Euler – Cauchy (E – C) Equations

Recall that we recently learned how to solve 2nd-order Linear Homogeneous Equation where all the coefficients were constants, i.e. ...

$$\alpha y'' + \beta y' + \gamma y = 0 \text{ , where } \alpha, \beta, + \gamma \text{ are constants.}$$

Now we will consider equations in similar ilk, but this time, not all of $\alpha, \beta, + \gamma$ will be constants. Instead, coefficients $\alpha, \beta, + \gamma$ may be a combination of constants and (explicit) functions of x. Specifically, let us consider 2nd-order Linear Homogeneous Equations of the type:

$$\underbrace{ax^2}_{\alpha} y'' + \underbrace{bx}_{\beta} y' + \underbrace{c}_{\gamma} y = 0 \text{ , where } a, b, c \text{ are constants}$$

NOTE: $\alpha = ax^2$, $\beta = bx$, and $c = \gamma$ in this case.

Equations of this type are called Euler-Cauchy equations.

NOTE: These equations are also referred to as Cauchy-Euler, Euler, or equidimensional equations

The reason why we can refer to these equations in terms of being "equidimensional" is because the degree of the coefficient (function) in each term of an E-C equation matches the order (of the derivative of y) in the corresponding term!

$$a\,x^{\overset{\text{degree } 2}{2}}\,y^{\overset{\text{order } 2}{('')}} + b\,\underset{}{\overset{\text{degree}=1}{x}}\,y^{\overset{\text{order}=1}{(')}} + \overset{\text{degree}=0}{c}\,\overset{\text{order } 0}{y} = 0$$

In general, E-C equations follow this pattern...

$$\overbrace{a_n x^n \cdot \frac{d^n y}{dx^n}}^{\text{same}} + \overbrace{a_{n-1} x^{n-1} \cdot \frac{d^{n-1} y}{dx^{n-1}}}^{\text{same}} + \dots + \overbrace{a_2 x^2 \frac{d^2 y}{dx^2}}^{\text{same}} + \overbrace{a_1 x \frac{dy}{dx}}^{\text{same}} + \overbrace{a_0 y}^{\text{same}} = 0$$

It turns out that if we select a "suitable" solution to these ODEs, we can solve them similar to how we solved 2^{nd}-order Linear Homogeneous ODEs with all constant coefficients. It also turns out that it is well-known that if we let $y = y(x) = x^r$, that this function is a "suitable" solution to our (general) E-C equations. We will show that by substituting $y = x^r$ (and its 1^{st} and 2^{nd} derivatives) into this general ODE that we will end up with 3 cases to consider based upon what the values of "r" look like. We will derive these 3 cases and state a summary to reference.

Verification of $y=x^r$ as a solution to $ax^2y''+bxy'+cy=0$

Let $y=x^r$. Then, $y'=rx^{r-1}$ and $y''=r(r-1)x^{r-2}$.

$$\therefore\ ax^2y''+bxy'+cy=0$$

$$\Rightarrow ax^2\left[r(r-1)\cdot x^{r-2}\right]+bx\left[rx^{r-1}\right]+c\left[x^r\right]=0$$

$$\Rightarrow a\cancel{x^2}\left[\frac{r(r-1)\cdot x^r}{\cancel{x^2}}\right]+b\cancel{x}\left[\frac{rx^r}{\cancel{x}}\right]+c\left[x^r\right]=0$$

$$\Rightarrow ar(r-1)\left[x^r\right]+br\left[x^r\right]+c\left[x^r\right]=0$$

$$\Rightarrow \left[ar(r-1)+br+c\right]\cdot x^r=0 \quad [\star]$$

$$\Rightarrow ar^2-ar+br+c=0\ ;\ x\neq 0\ \left(\text{i.e. } x=(-\infty,0)\cup(0,\infty)\right)$$

$$\Rightarrow ar^2+(b-a)r+c=0 \quad [1]$$

$$\Rightarrow r^2+\left(\frac{b}{a}-1\right)r+\frac{c}{a}=0$$

$$\Rightarrow r^2+(A-1)r+B=0\ \text{, where } A=\frac{b}{a} \text{ and } B=\frac{c}{a} \quad [2]$$

NOTE 1: Equations [1] and [2] are called indicial equations for the E-C equation. (You don't need to use both equations as one will suffice. I stated this equation in 2 different ways because you will see it stated differently based upon what resource you refer to.) In practice problem, equation [1] will be primarily used.

NOTE 1: (cont'd): You can simply call [1] or [2] an auxillary or characteristic equation for E-C ODEs!

NOTE 2: The term "indicial" is used here because E-C equations are normally used as an introduction to solving ODEs by using power series (i.e. the Frobenius method). The "indicial" equation essentially provides the foundation of establishing an index for the coefficients that can be derived (and used to ultimately find solutions) in order to find solutions to such ODEs. You can think of the word "indicial" as "index".

Therefore, the solutions to our E-C ODE will be based upon what the values of r look like! Note that [1] and [2] are quadratic equations, and, thus, can be solved via the quadratic formula. the what our roots "r" will look like will based upon whether the discriminant of the quadratic formula, b^2-4ac, is positive, negative or zero. Using [1], we will explore each case to see what r, and, thus, $y=x^r$ will be.

NOTE: The indicial equation for 2^{nd}-order E-C equations is specific to order 2. The indicial equation for a 3^{rd}-order E-C is different than one for order 2. Therefore, it is suggested that if you don't want to keep with memorizing indicial equations for E-C. for orders 2, 3, 4, etc., just do the substitution of y, y', y'', $\ldots, y^N$ into the ODE and derive the ODE organically.

$$\therefore\ ar^2 + (b-a)r + c = 0 \quad \Rightarrow r_{1,2} = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a}$$

Case 1 $\left((b-a)^2 - 4ac > 0\right)$

If $(b-a)^2 - 4ac > 0 \Rightarrow r_1$ and r_2 are 2 real, distinct numbers. So, $r_1 \neq r_2$. Thus, by principle of superposition and the concept of a basis of solutions, $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$, and, our general solution for this ODE will be...

$$\boxed{y(x) = c_1 y_1 + c_2 y_2 = c_1 x^{r_1} + c_2 x^{r_2}} \quad [3]$$

Case 2 $\left((b-a)^2 - 4ac = 0 \right)$

If $(b-a)^2 - 4ac = 0$, then $(b-a)^2 = 4ac$ and $r_1 = r_2 = r = \frac{-(b-a)}{2a}$.

Thus, r is a double root! Since we are dealing with a 2^{nd}-order ODE, we know we need 1 additional, linearly independent solution to our E-C equation to form a basis of solutions for it so that we can express the general solution for this ODE. We will use the Reduction of Order Method like we did for 2^{nd}-order Linear Homogeneous ODEs to accomplish this goal! (UGH!!!)

So, we let $y_1 = x^{r_1}$ be a solution to our E-C ODE. (This means that $ar_1^2 + (b-a)r_1 + c = 0$ and $ax^2 y_1'' + bxy_1' + cy_1 = 0$ as well)!

For simplicity, we let $r_1 = r$. So, $y_1 = x^r$ is a solution to our E-C ODE. Thus, we will let $y_2 = u(x) \cdot x^r = ux^r$.

$$\therefore \quad y_2' = u'x^r + u[rx^{r-1}] \quad \text{and} \quad y_2'' = \left(u''x^r + u'[rx^{r-1}]\right) + \left(u'rx^{r-1} + ru(r-1)x^{r-2}\right)$$

$$\therefore y_2' = u'x^r + rux^{r-1} \quad \text{and} \quad y_2'' = u''[x^r] + u'[2rx^{r-1}] + u[r(r-1)x^{r-2}]$$

Case 2 ($(b-a)^2 - 4ac = 0$) : cont'd

Substituting y_2, y_2', and y_2'' into our E-C ODE yields...

$$ax^2 y_2'' + bxy_2' + cy_2 = 0$$

$$\Rightarrow \left\{ \begin{array}{l} ax^2\left[u''(x^r) + u'(2rx^{r-1}) + u(r(r-1)x^{r-2})\right] \\ + bx\left[\qquad u'(x^r) + u(rx^{r-1})\right] \\ + c\left[\qquad\qquad u(x^r)\right] \end{array} \right\} = 0$$

$$\Rightarrow \left[\begin{array}{l} u''(ax^2 \cdot x^r) + u'(2ar \cdot x^r \cdot x) + u(ar(r-1) \cdot x^r) \\ \qquad + u'(b \cdot x^r \cdot x) + u(br \cdot x^r) \\ \qquad\qquad + u(c \cdot x^r) \end{array} \right] = 0$$

Factor x^r out of every term and dividing by x^r, where $x \neq 0$, yields..

$$\left[\begin{array}{l} u''(ax^2) + u'(2arx) + \cancelto{0}{u(ar(r-1))} \\ \qquad + u'(bx) + \cancel{u(br)} \\ \qquad\qquad \cancel{+ u(c)} \end{array} \right] = 0 \quad [\nabla]$$

NOTE: The "u" terms in our equation above if combined will become $(ar(r-1) + br + c)u$. Note that [☆] on page 3 of these notes states

Case 2 ($(b-a)^2 - 4ac = 0$) : cont'd - 2

$[ar(r-1)+br+c]\cdot x^r = 0$. Since we are only considering when $x \neq 0$; this implies that $ar(r-1)+br+c = 0$. But this our indicial / auxillary / characteristic (or whatever you want to call it) equation for our E-C ODEs ! Thus, we know that $ar(r-1)+br+c=0$ and equation [∇] simplifies to ...

$$u''(ax^2) + u'(2ar+b)x = 0$$

... which is a 2^{nd}-order Linear ODE with (some) non-constant coefficients. Using the substitution techniques we learned when introducing 2^{nd}-order ODEs, we let $v = u' \Rightarrow v' = u''$ which will turn our current ODE into a 1^{st}-order Linear ODE !

$$\therefore\ v'(ax^2) + v(2ar+b)x = 0$$

(This equation is 1^{st}-order Linear + Separable)

So, equation [∇] will simplify to $v'(ax^2)+v(2ar+b)x = 0$

Recall that $r = \frac{-(b-a)}{2a}$ in this case. So, $2ar+b = \cancel{2a}\left[\frac{-(b-a)}{\cancel{2a}}\right]+b = \cancel{-b}+a+\cancel{b} = a$. Therefore, $v'(ax^2)+v(a)x = 0$.

$$\therefore\ v'(ax^2)+(av)x = 0 \Rightarrow v' + \frac{1}{x}v = 0$$

(This equation is 1^{st}-order Linear + Separable as well)

Case 2 $\left((b-a)^2-4ac=0\right)$: cont'd_3

Choosing to solve this equation via Separable techniques...

$$v'+\frac{1}{x}v=0 \Rightarrow v'=-\frac{1}{x}v \Rightarrow \frac{dv}{dx}=-\frac{1}{x}v \Rightarrow \frac{dv}{v}=\frac{-dx}{x}$$

$$\therefore \int\frac{dv}{v}=\int-\frac{dx}{x} \Rightarrow \ln|v|=-\ln|x|+C_1 \Rightarrow \ln|v|+\ln|x|=C_1$$

$$\therefore \ln|vx|=C_1 \Rightarrow e^{\ln|vx|}=e^{C_1} \Rightarrow |vx|=e^{C_1} \Rightarrow vx=\pm e^{C_1}$$

Let $C_2=\pm e^{C_1}$. Then, $vx=C_2 \Rightarrow v=\frac{C_2}{x}$, $x>0$.

But $v=u' \Rightarrow u'=\frac{C_2}{x} \Rightarrow \frac{du}{dx}=\frac{C_2}{x} \Rightarrow du=C_2\cdot\frac{1}{x}\cdot dx$

$$\therefore \int du=C_2\int\frac{1}{x}\,dx \Rightarrow u=C_2\cdot\ln|x|+C_3=C_2\cdot\ln(x)+C_3,\ x>0.$$

For simplicity, we let $u=u(x)=\ln(x)$. Then $y_2=uy_1=x^r\cdot\ln(x)$.

So, $y_1(x)=x^r$ and $y_2(x)=x^r\cdot\ln(x)$. Thus, our general solution for E-C equations of Case 2 type will be...

$$\boxed{y(x)=c_1y_1+c_2y_2=c_1x^r+c_2x^r\ln(x)=x^r\left(c_1+c_2\ln(x)\right)}$$

Case 3 ($(b-a)^2 - 4ac < 0$)

If $(b-a)^2 - 4ac < 0 \Rightarrow r_1$ and r_2 are complex conjugates. So, let $r = \lambda \pm wi$, where $r_1 = r_+ = \lambda + wi$ and $r_2 = r_- = \lambda - wi$, where $\lambda, w \in \mathbb{R}$

$\therefore\ y = y(x) = x^{\lambda \pm wi}$ is a general solution for our E-C, but it is in complex form and not real useful in modeling applications of real-valued problems. Just like we found another way rewrite the solution $y(x) = e^{\lambda \pm wi}$ for 2^{nd}-order Linear Homogeneous ODEs, we will do the same thing here and leverage some of the derivation we did there to save time (and space) here.

Note that $x^{\lambda \pm wi} = x^{\lambda}\left(x^{\pm wi}\right)$ and recall that $x = e^{\ln(x)}$

$\therefore\ x^{\lambda \pm wi} = x^{\lambda}\left(x^{\pm wi}\right) = x^{\lambda}\left(e^{\ln(x)\cdot \pm wi}\right) = x^{\lambda}\left(e^{\pm i\theta}\right)$, where $\theta = w\ln(x)$. Recall that Euler's formula states that

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \quad \text{and} \quad e^{-i\theta} = \cos(\theta) - i\sin(\theta)$$

So, we could let $y_1 = x^{\lambda}\left(\cos(\theta) + i\sin(\theta)\right)$ and $y_2 = x^{\lambda}\left(\cos(\theta) - i\sin(\theta)\right)$, but, again, we need to rewrite $y_1 + y_2$ in terms of real-valued functio

Case 3 ($(b-a)^2 - 4ac < 0$) : cont'd

Using the principle of superposition and the same choices for linear combinations of y_1 + y_2 in there current form, if we find $y_1 + y_2$ and $i[y_1 - y_2]$ and added these 2 equations together, we would get...

$$x^{\lambda}\cos(\theta) + x^{\lambda}\sin(\theta) \quad ; \quad \theta = w\ln(x)$$

$$\Rightarrow \quad x^{\lambda}\cos(w\ln(x)) + x^{\lambda}\sin(w\ln(x))$$

$\therefore$ A general solution (that is expressed in real-valued functions) for $y(x)$ for Case 3 would be...

$$\boxed{\begin{aligned} y(x) &= c_1 y_1 + c_2 y_2 = c_1 x^{\lambda}\cos(w\ln(x)) + c_2 x^{\lambda}\sin(w\ln(x)) \\ \therefore\ y(x) &= x^{\lambda}\left[c_1\cos(w\ln(x)) + c_2\sin(w\ln(x))\right] \end{aligned}}$$

Summary of Cases (1) – (3) for E-C ODEs (2nd-order)

ODE: $ax^2y'' + bxy' + cy = 0$ or $ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$

Suitable solution: $y(x) = x^r$; $x = (-\infty, 0) \cup (0, \infty)$.

NOTE: "r" can be determined by either one of the indicial equations listed below. The Cases stated afterwards are determined based upon the discriminant of the indicial equation used.

Indicial Equation (Option 1)

* $ar^2 + (b-a)r + c = 0$
* $r_{1,2} = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a}$

Indicial Equation (Option 2)

* $r^2 + (A-1)r + B = 0$
  - $A = \frac{b}{a}$
  - $B = \frac{c}{a}$
* $r_{1,2} = \frac{-(A-1) \pm \sqrt{(A-1)^2 - 4B}}{2}$

Indicial Equation (Option 3)

* $ar(r-1) + br + c = 0$
* $r_{1,2} = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a}$

Discriminant for all Cases

$(b-a)^2 - 4ac = (A-1)^2 - 4B$

Case 1 ($(b-a)^2 - 4ac = (A-1)^2 - 4B > 0$)

* r_1, r_2 are 2 real, distinct solutions $\Rightarrow r_1 \neq r_2$
* General Solution: $\boxed{y(x) = c_1 x^{r_1} + c_2 x^{r_2}}$

Case 2 $\left((b-a)^2-4ac=(A-1)^2-4B=0\right)$

* $r_1=r_2=r$ (double root) $\Rightarrow$ reduction of order used to find $y_2=uy_1$

* General solution: $\boxed{y(x)=c_0x^r+c_1x^r\ln(x)}$

<u>NOTE</u>: If we had a 3^{rd}-order E-C equation, our general solution would be...

$$y(x)=c_0x^r+c_1x^r\ln(x)+c_2x^r[\ln(x)]^2$$

So, in general; an n^{th}-order E-C equation would have a general solution...

$$\boxed{y(x)=c_0x^r+c_1x^r\ln(x)+c_2x^r[\ln(x)]^2+\dots+c_nx^r[\ln(x)]^n}$$

Case 3 $\left((b-a)^2-4ac=(A-1)^2-4B<0\right)$

* $r=\lambda\pm wi$, where $r_1=r_+=\lambda+wi$ and $r_2=r_-=\lambda-wi$

* General solution:
$$\boxed{\begin{aligned} y(x)&=x^\lambda\left[c_1\cos(w\ln(x))+c_2\sin(w\ln(x))\right]\\ y(x)&=c_1x^\lambda\cos(w\ln(x))+c_2x^\lambda\sin(w\ln(x))\end{aligned}}$$

Now we will do a few examples in order to get practice on what our general solutions (and particular solutions for IVP-type problems) for each of our Cases (1) – (3).

Case 1 Examples: Solve the following E-C ODEs.

a) $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - 4y = 0$

Indicial eqn: $ar^2 + (b-a)r + c = 0$, where $a = 1$, $b = -2$, $c = -4$.

$\therefore r^2 + (-2-1)r - 4 = 0 \Rightarrow r^2 - 3r - 4 = 0 \Rightarrow (r+1)(r-4) = 0$

$\therefore r = -1, 4$ (Case 1; 2 real distinct roots).

$\therefore$ General solution: $\boxed{y(x) = c_1 x^{-1} + c_2 x^4 = \frac{c_1}{x} + c_2 x^4}$

b) $x^2 y'' - 20y = 0$

Indicial eqn: $ar^2 + (b-a)r + c = 0$, where $a = 1$, $b = 0$, $c = -20$

$\therefore r^2 + (0-1)r - 20 = 0 \Rightarrow r^2 - r - 20 = 0 \Rightarrow (r-5)(r+4) = 0$

$\therefore r = 5, -4 \Rightarrow$ General solution: $\boxed{\begin{array}{l} y(x) = c_1 x^5 + c_2 x^{-4} \\ y(x) = c_1 x^5 + \frac{c_2}{x^4} \end{array}}$

c) $x^2y'' - 3xy' - 21y = 0$

Indicial eqn: $ar^2 + (b-a)r + c = 0$, where $a=1$, $b=-3$, $c=-21$

$\therefore\ r^2 + (-3-1)r + (-21) = 0 \Rightarrow r^2 - 4r - 21 = 0 \Rightarrow (r+3)(r-7) = 0$

$\therefore\ r = -3, 7 \Rightarrow$ General sol'n: $\boxed{y(x) = c_1 x^{-3} + c_2 x^7}$

d) $x^2y'' - 20xy' = 0$

Indicial eqn: $ar^2 + (b-a)r + c = 0$, where $a=1$, $b=-20$, $c=0$

$\therefore\ r^2 + (-20-1)r + 0 = 0 \Rightarrow r^2 - 21r = 0 \Rightarrow r(r-21) = 0$

$\therefore\ r = 0, 21.$

$\therefore$ General solution: $y(x) = c_1 x^0 + c_2 x^{21} = c_1 + c_2 x^{21}$

$\therefore\ \boxed{y(x) = c_1 + c_2 x^{21}}$

e). $x^4 y^{(4)} + 6x^3 y''' = 0$

NOTE: This is a 4^{th}-order E-C, so you will have to do the "traditional way" in order derive your indicial equation.

Let $y = x^r$. Then, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$, $y''' = r(r-1)(r-2)x^{r-3}$,

and $y^{(4)} = r(r-1)(r-2)(r-3)x^{r-4}$

$\therefore\ xy^{(4)} + 6y''' = 0 \Rightarrow x\left[r(r-1)(r-2)(r-3)\cdot x^{r-4}\right] + 6\left[r(r-1)(r-2)x^{r-3}\right] = 0$

$\therefore\ r(r-1)(r-2)(r-3)\cdot x^{r-3} + 6\left[r(r-1)(r-2)x^{r-3}\right] = 0$

$\Rightarrow x^{r-3}\left[r(r-1)(r-2)(r-3) + 6r(r-1)(r-2)\right] = 0$

$\Rightarrow r(r-1)(r-2)(r-3) + 6r(r-1)(r-2) = 0$; $x \neq 0 \Rightarrow x^{r-3} \neq 0$.

$\therefore\ r(r-1)(r-2)\left[r-3+6\right] = 0 \Rightarrow r(r-1)(r-2)(r+3) = 0$

$\therefore\ r = 0, 1, 2, -3$

$\therefore\ y(x) = c_1x^0 + c_2x^1 + c_3x^2 + c_4x^{-3}$

$$\Rightarrow \boxed{y(x) = c_1 + c_2x + c_3x^2 + \frac{c_4}{x^3}}$$

f) $x^2y'' + 3xy' = 0$; $y(1) = 0$ and $y'(1) = 4$. 17

Indicial eqn : $ar^2 + (b-a)r + c = 0$, where $a=1, b=3, c=0$

$\therefore r^2 + (3-1)r + 0 = 0 \Rightarrow r^2 + 2r = 0 \Rightarrow r(r+2) = 0 \Rightarrow r = 0, -2.$

$\therefore$ General solution: $y(x) = c_1 x^0 + c_2 x^{-2} = c_1 + c_2 x^{-2}$

Applying I.C. $y(1) = 0$

$y(1) = 0 \Rightarrow c_1 + c_2(1)^{-2} = 0 \Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$

Applying I.C. $y'(1) = 4$

$y(x) = c_1 + c_2 x^{-2} \Rightarrow y'(x) = -2c_2 x^{-3} = \dfrac{-2c_2}{x^3}$

$\therefore y'(1) = 4 \Rightarrow \dfrac{-2c_2}{(1)^3} = 4 \Rightarrow -2c_2 = 4 \Rightarrow \boxed{c_2 = -2}$

$\therefore \boxed{c_1 = -c_2 = 2}$

$\therefore \boxed{y(x) = 2 - \dfrac{2}{x^2} = 2 - 2x^{-2}}$ → Final answer!

Case 2 Examples : Solve the following E-C ODEs.

a) $4x^2 \frac{d^2y}{dx^2} + 8x \frac{dy}{dx} + y = 0$

Indicial eqn: $ar^2 + (b-a)r + c = 0$, where $a = 4, b = 8, c = 1$.

$\therefore 4r^2 + (8-4)r + 1 = 0 \Rightarrow 4r^2 + 4r + 1 = 0 \Rightarrow (2r+1)(2r+1) = 0$

$\therefore 2r + 1 = 0 \Rightarrow r = -\frac{1}{2}$ (double root)

$\therefore$ General sol'n: $y(x) = c_1 x^r + c_2 x^r \ln(x)$

$\Rightarrow y(x) = c_1 x^{-1/2} + c_2 x^{-1/2} \ln(x)$

$\Rightarrow \boxed{y(x) = \frac{c_1}{\sqrt{x}} + \frac{c_2}{\sqrt{x}} \ln(x) \;;\; x > 0}$

b) $xy'' + y' = 0$

Indicial eqn: $ar^2 + (b-a)r + c = 0$, where $a = 1, b = 1, c = 0$.

$\therefore r^2 + 0r + 0 = 0 \Rightarrow r^2 = 0 \Rightarrow r = 0$

$\therefore$ General sol'n: $y(x) = c_1 x^r + c_2 x^r \ln(x)$

$\Rightarrow y(x) = c_1 x^0 + c_2 x^0 \ln(x)$

$\Rightarrow \boxed{y(x) = c_1 + c_2 \ln(x)}$

c) $x^2y'' + 5xy' + 4y = 0$

Indicial eqn : $ar^2 + (b-a)r + c = 0$, where $a=1$, $b=5$, $c=4$.

$\therefore\ r^2 + (5-1)r + 4 = 0 \Rightarrow r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \Rightarrow r+2 = 0$

$\therefore\ r = -2$ (double root)

$\therefore$ General sol'n : $y(x) = c_1 x^r + c_2 x^r \ln(x)$

$$\Rightarrow \boxed{y(x) = c_1 x^{-2} + c_2 x^{-2} \ln(x)}$$

d) $4x^2y'' + y = 0$

Indicial eqn : $ar^2 + (b-a)r + c = 0$, where $a=4$, $b=0$, $c=1$

$\therefore\ 4r^2 + (0-4)r + 1 = 0 \Rightarrow 4r^2 - 4r + 1 = 0 \Rightarrow (2r-1)^2 = 0$

$\therefore\ 2r - 1 = 0 \Rightarrow r = 1/2$ (double root)

$\therefore$ General sol'n : $y(x) = c_1 x^r + c_2 x^r \ln(x)$

$$\Rightarrow y(x) = c_1 x^{1/2} + c_2 x^{1/2} \ln(x)$$

$$\Rightarrow \boxed{y(x) = c_1 \sqrt{x} + c_2 \sqrt{x} \ln(x) \ ; x > 0}$$

e) $9x^2y'' + 3xy' + y = 0$; $y(1) = 1$ and $y'(1) = 0$

Indicial eqn: $ar^2 + (b-a)r + c = 0$, where $a = 9$, $b = 3$, $c = 1$

$\therefore 9r^2 + (3-9)r + 1 = 0 \Rightarrow 9r^2 - 6r + 1 = 0 \Rightarrow (3r-1)^2 = 0$

$\therefore 3r - 1 = 0 \Rightarrow r = \frac{1}{3}$ (double root).

$\therefore$ General sol'n: $y(x) = c_1 x^r + c_2 x^r \ln(x)$

$\Rightarrow y(x) = c_1 x^{1/3} + c_2 x^{1/3} \ln(x)$

Applying I.C. $y(1) = 1$

$y(1) = 1 \Rightarrow c_1 (1)^{1/3} + c_2 (1)^{1/3} \cancel{\ln(1)}^{0} = 1$

$\Rightarrow c_1 + c_2(0) = 1 \Rightarrow \boxed{c_1 = 1}$

Applying I.C. $y'(1) = 0$

$y(x) = c_1 x^{1/3} + c_2 x^{1/3} \ln(x) \Rightarrow y'(x) = \frac{1}{3} c_1 x^{-2/3} + c_2 \left[\frac{1}{3} x^{-2/3} \ln(x) + x^{1/3} \cdot \frac{1}{x}\right]$

$\therefore y'(x) = \frac{1}{3} c_1 x^{-2/3} + \frac{1}{3} c_2 x^{-2/3} \ln(x) + x^{-2/3} c_2$

$y'(x) = x^{-2/3} \left[\frac{1}{3} c_1 + \frac{1}{3} c_2 \ln(x) + c_2\right]$

$\therefore y'(1) = 0 \Rightarrow (1)^{-2/3} \left[\frac{1}{3}(1) + \frac{1}{3} c_2 \cancel{\ln(1)}^{0} + c_2\right] = 0$

$\Rightarrow \frac{1}{3} + c_2 = 0 \Rightarrow \boxed{c_2 = -\frac{1}{3}}$

$\therefore$ Final answer: $\boxed{y(x) = x^{1/3} - \frac{1}{3} x^{1/3} \ln(x) = \sqrt[3]{x}\left[1 - \ln(\sqrt[3]{x})\right]}$

Case 3 Examples: Solve the following E-C ODEs

a) $4x^2y'' + 17y = 0$

Indicial eqn: $ar^2 + (b-a)r + c = 0$, where $a = 4, b = 0, c = 17$

$\therefore\ 4r^2 + (0-4)r + 17 = 0 \Rightarrow 4r^2 - 4r + 17 = 0$

$$\therefore\ r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{16 - 4(4)(17)}}{2(4)} = \frac{4 \pm \sqrt{16(-16)}}{8} =$$

$$\frac{4 \pm 4 \cdot 4i}{8} = \tfrac{1}{2} \pm 2i \quad \text{(complex conjugates)}$$

$\therefore$ General sol'n: $y(x) = x^{\lambda}\left[c_1 \cos(\omega \ln(x)) + c_2 \sin(\omega \ln(x))\right]$

where $\lambda \pm \omega i = \tfrac{1}{2} \pm 2i$

$$\therefore\ y(x) = x^{1/2}\left[c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))\right]$$

$$\Rightarrow \boxed{y(x) = \sqrt{x}\left[c_1 \cos(\ln(x^2)) + c_2 \sin(\ln(x^2))\right], \quad x > 0}$$

NOTE: $y'(x) = \left(\frac{c_1}{2\sqrt{x}} + c_2\right)\cos(2\ln(x)) + \left(c_1\sqrt{x} + \frac{c_2}{2\sqrt{x}}\right)\sin(2\ln(x))$; $x > 0$

We will use this fact later in example (d)!

b) $x^2y'' - xy' + 5y = 0$

Indicial eqn: $ar^2 + (b-a)r + c = 0$, where $a = 1$, $b = -1$, $c = 5$.

$\therefore\ r^2 + (-1-1)r + 5 = 0 \Rightarrow r^2 - 2r + 5 = 0 \Rightarrow r^2 - 2r + 1 + 5 - 1 = 0$

$\therefore\ (r-1)^2 + 4 = 0 \Rightarrow (r-1)^2 = -4 \Rightarrow r - 1 = \pm 2i \Rightarrow r = 1 \pm 2i$

General solution: $y(x) = x^{\lambda}\left[c_1 \cos(w \ln(x)) + c_2 \sin(w \ln(x))\right]$

where $r = \lambda \pm wi = 1 \pm 2i$

$\therefore\ y(x) = x\left[c_1 \cos(2\ln(x)) + c_2 \sin(2\ln(x))\right]$

c) $4x^2y'' + 17y = 0$; $y(1) = -1$ and $y'(1) = 0$

NOTE: From Ex (a): $y(x) = \sqrt{x}\left[c_1 \cos(\ln(x^2)) + c_2 \sin(\ln(x^2))\right]$; $x > 0$

and $y'(x) = \left(\frac{c_1}{2\sqrt{x}} + \frac{2c_2}{\sqrt{x}}\right)\cos(2\ln(x)) + \left(\frac{2c_1}{\sqrt{x}} - \frac{c_2}{2\sqrt{x}}\right)\sin(2\ln(x))$; $x > 0$

Applying $y(1) = -1$: $\underbrace{\sqrt{1}}_{1}\left[c_1 \underbrace{\cos(0)}_{1} + c_2 \underbrace{\sin(0)}_{0}\right] = -1 \Rightarrow c_1 = -1$

Applying $y'(1) = 0$: $\left(\frac{\overset{-1}{c_1}}{2\sqrt{1}} + \frac{2c_2}{\sqrt{1}}\right)\underbrace{\cos(0)}_{1} + \left(\frac{2c_1}{\sqrt{1}} - \frac{c_2}{\underset{2}{2\sqrt{1}}}\right)\underbrace{\sin(0)}_{0} = 0$

$\therefore\ -\frac{1}{2} + 2c_2 = 0 \Rightarrow 2c_2 = \frac{1}{2} \Rightarrow \boxed{c_2 = \frac{1}{4}}$

$\therefore$ sol'n: $\boxed{y(x) = \sqrt{x}\left[-\cos(\ln(x^2)) + \frac{1}{4}\sin(\ln(x^2))\right];\ x > 0}$

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d) $4x^2y'' + 5y = 0$; $y(e^{\pi}) = 5$ and $y'(e^{\pi}) = -3$

Indicial eqn: $ar^2 + (b-a)r + c = 0$, where $a=4$, $b=0$, $c=5$.

$\therefore\ 4r^2 + (0-4)r + 5 = 0 \Rightarrow 4r^2 - 4r + 5 = 0$

$$\therefore\ r_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(-4) \pm \sqrt{16-4(4)(5)}}{2(4)} = \frac{4 \pm \sqrt{16(1-5)}}{8} = \frac{4 \pm 2\cdot 2i}{8}$$

$$\frac{1}{2} \pm \frac{1}{2}i = \lambda \pm wi \Rightarrow \lambda = \tfrac{1}{2} \text{ and } w = \tfrac{1}{2}.$$

General sol'n: $y(x) = x^{\lambda}\left[c_1\cos(w\ln(x)) + c_2\sin(w\ln(x))\right]$

$\Rightarrow y(x) = x^{1/2}\left[c_1\cos(\tfrac{1}{2}\ln(x)) + c_2\sin(\tfrac{1}{2}\ln(x))\right]$

$\Rightarrow \boxed{y(x) = \sqrt{x}\left[c_1\cos(\tfrac{1}{2}\ln(x)) + c_2\sin(\tfrac{1}{2}\ln(x))\right]}$

$\therefore\ y'(x) = \tfrac{1}{2}x^{-1/2}\left[c_1\cos(\tfrac{1}{2}\ln(x)) + c_2\sin(\tfrac{1}{2}\ln(x))\right]$

$\quad + x^{1/2}\left[-c_1\sin(\tfrac{1}{2}\ln(x))\cdot\frac{1}{2x} + c_2\cos(\tfrac{1}{2}\ln(x))\cdot\tfrac{1}{2}x\right]$

will use to find c_1 + c_2 !!

$'' = \frac{1}{2\sqrt{x}}\left[c_1\cos(\tfrac{1}{2}\ln(x)) + c_2\sin(\tfrac{1}{2}\ln(x))\right]$

$\quad + \frac{x^{1/2}}{2x}\left[c_2\cos(\tfrac{1}{2}\ln(x)) - c_1\sin(\tfrac{1}{2}\ln(x))\right]$

$$\boxed{y'(x) = \frac{1}{2\sqrt{x}}\left[(c_1+c_2)\cos(\tfrac{1}{2}\ln(x)) + (c_2-c_1)\sin(\tfrac{1}{2}\ln(x))\right]}$$

d) (cont'd)

Applying I.C. $y(e^{\pi}) = 5$

$$y(e^{\pi}) = 5 \Rightarrow \sqrt{e^{\pi}}\left[c_1 \cos\left(\tfrac{1}{2}\cancelto{\pi}{\ln(e^{\pi})}\right) + c_2 \sin\left(\tfrac{1}{2}\cancelto{\pi}{\ln(e^{\pi})}\right)\right] = 5$$

$$\Rightarrow e^{\pi/2}\left[c_1 \cancelto{0}{\cos\left(\tfrac{\pi}{2}\right)} + c_2 \cancelto{1}{\sin\left(\tfrac{\pi}{2}\right)}\right] = 5$$

$$\Rightarrow e^{\pi/2}[c_2] = 5 \Rightarrow \boxed{c_2 = 5e^{-\pi/2}}$$

Applying I.C. $y'(e^{\pi}) = -3$

$$y'(e^{\pi}) = -3 \Rightarrow \frac{1}{2\sqrt{e^{\pi}}}\left[(c_1+c_2)\cos\left(\tfrac{1}{2}\cancelto{\pi}{\ln(e^{\pi})}\right) + (c_2-c_1)\sin\left(\tfrac{1}{2}\cancelto{\pi}{\ln(e^{\pi})}\right)\right] = -3$$

$$\Rightarrow \frac{1}{2e^{\pi/2}}\left[(c_1+c_2)\cancelto{0}{\cos\left(\tfrac{\pi}{2}\right)} + (c_2-c_1)\cancelto{1}{\sin\left(\tfrac{\pi}{2}\right)}\right] = -3$$

$$\Rightarrow \tfrac{1}{2}e^{-\pi/2}\left[(c_2-c_1)\right] = -3$$

$$\Rightarrow c_2 - c_1 = -6e^{\pi/2}$$

$$\Rightarrow c_1 = c_2 + 6e^{\pi/2} \Rightarrow \boxed{5e^{-\pi/2} + 6e^{\pi/2} = c_1}$$

Final answer: $\boxed{y(x) = \sqrt{x}\left[\left(5e^{-\pi/2} + 6e^{\pi/2}\right)\cos\left(\tfrac{1}{2}\ln(x)\right) + 5e^{-\pi/2}\sin\left(\tfrac{1}{2}\ln(x)\right)\right]}$