \dot{x}(0) = 0$$

By using Laplace transform

$$\mathcal{L}(m\ddot{x} + kx) = s^2X(s) - sx(0) - \dot{x}(0) + kX(s) = 0$$

$$(s^2 + 100)X(s) = -0.05s$$

$$X(s) = \frac{-0.05s}{s^2 + 100}$$

and

$$x(t) = \mathcal{L}^{-1}\big(X(s)\big) = -0.05\cos(10t)$$

2. A system with input x(t) and output y(t) is defined by the following differential equation:

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=x(t)$$

Find the impulse response *h(t)* and the unit-step response *s(t)*.

If $Y(s) = \mathcal{L}[y(t)]$ and $X(s) = \mathcal{L}[x(t)]$, then

$$Y(s)\,[s^2 + 3s + 2] = X(s)$$

To find impulse response, we let x(t) = (t), and X(s) =1, then Y(s) is equal to H(s):

$$Y(s)=H(s)=\frac{1}{s^2+3s+2}=\frac{1}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}$$

We find

$$A=H(s)(s+1)\big|_{s=-1}=\frac{1}{-1+2}=1$$

and

$$B=H(s)(s+2)\big|_{s=-2}=\frac{1}{-2+1}=-1$$

therefore:

$$h(t)=\left[e^{-t}-e^{-2t}\right]\cdot u(t)$$

Similarly, unit step response is:

$$S(s)=\frac{H(s)}{s}=\frac{1}{s\cdot(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

and A=0.5, B= -1, C=0.5, therefore:

$$s(t)=0.5\cdot u(t)-e^{-t}\cdot u(t)+0.5e^{-2t}\cdot u(t)$$

3. Consider a second order differential equation,

$$\frac{d^2y(t)}{dt}+3\frac{dy(t)}{dt}+2y(t)=x(t)$$

with initial conditions $y(0) = 1$ and $\frac{dy(t)}{dt}|_{t=0} = 0$ and x(t) = u(t).

- Find the complete response y(t)
- Find the steady state response and the transient response.

The Laplace transform of the differential equation gives

$$[s^2Y(s) - sy(0) - \frac{dy(t)}{dt}|_{t=0}] + 3[sY(s) - y(0)] + 2Y(s) = X(s)$$
$$Y(s)(s^2 + 3s + 2) - (s + 3) = X(s)$$

so we have that

$$\begin{aligned} Y(s) &= \frac{X(s)}{(s+1)(s+2)} + \frac{s+3}{(s+1)(s+2)} \\ &= \frac{1+3s+s^2}{s(s+1)(s+2)} = \frac{B_1}{s} + \frac{B_2}{s+1} + \frac{B_3}{s+2} \end{aligned}$$

We find $B_1 = 0.5$, $B_2 = 1$, and $B_3 = -0.5$.

therefore:

$$y(t) = \lfloor 0.5 + e^{-t} - 0.5e^{-2t} \rfloor \, u(t)$$

steady state response is

$$y(t) = 0.5\, u(t)$$

and transient response is

$$y(t) = \lfloor e^{-t} - 0.5e^{-2t} \rfloor \, u(t)$$

4. The Laplace transform of the response is:

$$S(s) = H(s)X(s) = \frac{s}{s(s^2+s+1)} = \frac{1}{(s+1/2)^2+3/4}$$

since (take a look at page 199)

$$\mathcal{L}\left[\mathrm{A}e^{-\alpha t}\sin(\Omega_0 t \cdot u(t))\right] = \frac{\mathrm{A}\Omega_0}{(\mathrm{s}+\alpha)^2+\Omega_0^2}$$

Therefore, the Inverse Laplace transform of the response is:

$$s(t) = \frac{2}{\sqrt{3}}e^{-0.5t}\sin\left(\sqrt{3}t/2\right)u(t)$$

a) $y_1(t) = s(t) - s(t-1)$

b) $y_2(t) = h(t) - h(t-1) = \mathrm{d}(s(t) - s(t-1))/\mathrm{d}t$

5. General solution:

$$Y(s) = \big(X(s) - G(s)\,Y(s)\big)F(s) = \frac{F(s)}{1+F(s)G(s)}X(s)$$

$$H(s) = \frac{F(s)}{1+F(s)\cdot G(s)}$$

In this particular case system output is:

Y(s) = (X(s) – K Y(s)) H(s)
= X(s) H(s) – K H(s) Y(s)

and

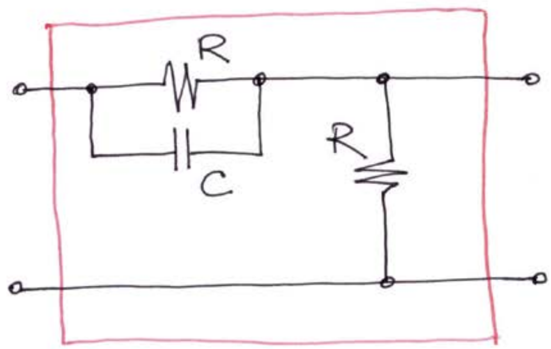
$$\frac{Y(s)}{X(s)} = \frac{H(s)}{1+K\cdot H(s)} = \frac{2}{s+2K-1}$$

In order to have the pole in the left-hand s-plane we need 2K – 1 > 0 → K > 0.5

For example, K = 1 → pole at s = -1 and impulse response

$$g(t) = 2e^{-t}u(t)$$

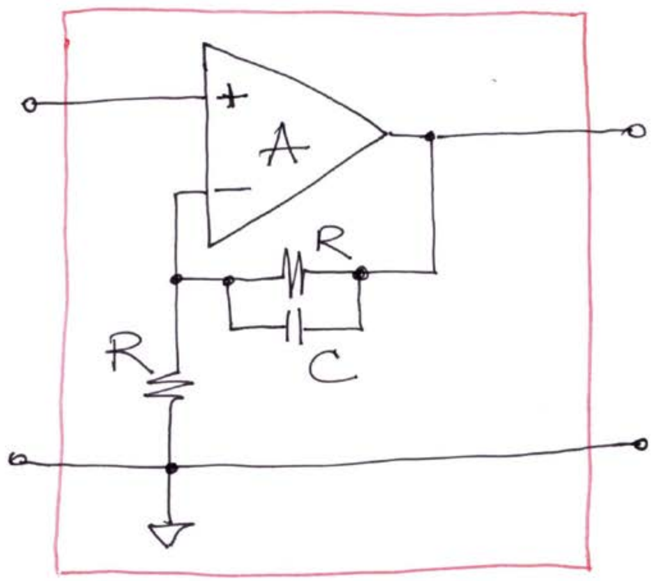
6. a) What is the transfer function of the following circuit:



$$H(s) = \frac{R}{R+R\,||\,\frac{1}{Cs}} = \frac{R}{R+\frac{R}{RCs+1}} = \frac{RCs+1}{RCs+2} = \frac{s+\frac{1}{RC}}{s+\frac{2}{RC}}$$

b) What is the transfer function of the following
Hints:

- you can use solutions of problem #5 and #6a
- to simplify the result you can assume that A → ∞



Since

$$H(s) = \frac{F(s)}{1+F(s)\cdot G(s)}$$

$F(s) = A$ and $G(s) = \frac{s+\frac{1}{RC}}{s+\frac{2}{RC}}$

$$H(s) = \frac{A}{1+A\left(\frac{s+\frac{1}{RC}}{s+\frac{2}{RC}}\right)} \quad for\ A \to \infty\ H(s) = \frac{s+\frac{2}{RC}}{s+\frac{1}{RC}}$$

c) Find and plot the unit-step response *s*(t) of the system?

$$S(s) = \frac{1}{s} \cdot \frac{s+\frac{2}{RC}}{s+\frac{1}{RC}} = \frac{A}{s} + \frac{B}{s+\frac{1}{RC}} = \frac{2}{s} - \frac{1}{s+\frac{1}{RC}} =$$

$$s(t) = (2 - e^{-\frac{t}{RC}}) \cdot u(t)$$