

SIGNALS AND SYSTEMS USING MATLAB

Chapter 9 — Discrete-time Signals and Systems

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Discrete-time signals

A discrete-time signal $x[n]$ is a function of an integer sample index n

$$\boxed{\begin{array}{ccc} x[\cdot] : I & \rightarrow & R \\ & n & x[n] \end{array}} \quad (C)$$

Example: Continuous-time signal

$$x(t) = 3 \cos(2\pi t + \pi/4), \quad -\infty < t < \infty$$

Sampling: Nyquist sampling rate condition

$$T_s \leq \frac{\pi}{\Omega_{max}} = \frac{\pi}{2\pi} = 0.5 \text{ sec/sample}$$

For $T_s = 0.5$ sec/sample we obtain

$$x[n] = 3 \cos(2\pi t + \pi/4)|_{t=0.5n} = 3 \cos(\pi n + \pi/4) \quad -\infty < n < \infty$$

a function of the integer n

Periodic and aperiodic signals

- Signal $x[n]$ is periodic if
 - defined in $-\infty < n < \infty$, and
 - there is integer $N > 0$, the **fundamental period** of $x[n]$ such that

$$x[n + kN] = x[n] \quad \text{for any integer } k$$

- Aperiodic signal does not satisfy one or both of the above conditions
- Periodic discrete-time sinusoids, of fundamental period N :

$$x[n] = A \cos \left(\frac{2\pi m}{N} n + \theta \right) \quad -\infty < n < \infty$$

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Example: Continuous-time vs discrete-time sinusoids

$$x(t) = \cos(t + \pi/4), \quad -\infty < t < \infty, \text{ periodic of fundamental period } T_0 = 2\pi$$

Nyquist condition (i) $T_s \leq \frac{\pi}{\Omega_0} = \pi$

periodic sampled signal $x(t)|_{t=nT_s} = \cos(nT_s + \pi/4)$, fundamental period N if
 $\cos((n + N)T_s + \pi/4) = \cos(nT_s + \pi/4) \Rightarrow (ii) NT_s = 2k\pi$

For sinusoid with fundamental period $N = 10$, then

- (ii) $T_s = k\pi/5$, for k satisfying
- (i) $0 < T_s = k\pi/5 \leq \pi$ so that $0 < k \leq 5$

$k = 1, 3$ so that $N = 10$, $\omega = 2\pi k/10$, k, N not divisible by each other
 $k = 2, 4$ give $\omega = 2\pi/5, 2\pi 2/5, 5$ as fundamental period
 $k = 5$ give 2 as fundamental period, $\omega = 2\pi/2$

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- Sampling

$$x(t) = A \cos(\Omega_0 t + \theta) \quad -\infty < t < \infty, \text{ fundamental period } T_0$$

results in **periodic discrete sinusoid**

$$x[n] = A \cos(\Omega_0 T_s n + \theta) = A \cos\left(\frac{2\pi T_s}{T_0} n + \theta\right)$$

provided that

$$(i) \frac{T_s}{T_0} = \frac{m}{N}, \quad (ii) T_s \leq \frac{\pi}{\Omega_0} = \frac{T_0}{2} \quad (\text{Nyquist condition})$$

- $z[n] = x[n] + y[n]$, periodic $x[n]$ with fundamental period N_1 , periodic $y[n]$ with fundamental period N_2

$z[n]$ is periodic if

$$\frac{N_2}{N_1} = \frac{p}{q} \quad p, q \text{ integers not divisible by each other}$$

Fundamental period of $z[n]$ is $qN_2 = pN_1$

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Finite-energy and finite-power signals

Discrete-time signal $x[n]$

$$\text{Energy: } \varepsilon_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$\text{Power: } P_x = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2$$

- $x[n]$ is **finite energy** or **square summable** if $\varepsilon_x < \infty$
- $x[n]$ is **absolutely summable** if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- $x[n]$ is **finite power** if $P_x < \infty$

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Example: “Causal” sinusoid

$$x(t) = \begin{cases} 2 \cos(\Omega_0 t - \pi/4) & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is sampled using $T_s = 0.1$ to obtain

$$x[n] = x(t)|_{t=0.1n} = 2 \cos(0.1\Omega_0 n - \pi/4) \quad n \geq 0$$

and zero otherwise

- $\Omega_0 = \pi$

$$x[n] = 2 \cos(2\pi n/20 - \pi/4), \quad n \geq 0, \quad 0 \text{ otherwise} \quad \text{repeats every } N_0 = 20, \text{ for } n \geq 0$$

$$P_x = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=0}^N |x[n]|^2 = \lim_{N \rightarrow \infty} \frac{N}{2N+1} \underbrace{\left[\frac{1}{N_0} \sum_{n=0}^{N_0-1} |x[n]|^2 \right]}_{\text{average of } N_0 \text{ samples}} = \frac{1}{2N_0} \sum_{n=0}^{N_0-1} |x[n]|^2$$

$$= \frac{4}{40} 0.5 \left[\sum_{n=0}^{19} 1 + \sum_{n=0}^{19} \cos \left(\frac{2\pi n}{10} - \pi/2 \right) \right] = \frac{2}{40} [20 + 0] = 1$$

- $\Omega_0 = 3.2$ rad/sec (an upper approximation of π), $x[n]$ does not repeat periodically after $n = 0$, frequency $3.2/10 \neq 2\pi m/N$ for integers m and N

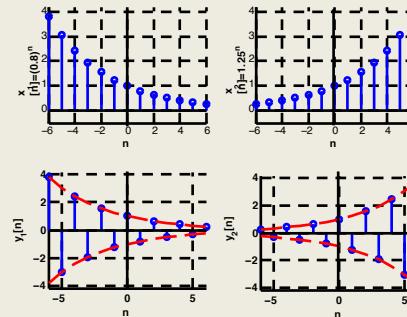
$$P_x = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2$$

Basic discrete-time signals

- Complex exponential

$$x[n] = |A|e^{j\theta}(|\alpha|e^{j\omega_0})^n = |A||\alpha|^n e^{j(\omega_0 n + \theta)}$$

$$= |A| |\alpha|^n [\cos(\omega_0 n + \theta) + j \sin(\omega_0 n + \theta)] \quad \omega_0 : \text{discrete frequency in radians}$$



Real exponential $x_1[n] = 0.8^n$, $x_2[n] = 1.25^n$ (top) and modulated $y_1[n] = x_1[n] \cos(\pi n)$ and $y_2[n] = x_2[n] \cos(\pi n)$

Example: $x(t) = e^{-at} \cos(\Omega_0 t)u(t)$, determine $a > 0$, Ω_0 and T_s to get

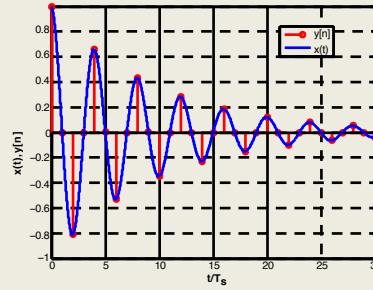
$$y[n] = 0.9^n \cos(\pi n/2) \quad n \geq 0 \text{ and zero otherwise}$$

$$(i) 0.9 = e^{-aT_s}, \quad (ii) \pi/2 = \Omega_0 T_s$$

$$(iii) T_s \leq \frac{\pi}{\Omega_{max}}, \quad \Omega_{max} = N\Omega_0, \quad N \geq 2 \quad x(t) \text{ not band-limited}$$

$$N = 2 \Rightarrow T_s = 0.25, \quad \Omega_0 = 2\pi$$

$$0.9 = e^{-a/4} \Rightarrow a = -4 \log 0.9$$



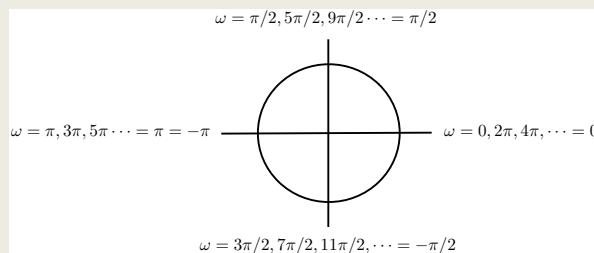
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Discrete-time sinusoids

Special case of complex exponential

$$x[n] = |A|e^{j(\omega_0 n + \theta)} = |A| \cos(\omega_0 n + \theta) + j|A| \sin(\omega_0 n + \theta)$$

- Periodic if $\omega_0 = 2\pi m/N$ (rad), integers m and $N > 0$ not divisible
- ω (radians) repeats periodically with 2π as fundamental period



- To avoid this ambiguity let $-\pi < \omega \leq \pi$

Example: signal $\sin(3\pi n) = \sin(\pi n)$; $\sin(1.5\pi n) = \sin(-0.5\pi n) = -\sin(0.5\pi n)$

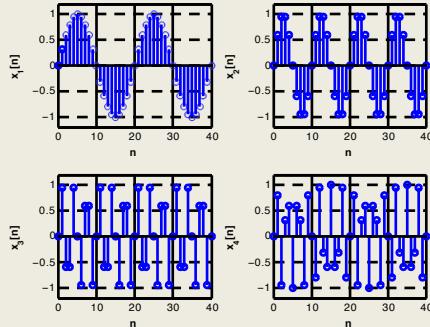
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Example: Consider the four sinusoids

$$x_1[n] = \sin(0.1\pi n) = \sin\left(\frac{2\pi}{20}n\right), \quad x_2[n] = \sin(0.2\pi n) = \sin\left(\frac{2\pi}{20}2n\right),$$

$$x_3[n] = \sin(0.6\pi n) = \sin\left(\frac{2\pi}{20}6n\right), \quad x_4[n] = \sin(0.7\pi n) = \sin\left(\frac{2\pi}{20}7n\right)$$

periodic of fundamental periods 20



Discrete-time unit-step and unit-sample signals

- Definitions

$$\text{Unit-step } u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\text{Unit-sample } \delta[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

- Connection

$$\delta[n] = u[n] - u[n - 1]$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n - k] = \sum_{m=-\infty}^n \delta[m]$$

- Generic representation of discrete-time signals

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k]$$

Discrete-time systems

Dynamic systems $S \{.\}$

$y[n] = S\{x[n]\}$

- Linear
- Time-invariant
- Stable
- Causal

System S is

- **Linear:** for inputs $x[n]$ and $v[n]$, and constants a and b , applies
superposition

$$S\{ax[n] + bv[n]\} = aS\{x[n]\} + bS\{v[n]\}$$

- **Time-invariant:** $x[n] \rightarrow y[n] = S\{x[n]\}$
 $x[n \pm M] \rightarrow y[n \pm M] = S\{x[n \pm M]\}$

Example: A square-root computation System

$$y[n] = 0.5 \left[y[n-1] + \frac{\alpha}{y[n-1]} \right] \quad n > 0$$

$$y[1] = 0.5 \left[y[0] + \frac{\alpha}{y[0]} \right], \quad y[2] = 0.5 \left[y[1] + \frac{\alpha}{y[1]} \right], \quad y[3] = 0.5 \left[y[2] + \frac{\alpha}{y[2]} \right], \quad \dots$$

Non-linear system: square root of 2 (top); square root of 4 compared with twice the square root of 2 (middle); sum of previous responses compared with response of square root of 2 + 4 (bottom)

Recursive and non-recursive systems

- Recursive/infinite impulse response (IIR) system

$$y[n] = - \sum_{k=1}^{N-1} a_k y[n-k] + \sum_{m=0}^{M-1} b_m x[n-m] \quad n \geq 0 \quad \text{initial conditions } y[-k], k = 1, \dots, N-1$$

- Non-recursive/finite impulse response (FIR) system

$$y[n] = \sum_{m=0}^{M-1} b_m x[n-m]$$

Example: Moving-average (MA) discrete system

$$y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2]), \quad \text{input } x[n], \text{ output } y[n]$$

- **Linearity:** system is linear

$$\begin{aligned} \{x_i[n], i = 1, 2\} &\rightarrow \{y_i[n], i = 1, 2\} \\ ax_1[n] + bx_2[n] &= \frac{1}{3}[(ax_1[n] + bx_2[n]) + (ax_1[n-1] + bx_2[n-1]) + (ax_1[n-2] + bx_2[n-2])] \\ &= ay_1[n] + by_2[n] \end{aligned}$$

- **Time invariance** system is time-invariant

$$\text{input } x_1[n] = x[n-N] \rightarrow \frac{1}{3}(x[n-N] + x[n-N-1] + x[n-N-2]) = y[n-N]$$

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Example: Autoregressive moving average (ARMA) system

$$y[n] = 0.5y[n-1] + x[n] + x[n-1] \quad n \geq 0, y[-1]$$

Initial condition $y[-1] = -2$, and the input $x[n] = u[n]$, recursively

$$\begin{aligned} y[0] &= 0.5y[-1] + x[0] + x[-1] = 0, \quad y[1] = 0.5y[0] + x[1] + x[0] = 2, \\ y[2] &= 0.5y[1] + x[2] + x[1] = 3, \quad \dots \end{aligned}$$

Initial condition $y[-1] = -2$, and input $x[n] = 2u[n]$ (doubled), response

$$\begin{aligned} y_1[0] &= 0.5y_1[-1] + x[0] + x[-1] = 1, \quad y_1[1] = 0.5y_1[0] + x[1] + x[0] = 4.5, \\ y_1[2] &= 0.5y_1[1] + x[2] + x[1] = 6.25, \quad \dots \end{aligned}$$

$$y_1[n] \neq 2y[n] \quad (\text{system non-linear})$$

If initial condition $y[-1] = 0$ the system is linear

Steady-state: if $x[n] = u[n]$, any $y[-1]$, assuming as $n \rightarrow \infty$ $Y = y[n] = y[n-1]$ and since $x[n] = x[n-1] = 1$, then

$$Y = 0.5Y + 2 \quad \text{or} \quad Y = 4 \quad \text{independent of IC}$$

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Convolution sum

$h[n]$ impulse response of LTI system: input $\delta[n]$, zero IC

$$\text{generic representation } x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$\text{output } y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{m=-\infty}^{\infty} x[n-m]h[m] \text{ convolution sum}$$

- Non-recursive or FIR system: output obtained by convolution sum

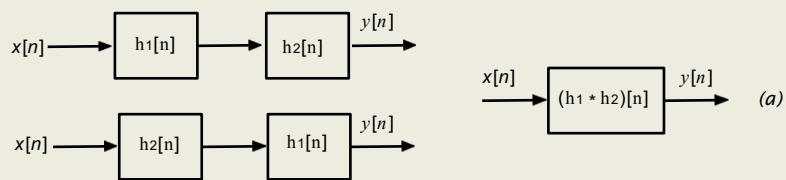
$$y[n] = \sum_{k=0}^{N-1} b_k x[n-k]$$

$$h[n] = \sum_{k=0}^{N-1} b_k \delta[n-k] = b_0 \delta[n] + b_1 \delta[n-1] + \cdots + b_{N-1} \delta[n-(N-1)]$$

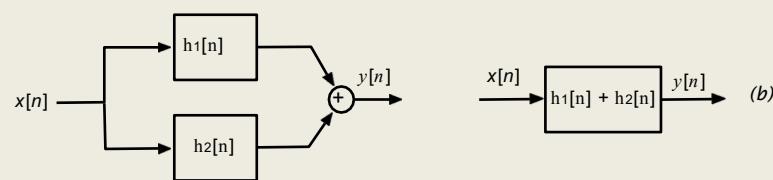
$$h[n] = b_n, \quad n = 0, \dots, N-1 \Rightarrow y[n] = \sum_{k=0}^{N-1} h[k]x[n-k]$$

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System interconnection



(a)



(b)

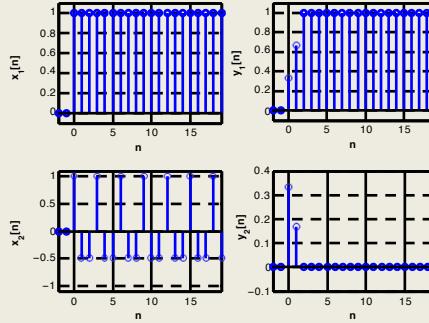
Cascade (a) and parallel (b) connections of LTI systems with impulse responses $h_1[n]$ and $h_2[n]$. Equivalent systems on the right. Notice the interchange of systems in the cascade connection.

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Example: FIR system

$$\text{I/O equation } y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$$

$$\text{impulse response } h[n] = \frac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2]) \Rightarrow y[n] = (h * x)[n]$$



Convolution sums for a moving averaging system $y[n] = (x[n] + x[n-1] + x[n-2])/3$

- with inputs $x_1[n] = u[n]$ (top) and $x_2[n] = \cos(2\pi n/3)u[n]$ (bottom).

Example: Autoregressive system

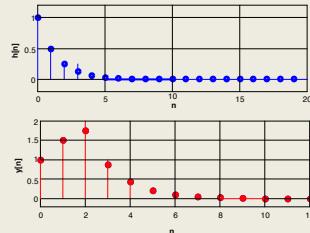
$$y[n] = 0.5y[n-1] + x[n] \quad n \geq 0 \quad \text{first order difference equation}$$

impulse response $x[n] = \delta[n]$, $y[n] = h[n]$, initial condition $y[-1] = h[-1] = 0$

$$\begin{aligned} h[0] &= 0.5h[-1] + \delta[0] = 1, & h[1] &= 0.5h[0] + \delta[1] = 0.5, \\ h[2] &= 0.5h[1] + \delta[2] = 0.5^2, & h[3] &= 0.5h[2] + \delta[3] = 0.5^3, \dots & h[n] &= 0.5^n \end{aligned}$$

Input $x[n] = u[n] - u[n-3]$ using convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} (u[k] - u[k-3])0.5^{n-k}u[n-k] = \begin{cases} 0 & n < 0 \\ 0.5^n(2^{n+1} - 1) & n = 0, 1, 2 \\ 7(0.5^n) & n \geq 3. \end{cases}$$



Impulse response $h[n]$ (top), and response $y[n]$ due to $x[n] = u[n] - u[n-3]$ (bottom)

Causality and BIBO stability

- **Causality:** System S is **causal** if:
 - input $x[n] = 0$, and no initial conditions, the output is $y[n] = 0$,
 - present output $y[n]$ does not depend on future inputs
- **BIBO stability:** for bounded $x[n]$, $|x[n]| < M < \infty$, the output of BIBO stable system $y[n]$ is also bounded, $|y[n]| < L < \infty$, for all n .
- **LTI systems:**
 - Causality: $h[n] = 0$ for $n < 0$
 - BIBO stability
$$\sum_k |h[k]| < \infty, \quad \text{absolutely summable}$$

Example:

- Causal non-linear time-invariant system

$$y[n] = x^2[n],$$

- Non-causal LTI system

$$y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1]).$$

Example: Deconvolution

Assume input $x[n]$ and output $y[n]$ of a causal LTI system are given, find impulse response $h[n]$ of the system

$$\begin{aligned} y[n] &= \sum_{m=0}^n h[n-m]x[m] = h[n]x[0] + \sum_{m=1}^n h[n-m]x[m] \\ h[n] &= \frac{1}{x[0]} \left[y[n] - \sum_{m=1}^n h[n-m]x[m] \right], \quad x[0] \neq 0 \\ h[0] &= \frac{1}{x[0]}y[0], \quad h[1] = \frac{1}{x[0]}(y[1] - h[0]x[1]), \quad h[2] = \frac{1}{x[0]}(y[2] - h[0]x[2] - h[1]x[1]) \quad \dots \end{aligned}$$

Let $y[n] = \delta[n]$, $x[n] = u[n]$:

$$h[0] = 1, \quad h[1] = -1, \quad h[2] = 0, \quad h[3] = 0, \quad \dots$$

$$h[n] = \delta[n] - \delta[n-1] \quad \text{length of } h[n] = (\text{length of } y[n]) - (\text{length of } x[n]) + 1$$

Example: Autoregressive system

$$y[n] = 0.5y[n-1] + x[n]$$

Impulse response

$$\begin{aligned} h[0] &= 0.5h[-1] + \delta[0] = 1, \\ h[1] &= 0.5h[0] + \delta[1] = 0.5, \\ h[2] &= 0.5h[1] + \delta[2] = 0.5^2 \\ h[3] &= 0.5h[2] + \delta[3] = 0.5^3, \quad \dots \\ &\Rightarrow h[n] = 0.5^n u[n] \\ \sum_{n=-\infty}^{\infty} |h[n]| &= \sum_{n=0}^{\infty} 0.5^n = \frac{1}{1-0.5} = 2 \end{aligned}$$

$h[n]$ is absolutely summable, so system is BIBO stable

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Two-dimensional Discrete Signals

A discrete two-dimensional signal $x[m, n]$ is a mapping of integers $[m, n]$ into real values, e.g., a discrete image is a mapping of space locations $0 \leq m \leq M$, $0 \leq n \leq N$ into positive pixel values.

Generic representation

A signal $x[m, n], [m, n] \in [M_1, N_1] \times [M_2, N_2]$, $M_1 < M_2$, $N_1 < N_2$ is represented as

$$x[m, n] = \sum_{k=M_1}^{M_2} \sum_{\ell=N_1}^{N_2} x[k, \ell] \delta[m - k, n - \ell]$$

using the two-dimensional impulse

$$\delta[m, n] = \begin{cases} 1 & [m, n] = [0, 0] \\ 0 & [m, n] \neq [0, 0] \end{cases}$$

Example: Two-dimensional unit-step and ramp with support in the first quadrant:

$$u_1[m, n] = \begin{cases} 1 & m \geq 0, n \geq 0 \\ 0 & \text{otherwise} \end{cases} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \delta[m - k, n - \ell]$$

$$r_1[m, n] = \begin{cases} mn & m \geq 0, n \geq 0 \\ 0 & \text{otherwise} \end{cases} = \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} k\ell \delta[m - k, n - \ell].$$

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