LaPlace Transform: Its Definition, Purpose, and Basic Properties

At this point in our course, we have studied how to solve various types of Ordinary Differential Equations using various methods and/or techniques based upon the type of equation we were dealing with. We will continue to learn methods and techniques to help us solve ODEs more efficiently. One way we can make this process of solving ODEs more efficient (and less painful) is to consider employing a technique called performing integral transformations on our ODEs. To be specific, we can transform our equation from looking one way (e.g. having terms in an equation that act like functions of time) into another way that might be more useful to us (e.g. having terms in an equation that acts like functions of frequency). The LaPlace Transform is one such integral transformation that does exactly what has been previously described. There are other integral transforms such as Fourier Transform, Z-Transform, etc., but the LaPlace Transform is probably one of the easiest integral transforms to understand. The goal of this set of notes is to introduce you to the LaPlace Transform's definition, communicate its primary purpose, provide examples on how the LaPlace Transform for several functions are derived, illuminate important properties of LaPlace Transforms, and (finally) show how we can express an ODE in its LaPlace Transform equivalent. The end-goal of using the LaPlace Transform will be to express an ODE in this equivalent form so that we can read charts and tables with these transforms listed in them (along with some possible algebraic manipulation) to find solutions to our ODEs. Therefore, by leveraging these transformations, we can avoid lengthy and laborious calculations to solve many of our ODE types we have dealt with in this course so far. LaPlace Transforms are especially useful in solving Nonhomogeneous ODEs that model "real-world" applications!! Now, let's get started!

<u>LaPlace Transform (Definition)</u> : Let $f(t)$ be a "suitable" continuous or piecewise continuous function $^{\circledast}$. The LaPlace Transform of $f(t)$, denoted by either $F(s)$ or $\mathcal{L}\{f(t)\}$, is the function given by...

$$\boxed{F(s) = \mathcal{L}\{f(t)\}\big|_s = \int_0^\infty f(t)\,e^{-st}\,dt}\,.$$

This definition looks like a fairly simple one, but there is a lot of details about this definition that we need to unpack to fully understand what we are dealing with in understanding this integral transform. Several comments will be made to unpack these details.

- We are changing the form of a function in terms of "t", where t is assumed to be a real number, into a function in terms of "s", where s can be real or complex (i.e. $s = \lambda \pm wi$)
- If $f(t)$ is a "suitable" function, one property $f(t)$ possesses is that $\lim_{t\to\infty}\left[f(t)\cdot e^{-st}\right] = 0$ (i.e. <u>$f(t)$ will not decrease as fast as e^{-st} as $t\to\infty$</u>). Re-expressing this concept more formally, if we know that $|f(t)| \le Me^{s_0 t}$ whenever $t \ge T$ (i.e. provided that M, T, + s_0 are finite, real-number constants, if the graph of $|f(t)|$ does not lie above the graph of $Me^{s_0 t}$ whenever $t \ge T$ and $s > s_0$), then

the function $f(t)$ is said to be of exponential order s_0. This just means that whatever $f(t)$ is in our LaPlace Transform, $f(t)$ won't decrease faster than some (arbitrary) exponential function $Me^{-s_0 t}$ and $Me^{-s_0 t}$ won't decrease faster than e^{-st}!

- Functions of type $f(t)$ such that $\lim_{t\to\infty}\left[f(t)\cdot e^{-st}\right]=0$ are functions that are considered "well-behaved" and don't "grow out of control". The phrase "well-behaved" means that the rate of change of $f(t)$ is slower than e^{-st}. If $f(t)$ could "grow out of control", then $f(t)$ would have a rate of change higher than e^{-st}.

- Functions that have the property of having a finite number of functions that can be used to express the derivative of that function of any order N are good candidates for being LaPlace-Transformable functions. Note that these functions are also the type of functions $g=g(t)$ for ODEs of the type $a_0 y^{(N)} + a_1 y^{(N-1)} + \dots + a_{N-1} y' + a_N y = g(t)$ where Method of Undetermined Coefficients can be used to find $y_p(t)$!

- For LaPlace Transforms, values $t<0$ are irrevelant. For example, one major application of using LaPlace Transforms is to evaluate linear time-invariant (LTI) systems (i.e. systems where the output of the system does not depend upon when the input for the system was applied). Therefore, the time at which an input was applied to a system would be considered time $t=0$ and nothing that happened before time $t=0$ (i.e. $t<0$) would matter. Mathematically speaking, if we considered times $t<0$, then note that the improper integral $\int_{-\infty}^{0} e^{-st} f(t)\, dt$ would diverge because $\lim_{t \to -\infty} \left[e^{-st} f(t) \right] = \infty$ if $f(t)$ has a slower rate of change than e^{-st} in this case. Thus, the "e^{-st}" and "$f(t)$" functions would changes roles in the LaPlace Transform. This isn't desirable!

- In general, the LaPlace Transform is an integral transform of the type... $\int_{0}^{\infty} K(s,t) \cdot f(t)\, dt$, where $K(s,t)$ is selected to be e^{-st} (i.e. $K(s,t) = e^{-st}$). If we chose $K(s,t)$ to equal something else, then we would have a different type of Transform. Thus, the definition of what "suitable functions" $f(t)$ would be would also change!

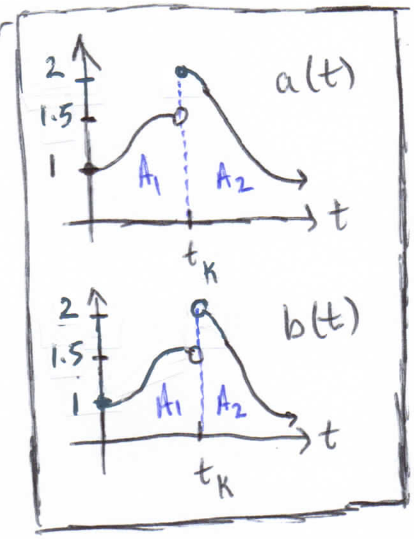
- By definition, functions that are continuous on $t = [0, \infty)$ meet the condition that $\lim_{t \to t_0} \left[f(t) \right] = f(t_0)$ for all $t_0 = (0, \infty)$ and $\lim_{t \to 0^+} \left[f(t) \right] = f(0)$ when $t = 0$. Functions that are considered to be piecewise-continuous are piecewise functions that have a finite number of jump discontinuities on $t = [0, \infty)$ AND it must be true that at each jump discontinuity location t_K that $\lim_{t \to t_K^-} \left[f(t) \right] = L \in \mathbb{R}$, $\lim_{t \to t_K^+} \left[f(t) \right] = M \in \mathbb{R}$, and $L \neq M$ (i.e. the limit to the left and right of $t = t_K$ must exist, but each limit will not equal the same real number).

- For a function $f(t)$ to be LaPlace-Transformable, one of the properties it must possess is that it is either continuous or (at least) piecewise-continuous on $t = [0, \infty)$ (or some subset of $t = [0, \infty)$ if the domain of $f(t)$ is not $t = [0, \infty)$ but is within $[0, \infty)$!

- A function $f(t)$ will be LaPlace-Transformable if and only if ① $f(t)$ is either a continuous or piecewise-continuous function, ② $f(t)$ satisfies $\lim_{t \to \infty} \left[e^{-st} \cdot f(t) \right] = 0$ (i.e. $f(t)$ is of exponential order s_0, where $s > s_0$), and ③ $f(t)$ is well-defined (unambiguously).

- The LaPlace Transform of a function $f(t)$ is not unique.

Recall that $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$, so this improper integral can represent the area under the curve of $f(t)$. If we want to find $\mathcal{L}\{a(t)\}$ and $\mathcal{L}\{b(t)\}$, where graphs of $a(t)$ and $b(t)$ are shown to your right and $a(t)$ and $b(t)$ are the same except for at $t = t_K$, then notice that both $a(t)$ and $f(t)$ will have the same area under its respective curves! Thus, $\mathcal{L}\{a(t)\} = \mathcal{L}\{b(t)\}$ although $a(t)$ and $b(t)$ are (technically) different functions! So, we could have several function $f(t)$ share the same LaPlace Transform $F(s)$! The reason why this can happen is because jump discontinuities do not contribute to the area under the curve of a function (i.e. $\int_{t_K}^{t_K} e^{-st} a(t)\,dt = \int_{t_K}^{t_K} e^{-st} b(t) = 0$). So using $f(t)$, we can say that



$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \cancelto{A_1}{\int_0^{t_K} e^{-st} f(t)\,dt} + \cancelto{0}{\int_{t_K}^{t_K} e^{-st} f(t)\,dt} + \cancelto{A_2}{\int_{t_K}^{\infty} e^{-st} f(t)\,dt} =$$

$$\cancelto{A_1}{\int_0^{t_K} e^{-st} f(t)\,dt} + \cancelto{A_2}{\int_{t_K}^{\infty} e^{-st} f(t)\,dt} = A_1 + A_2 \,!$$

Summary about LaPlace Transform and Its Implied Concepts

- LaPlace Transform defined to be $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$.
- $f(t) \rightarrow F(s)$ transform $f(t)$ from real numbers to either real or complex numbers. (However, transformation of complex # → complex # possible).
- Function $f(t)$ is LaPlace Transformable (and, thus, a "suitable" function for this transformation) if and only if ① $f(t)$ is a continuous or piecewise continuous function on $t = [0, \infty)$ or any subset of $t = [0, \infty)$ if the domain of $f(t)$ is within $t = [0, \infty)$, ② $\lim_{t \to \infty} \left[e^{-st} f(t)\right] = 0$ (i.e. $f(t)$ is of exponential order s_0, where $s > s_0$), and ③ $f(t)$ is well-defined (i.e. it is not confusing to know $f(t_0)$ comes out to be for a given value $t = t_0$).
- One property of suitable functions $f(t)$ is that $f(t) \cdot e^{-st}$ ends up being well-behaved and does not grow out of control as $t \to \infty$.
- If $f(t)$ is a piecewise function, then $f(t)$ must be piecewise-continuous in order to have a LaPlace Transform.
- The LaPlace Transform of a function is not unique. Thus, several functions can possess/share the same LaPlace Transform $F(s)$!

- For LaPlace Transforms, values $t<0$ are irrevelant.
- For taking the LaPlace Transform of a piecewise-continuous function with a finite number of jump discontinuities, the jump discontinuities do not contribute to the LaPlace Transform final result.

Now that we have a basic understanding of what the LaPlace Transform is as well as what implied concepts are rolled into this definition, we will now turn to establishing the LaPlace Transform for various functions that we normally need to express in this form when solving ODEs. After establishing these LaPlace Transforms (through examples), we will list a collection of LaPlace Transforms and their corresponding "suitable" functions $f(t)$ as a reference before moving on to establish other higher-level LaPlace Transform properties.

Ex. 1 Find $\mathcal{L}\{f(t)\} = F(s)$, if $f(t) = 1$, where $s > 0$ is a constant.

$$\therefore\ F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}\,dt = \lim_{b\to\infty}\left[\left(\frac{e^{-st}}{-s}\right)\Big|_0^b\right] =$$

$$\lim_{b\to\infty}\left[\frac{e^{-sb}}{-s} - \frac{e^{-s(0)}}{-s}\right] = \lim_{b\to\infty}\left[\frac{1}{s}\left(\cancelto{1}{e^{-s(0)}} - e^{-sb}\right)\right] = \frac{1}{s}\left(1 - \cancelto{0}{e^{-s(\infty)}}\right)$$

$$= \frac{1}{s}$$

$$\therefore\ \boxed{\mathcal{L}\{1\} = \frac{1}{s}}$$

Ex. 2: Find $\mathcal{L}\{f(t)\} = F(s)$ for $f(t) = t, t^2, t^3$, and t^4. Use these results to come up with a pattern to guess what $\mathcal{L}\{t^n\}$ would be if $n =$ positive integer.

a) $\mathcal{L}\{t\} = ?$

$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty t e^{-st}\,dt$. Using Integration By Parts, we let $\int_0^\infty u\,dv = [uv]\Big|_0^\infty - \int v\,du$, where $u = t$ and $dv = e^{-st}\,dt$.

$$\therefore\ \begin{aligned} u &= t & dv &= e^{-st}\,dt \\ du &= dt & v &= -\frac{1}{s}e^{-st} \end{aligned} \quad \Rightarrow \int_0^\infty u\,dv = \left[t\cdot -\frac{1}{s}e^{-st}\right]\Big|_0^\infty - \int_0^\infty -\frac{1}{s}e^{-st}\,dt =$$

$$\lim_{b\to\infty}\left[\left(\frac{-t}{e^{st}}\right)\Big|_0^b\right] + \frac{1}{s}\cancelto{\frac{1}{s} = \mathcal{L}\{1\}}{\int_0^\infty e^{-st}\,dt} = \lim_{b\to\infty}\left[\left(\frac{-b}{e^{sb}} - \cancelto{0}{\frac{-(0)}{e^{s(0)}}}\right)\right] + \frac{1}{s^2}$$

Ex. 2 : (cont'd)

$$\therefore \int_0^\infty u\,dv = \int_0^\infty e^{-st}\cdot t\cdot dt = \lim_{b\to\infty}\left[\frac{-b}{e^{sb}}\right] + \frac{1}{s^2} \overset{H}{=} \lim_{b\to\infty}\left[\frac{-1}{se^{sb}}\right] + \frac{1}{s^2} =$$

$$-\frac{1}{s\underbrace{e^{s(\infty)}}_{\infty}} + \frac{1}{s^2} = -\cancelto{0}{\frac{1}{\infty}} + \frac{1}{s^2} = \frac{1}{s^2} \Rightarrow \mathcal{L}\{t\} = \frac{1}{s^2}$$

b) $\mathcal{L}\{t^2\} = ?$; $F(s) = \mathcal{L}\{t^2\} = \int_0^\infty e^{-st}\cdot f(t)\,dt = \int_0^\infty t^2\cdot e^{-st}\,dt.$

Using Tabular Integration (for efficiency), we can evaluate this improper integral.

u	sign	dv
t^2	$+$	$e^{-st}\,dt$
$2t$	$-$	$-\frac{1}{s}e^{-st}$
2	$+$	$\frac{1}{s^2}e^{-st}$
0		$-\frac{1}{s^3}e^{-st}$

$\Rightarrow$ $\int_0^\infty t^2\cdot e^{-st}\,dt = [uv]_0^\infty - \int_0^\infty v\,du$, where

$u = t^2$ and $dv = e^{-st}\cdot dt$

$$\therefore \int_0^\infty t^2\cdot e^{-st}\,dt = \lim_{b\to\infty}\left[\left(\frac{-t^2}{se^{st}} - \frac{2t}{s^2e^{st}} - \frac{2}{s^3e^{st}}\right)\Bigg|_0^b\right]$$

$$= \lim_{b\to\infty}\left[\left(\frac{-b^2}{se^{sb}} - \frac{2b}{s^2e^{sb}} - \frac{2}{s^3e^{sb}}\right) - \left(\cancelto{0}{\frac{-0^2}{se^{s(0)}}} - \cancelto{0}{\frac{2(0)}{s^2e^{s(0)}}} - \frac{2}{s^3\underbrace{e^{s(0)}}_{1}}\right)\right]$$

$$= \lim_{b\to\infty}\left[\frac{-b^2}{se^{sb}} - \frac{2b}{s^2e^{sb}}\right] - \frac{2}{s^3}\cdot \cancelto{0}{\lim_{b\to\infty}\left[e^{-sb}\right]} + \cancelto{2/s^3}{\lim_{b\to\infty}\left[\frac{2}{s^3}\right]}$$

$$\overset{H}{=} \cancelto{0}{\lim_{b\to\infty}\left[\frac{-2}{s^3e^{sb}}\right]} - \cancelto{0}{\lim_{b\to\infty}\left[\frac{2}{s^3e^{sb}}\right]} - 0 + 2/s^3 = \frac{2}{s^3}$$

$\therefore$ $\boxed{\mathcal{L}\{t^2\} = \frac{2}{s^3}}$

Ex. 2 : (cont'd – 2)

c) $\mathcal{L}\{t^3\} = ?$; $F(s) = \mathcal{L}\{t^3\} = \int_0^{\infty} e^{-st} \cdot t^3 \, dt$

Using Tabular Integration (again, for efficiency), we can evaluate the improper integral.

u		dv
t^3	$+$	$e^{-st}\,dt$
$3t^2$	$-$	$-\frac{1}{s}e^{-st}$
$6t$	$+$	$\frac{1}{s^2}e^{-st}$
6	$-$	$-\frac{1}{s^3}e^{-st}$
0		$\frac{1}{s^4}e^{-st}$

$$\Rightarrow F(s) = \int_0^{\infty} e^{-st} \cdot t^3 \, dt = \lim_{b\to\infty}\left[\left(\frac{-t^3}{se^{st}} - \frac{3t^2}{s^2e^{st}} - \frac{6t}{s^3e^{st}} - \frac{6}{s^4e^{st}}\right)\Bigg|_0^b\right] =$$

$$\lim_{b\to\infty}\left[\left(\frac{-b^3}{se^{sb}} - \frac{3b^2}{s^2e^{sb}} - \frac{6b}{s^3e^{sb}} - \frac{6}{s^4e^{sb}}\right) - \left(\cancelto{0}{\frac{-(0)^3}{e^{s(0)}}} - \cancelto{0}{\frac{3(0)^2}{s^2e^{s(0)}}} - \cancelto{0}{\frac{6(0)}{s^4e^{s(0)}}} - \frac{6}{s^4 \underbrace{e^{s(0)}}_{1}}\right)\right]$$

$$= \lim_{b\to\infty}\left[\frac{-b^3}{se^{sb}} - \frac{3b^2}{s^2e^{sb}} - \frac{6b}{s^3e^{sb}}\right] - \frac{6}{s^4} \cdot \cancelto{0}{\lim_{b\to\infty}\left[\frac{1}{e^{sb}}\right]} + \frac{6}{s^4}$$

$$\overset{H}{=} \lim_{b\to\infty}\left[\frac{-6}{s^4e^{sb}}\right] - \lim_{b\to\infty}\left[\frac{6}{s^4e^{sb}}\right] - \lim_{b\to\infty}\left[\frac{6}{s^4e^{sb}}\right] - 0 + \frac{6}{s^4}$$

$$= -\frac{6}{s^4}\cancelto{0}{\lim_{b\to\infty}\left[\frac{1}{e^{sb}}\right]} - \frac{6}{s^4}\cancelto{0}{\lim_{b\to\infty}\left[\frac{1}{e^{sb}}\right]} - \frac{6}{s^4}\cancelto{0}{\lim_{b\to\infty}\left[\frac{1}{e^{sb}}\right]} + \frac{6}{s^4} = \frac{6}{s^4}$$

$$\therefore \boxed{\mathcal{L}\{t^3\} = \frac{6}{s^4}}$$

Ex. 2 : (cont'd_3)

d) $\mathcal{L}\{t^4\} = ?$; $F(s) = \mathcal{L}\{t^4\} = \int_0^\infty e^{-st} \cdot t^4 \, dt$

Using Tabular Integration by parts, we can evaluate our improper integral.

u		dv
t^4	$+$	$e^{-st}\,dt$
$4t^3$	$-$	$-\frac{1}{s}e^{-st}$
$12t^2$	$+$	$\frac{1}{s^2}e^{-st}$
$24t$	$-$	$-\frac{1}{s^3}e^{-st}$
24	$+$	$\frac{1}{s^4}e^{-st}$
0		$-\frac{1}{s^5}e^{-st}$

$\therefore \int_0^\infty e^{-st} \cdot t^4 \, dt =$

$$\lim_{b\to\infty}\left[\left(-\frac{t^4}{se^{st}} - \frac{4t^3}{s^2e^{st}} - \frac{12t^2}{s^3e^{st}} - \frac{24t}{s^4e^{st}} - \frac{24}{s^5e^{st}}\right)\Bigg|_0^b\right]$$

$$= \lim_{b\to\infty}\left\{\left(\frac{-b^4}{se^{sb}} - \frac{4b^3}{s^2e^{sb}} - \frac{12b^2}{s^3e^{sb}} - \frac{24b}{s^4e^{sb}} - \frac{24}{s^5e^{sb}}\right) - \left(\cancelto{0}{\frac{-(0)^4}{se^{s(0)}}} - \cancelto{0}{\frac{4(0)^3}{s^2e^{s(0)}}} - \cancelto{0}{\frac{12(0)^2}{s^3e^{s(0)}}} - \cancelto{0}{\frac{24(0)}{s^4e^{s(0)}}} - \frac{24}{s^5\cancelto{1}{e^{s(0)}}}\right)\right\}$$

$$= \lim_{b\to\infty}\left[\frac{-b^4}{se^{sb}} - \frac{4b^3}{s^2e^{sb}} - \frac{12b^2}{s^3e^{sb}} - \frac{24b}{s^4e^{sb}}\right] - \lim_{b\to\infty}\left[\frac{24}{s^5e^{sb}} - \frac{24}{s^5}\right]$$

$$= \lim_{b\to\infty}\left[\frac{-b^4}{se^{sb}} - \frac{4b^3}{s^2e^{sb}} - \frac{12b^2}{s^3e^{sb}} - \frac{24b}{s^4e^{sb}}\right] - \frac{24}{s^5}\cancelto{0}{\lim_{b\to\infty}\left[\frac{1}{e^{sb}}\right]} + \cancelto{\frac{24}{s^5}}{\lim_{b\to\infty}\left[\frac{24}{s^5}\right]}$$

$$\overset{H}{=} \lim_{b\to\infty}\left[\frac{-24}{s^5e^{sb}} - \frac{24}{s^5e^{sb}} - \frac{24}{s^5e^{sb}} - \frac{24}{s^5e^{sb}}\right] - 0 + \frac{24}{s^5}$$

$$= -\frac{24}{s^5}\left[4\cancelto{0}{\lim_{b\to\infty}\left(\frac{1}{e^{sb}}\right)}\right] + \frac{24}{s^5} = \cancelto{0}{-\frac{24}{s^5}(0)} + \frac{24}{s^5} = \frac{24}{s^5}$$

$\therefore \mathcal{L}\{t^4\} = \frac{24}{s^5}$

e) $\mathcal{L}\{t^n\} = ?$

Note from examples (a) – (d) that . . .

- $\mathcal{L}\{1\} = \frac{1}{s} = \frac{0!}{s^{0+1}} = \frac{n!}{s^{n+1}}$, if $n = 0$
- $\mathcal{L}\{t^1\} = \frac{1}{s^2} = \frac{1}{s^{1+1}} = \frac{1!}{s^{1+1}} = \frac{n}{s^{n+1}}$, if $n = 1$
- $\mathcal{L}\{t^2\} = \frac{2}{s^3} = \frac{2 \cdot 1}{s^{2+1}} = \frac{2!}{s^{2+1}} = \frac{n!}{s^{n+1}}$, if $n = 2$
- $\mathcal{L}\{t^3\} = \frac{6}{s^4} = \frac{3 \cdot 2 \cdot 1}{s^{3+1}} = \frac{3!}{s^{3+1}} = \frac{n!}{s^{n+1}}$, if $n = 3$
- $\mathcal{L}\{t^4\} = \frac{24}{s^5} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{s^{4+1}} = \frac{4!}{s^{4+1}} = \frac{n!}{s^{n+1}}$, if $n = 4$

$$\therefore \boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \;;\; n = \text{non-negative integer} \quad (*)}$$

(*) NOTE: In tables of LaPlace Transforms, it is usually stated that $\mathcal{L}\{1\} = \frac{1}{s}$ and $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$, if n = positive integer.

Ex. 3: Find the Laplace Transform for the functions $f(t) = e^{\alpha t}$ and $f(t) = e^{i\alpha t}$, where α is a constant.

a) $\mathcal{L}\{e^{\alpha t}\} = ?$; $F(s) = \int_0^\infty e^{-st} \cdot e^{\alpha t}\, dt = \int_0^\infty e^{(-s+\alpha)t} \cdot dt$, $s = \text{const.}$

$$\therefore F(s) = \int_0^\infty e^{-(s-\alpha)t} \cdot dt = \lim_{b\to\infty}\left[\left(\frac{e^{-(s-\alpha)t}}{-(s-\alpha)}\right)\Bigg|_0^b\right] =$$

$$\lim_{b\to\infty}\left[\frac{e^{-(s-\alpha)b}}{-(s-\alpha)} - \frac{\cancelto{1}{e^{-(s-\alpha)(0)}}}{-(s-\alpha)}\right] = \lim_{b\to\infty}\left[\frac{1}{s-\alpha} - \frac{e^{-(s-\alpha)b}}{s-\alpha}\right] =$$

$$\cancelto{\frac{1}{s-\alpha}}{\lim_{b\to\infty}\left[\frac{1}{s-\alpha}\right]} - \lim_{b\to\infty}\left[\frac{\cancelto{0}{e^{-(s-\alpha)(\infty)}}}{s-\alpha}\right] = \frac{1}{s-\alpha} - 0 = \frac{1}{s-\alpha}$$

$$\therefore \boxed{\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha}}$$

b) $\mathcal{L}\{e^{i\alpha t}\} = ?$; $F(s) = \int_0^\infty e^{-st} \cdot e^{i\alpha t}\, dt = \int_0^\infty e^{(-s+i\alpha)t}\, dt$

$$= \int_0^\infty e^{-(s-i\alpha)t}\, dt$$

Let $z = s - i\alpha$, where $z = \text{const.}$ Then, $e^{-(s-i\alpha)t} = e^{-zt}$

$$\therefore F(z) = \int_0^\infty e^{-zt}\, dt = \int_0^\infty e^{-zt} \cdot 1\, dt = \mathcal{L}\{1\}\big|_z = \frac{1}{z} = \frac{1}{s-i\alpha} = F(s)$$

$$\therefore \boxed{\mathcal{L}\{e^{i\alpha t}\} = \frac{1}{s-i\alpha}}$$

Ex. 4 : Find the LaPlace Transform for $f(t) = \sin(wt)$, where w = constant.

$F(s) = \mathcal{L}\{\sin(wt)\} = \int_0^\infty e^{-st} \cdot \sin(wt)\, dt$. For efficiency, we will evaluate this improper integral by using tabular integration, but we will adjust this method to deal with the "circular" integral $\int e^{-st} \sin(wt)\, dt$ ⊛. Let $u = \sin(wt)$ and $dv = e^{-st} dt$. ⊛

⊛ NOTE: This method used to perform augmented tabular integration is a very common way to perform the method of tabular integration. The method done here assumes that you let your trig. function be u (i.e. the function you take successive derivatives of) and you let your exponential function be dv (i.e. the function you want to take successive antiderivatives of). However this method will still work if you let u = exponential function and dv = trig. function

u		dv
$u_1 = \sin(wt)$	+	$e^{-st} = dv_1$
$u_2 = w\cos(wt)$	−	$-\frac{1}{s} e^{-st} = dv_2$
$u_3 = -w^2 \sin(wt)$	$\xrightarrow{+}$	$\frac{1}{s^2} e^{-st} = dv_3$

$$\int u\, dv = u_1 dv_2 - u_2 dv_3 + \int u_3\, dv_3$$

Note: we continue to collect terms by multiplying down diagonally until we see that by multiplying across, we can have an integrand that will recover our original integral!

$$\therefore \int e^{-st} \sin(wt)\, dt = \int u\, dv = \frac{-\sin(wt)}{s e^{st}} - \frac{w\cos(wt)}{s^2 e^{st}} + \int -\frac{w^2 \sin(wt)}{s^2 e^{st}}\, dt$$

$$\Rightarrow \left(\cancel{1}^{\frac{s^2}{s^2}} + \frac{w^2}{s^2}\right) \int e^{-st} \sin(wt)\, dt = \frac{-\sin(wt)}{s e^{st}} - \frac{w\cos(wt)}{s^2 e^{st}}$$

$$\Rightarrow \int e^{-st} \sin(wt)\, dt = \left(\frac{s^2}{s^2 + w^2}\right)\left[\frac{-\sin(wt)}{s e^{st}} - \frac{w\cos(wt)}{s^2 e^{st}}\right] + C$$

Ex.4 : (cont'd)

$$\therefore \int_0^\infty e^{-st}\sin(wt)\,dt = \left(\frac{s^2}{s^2+w^2}\right)\lim_{b\to\infty}\left[\left(\frac{-\sin(wt)}{se^{st}} - \frac{w\cos(wt)}{s^2e^{st}}\right)\Bigg|_0^b\right] =$$

$$\left(\frac{s^2}{s^2+w^2}\right)\lim_{b\to\infty}\left[\left(\frac{-\sin(wb)}{se^{sb}} - \frac{w\cos(wb)}{s^2e^{sb}}\right) - \left(\cancelto{0}{\frac{-\sin(0)}{se^0}} - \frac{w\cancelto{1}{\cos(0)}}{s^2\cancelto{1}{e^0}}\right)\right]$$

$$= \left(\frac{s^2}{s^2+w^2}\right)\left[\cancelto{0}{\lim_{b\to\infty}\left(\frac{-\sin(wb)}{se^{sb}}\right)} + \cancelto{0}{\lim_{b\to\infty}\left(\frac{-w\cos(wb)}{se^{sb}}\right)} + \cancelto{w/s^2}{\lim_{b\to\infty}\left(\frac{w}{s^2}\right)}\right]$$

$$= \frac{s^2}{s^2+w^2}\cdot\frac{w}{s^2} = \frac{w}{s^2+w^2} \Rightarrow \boxed{\mathcal{L}\{\sin(wt)\} = \frac{w}{s^2+w^2}}$$

Ex.5 : Show that $\mathcal{L}\{\sin(wt)\} = \frac{w}{s^2+w^2}$ can also be established by noting that Euler's Formula states that $e^{iwt} = \cos(wt) + i\sin(wt)$.

Sol'n : If $e^{iwt} = \cos(wt) + i\sin(wt)$, then $e^{-iwt} = \cos(wt) - i\sin(wt)$. Recall that the right linear combination of e^{iwt} and e^{-iwt} can yield us either $\sin(wt)$ or $\cos(wt)$. If we find $e^{iwt} - e^{-iwt}$, we will see that...

$$e^{iwt} - e^{-iwt} = \cos(wt) + i\sin(wt) - \cos(wt) + i\sin(wt) = 2i\sin(wt)$$

$$\Rightarrow \sin(t) = \frac{e^{iwt}-e^{-iwt}}{2i} = \frac{1}{2i}\left[e^{iwt}-e^{-iwt}\right]. \text{ So } \mathcal{L}\{\sin(wt)\} = \mathcal{L}\left\{\frac{e^{iwt}-e^{-iwt}}{2i}\right\}$$

$$\therefore \mathcal{L}\left\{\frac{e^{iwt}-e^{-iwt}}{2i}\right\} = \frac{1}{2i}\int_0^\infty e^{-st}\cdot\left[e^{iwt}-e^{-iwt}\right]dt = \frac{1}{2i}\int_0^\infty e^{-(s-iw)t}\,dt - \frac{1}{2i}\int_0^\infty e^{-(s+iw)t}\,dt \;\star$$

$$\frac{\mathcal{L}\{1\}\big|_{s-iw}}{2i} - \frac{\mathcal{L}\{1\}\big|_{s+iw}}{2i} = \frac{\frac{1}{s-iw} - \frac{1}{s+iw}}{2i} = \frac{\cancel{s}+iw-\cancel{s}+iw}{(s^2+w^2)(2i)} = \frac{\cancelto{1}{2}iw}{(s^2+w^2)\cancelto{1}{(2i)}} = \boxed{\frac{w}{s^2+w^2}} \;\star$$

Ex. 6 : Find the LaPlace Transform of $u(t) = step(t)$.

NOTE: $u(t) = step(t)$ is usually as either ...

- $step(t) = \begin{cases} 0 \, ; \, t<0 \\ 1 \, ; \, t \geq 0 \end{cases}$ or • $step(t) = \begin{cases} 0 \, ; \, t<0 \\ 1 \, ; \, t>0 \end{cases}$.

Since both functions are piecewise-continuous and have a finite # of discontinuities within $t = [0, \infty)$, the LaPlace Transform for either function $u(t) = step(t)$ will be the same. This is an example of why the LaPlace Transform for a function $f(t)$ (if it is a suitable one) is not unique!

Sol'n: $\mathcal{L}\{u(t)\} = \int_0^\infty e^{-st} u(t)\, dt$ (*) $= \int_0^0 e^{-st} \cdot \cancel{u(t)}^{1} + \int_0^\infty e^{-st} \cdot \cancel{u(t)}^{1}\, dt =$

$0 + \int_0^\infty e^{-st} \cdot (1)\, dt = \mathcal{L}\{1\} = \frac{1}{s}$

$\therefore \boxed{\mathcal{L}\{step(t)\} = \mathcal{L}\{u(t)\} = \frac{1}{s}}$ (*)

(*) NOTE: Since values $t<0$ are irrevelant in the LaPlace Transform, we ignore the portion of $step(t)$ where $t<0$.

Ex. 7: Find the Laplace Transform $u_\alpha(t) = \text{step}_\alpha(t) = \text{step}(t-\alpha)$, where $\alpha \geq 0$ and $s > 0$.

Sol'n: $\mathcal{L}\{\text{step}(t-\alpha)\} = \int_0^\infty e^{-st} \cdot u_\alpha(t)\,dt$, where $u_\alpha(t) = \begin{cases} 0; & t < \alpha \\ 1; & t \geq \alpha \end{cases}$

$$\therefore \mathcal{L}\{\text{step}(t-\alpha)\} = \mathcal{L}\{u_\alpha(t)\} = \int_0^\alpha \cancel{e^{-st}(0)}^{\,0}\,dt + \int_\alpha^\infty e^{-st}(1)\,dt =$$

$$\lim_{b\to\infty}\left(\left[C\right]_0^b\right) + \lim_{b\to\infty}\left(\left[\frac{e^{-st}}{-s}\right]_\alpha^b\right) = \lim_{b\to\infty}\left(\cancel{C-C}^{\,0}\right) + \lim_{b\to\infty}\left[\frac{e^{-sb}}{-s} - \frac{e^{-s\alpha}}{-s}\right] =$$

$$0 + \cancel{\frac{e^{-s(\infty)}}{-s}}^{\,0} + \frac{e^{-s\alpha}}{+s} = 0 + 0 + \frac{e^{-s\alpha}}{s} = \frac{e^{-s\alpha}}{s}$$

$$\therefore \boxed{\mathcal{L}\{\text{step}(t-\alpha)\} = \frac{e^{-s\alpha}}{s}} \quad (\star\star)$$

NOTE: If $\alpha = 0$, then $\mathcal{L}\{\text{step}(t-\alpha)\} = \mathcal{L}\{\text{step}(t)\} = \frac{e^{-s(0)}}{s} = \frac{1}{s}$.

This is the same result as Ex. 6! Therefore, we can collapse identities $(\star)$ and $(\star\star)$ into the single identity below!

$$\boxed{\begin{array}{l}\mathcal{L}\{\text{step}(t)\} = \mathcal{L}\{\text{step}(t-\alpha)\} = \mathcal{L}\{u_\alpha(t)\} = \dfrac{e^{-s\alpha}}{s}\ ; \alpha \geq 0, \\ \text{where } \alpha, s \text{ are constants, and } s > 0.\end{array}}$$

Now that we have establish some LaPlace Transform Identities that are basic, we now will look at some other useful identities that are actually properties of the LaPlace Transform operation. These fundamental identities consist of the Linearity Properties and Transformation / Translation Properties of the LaPlace Transform.

Linearity Properties of $\mathcal{L}$

Let $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$. Also, let α, β be constants.

(1) $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \cdot \mathcal{L}\{f(t)\} + \beta \cdot \mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s)$

(2) $\mathcal{L}\{c_1 f_1(t) + \dots + c_n f_n(t)\} = c_1 F_1(s) + \dots + c_n F_n(s)$

Ex. 8. : Let $f(t) = 3t^4 - 6\sin(5t)$. Find $\mathcal{L}\{f(t)\} = F(s)$.

$\therefore \mathcal{L}\{f(t)\} = \mathcal{L}\{3t^4 - 6\sin(5t)\} = \mathcal{L}\{3t^4\} - \mathcal{L}\{6\sin(5t)\} =$

$3\mathcal{L}\{t^4\} - 6\mathcal{L}\{\sin(5t)\} = 3\left[\frac{\overset{24}{\cancel{4!}}}{s^{4+1}}\right] - 6\left[\frac{5}{s^2+(5)^2}\right] = \frac{72}{s^5} - \frac{30}{s^2+25}$

$$\therefore \boxed{\mathcal{L}\{3t^4 - 6\sin(5t)\} = F(s) = \frac{72}{s^5} - \frac{30}{s^2+25}}$$

Ex. 9: Let $f(t) = 5t^2 - 7\sinh(\pi t) + 10\,u_6(t) - 12$

$$\therefore \mathcal{L}\{f(t)\} = F(s) = \mathcal{L}\{5t^2 - 7\sinh(\pi t) + 10\,u_6(t) - 12\}$$

$$= 5\,\mathcal{L}\{t^2\} - 7\,\mathcal{L}\{\sinh(\pi t)\} + 10\,\mathcal{L}\{\text{step}(t-6)\} - 12\,\mathcal{L}\{1\}$$

$$= 5\left[\frac{2!}{s^3}\right] - 7\left[\frac{\pi}{s^2-\pi^2}\right] + 10\left[e^{-6s}\cdot\frac{1}{s}\right] - 12\left(\frac{1}{s}\right)$$

$$\therefore \boxed{\mathcal{L}\{f(t)\} = \frac{10}{s^3} - \frac{7\pi}{s^2-\pi^2} + \frac{10e^{-6s}}{s} - \frac{12}{s}}$$

Transformation / Translation Identities of $\mathcal{L}$

Let $F(s) = \mathcal{L}\{f(t)\}$ + $G(s) = \mathcal{L}\{g(t)\}$. Also, let $s, \alpha, \beta,$ + γ be constants such that $s > \alpha$.

(1) $\mathcal{L}\{e^{\alpha t} f(t)\} = F(s-\alpha)$

NOTE: Multiplying a function $f(t)$ by $e^{\alpha t}$ results in a horizontal shift (e.g. a shift in time) of α units. If $\alpha > 0$, then we have a horizontal shift to the right. If $\alpha < 0$, then we have a horizontal shift to the left.

(2) $\mathcal{L}\{f(t-\alpha)\} = e^{-s\alpha} F(s)$

NOTE: A horizontal shift for $f(t)$ results in $F(s)$ being vertically stretched or compressed by $e^{-s\alpha}$ units! If $s\alpha > 0$, then $0 < e^{-s\alpha} < 1 \Rightarrow F(s)$ is vertically compressed. If $s\alpha < 0$, then $e^{-s\alpha} > 1 \Rightarrow F(s)$ is vertically stretched.

(3) $\mathcal{L}\{f(\alpha t)\} = \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right)$; $\alpha > 0$

NOTE: A horizontal compression (i.e. $\alpha > 1$) of $f(t)$ results in $F(s)$ being horizontal stretch by a factor of α and vertically compressed by a factor of α. Similarly, a horizontal stretch (i.e. $0 < \alpha < 1$) of $f(t)$ results in $F(s)$ being horizontally compressed by a factor of α and vertically stretched by a factor of α!

Ex. 10: Find the LaPlace Transform for the following functions.

a) $f(t) = e^{-2t} \cdot \sin(\frac{4}{\pi} t)$

NOTE: $\mathcal{L}\{\sin(\frac{4}{\pi}t)\} = \dfrac{\frac{4}{\pi}}{s^2 + (\frac{4}{\pi})^2} = \dfrac{\frac{4}{\pi}}{s^2 + \frac{16}{\pi^2}} = \dfrac{4\pi}{\pi s^2 + 16}$

Also, $\mathcal{L}\{e^{-2t} \cdot f(t)\} = F(s-(-2)) = F(s+2)$

$\therefore \mathcal{L}\{e^{-2t} \cdot \sin(\frac{4}{\pi}t)\} = \left[\dfrac{4\pi}{\pi s^2 + 16}\right]\Bigg|_{s+2} = \dfrac{4\pi}{\pi(s+2)^2 + 16}$

$\therefore \boxed{\mathcal{L}\{e^{-2t} \cdot \sin(\frac{4}{\pi}t)\} = F(s+2) = \dfrac{4\pi}{\pi(s+2)^2 + 16}}$

b) $f(t) = (t+4)^3 + 6$

NOTE: $\mathcal{L}\{(t+4)^3\} = \mathcal{L}\{(t-(-4))^3\} = e^{-s(-4)} \mathcal{L}\{t^3\} = e^{4s} \cdot \dfrac{\cancel{3!}\,6}{s^4}$

Also, $\mathcal{L}\{6\} = 6\,\mathcal{L}\{1\} = 6 \cdot \frac{1}{s} = \frac{6}{s}$

$\therefore \mathcal{L}\{(t+4)^3 + 6\} = \mathcal{L}\{(t+4)^3\} + \mathcal{L}\{6\} = \boxed{\dfrac{6e^{4s}}{s^4} + \dfrac{6}{s}}$

Table 1 : Table of Basic LaPlace Transform Identities

$f(t)$	$F(s)=\mathcal{L}\{f(t)\}$	Restrictions
1	$\frac{1}{s}$	$s>0$
t^n	$\frac{n!}{s^{n+1}}$	$s>0$, $n=1,2,\ldots$
$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$	$s>0$
t^α	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ ⊛	$\alpha>-1$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$	$s>\alpha$
$e^{i\alpha t}$	$\frac{1}{s-i\alpha}$	$s>0$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$s>0$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$s>0$
$\text{step}(t)$, $\text{step}(t-\alpha)$	$\frac{e^{-\alpha s}}{s}$	$\alpha\geq 0$

⊛ $\Gamma(x)$ = Gamma Function $= \int_0^\infty u^{x-1} e^{u} \cdot du \Rightarrow \Gamma(x+1) = \int_0^\infty u^\alpha e^{-u}\, du$

where $u=st$, $x=\text{constant}$; $\alpha=\text{constant}$

Table 1 : Table of Basic LaPlace Transform Identities (cont'd)

$f(t)$	$F(s)=\mathcal{L}\{f(t)\}$	Restrictions
$\sinh(at)$	$\dfrac{a}{s^2-a^2}$	$s>0$
$\cosh(at)$	$\dfrac{s}{s^2-a^2}$	$s>0$
$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$	$s>a$
$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$	$s>a$
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$	$s>a$
$\text{step}(t)\cdot f(t-\alpha)$	$e^{-\alpha s}F(s)$	$s>\alpha$
$e^{\alpha t}f(t)$	$F(s-\alpha)$	$s>\alpha$
$f(\alpha t)$	$\frac{1}{\alpha}F\left(\frac{s}{\alpha}\right)$	$\alpha>0$
$(-t)^n f(t)$	$F^{(n)}(s)$	$s>0$