Solving Non-Homogeneous 2^{nd}-Order Linear ODEs via Method of Undetermined Coefficients

At this point in our study of solving ODEs, we have built the foundation of knowledge to be able to solve 2^{nd}-order Linear Homogeneous equations. We also know that the general solution to these type of ODEs depend upon the roots of the characteristic/auxillary equation $ar^2 + br + c = 0$! Furthermore, we know what our general solution to these type of ODEs look like based upon if the discriminant of our quadratic equation, $b^2 - 4ac$, is positive, zero, or negative. Finally, we would naturally assume that the application of these properties of 2^{nd} order Linear Homogeneous ODEs (along with the ability to find solutions from the fundamental set of solutions created from y_1 + y_2, where these functions are linearly independent functions that also satisfy these types of ODEs) would carry over to non-homogeneous equations seamlessly, but not so!

• It turns out that the superposition principle in its application to non-homogeneous equations does not "look" exactly the same as it does for its homogeneous counterpart. The primary issue is that we have to find a "suitable solution" for a non-homogeneous 2nd-order linear ODE first! It turns out that a (general) "suitable solution" for these types of equations is $y(x) = y_h(x) + y_p(x)$, where $y_h(x)$ = general solution to 2nd-order linear homogeneous ODE and $y_p(x)$ = particular solution to the 2nd-order linear non-homogeneous ODE.

<u>NOTE</u>: What we mean by "particular solution" is just a solution to our ODE that does not include arbitrary coefficients like c_1 & c_2 we normally include in our general solutions for 2nd-order linear Homogeneous Equations. We will learn methods to find out what these types of solutions are later in our study of this topic!

Why solutions for $ay''+by'+cy=g(x)$; $g(x)\neq 0$ are $y(x)=y_h+y_p$?

Suppose that y_a & y_b are particular solutions (i.e. solutions that don't include any arbitrary constants in it) to the ODE $ay''+by'+cy=g(x)$, where a, b, c are constants and $g(x)\neq 0$. Then, by the superposition principle, $y(x)=c_1y_a+c_2y_b$, where c_1 and c_2 can be real or complex numbers, should also be a solution to our non-homogeneous 2^{nd}-order linear ODE.

So, $y'(x)=y'=c_1y_a'+c_2y_b'$ and $y''(x)=y''=c_1y_a''+c_2y_b''$.

$$\therefore\ ay''+by'+cy=g(x)$$

$$\Rightarrow a[c_1y_a''+c_2y_b'']+b[c_1y_a'+c_2y_b']+c[c_1y_a+c_2y_b]=g(x)$$

$$\Rightarrow ac_1y_a''+ac_2y_b''+bc_1y_a'+bc_2y_b'+cc_1y_a+cc_2y_b=g(x)$$

$$\Rightarrow c_1[ay_a''+by_a'+cy_a]+c_2[ay_b''+by_b'+cy_b]=g(x)$$

NOTE: If y_a & y_b are solutions to $ay''+by'+cy=g(x)$, then $ay_a''+by_a'+cy_a=g(x)$ and $ay_b''+by_b'+cy_b=g(x)$!

$$\therefore\ c_1[ay_a''+by_a'+cy_a]+c_2[ay_b''+by_b'+cy_b]=g(x)$$

$$\Rightarrow c_1g(x)+c_2g(x)=g(x) \Rightarrow (c_1+c_2)g(x)=g(x) \Rightarrow \boxed{c_1+c_2=1}$$

NOTE: The fact that $c_1 + c_2 = 1$ must be true for our 2^{nd}-order Linear Non-homogeneous ODE to have a solution of the form $y(x) = y = c_1 y_a + c_2 y_b$ goes against the principle of Superposition $\Rightarrow$ CONTRACTION !!! THUS, IT IS NOT TRUE THAT IF y_a + y_b ARE PARTICULAR SOLUTIONS TO OUR NON-HOMOGENEOUS ODE IN QUESTION THAT ANY LINEAR COMBINATION OF y_a + y_b WILL ALSO BE A SOLUTION !!

The previous results leads us to conclude the following statements based on this observation.

(a) There is not a (simple) set of 2 linearly independent functions that will create a fundamental set of solutions (i.e. a basis) for our non-homogeneous ODE via the Superposition Principle.

(b) There are specific combinations (or just a single combination) of c_1 and c_2 for $y = y(x) = c_1 y_a + c_2 y_b$ that will be a solution to our 2^{nd}-order non-homogeneous Linear ODE in question.

(c) Since $c_1 + c_2 = 1 \Rightarrow c_2 = 1 - c_1$ must be true, it follows that $y(x) = c_1 y_a + (1 - c_1) y_b = c_1 (y_a - y_b) + 1 = c_1(y_a - y_b) + (c_1 + c_2)$ is a general set of particular solutions for our non-homogeneous ODE

$$\therefore \left(\underbrace{ay_a'' + by_a' + cy_a}_{g(x)}\right) - \left(\underbrace{ay_b'' + by_b' + cy_b}_{g(x)}\right) = g(x)$$

$$\Rightarrow g(x) - g(x) = g(x) \quad \Rightarrow \quad 0 = g(x)$$

ATTENTION!! The fact that $g(x) = 0$ has to be true in this case tells us that...

- $y_a - y_b$ is actually another way of expressing the homogeneous solution of $ay'' + by' + cy = 0$
- If we knew 2 particular solutions to $ay'' + by' + cy = g(x)$, where $g(x) \neq 0$, then the difference of these 2 particular solutions would actually be the same as the homogeneous solution to $ay'' + by' + cy = 0$

Let y_h = homogeneous solution to $ay'' + by' + cy = 0$. Then, it follows that $y_a - y_b = y_h$. If we let $y = y_a$ and $y_b = y_p$, then

$$y_a - y_b = y_h \Rightarrow y - y_p = y_h \Rightarrow \boxed{y = y_h + y_p}$$, where both

y and y_p are particular solutions to $ay'' + by' + cy = g(x)$; $g(x) \neq 0$.

From observation (c) on the previous page, we note that if we knew y_a and y_b, then the difference of these particular solutions for our non-homogeneous ODE in question form a function that could serve as a basis of solutions for our 2nd-order Linear Non-homogeneous ODE. Therefore, let's assume that $y_a - y_b$ is a solution to $ay'' + by' + cy = g(x)$, where a, b, c are constants but $g(x)$ could be any function (including $g(x) = 0$). Let $y = y_a - y_b$ in this case.

So, $y' = y_a' - y_b'$ and $y'' = y_a'' - y_b''$.

$\therefore\ ay'' + by' + cy = g(x)$

$\Rightarrow a[y_a'' - y_b''] + b[y_a' - y_b'] + c[y_a - y_b] = g(x)$

$\Rightarrow ay_a'' - ay_b'' + by_a' - by_b' + cy_a - cy_b = g(x)$

$\Rightarrow (ay_a'' + by_a' + cy_a) - (ay_b'' + by_b' + cy_b) = g(x)$

Recall that since y_a and y_b are particular solutions to $ay'' + by' + cy = g(x)$ $\Rightarrow ay_a'' + by_a' + cy_a = g(x)$ and $ay_b'' + by_b' + cy_b = g(x)$ as well !!

∴ So; $y = y_h + y_p$ gives us a way to find a particular solution, "y" for $ay'' + by' + cy' = g(x)$; $g(x) \neq 0$, if we know the homogeneous solution y_h and already know (or make a good guess) one particular solution y_p of our ODE in question!

The goal of this set of notes is to provide you with a method to find y_p, provided that we can use what we learned about solving 2nd-order Linear Homogeneous ODEs with (all) constant coefficients to find y_h.

Method for Finding $y = y_h + y_p$ for $ay'' + by' + cy = g(x) \neq 0$

(1) Find y_h by solving $ay'' + by' + cy = 0$.

(2) Make an educated guess for find y_p by doing the following:

a) Use the chart on the next page as a guide to the kind of "general" function for y_p as an initial guess

b) Use either the Modification Rule and/or Sum Rule (stated after the chart on the next page) to modify your first guess

c) Use systems of equation to solve for the unknown coefficients.

Chart of Trial Particular Solutions for y_p

Type of Function	Example ($g(x)$)	General Form of y_p (1st guess)
Constant	2	A
Linear	$4x-3$	$Ax+B$
Quadratic	$-5x^2-14$	Ax^2+Bx+C
Cubic	$4x^3-x^2+x-15$	Ax^3+Bx^2+Cx+D
n^{th}-degree polynomial	$a_nx^n+a_{n-1}x^{n}+\ldots+a_1x+a_0$	$K_nx^n+K_{n-1}x^{n-1}+\ldots+K_1x+K_0$
sine	$\sin(\omega x)$	$A\cos(\omega x)+B\sin(\omega x)$
cosine	$\tan(\omega x)$	$A\cos(\omega x)+B\sin(\omega x)$
exponential	$e^{\lambda x}$	$Ae^{\lambda x}$
(Linear)(exponential)	$(3x-2)e^{\lambda x}$	$(Ax+B)e^{\lambda x}$
(Quadratic)(exponential)	$(x^2+x-2)e^{\lambda x}$	$(Ax^2+Bx+C)e^{\lambda x}$
(exponential)(sine)*	$e^{\lambda x}(\sin(\omega x))$	$Ae^{\lambda x}\cos(\omega x)+Be^{\lambda x}\sin(\omega x)$
(Quadratic)(sine)*	$4x^2(\sin(\omega x))$	$(Ax^2+Bx+C)\cos(\omega x)+(Ex^2+Fx+G)\sin(\omega$
(Linear)(sine)*	$xe^{\lambda x}\cos(\omega x)$	$(Ax+B)\cos(\omega x)+(Cx+E)\sin(\omega x)$

* Function can be sine or cosine functions

NOTE: If you are dealing with an ODE $ay'' + by' + cy = g(x)$, where a, b, c are constants AND $g(x) \neq 0$ AND $g(x)$ is a function unlike any of the types listed in the chart on the previous page, you will need to use another process called Variation on Parameters to solve these type of ODEs. We will learn a bit about the process in our next set of notes.

Method of Undetermined Coefficients Rule for Chart of Trial Particular Solutions

NOTE: Follow these rules in the order they are presented

(A) Basic Rule: If $g(x)$ is one like in the "Example $(g(x))$" column in the chart on the previous page AND there exists no terms in your solution to y_h that are of the same type as $g(x)$, then ...

(i) Let y_p be equal to corresponding function in the "General Form of y_p (1st guess)" column. Find y_p' + y_p''. Sub. results into ODE.

(ii) Combine like terms in resulting ODE. Solve for undetermined coeffs

(B) If a term in your choice of y_p in (i) above happens to be a solution to y_h for the corresponding homogeneous equation, then multiply your choice for y_p (for that term) by x^m, where m = smallest positive integer to eliminate that duplication.(*) This is known as the Modification Rule.

(*) In this case the main idea is to have no matching terms of y_h + y_p !!!

:(cont'd)

(C) <u>Sum Rule</u>: If $g(x)$ is a sum of functions listed in the "Example ($g(x)$)" column in the chart on pg. 8, then let y_p be the sum of these choices

<u>NOTE</u>: For (B) + (C) above, after deciding on what y_p should be (with undetermined coefficients), following steps (i) and (ii) of (A) to find the undetermined coefficients.

Now we will do examples of the following types:

- Basic Rule applied only ; $g(x)$ = single function
- Basic Rule applied ; $g(x)$ = single function ; y_p modified
- Sum Rule applied ; no modification of y_p needed
- Sum Rule applied ; modification of y_p needed on 1 or more terms
- 1st order ODE ; sum rule applied ; modification of y_p needed
- 1st order ODE ; sum rule ; no modification of y_p needed

For each example that follows, solve the ODE for a particular solution $y = y_h + y_p$. Find arbitrary constants for IVP problems.

Ex 1a : Basic Rule Applied only ; $g(x)$ = single function

- $y'' - y' - 2y = \sin(2x)$

Find y_h

Char eqn: $r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0 \Rightarrow r = 2, -1$

$\therefore y_h = c_1 e^{2x} + c_2 e^{-x}$

Find y_p

Since $y_h = c_1 e^{2x} + c_2 e^{-x}$ and $g(x) = \sin(2x)$ do not have any terms in common, we let $\boxed{y_p = A\sin(2x) + B\cos(2x)}$.

$\therefore y_p' = 2A\cos(2x) - 2B\sin(2x)$

$\therefore y_p'' = -4A\sin(2x) - 4B\cos(2x)$

Now sub y_p' + y_p'' into the ODE ...

$\therefore y_p'' - y_p' - 2y_p = \sin(2x) \Rightarrow -4A\sin(2x) - 4B\cos(2x$

$\Rightarrow [-4A\sin(2x) - 4B\cos(2x)] - [2A\cos(2x) - 2B\sin(2x)] - 2[A\sin(2x) + B\cos(2x)]$

$= \sin(2x)$

$\Rightarrow -4A\sin(2x) - 4B\cos(2x) - 2A\cos(2x) + 2B\sin(2x) - 2A\sin(2x) - 2B\cos(2x) = \sin(2$

$\Rightarrow \sin(2x)[\overset{2B-6A}{\cancel{-4A+2B-2A}}] + \cos(2x)[\overset{-2A-6B}{\cancel{-4B-2A-2B}}] = 1\cdot\sin(2x) + 0\cdot\cos(2x)$

$\Rightarrow 2B - 6A = 1$ and $-2A - 6B = 0 \Rightarrow 2B = 1 + 6A$ and $A = -3B$

Ex. 1a (cont'd):

$\therefore 2B = 1 + 6A \Rightarrow 2B = 1 + 6(-3B) = 1 - 18B \Rightarrow \underbrace{2B + 18B}^{20B} = 1 \Rightarrow \boxed{B = \frac{1}{20}}$

$\therefore A = -3B = -3\left(\frac{1}{20}\right) = \frac{-3}{20}$

$\therefore$ Final answer: $y(x) = c_1 e^{2x} + c_2 e^{-x} - \frac{3}{20}\sin(2x) + \frac{1}{20}\cos(2x)$

Ex. 1b: Same type as Ex. 1a

- $y'' - 2y' - 2y = 4x^2$; $y(0) = 2$ and $y'(0) = 0$

Find y_h

Char eqn: $r^2 - 2r - 2 = 0 \Rightarrow (r^2 - 2r + 1) - 1 - 2 = 0 \Rightarrow (r-1)^2 - 3 = 0$

$\therefore (r-1)^2 = 3 \Rightarrow r - 1 = \pm\sqrt{3} \Rightarrow r = 1 \pm \sqrt{3} = \lambda \pm wi$

$\therefore y_h = e^{\lambda x}\left[c_1 \cos(wx) + c_2 \sin(wx)\right] \Rightarrow \boxed{y_h = e^{x}\left[c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\right]}$

Find y_p: Since y_h and $g(x)$ do not have any terms that are in common, we will let $y_p = Ax^2 + Bx + C$. Thus, $y_p' = 2Ax + B$ and $y_p'' = 2A$

Now we will sub in y_p, y_p', and y_p'' into our ODE

$\therefore y_p'' - 2y_p' - 2y_p = 4x^2 \Rightarrow [2A] - 2[2Ax + B] - 2[Ax^2 + Bx + C] = 4x^2$

$\therefore 2A - 4Ax - 2B - 2Ax^2 - 2Bx - 2C = 4x^2$

$\Rightarrow x^2[-2A] + x[-4A - 2B] + [2A - 2B - 2C] = 4x^2 + 0x + 0$

Find y_p (cont'd)

$\therefore -2A = 4$, $-4A - 2B = 0$, and $2A - 2B - 2C = 0$

$\Rightarrow \boxed{A = -2}$, $2A + B = 0$, and $A - B - C = 0$

$\therefore B = -2A = -2(-2) = 4 \Rightarrow \boxed{B = 4}$

$\therefore A - B - C = 0 \Rightarrow -2 - 4 - C = 0 \Rightarrow \boxed{C = -6}$

$\therefore \boxed{y = y_h + y_p = e^x\left[c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)\right] - 2x^2 + 4x - 6}$

$\therefore y' = e^x\left[c_1\cos(\sqrt{3}x) + c_2\sin(\sqrt{3}x)\right] + e^x\left[-\sqrt{3}c_1\sin(\sqrt{3}x) + \sqrt{3}c_2\cos(\sqrt{3}x)\right] - 4x + 4$

Find c_1 + c_2

Applying $y(0) = 2$: $\overset{1}{\cancel{e^0}}\left[c_1\overset{1}{\cancel{\cos(0)}} + c_2\overset{0}{\cancel{\sin(0)}}\right] - 2\overset{0}{\cancel{(0)^2}} - 4\overset{0}{\cancel{(0)}} - 6 = 2$.

$\therefore c_1 - 6 = 2 \Rightarrow \boxed{c_1 = 8}$

Applying $y'(0) = 0$: $\overset{1}{\cancel{e^0}}\left[c_1\overset{1}{\cancel{\cos(0)}} + c_2\overset{0}{\cancel{\sin(0)}}\right] + \overset{1}{\cancel{e^0}}\left[-\sqrt{3}c_1\overset{0}{\cancel{\sin(0)}} + \sqrt{3}c_2\overset{1}{\cancel{\cos(0)}}\right] - 4\overset{0}{\cancel{(0)}} + 4 =$

$\therefore \overset{8}{\cancel{c_1}} + \sqrt{3}c_2 + 4 = 0 \Rightarrow 12 + \sqrt{3}c_2 = 0 \Rightarrow \sqrt{3}c_2 = -12 \Rightarrow \boxed{c_2 = \frac{-12}{\sqrt{3}} = -4\sqrt{3}}$

Final answer: $\boxed{y = e^x\left[8\cos(\sqrt{3}x) - \frac{12}{\sqrt{3}}\sin(\sqrt{3}x)\right] - 2x^2 + 4x - 6}$

Ex. 2 : Basic Rule applied ; $g(x)$ = single function ; y_p modified

- $y'' - 5y' + 6y = 4e^{2x}$

Find y_h

Char eqn : $r^2 - 5r + 6 = 0 \Rightarrow (r-2)(r-3) = 0 \Rightarrow r = 2, 3$

$\therefore y_h = c_1 e^{2x} + c_2 e^{3x}$

Find y_p : Since y_h and $g(x) = 4e^{2x}$ both have a term with "e^{2x}" in it, we will need to modify our initial guess for y_p to eliminate this duplication of terms.

1st guess : $y_p = Ae^{2x}$; Modified guess : $y_p = Axe^{2x}$

$\therefore y_p' = A[(1)e^{2x} + x(2e^{2x})] = A[e^{2x} + 2xe^{2x}] = Ae^{2x} + 2Axe^{2x}$

$\therefore y_p'' = A[2e^{2x} + 2((1)e^{2x} + x(2e^{2x}))] = A[2e^{2x} + 2e^{2x} + 4xe^{2x}]$

$\Rightarrow y_p'' = 4Ae^{2x} + 4Axe^{2x} = 4Ae^{2x}(1+x)$

Now we sub y_p, y_p', and y_p'' into our ODE

$\therefore y_p'' - 5y_p' + 6y_p = 4e^{2x} \Rightarrow$

$\Rightarrow [4Ae^{2x} + 4Axe^{2x}] - 5[Ae^{2x} + 2Axe^{2x}] + 6[Axe^{2x}] = 4e^{2x}$

$\Rightarrow 4Ae^{2x} + 4Axe^{2x} - 5Ae^{2x} - 10Axe^{2x} + 6Axe^{2x} = 4e^{2x}$

$\Rightarrow xe^{2x}[\cancelto{0}{4A - 10A + 6A}] + e^{2x}[\cancelto{-A}{4A - 5A}] = 0xe^{2x} + 4e^{2x}$

Ex. 2 : (cont'd)

$\therefore$ $\dfrac{xe^{2x}}{0=0}$ $\qquad$ $\dfrac{e^{2x}}{-A=4}$ $\Rightarrow A=-4$

$\therefore$ $y_p = -4xe^{2x}$ $\Rightarrow$ $\boxed{y = y_h + y_p = c_1e^{2x} + c_2e^{3x} - 4xe^{2x}}$ (Final answer)

Ex. 3a : Sum Rule applied ; no modification of y_p needed

- $y'' - 6y' + 25y = 2\sin(t/2) - \cos(t/2)$

Find y_h

Char. eqn: $r^2 - 6r + 25 = 0$ $\Rightarrow r_{1,2} = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(25)}}{2(1)}$

$\therefore r_{1,2} = \dfrac{6 \pm \sqrt{-64}}{2} = \dfrac{6 \pm 8i}{2} = 3 \pm 4i = \lambda \pm wi$

$\therefore y_h = e^{\lambda t}\left[c_1\cos(wt) + c_2\sin(wt)\right]$ $\Rightarrow y_h = e^{3t}\left[c_1\cos(3t) + c_2\sin(3t)\right]$

$\Rightarrow \boxed{y_h = c_1e^{3t}\cos(3t) + c_2e^{3t}\sin(3t)}$

Find y_p : Since y_h and $g(x)$ do not have common terms (i.e. $\sin(t/2)$, $\cos(t/2)$, $e^{3t}\cos(3t)$, and $e^{3t}\sin(3t)$ are not like terms), we will let

$\boxed{y_p = A\sin(t/2) + B\cos(t/2)}$

$\Rightarrow y_p' = \dfrac{A}{2}\cos(t/2) - \dfrac{B}{2}\sin(t/2)$ and $y_p'' = -\dfrac{A}{4}\sin(t/2) - \dfrac{B}{4}\cos(t/2)$

Now we will sub y_p, y_p', and y_p'' into our ODE.

Ex. 3a: (cont'd)

$$\therefore\ y_p'' - 6y_p' + 25y_p = 2\sin\left(\tfrac{t}{2}\right) - \cos\left(\tfrac{t}{2}\right)$$

$$\Rightarrow \left\{ \begin{array}{l} \left[-\frac{A}{4}\sin\left(\frac{t}{2}\right) - \frac{B}{4}\cos\left(\frac{t}{2}\right)\right] - 6\left[\frac{A}{2}\cos\left(\frac{t}{2}\right) - \frac{B}{2}\sin\left(\frac{t}{2}\right)\right] \\ + 25\left[A\sin\left(\frac{t}{2}\right) + B\cos\left(\frac{t}{2}\right)\right] \end{array} \right\} = 2\sin\left(\tfrac{t}{2}\right) - \cos\left(\tfrac{t}{2}\right)$$

$$\Rightarrow \left\{ \begin{array}{l} -\frac{A}{4}\sin\left(\frac{t}{2}\right) - \frac{B}{4}\cos\left(\frac{t}{2}\right) - 3A\cos\left(\frac{t}{2}\right) + 3B\sin\left(\frac{t}{2}\right) \\ 25A\sin\left(\frac{t}{2}\right) + 25B\cos\left(\frac{t}{2}\right) \end{array} \right\} = 2\sin\left(\tfrac{t}{2}\right) - \cos\left(\tfrac{t}{2}\right)$$

$$\Rightarrow \sin\left(\tfrac{t}{2}\right)\left[-\frac{A}{4} + 3B + 25A\right] + \cos\left(\tfrac{t}{2}\right)\left[-\frac{B}{4} - 3A + 25B\right] = 2\sin\left(\tfrac{t}{2}\right) - \cos\left(\tfrac{t}{2}\right)$$

$\therefore$ $\underline{\sin\left(\tfrac{t}{2}\right)}$

$$3B + 25A - \frac{A}{4} = 2$$

$$\Rightarrow 12B + 100A - A = 8$$

$$\Rightarrow 12B + 99A = 8 \quad (1)$$

$\underline{\cos\left(\tfrac{t}{2}\right)}$

$$25B - \frac{B}{4} - 3A = -1$$

$$\Rightarrow 100B - B - 12A = -4$$

$$\Rightarrow 99B - 12A = -4$$

$$\therefore\ \begin{array}{l} 12B + 99A = 8 \\ 99B - 12A = -4 \end{array} \Rightarrow \begin{array}{l} 12(12B + 99A) = 12(8) \\ 99(99B - 12A) = 99(-4) \end{array} \Rightarrow \begin{array}{l} 144B + \cancel{1188A} = 96 \\ +9801B - \cancel{1188A} = -396 \end{array}$$

$$\therefore\ 9945B = -300 \Rightarrow B = \frac{-300}{9945} = \frac{-20}{663} \Rightarrow \boxed{B = \frac{-20}{663}}$$

$$\therefore\ 12B + 99A = 8 \Rightarrow \overset{4}{\cancel{12}}\left(\frac{-20}{\underset{221}{\cancel{663}}}\right) + 99A = 8 \Rightarrow 99A = 8 + \frac{80}{221} = \frac{1848}{221} \Rightarrow A = \frac{1848}{(221)(99)}$$

$$\Rightarrow \boxed{A = \frac{56}{663}}$$

$$\therefore\ \boxed{\text{Final Answer: } y = y_h + y_p = c_1 e^{3t}\cos(3t) + c_2 e^{3t}\sin(3t) + \frac{56}{663}\sin\left(\tfrac{t}{2}\right) - \frac{20}{663}\cos\left(\tfrac{t}{2}\right)}$$

Ex. 3b: Same type as Ex. 3a

- $y'' - 5y' = (x-1)\sin(x) + (x+1)\cos(x)$

Find y_h

Char. eqn: $r^2 - 5r = 0 \Rightarrow r(r-5) = 0 \Rightarrow r = 0, 5$

$\therefore y_h = c_1 e^{0x} + c_2 e^{5x} \Rightarrow \boxed{y_h = c_1 + c_2 e^{5x}}$

Find y_p: Since y_h and $g(x)$ do not have any common terms, we will

let $y_p = \underbrace{(Ax+B)\sin(x) + (Cx+D)\cos(x)}_{\text{for } (x-1)\sin(x)} + \underbrace{(Ex+F)\sin(x) + (Gx+H)\cos(x)}_{\text{for } (x+1)\cos(x)}$

$\Rightarrow y_p = x\cdot\sin(x)[A+E] + x\cdot\cos(x)[C+G] + \sin(x)[B+F] + \cos(x)[D+H]$

$\Rightarrow y_p = x\cdot\sin(x)[J] + x\cdot\cos(x)[K] + \sin(x)[L] + \cos(x)[M]$,

where $J = A+E$, $K = C+G$, $L = B+F$, and $M = D+H$.

$\therefore y_p = (Jx+L)\sin(x) + (Kx+M)\cos(x)$

Since the choices for y_p consists of the same type of terms, we could have just let $y_p = (Ax+B)\sin(x) + (Cx+D)\cos(x)$ from the beginning to avoid all the combining of like terms

$\therefore y_p' = J\sin(x) + (Jx+L)\cos(x) + K\cos(x) - (Kx+M)\sin(x)$

$y_p' = \sin(x)[J - Kx - M] + \cos(x)[Jx + L + K]$

$\boxed{y_p' = \sin(x)[N - Kx] + \cos(x)[Jx+P]}$, where $N = J - M$ and $P = L+K$

$\therefore y_p'' = \cos(x)[N-Kx] + \sin(x)[-K] - \sin(x)[Jx+P] + \cos(x)[J]$

Ex. 3b: (cont'd)

$$\therefore y_p'' = \cos(x)\left[N - Kx + J\right] + \sin(x)\left[-K - Jx - P\right]$$

$$\boxed{y_p'' = \cos(x)\left[Q - Kx\right] - \sin(x)\left[Jx + R\right]}, \text{ where } Q = N + J \text{ and } R = K + P.$$

Now, we will sub y_p, y_p', and y_p'' into our ODE.

$$\therefore\ y_p'' - 5y_p' = (x-1)\sin(x) + (x+1)\cos(x)$$

$$\Rightarrow \left\{\begin{array}{l} \left(\cos(x)\left[Q - Kx\right] - \sin(x)\left[Jx + R\right]\right) \\ -5\left(\sin(x)\left(N - Kx\right) + \cos(x)\left(Jx + P\right)\right) \end{array}\right\} = (x-1)\sin(x) + (x+1)\cos(x)$$

$$\Rightarrow \left\{\begin{array}{l} Q\cos(x) - Kx\cdot\cos(x) - Jx\cdot\sin(x) - R\sin(x) \\ -5N\sin(x) + 5Kx\cdot\sin(x) - 5Jx\cdot\cos(x) - 5P\cos(x) \end{array}\right\} = (x-1)\sin(x) + (x+1)\cos(x)$$

$$\Rightarrow \left\{\begin{array}{l} x\cdot\cos(x)\left[-K - 5J\right] + x\cdot\sin(x)\left[-J + 5K\right] \\ +\cos(x)\left[Q - 5P\right] + \sin(x)\left[-R - 5N\right] \end{array}\right\} = x\cdot\sin(x) - \sin(x) + x\cdot\cos(x) + \cos(x)$$

$\therefore$ $x\cdot\cos(x)$	$x\cdot\sin(x)$	$\cos(x)$	$\sin(x)$
$-K - 5J = 1$ $\Rightarrow K + 5J = -1$ ①	$5K - J = 1$ ②	$Q - 5P = 1$ ③	$-R - 5N = -1$ $R + 5N = 1$ ④

NOTE: Our preferred form for yp has coefficients J, K, L, and M. Since we have 2 equations with K and J, we need to solve for these coefficients first. Next, we need to rewrite equations ③ + ④ in terms of J, K, L, and/or M to find L+M

$$\begin{cases} K+5J=-1 \\ 5K-J=1 \end{cases} \Rightarrow \begin{array}{r} K+\cancel{5J}=-1 \\ +\ 25K-\cancel{5J}=5 \\ \hline 26K=4 \end{array} \Rightarrow K=\frac{4}{26}=\frac{2}{13} \quad \boxed{K=\frac{2}{13}}$$

$$\therefore 5K-J=1 \Rightarrow 5\left(\frac{2}{13}\right)-J=1 \Rightarrow \frac{10}{13}-J=\frac{13}{13} \Rightarrow \frac{10}{13}-\frac{13}{13}=J \Rightarrow \boxed{J=\frac{-3}{13}}$$

Now, recall that $N=J-M$, $P=L+K$, $Q=N+J$, and $R=K+P$.

For equation (3) $\Rightarrow N+J-5(L+K)=1 \Rightarrow N+J-5L-5K=1$

$$\therefore N-5L=5K-J \Rightarrow N-5L=5\left(\frac{2}{13}\right)-\left(\frac{-3}{13}\right)=\frac{10+3}{13}=1 \Rightarrow \boxed{N-5L=1}$$

For equation (4) $\Rightarrow (K+P)+5N=1 \Rightarrow (K+(L+K))+5N=1$

$$\therefore 2K+L+5N=1 \Rightarrow 5N+L=1-2K \Rightarrow 5N+L=1-2\left(\frac{2}{13}\right)=\frac{9}{13}$$

$$\therefore \begin{cases} N-5L=1 \\ 5N+L=\frac{9}{13} \end{cases} \Rightarrow \begin{array}{r} N-\cancel{5L}=1 \\ +\ 25N+\cancel{5L}=\frac{45}{13} \\ \hline 26N=\frac{45}{13} \end{array} \Rightarrow N=\frac{45}{(13)(26)}=\frac{45}{338}$$

$$\therefore N=\frac{45}{338}$$

We don't need this for y_p, but we can use it to find L and M for y_p

$$\therefore N-5L=1 \Rightarrow N-1=5L \Rightarrow \frac{N-1}{5}=L \Rightarrow L=\frac{\frac{45}{338}-1}{5}=\frac{-293}{1690}$$

$$\therefore \boxed{L=\frac{-293}{1690}}$$

$$\therefore \text{Recall that } N=J-M \Rightarrow M=J-N=\frac{-3}{13}-\frac{45}{338} \Rightarrow \boxed{M=\frac{-123}{338}}$$

$$\therefore \boxed{y=y_h+y_p=c_1+c_2e^{5x}-\left(\frac{3}{13}x+\frac{293}{1690}\right)\sin(x)+\left(\frac{2}{13}x-\frac{123}{338}\right)\cos(x)}$$

Final answer

Ex. 4 : Sum Rule applied ; modify y_p to 1 or more terms

- $y'' = 9x^2 + 2x - 1$

NOTE : This equation can be solved easily by just integrating twice. The purpose of this example is to demonstrate how to correctly modify y_p when y_h possess terms that are common to (some of) y_p's terms!

Find y_h : $y'' = 0 \Rightarrow y' = c_1 \Rightarrow y = c_1 x + c_2 \Rightarrow \therefore \boxed{y_h = c_1 x + c_2}$

Find y_p : Our initial guess for y_p would probably be $y_p = Ax^2 + Bx + C$. However, the "Bx" and "C" terms are common to "$c_1 x$" and "c_2" from y_h and $g(x) = 9x^2 + 2x - 1$, respectively. Therefore, we need to make sure that we modify our y_p so that it does not have "x" or constant terms. Thus, we need to multiply our 1st guess for y_p by "x^2" to make this happen.

$\therefore$ y_p (modified) : $y_p = x^2(Ax^2 + Bx + C) = Ax^4 + Bx^3 + Cx^2$

$\therefore y_p' = 4Ax^3 + 3Bx^2 + 2Cx$ and $y_p'' = 12Ax^2 + 6Bx + 2C$

Now we will sub y_p'' into our ODE.

$\therefore y_p'' = 9x^2 + 2x - 1 \Rightarrow 12Ax^2 + 6Bx + 2C = 9x^2 + 2x - 1$

$\therefore$

x^2	x	constants
$12A = 9$	$6B = 2$	$2C = -1$
$A = \frac{9}{12}$	$\boxed{B = \frac{1}{3}}$	$\boxed{C = -\frac{1}{2}}$
$\boxed{A = \frac{3}{4}}$		

$\therefore y = y_h + y_p$

$\Rightarrow \boxed{y = c_1 x + c_2 + \frac{3}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2}$

Final answer