Word Problem Modeling w/ 1^{st}-ORDER DEs

In mathematics, the word "modeling" is just the process of deriving/developing a set of equations and/or formulas that describes some process or phenomenon of interest. The end result of coming up with these models usually serve as the basis for being able to understand and solve "word problems" often seen in math courses.

Our goal for these notes is not to only expose ourselves to "end-result" models for various situations that are often modeled with 1^{st}-order ODEs (such as Population/Logistics Dynamics, Mixtures (of a fluid solution), Spread of Diseases, Draining Tank, Series (Electrical) Circuits, and Radioactive Decay to name a few), but we want to also understand how these models are created in the first place. In the problems that we will do in these notes, our primary focus will be on using known, generic models of situations (that are many times overly simplified) in order to augment these models to account for phenomena not originally considered in the base model. Our focus will also be on recognizing the types of ODEs these models result in being so that the connection between these problems and the equation techniques we have learned so far can be made.

Overview of Situations Modeled w/ 1st-order ODEs

Mixtures: $\dfrac{dA}{dt} = \begin{pmatrix}\text{rate of substance}\\ \text{entering container}\end{pmatrix} - \begin{pmatrix}\text{rate of substance}\\ \text{leaving container}\end{pmatrix} = R_1 - R_2$

where ...

- R_1 = rate of substance entering container

  $R_1 = \begin{pmatrix}\text{rate of fluid/solution entering}\\ \text{the container with (dissolved)}\\ \text{substance to be measured}\end{pmatrix}\begin{pmatrix}\text{concentration of}\\ \text{substance in solution/}\\ \text{fluid per unit of measure}\\ \text{flowing in container}\end{pmatrix}$

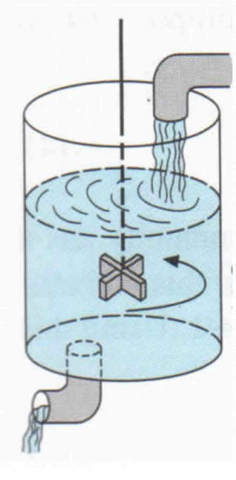
- R_2 = rate of substance leaving container

  $R_2 = \begin{pmatrix}\text{rate of fluid/solution leaving}\\ \text{the container with (dissolved)}\\ \text{substance to be measured}\end{pmatrix}\begin{pmatrix}\text{concentration of substance}\\ \text{in solution/fluid per unit}\\ \text{of measure flowing out container}\end{pmatrix}$

- $A = A(t)$ = amount of substance (that is dissolved in fluid/solution) in the container at time t

- t = time (usually in minutes or seconds)

Mixing 2 (fluid) solutions that both have different concentrations of a particular substance (e.g. salt) gives rise to a 1st-order ODE for the amount of this substance, $A(t)$, contained in the mixture.



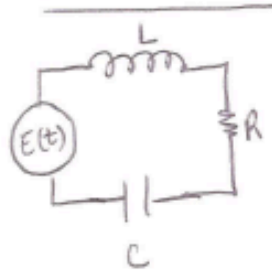
• Population Growth : $\frac{dP}{dt} = kP$, where $P = P(t) = P_0 e^{kt}$, $k > 0$, $P_0 \in \mathbb{R}$

• Radioactive Decay : $\frac{dD}{dt} = kD$; where $D = D(t) = D_0 e^{kt}$, $k < 0$, $D_0 \in \mathbb{R}$

• Spread of Disease : $\frac{dx}{dt} = kxy = kx(n+1-x)$, where $x = x(t)$ = # people infected by disease, $y = y(t)$ = # people not yet infected (by disease) , n = # of people in given population , $x + y = n+1$ (i.e. relationship between $x, y,$ and n if 1 extra person/host introduced to population), and $k > 0$.

• Chemical Reactions : $\frac{dX}{dt} = k(\alpha - X)(\beta - X)$, where $X(t)$ = amount of substance A remaining at time t (that has not decomposed into smaller molecules) of substance B, α = given (initial) amount of substance A , β = given (initial) amount of substance B, + $k < 0$.

• Series Circuits :



* $L \frac{di}{dt} = L \frac{d^2q}{dt^2}$ (inductor)
* $iR = R \frac{dq}{dt}$ (resistor)
* $\frac{1}{C} q$ (capacitor)

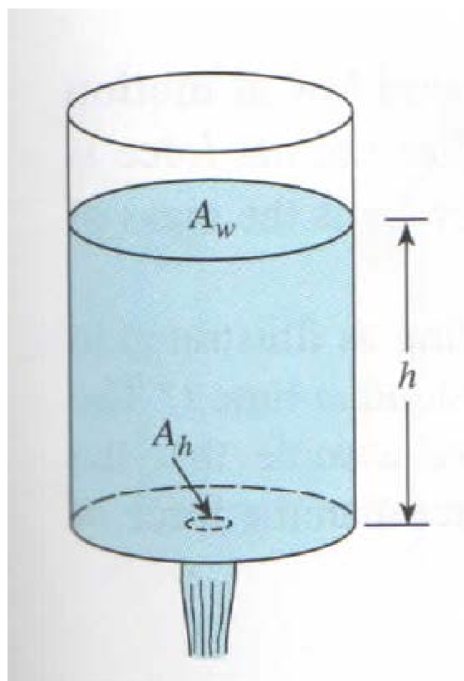
* $i(t)$ = current of circuit at time t
* $q(t)$ = charge across capacitor at time t
* $i(t) = \frac{dq}{dt}$

Kirchhoff's Voltage Law (KVL) states that voltage $E(t)$ on a closed (circuit) loop must equal the sum of the voltage drops in the loop.

$$\therefore \underbrace{L \frac{d^2q}{dt^2}}_{\text{volt drop across inductor}} + \underbrace{R \frac{dq}{dt}}_{\text{volt drop across resistor}} + \underbrace{\frac{1}{C} q}_{\text{volt drop across capacitor}} = \underbrace{E(t)}_{\text{volt source or electromotive force of circuit}}$$

- **Draining a Tank** : $\dfrac{dh}{dt} = -\dfrac{A_h}{A_w}\sqrt{2gh}$

In hydrodynamics, Torricelli's law states that the speed $v = \sqrt{2gh}$ of efflux of water through a sharp-edged hole at the bottom of a tank filled to a depth of h is the same as the speed that a body (i.e. drop of water in this case) would acquire in falling freely from height h. (Note that g = acceleration due to gravity and A_h = area of hole in bottom of tank and A_w = constant area of the upper surface of the water in the tank).



Logistics Model (for Population) : $\frac{dQ}{dt} = \beta_0 Q - \gamma Q^2$, where...

- β_0 = ideal birth rate / organism of the population / community studied
- $Q = Q(t)$ = quantity / population of organisms in community at time t
- t = time (usually in hours, months, or years)
- $\beta_0 Q$ = # of births in community of organisms being studied
- $\gamma = \gamma_B + \gamma_D = \left(\begin{array}{l}\text{positive constant related to fraction}\\ \text{of population added to}\\ \text{community through births}\end{array}\right) + \left(\begin{array}{l}\text{positive constant related to}\\ \text{fraction of population subtracted}\\ \text{from community through deaths}\end{array}\right)$
- δ = fraction of population that dies within each unit of time

  $\delta = \gamma_D Q = f(Q)$ (i.e. δ is a function of Q)
- $\beta = \beta_0 - \gamma_B Q = f(Q)$
- $K = \frac{\beta_0}{\gamma}$ = carrying capacity
- $\beta Q = [\beta_0 - \gamma_B Q]Q$ = # birth per unit time
- $\delta Q = [\gamma_D Q]Q$ = # death per unit time
- $\beta Q - \delta Q = [\beta_0 - \gamma_B Q]Q - [\gamma_D Q]Q = \beta_0 Q - \gamma_B Q^2 - \gamma_D Q^2 = \beta_0 Q - \gamma Q^2$

If we take our (basic) model for population growth (or decay), $\frac{dQ}{dt} = kQ$, and account for things such as deaths (due to normal life expectancy), deaths due to limited supply of food and other resources, decrease in birth rate due to limited supply of food and other resources, and the ideal birth rate (of an organism in this community), then what we have is a Logistics Model for our population growth (or decay). The solution to this ODE is $Q(t) = \frac{K Q_0}{Q_0 + (K - Q_0)e^{-\beta_0 t}}$.

Now we will do several examples in order to get a good idea of how to model, augment, and (sometimes) solve word problems.

Population / Logistics Dynamics problem(s)

Ex. 1: Recall that for a population P, the ODE that models (simple) population growth/decay is $\frac{dP}{dt} = kP$, where $k > 0$ (pop. growth) and $k < 0$ (pop. decay). Assuming the assumptions that go with this model, determine an ODE of a growing population $P = P(t)$ of a country when individuals are allowed to ...

a) immigrate into this country at constant rate "r".

If immigrates are adding to the population at a constant rate "r" $\Rightarrow P_i(t) = rt$ is a formula that tracks the amount of people added based on immigration. So, $\frac{dP_i}{dt} = r$. Also, if $P_b(t)$ = pop. growth via births in population, then $\frac{dP_b}{dt} = kP_b$. Therefore, $P(t) = P_b + P_i \Rightarrow \frac{dP}{dt} = kP_b + r$ will be the ODE that describes pop. growth due to births and immigration in this example.

b) emigrate out of this country at constant rate "r".

If people are emigrating out the country at a constant rate, then $P_e(t) = -rt$ $\Rightarrow \frac{dP_e}{dt} = -r$. It follows that $\frac{dP}{dt} = kP_b - r$ will be the ODE that models this situation.

Ex. 2: Note that our model $\frac{dP}{dt} = KP$ does not take into account deaths in the population of a community. (Therefore, the growth rate = birth rate) In another model of a changing population of a community, it is assumed that the rate in which the population changes is a <u>net rate</u> (i.e. net rate of pop. = (rate of births) − (rate of deaths) of the community).

Determine an ODE that predicts the population $P(t)$ if both the birth rate and death rate are proportional to the population present at time t.

<u>NOTE</u>: Use K_B and K_D for the constant of proportionality for births and deaths in this community, respectively.

Let $P(t) = P_B(t) - P_D(t)$, where $P_B(t)$ = pop. (increase) due to births and $P_D(t)$ = pop. (decrease) due to deaths. If both the birth and death rate is proportional to current pop. at time t, then $\frac{dP}{dt} = K_B P - K_D P = [K_B - K_D] \cdot P$

Answer: $\boxed{\frac{dP}{dt} = K_B P - K_D P = [K_B - K_D] \cdot P}$

<u>Ex. 3</u>: Using the concept of net rate of a population from Ex. 2 above, determine an ODE that will predict a population $P(t)$ if the birth rate is proportional to the population present at time t, but the death rate is proportional to the square of the population present at time t.

$\therefore \boxed{\frac{dP}{dt} = K_B P - K_D P^2}$

Ex. 4: Recall that the ODE $\frac{dQ}{dt} = \beta_0 Q - \gamma Q^2$ models the population growth of a community where (ideal) birth rates, decreases in birth rate due to limits on food and other resources, and (increase) in deaths also due to limits on food and other resources (as population Q grows) are accounted for in this model.

a) Verify that $Q(t) = \dfrac{K Q_0}{Q_0 + (K - Q_0)e^{-\beta_0 t}}$, where $Q_0 = Q(0)$ is a solution to the ODE $\frac{dQ}{dt} = \beta_0 Q - \gamma Q^2$.

NOTE: K = carrying capacity $= \frac{\beta_0}{\gamma} \Rightarrow \beta_0 = K\gamma$. So, $\beta_0 Q - \gamma Q^2 = K\gamma Q - \gamma Q^2$.

$\therefore K\gamma Q - \gamma Q^2 = \gamma Q[K - Q]$. So, $\frac{dQ}{dt} = \beta_0 Q - \gamma Q^2 \Rightarrow \frac{dQ}{dt} = \gamma Q[K - Q]$

$$\Rightarrow \frac{dQ}{\gamma Q[K-Q]} = dt \Rightarrow \frac{dQ}{Q[K-Q]} = \gamma\, dt \Rightarrow \int \frac{dQ}{Q[K-Q]} = \int \gamma\, dt$$

$\therefore \dfrac{1}{Q[K-Q]} = \dfrac{A}{Q} + \dfrac{B}{K-Q}$, where $A = \left[\dfrac{1}{K-Q}\right]_{Q=0} = \dfrac{1}{K}$ and $B = \left[\dfrac{1}{Q}\right]_{Q=K} = \dfrac{1}{K}$

$$\therefore \int \left[\frac{\frac{1}{K}}{Q} + \frac{\frac{1}{K}}{K-Q}\right] dQ = \int \gamma\, dt \Rightarrow \frac{1}{K}\Big[\ln|Q| - \ln|K-Q|\Big] = \gamma t + C$$

$\therefore \ln\left|\dfrac{Q}{K-Q}\right| = K\gamma t + D$, where $KC = D$, $\Rightarrow e^{\ln\left|\frac{Q}{K-Q}\right|} = e^{K\gamma t}\cdot A$; $A = e^{D}$.

$$\therefore \frac{Q}{K-Q} = e^{K\gamma t}\cdot A \Rightarrow Q = Ke^{K\gamma t}\cdot A - Qe^{K\gamma t}\cdot A \Rightarrow Q[1 + e^{K\gamma t}A] = Ke^{K\gamma t}\cdot A$$

Ex. 4: (cont'd)

$$\therefore Q = \frac{\varkappa e^{\varkappa \gamma t}\cdot A}{1+e^{\varkappa\gamma t}\cdot A} = \frac{\varkappa e^{\varkappa\left(\frac{\beta_0}{\varkappa}\right)t}\cdot A}{1+e^{\varkappa(\beta_0/\varkappa)t}\cdot A} = \frac{\varkappa e^{\beta_0 t}\cdot A}{1+e^{\beta_0 t}\cdot A}\cdot\frac{e^{-\beta_0 t}}{e^{-\beta_0 t}} = \frac{A\varkappa}{e^{-\beta_0 t}+A}$$

NOTE: $Q_0 = Q(0) = \dfrac{A\varkappa}{1+A} \Rightarrow Q_0(1+A) = A\varkappa \Rightarrow Q_0 + Q_0 A = A\varkappa \Rightarrow Q_0 = A\varkappa - Q_0 A$

$\therefore Q_0 = A[\varkappa - Q_0] \Rightarrow A = \dfrac{Q_0}{\varkappa - Q_0}$. Thus, $Q = \left(\left[\dfrac{Q_0}{\varkappa-Q_0}\right]\varkappa\right) \div \left(e^{-\beta_0 t} + \dfrac{Q_0}{\varkappa - Q_0}\right)$

$$\therefore Q = Q(t) = \frac{\left[\frac{Q_0}{\varkappa-Q_0}\right]\varkappa}{e^{-\beta_0 t}+\left[\frac{Q_0}{\varkappa-Q_0}\right]}\cdot\frac{\varkappa-Q_0}{\varkappa-Q_0} = \frac{Q_0\varkappa}{(\varkappa-Q_0)e^{-\beta_0 t}+Q_0} \Rightarrow \boxed{Q(t) = \frac{Q_0\varkappa}{(\varkappa-Q_0)e^{-\beta_0 t}+Q_0}}$$

b) Now, assume that $Q_0 = 4$, $\beta_0 = 5/4$, and the carry capacity $\varkappa$ of this population (with respect to its environment) is 8,000,000. If $Q(t)$ = # of rabbits in this population at time t (in years), then how many rabbits (approximately) does this model predict will be in this (isolated and protected) population . . .

i) At the end of the 1st 6 months?

$$\therefore Q(\tfrac{1}{2}) = \frac{(4)(8{,}000{,}000)}{(8{,}000{,}000-4)e^{-5/4(1/2)}+4} = 7.472980586 \approx 7 \text{ rabbits}$$

ii) At the end of the second year?

$$Q(2) = \frac{(4)(8{,}000{,}000)}{(8{,}000{,}000-4)\cdot e^{-5/4(2)}+4} = 48.729703 \approx 49 \text{ rabbits}$$

Spread of Disease problem(s)

Ex. 1: Suppose a student carrying a flu virus returns to an isolated college campus of 1000 students. Determine an ODE that will predict the number of people, $x(t)$, who have contracted the flu if the rate at which the virus spreads is proportional to the number of interactions between the number of students with the flu and the number of students who have not yet been exposed to it. (Assume that time t is in days).: Recall that the ODE $\frac{dx}{dt} = kxy = kx(n+1-x)$ models the spread of disease similar to a situation like this one. Note that $n = 1000$. Therefore, $\frac{dx}{dt} = kx(1000+1-x) = kx(1001-x)$.

$\therefore \frac{dx}{dt} = kx(1001-x)$, where $x = x(t) =$ # students that have virus, $k =$ const. of proportionality that is > 0!

NOTE! $\frac{dx}{dt} = kx(1001-x) \Rightarrow \frac{dx}{kx(1001-x)} = dt$

Ex. 2: For the resulting ODE you arrived at in Ex. 1 above, what type of ODE is it? How do you know? Can we solve this ODE for a general solution using the techniques that we learned so far?

Since our ODE in Ex. 1 can be written in the form $M(x) \cdot dx = N(t) \cdot dt$, it follows that this ODE is separable! Yes, we can solve separable equations for their general solutions based upon what we've learned so far!

Spread of Disease problem(s) : (cont'd)

Ex. 3: If your answer to Ex. 2 was that your ODE in Ex. 1 was a recognizable ODE that can be solved using the techniques learned so far, solve the ODE for a general solution for $x(t)$ in terms of k = constant of proportionality = rate of which virus is spread in population and t = time (in days). So, $\frac{dx}{kx(1001-x)} = dt \Rightarrow \int \frac{dx}{x(1001-x)} = \int k\, dt$.

NOTE: $\frac{dx}{x(1001-x)} = \left[\frac{A}{x} + \frac{B}{1001-x}\right] dx$, where $A = \left[\frac{1}{1001-x}\right]_{x=0} = \frac{1}{1001}$

and $B = \left[\frac{1}{x}\right]_{x=1001} = \frac{1}{1001}$. Thus, $\frac{dx}{x(1001-x)} = \frac{1}{1001}\left[\frac{1}{x} - \frac{1}{x-1001}\right]$

$$\therefore \int \frac{dx}{x(1001-x)} = \int k\, dt \Rightarrow \frac{1}{1001}\Big[\ln|x| - \ln|x-1001|\Big] = kt + C$$

$$\therefore \ln\left|\frac{x}{x-1001}\right| = (1001k)t + D, \text{ where } D = 1001C$$

$$\Rightarrow e^{\ln\left|\frac{x}{x-1001}\right|} = e^{(1001k)t} \cdot A, \text{ where } A = e^{D} \Rightarrow \frac{x}{x-1001} = Ae^{(1001k)t}$$

$$\therefore x = Axe^{(1001k)t} - 1001Ae^{(1001k)t} \Rightarrow x\left[1 - Ae^{(1001k)t}\right] = -1001Ae^{(1001k)t}$$

$\therefore x = \frac{-1001Ae^{(1001k)t}}{\left[1 - Ae^{(1001k)t}\right]}$. Note: $x(0) = \frac{-1001A}{1-A} = x_0 \Rightarrow -1001A = x_0 - Ax_0$

$\therefore A[x_0 - 1001] = x_0 \Rightarrow \boxed{A = \frac{x_0}{x_0 - 1001}}$. Thus, $x = \frac{-1001\left[\frac{x_0}{x_0-1001}\right] \cdot e^{(1001k)t}}{\left[1 - \left(\frac{x_0}{x_0-1001}\right)e^{(1001k)t}\right]}$

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$$\therefore\ X = X(t) = \frac{-1001\left[\frac{X_0}{X_0-1001}\right]e^{(1001K)t}}{\left[1-\left(\frac{X_0}{X_0-1001}\right)e^{(1001K)t}\right]} \cdot \frac{(X_0-1001)\cdot e^{-(1001K)t}}{(X_0-1001)\cdot e^{-(1001K)t}}$$

$$'' = \frac{-1001\,X_0}{(X_0-1001)e^{-(1001K)t} - X_0} \cdot \frac{-1}{-1}$$

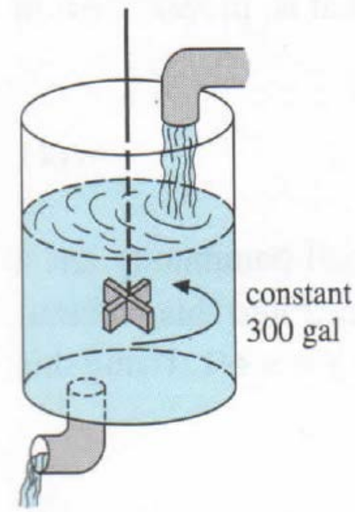
$$\boxed{X(t) = \frac{1001\,X_0}{X_0-(X_0-1001)\cdot e^{-(1001K)t}}}$$

, where X_0 = # student initially had the flu virus

t = time (in days)

Mixture problem(s)

Ex. 1: Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt has been dissolved. Pure water is pumped into the tank at a rate of 3 gal/min and when the solution is well-stirred, it is pumped out at the same rate. Determine an ODE for the amount $A(t)$ of salt in the tank at time t.



Recall that $\frac{dA}{dt} = \begin{pmatrix}\text{rate of substance}\\ \text{entering tank}\end{pmatrix} - \begin{pmatrix}\text{rate of substance}\\ \text{leaving tank}\end{pmatrix} = R_1 - R_2$,

where ...

$$R_1 = \begin{pmatrix}\text{rate of solution entering}\\ \text{tank w/ dissolved substance}\\ \text{to be measured}\end{pmatrix}\begin{pmatrix}\text{Concentration of substance in}\\ \text{solution per unit of measure}\\ \text{flowing in tank}\end{pmatrix}, \text{ and}$$

$$R_2 = \begin{pmatrix}\text{rate of solution leaving}\\ \text{tank w/ dissolved substance}\\ \text{to be measured}\end{pmatrix}\begin{pmatrix}\text{concentration of substance in}\\ \text{solution per unit of measure}\\ \text{flowing out tank}\end{pmatrix}$$

What is R_1?: Since we are pumping pure water into the tank (i.e. solution contains 0 lb./gal into tank $\Rightarrow R_1 = (3\ \text{gal}/\text{min})(0\ \text{lb}/\text{gal}) = 0\ \text{lb}/\text{gal}$ salt solution.

What is R_2?: Since fluid is pumped out of the tank at the same rate it enters the tank, there will be 300 gals of water (solution) in the tank at all times. Also, note that if $A = A(t) = \#$ lbs. of salt in the tank at any time t, then $\frac{A}{300}$ will be the concentration of salt per gallon flowing out the tank.

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$$\therefore R_2 = \left(3\ \text{gal/min}\right)\left(\frac{A}{300}\ \text{lb/gal}\right) = \frac{A}{100}\ \text{lb/min}$$

$$\therefore \frac{dA}{dt} = R_1 - R_2 = 0 - \frac{A}{100} = -\frac{A}{100} \Rightarrow \boxed{\frac{dA}{dt} = -\frac{A}{100}}$$

Ex. 2 : Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt has been dissolved (just as in Ex. 1). Another brine solution is pumped into the tank at a rate of 3 gal/min, and when the solution is well stirred, it is pumped out at a slower rate of 2 gal/min. If the concentration of the solution entering (the mixing tank) is 2 lbs/gal, determine an ODE for the amount $A(t)$ of salt in the tank at time t. Using the same approach we did in Ex. 1 on the previous page, we see that R_1 and R_2 are...

$$R_1 = \left(3 \text{ gal/min}\right)\left(2 \text{ lbs/gal}\right) = 6 \text{ lb/min (of salt)}$$

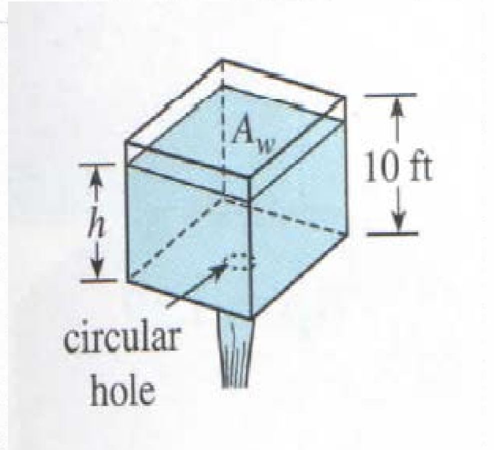
$$R_2 = \left(2 \text{ gal/min}\right)\left(\frac{A}{300+t}\right)$$, where $300+t$ come from the fact that

since there is $3 \text{ gal/min} - 2 \text{ gal/min} = 1 \text{ gal/min}$ addition fluid to the tank, this means that after t minutes of mixing solutions in this tank, the amount of solution in the tank will be $300+t$ gals!

$$\therefore \frac{dA}{dt} = R_1 - R_2 = 6 - \frac{2A}{300+t} \Rightarrow \boxed{\frac{dA}{dt} = 6 - \frac{2A}{300+t}}$$

Draining a Tank problem(s)

Ex.1 : Suppose water is leaking from a tank through a circular hole of area A_h at its bottom. When water leaks through a hole, friction and contraction of the water stream near the hole reduces the volume of the water leaving the tank



per second to $c \cdot A_h \sqrt{2gh}$, where $c = (0,1)$ is an empirical constant (i.e. a constant that can be determined through experimentation). Determine an ODE for the height "h" of water at time t for the cubical tank shown above.

NOTE: Radius of circular hole = 2 in$^{(*)}$, $g = 32\, ft/s^2$, and A_w = constant area of the upper water surface. ((*) what is 2 in. in ft.)?

Recall that an ODE that describes the draining of a tank (in terms of the rate in which the volume of the tank decreases) that incorporates Torricelli's law is $\frac{dV}{dt} = -\frac{A_h}{A_w}\sqrt{2gh}$. We are told that in this special situation that $A_h\sqrt{2gh}$ should be changed to $c \cdot A_h\sqrt{2gh}$. Thus, $\frac{dV}{dt} = -c \cdot \frac{A_h}{A_w}\sqrt{2gh}$. Since the radius of circular hole at the bottom of the tank is $r = 2 \text{ in} = \frac{2 \text{ in}}{} \left| \frac{1 \text{ ft}}{12 \text{ in}} \right. = \frac{1}{6} \text{ ft} \Rightarrow A_h = \pi r^2 = \pi\left(\frac{1}{6}\right)^2 = \frac{\pi}{36} \text{ ft}^2$

$\therefore\ A_h = \frac{\pi}{36}\,ft^2,\quad g = 32\,ft/s^2,\quad A_w = (10)^2 = 100\,ft^2$ ⊛

⊛ NOTE: Our tank is cubical, so all sides of tank are the same (i.e. 10 ft) according to the figure that illustrates our problem on the previous page.

$$\therefore\ \frac{dV}{dt} = -c\cdot\frac{\frac{\pi}{36}}{100}\sqrt{2(32)h} = \frac{-\pi c}{36\cdot 100}\cdot\sqrt{\cancel{64}^{\,8}}\,\sqrt{h} = \frac{-\cancel{8}^{\,2}\pi c}{\cancel{36}_{\,9}\cdot 100}\sqrt{h} = \frac{-\cancel{2}^{\,1}\pi\sqrt{h}\cdot c}{9\cdot\cancel{100}_{\,50}}$$

$$= \frac{-c\pi\sqrt{h}}{450}\ ft^3/sec$$

$$\therefore\ \boxed{\frac{dV}{dt} = \frac{-c\pi}{450}\sqrt{h}\ ft^3/sec}$$

Radioactive Decay problem(s)

Ex. 1 : Assume that $A(t) = A_0 e^{-rt}$ is the amount of some radioactive substance at time t having a half life $\tau_{1/2}$.

NOTE: Recall that the relationship between the growth/decay rate "r" (i.e. constant of proportionality) and the k-life (if $k=(0,1)$) or k-time (if $k>1$) is ...

$$r\tau_k = \ln(k) \text{ , where...}$$

$* \; k=(0,1) \Rightarrow \tau_k$ = k-life of substance
$\Rightarrow \ln(k) < 0$

$* \; k=(1,\infty) \Rightarrow \tau_k$ = k-time of substance
$\Rightarrow \ln(k) > 0$

and k=2 (double), k=3 (triple), etc.

a) Verify that, for each value of t (not just t=0), that $A(t+\tau_{1/2}) = \frac{1}{2}A(t)$.

Let $A(t) = A_0 e^{-rt}$. Since $\tau_{1/2}$ = half-life of a substance, by definition, this means that at time $t = \tau_{1/2}$, $A(\tau_{1/2}) = \frac{1}{2}A_0$, where A_0 = initial amount of substance. Thus, $A(t+\tau_{1/2}) = A_0 e^{-r(t+\tau_{1/2})}$. Therefore, ...

$$A(t+\tau_{1/2}) = A_0 e^{-rt} \cdot e^{-r\tau_{1/2}} = e^{-rt}\left[A_0 e^{-r(\tau_{1/2})}\right] = e^{-rt}\left[A(\tau_{1/2})\right] = e^{-rt}\left[\tfrac{1}{2}A_0\right] =$$

$$\tfrac{1}{2}A_0 e^{-rt} = \tfrac{1}{2}\left[A_0 e^{-rt}\right] = \tfrac{1}{2}A(t). \text{ Thus, } \boxed{A(t+\tau_{1/2}) = \tfrac{1}{2}A(t)}.$$

Ex. 1 : cont'd

b) Verify that the formula $A(t) = A_0 e^{-rt}$ can be rewritten as ...

$$A(t) = A_0 \left(\tfrac{1}{2}\right)^{t/\tau_{1/2}}$$

NOTE! $r\tau_{1/2} = \ln(\tfrac{1}{2}) \Rightarrow r = \dfrac{\ln(\frac{1}{2})}{\tau_{1/2}} = \dfrac{\ln(1) - \ln(2)}{\tau_{1/2}} = \dfrac{-\ln(2)}{\tau_{1/2}}$.

$$\therefore A(t) = A_0 e^{rt} = A_0 e^{\left(\frac{-\ln(2)}{\tau_{1/2}}\right)t} = A_0 \left[e^{-\ln(2)}\right]^{t/\tau_{1/2}} = A_0 \left[e^{\ln(\frac{1}{2})}\right]^{t/\tau_{1/2}} = A_0 \left(\tfrac{1}{2}\right)^{t/\tau_{1/2}}.$$

$$\therefore \boxed{A(t) = A_0 e^{-rt} = A_0 \left(\tfrac{1}{2}\right)^{t/\tau_{1/2}}}$$

Ex. 2: Cesium-137 is a radioactive isotope of cesium with a half-life of 30 years. Do the following:

a) Find the decay constant "r" for Cesium-137.

$\tau_{1/2} = 30$ yrs $\Rightarrow r\tau_{1/2} = \ln(\tfrac{1}{2}) = -\ln(2) \Rightarrow r = \dfrac{-\ln(2)}{\tau_{1/2}} = \dfrac{-\ln(2)}{30} \approx -0.023104906$

b) Find (approx.) how much cesium-137 will be left in an unopened bottle 100 years from now.

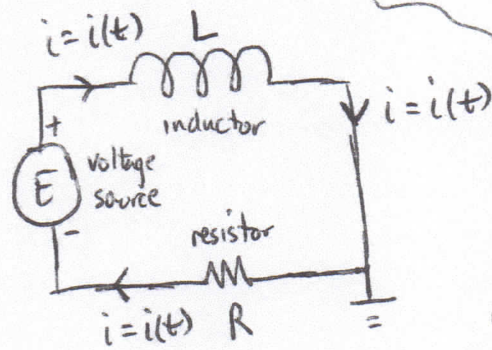
$$A(100) = A_0 e^{-\left(\frac{\ln(2)}{30}\right)(100)} = A_0 \left[e^{\ln(\frac{1}{2})}\right]^{\frac{100}{30}} = A_0 \left[\tfrac{1}{2}\right]^{10/3} \approx 0.0992125657\, A_0$$

$= 9.92\%$ of the original amount of Cesium-137.

(Electrical) Series Circuit problem(s)

Ex. 1: A series circuit contains a resistor and an inductor as shown below. Determine an ODE for the current $i(t)$ if the resistance is R, the inductance is L, and the (impressed) voltage is $E(t)$.

NOTE: Recall that according to Kirchhoff's Voltage Law (i.e. KVL), voltage $E(t)$ on a closed circuit loop must equal the sum of the voltage drops in the loop.



Also, note that ...

* Volt. drop across inductor: $L\frac{di}{dt}$
* Volt. drop across resistor: iR

$\therefore$ Voltage Source = (Volt. Drop Across Inductor) + (Volt. Drop Across Resistor)

$\Rightarrow E = E(t) = L\frac{di}{dt} + iR$, where L, R are constants!!

$\Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$ (NOTE: This ODE is 1st order Linear where $p = R/L$ and $r = E/L$)

NOTE: Solution to this equation (for $i = i(t)$) is $i(t) = e^{-h}\left[\int e^{h} \cdot r \cdot dx + C\right]$, where $h = \int p \cdot dt = \int \frac{R}{L}\,dt$!!!