8.7 Improper Integrals

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

$$\int_0^{\infty} \frac{1}{x^2}\,dx, \quad \int_0^{\infty} e^{-x}\,dx$$

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$$\int_0^2 \frac{1}{x-1}\,dx$$

Type I:

(1) If $f(x)$ is continuous on $[a, \infty)$

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then $\int_a^{\infty} f(x)\,dx$

$$= \lim_{b\to\infty} \int_a^b f(x)\,dx$$

(2) If $f(x)$ is continuous on $(-\infty, b]$

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then $\int_{-\infty}^{b} f(x)\,dx$

$$= \lim_{a\to-\infty} \int_a^b f(x)\,dx$$

(3) If $f(x)$ is continuous on $(-\infty, \infty)$, then

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$$\int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

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If the limit is finite (or exists), then we say the improper integral converges (or convergent)

If the limit fails to exists, then it is divergent or diverges.

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$\boxed{\text{Type II:}}$

① If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then $\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx$

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② If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx$$

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③ If $f(x)$ is discontinuous at c where $a < c < b$ and continuous on $[a, c) \cup (c, b]$, then

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$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

If the limit exists, then the improper integral is

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<u>convergent</u> and If the limit doesnot exists, then <u>divergent</u>

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Ex $\int_0^\infty e^{-x}\,dx$

$$= \lim_{a \to \infty} \int_0^a e^{-x}\,dx$$

$$= \lim_{a \to \infty} -e^{-x}\Big|_0^a$$

$$= \lim_{a \to \infty} -e^{-a} + e^{-0}$$

$$= 0 + 1 = 1$$

$\Rightarrow$ convergent

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(ex) $\int_1^\infty \frac{\ln x}{x^{2}}\,dx$

$\lim_{a\to\infty} \int_1^a \frac{\ln x}{x^2}\,dx$

$u = \ln x \qquad dv = \frac{1}{x^2}dx \to x^{-2}$

$du = \frac{1}{x}dx \qquad v = -\frac{1}{x}$

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$\lim_{a\to\infty}\left[-\frac{\ln x}{x}\Big|_1^a + \int_1^a \frac{1}{x^2}\,dx \right]$

$= \lim_{a\to\infty}\left[-\frac{\ln a}{a} + \frac{\overset{0}{\cancel{\ln 1}}}{1} + \left[-\frac{1}{x}\right]_1^a \right]$

$= \lim_{a\to\infty}\left[-\frac{\ln a}{a} - \frac{1}{a} + 1 \right] -$

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$\frac{\ln a}{a} \left(\frac{\infty}{\infty}\right)$

$\underline{\underline{L'H}} \quad \frac{\frac{1}{a}}{1} = \frac{1}{a}$

$\lim_{a\to\infty} \frac{1}{a} = 0$

$\to = 0 - 0 + 1 = \boxed{1}$

convergent

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(Ex) $\int_2^\infty \frac{1}{x}\,dx$

$\lim_{a\to\infty} \int_2^a \frac{1}{x}\,dx$

$= \lim_{a\to\infty} \ln|x| \Big|_2^a$

$= \lim_{a\to\infty} \ln|a| - \ln|2| = \infty$ divergent

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(Ex) $\int_2^\infty \frac{1}{x^2}\,dx$

$\lim_{a\to\infty} \int_2^a \frac{1}{x^2}\,dx$

$= \lim_{a\to\infty} -\frac{1}{x}\Big|_2^a$

$= \lim_{a\to\infty} -\frac{1}{a} + \frac{1}{2} = 0 + \frac{1}{2}$

$= \frac{1}{2}$

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$\int_a^\infty \frac{1}{x^p}\,dx$

$p > 1 \to$ converges

$p \le 1 \to$ divergent

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(Ex) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$

$\displaystyle\int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx$

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$\displaystyle\int_{0}^{\infty} \frac{1}{x^2+1}\,dx$

$\displaystyle= \lim_{a\to\infty} \int_{0}^{a} \frac{1}{x^2+1}\,dx$

$\displaystyle= \lim_{a\to\infty} \tan^{-1}x \Big|_{0}^{a}$

$\displaystyle= \lim_{a\to\infty} \tan^{-1}a - \tan^{-1}0$

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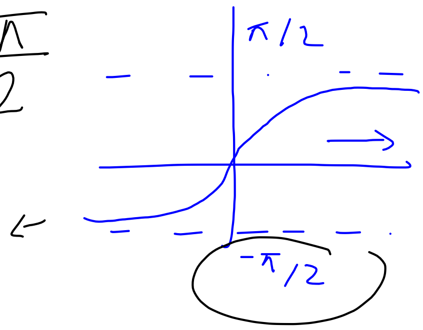
$= \tan^{-1}(\infty) - 0$

$= \dfrac{\pi}{2}$



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$\displaystyle\int_{-\infty}^{0} \frac{1}{x^2+1}\,dx$

$\displaystyle= \tan^{-1}a \Big|_{-\infty}^{0}$

$= \tan^{-1}(0) - \tan^{-1}(-\infty)$

$= 0 - \left(-\dfrac{\pi}{2}\right) = \dfrac{\pi}{2}$

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$\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \frac{\pi}{2} + \frac{\pi}{2}$

$= \boxed{\pi}$

Convergent

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$\displaystyle\int_{0}^{1} \frac{1}{\sqrt{x}}\,dx$

$\displaystyle= \lim_{a\to 0^+} \int_{a}^{1} \frac{1}{\sqrt{x}}\,dx \quad \to x^{-1/2}$

$\displaystyle= \lim_{a\to 0^+} 2\sqrt{x} \Big|_{a}^{1}$

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$$= \lim_{a \to 0^+} \left[2\sqrt{1} - 2\sqrt{a} \right]$$

$$= 2 - 2\sqrt{0}$$

$$= \boxed{2}$$

Convergent

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Ex) $\int_0^1 \frac{1}{1-x}\,dx$

$$\lim_{b \to 1^-} \int_0^b \frac{1}{1-x}\,dx$$

$$= \lim_{b \to 1^-} -\ln|1-x| \Big|_0^b$$

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$$= \lim_{b \to 1^-} \left[-\ln|1-b| + \ln|1-\cancel{0}| \right]$$

0

$$= \infty$$

$\Rightarrow$ divergent

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