CS 214
Introduction to Discrete Structures

Chapter 4

Sets, Combinatorics, and Probability

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Chapter sections and objectives

- 4.1 Sets
  - Use the notation of set theory
  - Find the power set of a finite set
  - Find the union, intersection, difference, complement, and Cartesian product of sets
  - Identify binary and unary operations on sets
  - Prove set identities
  - Recognize that not all sets are countable
- 4.2 Counting
  - Apply the multiplication and addition principles
  - Use decision trees to solve counting problems

- 4.3 Principle of Inclusion and Exclusion
  - Use the principle of inclusion and exclusion to find the number of elements in the union of sets
  - Use the pigeonhole principle to decide when certain common events must occur
- 4.4 Permutations and Combinations
  - Use the formulas for permutations and combinations
  - Find the number of distinct permutations of objects that are not all distinct
- 4.5 Binomial Theorem
- 4.6 Probability
  - Find the probability of an event
  - Compute the expected value of an event

Sections 4.3–4.6 not covered in CS 214

Sample problem

You survey the 87 subscribers to your newsletter in preparation for the release of your new software product. The results of your survey reveal that 68 have a Windows system available to them, 34 have a Unix system available, and 30 have access to a Mac. In addition,
19 have access to both Windows and Unix systems,
11 have access to both Unix systems and Macs,
and 23 can use both Macs and Windows.

How many have access to all three types of systems?

4.1 Sets

Set definitions, notation, and properties

- Definition
  - Set; collection of objects
  - Object in set called element
  - Elements in a set often share some common property
- Notation
  - Uppercase letters denote set, e.g., set S
  - Lowercase letters denote elements, e.g., element k
  - Braces { } indicate set, e.g., $S = \{\mathrm{k, s, p}\}$
  - Symbol $\in$ denotes set membership, e.g., $\mathrm{k} \in S$, $\mathrm{m} \notin S$
  - e.g, $A = \{\text{violet, chartreuse, burnt umber}\}$, $\text{chartreuse} \in A$, $\text{magenta} \notin A$
  - e.g, $A = \{1, 3, 5, 7, \ldots\}$, $9 \in A$, $8 \notin A$

Example 1

- Properties of sets
  - Uniqueness; an element may only occur once in a set, e.g., $A = \{0, 1\}$ and $B = \{0, 1, 1\}$ are the same set
  - Non-order; set elements have no order, e.g., $A = \{0, 1\}$ and $B = \{1, 0\}$ are the same set
  - Equality; sets containing the same elements are equal, e.g., if $A = \{\text{b, j, s}\}$ and $B = \{\text{j, s, b}\}$ then $A = B$ i.e., $A = B$ means $(\forall x)[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$
  - Sets may be finite or infinite (number of elements)

Sets of sets and empty sets

- Sets may be elements of sets
  - e.g., $\{0, \{1\}, \{2, \{3\}\}, 4\}$
- Sets may have no elements, i.e., “empty set”
  - Written $\varnothing$, or sometimes $\{ \}$, but not $\{\varnothing\}$

Number sets

- $\mathbb{N}$ = "natural numbers"; set of all nonnegative integers ($0 \in \mathbb{N}$), i.e., $\{0, 1, 2, 3, \ldots\}$, e.g., $86 \in \mathbb{N}$
- $\mathbb{Z}$ = set of all integers, i.e. $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, e.g., $+86 \in \mathbb{Z}$
- $\mathbb{Q}$ = set of all rational numbers, e.g., $86/1 \in \mathbb{Q}$
- $\mathbb{R}$ = set of all real numbers, e.g., $86\pi \in \mathbb{R}$
- $\mathbb{C}$ = set of all complex numbers, e.g., $86 + 12i \in \mathbb{C}$

Specifying sets

- Specifying (identifying elements of) sets
  - Exhaustively list all elements
  - Partially list elements to show pattern
  - Use recursion to describe how to generate elements
  - Describe property P that characterizes elements, informally using words
  - Describe property P that characterizes elements, formally using equations and/or logic
- Property P
  - Set whose elements have property P: $\{x \mid P(x)\}$
  - $P(x)$ is unary predicate
  - $S = \{x \mid P(x)\}$ means $(\forall x)[(x \in S \rightarrow P(x)) \wedge (P(x) \rightarrow x \in S)]$

Examples of specifying sets

From characterizing property to exhaustive list

5a. $\{x \mid x \in \mathbb{N} \text{ and } x^2 < 25\}$
$x < 0 \rightarrow x \notin \mathbb{N}$
$\{0, 1, 2, 3, 4\}$

8a. $\{x \mid x \in \mathbb{N} \text{ and } (\exists q)(q \in \{2, 3\} \text{ and } x = 2q)\}$
$\{4, 6\}$

9. $A = \{2, 4, 8, \ldots\}$
Is $16 \in A$?
Yes: $A = \{x \mid x = 2^n \text{ for } n \text{ a positive integer}\}$
No: $A = \{x \mid x = 2 + n(n - 1) \text{ for } n \text{ a positive integer}\}$
Not enough information

Exercises 5, 8, 9

From characterizing property to exhaustive list

$A = \{x \mid (\exists y)(y \in \{0, 1, 2\} \text{ and } x = y^3)\}$
$A = \{x \mid x \text{ is a cube of } 0, 1, \text{ or } 2\}$
$A = \{0, 1, 8\}$

$B = \{x \mid x \in \mathbb{N} \text{ and } (\exists y)(y \in \mathbb{N} \text{ and } x \leq y)\}$
$B = \{x \mid x \text{ is nonnegative integer} \leq \text{some nonnegative integer}\}$
$B = \mathbb{N}$

$C = \{x \mid x \in \mathbb{N} \text{ and } (\forall y)(y \in \mathbb{N} \rightarrow x \leq y)\}$
$C = \{x \mid x \text{ a nonnegative integer} \leq \text{all nonnegative integers}\}$
$C = \{0\}$

Example 2

From exhaustive list to characterizing property

a. $\{1, 4, 9, 16\}$
$\{x \mid x \text{ is one of the first four perfect squares}\}$
$\{x \mid x = y^2 \wedge 1 \leq y \leq 4\}$

b. $\{\text{butcher, baker, candlestick maker}\}$
$\{x \mid x \text{ is one of the “Three Men in a Tub”}\}$

c. $\{2, 3, 5, 7, 11, 13, 17, \ldots\}$
$\{x \mid x \text{ is a prime number}\}$
$\{x \mid ((\exists y)(y \neq 1 \wedge y \neq x \wedge y|x))'\}$

Practice 2

Subsets

- Subset
  - Set A is a subset of set B iff every element of A is an element of B
  - e.g., $A = \{2, 3, 5, 12\}$, $B = \{2, 3, 4, 5, 9, 12\}$
    A is a subset of B
  - Notation: $A \subseteq \mathrm{B}$
  - $A \subseteq B$ iff $(\forall x)(x \in A \rightarrow x \in B)$
- Proper subset
  - Set A is a proper subset of B iff A is a subset of B and there is at least one element of B not in A
  - Notation: $A \subset B$
  - $A \subset B$ iff $A \subseteq B$ and $A \neq B$
  - $A \subset B \rightarrow A \subseteq B$ as in example above

Example subsets

$A = \{1, 7, 9, 15\}$
$B = \{7, 9\}$
$C = \{7, 9, 15, 20\}$

True

$B \subseteq C$	$B \subseteq A$	$B \subset A$	$A \nsubseteq C$
$15 \in C$	$\{7, 9\} \subseteq B$	$\{7\} \subset A$	$\varnothing \subseteq C$

False

$C \subseteq B$	$\{7, 9\} \subset B$	$A \subseteq B$	$\varnothing \subset \varnothing$

Example 3

$A = \{x \mid x \in \mathbb{N} \text{ and } x \geq 5\}$ $= \{5, 6, 7, 8, \ldots\}$

$B = \{10, 12, 16, 20\}$

$C = \{x \mid (\exists y)(y \in \mathbb{N} \text{ and } x = 2y)\}$ $= \{0, 2, 4, 6, \ldots\}$

a. $B \subseteq C$ true

c. $A \subseteq C$ false

e. $\{11, 12, 13\} \subseteq A$ true

g. $\{12\} \in B$ false (but $12 \in B$)

i. $\{x \mid x \in \mathbb{N} \text{ and } x < 20\} \nsubseteq B$ true

k. $\{\varnothing\} \subseteq B$ false (but $\varnothing \subseteq B$)

Practice 6

Relationship of number sets

- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$
- Each number set is a proper subset of the next
- e.g., $86 \in \mathbb{N}$, $+86 \in \mathbb{Z}$, $86/1 \in \mathbb{Q}$, $86.0 \in \mathbb{R}$, $86 + 0i \in \mathbb{C}$
- e.g., $-86 \in \mathbb{Z}$ but $\notin \mathbb{N}$, $86/5 \in \mathbb{Q}$ but $\notin \mathbb{Z}$, $86\pi \in \mathbb{R}$ but $\notin \mathbb{Q}$, $86 + 12i \in \mathbb{C}$ but $\notin \mathbb{R}$
- Mnemonic: "Nine Zulu Queens Ruled China."

Proving set relationships

- May want to prove
  - Subset $A \subseteq B$
  - Proper subset $A \subset B$
  - Equality $A = B$
- Proving subsets
  - If the elements of a set have some property, the elements of a subset of the set have the property
  - If $B = \{x \mid P(x)\}$ and $A \subseteq B$, then $(\forall x)(x \in A \rightarrow P(x))$
  - This can be used to prove $A \subseteq B$, $A \subset B$
- Proving equality
  - To prove $A = B$, prove $A \subseteq B$ and $B \subseteq A$

Example subset proof

$B = \{x \mid x \text{ is a multiple of } 4\}$
$A = \{x \mid x \text{ is a multiple of } 8\}$

Theorem
$A \subseteq B$

Proof
Let $x \in A$. We must show $x \in B$.
Because $x \in A$, $x = m \cdot 8$ for integer m.
$x = m \cdot 8 = m \cdot 2 \cdot 4 = k \cdot 4$, where $k = 2m$ an integer.
Therefore x is a multiple of 4, thus $x \in B$. ■

To prove $A \subset B$, prove $A \subseteq B$ and exhibit $x \in B$, $x \notin A$; e.g., $12 \in B$, $12 \notin A$.

Example 4

Example subset proof

$A = \{x \mid x \in \mathbb{R} \text{ and } x^2 - 4x + 3 = 0\} \quad = \{1, 3\}$
$B = \{x \mid x \in \mathbb{N} \text{ and } 1 \leq x \leq 4\} \quad = \{1, 2, 3, 4\}$

Theorem
$A \subset B$

Proof
Let $x \in A$. We must show $x \in B$.
Because $x \in A$, $x \in \mathbb{R}$ and $x^2 - 4x + 3 = 0$, so $x = 1$ or $x = 3$.
In both cases, $x \in \mathbb{N}$ and $1 \leq x \leq 4$, so $x \in B$, therefore $A \subseteq B$.
$4 \in B$ but $4 \notin A$. Thus $A \subset B$. ■

Practice 7

Example set equality proof

$A = \{x \mid x \in \mathbb{N} \text{ and } x^2 < 15\} \quad = \{0, 1, 2, 3\}$

$B = \{x \mid x \in \mathbb{N} \text{ and } 2x < 7\} \quad = \{0, 1, 2, 3\}$

Theorem

$A = B$

Proof

($A \subseteq B$) Let $x \in A$. Because $x \in A$, $x \in \mathbb{N}$ and $x^2 < 15$,
so $x = 0, 1, 2,$ or 3 ($4^2 = 16 > 15$).
For each case, $2x < 7$, so $x \in B$, therefore $A \subseteq B$.
($B \subseteq A$) Let $x \in B$. Because $x \in B$, $x \in \mathbb{N}$ and $2x < 7$,
so $x = 0, 1, 2,$ or 3 ($2 \cdot 4 = 8 > 7$).
For each case, $x^2 < 15$, so $x \in A$, therefore $B \subseteq A$.
$A \subseteq B$ and $B \subseteq A$, thus $A = B$. ■

Example 5

Power sets

- Definition
  - Set of all subsets (⊆) of a set
  - Notation $\wp(A)$
  - e.g., $A = \{0, 1\}$, $\wp(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
  - $\wp(A)$ is a set, elements of $\wp(A)$ are sets
  - $x \in \wp(A) \leftrightarrow x \subseteq A$
- Size of $\wp(A)$
  - If A has n elements, $\wp(A)$ has 2^n elements
  - Each element of A may be in or out of subset of A

Example 6, Practice 9

Example power set

$A = \{1, 2, 3\}$

	1	2	3	
$\wp(A) = \{ \varnothing,$	0	0	0	= 0
$\{3\},$	0	0	1	= 1
$\{2\},$	0	1	0	= 2
$\{2, 3\},$	0	1	1	= 3
$\{1\},$	1	0	0	= 4
$\{1, 3\}$	1	0	1	= 5
$\{1, 2\}$	1	1	0	= 6
$\{1, 2, 3\} \}$	1	1	1	= 7

Binary operations

- Samples
  - Binary operation; addition $x + y$, subtraction $3 - 5$
  - Unary operation: negation -8
- Concepts
  - Operations are defined on sets (above on $\mathbb{Z}$)
  - Binary operation operands are ordered pairs, e.g., $(3, 5)$
- Definition
  - Binary operation; $\circ$ is a binary operation on set S if for every ordered pair (x, y) of elements of S, $x \circ y$ exists, is unique, and is an element of S
  - Well-defined; result exists and is unique
  - Closed; result is in set S

Example valid binary operations

Operation	Symbol	Set	Exists	Unique	Closed
addition	$+$	$\mathbb{Z}$	yes	yes	yes
subtraction	$-$	$\mathbb{Z}$	yes	yes	yes
multiplication	$\cdot$	$\mathbb{Z}$	yes	yes	yes
conjunction	$\wedge$	wffs	yes	yes	yes
disjunction	$\vee$	wffs	yes	yes	yes
implication	$\rightarrow$	wffs	yes	yes	yes
equivalence	$\leftrightarrow$	wffs	yes	yes	yes

Examples 7, 8

Example invalid binary operations

- Candidate $\circ$ for operation fails if for some $x, y \in S$
  - $x \circ y$ does not exist (not defined)
  - $x \circ y$ not unique
  - $x \circ y$ not in S

	Operation	Symbol	Set	Exists	Unique	Closed
1	subtraction	$-$	$\mathbb{N}$	yes	yes	no
2	no name	$\circ$	$\mathbb{N}$	yes	no	yes
3	division	$\div$	$\mathbb{Z}$	no	yes	no
4	division	$\div$	$\mathbb{Z} - \{0\}$	yes	yes	no

1 $1 - 2 = -1 \notin \mathbb{N}$

3 $1 \div 0$ not defined

4 $1 \div 2 \notin \mathbb{Z}$ or $\mathbb{Z} - \{0\}$

2
$$x \circ y = \begin{cases} 1 & \text{if } x \geq 5 \\ 0 & \text{if } x \leq 5 \end{cases}$$

$$x = 5 \rightarrow x \circ y = 1 \text{ and } 0$$

Examples 9, 10, 11

Unary operations

- Samples
  - Binary operation; addition $x + y$, subtraction $3 - 5$
  - Unary operation: negation -8
- Concepts
  - Operations are defined on sets (above on $\mathbb{Z}$)
  - Unary operation operand is singleton, e.g., 3
- Definition
  - Unary operation; $\#$ is a unary operation on set S if for every element x of S,
    $x^{\#}$ exists, is unique, and is an element of S
  - Well-defined; result exists and is unique
  - Closed; result is in set S

Example valid and invalid unary operations

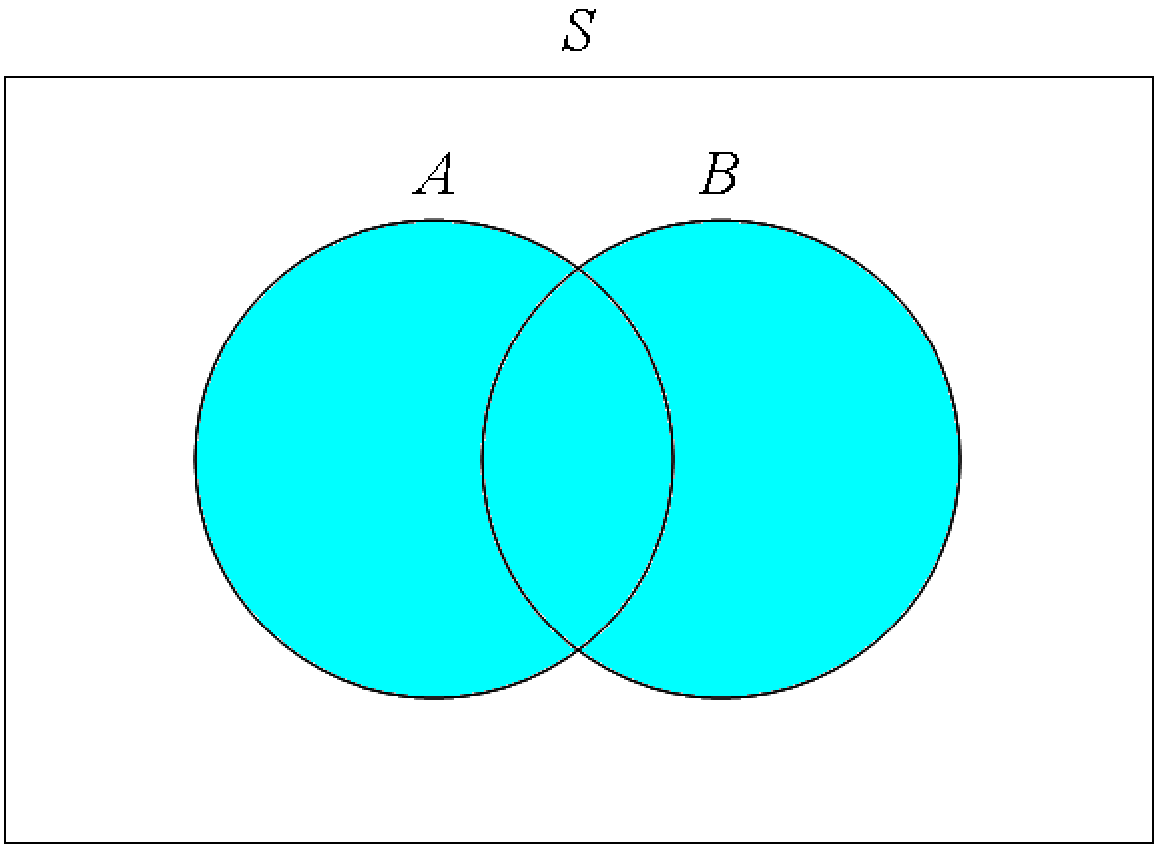
Operation	Symbol	Set	Exists	Unique	Closed
arithmetic negation	–	$\mathbb{Z}$	yes	yes	yes
arithmetic negation	–	$\mathbb{N}$	yes	yes	no
logical negation	′	wffs	yes	yes	yes

Examples 12, 13

Operations on sets

- Operations
  - Binary: union, intersection, difference
  - Unary: complement
  - Other: cartesian product
- Context for operations
  - Operations are defined within set S, aka universal set
  - Operands and results are elements of $\wp(S)$, i.e., subsets of S
  - Binary operations act on any two subsets of S to produce a subset of S
  - Unary operations act on any subset of S to produce a subset of S

Union concept



$A \cup B$

Figure 4.1

Union details

- Description
  - The union of sets A and B is the set containing all elements in either one or both of A and B.
- Details
  - Operands $A, B \in \wp(S)$; binary operation
  - Notation $A \cup B$
  - Definition $A \cup B = \{x \mid x \in A \vee x \in B\}$
  - Example $S = \mathbb{N}$

    $A = \{1, 3, 5, 7, 9\}$

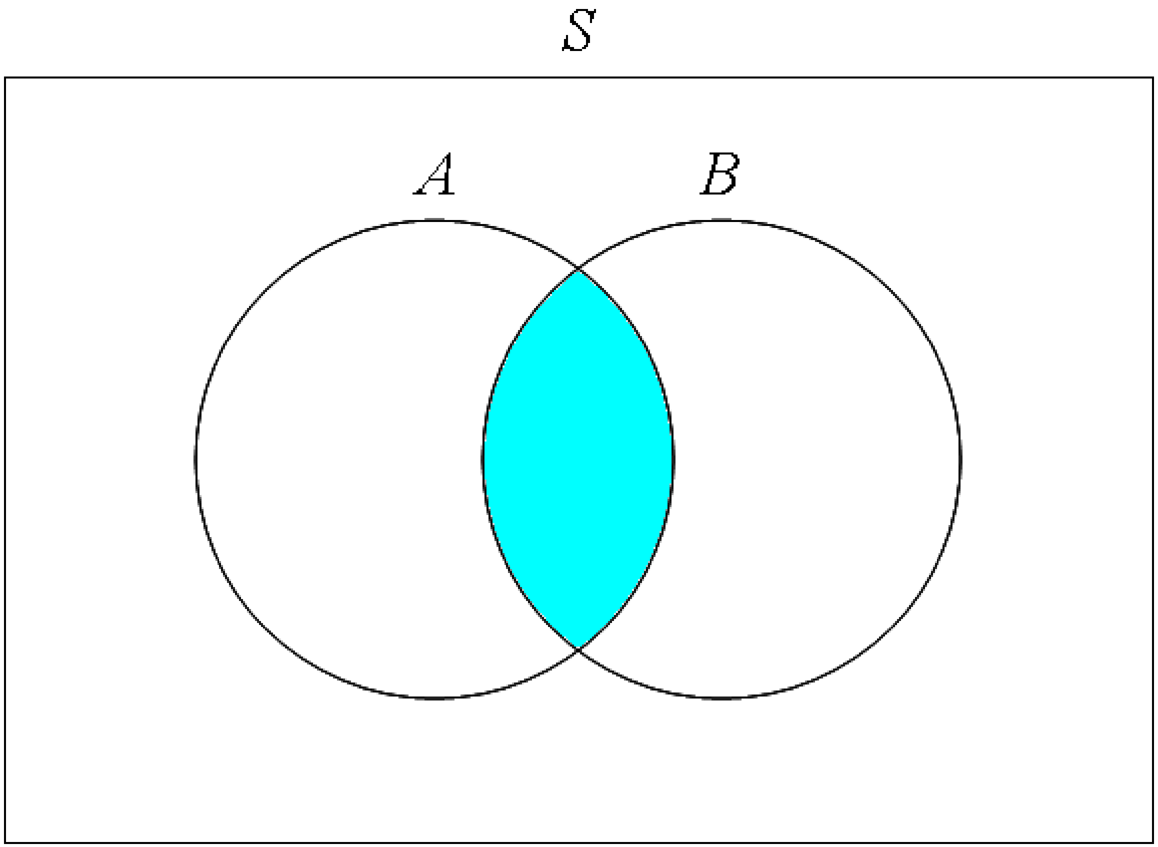
    $B = \{3, 5, 6, 10, 11\}$

    $A \cup B = \{1, 3, 5, 6, 7, 9, 10, 11\}$

    $A \cup B \in \wp(S)$, i.e., $\wp(\mathbb{N})$

Example 16

Intersection concept



$A \cap B$

Figure 4.2

Intersection details

- Description
  - The intersection of sets A and B is the set containing all elements in both A and B.
- Details
  - Operands $A, B \in \wp(S)$; binary operation
  - Notation $A \cap B$
  - Definition $A \cap B = \{x \mid x \in A \wedge x \in B\}$
  - Example $S = \mathbb{N}$

    $A = \{1, 3, 5, 7, 9\}$

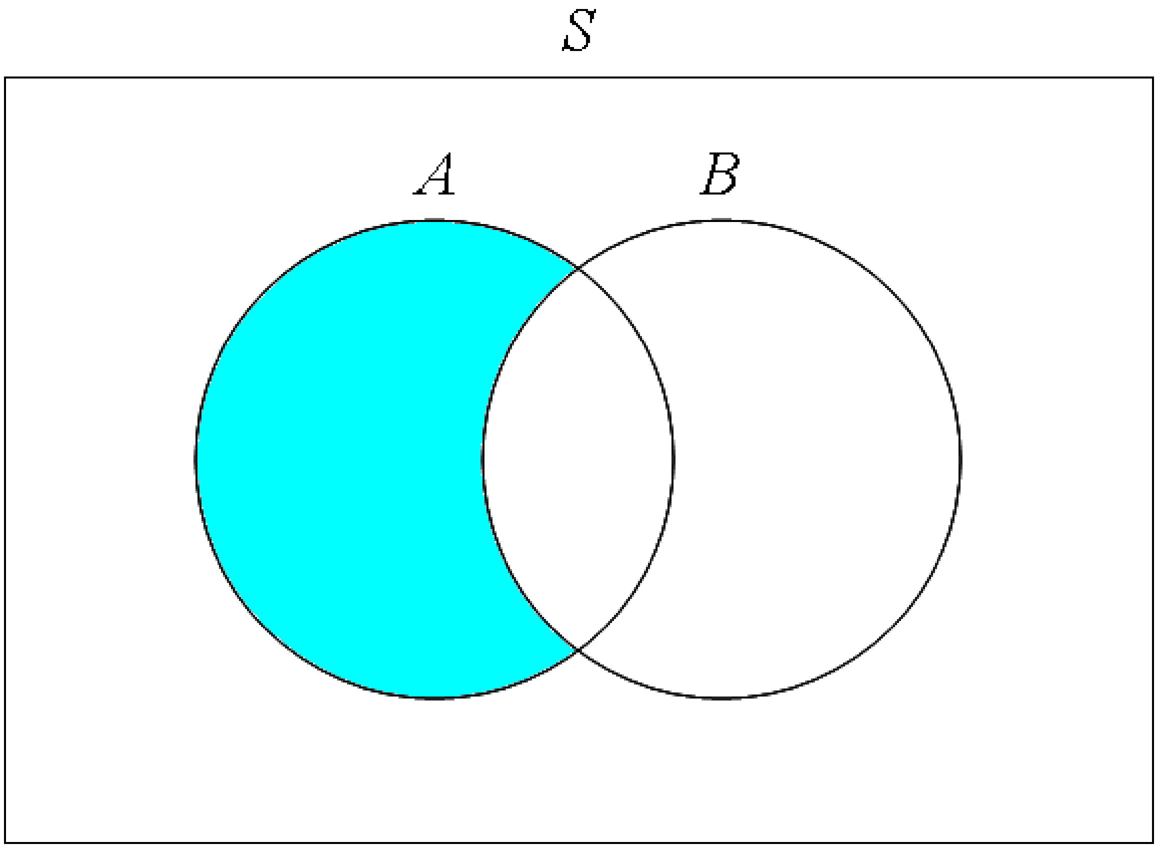
    $B = \{3, 5, 6, 10, 11\}$

    $A \cap B = \{3, 5\}$

    $A \cap B \in \wp(S)$, i.e., $\wp(\mathbb{N})$

Example 16

Difference concept



$A - B$

Practice 15

Difference details

- Description
  - The difference of sets A and B is the set containing all elements in A that are not in B.
- Details
  - Operands $A, B \in \wp(S)$; binary operation
  - Notation $A - B$
  - Definition $A - B = \{x \mid x \in A \wedge x \notin B\}$
  - Example $S = \mathbb{N}$

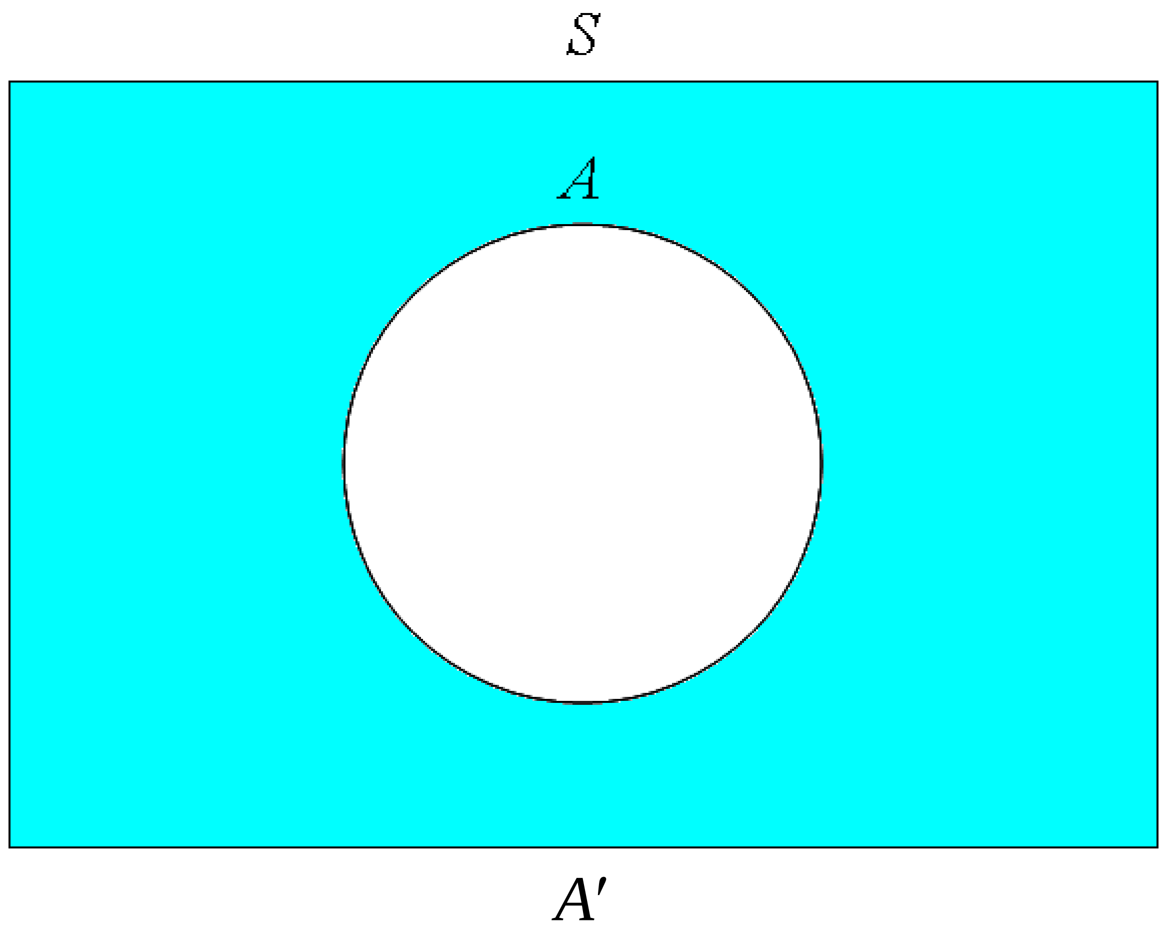
    $A = \{1, 3, 5, 7, 9\}$

    $B = \{3, 5, 6, 10, 11\}$

    $A - B = \{1, 7, 9\}$

    $A - B \in \wp(S)$, i.e., $\wp(\mathbb{N})$

Complement concept



Practice 14

Complement details

- Description
  - The complement of set A is the set containing all elements in universal set S that are not in A.
- Details
  - Operands $A \in \wp(S)$; unary operation
  - Notation A'
  - Definition $A' = \{x \mid x \in S \wedge x \notin A\}$
  - Example $S = \mathbb{N}$
    $A = \{0, 2, 4, 6, 8, \ldots\}$
    $A' = \{1, 3, 5, 7, 9, \ldots\}$
    $A, A' \in \wp(S)$, i.e., $\wp(\mathbb{N})$

Notes and definitions

- Set facts (always true)
  - $A \cap B \subseteq A \cup B$
  - $A - B = A \cap B'$
- Definitions
  - Two sets A and B are disjoint if they have no common elements, i.e., $A \cap B = \varnothing$
  - The cardinality of a set is the number of its elements; written $|A|$, read "size of A" or "cardinality of A"

Example set operations

$S = \mathbb{N}$

$A = \{x \mid x \text{ is an even nonnegative integer}\} = \{0, 2, 4, 6, 8, \ldots\}$

$B = \{x \mid (\exists y)(y \in \mathbb{N} \text{ and } x = 2y + 1\} = \{1, 3, 5, 7, 9, \ldots\}$

$C = \{x \mid (\exists y)(y \in \mathbb{N} \text{ and } x = 4y\} = \{0, 4, 8, 12, 16, \ldots\}$

a. $A \cap B = \emptyset$, i.e., A and B are disjoint
b. $A \cup B = \mathbb{N}$
c. $A' = B$ (because $S = \mathbb{N}$)
d. $C \subseteq A$
e. $A \cup C = A$
f. $A - C = \{x \mid (\exists y)(y \in \mathbb{N} \text{ and } x = 4y + 2\}$

Example 18

Example set operations

$S = \{1, 2, 3, \ldots, 10\}$
$A = \{1, 2, 3, 5, 10\}$
$B = \{2, 4, 7, 8, 9\}$
$C = \{5, 8, 10\}$

a. $A \cup B = \{1, 2, 3, 4, 5, 7, 8, 9, 10\}$
b. $A - C = \{1, 2, 3\}$
c. $B' \cap (A \cup C) = \{1, 3, 5, 10\}$

$A \cup C = \{1, 2, 3, 5, 8, 10\}$
$B' = \{1, 3, 5, 6, 10\}$
$B' \cap (A \cup C) = \{1, 3, 5, 10\}$

Practice 16

Cartesian product

- Description
  - The cartesian product of sets A and B is the set of all ordered pairs consisting of one element from A and one element from B.
- Details
  - Operands $A, B \in \wp(S)$
  - Notation $A \times B$, read "A cross B"
  - Definition $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$
  - Example $S = \mathbb{N}$

    $A = \{1, 2\}$

    $B = \{3, 4\}$

    $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$

    $B \times A = \{(3, 1), (3, 2), (4, 1), (4, 2)\}$

René Descartes
1596–1650

Practice 17

- Notes
  - Cartesian product aka "cross product"
  - Not a binary operation as formally defined earlier, because result (set of ordered pairs) $\notin \wp(S)$
- Additional definitions
  - $A^2 = A \times A \qquad\qquad = \{(x_1, x_2) \mid x_1, x_2 \in A\}$
  - $A^n = A \times A \times \ldots \times A = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in A\}$, i.e., ordered n-tuples
- Examples

$$A = \{1, 2\}$$

$$A^2 = A \times A \qquad = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

$$A^3 = A \times A \times A = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2)\ (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$$

Practice 17

Set identities

Commutative	1a.	$A \cup B = B \cup A$
	1b.	$A \cap B = B \cap A$
Associative	2a.	$(A \cup B) \cup C = A \cup (B \cup C)$
	2b.	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributive	3a.	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	3b.	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity	4a.	$A \cup \varnothing = A$
	4b.	$A \cap S = A$
Complement	5a.	$A \cup A' = S$
	5b.	$A \cap A' = \varnothing$
De Morgan's	6a.	$(A \cup B)' = A' \cap B'$
	6b.	$(A \cap B)' = A' \cup B'$

where S is the universal set and $\varnothing$ is the empty set.

Types of set proofs

- Subset proof
  - Show $A \subseteq B$
  - Specific sets with known properties
  - Assume $x \in A$, show $x \in B$, proves $A \subseteq B$
  - For $A \subset B$, also exhibit $y \in B$, $\notin A$
- Set equality proof
  - Show $A = B$
  - Specific sets with known properties
  - Assume $x \in A$, show $x \in B$, proves $A \subseteq B$
  - Assume $x \in B$, show $x \in A$, proves $B \subseteq A$
  - $A \subseteq B$ and $B \subseteq A$ prove $A = B$

- Set identity proof using inclusion
  - Show set expression = set expression, e.g., $(A \cup B)' = A' \cap B'$
  - Specific properties of sets not known
  - Reason about sets using definitions
  - Assume $x \in$ left side, show $x \in$ right side
  - Assume $x \in$ right side, show $x \in$ left side
  - Left ⊆ right and right ⊆ left prove left = right
- Set identity proof, using identities
  - Use for set expression = set expression, e.g., $(A \cup B)' = A' \cap B'$
  - Specific properties of sets not known
  - Reason about sets using identities
  - Transform left side into right side

Example set identity proof, using inclusion

Distributive identity $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

To show $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
show $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
and $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Theorem $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
Proof Let $x \in A \cup (B \cap C)$.

$x \in A \cup (B \cap C)$	
$\rightarrow x \in A \text{ or } x \in (B \cap C)$	def of ∪
$\rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C)$	def of ∩
$\rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)$	distributive
$\rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C)$	def of ∪
$\rightarrow x \in (A \cup B) \cap (A \cup C)$ ■	def of ∩

Example 19

Example set identity proof, using inclusion

De Morgan $(A \cap B)' = A' \cup B'$

To show $(A \cap B)' = A' \cup B'$,
show $(A \cap B)' \subseteq A' \cup B'$
and $A' \cup B' \subseteq (A \cap B)'$.

Theorem $(A \cap B)' = A' \cup B'$
Proof Let $x \in (A \cap B)'$.

	$x \in (A \cap B)'$	
$\leftrightarrow$	$x \notin (A \cap B)$	def of $'$
$\leftrightarrow$	$x \notin A$ or $x \notin B$	def of $\cap$
$\leftrightarrow$	$x \in A'$ or $x \in B'$	def of $'$
$\leftrightarrow$	$x \in A' \cup B'$ ■	def of $\cup$

Exercise 82b

Example set identity proof, using identities

Theorem

$[A \cup (B \cap C)] \cap ([A' \cup (B \cap C)] \cap (B \cap C)') = \varnothing$

Proof

	$[A \cup (B \cap C)] \cap ([A' \cup (B \cap C)] \cap (B \cap C)')$	
=	$([A \cup (B \cap C)] \cap [A' \cup (B \cap C)]) \cap (B \cap C)'$	associative
=	$([(B \cap C) \cup A] \cap [(B \cap C) \cup A']) \cap (B \cap C)'$	commutative
=	$[(B \cap C) \cup (A \cap A')] \cap (B \cap C)'$	distributive
=	$[(B \cap C) \cup \varnothing] \cap (B \cap C)'$	complement
=	$(B \cap C) \cap (B \cap C)'$	identity
=	$\varnothing$ ■	complement

Example 20

Countable sets

- Elements can be mapped one-to-one to $\mathbb{N} - \{0\}$
  - $\mathbb{N} - \{0\} = \{1, 2, 3, 4, \ldots\}$, aka "counting" numbers
- All finite sets are countable
  - Number elements in arbitrary order
- Some infinite sets may be countable
  - AKA countably infinite, denumerable
  - To prove set A is denumerable,
    give a one-to-one mapping
    from the elements of $\mathbb{N} - \{0\}$ to the elements of A
  - Mapping AKA "numbering" or "enumeration"
  - Numbering must reach every element of A eventually

Example countable sets

$\mathbb{N}$ is denumerable.

$$\mathbb{N} = \{\ 0,\ 1,\ 2,\ 3,\ 4,\ \ldots\}$$

$$\uparrow\ \uparrow\ \uparrow\ \uparrow\ \uparrow$$

$$1\ \ 2\ \ 3\ \ 4\ \ 5\ \ \ldots$$

Example 21

The set of even positive integers is denumerable.

$$E = \{x \mid (\exists k)(k \in \mathbb{N} \text{ and } x = 2k + 2\}$$

$$E = \{\ 2,\ 4,\ 6,\ 8,\ \ \ldots\}$$

$$\uparrow\ \uparrow\ \uparrow\ \uparrow$$

$$1\ \ 2\ \ 3\ \ 4\ \ \ \ldots$$

Practice 20

Example countable set

$\mathbb{Z}$ is denumerable.

$$\begin{array}{ccccccccc} \mathbb{Z} = \{\ldots, & -3, & -2, & -1, & 0, & 1, & 2, & 3, & \ldots\} \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ \ldots & 7 & 5 & 3 & 1 & 2 & 4 & 6 & \ldots \end{array}$$

Exercise 98

Example countable set

$\mathbb{Q}$ is the set of rational numbers;

$\mathbb{Q}^+$ is set of positive rational numbers, e.g., 1/2, 25/32, …

$\mathbb{Q}^+ = \{x/y \mid x \in \mathbb{N} - \{0\} \text{ and } y \in \mathbb{N} - \{0\}\}$

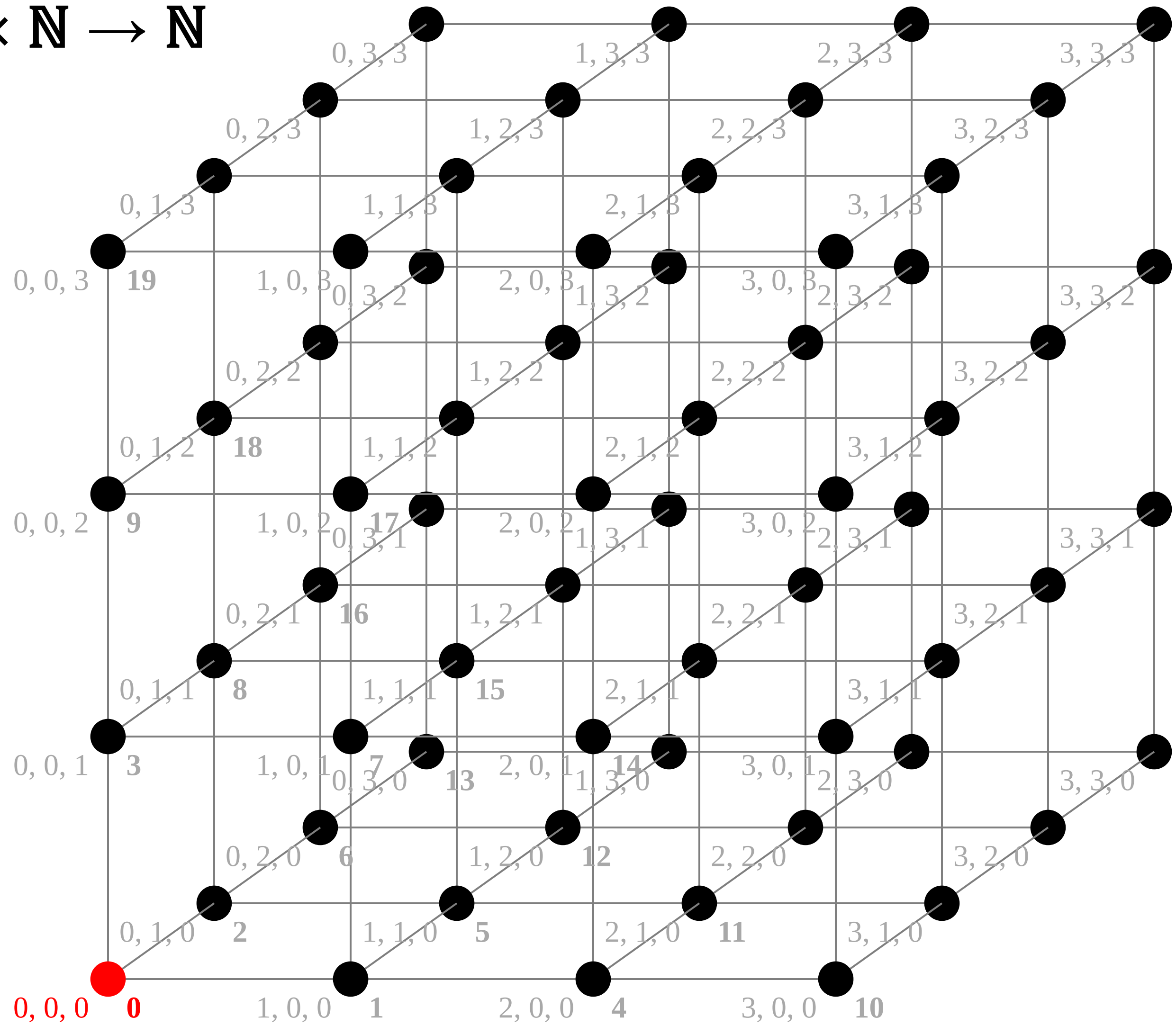
$= \{1/1, 1/2, 1/3, \ldots, 2/1, 2/2, 2/3, \ldots, 3/1, 3/2, 3/3, \ldots\}$

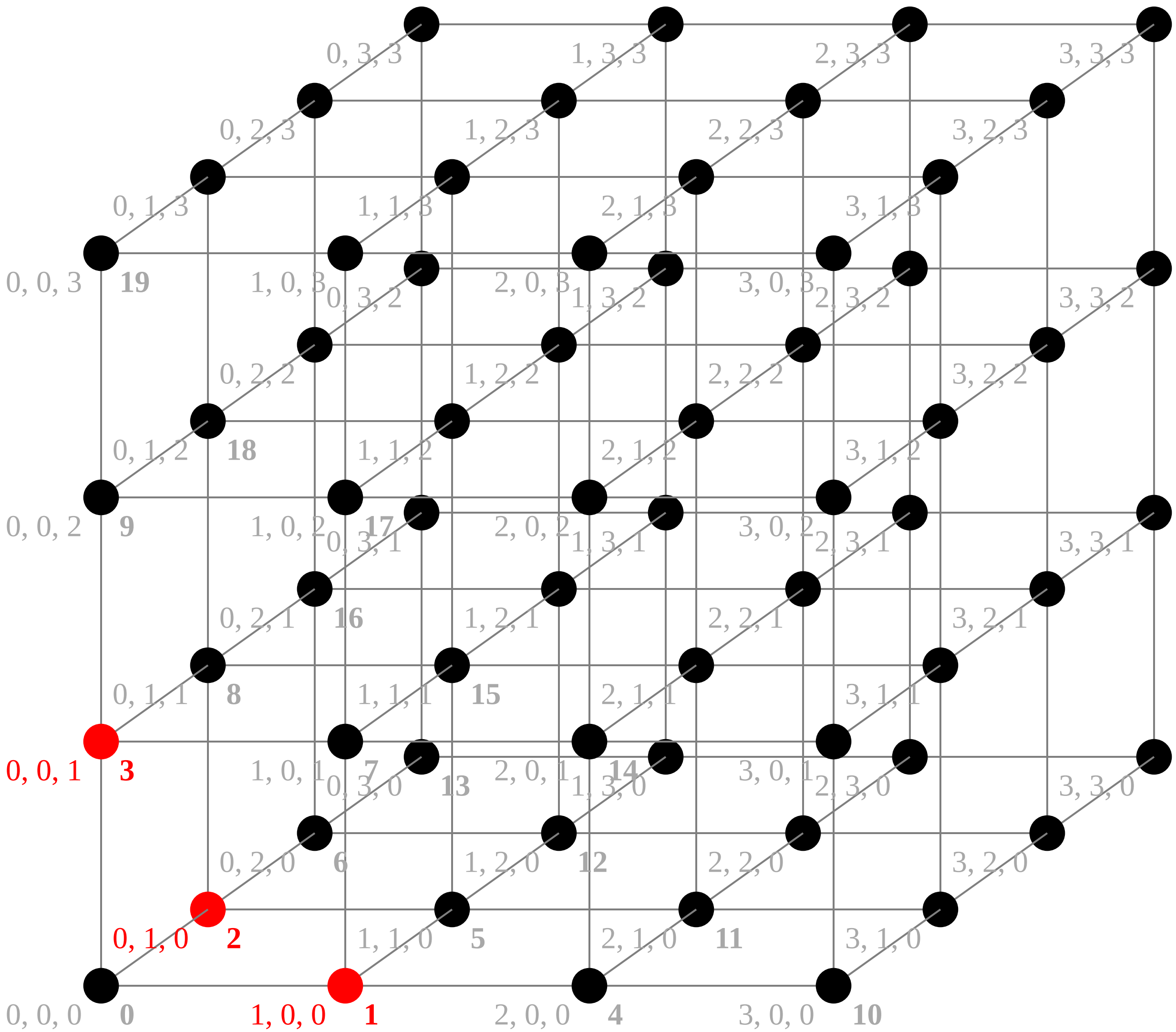
$\mathbb{Q}^+$ is denumerable.

1 1/1	3 1/2	4 1/3	10 1/4	11 1/5	…
2 2/1	5 2/2	9 2/3	12 2/4	2/5	…
6 3/1	8 3/2	13 3/3	3/4	3/5	…
7 4/1	14 4/2	4/3	4/4	4/5	…
15 5/1	5/2	5/3	5/4	5/5	…
…	…	…	…	…	…

Example 22

$\mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

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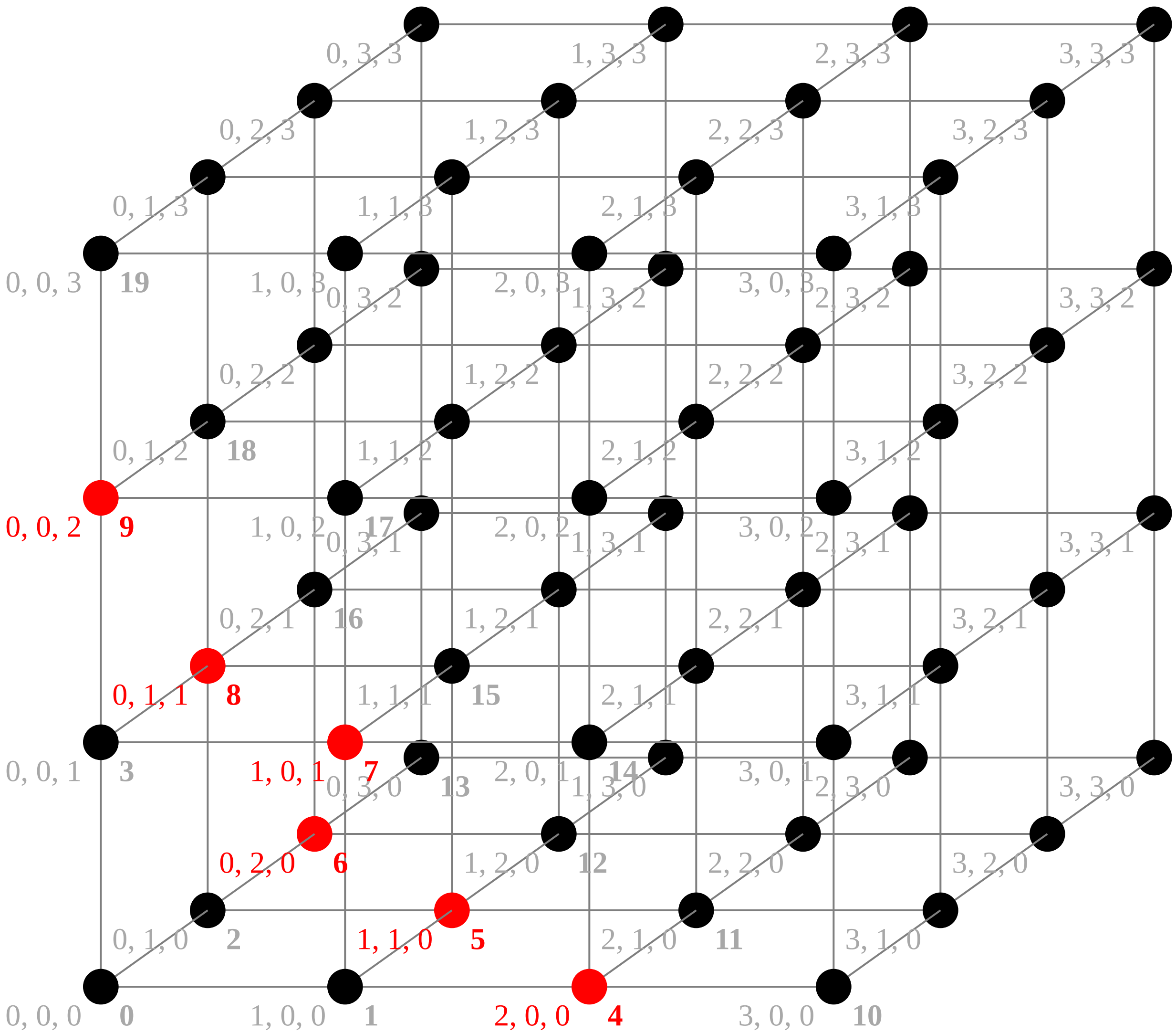
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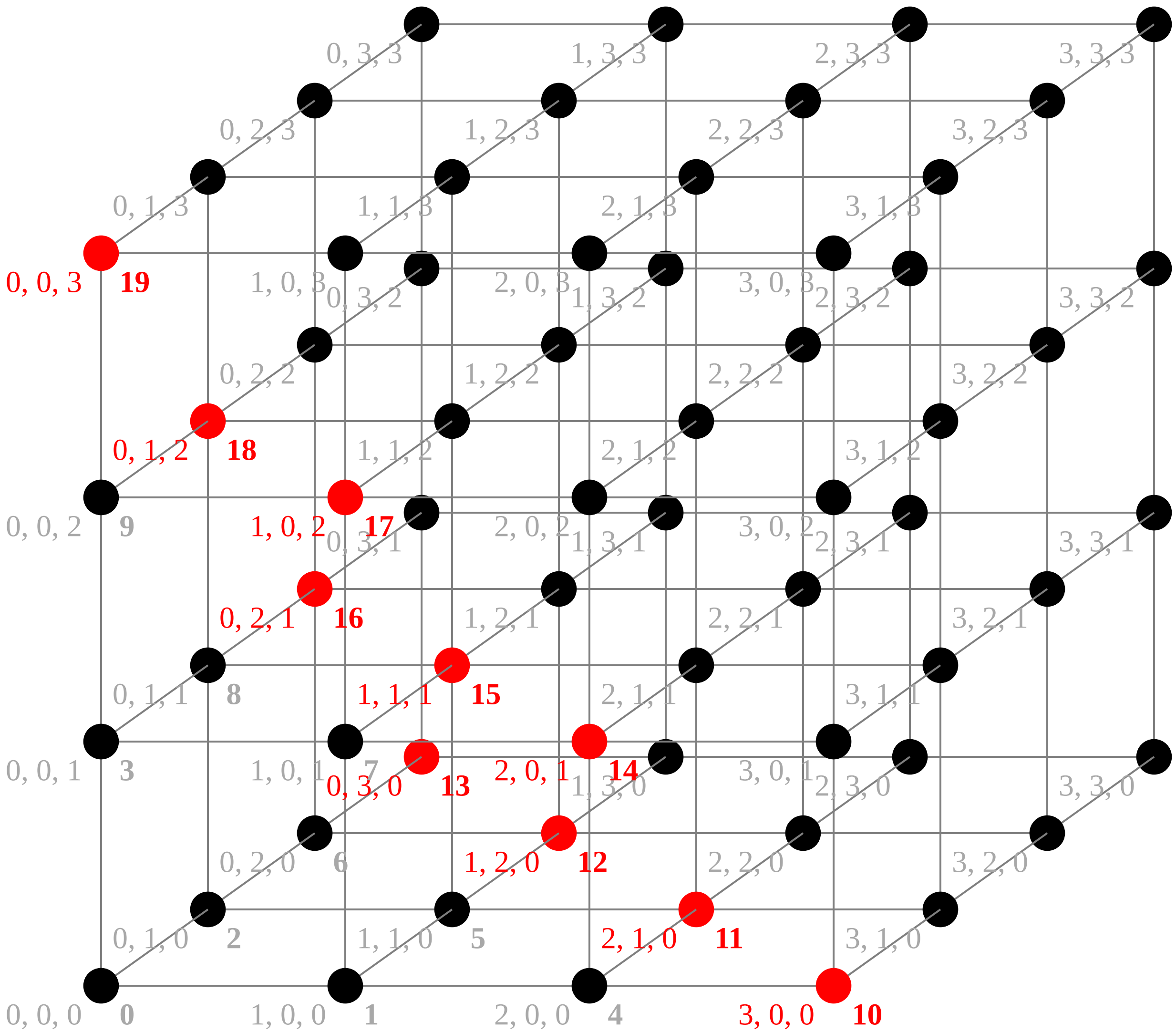
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Uncountable sets

- Elements of an uncountable set
  can not be mapped one-to-one to $\mathbb{N} - \{0\}$
- All uncountable sets are infinite,
  but not vice versa
- e.g., $\mathbb{R}$

Example uncountable set

$\mathbb{R}$ is uncountable, i.e., uncountably infinite.

Theorem
$R = \{x \mid x \in \mathbb{R} \text{ and } 0 \leq x < 1\}$ is uncountable; $R \subset \mathbb{R}$.

Proof (by contradiction)
Assume by way of contradiction that R is countable;
then there is a mapping from $\mathbb{N} - \{0\}$ to R.

$1 \longrightarrow 0.342134 \quad \ldots$

$2 \longrightarrow 0.257001 \quad \ldots$

$3 \longrightarrow 0.546122 \quad \ldots$

$4 \longrightarrow 0.716525 \quad \ldots$

$\ldots$

Construct number $p = 0.p_1p_2p_3\ldots$ digit by digit as follows:

$$p_i = \begin{cases} 5 \text{ if digit } i \text{ of the } i\text{th real number in the mapping} \neq 5 \\ 6 \text{ if digit } i \text{ of the } i\text{th real number in the mapping} = 5 \end{cases}$$

$1 \rightarrow 0.342134\ \ldots$
$2 \rightarrow 0.257001\ \ldots$
$3 \rightarrow 0.546122\ \ldots$
$4 \rightarrow 0.716525\ \ldots$
$\ldots$


Georg Cantor
1845-1918

e.g. $p = 0.5656\ldots$

$0 \leq p < 1$ so $p \in R$, but p differs from every real number in the mapping, contradicting the assumption that a mapping from $\mathbb{N}$ to R exists. Thus R is uncountable. ■

Example 23

Diagonalization

- Powerful and important proof technique
- Famous diagonalization proofs
  - $\mathbb{R}$ is uncountable (Cantor)
  - The Halting problem is uncomputable (Turing)
  - In any formal system containing arithmetic,
    there are true statements that are unprovable (Gödel)

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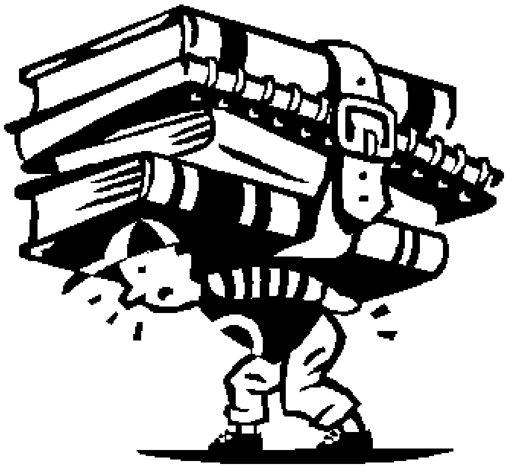
Section 4.1 homework assignment

See homework list for specific exercises.



4.2 Counting

Combinatorics and counting

- Combinatorics; mathematics of counting
- Applications
  - How many rows in a truth table?
  - How many elements in a power set?
  - How much computation does an algorithm require?
  - How many execution paths through a program?
  - How much storage does a database require?
  - How many users can a computer system support?
- Methods
  - Multiplication principle, with and without repetitions
  - Addition principle
  - Decision trees

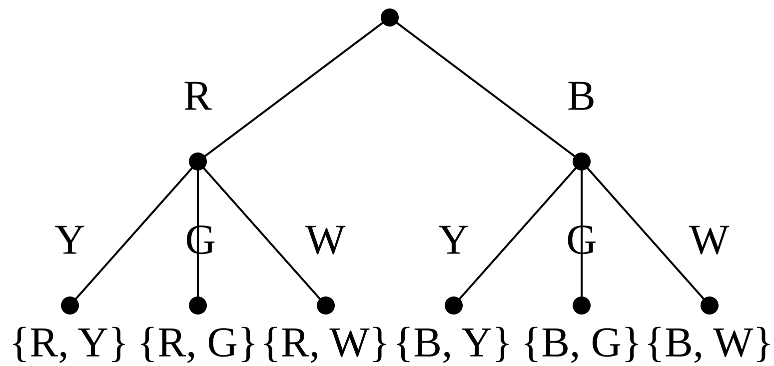
Multiplication principle example

A child may choose one of two jelly beans (red or black), and one of three gummy bears (yellow, green, or white).

Choose jelly bean

Choose gummy bear

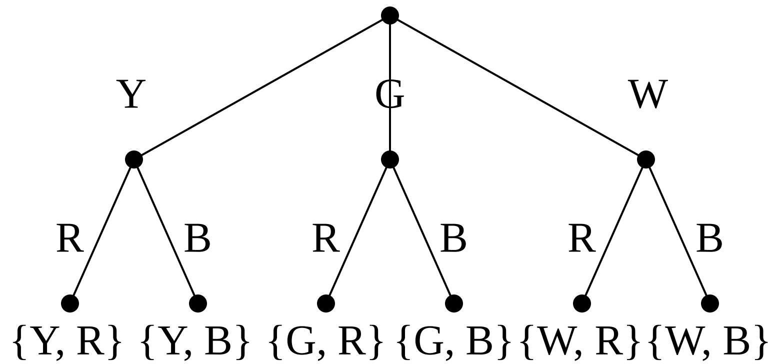
$2 \cdot 3 = 6$ outcomes



Choose gummy bear

Choose jelly bean

$3 \cdot 2 = 6$ outcomes



Example 24

Multiplication principle

- Multiplication principle
  - If there are n_1 possible outcomes for a first event, and n_2 possible outcomes for a second event, there are $n_1 \cdot n_2$ possible outcomes for the sequence of the two events.
- Comments
  - Assumes events are independent
  - Applies to any number of successive events (by induction)

Multiplication principle examples

Last part of a U.S. telephone number has 4 digits, e.g., 1-800-555-1212.
How many 4 digit numbers are there?

$10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10{,}000$

Example 25

Repetition of digits allowed in previous example.
When successive events are from the same set, question of repetitions must be considered.
How many 4 digit numbers are there if digits may not repeat?

$10 \cdot 9 \cdot 8 \cdot 7 = 5{,}040$

Example 26

Multiplication principle examples

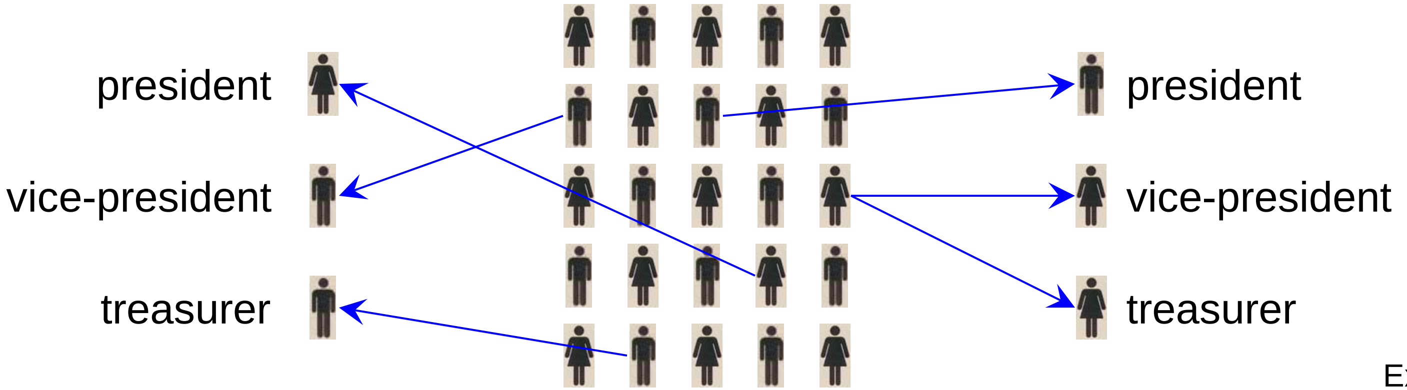
How many ways are there to choose 3 officers from a club of 25 people, if each person may hold at most 1 office?

$25 \cdot 24 \cdot 23 = 13{,}800$

Example 27

How many ways are there to choose 3 officers from a club of 25 people, if a person may hold > 1 office?

$25 \cdot 25 \cdot 25 = 25^3 = 15{,}625$



Example 27

Multiplication principle example

How many possible binary logical connectives are there?
Binary $\rightarrow$ two operands $\rightarrow$ 4 rows in truth table;
each row in result column has 2 possible values, T and F.

$2 \cdot 2 \cdot 2 \cdot 2 = 2^4 = 16$ possible truth tables,
i.e., 16 possible binary logical connectives.

A	B	F	$\wedge$	$\circ$	$\circ$	$\circ$	$\circ$	$\oplus$	$\vee$	$\downarrow$	$\leftrightarrow$	$\circ$	$\circ$	$\circ$	$\rightarrow$	$\mid$	T
T	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T
T	F	F	F	T	T	F	F	T	T	F	F	T	T	F	F	T	T
F	T	F	F	F	F	T	T	T	T	F	F	F	F	T	T	T	T
F	F	F	F	F	F	F	F	F	F	T	T	T	T	T	T	T	T

F FALSE, $\wedge$ AND, $\oplus$ XOR, $\vee$ OR, $\downarrow$ NOR,
$\leftrightarrow$ EQUIVALENCE, $\rightarrow$ IMPLIES, | NAND, T TRUE

Exercise 13

Size of cartesian product

Let A and B be sets.
By the multiplication principle,
$|A \times B| = |A| \cdot |B|$

Example
$A = \{\text{R, B}\}$ $|A| = 2$
$B = \{\text{Y, G, W}\}$ $|B| = 3$

$A \times B = \{(\text{R, Y}), (\text{R, G}), (\text{R, W}), (\text{B, Y}), (\text{B, G}), (\text{B, W})\}$
$|A \times B| = |A| \cdot |B| = 2 \cdot 3 = 6$

Example 29

Addition principle example

A diner may select 1 dessert from 3 pies and 4 cakes.
How many ways can this be done?
Two events (pie and cake) but only one will occur.

$3 + 4 = 7$

Addition principle

- Addition principle
  - If A and B are disjoint events with n_1 and n_2 possible outcomes respectively, there are $n_1 + n_2$ possible outcomes for event A or B.
- Comments
  - Useful when counting disjoint outcomes
  - Applies to any number of disjoint events (by induction)

Addition principle example

A customer wants to purchase a vehicle from a dealer with 23 autos and 14 trucks in stock.
How many choices does the customer have?

$23 + 14 = 37$

A and B must be disjoint sets.
23 autos $+$ 14 trucks $+$ 17 red vehicles $\neq 54$ choices, if any of the autos or trucks are red.

Example 30

Size of set union

Let A and B be disjoint sets.
By the addition principle,
$|A \cup B| = |A| + |B|$

Example
$A = \{\text{Bach, Handel}\}$ $|A| = 2$
$B = \{\text{Kraftwerk, Dragonforce, Primal Fear}\}$ $|B| = 3$

$A \cup B = \{\text{Bach, Handel, Kraftwerk, Dragonforce, Primal Fear}\}$
$|A \cup B| = |A| + |B| = 2 + 3 = 5$

Example 31

Size of set difference

Let A and B be finite sets; then these equations hold:

a. $|A - B| = |A| - |A \cap B|$

b. $|A - B| = |A| - |B|$ if $B \subseteq A$

Lemma below used in proof of equation a to follow.

Lemma

$$
\begin{aligned}
& (A - B) \cup (A \cap B) & & \\
= \ & (A \cap B') \cup (A \cap B) & & (A - B) = A \cap B' \text{ (p. 195)} \\
= \ & A \cap (B' \cup B) & & \text{distributive} \\
= \ & A \cap S & & \text{complement} \\
= \ & A & & \text{identity}
\end{aligned}
$$

In sets, if $A = B$ then $|A| = |B|$ (but not vice versa), so lemma shows $|(A - B) \cup (A \cap B)| = |A|$.

Theorem

$|A - B| = |A| - |A \cap B|$

Proof

$\lvert(A - B) \cup (A \cap B)\rvert = \lvert A\rvert$	lemma
$\lvert A - B\rvert + \lvert A \cap B\rvert = \lvert A\rvert$	Example 31
$\lvert A - B\rvert = \lvert A\rvert - \lvert A \cap B\rvert$ ■	algebra

$A - B$ and $A \cap B$ disjoint sets, so Example 31 applies.
$A - B$ elements in A not in B; $A \cap B$ elements in both A and B

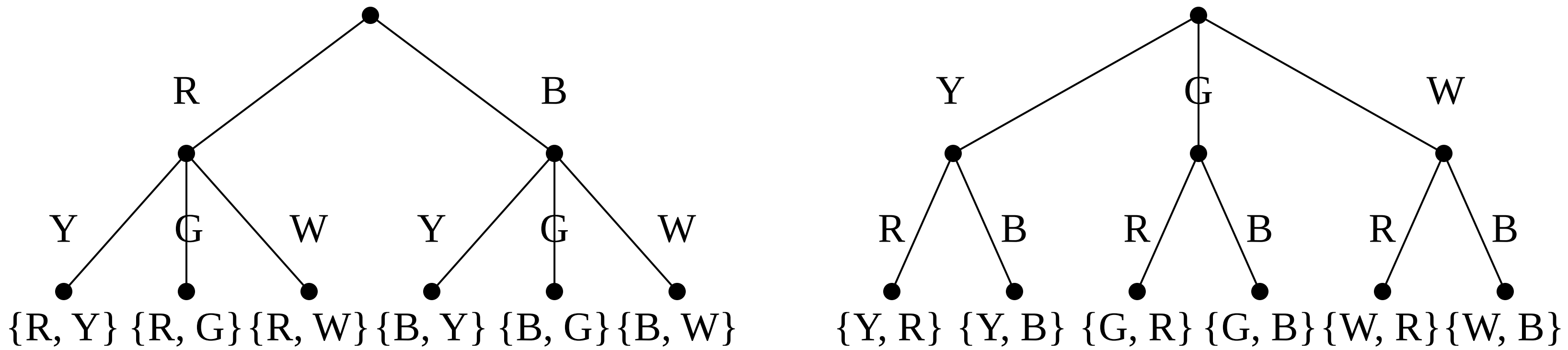
Equation b follows from equation a,
because if $B \subseteq A$, then $A \cap B = B$.

Example 32

Both principles example

In the candy choosing example (Example 24),
how many ways can the choosing process proceed?

$$\underbrace{2 \cdot 3}_{\text{Choose jelly bean first}} + \underbrace{3 \cdot 2}_{\text{Choose gummy bear first}} = 12$$



Example 33

Both principles examples

How many 3 digit integers (numbers between 100 and 999 inclusive) are even?

ends with 0 $(9 \cdot 10 \cdot 1) +$
ends with 2 $(9 \cdot 10 \cdot 1) +$
ends with 4 $(9 \cdot 10 \cdot 1) +$
ends with 6 $(9 \cdot 10 \cdot 1) +$
ends with 8 $(9 \cdot 10 \cdot 1) = 450$ both principles

$9 \cdot 10 \cdot 5 = 450$ multiplication

Example 36

Both principles example

Suppose the last four digits of a telephone number must include at least one repeated digit.
How many such numbers are there?

Solving via addition problematic; repetition at any digit.

Note that the set of numbers with repetitions and the set of numbers without repetitions are disjoint.

Four digit telephone numbers (all possible)
$10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10{,}000$
Four digit telephone numbers without repetitions
$10 \cdot 9 \cdot 8 \cdot 7 = 5{,}040$
Four digit telephone numbers with at least one repetition
$10{,}000 - 5{,}040 = 4{,}960$

Example 37

Both principles example

IP addresses consist of 4 bytes, e.g., 128.12.15.26, divided into two parts, netid and hostid.

Class	Leading bits	Netid	Hostid
A	0	7 bits to finish 1 byte	3 bytes
B	10	14 bits to finish 2 bytes	2 bytes
C	110	21 bits to finish 3 bytes	1 bytes

```
A 0-------.--------.--------.--------
B 10------.--------.--------.--------
C 110-----.--------.--------.--------
```

How many IP addresses are possible?

(A netids) · (A hostids) + (B netids) · (B hostids) + (C netids) · (C hostids)

$2^7 \cdot 2^{24} + 2^{14} \cdot 2^{16} + 2^{21} \cdot 2^8 =$

$2^{31} + 2^{30} + 2^{29} =$

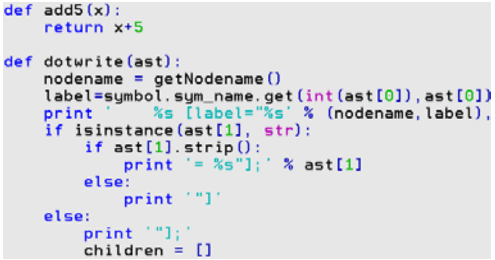
$2{,}147{,}483{,}648 + 1{,}073{,}741{,}824 + 536{,}870{,}912 = 3{,}758{,}096{,}384$

Example 38

Both principles example

Identifiers in Python programming language:
First character `_`, `A–Z`, `a–z` (53 symbols)
Other characters `0–9`, `A–Z`, `a–z` (63 symbols)
Reserved keywords (`and`, `if`, …) disallowed (31 keywords)

How many Python identifiers of 8 characters or less?



```
def add5(x):
    return x+5

def dotwrite(ast):
    nodename = getNodename()
    label=symbol.sym_name.get(int(ast[0]),ast[0])
    print '    %s [label="%s' % (nodename, label),
    if isinstance(ast[1], str):
        if ast[1].strip():
            print '= %s"];' % ast[1]
        else:
            print '"]'
    else:
        print '"];'
        children = []
```

Length 1: 53
Length 2: $53 \cdot 63$
Length 3: $53 \cdot 63 \cdot 63 = 53 \cdot 63^2$
. . .
Length 8: $53 \cdot 63 \cdot 63 \cdot 63 \cdot 63 \cdot 63 \cdot 63 \cdot 63 = 53 \cdot 63^7$

$$53 + 53 \cdot 63 + 53 \cdot 63^2 + \ldots + 53 \cdot 63^7 - 31 = 212{,}133{,}167{,}002{,}849$$

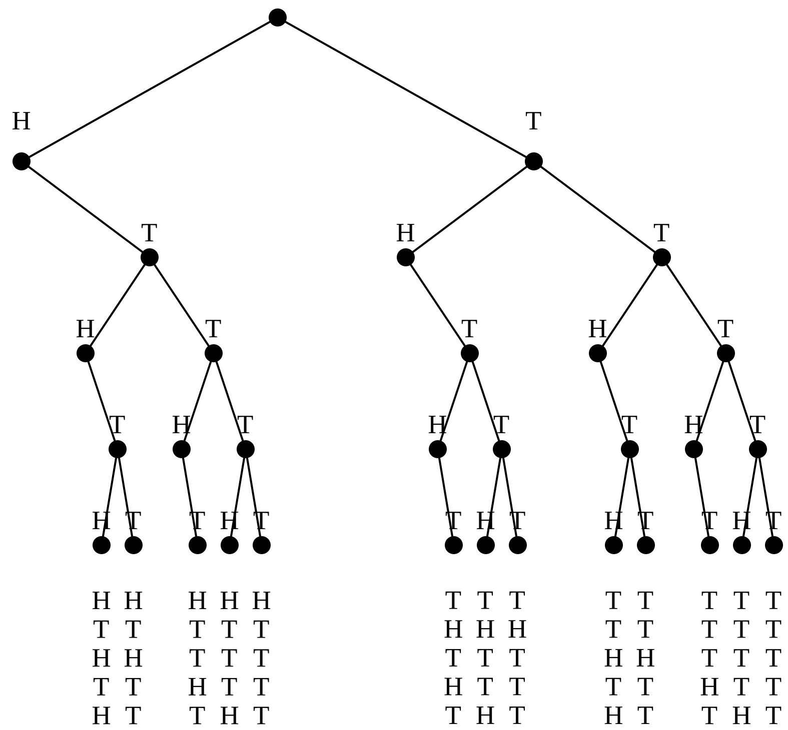
Example 9.3.4, p. 543, S. S. Epp, *Discrete Mathematics with Applications, Fourth Edition*, Brooks/Cole, Boston MA, 2011.

Decision trees

- Decision tree
  - Diagram illustrating the outcomes resulting from a sequence of possible choices.
- Comments
  - When all choices available at each decision, equivalent to multiplication principle (Example 24)
  - Can solve problems when not all choices available at each decision, i.e., where multiplication principle does not apply

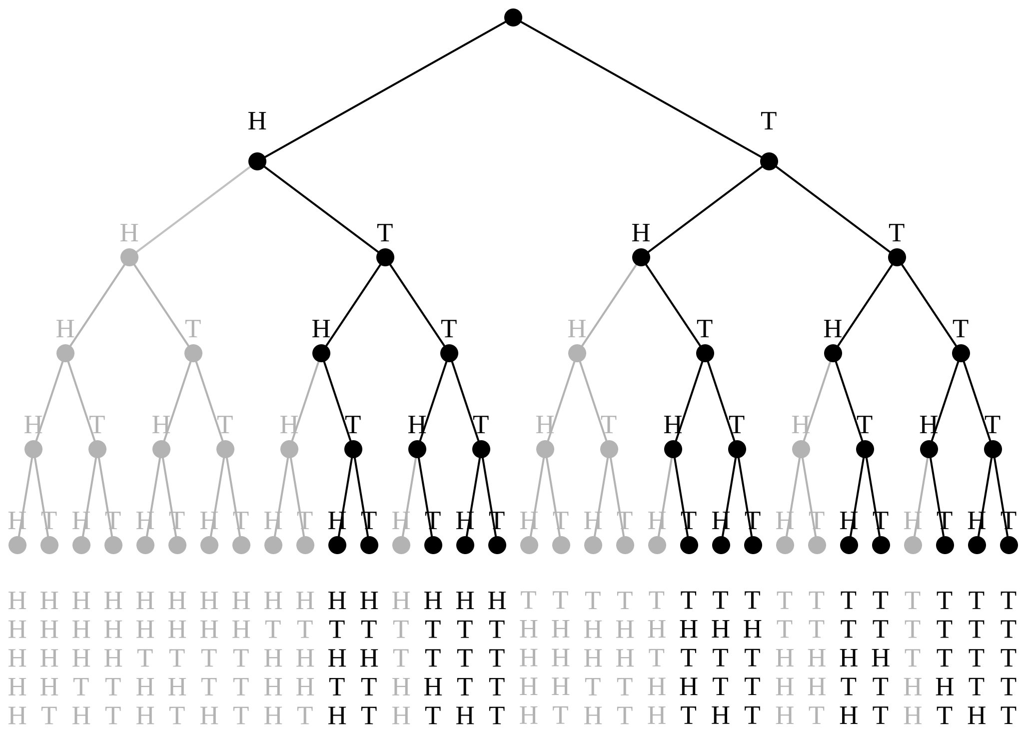
Example decision tree

How many ways can a fair coin be tossed five times without having two heads in a row? 13



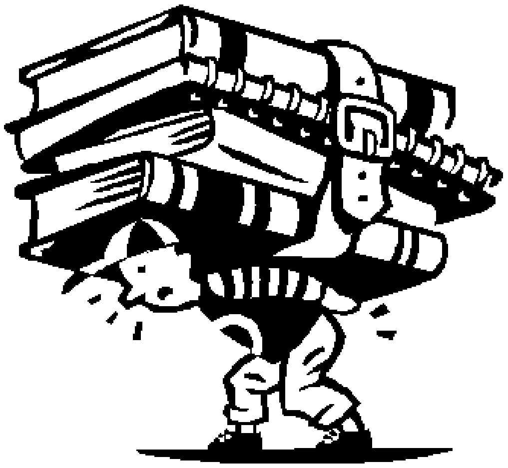
Example 39, Practice 24

How many ways can a fair coin be tossed five times without having two heads in a row? 13 out of 32



Section 4.2 homework assignment

See homework list for specific exercises.



4.3 Principle of Inclusion and Exclusion; Pigeonhole Principle

Principle of inclusion and exclusion

- Useful in solving certain combinatorics problems
- Background
  - Let sets A, B be subsets of universal set S
  - $A - B$, $B - A$, and $A \cap B$ are disjoint
  - e.g., $x \in A - B \rightarrow x \notin B \wedge x \notin B - A \wedge x \notin A \cap B$
  - $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

Practice 26

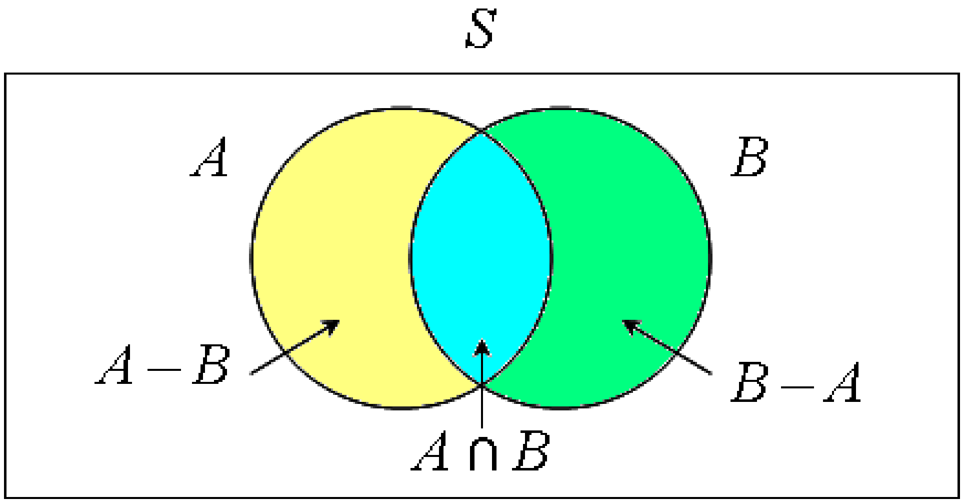


Figure 4.6

Goal: an expression for $|A \cup B|$

$A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$ (by Practice 26)
so $|A \cup B| = |(A - B) \cup (B - A) \cup (A \cap B)|$

$|(A - B) \cup (B - A) \cup (A \cap B)| = |A - B| + |B - A| + |A \cap B|$
(by Example 31) so $|A \cup B| = |A - B| + |B - A| + |A \cap B|$

Substituting into above using these equations
(from Example 32)
$|A - B| = |A| - |A \cap B|$
$|B - A| = |B| - |A \cap B|$

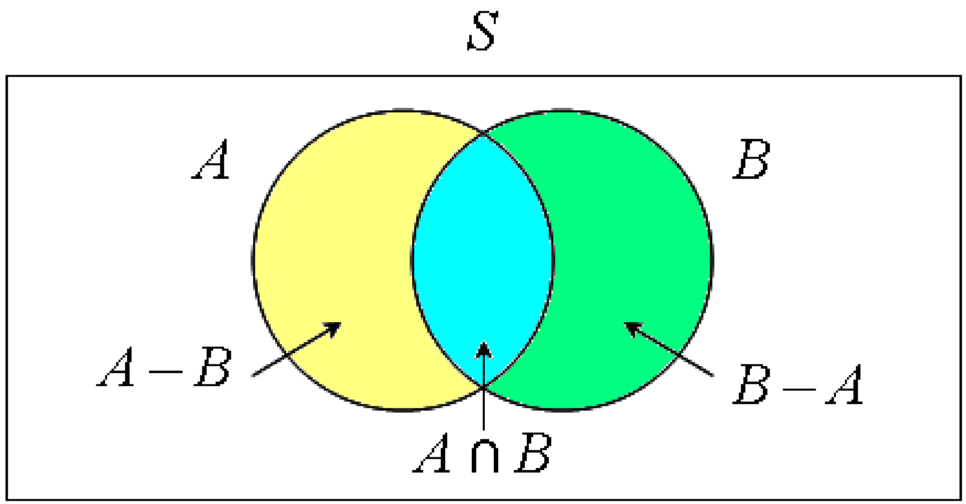
gives $|A \cup B| = |A| - |A \cap B| + |B| - |A \cap B| + |A \cap B|$

Simplifying gives $|A \cup B| = |A| + |B| - |A \cap B|$

Principle of inclusion and exclusion (two-set version)

Principle of inclusion and exclusion (two-set version)

$|A \cup B| = |A| + |B| - |A \cap B|$



If A and B disjoint then $|A \cap B| = 0$
so $|A \cup B| = |A| + |B| - 0 = |A| + |B|$

Practice 27

Principle of inclusion and exclusion example

A pollster queries 35 voters, all of whom support referendum 1, referendum 2, or both.
14 voters support referendum 1, 26 support referendum 2.
How many support both?

$|A \cup B| = |A| + |B| - |A \cap B|$
$35 = 14 + 26 - |A \cap B|$
$|A \cap B| = 5$

Example 40

Principle of inclusion and exclusion, three sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

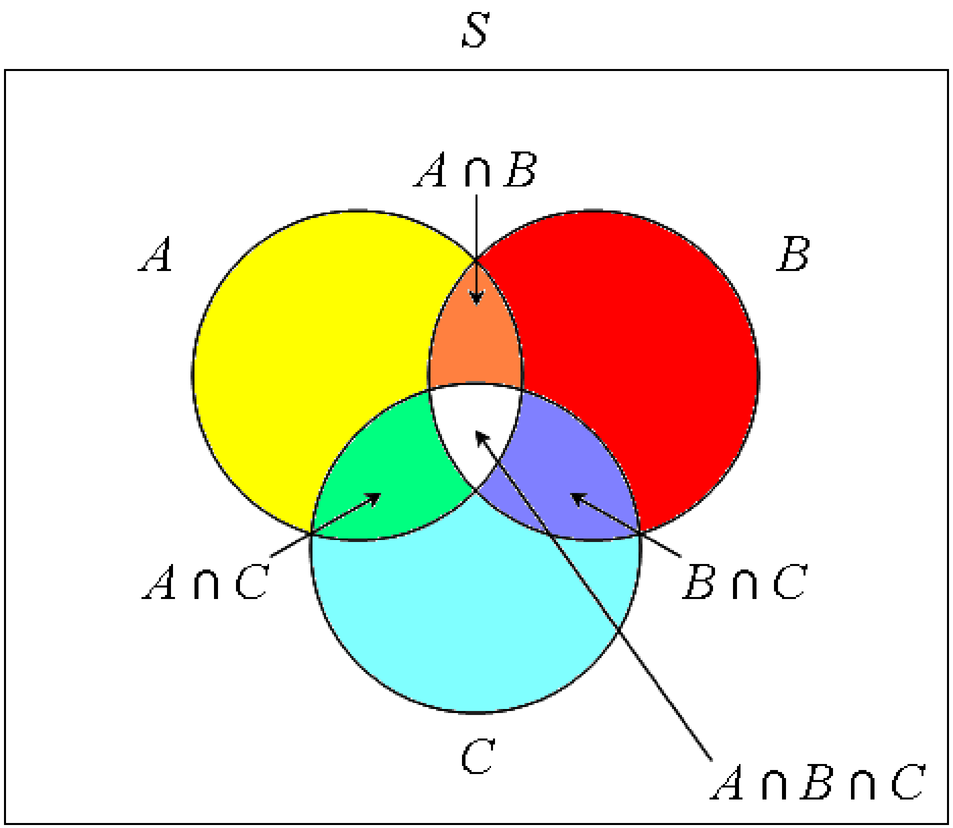


Figure 4.7

Principle of inclusion and exclusion example

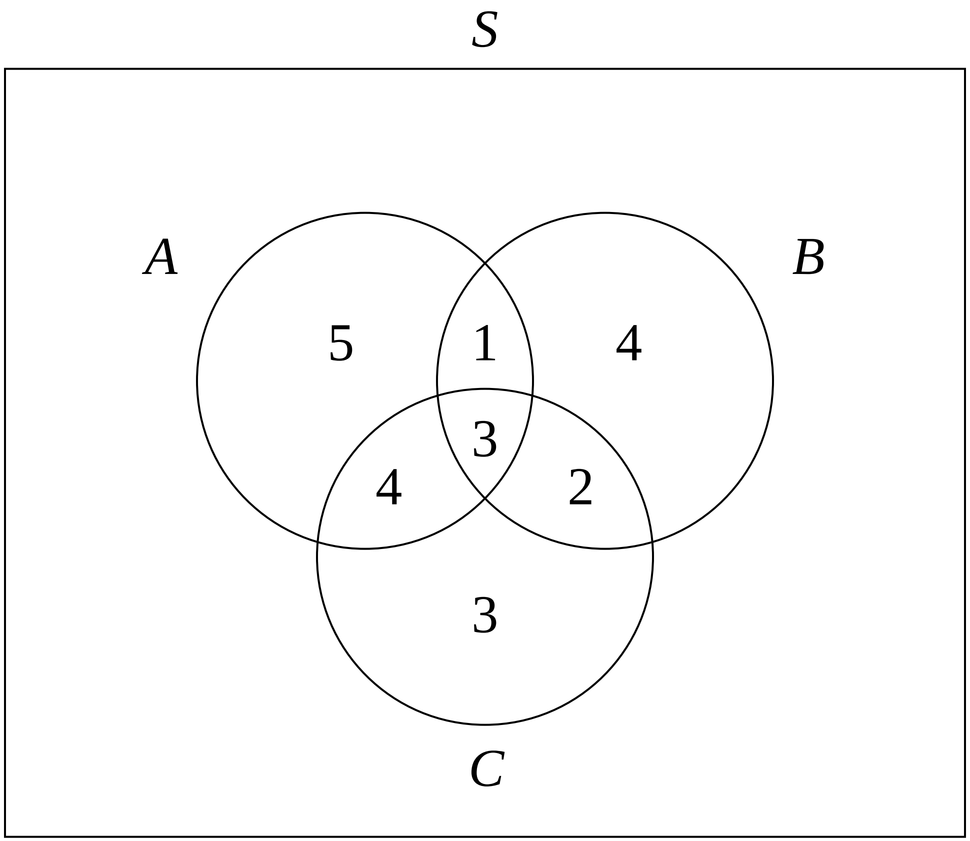
A group of students plan to order pizza.
13 will eat sausage, A
10 will eat pepperoni, B
12 will eat extra cheese, C
4 will eat both sausage and pepperoni, $A \cap B$
5 will eat both pepperoni and extra cheese, $B \cap C$
7 will eat both sausage and extra cheese, $A \cap C$
3 will eat all three. $A \cap B \cap C$
How many students are in the group?

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$|A \cup B \cup C| = 13 + 10 + 12 - 4 - 5 - 7 + 3 = 22$$
$$|A \cup B \cup C| = 22$$

Example 41

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$x = 13 + 10 + 12 - 4 - 7 - 5 + 3 = 22$$



Sample problem solution

You survey the 87 subscribers to your newsletter in preparation for the release of your new software product. The results of your survey reveal that 68 have a Windows system available to them, 34 have a Unix system available, and 30 have access to a Mac. In addition,
19 have access to both Windows and Unix systems,
11 have access to both Unix systems and Macs,
and 23 can use both Macs and Windows.

How many have access to all three types of systems?

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$87 = 68 + 34 + 30 - 19 - 11 - 23 + |A \cap B \cap C|$$
$$|A \cap B \cap C| = 8$$

Exercise 11

General principle of inclusion and exclusion

Pattern so far:

Two sets; add sizes of two sets singly,
subtract size of the intersection of the two sets.

$|A \cup B| = |A| + |B| - |A \cap B|$

Three sets; add sizes of the three sets singly,
subtract sizes of the pairwise intersections of the sets,
add size of the intersection of the three sets.

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

General formula for n sets

$$
\begin{aligned}
|A_1 \cup \ldots \cup A_n| &= \sum_{1 \le i \le n} |A_i| \\
&- \sum_{1 \le i < j \le n} |A_i \cap A_j| \\
&+ \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| \\
&- \ldots \\
&+ (-1)^{n+1} |A_i \cap \ldots \cap A_k|
\end{aligned}
$$

where, e.g., $\sum_{1 \le i < j \le n} |A_i \cap A_j|$

means to add the number of elements in all the intersections of the form $A_i \cap A_j$ where i and j take on all values from 1 to n with $i < j$.

Pigeonhole principle

- Principle
  - If more than k items are placed into k bins, then at least 1 bin contains > 1 item.
- Comments
  - "Bins" may be sets, "items" may be elements
  - Useful for solving combinatorics problems
  - May be invoked in proofs when appropriate

http://en.wikipedia.org/wiki/Pigeonhole_principle

Pigeonhole principle examples

How many people must be in a room to guarantee that two people will have last names beginning with the same letter?

There are 26 letters of the alphabet (bins).
With 27 people, at least two must have the same last initial.

Example 43

How many times must a single die be rolled to guarantee getting the same value twice?

There are 6 possible values.
With 7 rolls, at least two rolls must have the same value.

Practice 29

Pigeonhole principle example

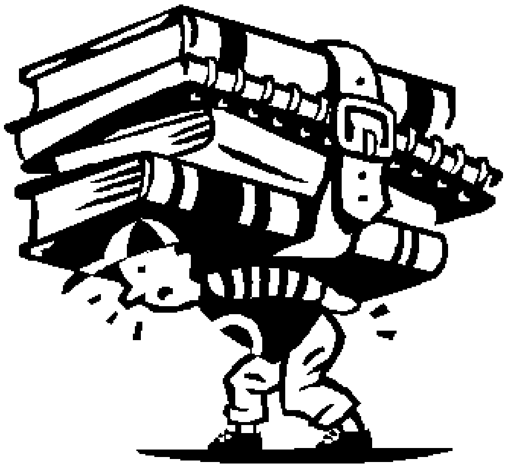
Theorem
In any set of $n + 1$ integers, where n is a positive number, there are at least two integers with the same remainder when divided by n.

Proof
When dividing by n, there are n possible remainders, 0 to $n - 1$. In a set of $n + 1$ such remainders at least two must be the same by the pigeonhole principle. ■

Exercise 27

Section 4.3 homework assignment

See homework list for specific exercises.



4.4 Permutations and Combinations

Permutations

- Description
  - Permutation; ordered arrangement of distinct objects
  - e.g., permutations of two digits from $\{0, 1\}$: $01, 10$
  - Order matters, e.g., 01 different from 10
  - e.g., permutations of three letters from $\{x, y, z\}$: $xyz, xzy, yxz, yzx, zxy, zyx$
- Details
  - Quantity of interest: number of permutations of r distinct objects chosen from n distinct objects
  - Notation: $P(n, r)$
  - Formula:

$$P(n, r) = \frac{n!}{(n-r)!} \text{ for } 0 \le r \le n$$

Permutations formula

Example of the formula

4 digit telephone number, with no repeated digits

$$P(10, 4) = 10 \cdot 9 \cdot 8 \cdot 7$$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{10!}{6!} = \frac{10!}{(10-4)!}$$

More generally

$$P(n, r) = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

$$= \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-r+1) \cdot (n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

Permutations examples

The number of permutations of three objects a, b, and c is

$$P(3, 3) = \frac{3!}{(3-3)!} = \frac{3!}{0!} = 3! = 6$$

The 6 permutations of those objects are
abc, acb, bac, bca, cab, cba

Example 47

The value of $P(7, 3)$ is

$$P(7, 3) = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = 210$$

Example 45

Permutations example

How many three letter strings can be formed from the letters in "compiler" if no letters can be repeated?

$$P(8,3) = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 336$$

com, cop, coi, col, coe, cor,
cmo, cmp, cmi, cml, cme, cmr,
cpo, cpm, cpi, cpl, cpe, cpr,
cio, cim, cip, cil, cie, cir,
clo, clm, clp, cli, cle, clr,
cro, crm crp, cri, crl, cre,
…

Example 48

Permutations special cases

Special case values of the permutation formula:

$$P(n,0) = \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$$

1 ordered arrangement of 0 objects

$$P(n,1) = \frac{n!}{(n-1)!} = n$$

n ordered arrangements of 1 object

$$P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

$n!$ ordered arrangement of n objects

Example 46

Permutations examples

Ten athletes compete in an Olympic event;
gold, silver, and bronze medals are awarded.
How many ways can the awards be made?

$$P(10, 3) = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 720$$

Example 49

How many ways can a president and vice-president be selected from a group of 20 people?

$$P(20, 2) = \frac{20!}{18!} = 380$$

Practice 30

Permutations example

A library has 4 books on operating systems, 7 on programming, and 3 on data structures. How many ways can these books be arranged, given that all books on a subject must be together?

$P(3,3) = 3!$ ways to arrange 3 subjects

$P(4,4) = 4!$ ways to arrange 4 operating systems books

$P(7,7) = 7!$ ways to arrange 7 programming books

$P(3,3) = 3!$ ways to arrange 3 data structures books

$3! \cdot 4! \cdot 7! \cdot 3! = 4{,}354{,}560$ total arrangements

Example 50

Permutations of non-distinct objects

If the permutations are drawn from a collection of objects that includes non-distinct objects, the formula is different.

How many permutations can be made from the letters in

a. FLORIDA

$$P(7, 7) = 7! = 5{,}040$$

all distinct

b. MISSISSIPPI

$$\frac{11!}{\underbrace{4!}_{\text{number of Ss}} \cdot \underbrace{4!}_{\text{number of Is}} \cdot \underbrace{2!}_{\text{number of Ps}} \cdot \underbrace{1!}_{\text{number of Ms}}} = 34{,}650$$

not all distinct

In general, if there are $n = n_1 + n_2 + \ldots + n_k$ objects, where each collection of n_i objects are indistinguishable, the number of distinct permutations is

$$\frac{n!}{n_1! \cdot n_2! \cdot \text{K} \cdot n_k!}$$

Example 57

Combinations

- Description
  - Combination; unordered selection of distinct objects
  - e.g., combinations of two digits from $\{0, 1\}$: 01
  - Order does not matter, e.g., 01 same as 10
  - e.g., combinations of two letters from $\{x, y, z\}$: xy, xz, yz
- Details
  - Quantity of interest: number of combinations of r distinct objects chosen from n distinct objects
  - Notation: $C(n, r)$
  - Formula: $C(n,r) = \dfrac{P(n,r)}{r!} = \dfrac{n!}{r!(n-r)!}$ for $0 \le r \le n$

Combinations examples

How many ways can two letters be chosen from three, where order does not matter?

e.g., combinations of two letters from $\{x, y, z\}$: xy, xz, yz

$$C(3,2) = \frac{3!}{2!(3-2)!} = \frac{3!}{2!\cdot 1!} = \frac{3\cdot 2\cdot 1}{2\cdot 1\cdot 1} = \frac{6}{2} = 3$$

How many combinations of 3 objects chosen from 7?

$$C(7,3) = \frac{7!}{3!(7-3)!} = \frac{7!}{3!\cdot 4!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 7\cdot 5 = 35$$

Example 51

Combinations special cases

Special case values of the combination formula:

$$C(n, 0) = \frac{n!}{0!(n-0)!} = \frac{n!}{1 \cdot n!} = 1$$

1 unordered selection of 0 objects

$$C(n, 1) = \frac{n!}{1!(n-1)!} = n$$

n unordered selections of 1 object

$$C(n, n) = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 0!} = 1$$

1 unordered selection of n objects

Example 52

Combinations examples

How many 5-card poker hands can be dealt from a 52 card deck?
Order does not matter, thus combinations.

$$C(52, 5) = \frac{52!}{5! \cdot 47!} = 2{,}598{,}960$$

Example 53

If 10 athletes compete in an Olympic event, how many different sets of medalists are there?
Order does not matter, thus combinations.

$$C(10, 3) = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = 120$$

Example 54

Combinations examples

A committee of 8 students is to be selected from a class consisting of 19 freshman and 34 sophomores?

a. In how many ways can 3 freshman and 5 sophomores be selected?

$$\underbrace{C(19,3)}_{\text{choose 3 freshmen}} \cdot \underbrace{C(34,5)}_{\text{choose 5 sophomores}} = \frac{19!}{3! \cdot 16!} \cdot \frac{34!}{5! \cdot 29!} = 969 \cdot 278{,}256 = 269{,}630{,}064$$

b. In how many ways can a committee with exactly 1 freshman be selected?

$$C(19,1) \cdot C(34,7) = \frac{19!}{1! \cdot 18!} \cdot \frac{34!}{7! \cdot 27!} = 19 \cdot 5{,}379{,}616 = 102{,}212{,}704$$

Example 55

c. In how many ways can a committee with at most 1 freshman be selected?

$$C(19,1)\cdot C(34,7)+C(19,0)\cdot C(34,8)=\frac{19!}{1!\cdot 18!}\cdot\frac{34!}{7!\cdot 27!}+\frac{34!}{8!\cdot 26!}$$

$$=19\cdot 5{,}379{,}616+18{,}156{,}204$$

$$=120{,}368{,}908$$

committee with 0 freshmen

committee with 1 freshman

d. In how many ways can a committee with at least 1 freshman be selected?

Method 1

number of freshmen on committee

$$
\begin{array}{ll}
1 & C(19,1)\cdot C(34,7)+ \\
2 & C(19,2)\cdot C(34,6)+ \\
3 & C(19,3)\cdot C(34,5)+ \\
4 & C(19,4)\cdot C(34,4)+ \\
5 & C(19,5)\cdot C(34,3)+ \\
6 & C(19,6)\cdot C(34,2)+ \\
7 & C(19,7)\cdot C(34,1)+ \\
8 & C(19,8)\cdot C(34,0)=868{,}166{,}506
\end{array}
$$

Method 2

$$C(53,8)-C(34,8)=868{,}166{,}506$$

committees with 0 freshmen

all committees

Combinations examples

In how many ways can you seat 11 men and 8 women in a row if no 2 women are to sit together?

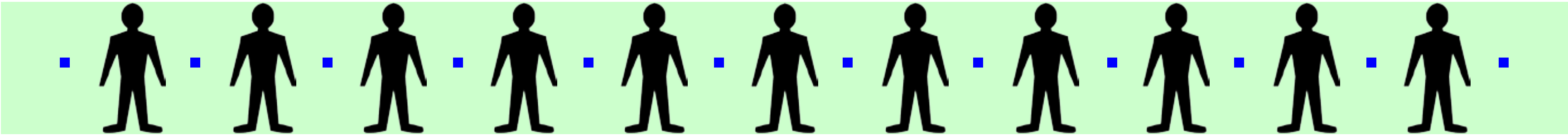
$$P(11, 11) \cdot P(8, 8) \cdot C(12, 8) = 796{,}675{,}461{,}120{,}000$$

select slots for women between men or at ends of row

ways to arrange women

ways to arrange men

Exercise 14



A computer network has 60 switching nodes.
In how many ways can exactly 2 nodes fail?

$$C(60, 2) = 1{,}770$$

Exercise 53

Combinations example

14 copies of a code module are to be executed in parallel on identical processors grouped into 2 clusters, A with 16 processors and B with 32 processors.

How many ways can modules be assigned to processors if exactly 2 modules are to execute on cluster B?

$$C(32,2) \cdot C(16,12) = 902{,}720$$

choose $14 - 2 = 12$ processors from A

choose 2 processors from B

Exercise 47

Duplicates

Care must be taken to avoid counting an object more than once or not at all.

A committee of 8 students is to be selected from a class consisting of 19 freshman and 34 sophomores.

In how many ways can a committee with at least 1 freshman be selected?

$$C(19,1)\cdot C(52,7) = \frac{19!}{1!\cdot 18!}\cdot\frac{52!}{7!\cdot 45!} = \text{too many}$$

choose rest of committee

choose freshman

This incorrect solution counts freshman more than once, in both $C(19, 1)$ and $C(52, 7)$.

Example 56

Duplicates example

A committee of 2 to be chosen from 4 math majors and 3 physics majors must include at least 1 math major.

Correct solution

$$C(7,2) - C(3,2) = 18$$

committees with only physics majors

all possible committees

Incorrect solution

$$C(4,1) \cdot C(6,1) = 24$$

choose rest of committee

choose math major

Practice 33

Computing factorial

Values of factorial function grow quickly.

158 digits

$$C(100, 25) = \frac{100!}{25! \cdot 75!} = 242{,}519{,}269{,}720{,}337{,}000{,}000{,}000$$

26 digits 110 digits

Values of this size can overflow program variables.
Canceling common factors beforehand may help.

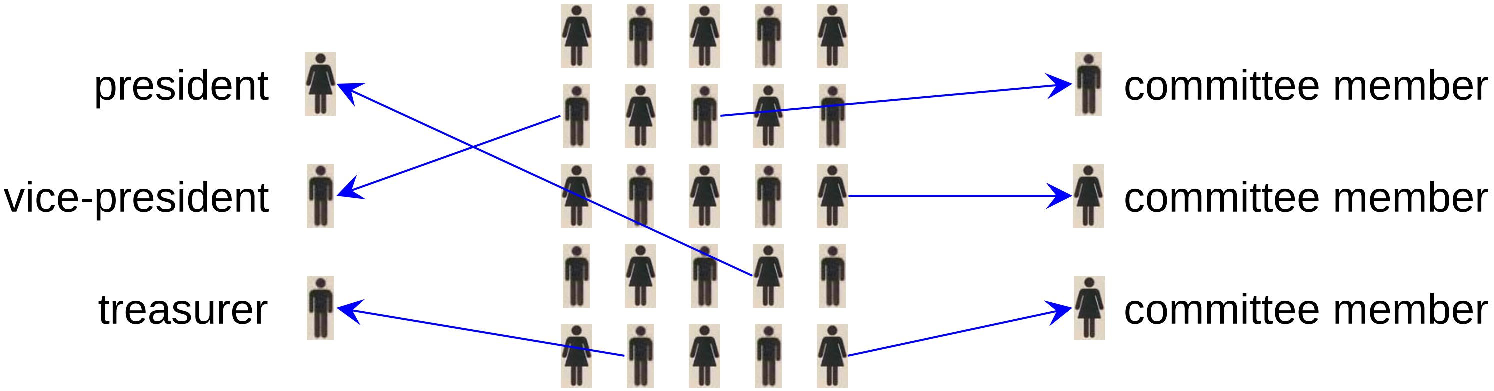
$$C(100, 25) = \frac{100!}{25! \cdot 75!} = \frac{100 \cdot 99 \cdot 98 \ldots 76}{25!}$$

$$= \frac{4 \cdot 99 \cdot 98 \cdot 97 \cdot 4 \cdot 95 \cdot 94 \cdot 93 \cdot 4 \cdot \cdots \cdot 4 \cdot 79 \cdot 78 \cdot 77 \cdot 4}{18!}$$

$$= 242{,}519{,}269{,}720{,}337{,}000{,}000{,}000$$

Permutations or combinations?

- Permutations, order does matter, e.g., president, vice-president, and treasurer
- Combinations; order does not matter, e.g., 3 person committee



Comparing permutations and combinations

Permutations

$P(n,r)$	1	2	3	4	5	6	7	8	9	10
1	1									
2	2	2								
3	3	6	6							
4	4	12	24	24						
5	5	20	60	120	120					
6	6	30	120	360	720	720				
7	7	42	210	840	2520	5040	5040			
8	8	56	336	1680	6720	20160	40320	40320		
9	9	72	504	3024	15120	60480	181440	362880	362880	
10	10	90	720	5040	30240	151200	604800	1814400	3628800	3628800

Combinations

$C(n,r)$	1	2	3	4	5	6	7	8	9	10
1	1									
2	2	1								
3	3	3	1							
4	4	6	4	1						
5	5	10	10	5	1					
6	6	15	20	15	6	1				
7	7	21	35	35	21	7	1			
8	8	28	56	70	56	28	8	1		
9	9	36	84	126	126	84	36	9	1	
10	10	45	120	210	252	210	120	45	10	1

?

Figure 4.9

Permutations or combinations with repetitions

- Permutation and combination formulas $P(n, r)$ and $C(n, r)$ assume objects are arranged or selected without repetitions
- Permutations and combinations can also be considered with repetitions
- e.g., 2 letters from $\{x, y, z\}$

	without repetitions	with repetitions
permutations	xy, xz, yz, yx, zx, zy	xy, xz, yz, yx, zx, zy, xx, yy, zz
combinations	xy, xz, yz	xy, xz, yz, xx, yy, zz

Formulas

- Permutation or combination of r objects chosen from n objects, with repetitions

Permutations n^r

Combinations $C(r+n-1, r) = \dfrac{(r+n-1)!}{r!(r+n-1-r)!} = \dfrac{(r+n-1)!}{r!(n-1)!}$

	without repetitions	with repetitions
permutations	$P(3,2) = 6$	$3^2 = 9$
combinations	$C(3, 2) = 3$	$C(2 + 3 - 1, 2)$ $= C(4, 2) = 6$

Combinations with repetitions example

Six children choose one lollipop each from among a selection of red, yellow, and green lollipops. In how many ways can this be done? We don't care which child gets which.

"don't care which child" $\rightarrow$ order does not matter $\rightarrow$ combinations
"selection of red, …" $\rightarrow$ many of each color $\rightarrow$ with repetitions
$r = 6$ (number of children) and $n = 3$ (number of lollipop colors)

$$C(r+n-1, r) = C(6+3-1, 6)$$
$$= \frac{(6+3-1)!}{6!(6+3-1-6)!}$$
$$= \frac{8!}{6! \cdot 2!} = 28$$

Practice 35

Counting techniques

To count	Technique
Subsets of n-element set	Use 2^n formula
Outcomes of successive independent events	Multiplication principle; multiply number of outcomes for each event
Outcomes of disjoint events	Addition principle; add number of outcomes for each event
Outcomes given specific choices at each step	Create decision tree and count number of paths
Elements in overlapping sections of related sets	Use principle of inclusion and exclusion formula
Ordered arrangements of r out of n distinct objects	Use $P(n, r)$ formula
Ordered arrangements of r out of n objects with indistinguishable collections	Use $n!/(n_1! \cdot n_2! \cdot \ldots \cdot n_k!)$ formula
Unordered selections of r out of n distinct objects	Use $C(n, r)$ formula
Ordered arrangements of r out of n distinct objects with repetitions	Use n^r formula
Unordered selections of r out of n distinct objects with repetitions	Use $C(r + n - 1, r)$ formula

Table 4.2

Generating permutations or combinations

- Formulas $P(n, r)$ and $C(n, r)$ count permutation and combinations
- Some applications require that permutations or combinations be generated
  - e.g., produce all possible lottery tickets
  - e.g., list all possible subcommittee memberships

Permuations of a, b, c

$P(3, 3) = 3! = 6$

Permutations: abc, acb, bac, bca, cab, cba

Lexicographic order; order in which permutations would be found in a dictionary; i.e., sorted by object, left to right

Finding the next permutation

- Algorithm generates permutations of n integers, $P(n, n)$
- Start with permutation $1234\ldots n$
- From each permutation generate next, $P(n, n) - 1 = n! - 1$ times

Procedure to generate the next permutation

1. Scan right to left for first break in increasing sequence d_i
2. Scan right to left for smallest value greater than d_i, found value is d_j
3. Swap d_i and d_j
4. Reverse digits to right of location of d_j (formerly d_i)

1.	1 2 3 **4** 5	$d_i = 4 \quad i = 4$
2.	1 2 3 4 **5**	$d_j = 5 \quad j = 5$
3.	1 2 3 **5** **4**	
4.	1 2 3 5 4	Example 60
1.	1 2 **3** 5 4	$d_i = 3 \quad i = 3$
2.	1 2 3 5 **4**	$d_j = 4 \quad j = 5$
3.	1 2 **4** 5 **3**	
4.	1 2 4 **3** **5**	Example 60

1.	**2** 5 4 3 1	$d_i = 2 \quad i = 1$
2.	2 5 4 **3** 1	$d_j = 3 \quad j = 4$
3.	**3** 5 4 **2** 1	
4.	3 **1** **2** **4** **5**	Example 61
1.	5 **1** 4 3 2	$d_i = 1 \quad i = 2$
2.	5 1 4 3 **2**	$d_j = 2 \quad j = 5$
3.	5 **2** 4 3 **1**	
4.	5 2 **1** **3** **4**	Practice 37

Algorithm *PermGenerator*(integer $n \geq 2$)
// Generates in lexicographic order all permutations of integers $\{1, \ldots, n\}$

Local variables:
integers i, j // indices of permutation elements
integer k // for loop counter
integers $d_1, d_2, \ldots, d_n$ // left to right elements of permutation

```
// create and write out first permutation
for k = 1 to n do
    d_k = k
end for
write d_1 d_2 … d_n

// create and write out remaining permutations
for k = 2 to n! do
    // look right to left for first break in increasing sequence
    i = n – 1
    j = n
    while d_i > d_j do  // still increasing right to left
        i = i – 1
        j = j – 1
    end while
    // d_i < d_j, need to replace d_i with next largest integer
```

Part 1 of 2

```
        // look right to left for smallest value greater than d_i
        j = n
        while d_i > d_j do
            j = j – 1
        end while
        // now d_j is smallest value > d_i

        swap d_i and d_j

        // reverse the digits to the right of index i
        i = i + 1
        j = n
        while i < j do
            swap d_i and d_j
            i = i + 1
            j = j – 1
        end while

        write d_1d_2…d_n
    end for
end function PermGenerator
```

Part 2 of 2

Finding the next combination

- Algorithm generates combinations of r integers out of n, $C(n, r)$
- Start with combination $1234\ldots r$
- From each combination generate next, $C(n, r) - 1$ times

Procedure to generate the next combination

1. Scan right to left for first digit not at max value d_i (max value of digit k is $n - r + k$)
2. Increment d_i
3. Set digits to right of d_i to min value (min value of digit k is value of digit $(k - 1) + 1$)

Example 62

1. 2 3 4 **6**
2. 2 3 4 **7**
3. 2 3 4 7

Example 62

1. 2 3 **4** 7
2. 2 3 **5** 7
3. 2 3 5 **6**

Example 62

1. 2 **3** 6 7
2. 2 **4** 6 7
3. 2 4 **5** **6**

Practice 38

1. 2 4 **5** 8 9
2. 2 4 **6** 8 9
3. 2 4 6 **7** **8**

Algorithm *CombGenerator*(integer $n \geq 2$, integer $r \geq 1$)
// Generates in lexicographic order all combinations of r integers from $\{1, \ldots, n\}$

Local variables:
integers i, j // indices of combination elements
integer k // for loop counter
integer max // maximum allowable value for a digit
integers $d_1, d_2, \ldots, d_r$ // left to right elements of combination

```
// create and write out first combination
for k = 1 to r do
    d_k = k
end for
write d_1 d_2 … d_r

// create and write out remaining combinations
for k = 2 to C(n, r) do
    // look right to left for first non-max value
    max = n
    i = r
    while d_i = max do  // look left
        i = i – 1
        max = max – 1
    end while
    // d_i < max, need to increment d_i
```

Part 1 of 2

```
            d_i = d_j + 1

            // reset values right of d_i
            for j = i + 1 to r do
                  d_j = d_(j–1) + 1
            end for

            write d_1 d_2 … d_n
      end for
end function CombGenerator
```

$d_i = d_j + 1$

// reset values right of d_i
for $j = i + 1$ to r **do**
$d_j = d_{j-1} + 1$
end for

write $d_1 d_2 \ldots d_n$
end for
end function CombGenerator

Part 2 of 2

Section 4.4 homework assignment

See homework list for specific exercises.

Unless specifically asked to, don't compute values of expressions such as "$P(50, 20)$" or "$C(32, 10)$".

4.5 Binomial Theorem

Raising a binomial to a power

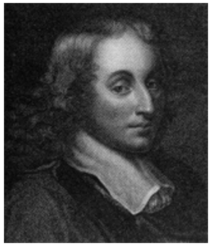
n	$(a+b)^n$
0	$(a+b)^0 = 1$
1	$(a+b)^1 = a + b$
2	$(a+b)^2 = a^2 + 2ab + b^2$
3	$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
4	$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
5	$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
…	…

Practice 39

Pascal's triangle: binomial coefficients

Arranging coefficients of terms in binomial expansions in a triangular shape …

0						1						
1					1		1					
2				1		2		1				
3			1		3		3		1			
4		1		4		6		4		1		
5	1		5		10		10		5		1	



Blaise Pascal
1623-1662

… reveals a pattern in the values:
each value is the sum of the two values above it.

Pascal's triangle, combinations

The values in Pascal's triangle match combination values.

0						$C(0, 0)$						
1					$C(1, 0)$		$C(1, 1)$					
2				$C(2, 0)$		$C(2, 1)$		$C(2, 2)$				
3			$C(3, 0)$		$C(3, 1)$		$C(3, 2)$		$C(3, 3)$			
4		$C(4, 0)$		$C(4, 1)$		$C(4, 2)$		$C(4, 3)$		$C(4, 4)$		
5	$C(5, 0)$		$C(5, 1)$		$C(5, 2)$		$C(5, 3)$		$C(5, 4)$		$C(5, 5)$	

Does the "sum of two above" rule apply to combinations?
Pascal's formula:
$C(n, k) = C(n - 1, k - 1) + C(n - 1, k)$ for $1 \leq k \leq n - 1$

Proof of Pascal's formula

$C(n-1,k-1)+C(n-1,k)$

$= \frac{(n-1)!}{(k-1)![n-1-(k-1)]!} + \frac{(n-1)!}{k!(n-1-k)!}$ formula for $C(n, k)$

$= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!}$ simplify denominator

$= \frac{k(n-1)!}{k!(n-k)!} + \frac{(n-1)!(n-k)}{k!(n-k)!}$ multiply first term by k/k and second by $(n-k)/(n-k)$

$= \frac{k(n-1)! + (n-1)!(n-k)}{k!(n-k)!}$ add fractions

$= \frac{(n-1)![k+(n-k)]}{k!(n-k)!}$ factor out $(n-1)!$ in numerator

$= \frac{n(n-1)!}{k!(n-k)!}$ simplify numerator

$= \frac{n!}{k!(n-k)!}$ definition of $n!$

$= C(n,k)$ ■ formula for $C(n, k)$

Binomial theorem

Relates values in Pascal's triangle to coefficients of binomial expansion.

Binomial theorem. For every nonnegative integer n,

$$
\begin{aligned}
(a+b)^n &= C(n,0)a^n b^0 + c(n,1)a^{n-1}b^1 + \ldots \\
&\quad + C(n,k)a^{n-k}b^k + \ldots \\
&\quad + C(n,n-1)a^1 b^{n-1} + C(n,n)a^0 b^n \\
&= \sum_{k=0}^{n} C(n,k)a^{n-k}b^k
\end{aligned}
$$

for example,

$$
\begin{aligned}
(a+b)^3 &= C(3,0)a^3b^0 + C(3,1)a^2b^1 + C(3,2)a^1b^2 + C(3,3)a^0b^3 \\
&= a^3 + 3a^2b + 3ab^2 + b^3
\end{aligned}
$$

Proof of the binomial theorem

"For every nonnegative integer n," suggests induction.

Proof (by induction on n)
Basis. $P(0)$: $(a + b)^0 = C(0, 0)a^0b^0 = 1$

Inductive hypothesis. $P(k)$: Assume $(a + b)^k =$
$C(k, 0)a^kb^0 + C(k, 1)a^{k-1}b^1 + \ldots + C(k, k-1)a^1b^{k-1} + \mathrm{C}(k, k)a^0b^k$

Inductive step. $P(k + 1)$: Show $(a + b)^{k+1} =$
$C(k + 1, 0)a^{k+1}b^0 + C(k + 1, 1)a^kb^1 + C(k + 1, 2)a^{k-1}b^2 + \ldots$
$+ C(k + 1, k)a^1b^k + \mathrm{C}(k + 1, k + 1)a^0b^{k+1}$

$(a + b)^{k+1}$ — left side of $P(k + 1)$

$= (a + b)^k(a + b)$ — prop of exponents

$= (a + b)^k a + (a + b)^k b$ — distributivity

$= [C(k, 0)a^k b^0 + C(k, 1)a^{k-1}b^1 + \ldots +$
$C(k, k-1)a^1 b^{k-1} + C(k, k)a^0 b^k]a +$
$[C(k, 0)a^k b^0 + C(k, 1)a^{k-1}b^1 + \ldots +$
$C(k, k-1)a^1 b^{k-1} + C(k, k)a^0 b^k]b$ — substitution of IH

$= C(k, 0)a^{k+1}b^0 + C(k, 1)a^k b^1 + \ldots +$
$C(k, k-1)a^2 b^{k-1} + C(k, k)a^1 b^k +$
$C(k, 0)a^k b^1 + C(k, 1)a^{k-1}b^2 + \ldots +$
$C(k, k-1)a^1 b^k + C(k, k)a^0 b^{k+1}$ — multiply through

$= C(k, 0)a^{k+1}b^0 + [C(k, 0) + C(k, 1)]a^k b^1 +$
$[C(k, 1) + C(k, 2)]a^{k-1}b^2 + \ldots +$
$[C(k, k-1) + C(k, k)]a^1 b^k + C(k, k)a^0 b^{k+1}$ collect terms

$= C(k, 0)a^{k+1}b^0 + C(k + 1, 1)a^k b^1 +$
$C(k + 1, 2)a^{k-1}b^2 + \ldots +$
$C(k + 1, k)a^1 b^k + C(k, k)a^0 b^{k+1}$ using Pascal's formula

$= \mathrm{C}(k + 1, 0)a^{k+1}b^0 + C(k + 1, 1)a^k b^1 +$
$C(k + 1, 2)a^{k-1}b^2 + \ldots +$
$C(k + 1, k)a^1 b^k + C(k + 1, k + 1)a^0 b^{k+1}$ ■

$C(k, 0) = \mathrm{C}(k + 1, 0) = 1$
$C(k, k) = \mathrm{C}(k + 1, k + 1) = 1$
right side of $\mathrm{P}(k + 1)$

Example applications of the binomial theorem

Using the binomial theorem, expand:

$$\begin{aligned}(x-3)^4 &= C(4,0)x^4(-3)^0 + C(4,1)x^3(-3)^1 + C(4,2)x^2(-3)^2 + \\ &\quad C(4,3)x^1(-3)^3 + C(4,4)x^0(-3)^4 \\ &= x^4 + 4x^3(-3) + 6x^2(9) + 4x(-27) + 81 \\ &= x^4 - 12x^3 + 54x^2 - 108x + 81\end{aligned}$$

Example 63

$$\begin{aligned}(x+1)^5 &= C(5,0)x^5 1^0 + C(5,1)x^4 1^1 + C(5,2)x^3 1^2 + \\ &\quad C(5,3)x^2 1^3 + C(5,4)x^1 1^4 + C(5,5)x^0 1^5 \\ &= x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1\end{aligned}$$

Practice 40

Using the binomial theorem, find the fifth term in $(x + y)^7$:

$$
\begin{aligned}
(x + y)^7 = \; & C(7, 0)x^7y^0 + C(7, 1)x^6y^1 + C(7, 2)x^5y^2 + \\
& C(7, 3)x^4y^3 + C(7, 4)x^3y^4 + C(7, 5)x^2y^5 + \\
& C(7, 6)x^1y^6 + C(7, 7)x^0y^7
\end{aligned}
$$

Practice 41

In general, for $(a + b)^n$, the kth term is $C(n, k - 1)a^{n-k+1}b^{k-1}$

Section 4.5 homework assignment

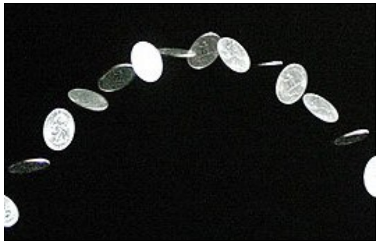
See homework list for specific exercises.

4.6 Probability

Probability examples

What is the probability of

Getting “heads” when a coin is tossed?
Possible outcomes $\{H, T\}$
Outcomes of interest $\{H\}$
Probability $1/2$



Getting a 3 with a roll of a die?
Possible outcomes $\{1, 2, 3, 4, 5, 6\}$
Outcomes of interest $\{3\}$
Probability $1/6$

Example 65

Drawing either A♣ or Q♥ from a standard deck of cards?
Possible outcomes {A♣, 2♣, …, A♥, …, A♦, …, A♠, …}
Outcomes of interest {A♣, Q♥}
Probability 2/52

Finite probability

- Concepts
  - How likely is a certain outcome (or set of outcomes)?
  - Counting techniques often apply
- Definitions
  - Sample space; finite set of all possible outcomes
    - Denoted S
    - Assumption: S finite
    - Assumption: all possible outcomes equally likely
  - Event; subset of the sample space, denoted E
  - Probability of an event, denoted $P(E)$

$$P(E) = \frac{|E|}{|S|}$$

Probability examples

Two fair coins are tossed at the same time.
What is the probability of getting 2 heads?

$S = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$ $|S| = 4$

$E = \{\mathrm{HH}\}$ $|E| = 1$

$$P(E) = \frac{|E|}{|S|} = \frac{|\{\mathrm{HH}\}|}{|\{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}|} = \frac{1}{4}$$

Example 66

What is the probability of drawing an ace from a standard card deck?

$S = \{\mathrm{A}\clubsuit, 2\clubsuit, \ldots, \mathrm{A}\heartsuit, \ldots, \mathrm{A}\diamondsuit, \ldots, \mathrm{A}\spadesuit, \ldots\}$ $|S| = 52$

$E = \{\mathrm{A}\clubsuit, \mathrm{A}\heartsuit, \mathrm{A}\diamondsuit, \mathrm{A}\spadesuit\}$ $|E| = 4$

$$P(E) = \frac{|E|}{|S|} = \frac{4}{52}$$

Practice 42

Probability example

A family has 3 children; boys and girls are equally likely.

$S = \{\text{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG}\}$ $|S| = 8$

What is probability that the oldest child is a boy?

$E = \{\text{BBB, BBG, BGB, BGG}\}$ $|E| = 4$

$P(E) = 4/8$

What is probability of 2 boys and 1 girl?

$E = \{\text{BBG, BGB, GBB}\}$ $|E| = 3$

$P(E) = 3/8$

What is probability of 3 girls?

$E = \{\text{GGG}\}$ $|E| = 1$

$P(E) = 1/8$

Exercises 81, 83, 86

Events as sets

- Range of values for $P(E)$
  - $0 \le P(E) \le 1$
  - $E = \emptyset \rightarrow P(E) = 0$; $E = S \rightarrow P(E) = 1$
- Set operations and counting techniques
  - Events are sets; set operations apply;
    e.g., let E_1 and E_2 be events (sets) from S
    - Outcomes from E_1 or E_2 or both is set $E_1 \cup E_2$
    - Outcomes from both E_1 and E_2 is set $E_1 \cap E_2$
    - Outcomes not in E_1 is set E_1'
  - Event sets have sizes; counting techniques apply
    - Multiplication or addition principles
    - Principle of inclusion and exclusion
    - Combinations

Probability example

Employees from testing, development, and marketing participate in a drawing in which one name is chosen.

	Men	Women
Testing	2	3
Development	16	7
Marketing	6	8

$S = \{x \mid x \text{ is an employee name}\}$
$|S| = 42$

W = event that name drawn is woman
$|W| = 3 + 7 + 8 = 18$
$P(W) = |W|/|S| = 18/42$

Example 67

	Men	Women
Testing	2	3
Development	16	7
Marketing	6	8

M = event that name drawn from marketing
$|M| = 14$
$P(M) = |M|/|S| = 14/42$

$W \cap M$ = event that name drawn is woman and marketing
$|W \cap M| = 8$
$P(W \cap M) = 8/42$

$W \cup M$ = event that name drawn is woman or marketing
$|W \cup M| = 3 + 7 + 14 = 24$ by inspection
$|W \cup M| = |W| + |M| - |W \cap M| = 18 + 14 - 8 = 24$ by PI&E
$P(W \cup M) = 24/42$

Probability example

At a party, each card in a standard deck is torn in half and the halves are placed in a box.
Two guests each take a half-card from the box.
What is the probability they draw halves of the same card?

$52 \cdot 2 = 104$ half-cards in the box
S = set of possible pairs of half-cards that can be chosen
$|S| = C(104, 2)$ ways to choose 2 items from 104
H = event that halves match
$|H| = 52$

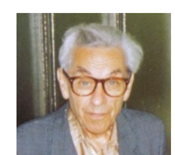
$$P(H) = \frac{|H|}{|S|} = \frac{52}{C(104, 2)} = \frac{52}{\left(\dfrac{104!}{2! \cdot 102!}\right)} = \frac{52}{52 \cdot 103} = \frac{1}{103}$$

Example 68

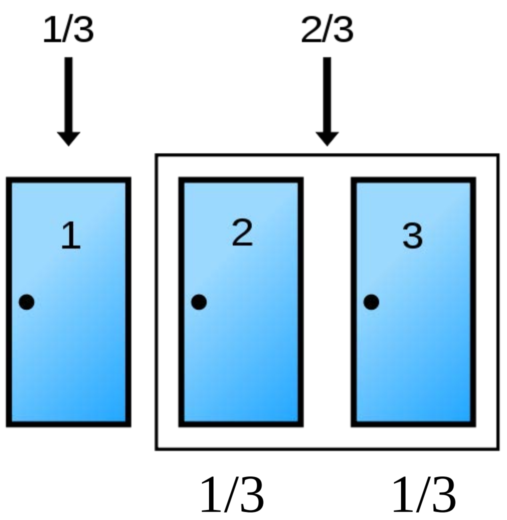
Monty Hall

- *Let's Make A Deal* game show final challenge has three doors
- Behind one door is a valuable prize (e.g., car)
- Behind the other two doors are "zonk" gifts (e.g., goat)
- Contestant allowed to choose one door
- Host opens one door contestant did not choose, showing a goat
- Host offers contestant chance to switch to the other unopened door
- What should the contestant do?

Marilyn vos Savant: Switch doors.
Many professional mathematicians: It doesn't matter.



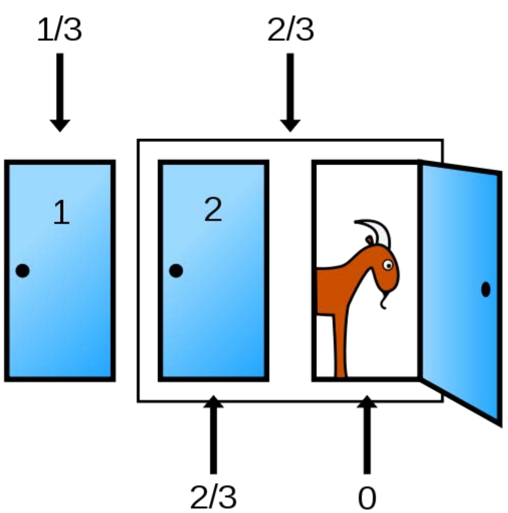
Solution 1 (analytic)



Initial selection

P(car behind door 1) = 1/3

P(car behind door 2 or door 3) = 2/3



After goat is revealed

P(car behind door 1) = 1/3

P(car behind door 2) = 2/3

P(car behind door 3) = 0

Solution 2 (analytic)

Door 1	Door 2	Door 3	You choose	Host opens	P	Result if switch	Result if stay
Car	Goat	Goat	Door 1	Door 2	1/18	Goat	Car
				Door 3	1/18	Goat	Car
			Door 2	Door 1	0	Not allowed	
				Door 3	1/9	Car	Goat
			Door 3	Door 1	0	Not allowed	
				Door 2	1/9	Car	Goat
Goat	Car	Goat	Door 1	Door 2	0	Not allowed	
				Door 3	1/9	Car	Goat
			Door 2	Door 1	1/18	Goat	Car
				Door 3	1/18	Goat	Car
			Door 3	Door 1	1/9	Car	Goat
				Door 2	0	Not allowed	
Goat	Goat	Car	Door 1	Door 2	1/9	Car	Goat
				Door 3	0	Not allowed	
			Door 2	Door 1	1/9	Car	Goat
				Door 3	0	Not allowed	
			Door 3	Door 1	1/18	Goat	Car
				Door 2	1/18	Goat	Car

P(Car | Switch) = (6 · 1/9) = 2/3 P(Car | Stay) = (6 · 1/18) = 1/3
P(Goat | Switch) = (6 · 1/18) = 1/3 P(Goat | Stay) = (6 · 1/9) = 2/3

Solution 3 (Simulation)

Process steps

(2) Contestant chooses (random, uniform)

(1) Car placed (random, uniform)

	Door 1	Door 2	Door 3
Door 1	2, 3	3	2
Door 2	3	1, 3	1
Door 3	2	1	1, 2

(3) Host opens (random, uniform)

(4) Contestant switches $\in$ { Yes, No } (random, uniform)

(5) Prize determined based on car location, final choice

Model implementation and results of 2,000,000 trials

```
MontyHall <- function(switch) {
  doors      <- c(1, 2, 3)                  # Three possible doors
  car        <- sample(doors, size=1)       # Door car is behind
  contestant <- sample(doors, size=1)       # Door contestant chooses
  goats      <- doors[doors != car]         # Host can't open door car is behind
  avail      <- goats[goats != contestant]  # Host can't open door contestant chooses

  if (length(avail) == 1) { host <- avail }  # Host chooses only available door
  else { host <- sample(avail, size=1) }     # Host chooses one of the available doors

  result <- 1                                             # Assume contestant wins
  if (switch) { if (car == contestant) { result <- 0 } }  # Contestant switched and lost
  else { if (car != contestant) { result <- 0 } }         # Contestant did not switch and lost

  return(result)
}

>
> trials  <- 1000000
> Results <- replicate(trials, MontyHall(FALSE)) # FALSE means contestant does not switch doors
> sum(Results)/trials
[1] 0.333758
>
>
> trials  <- 1000000
> Results <- replicate(trials, MontyHall(TRUE))  # TRUE means contestant does switch doors
> sum(Results)/trials
[1] 0.66623
>
```

Probability observations

	Observation	Justification
1	$0 \le P(E) \le 1$	$E \subseteq S$, so $0 \le \lvert E \rvert$ and $\lvert E \rvert \le \lvert S \rvert$
2	Probability of an impossibility is 0	$E = \emptyset$, so $\lvert E \rvert = 0$
3	Probability of a certainty is 1	$E = S$, so $\lvert E \rvert = \lvert S \rvert$
4	$P(E') = 1 - \mathrm{P}(E)$	$\lvert E' \rvert = \lvert S \rvert - \lvert E \rvert$
5	$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$	Principle of inclusion and exclusion
6	If E_1 and E_2 are disjoint events (sets), $P(E_1 \cup E_2) = P(E_1) + P(E_2)$	Follows from observation 5

Table 4.3

Outcomes not equally likely

- Earlier definition of $P(E)$ applies when all outcomes equally likely; what if they aren't?
- Approaches to handling
  - Outcome repetitions; simpler, less useful
  - Probability distributions; not as simple, more useful

Outcome repetitions example

A fair die is rolled.

$S = \{1, 2, 3, 4, 5, 6\}$ $\quad |S| = 6$

$T = \{3\}$ event of rolling 3 $\quad |T| = 1$

$$P(T) = \frac{|T|}{|S|} = \frac{1}{6}$$

The die is loaded so 4 is 3 times as likely as 1, 2, 3, 5, or 6. To reflect this, the sample space can be revised.

$S = \{1, 2, 3, 4_1, 4_2, 4_3, 5, 6\}$ $\quad |S| = 8$

$T = \{3\}$ $\quad |T| = 1$

$F = \{4_1, 4_2, 4_3\}$ $\quad |F| = 3$

$$P(T) = \frac{|T|}{|S|} = \frac{1}{8} \qquad P(F) = \frac{|F|}{|S|} = \frac{3}{8}$$

Example 69

Probability distribution

- Assign probability distribution to sample space
  - Drop assumption all outcomes equally likely
  - Assign probability to each outcome in sample space

Example

$S = \{H, T\}$ $P(H) = 1/3$, $P(T) = 2/3$ (unfair coin)

x_i	H	T	outcome
$p(x_i)$	$1/3$	$2/3$	probability of outcome

- Notation
  - Number of outcomes in sample space is k, i.e., $|S| = k$
  - Outcome is x_i, i.e., $S = \{x_1, x_2, \ldots, x_i, \ldots, x_k\}$
  - Assigned probability of outcome x_i is $p(x_i)$
- Rules

1. $0 \leq p(x_i) \leq 1$ — Each probability is between 0 and 1
2. $\sum_{i=1}^{k} p(x_i) = 1$ — The probabilities sum to 1

Now consider event $E \subseteq S$;
E may be a single outcome, or a set of outcomes.
The probability of event E is

$$P(E) = \sum_{x_i \in E} p(x_i)$$

The earlier definition of $P(E) = |E|/|S|$ is a special case of this formula where $p(x_i) = 1/|S|$ for each $x_i \in E$.

Probability distribution example

For loaded die of Example 67, the probability distribution is

x_i	1	2	3	4	5	6
$p(x_i)$	1/8	1/8	1/8	3/8	1/8	1/8

Probability of event E $$P(E) = \sum_{x_i \in E} p(x_i)$$

Probability of rolling 3 $$P(3) = \frac{1}{8}$$

Probability of rolling 4 $$P(4) = \frac{3}{8}$$

Probability of rolling 2 or 4 $$P(E) = P(2) + P(4) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8}$$

Example 70

Probability distribution example

Sample space $S = \{a, b, c\}$; $p(a) = 0.2$, $p(b) = 0.3$

x_i	a	b	c
$p(x_i)$	0.2	0.3	0.5

Probability of outcome c

$$P(c) = 1 - \big(P(a) + P(b)\big) = 1 - (0.2 + 0.3) = 0.5$$

Probability of event a or c

$E = \{a, c\}$

$$P(E) = p(a) + p(c) = 0.2 + 0.5 = 0.7$$

Practice 44

Probability distribution examples

A loaded die has this probability distribution

x_i	1	2	3	4	5	6
$p(x_i)$	0.2	0.05	0.1	0.2	0.3	0.15

E_1 roll is odd $P(E_1) = P(1) + P(3) + P(5) = 0.6$

E_2 roll is 3 or 6 $P(E_2) = P(3) + P(6) = 0.25$

E_3 roll is ≥ 4 $P(E_3) = P(4) + P(5) + P(6) = 0.65$

E_2 and E_3 $P(E_2 \cap E_3) = P(6) = 0.15$

E_1 or E_3 $P(E_1 \cup E_3) = P(E_1) + P(E_3) - P(E_1 \cap E_3) = 0.95$

Exercise 69

Conditional probability

A fair coin is tossed twice.

$S = \{\text{HH, HT, TH, TT}\}$	$\|S\| = 4$	
$T = \{\text{TT}\}$	$\|T\| = 1$	first and second toss T
$E_1 = \{\text{TH, TT}\}$	$\|E_1\| = 2$	first toss is T
$E_2 = \{\text{HT, TT}\}$	$\|E_2\| = 2$	second toss is T
$E_1 \cap E_2 = \{\text{TT}\}$		first and second toss T

$$P(T) = \frac{|T|}{|S|} = \frac{|E_1 \cap E_2|}{|S|} = \frac{1}{4}$$

Suppose we know first toss is T; how likely is two tails?

Sample space has only outcomes where first toss was T, i.e., event $E_1 = \{\mathrm{TH}, \mathrm{TT}\}$ occurred, so sample space is E_1.

$E_2|E_1$ denotes probability E_2 occurs, given E_1 occurs.

$$P(E_2 \mid E_1) = \frac{|E_1 \cap E_2|}{|E_1|} = \frac{1}{2}$$

In terms of probabilities

$$P(E_2 \mid E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{1/4}{2/4} = \frac{1}{2}$$

E_1 E_2 $E_1 \cap E_2$

x_i	HH	HT	TH	TT
$p(x_i)$	1/4	1/4	1/4	1/4

Conditional probability formula

Given events E_1 and E_2 and sample space S, the conditional probability of E_2 given E_1, denoted $P(E_2|E_1)$, is

$$P(E_2 \mid E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)}$$

Conditional probability example

In a drug study on a group of patients,
17% responded positively to compound A, $P(A) = 0.17$
34% responded positively to compound B, $P(B) = 0.34$
8% responded positively to both. $P(A \cap B) = 0.08$

What is the probability that a patient responds positively to B, given a positive response to A?

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{0.08}{0.17} \approx 0.47$$

Example 71

What is the probability that a patient responds positively to A, given a positive response to B?

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.08}{0.34} \approx 0.24$$

What is the probability that a patient responds positively to either A or B?

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.17 + 0.34 - 0.08 = 0.43$$

What is the probability that a patient does not respond positively to either A or B?

$$P(A \cup B)' = 1 - P(A \cup B) = 1 - 0.43 = 0.57$$

Exercise 73

Conditional probability example

A fair coin is tossed twice.
What is the probability of getting two H,
given that at least one of the tosses is H?

$S = \{\mathrm{HH, HT, TH, TT}\}$ $|S| = 4$

$E_1 = \{\mathrm{HH, HT, TH}\}$ $|E_1| = 3$ at least one H
$E_2 = \{\mathrm{HH}\}$ $|E_2| = 1$ two H
$E_1 \cap E_2 = \{\mathrm{HH}\}$ $|E_1 \cap E_2| = 1$

$$P(E_2 \mid E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Practice 45

Independent events

If $P(E_2|E_1) = P(E_2)$, then E_2 is as likely to occur whether E_1 occurs or not.

In such cases, E_1 and E_2 are independent events.

Independence implies these equations and vice versa.

$$P(E_2 \mid E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} = P(E_2)$$

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

Independence goes both ways.

$$P(E_2 \mid E_1) = P(E_2) \leftrightarrow P(E_1 \mid E_2) = P(E_1)$$

Independent events example

A fair coin is tossed twice; the events H as the first toss (E_1) and T as the second (E_2) are independent.

$S = \{\text{HH, HT, TH, TT}\}$
$E_1 = \{\text{HH, HT}\}$
$E_2 = \{\text{HT, TT}\}$

$$P(E_1 \cap E_2) = \frac{1}{4} \quad P(E_1) = \frac{1}{2} \quad P(E_2) = \frac{1}{2}$$

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

Example 72

Probability rules

- "Addition rule" for probability
  - Probability of E_1 or E_2: $P(E_1 \cup E_2) = P(E_1) + P(E_2)$
  - Applies iff the events are disjoint
  - Otherwise use principle of exclusion and inclusion
- "Multiplication rule" for probability
  - Probability of E_1 and E_2: $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$
  - Applies iff the events are independent
  - Otherwise use conditional probability formula

Weighted average

A student takes 3 tests; set of grades is $S = \{g_1, g_2, g_3\}$.
Average test grade is

$$A(g) = \frac{g_1 + g_2 + g_3}{3}$$

This assumes the grades are weighted evenly.

$$A(g) = \frac{1}{3}(g_1 + g_2 + g_3) = g_1\left(\frac{1}{3}\right) + g_2\left(\frac{1}{3}\right) + g_3\left(\frac{1}{3}\right)$$

If the grades are weighted differently, e.g., g_3 is $2\times$,

$$A(g) = \frac{1}{4}(g_1 + g_2 + 2 \cdot g_3) = g_1\left(\frac{1}{4}\right) + g_2\left(\frac{1}{4}\right) + g_3\left(\frac{1}{2}\right)$$

Consider S (the set of grades) as the sample space and assign a probability distribution using the weights

x_i	g_1	g_2	g_3
$p(x_i)$	1/4	1/4	2/4

Average grade

$$A(g) = \sum_{i=1}^{3} x_i \cdot p(x_i)$$

This is an example of a weighted average.

Expected value

If the sample space values are not numerical,
define function $X: S \rightarrow \mathbb{R}$ to associate a numerical value with each element of the sample space.
Such a function is a random variable.

Given sample space $S = \{x_1, x_2, \ldots, x_n\}$,
with random variable X and probability distribution p,
the weighted average or expected value of X is

$$E(X) = \sum_{i=1}^{n} X(x_i) \cdot p(x_i)$$

Expected value example

A fair coin is tossed three times.
X gives the number of heads in each outcome.

x_i	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$X(x_i)$	3	2	2	1	2	1	1	0
$p(x_i)$	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8

The expected number of heads in three tosses is

$$E(X) = \sum_{i=1}^{8} X(x_i) \cdot p(x_i)$$

$$= 3\left(\frac{1}{8}\right) + 2\left(\frac{1}{8}\right) + 2\left(\frac{1}{8}\right) + 1\left(\frac{1}{8}\right) + 2\left(\frac{1}{8}\right) + 1\left(\frac{1}{8}\right) + 1\left(\frac{1}{8}\right) + 0\left(\frac{1}{8}\right)$$

$$= 12\left(\frac{1}{8}\right) = 3/2 = 1.5$$

Example 74

Suppose now that the coin is weighted so that $P(H) = 3/4$ and $P(T) = 1/4$.

x_i	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$X(x_i)$	3	2	2	1	2	1	1	0
$p(x_i)$	27/64	9/64	9/64	3/64	9/64	3/64	3/64	1/64

The expected number of heads in three tosses is now

$$
\begin{aligned}
E(X) &= \sum_{i=1}^{8} X(x_i) \cdot p(x_i) \\
&= 3\left(\frac{27}{64}\right) + 2\left(\frac{9}{64}\right) + 2\left(\frac{9}{64}\right) + 1\left(\frac{3}{64}\right) + 2\left(\frac{9}{64}\right) + 1\left(\frac{3}{64}\right) + 1\left(\frac{3}{64}\right) + 0\left(\frac{1}{64}\right) \\
&= 144/64 = 2.25
\end{aligned}
$$

Expected value example

A computer folder holds 12 files, 3 infected with viruses.
If an infected file is selected, it is cleaned, then another file is selected.
What is the average number of files selected to find a virus-free file?

x_i	F	VF	VVF	VVVF
$X(x_i)$	1	2	3	4
$p(x_i)$	(9/12)	(3/12)(9/11)	(3/12)(2/11)(9/10)	(3/12)(2/11)(1/10)(9/9)

$$E(X) = 1\left(\frac{9}{12}\right) + 2\left(\frac{3}{12}\cdot\frac{9}{11}\right) + 3\left(\frac{3}{12}\cdot\frac{2}{11}\cdot\frac{9}{10}\right) + 4\left(\frac{3}{12}\cdot\frac{2}{11}\cdot\frac{1}{10}\cdot\frac{9}{9}\right)$$
$$= \frac{1716}{1320} = 1.3$$

Exercise 96

Average case analysis for algorithms

- Average case analysis of algorithms
  - Can be done using expected value formula
  - Requires some knowledge of algorithm, inputs
- Procedure
  - Identify “unit of work”, i.e., basic operation
  - Define sample space S as set of all possible inputs
  - Let random variable X give number of operations for an input
  - Let probability distribution p give probability of an input
  - Compute expected value over all inputs

Average case analysis example

Algorithm: sequential search (§ 2.6), assume target on list.
Basic operation: comparison of target with list element

x_i	L_1	L_2	…	L_n
$X(x_i)$	1	2	…	n
$p(x_i)$	$1/n$	$1/n$	…	$1/n$

location of target in list

number of comparisons to find

probability of finding at L_i

The expected number of comparisons in average case is

$$
\begin{aligned}
E(X) &= \sum_{i=1}^{n} X(x_i) \cdot p(x_i) \\
&= \sum_{i=1}^{n} i \cdot \left(\frac{1}{n}\right) = \frac{1}{n}\sum_{i=1}^{n} i \\
&= \frac{1}{n}(1+2+\ldots+n) = \frac{1}{n}\frac{n(n+1)}{2} = \frac{n+1}{2}
\end{aligned}
$$

Example 76

Section 4.6 homework assignment

See homework list for specific exercises.

End

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