# Laplace Transform Examples

CPE 381 Foundations of Signals & Systems for Computer Engineers

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# One-sided Laplace Transform

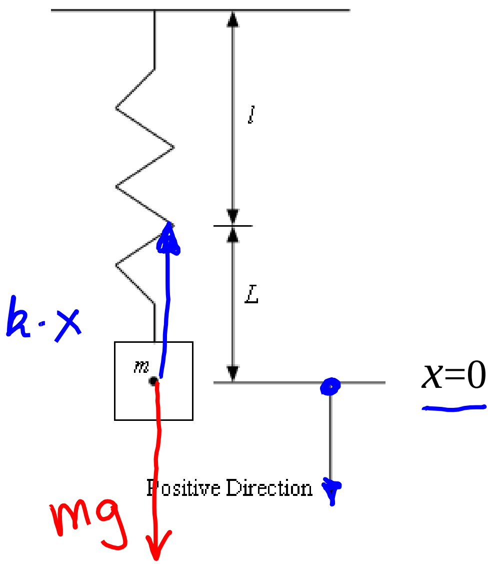
| Function of Time | Function of $s$, ROC |
|---|---|
| $\delta(t)$ | 1, whole $s$-plane |
| $u(t)$ | $\frac{1}{s}$, $\mathcal{R}e[s] > 0$ |
| $r(t)$ | $\frac{1}{s^2}$, $\mathcal{R}e[s] > 0$ |
| $e^{-at}u(t),\ a > 0$ | $\frac{1}{s+a}$, $\mathcal{R}e[s] > -a$ |
| $\cos(\Omega_0 t)u(t)$ | $\frac{s}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$ |
| $\sin(\Omega_0 t)u(t)$ | $\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{R}e[s] > 0$ |
| $e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$ | $\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$ |
| $e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$ | $\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{R}e[s] > -a$ |
| $2A\, e^{-at}\cos(\Omega_0 t + \theta)u(t),\ a > 0$ | $\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{R}e[s] > -a$ |

# One-sided Laplace Transform

| | | | |
|---|---|---|---|
| Causal functions and constants: | $\alpha f(t)$ | $\Leftrightarrow$ | $\alpha F(s)$ |
| Linearity: | $\alpha f(t) + \beta g(t)$ | $\Leftrightarrow$ | $\alpha F(s) + \beta G(s)$ |
| Time shifting: | $f(t - \alpha)$ | $\Leftrightarrow$ | $e^{-\alpha s} F(s)$ |
| Frequency shifting: | $e^{\alpha t} f(t)$ | $\Leftrightarrow$ | $F(s - \alpha)$ |
| Multiplication by t: | $tf(t)$ | $\Leftrightarrow$ | $-\frac{dF(s)}{ds}$ |
| Derivative: | $\frac{df(t)}{dt}$ | $\Leftrightarrow$ | $sF(s) - f(0-)$ |
| Second derivative: | $\frac{d^2 f(t)}{dt^2}$ | $\Leftrightarrow$ | $s^2 F(s) - sf(0-) - f^{(1)}(0)$ |
| Integral: | $\int_{0-}^{t} f(t')dt$ | $\Leftrightarrow$ | $\frac{F(s)}{s}$ |
| Expansion/Contraction: | $f(\alpha t) \alpha \neq 0$ | $\Leftrightarrow$ | $\frac{1}{\lvert a \rvert} F(\frac{s}{\alpha})$ |

# Suspended Weight #1

- Write differential equation describing displacement $x$ of suspended weight $m$ on spring with elastic constant $k$.




At any time, sum of all forces is equal to zero

$$m\ddot{x} + c\dot{x} + kx = 0$$

With initial conditions

$$x(0)[m] \; and \;\; \dot{x}(0)$$

By using Laplace transform

$$\mathcal{L}(m\ddot{x} + c\dot{x} + kx) = ms^2X(s) - msx(0) - m\dot{x}(0) + csX(s) - cx(0) + kX(s) = 0$$

$$(ms^2 + cs + k)X(s) = msx(0) + cx(0)$$

$$X(s) = \frac{msx(0) + cx(0)}{ms^2 + cs + k} = \frac{sx(0) + \frac{c}{m}x(0)}{s^2 + \frac{c}{m}s + \frac{k}{m}}$$

# Suspended Weight #2

**Example #1:**

$$m = 1\ [kg], k = 2\left[\frac{kg}{s^2}\right], c = 2\left[\frac{kg}{s}\right], x(0) = 1\ [m], and\ \dot{x}(0) = 0\ \left[{}^{m}/{}_{s^2}\right]$$

$$X(s) = \frac{s+2}{s^2+2s+17} = \frac{s+2}{(s+1)^2+16}$$

$$s^2 + 2s + 1 + 16$$

$$= \frac{s+1}{\underbrace{(s+1)^2+16}} + \frac{1}{\Omega_0}\,\frac{\Omega_0}{\underbrace{(s+1)^2+16}_{\Omega_0^2}}$$

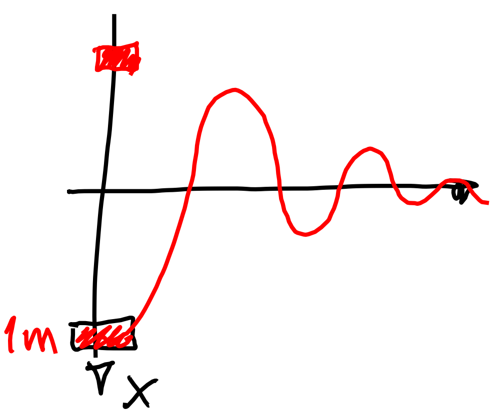
$$e^{-t}\sin(4t)$$

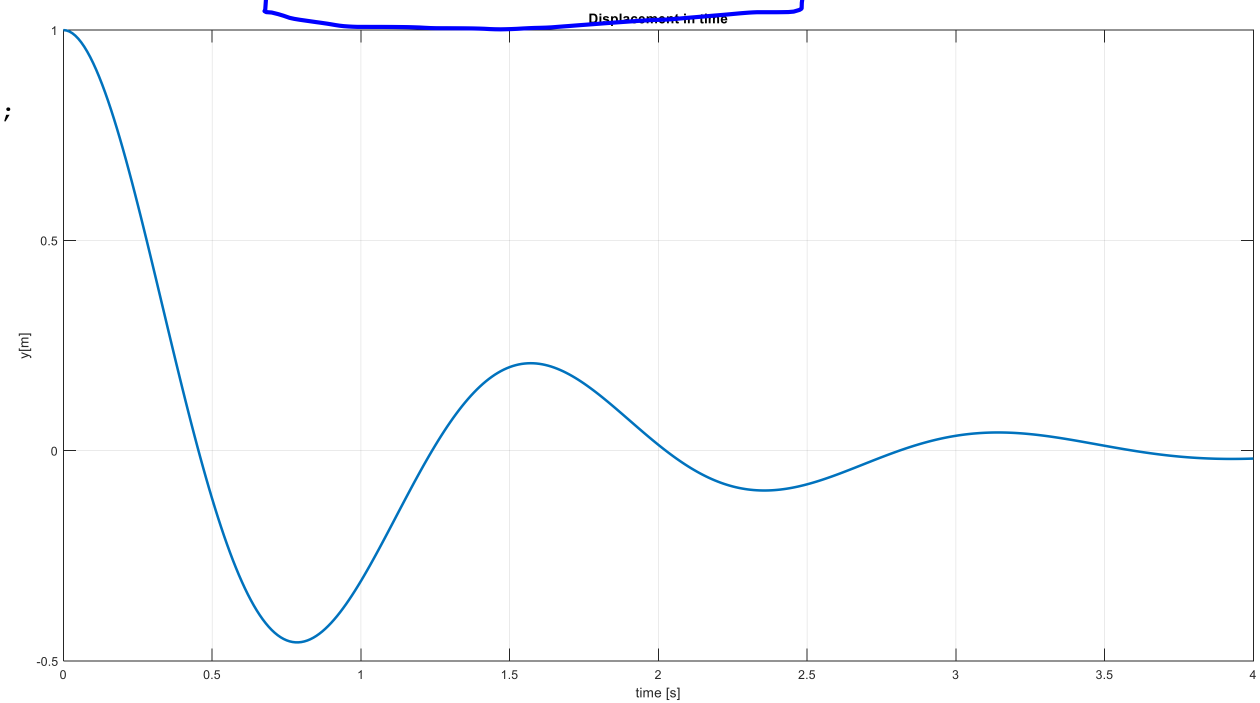
$$\Omega_0 = 4$$

and

$$x(t) = \mathcal{L}^{-1}\big(X(s)\big) = e^{-t}\left(\cos(4t) + \frac{1}{4}\sin(4t)\right)$$

```
% Example:differential equation
t=0:0.01:4;
x=exp(-t).*(cos(4*t)+0.25*sin(4*t));
plot(t,x),
title('Displacement in time'),
xlabel('time [s]'),ylabel('y[m]')
grid
```



# Suspended Weight #3

**Example #2**: A 1 kg weight is hung on the end of a vertically suspended spring, thereby stretching the spring *L*=10 cm. The weight is raised 5 cm above its equilibrium position and released from rest at time *t*=0. Find the displacement *x* of the weight from its equilibrium position at time *t.* Use g=10m/s$^2$.

$$F = kL, \qquad k = \frac{F}{L} = \frac{mg}{L} = \frac{1[kg]\,10\left[\frac{m}{s^2}\right]}{0.1[m]} = 100\left[\frac{kg}{s^2}\right]$$

At any time, sum of all forces is equal to zero

$$m\ddot{x} + kx = 0$$

With initial conditions

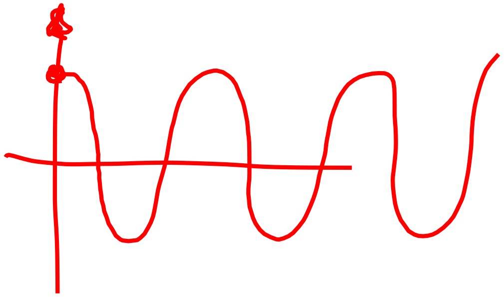
$$x(0) = -0.05[m] \;\; \dot{x}(0) = 0$$

By using Laplace transform

$$\mathcal{L}(m\ddot{x} + kx) = s^2X(s) - sx(0) - \dot{x}(0) + kX(s) = 0$$

$$(s^2 + 100)X(s) = -0.05s$$

$$X(s) = \frac{-0.05s}{s^2 + 100}$$



and

$$x(t) = \mathcal{L}^{-1}\big(X(s)\big) = -0.05\cos(10t)$$

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$$(s^2 + 100)X(s) = -0.05s$$

$$X(s) = \frac{-0.05s}{s^2 + 100}$$

and

$$x(t) = \mathcal{L}^{-1}\big(X(s)\big) = -0.05\cos(10t)$$

# Differential Equations

- A system with input x(t) and output y(t) is defined by the following differential equation:

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=x(t)$$

Find the impulse response *h(t)* and the unit-step response *s(t)*.

If $Y(s) = \mathcal{L}[y(t)]$ and $X(s) = \mathcal{L}[x(t)]$, then

$$Y(s)\,[s^2 + 3s + 2] = X(s)$$

$x(t)=\delta(t)$ $\quad X(s)=1$

$y(t)=h(t)$

$Y(s)=H(s)\cdot X(s)$

To find impulse response, we let x(t) = (t), and X(s) =1, then Y(s) is equal to H(s):

$$Y(s)=H(s)=\frac{1}{s^2+3s+2}=\frac{1}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}$$

We find

$$A=H(s)(s+1)\big|_{s=-1}=\frac{1}{-1+2}=1$$

and

$$B=H(s)(s+2)\big|_{s=-2}=\frac{1}{-2+1}=-1$$

$\frac{1}{s+1}\Big|_{s=-2}$

$Y(s)=\frac{1}{s+1}-\frac{1}{s+2}=H(s)$

$h(t)=\mathcal{L}^{-1}[\quad]$

therefore:

$$h(t)=\left[e^{-t}-e^{-2t}\right]\cdot u(t)$$

$x(t)=u(t) \Rightarrow X(s)=\frac{1}{s}$

$Y(s)=H(s)\cdot\frac{1}{s}$

Similarly, unit step response is:

$$S(s)=\frac{H(s)}{s}=\frac{1}{s\cdot(s+1)(s+2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}$$

$y(t)=\mathcal{L}^{-1}\{Y(s)\}=\mathcal{L}^{-1}\{H(s)\cdot\frac{1}{s}\}$

and A=0.5, B= -1, C=0.5, therefore:

$$s(t)=0.5\cdot u(t)-e^{-t}\cdot u(t)+0.5e^{-2t}\cdot u(t)$$

# Differential Equations

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$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

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To find impulse response, we let x(t) = (t), and X(s) =1, then Y(s) is equal to H(s):

$$Y(s) = H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

We find

$$A = H(s)(s+1)\big|_{s=-1} = \frac{1}{-1+2} = 1$$

and

$$B = H(s)(s+2)\big|_{s=-2} = \frac{1}{-2+1} = -1$$

therefore:

$$h(t) = \left[e^{-t} - e^{-2t}\right]\cdot u(t)$$

Similarly, unit step response is:

$$S(s) = \frac{H(s)}{s} = \frac{1}{s\cdot(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

and A=0.5, B= -1, C=0.5, therefore:

$$s(t) = 0.5\cdot u(t) - e^{-t}\cdot u(t) + 0.5e^{-2t}\cdot u(t)$$

# Differential Equations

- The Laplace transform of the response is:

$$S(s) = H(s)X(s) = \frac{s}{s(s^2+s+1)} = \frac{1}{(s+1/2)^2+3/4}$$

$\Omega_0$

$\Omega_0^2 = \Omega_0$

$\Omega_0 = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

since (take a look at page 199)

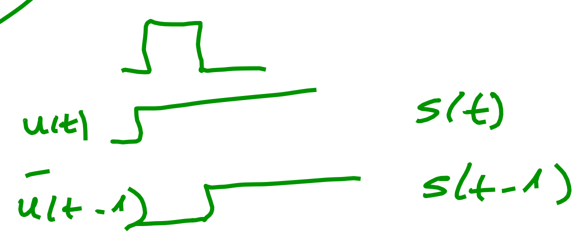
$$\mathcal{L}\left[\mathrm{A}e^{-\alpha t}\sin\left(\Omega_0 t\cdot u(t)\right)\right] = \frac{\mathrm{A}\Omega_0}{(s+\alpha)^2+\Omega_0^2}$$

Therefore, the Inverse Laplace transform of the response is:

$$s(t) = \frac{2}{\sqrt{3}}e^{-0.5t}\sin\left(\sqrt{3}t/2\right)u(t)$$

a) $y_1(t) = s(t) - s(t-1)$

b) $y_2(t) = h(t) - h(t-1) = \mathrm{d}(s(t) - s(t-1))/\mathrm{d}t$

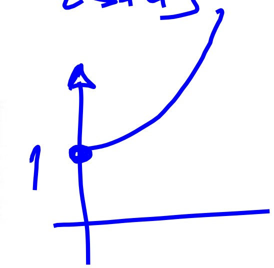


# Feedback Stabilization

An unstable system can be stabilized by using negative feedback with a gain $K$ in the feedback loop. For instance, consider an unstable system with transfer function

$$\boxed{H(s) = \frac{2}{s-1}} = 2 \cdot e^{1 \cdot t}$$

$\mathcal{L}\left[\frac{1}{s+a}\right] = e^{-at}$

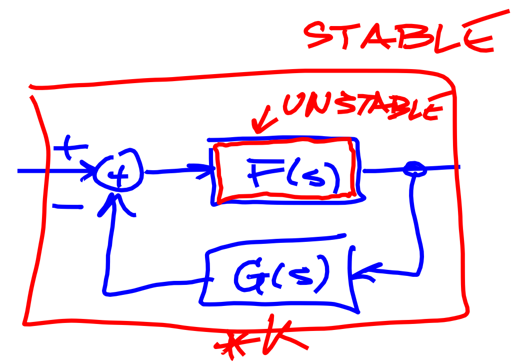



which has a pole in the right-hand $s$-plane, making the impulse response of the system $h(t)$ grow as $t$ increases. Use negative feedback with a gain $K > 0$ in the feedback loop, and put $H(s)$ in the forward loop. Draw a block diagram of the system. Obtain the transfer function $G(s)$ of the feedback system and determine the value of $K$ that makes the overall system BIBO stable (i.e., its poles in the open left-hand $s$-plane).

General solution:

$$Y(s) = \big(X(s) - G(s)\,Y(s)\big)F(s) = \frac{F(s)}{1 + F(s)G(s)}X(s)$$

$$\boxed{H(s) = \frac{F(s)}{1 + F(s) \cdot G(s)}}$$




In this particular case system output is:

Y(s) = ( X(s) – K Y(s) ) H(s)

= X(s) H(s) – K H(s) Y(s)

and

$$\frac{Y(s)}{X(s)} = \frac{H(s)}{1 + K \cdot H(s)} = \frac{2}{s + 2K - 1}$$

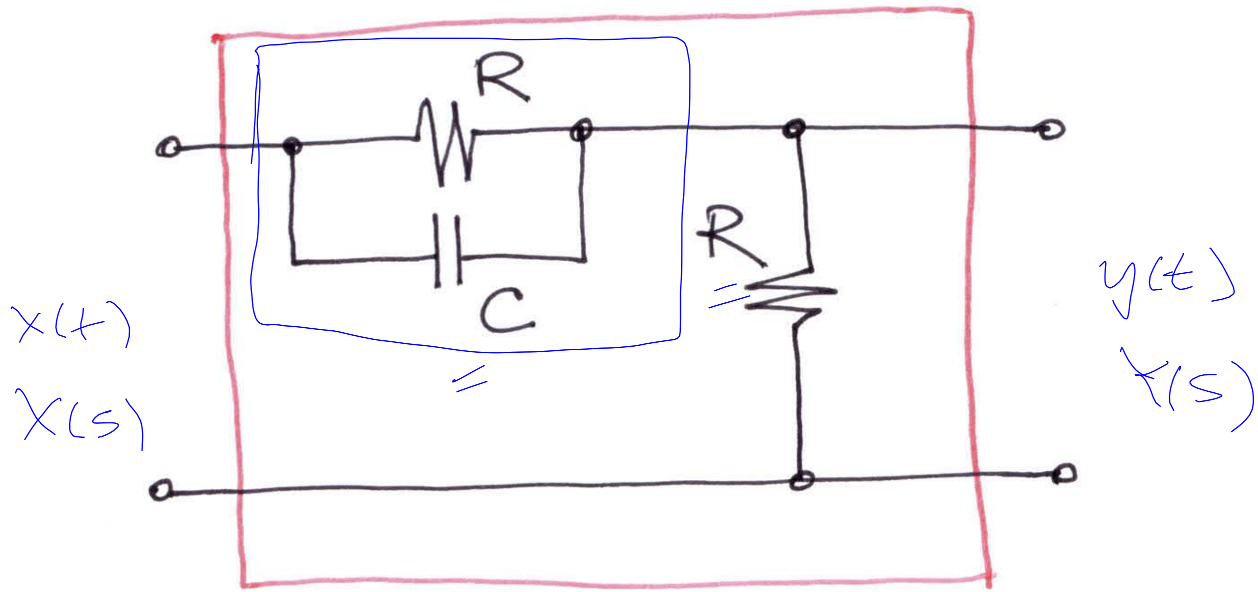
In order to have the pole in the left-hand s-plane we need 2K – 1 > 0 → K > 0.5

For example, K = 1 → pole at s = -1 and impulse response

$$g(t) = 2e^{-t}u(t)$$

# Transfer Functions

- What is the transfer function of the following circuit:



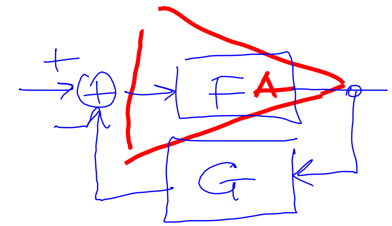

$$H(s) = \frac{R}{R + R \,||\, \frac{1}{Cs}} = \frac{R}{R + \frac{R}{RCs+1}} = \frac{RCs+1}{RCs+2} = \frac{s+\frac{1}{RC}}{s+\frac{2}{RC}}$$

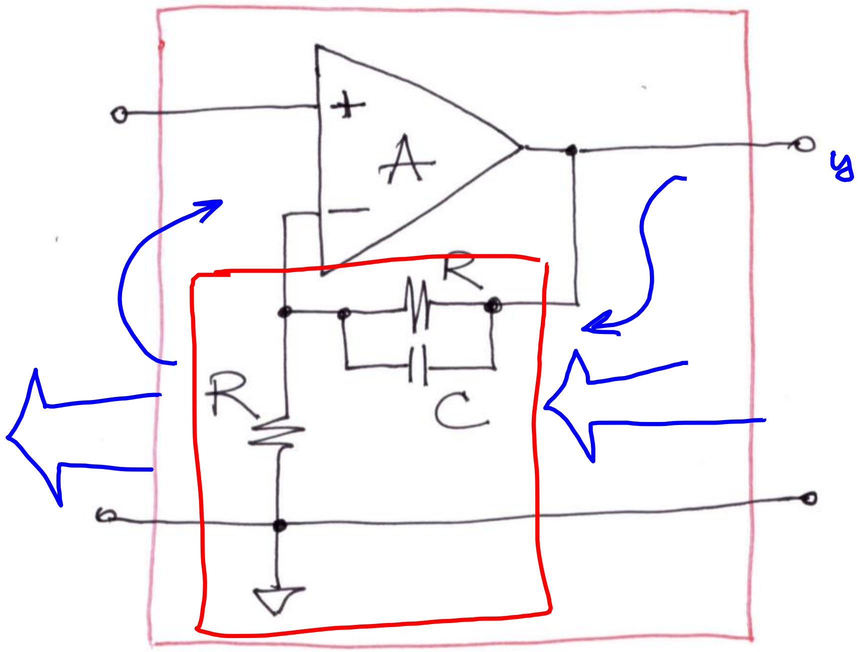
# Transfer Functions

b) What is the transfer function of the following negative feedback circuit

Hints:

- to simplify the result you can assume that A → ∞




Since

$$H(s) = \frac{F(s)}{1 + F(s) \cdot G(s)}$$

$F(s) = A$ and $G(s) = \frac{s+\frac{1}{RC}}{s+\frac{2}{RC}}$

$$H(s) = \frac{A}{1+A\left(\frac{s+\frac{1}{RC}}{s+\frac{2}{RC}}\right)} \quad for\ A \to \infty\ H(s) = \frac{s+\frac{2}{RC}}{s+\frac{1}{RC}}$$

$$\frac{A}{1+A(\ )} \,/A = \frac{1}{\frac{1}{A}+(\ )} = \frac{1}{G(s)}$$

$\frac{1}{A} \to 0$, $A \to \infty$, $(\ ) = G(s)$

# Transfer Functions

c) Find and plot the unit-step response *s*(t) of the system?

$$S(s) = \frac{1}{s} \cdot \frac{s+\frac{2}{RC}}{s+\frac{1}{RC}} = \frac{A}{s} + \frac{B}{s+\frac{1}{RC}} = \frac{2}{s} - \frac{1}{s+\frac{1}{RC}} =$$

$$s(t) = (2 - e^{-\frac{t}{RC}}) \cdot u(t)$$

$X(s) = \frac{1}{s}$ → $H(s)$ →

$Y(s) = \frac{1}{s} \cdot H(s)$

$s(t) = \mathcal{L}^{-1}[Y(s)]$

$t = \emptyset \quad (2 - e^{\emptyset}) = 1$

$t \to \infty \quad 2 - \frac{1}{e^{\infty}} = 2$