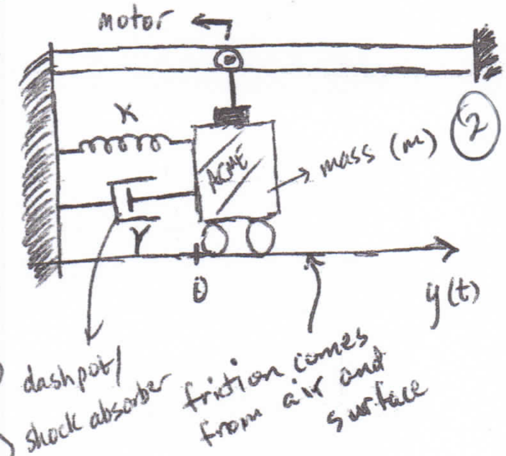
Application of 2^{nd}-Order Linear Equations: Spring-Mass Systems

At this point in our study of Ordinary Differential Equations, we have learned how to solve both homogeneous and non-homogeneous 2^{nd}-order linear ODEs. In the non-homogeneous case, we have addressed (through the Method of Undetermined Coefficients) how to solve such equations when the "forcing function" $g(x)$ (or $g(t)$) is a polynomial, exponential, sine, or cosine function. (Note that this list also implies any linear combination and/or product of these types of functions as well). With this knowledge under our belt, we have enough background to study one MAJOR APPLICATION of these type of ODEs: Spring-Mass Systems! Our goal in this set of notes is to do the following:

- Derive a simple spring-mass system to understand how these types of systems relate to 2^{nd}-order Linear ODEs with constant coefficients.
- Review Cases I – III for solving 2^{nd}-order Linear Homogeneous ODEs and see how these 3 cases translate to the 4 cases we will study for spring-mass systems. Concepts of Damping + Resonance will be discussed as well.
- Complete a set of examples showcasing various aspects of spring-mass systems.

Derive Model For (Simple) Spring-Mass Systems

Consider Fig. A of a simple spring-mass system. This system consists of a mass that is attached to both a spring (that follows the rules of Hooke's Law from Physics) and a dashpot (i.e. a shock absorber) and rolls on some surface. There is also a motor that is built into a horizontal pulley system (with wheels) that can pull or push the mass backwards or forwards (or both) with a force $F_0 = F_{other}$ = external force on system since there is an extension on the motor-pulley sub-system that is attached to the mass. Note that for simplicity the motor-pulley subsystem is assumed to have negligible friction so that the only resistive forces in this system due to friction will be the friction caused by air (i.e. air drag) and friction caused by the wheels under the mass making contact with surface they are rolling on.



- m = mass (kg)
- κ = spring constant $\left(\frac{kg}{sec^2}\right)$
- γ = damping coefficient of dashpot $\left(\frac{kg}{sec}\right)$
- $m, \kappa > 0$
- $\gamma \geq 0$
- F_0 = force acting on the spring-mass system with magnitude $|F_0|$

Fig. A

Now that we have all of these assumptions stated, we can start to model our spring-mass system. (Note that other assumptions will be stated, but we are choosing to wait to introduce those assumptions until it makes sense to do so in the derivation process.

It turns that with all of these assumptions, we could (simply) model our spring-mass system as a sum of forces balancing out (to zero).

$$\therefore \begin{pmatrix}\text{Total Force}\\ \text{of spring-mass}\\ \text{system}\end{pmatrix} = \begin{pmatrix}\text{Force produced}\\ \text{by dashpot to}\\ \text{help slow down}\\ \text{motion faster}\end{pmatrix} + \begin{pmatrix}\text{Force produced}\\ \text{by the resistance}\\ \text{of air on mass}\\ \text{as it moves}\end{pmatrix} + \begin{pmatrix}\text{Force}\\ \text{of}\\ \text{spring}\end{pmatrix} + \begin{pmatrix}\text{Force of}\\ \text{motor}\\ \text{acting}\\ \text{on}\\ \text{mass}\end{pmatrix}$$

$$\Rightarrow \quad F_{total} = F_{dashpot} + F_{air} + F_{spring} + F_{other}$$

<u>NOTE</u>: In Fig. A, we stated that the friction in this system only comes from the surface the mass rolls on and air. Thus, for simplicity, we will lump these forces together into a new, single force called

$$F_{resist} = F_{dashpot} + F_{air}$$

$$\therefore F_{total} = F_{resist} + F_{spring} + F_{other}$$

Now we observe the following facts about each force.

- $\boxed{F_{total} = \text{total force of system} = ma = m\frac{dv}{dt} = m\frac{d^2y}{dt^2}}$ (via Newton's law of Motion)
- F_{resist} will be proportional to how fast the mass is moving in this system since both surface friction and the (resistive) friction caused by the material and/or fluid in the dashpot/shock absorber will increase as the velocity of the mass increases. The constant of proportionality here will be called the dampening constant γ. Thus, $\boxed{F_{resist} = -\gamma v(t) = -\gamma \cdot \frac{dy}{dt}}$.

- F_{resist} (cont'd): Our force will have a negative sign because this force will work in the opposite direction of any external force F_{other} we apply to the system

- F_{spring} : If our system allows the mass "m" to move backwards and/or forwards without stretching too much, then this spring will follow Hooke's Law. Thus, $F_{spring} = -k(y_f - y_i)$, where y_i = position of mass at equilibrium position (i.e. position of mass when spring is at its natural length). The final position of the mass at some time t is y_f. Per our Fig. A, $y_i = 0$. If we let $y_f = y$, then we can rewrite our equation for F_{spring} to be $\boxed{F_{spring} = -ky}$, where k = spring constant that is proportional to the force and distance applied to mass "m" according to Hooke's Law.

Using all this info, we can rewrite our "force balance" equation for this system (in terms of y). Noting that F_{other} can be a force specified later to be non-zero quantity or zero, we see that...

$$F_{total} = F_{resist} + F_{spring} + F_{other} \Rightarrow m\frac{d^2y}{dt^2} = -\gamma\frac{dy}{dt} - ky + F_0$$

$$\therefore \boxed{m\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + ky = F_0}$$

} this is a 2nd-order Linear ODE will constant coefficients

For simplicity, we could rewrite this equation as ...

$$\boxed{my'' + \gamma y' + ky = F_0} \quad (\star)$$

We know the equation above has a characteristic equation of $mr^2 + \gamma r + k = 0$ where Cases I - III are determined by the discriminant $\gamma^2 - 4mk$. We will now study how these 3 cases translate to the 4 cases for spring-mass systems. Note that the value of our damping coefficient, γ, will control/identify each case.

The 4 Cases for Spring-Mass Systems

Recall that $mr^2 + \gamma r + k = 0$ is the characteristic equation for equation ($\star$) above. The roots of this equation can be found by the quadratic equation $r_{1,2} = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$. The 4 cases for spring-mass systems will be listed below. Afterwards, we will discuss the concepts, physical interpretation, and general solution for each case.

- Case 1 ($\gamma = 0$): No damping
- Case 2 ($\gamma = (0, 2\sqrt{mk})$): Underdamped
- Case 3 ($\gamma = 2\sqrt{mk}$): Critically-damped
- Case 4 ($\gamma > 2\sqrt{mk}$): Overdamped

<u>NOTE</u>: Before we get into discussing Cases 1-4 of spring-mass systems, we need to further explain what the concept of damping in a spring-mass system entails so that we have an intuitive understanding of why this parameter controls how the response of our spring-mass system (i.e. the position function for the mass, $y(t)$) will behave.

Recall that the dashpot is nothing more than a shock absorber for our spring-mass system. The purpose of this device is to reduce the amount of oscillation in our system to try to get our mass back to equilibrium as quickly as possible if our system experiences a sudden, impulse (force) $F_{other} = F_0$. (F_0 being an impulse (force) is just saying that the force F_0 occurs in a quick burst. A good example of what this is like is when a car rolls over a pothole in the road. That "bump" that you feel can be characterized as an impulse force). In the cases where our external force F_0 is continuous (i.e. F_0 is applied for a relatively long period of time without stopping), the dashpot's purpose is to "smooth out" the possible oscillations in the system so that the oscillations are not so violent and/or grow out of control (+ break something).

The damping coefficient, γ, for a material and/or fluid is just a dimensionless measure that quantifies oscillations in a spring-mass system will (possibly) decay over time after an external, impulse force F_0 is applied to the system. The value of γ varies based upon the type of material or fluid used to dampen the mass' motion. In short, the "thicker" the material or fluid is, the higher the value of γ is, and, thus, the less the mass will oscillate in the system.

The coefficient γ is usually found empirically (i.e. by observation via experiment) and takes things into account such as the temperature of the material/fluid, geometry/shape of the container that the material/fluid is housed in, etc.

<u>Case 1 ($\gamma = 0$): No damping</u>

If $\gamma = 0 \Rightarrow$ the discriminant $\gamma^2 - 4mk = -4mk$. Thus, our roots for $r_{1,2} = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$ are $r_{1,2} = \dfrac{\pm\sqrt{-4mk}}{2m} = \pm\dfrac{2\sqrt{mk}}{2m}\, i = \pm\sqrt{\dfrac{k}{m}}\, i$.

$\therefore\ r_{1,2} = \lambda \pm wi = 0 \pm \sqrt{\frac{k}{m}}\, i \Rightarrow \lambda = 0$ and $w = \sqrt{\frac{k}{m}}$

$\therefore\ y_h(t) = e^{\lambda t}\left[c_1 \cos(wt) + c_2 \sin(wt)\right] = \cancel{e^{0t}}^{\,1}\left[c_1 \cos\left(\sqrt{\tfrac{k}{m}}\, t\right) + c_2 \sin\left(\sqrt{\tfrac{k}{m}}\, t\right)\right]$

$\therefore\ y_h(t) = c_1 \cos(w_0 t) + c_2 \sin(w_0 t)$, where $w_0 = w = \sqrt{\frac{k}{m}}$ = natural freq. of system. (in radians/sec).

Case 1 ($\gamma=0$) : No damping (cont'd)

NOTE: $y_h(t) = c_1\cos(w_0 t) + c_2\sin(w_0 t)$ shows us that if our dashpot has no dampening affect on our system, our mass will oscillate continuously without decaying. Thus, our mass will never go back to equilibrium in this case. (See Exs. 2a + 3a for an example of what this case looks like).

Case 2 ($\gamma = (0, 2\sqrt{mk})$) : Underdamped

If $0 < \gamma < 2\sqrt{mk}$, then our discriminant $\gamma^2 - 4mk$ is still negative (i.e. $\gamma^2 - 4mk < 0$), but since $\gamma \neq 0$ and $\gamma > 0$, this guarantees that our mass "m" would eventually get back to equilibrium in our system (i.e. mass will be at position $y=0$ eventually as $t \to \infty$) although it make take the mass a (relatively long) time to do it. To see why this happens, note that if $0<\gamma<2\sqrt{mk}$, then $\gamma^2 - 4mk < 0 \Rightarrow$

$$4mk - \gamma^2 > 0 \Rightarrow r_{1,2} = \frac{-\gamma \pm \sqrt{4mk-\gamma^2}\, i}{2m} = -\frac{\gamma}{2m} \pm \frac{\sqrt{4mk-\gamma^2}}{2m} i$$

$$= \lambda \pm wi, \text{ where } \lambda = \frac{-\gamma}{2m} < 0 \text{ and } w = \frac{\sqrt{4mk-\gamma^2}}{2m} > 0.$$

$$\therefore y_h(t) = e^{\lambda t}\left[c_1\cos(wt) + c_2\cos(wt)\right]$$

$$\Rightarrow y_h(t) = e^{-\frac{\gamma}{2m}t}\left[c_1\cos\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t\right) + c_2\sin\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t\right)\right]$$

This is an exponentially decreasing function. This is what forces our oscillating response to eventually decay back to $y=0$!

Case 2 ($\gamma = (0, 2\sqrt{mk})$): Underdamped (cont'd).

See examples 2c + 3b for an example of what this case would look like.

CONNECTION: Relationship between the natural frequency and period of $y(t)$ for Cases 1 + 2 for spring-mass systems.

At this time, it would be good to stop and analyze how the oscillatory behavior of $y(t)$ for Case 1 vs. Case 2 is changing so that we can better understand ① the physical difference between Cases 1+2, ② how Case 3 (which we will discuss in detail later) relates to Cases 1+2, and ③ how the concept of resonance in a spring-mass system occurs only occurs when $\gamma = 0$ (i.e. Case 1 situation).

Recall that for Cases 1 + 2, $r_{1,2} = \dfrac{-\gamma \pm \sqrt{4mk - \gamma^2}\, i}{2m}$ yields the roots for our position function (solution) to our ODE $my'' + \gamma y' + ky = F_0$.

If $\gamma = 0$, then $4mk - \gamma^2 = 4mk$. If $0 < \gamma < 2\sqrt{mk}$, then

$$4mk - \gamma^2 < 4mk \Rightarrow \frac{\sqrt{4mk - \gamma^2}}{2m} < \frac{\sqrt{4mk}}{2m} \Rightarrow \omega_{\text{Case 2}} < \omega_{\text{Case 1}}.$$

So, the natural (angular) frequency for Case 1 is larger than the natural

CONNECTION : (cont'd)

(angular) frequency for Case 2. This lets us know the following:

(i) The (angular) frequency $w_{case\,1} = w_0$ (of the homogeneous solution $y_h(t)$ of the entire response $y(t) = y_h(t) + y_p(t)$) is the highest frequency possible for this spring-mass system given that mass "m" and spring constant "k" don't change and damping coefficient $\gamma \geq 0$! Frequency $w_{case\,1} = w_0$ occurs when $\gamma = 0$ (no damping).

(ii) when $\gamma = 0$, the mass will oscillate back and forth at the fastest rate possible for this system. As γ increases within $\gamma = (0, 2\sqrt{mk})$, the value of w will get smaller $\Rightarrow$ the rate at which the mass oscillates back and forth will get slower as γ increases within the interval $\gamma = (0, 2\sqrt{mk})$.

(iii) When $\gamma = 2\sqrt{mk}$, $4mk - \gamma^2 = 0$ $\Rightarrow r_{1,2} = \frac{-\gamma \pm \sqrt{0}}{2m} = \frac{-\gamma}{2m}$.

The natural frequency of the spring-mass system is zero (i.e. $w = 0$). The spring-mass system will not have a full oscillation in this case!

Case 3 ($\gamma = 2\sqrt{mk}$): Critically-damped

If $\gamma = 2\sqrt{mk}$, then $r_{1,2} = -\frac{\gamma}{2m}$ is the (double) root from the characteristic equation for our ODE. Thus, our (homogeneous) solution is...

$$\boxed{y_h(t) = (c_1 + c_2 t)e^{-\gamma/2m\, t} = c_1 e^{-\gamma/2m\, t} + c_2 t e^{-\frac{\gamma}{2m} t}}$$

NOTE: $y_h(t)$ does not have any sine or cosine functions in it, so this is why $y_h(t)$ ceases to have oscillatory behavior when $\gamma = 2\sqrt{mk}$.

In reality, when a spring-mass system is critically-damped it tries to oscillate the mass back and forth, but since $\gamma = 2\sqrt{mk}$, the mass does not make a full oscillation, and, thus, returns the mass back to equilibrium as fast as the properties of the spring-mass system will allow. Thus, the Critically-damped condition for a spring-mass system is essentially the borderline between oscillatory and exponential behavior for this system!

To illustrate this concept of Case 1 vs. Case 2 vs. Case 3 behavior of $y_h(t)$, we will set $m = k = 1$ and evaluate $\gamma^2 - 4mk$ for several values of γ.

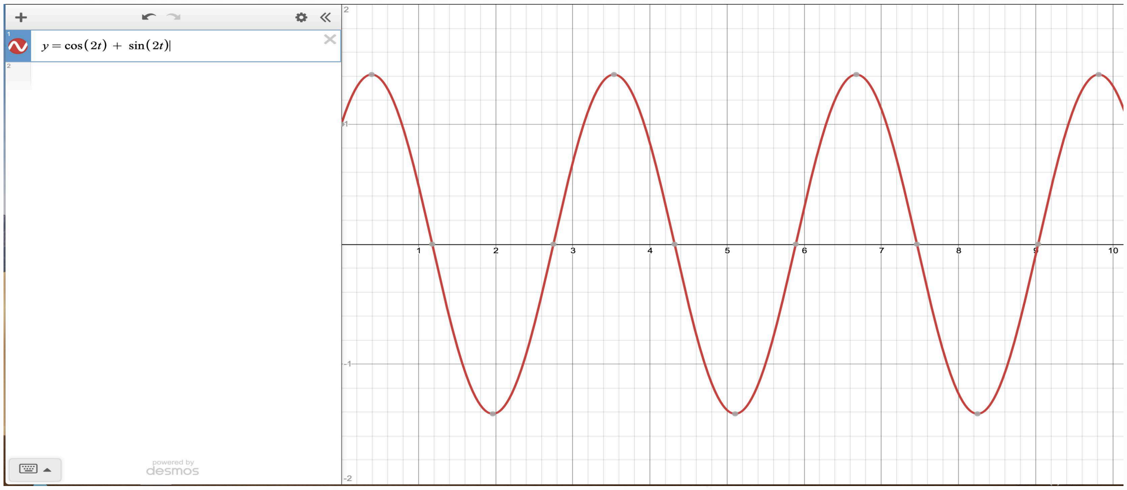
Let $m = k = 1$. Also, note that $w_0 = 2\pi f_0 = \frac{4mk - \gamma^2}{2m}$, where f_0 is the natural frequency of our spring-mass system in Hz (i.e. cycles/sec). Finally, recall that the period of a periodic/oscillatory function is $T_0 = \frac{1}{f_0}$. Now we will see how the frequency and period of our spring-mass system changes as γ increases on the interval $\gamma = [0, 2\sqrt{mk}] = [0, 2]$.

γ	$\gamma^2 - 4mk$	$4mk - \gamma^2$	w_0 (rad/sec)	f_0 (Hz)	Period (T) (secs)	Case #
0	-4	4	2	$\frac{1}{\pi} \approx 0.3183$	$\pi \approx 3.1416$	1
$\sqrt{2}$	-2	2	1	$\frac{1}{2\pi} \approx 0.1592$	$2\pi \approx 6.2832$	2
2	0	0	0	0	∞ (undefined)	3

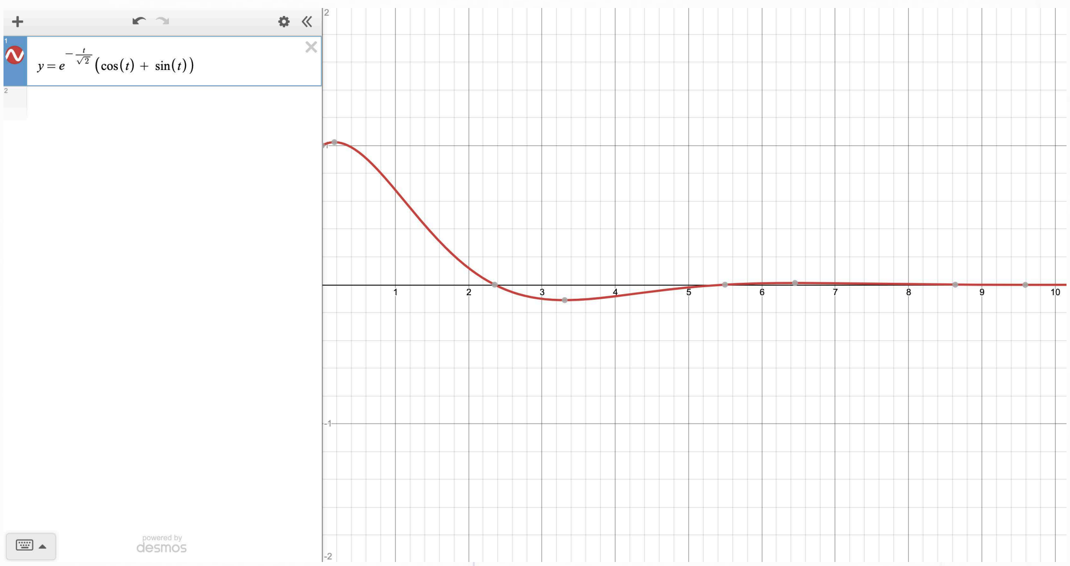
NOTE: For values $\gamma > 2\sqrt{mk}$ (i.e. $\gamma > 2$ in this case), our position function $y_h(t)$ is purely exponential, so $y(t)$ is no longer periodic. Thus, it makes no sense to find w_0, f_0, and period T for this situation (i.e. Case 4 situation for our spring-mass system)!

See the resulting graphs on the next page that shows $y_h(t)$ in each Case 1–3.

Case 1 ($\gamma = 0$, $w_0 = 2$, $f_0 = \frac{1}{\pi}$, period $= \pi$)



Case 2 ($\gamma = \sqrt{2}$, $w_0 = 1$, $f_0 = \frac{1}{2\pi}$, period $= 2\pi$)



Case 3 ($\gamma = 2$, $w_0 = 0$, $f_0 = 0$, period $= \infty$)

Case 4 ($\gamma > 2\sqrt{mk}$) : Overdamped

If $\gamma > 2\sqrt{mk}$, then the discriminant $\gamma^2 - 4mk > 0$. Therefore,

$r_{1,2} = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$ will be 2 real, distinct numbers, and, thus,

our position function $y_h(t)$ is purely exponential as commented on the previous page.

$$\therefore\ y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \text{ where } r_1, r_2 < 0.$$

NOTE: For these types of systems, $r_1, r_2 < 0$ will always be true. This is needed to ensure that the system is stable. If $r_{1,2} > 0$, then this system's position function $y(t) \to \infty$ as $t \to \infty$. We know that this situation is not practical because eventually either the dashpot and/or spring would break! Also, note that $r_{1,2} > 0$ violates the assumption that the spring is not stretched too much $\Rightarrow$ the spring would not behave according to Hooke's Law!

Revisiting Case 1 to Discuss the Concept of Resonance

Since Case 1 of a spring-mass system is the case where $y(t)$ will not approach 0 as $t \to \infty$, this is the case where the phenomenon of resonance of a spring-mass system could be experienced.

In spring-mass systems, there are 2 primary responses we are concerned about. These responses are called the transient (i.e. impulse) response and the steady-state response.

Translating the types of solutions we have dealt with in this course to what these responses mean in terms of spring-mass systems, note that ...

- y_h = homogeneous solution = transient/impulse response
- y_p = particular solution = steady-state response

Thus, y_h characterizes how our spring-mass system will respond when there is a quick, impulse (force) function F_0 applied to this system. (Recall the example of the car rolling over a pothole in the road). Our solution y_p characterizes how our spring-mass system will respond as $t \to \infty$ if our system is "excited" by either an impulse, constant, or continuously oscillating force F_0 !!

Now that we have properly defined what a transient and a steady-state response actually is for a spring-mass system, we will now define the concept of resonance.

<u>Def'n (Resonance)</u>: At least one oscillation/vibration of larger than normal amplitude that is produced by an external, oscillatory force (F_0) on a (spring-mass) system whose frequency is close to (or equals) that of the natural frequency of the (spring-mass) system and has an amplitude that is relatively small compared to the (larger than normal) amplitude response of the (spring-mass) system.

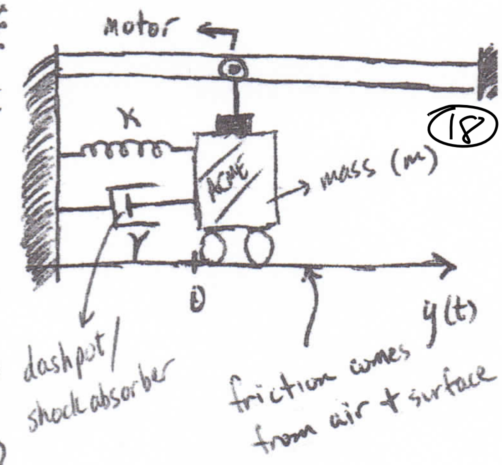
<u>NOTE</u>: For spring-mass systems, this phenomenon occurs when the max displacement of $y_p(t)$ (i.e. the amplitude of the steady-state response of our spring-mass system), when F_0 is oscillatory, is larger than the max displacement of $y_p(t)$ when F_0 is constant. Resonance for a (spring-mass) system reaches its peak when the external force F_0 oscillates at the same frequency as the natural frequency $w_0 = \sqrt{\frac{K}{m}}$ of our spring-mass system.

To clarify our definition of resonance with a concrete example, note that in example 4a that occurs later in this document, you will see that when $\gamma = 0$ and $F_0 = 2$, the position function (response) $y(t) = y_h(t) + y_p(t) = c_1 \cos(t) + c_2 \sin(t) + 2$, where $y_p(t) = 2$ and $y_h(t) = c_1 \cos(t) + c_2 \sin(t)$. You will also discover (in Ex. 4d) that the max displacement (i.e. amplitude) of $y(t)$ is C, where $C = \sqrt{c_1^2 + c_2^2}$. However, when $\gamma = 0$ and $F_0 = 12\sin(t)$ in Ex. 5a, the position function $y(t) = y_h(t) + y_p(t) = c_1 \cos(t) + c_2 \sin(t) - 6t\cos(t)$, where $y_h(t) = c_1 \cos(t) + c_2 \sin(t)$ and $y_p(t) = -6t \cdot \cos(t)$. The max displacement (i.e. amplitude) of $y(t)$ in this case is infinite because as $t \to \infty$, $|y_p(t)| = 6t\cos(t) \gg |y_h(t)| = c_1 \cos(t) + c_2 \sin(t)$, and $\lim_{t \to \infty} |y_p(t)| = \infty$.

Also, note that when $F_0 = 12\sin(t)$, $y_h(t) = c_1 \cos(t) + c_2 \sin(t)$ which implies that F_0 and $y_h(t)$ (i.e. the transient/impulse response of our spring-mass system) oscillate at the same frequency!

Now we will do 6 examples that practically showcase all that we have discussed.

Ex. 1: Consider the spring-mass system shown.



- m = mass = 12 kg
- κ = spring constant $\left(\frac{kg}{sec^2}\right)$
- γ = damping coefficient of dashpot $\left(\frac{kg}{sec}\right)$
- $m, \kappa > 0$
- $\gamma \geq 0$
- F_0 = force acting on the spring-mass system with magnitude $|F_0|$

a) Suppose that mass "m" is moved such that the spring within our spring-mass system is stretched 2m from its equilibrium by the device that is attached to the mass that has a motor that exerts a (single, one-time) force of 24 N (i.e. 24 $\frac{kg \cdot m}{s^2}$) before returning the mass back to the position it was in when the spring was in equilibrium.

Using Hooke's Law, find the spring constant, κ, and the equation that represents the force from the due to it being stretched or compressed (i.e. F_{spring}).

NOTE: F_{spring} will be in the opposite direction of force F_0. We also assume that $y(0) = 0$ per our figure above for the spring-mass system.

$F_{spring} = -\kappa(y_f - y_i)$. Since at time $t=0$, $y(0)=0$ (i.e. position of mass is at equilibrium at time $t=0$), this means that $y_i = y(0) = 0$. Also, $|F_{spring}| = 24$ N when y_f = position mass stretched from equilibrium = 2 m.

$$\therefore \kappa = \frac{|F_{spring}|}{y_f - y_i} = \frac{24 \text{ N}}{2 - 0 \text{ m}} = 12 \text{ N/m} = 12 \frac{kg}{sec^2} \Rightarrow \boxed{F_{spring} = -12y}, \text{ where}$$

$y = y(t) = y_f - y_i = y_f - 0 = y_f$!!

b) State the equation for F_{resist} = force due to the air resistance and friction in terms of γ. (NOTE: F_{resist} is proportional to the velocity of the mass "m" and is of opposite direction to force F_0. $F_{resist} = F_{dashpot}$).

$F_{resist} = -\gamma v(t) = -\gamma \cdot \frac{dy}{dt}$, where $v(t) = \frac{d}{dt}[y(t)]$. Thus, $\boxed{F_{resist} = -\gamma \frac{dy}{dt} = -\gamma \cdot y'}$.

Ex. 1 : (cont'd)

c) Recall that the total force F of this spring-mass system can be characterized by the equation ...

$$F_{resist} + F_{spring} + F_{other} = F \quad , \text{ where } F = ma = m \cdot \frac{d^2y}{dt^2} .$$

If we let $F_{other} = F_z$ (for now), express this equation that characterizes the total force for this spring-mass system in terms of $y = y(t)$.

$$F_{resist} + F_{spring} + F_{other} = F \Rightarrow -\gamma \frac{dy}{dt} - 12y + F_z = m\frac{d^2y}{dt^2}$$

$$\therefore \quad m\frac{d^2y}{dt^2} + \gamma \frac{dy}{dt} + 12y = F_z$$

or

$$my'' + \gamma \cdot y' + 12y = F_z \, .$$ Since $m = 12$ kg, we can say

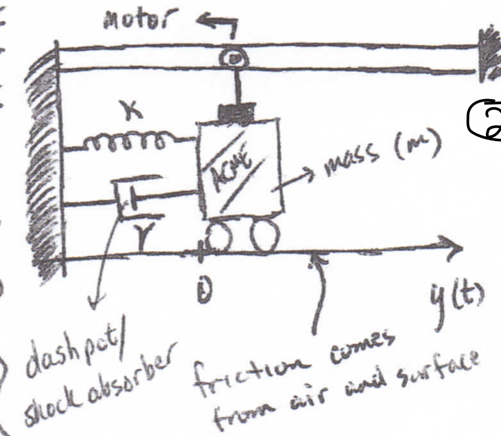
that ...

$$\boxed{12y'' + \gamma y' + 12y = F_z}$$

d) Let $F_z = 0$ for your equation in part (c) above. State this (new) equation and identify what type of ODE we classify this type of equation to be.

$$12y'' + \gamma y' + 12y = 0$$ (2^{nd}-order Linear Homogeneous ODE)

Ex. 2 : Consider the same spring-mass system as we did in Ex. 1. Using the equation we arrived at for Ex. 1d, where $F_0 = F_2 = 0$ and m and κ are the same as from Ex. 1 as well, we will consider a dashpot filled with a fluid and/or material that has the corresponding damping coefficients as shown in the table below.



- m = mass = 12 kg
- κ = spring constant ($\frac{kg}{sec^2}$)
- γ = damping coefficient ($\frac{kg}{sec}$) of dashpot + surface
- $m, \kappa > 0$
- $\gamma \geq 0$
- F_0 = force acting on the spring-mass system with magnitude $|F_0|$

a) Find a general solution for this spring-mass system (i.e. find a solution for $y(t)$) if we decide to use no fluid in our dashpot.(*)

$$12y'' + 0y' + 12y = 0 \Rightarrow 12y'' + 12y = 0$$

$$\therefore y'' + y = 0 \Rightarrow \underline{\text{Char. eqn.}}: r^2 + 1 = 0$$

$$\therefore r^2 = -1 \Rightarrow r = \pm i = 0 \pm 1i = \lambda \pm \omega i$$

So, $\lambda = 0$ and $\omega = 1$.

$$\therefore y(t) = y_h(t) = e^0\left[c_1 \cos(1 \cdot t) + c_2 \sin(1 \cdot t)\right]$$

$$\Rightarrow y(t) = y_h(t) = c_1 \cos(t) + c_2 \sin(t)$$

Name of Fluid (*)	Damping Coefficient (γ)
No Fluid	$\gamma = 0$
Fluid A	$\gamma = 18$
Fluid B	$\gamma = 24$
Fluid C	$\gamma = 26$

(*) The total friction of the dashpot is assumed to include the friction of the surface the wheels of mass "m" roll on as well.

b) Will the dashpot / surface of our spring-mass system play a role in damping the position of our mass "m" if a force $F_0 \neq 0$ was applied to this system?

No, since $\gamma = 0$, no damping will occur from dashpot! If any damping occurs, it would have to come from the spring.

Ex. 2 : (cont'd)

c) Find a general solution for this spring-mass system (i.e. find $y(t)$) if we decide to use Fluid A in our dashpot.* $12y'' + 18y' + 12y = 0$

$\therefore$ Char. eqn: $12r^2 + 18r + 12 = 0 \Rightarrow 2r^2 + 3r + 2 = 0 \Rightarrow r_{1,2} = \frac{-(3) \pm \sqrt{(3)^2 - 4(2)(2)}}{2(2)}$

$\therefore r_{1,2} = \frac{-3 \pm \sqrt{9-16}}{4} = \frac{-3 \pm \sqrt{-7}}{4} = \frac{-3 \pm \sqrt{7}i}{4} = -\frac{3}{4} \pm \frac{\sqrt{7}}{4} i$ (Under-damped Case)

$\therefore \boxed{y(t) = y_h(t) = e^{-\frac{3}{4}t}\left[c_1 \cos\left(\frac{\sqrt{7}}{4}t\right) + c_2 \sin\left(\frac{\sqrt{7}}{4}t\right)\right]}$

d) Find a general solution for this spring-mass system (i.e. find $y(t)$) if we decide to use Fluid B in our dashpot.* $12y'' + 24y' + 12y = 0$

Char. eqn: $12r^2 + 24r + 12 = 0 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0$

$\therefore r + 1 = 0 \Rightarrow r = -1$ (Critically-damped case).

NOTE: Since $r = -1$ is a double root $\Rightarrow e^{-t}$ is a solution to our ODE, but we need another solution to complete the basis of solutions for our ODE. From reduction of order, we know that te^{-t} would be this other, linearly-independent, solution (with respect to e^{-t}).

$\therefore \boxed{y(t) = y_h(t) = e^{-t}\left[c_1 + c_2 t\right] = c_1 e^{-t} + c_2 t e^{-t}}$

e) Find a general solution for this spring-mass system (i.e. find $y(t)$) if we decide to use Fluid C in our dashpot. (*) $12y'' + 26y' + 12y = 0$

$\therefore$ Char. eqn: $12r^2 + 26r + 12 = 0 \Rightarrow 6r^2 + 13r + 6 = 0 \Rightarrow (2r+3)(3r+2) = 0$

$\therefore 2r+3 = 0$ or $3r+2 = 0 \Rightarrow r = -\frac{3}{2}$ or $r = -\frac{2}{3}$ (Over-damped case)

$$\therefore \boxed{y(t) = y_h(t) = c_1 e^{-\frac{3}{2}t} + c_2 e^{-\frac{2}{3}t}}$$

f) Decide what the expected behavior of $y(t)$ should be based upon the different damping coefficients γ for Fluids A–C (and No Fluid).

(i) No Fluid ($\gamma = 0$)

(ii) Fluid A ($\gamma = 18$)

(iii) Fluid B ($\gamma = 24$)

(iv) Fluid C ($\gamma = 26$)

iii System will try to oscillate, but won't complete a full cycle of oscillation. As $t \to \infty$, system responds in more exponential fashion. (Critically-damped)

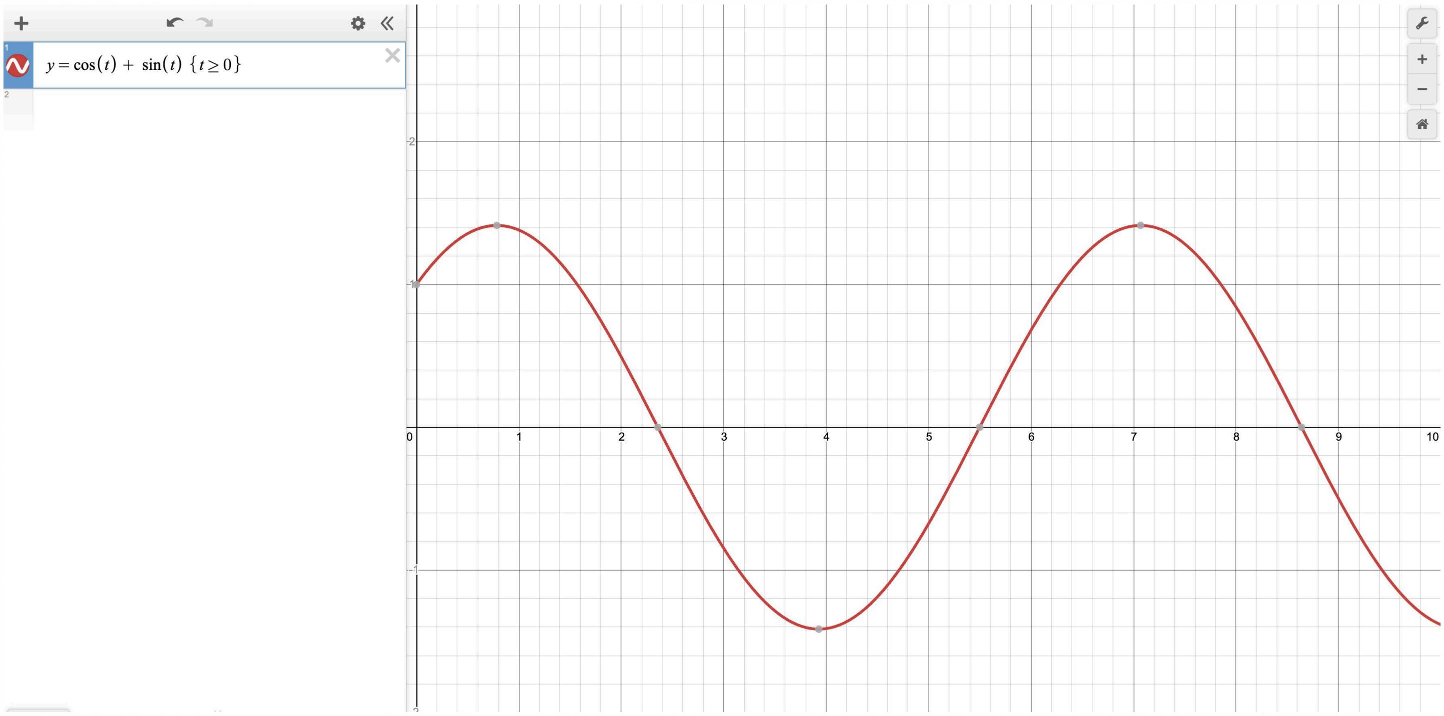
iv System will never oscillate. System responds exponentially from $t = 0$ to $t \to \infty$. (Over-damped)

i System will oscillate without end from $t = 0$ to $t \to \infty$. (No damping)

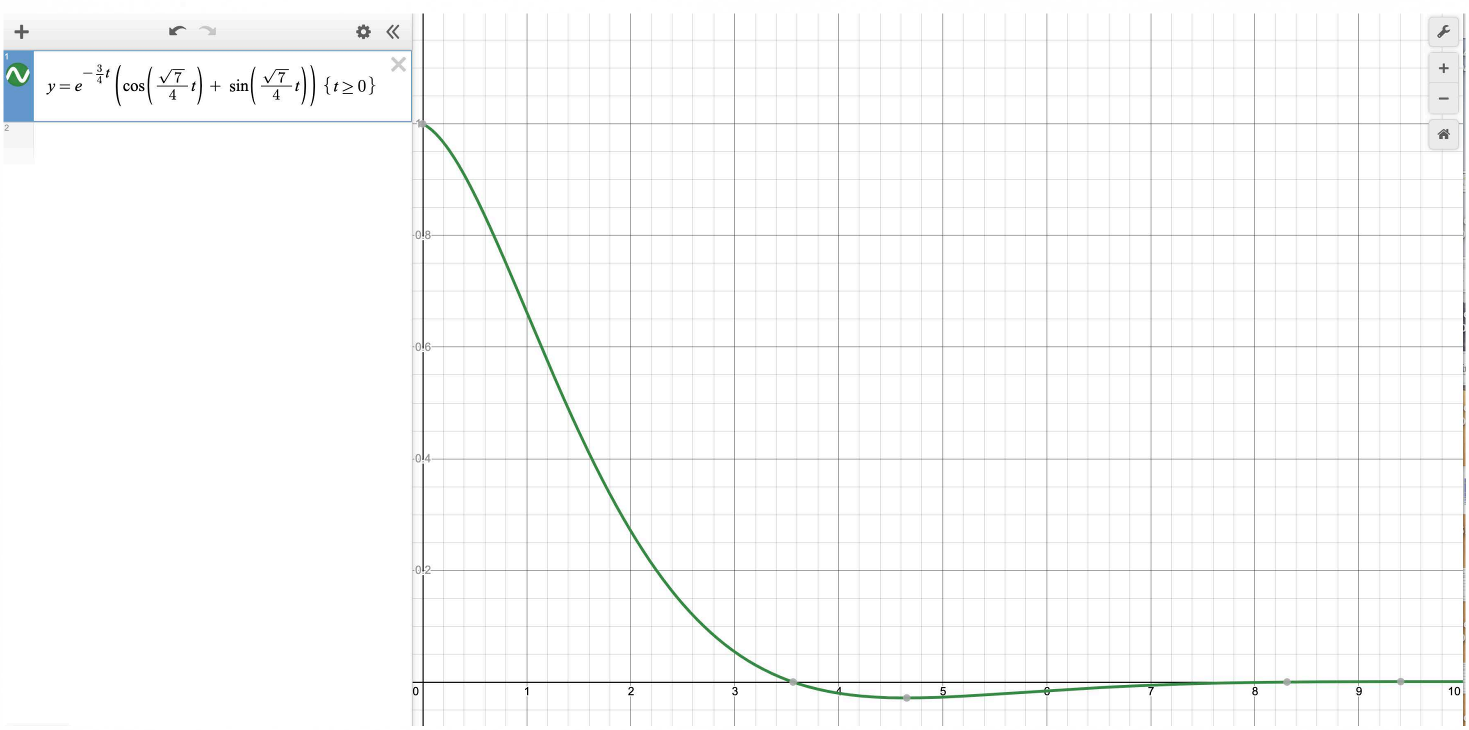
ii System will oscillate for a while, but it will eventually die out (at an exponential rate). (Under-damped).

Ex. 3: Graph the solutions for $y(t)$ from Ex. 2a, c, d, and e. Let $c_1 = c_2 = 1$.

a) Ex. 2a: $y(t) = y_h(t) = \cos(t) + \sin(t)$; $t \geq 0$

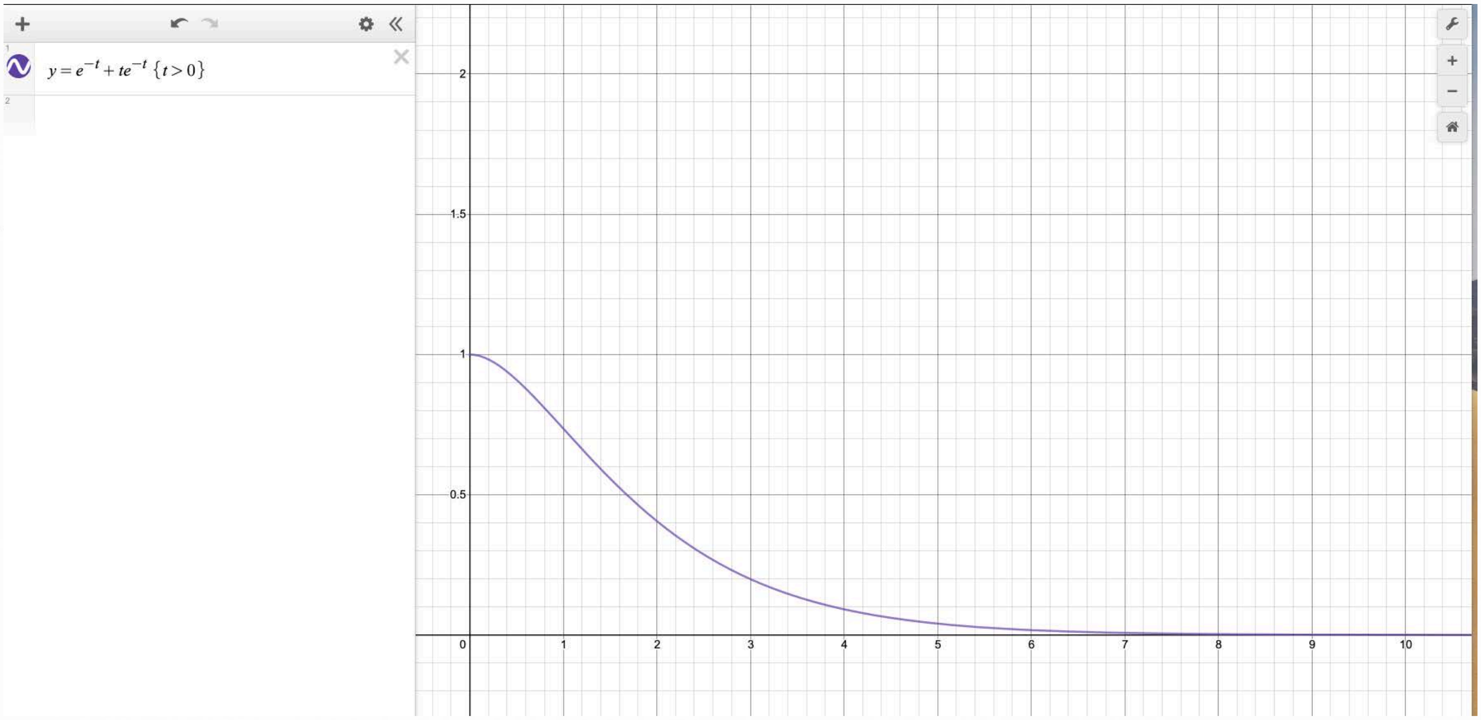


b) Ex. 2c: $y(t) = y_h(t) = e^{-\frac{3}{4}t}\left[\cos\left(\frac{\sqrt{7}}{4}t\right) + \sin\left(\frac{\sqrt{7}}{4}t\right)\right]$; $t \geq 0$

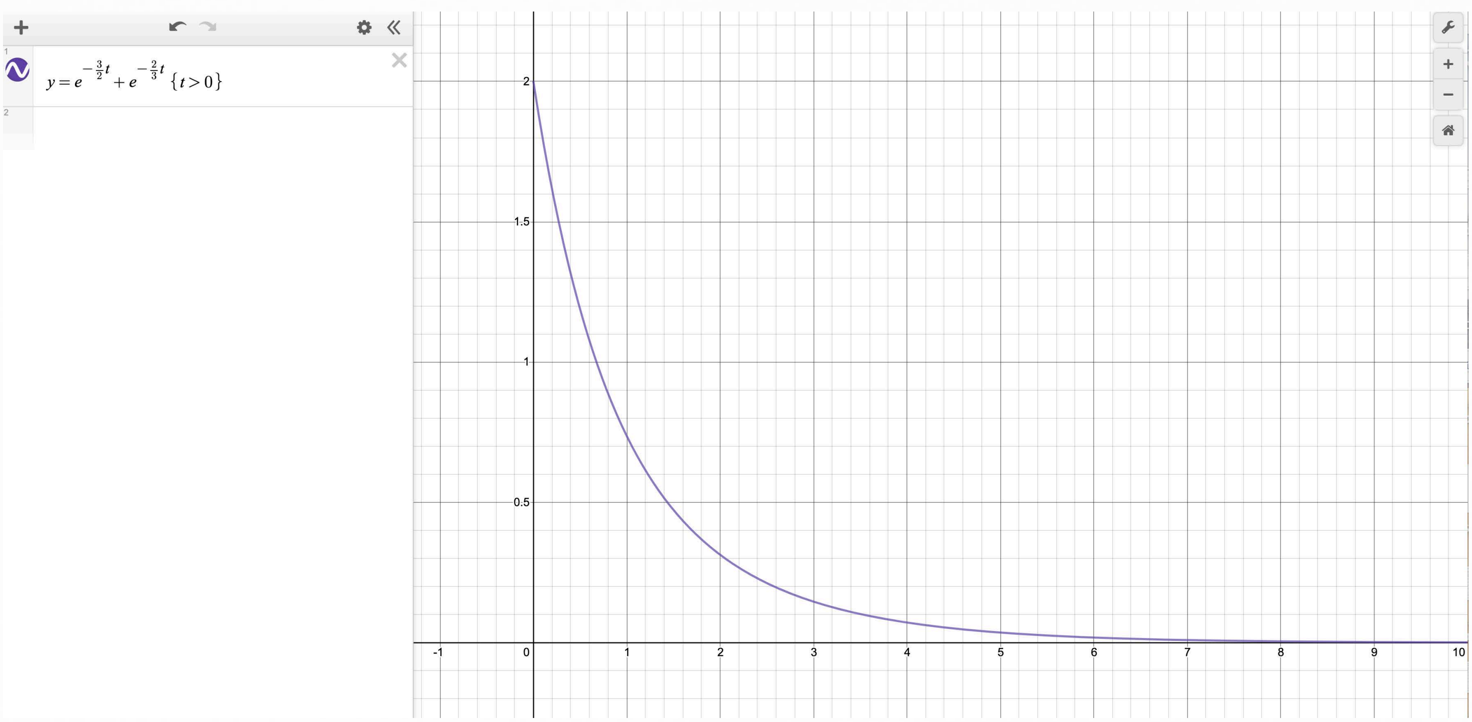


Ex. 3 : (Cont'd)

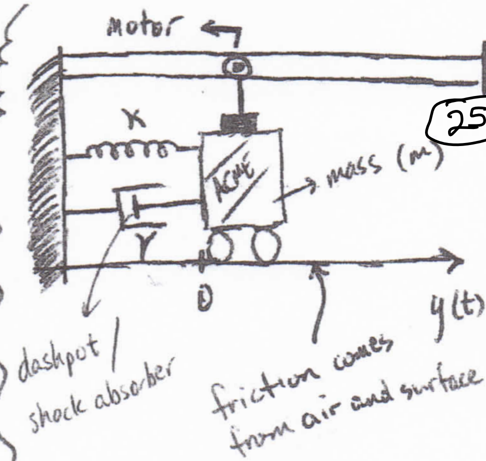
c) Ex. 2d : $y(t) = y_h(t) = e^{-t}\left[1 + t\right] = e^{-t} + te^{-t} \quad ; t \geq 0$



d) Ex. 2e : $y(t) = y_h(t) = e^{-\frac{3}{2}t} + e^{-\frac{2}{3}t} \quad ; t \geq 0$



Ex. 4: Consider the spring-mass system shown where $m = 12$ kg and $\kappa = 12$ kg/sec^2. Also, let $F_0 =$ 24 N $= 24$ kg$\cdot$m/sec^2 be a constant force applied to our spring-mass system via the motor moving the mass "m".



- m = mass = 12 kg
- κ = spring constant (kg/sec^2)
- γ = damping coefficient of dashpot (kg/sec)
- $m, \kappa > 0$
- $\gamma \geq 0$
- F_0 = force acting on the spring-mass system with magnitude $|F_0|$

a) If $\gamma = 0$ (i.e. no damping occurs due to the dashpot and surface the mass "m" moves on), find the general solution for $y(t)$ noting that (in this case) $y(t) = y_h(t) + y_p(t)$.

If $m = 12$ kg and $\kappa = 12$ kg/sec^2, then we know from Ex. 1c that the resulting ODE to model this situation is ... $12y'' + \gamma y' + 12y = F_z$. If $\gamma = 0$ and $F_z = 24$ N $= 24 \frac{\text{kg}\cdot\text{m}}{\text{sec}^2}$, then this equation becomes ... $12y'' + 12y = 24 \Rightarrow \boxed{y'' + y = 2}$ (✱). Our solution to this ODE is $y(t) = y_h(t) + y_p(t)$. Recall that $y_h(t) = c_1 \cos(t) + c_2 \sin(t)$ from Ex. 2a. We also know (from the Method of Undetermined Coefficients) that since $y_h(t)$ and $g(t) = F_z = 24$ do not have any like terms, we can let $y_p(t) = A$. Thus, $y_p' = y_p'' = 0$.

$\therefore 12y_p'' + 12y_p = 24 \Rightarrow 12(0) + 12(A) = 24 \Rightarrow 12A = 24 \Rightarrow \boxed{A = 2}$ (✱)

$\Rightarrow \boxed{y_p(t) = A = 2}$

$\therefore \underline{y(t) = y_h(t) + y_p(t) = c_1 \cos(t) + c_2 \sin(t) + 2}$

Note(✱): We could easily see that $y_p(t)$ was 2 in this case based on the characteristic equation!!

b) If $\gamma = 18$, find the general solution for $y(t)$ noting that $y(t) = y_h(t) + y_p(t)$.

From Ex. 1c with $m = 12$ kg, $k = 12\ \text{kg}/\text{sec}^2$, $\gamma = 18\ \text{kg}/\text{sec}$, and $F_z = 24\ \frac{\text{kg}\cdot\text{m}}{\text{sec}^2}$;

we see that $my'' + \gamma y' + 12y = F_z \Rightarrow 12y'' + 18y' + 12y = 24$. The

solution to this ODE will be $y(t) = y_h(t) + y_p(t)$, where $y_h(t)$ was found in

Ex. 2c to be ... $y_h(t) = e^{-\frac{3}{4}t}\left[c_1 \cos\left(\frac{\sqrt{7}}{4}t\right) + c_2 \sin\left(\frac{\sqrt{7}}{4}t\right)\right]$. Also, we

note that $y_h(t)$ and $g(t) = F_z = 24$ do not have any common terms, so

we let $y_p(t) = A \Rightarrow y_p' = y_p'' = 0$.

$\therefore\ 12y_p'' + 18y_p' + 12y_p = 24 \Rightarrow 12(0) + 18(0) + 12(A) = 24 \Rightarrow 12A = 24$

$\Rightarrow A = 2$

$\Rightarrow y_p(t) = 2$

Once again, we see that $y_p(t) = 2 \Rightarrow \boxed{y(t) = e^{-\frac{3}{4}t}\left[c_1 \cos\left(\frac{\sqrt{7}}{4}t\right) + c_2 \sin\left(\frac{\sqrt{7}}{4}t\right)\right] + 2}$

c) Based upon the solution $y(t) = y_h(t) + y_p(t)$ from Ex. 4a, what is the (new) equilibrium value for this spring-mass system? Show that this value for $y(t)$ can be found by knowing both F_0 and κ.

The new equilibrium value is now 2 (i.e. because of the constant force of $24 \frac{kg \cdot m}{sec^2}$ being applied to the mass in our spring-mass system, the new "at rest" position for the mass is at a position that stretches the spring in our system 2 m from its natural length).

$\boxed{\nabla}$: Recall that $1N = 1 \frac{kg \cdot m}{sec^2}$

$\boxed{\nabla}$

NOTE: $\text{Equilibrium position} = \frac{F_0}{\kappa} = \frac{F_z}{\kappa} = \frac{24 \frac{kg \cdot m}{sec^2}}{12 \ kg/sec^2} = 2 \ m$, where $F_0 = F_z = 24N$.

CONNECTION: The formula equilibrium position $= y_{eq} = \frac{F_0}{\kappa} = \frac{F_{other}}{\kappa}$ always. Note that when $F_0 = F_{other} = 0$ in our homogeneous examples in Exs. 1+2, $y_{eq} = 0$ since $y_{eq} = \frac{F_0}{\kappa} = \frac{0}{12} = 0$!

d) What is the behavior of $y(t)$ as $t \to \infty$ for Ex. 4a (i.e. $\lim_{t\to\infty} [y(t)]$)? What does this tell us about the dampening properties of our spring-mass system according to Ex. 4a (i.e. will our mass return to equilibrium and not move anymore at some point in time)?

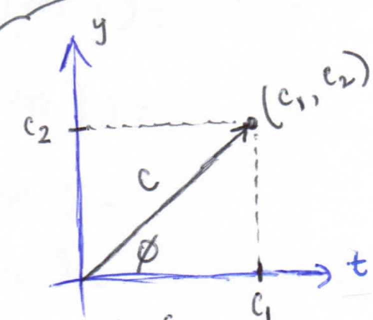
$$\lim_{t\to\infty} [y(t)] = \lim_{t\to\infty} [c_1 \cos(t) + c_2 \sin(t) + 2] = c_1 \cos(\infty) + c_2 \sin(\infty) + 2$$
$$= DNE + DNE + 2$$
$$= DNE$$

Since $\lim_{t\to\infty} [y(t)] = DNE$, this tells us that our spring-mass just oscillates back and forth within a finite range of (y-) values with no damping of motion!

see next page

Ex. 4d): Going deeper to find finite range of y-values $y(t)$ oscillates between

Recall from Trigonometry / Precalculus the Reduction Formula for the sum of a sine or cosine function can be rewritten as either a single sine function or single cosine function if we interpret this sum of sine and cosine functions in polar form (i.e. as a single magnitude "C" with phase angle ϕ) as shown in the figure below.



- $\tan(\phi) = \frac{c_2}{c_1}$
- $f = c_1\cos(t) + c_2\sin(t)$
- $C = \sqrt{c_1^2 + c_2^2}$
- $\cos(\phi) = \frac{c_1}{C} \Rightarrow c_1 = C\cos(\phi)$
- $\sin(\phi) = \frac{c_2}{C} \Rightarrow c_2 = C\sin(\phi)$

Since $c_1 = C\cos(\phi)$ and $c_2 = C\sin(\phi)$, we can say that...

$f = f(t) = c_1\cos(t) + c_2\sin(t)$

$\Rightarrow f(t) = C\cos(\phi)\cos(t) + C\sin(\phi)\sin(t)$

$\Rightarrow f(t) = C[\cos(\phi)\cos(t) + \sin(\phi)\sin(t)]$

NOTE: $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$

$\therefore f(t) = C[\cos(\phi - t)] = C\cos(t - \phi)$

Since $\cos(\theta)$ is an even function this implies that $\cos(-\theta) = \cos(\theta)$. Thus, $\cos(\phi - t) = \cos(-(t-\phi)) = \cos(t-\phi)$.

NOTE: $\cos(t - \frac{\pi}{2}) = \sin(t)$ and $\cos(t) = \sin(t + \frac{\pi}{2})$ are (Complementary) identities relating sine and cosine functions.

$\therefore f(t) = C\cos(t - \phi) = C\sin((t-\phi) + \frac{\pi}{2})$

$\Rightarrow f(t) = C\sin(t + (\frac{\pi}{2} - \phi)) = C\sin(t + \alpha)$,

where $\alpha = \frac{\pi}{2} - \phi$.

$\therefore \boxed{f(t) = C\cos(t - \phi) = C\sin(t + \alpha), \text{ where } \alpha = \frac{\pi}{2} - \phi}$

Ex. 4d) : Cont'd

So choosing to use the form $f(t) = C\cos(t-\phi)$, we recall that...

- $C = \sqrt{c_1^2 + c_2^2} \geq 0$
- $-1 \leq \cos(t-\phi) \leq 1$

$\Rightarrow -C \leq f(t) \leq C$ (i.e. finite range of y-values for $y(t)$ is $[-C, C]$).

- ϕ is an angle in standard position whose terminal side contains the point (c_1, c_2)
- $\phi = [0, 2\pi) \equiv [0, 360°)$

So $f(t) = C\cos(t-\phi)$ oscillates between $-C$ and C at a (starting) phase angle ϕ.

Back to our problem, since $y(t) = c_1\cos(t) + c_2\sin(t) + 2$, we can express $y(t)$ as... $y(t) = f(t) + 2 = C\cos(t-\phi) + 2$.

So, the range of values for $y(t)$ is $-C+2 \leq y(t) \leq C+2$. Thus, our y-values oscillate around $y(t) = 2$ as $t \to \infty$.

Ex. 4 : Cont'd _ 3

e) Based upon the solution $y(t) = y_h(t) + y_p(t)$ from Ex. 4b, what is the (new) equilibrium value for this spring-mass system? Is it the same as it is in Ex. 4c?

Since $y_p(t) = 2$ is the same as in Ex. 4c, it follows that the (new) equilibrium value for the spring-mass system in Ex. 4b is the same as it is in Ex. 4c.

f) What is the behavior of $y(t)$ as $t \to \infty$ (i.e. find $\lim_{t\to\infty}[y(t)]$)? What does this tell us about the dampening properties of our spring-mass system according to Ex. 4b (i.e. will our mass return to equilibrium and not move anymore at some point in time)?

NOTE: $c_1 \cos\left(\frac{\sqrt{7}}{4}t\right) + c_2 \sin\left(\frac{\sqrt{7}}{4}t\right) = \sqrt{c_1^2+c_2^2} \cdot \cos\left(\frac{\sqrt{7}}{4}t - \phi\right) = C \cdot \cos\left(\frac{\sqrt{7}}{4}t - \phi\right)$, where $C = \sqrt{c_1^2+c_2^2}$. Thus, $y(t)$ oscillates between $-C+2 \le y(t) \le C+2$. We can use the Squeeze Theorem from Calc A to evaluate $\lim_{t\to\infty}[y(t)]$.

$\therefore$ If $f(t) = C\cos\left(\frac{\sqrt{7}}{4}t - \phi\right)$, then $-C \le f(t) \le C$. But this implies that

$-e^{-\frac{3}{4}t}C \le e^{-\frac{3}{4}t} f(t) \le e^{-\frac{3}{4}t}C \Rightarrow -e^{-\frac{3}{4}t}C + 2 \le e^{-\frac{3}{4}t} f(t) + 2 \le e^{-\frac{3}{4}t}C + 2$.

But $y(t) = e^{-\frac{3}{4}t} f(t) + 2 \Rightarrow 2 - e^{-\frac{3}{4}t}C \le y(t) \le 2 + e^{-\frac{3}{4}t}C$.

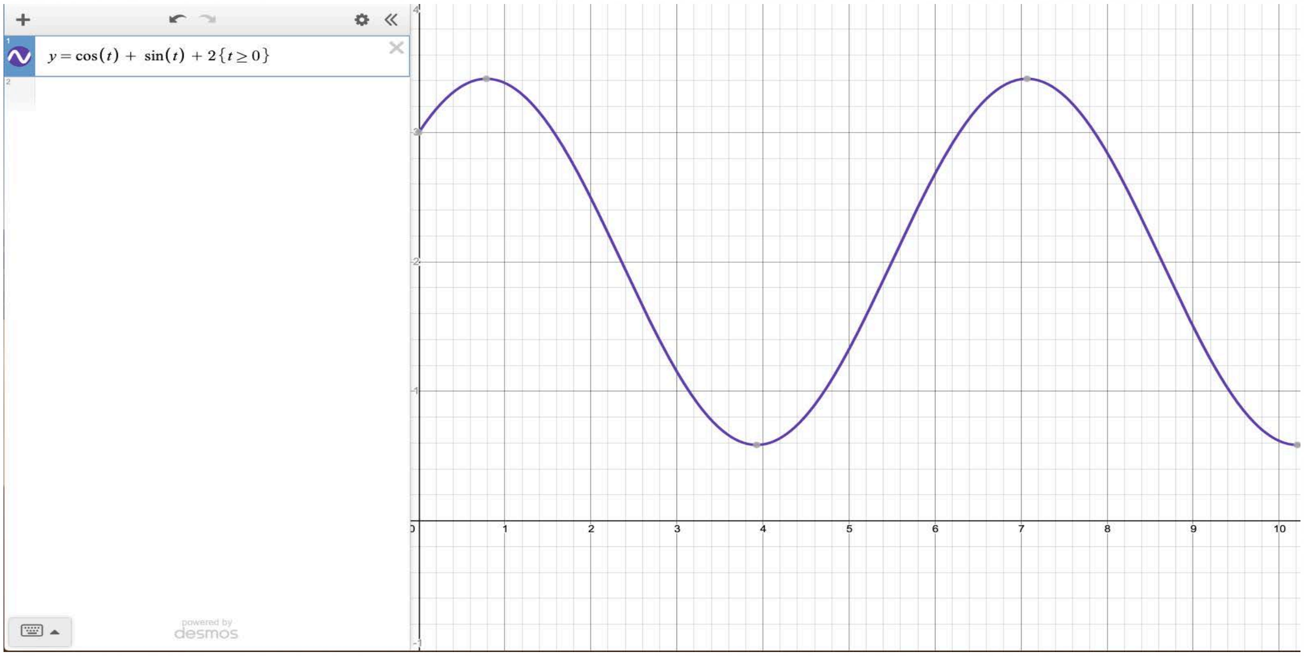
$\therefore \lim_{t\to\infty}\left[-e^{-\frac{3}{4}t}C + 2\right] \le \lim_{t\to\infty}[y(t)] \le \lim_{t\to\infty}\left[e^{-\frac{3}{4}t}C + 2\right]$

$\Rightarrow -\cancelto{0}{e^{-\frac{3}{4}(\infty)}}C + 2 \le \lim_{t\to\infty}[y(t)] \le \cancelto{0}{e^{-\frac{3}{4}(\infty)}}C + 2 \Rightarrow 2 \le \lim_{t\to\infty}[y(t)] \le 2$

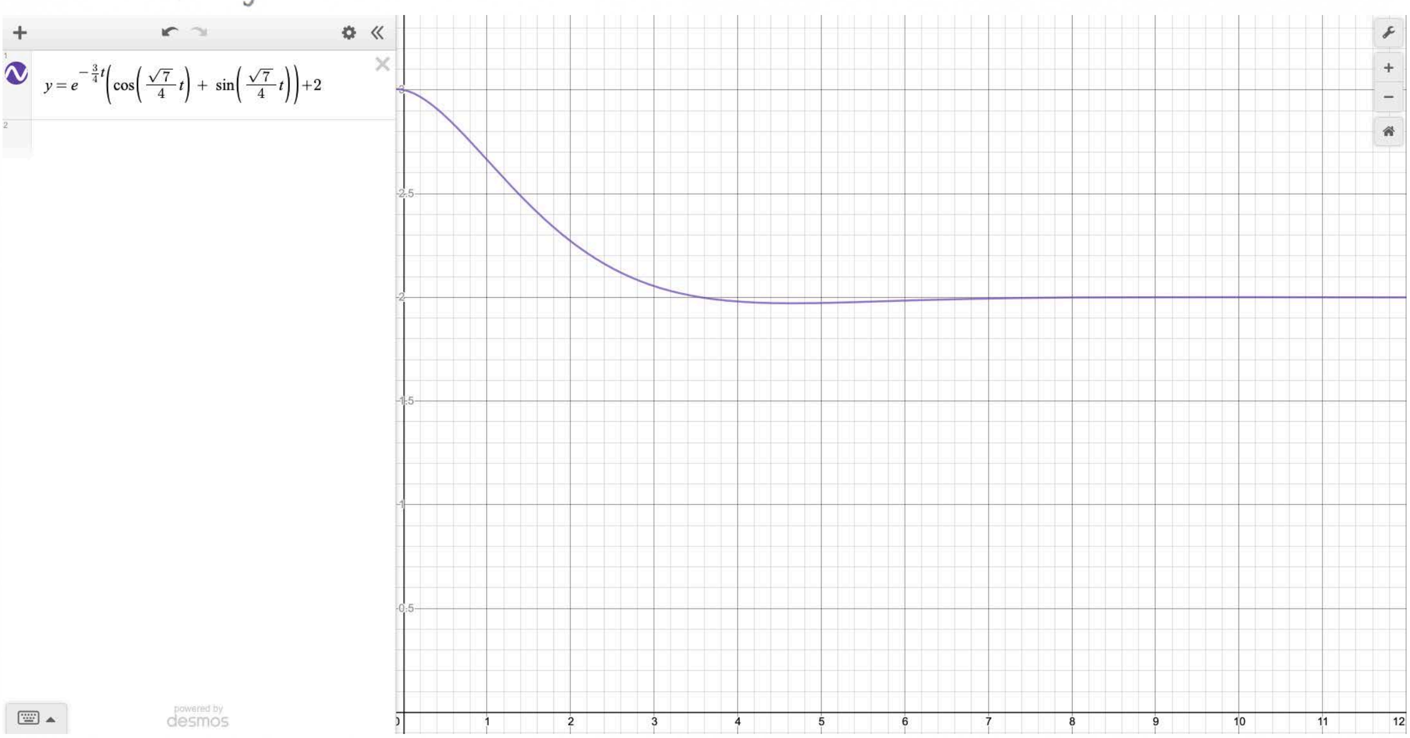
$\therefore$ By Squeeze Theorem, $\lim_{t\to\infty}[y(t)] = 2$!! So, our mass will return to $y_{eq} = 2m$ eventually.

Ex. 4 ; cont'd - 4

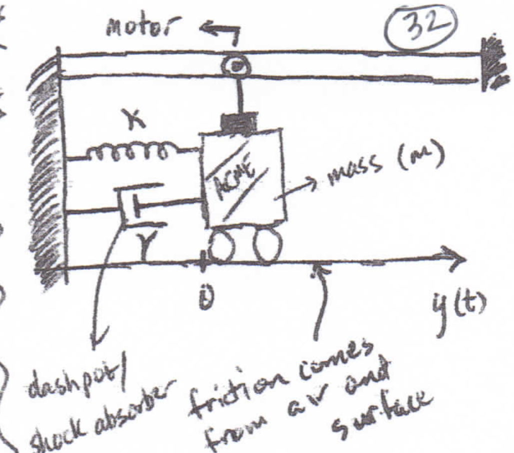
g) Graph the solution for $y(t)$ from Ex. 4a. Does this graph verify the answers you arrived at in Ex. 4e? (let $c_1 = c_2 = 1$). YES!



h) Graph the solution for $y(t)$ from Ex. 4b. Does this graph verify the answers you arrived at in Ex. 4f? (Let $c_1 = c_2 = 1$). YES!



Ex. 5: Consider the spring-mass system shown where $m = 12$ kg, $k = 12$ kg/sec^2, and $F_0 = 12\sin(t)$ is a continuous force function applied to our spring-mass system via the motor moving the mass "m".



- m = mass = 12 kg
- k = spring constant ($\frac{kg}{sec^2}$)
- γ = damping coefficient of dashpot ($\frac{kg}{sec}$)
- $m, k > 0$
- $\gamma \geq 0$
- F_0 = force acting on the spring-mass system with magnitude $|F_0|$

a) If $\gamma = 0$ (i.e. no damping occurs due to the dashpot and surface the mass "m" moves on), find the general solution for $y(t)$. Our ODE in this case would be $\boxed{12y'' + 12y = 12\sin(t)}$ and our solution to this ODE is $\boxed{y(t) = y_h(t) + y_p(t)}$, where the homogeneous solution $\boxed{y_h(t) = c_1\cos(t) + c_2\sin(t)}$.

NOTE: Since $y_h(t)$ and $F_0 = g(t)$ have like terms with $\sin(t)$ in it, we will have to adjust our initial guess of $y_p(t)$ (using Method of Undetermined Coefficients) from $y_p(t) = A\cos(t) + B\sin(t)$ to $\boxed{y_p(t) = At\cos(t) + Bt\sin(t)}$.

$$\therefore y_p' = A[\cos(t) - t\sin(t)] + B[\sin(t) + t\cos(t)] = A\cos(t) - At\sin(t) + B\sin(t) + Bt\cos(t)$$

$$\therefore y_p'' = -A\sin(t) - A[\sin(t) + t\cos(t)] + B\cos(t) + B[\cos(t) - t\sin(t)]$$

$$y_p'' = -2A\sin(t) - At\cos(t) + 2B\cos(t) - Bt\sin(t)$$

$$\therefore 12y_p'' + 12y_p = 12\sin(t) \Rightarrow y_p'' + y_p = \sin(t)$$

$$\therefore [-2A\sin(t) - \cancel{At\cos(t)} + 2B\cos(t) - \cancel{Bt\sin(t)}] + [\cancel{At\cos(t)} + \cancel{Bt\sin(t)}] = \sin(t)$$

$$\Rightarrow -2A\sin(t) + 2B\cos(t) = \sin(t) + 0\cos(t)$$

$$\Rightarrow -2A = 1 \text{ and } 2B = 0 \Rightarrow \boxed{A = -\tfrac{1}{2} \text{ and } B = 0}.$$

Ex. 5! cont'd a) Cont'd : $\therefore \boxed{y(t) = c_1\cos(t) + c_2\sin(t) - \frac{1}{2}t\cos(t)}$ is the solution to the ODE $12y'' + 12y = 12\sin(t)$!!

b) Is the spring-mass system under these conditions stable? Why or why not?

No, this system is not stable! This system will oscillate the mass out of control (unless the spring or dashpot breaks first) because $y_p(t) = -\frac{1}{2}t\cos(t)$ will approach $-\infty$ as $t \to \infty$! (Recall that $c_1\cos(t) + c_2\sin(t) = y_h(t) = C\cos(t-\phi)$, where $C = \sqrt{c_1^2 + c_2^2}$ and $\phi = [0, 2\pi)$, where ϕ depends on the values of c_1 and c_2 and what quadrant the ordered pair (c_1, c_2) lies in. So $y_h(t)$ will oscillate between $y = [-C, C]$, and, thus, is the stable part of this system. It is only the $y_p(t) = -\frac{1}{2}t\cos(t)$ function that grows out of control (i.e. $\lim_{t\to\infty} |y_p(t)| = \infty$) and makes the system unstable.

c) Find the (natural) resonance frequency of the spring-mass system in Ex. 5a.

NOTE 1: w_0 = resonant freq of spring-mass system $= \sqrt{\frac{k}{m}} = \sqrt{\frac{12\ \text{kg/sec}^2}{12\ \text{kg}}} = 1$ rad/sec.

NOTE 2: For $y_h(t) = c_1\cos(t) + c_2\sin(t) = e^{\lambda t}[c_1\cos(wt) + c_2\sin(wt)]$, where $\lambda = 0$ and $w = 1$, we see that $w = 1 = w_0 = \sqrt{\frac{k}{m}}$. So, the imaginary coefficient of the roots to the characteristic equation for this ODE, $r = \lambda \pm wi$, equals the spring-mass system resonant frequency!

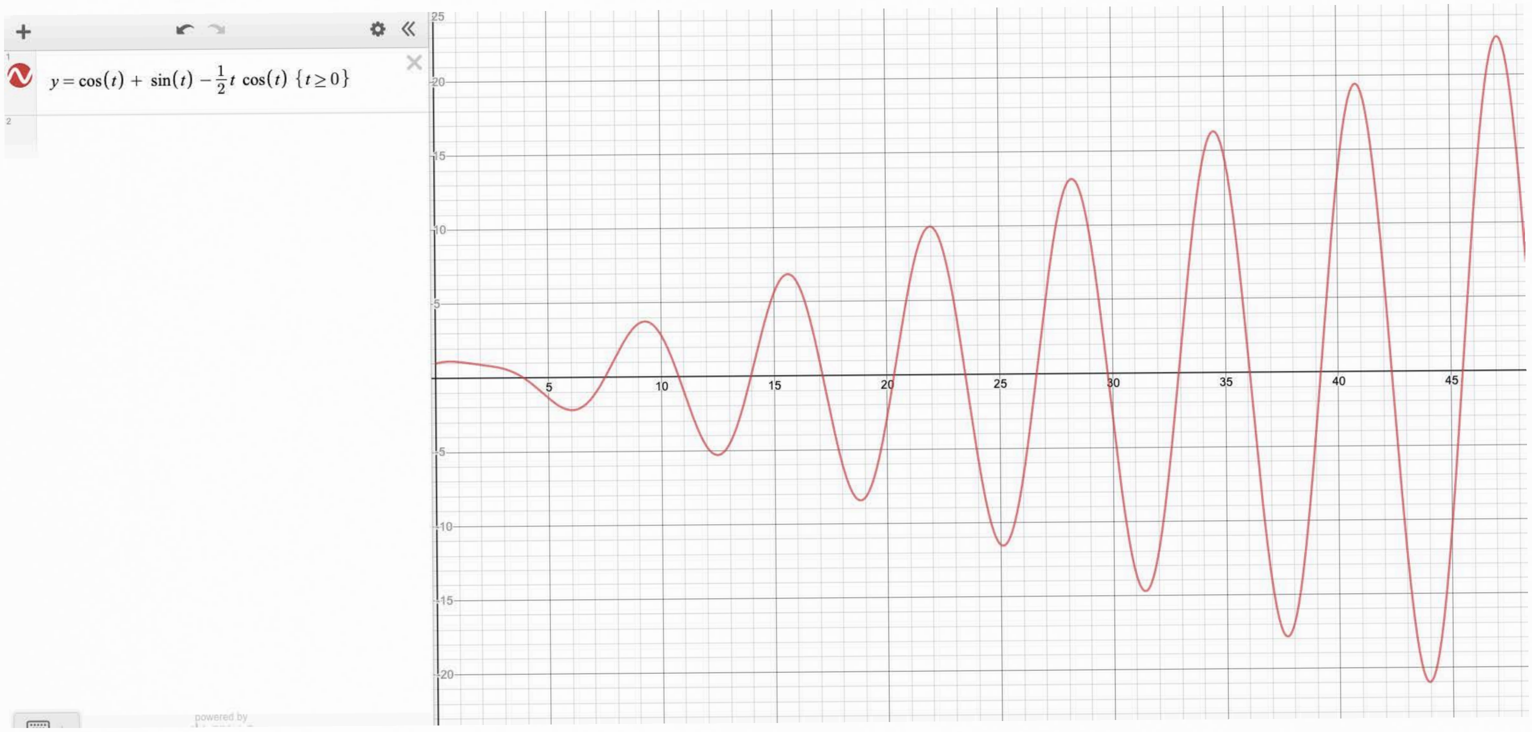
d) What is the frequency of the force function $F_0 = 12\sin(t)$?

$F_0 = 12\sin(t) = 12\sin(1\cdot t) \Rightarrow w_{F_0} = 1$ rad/sec

NOTE: $w_{F_0} = 2\pi f_{F_0} \Rightarrow f_{F_0} = \frac{w_{F_0}}{2\pi} = \frac{1}{2\pi}$ cycle/sec $= \frac{1}{2\pi}$ Hz.

Ex. 5 : cont'd – 3

e) Graph the function you arrived at for $y(t)$ in Ex. 5a. Does this graph verify your answer above in Ex. 5b? (Let $c_1 = c_2 = 1$).



f) What is the relationship between the frequency of the homogeneous solution $y_h(t)$ to this spring-mass system and the force function frequency of F_0? Are they the same or different? How do these frequencies work together to play a role in the stability of this spring-mass system?

The (resonant) frequencies of $y_h(t)$ and $F_0 = 12\sin(t)$ are the same! Since these frequencies are the same, our spring-mass system will experience resonance because $y_p(t) = -\frac{1}{2}t\cos(t)$ causes the max displacement of $y(t)$ to be $|-\frac{1}{2}t| = \frac{1}{2}t$ which approaches ∞ as $t \to \infty$!! If y_p's frequency did not match y_h's frequency, y_p's max displacement would be a finite value!

g) Now consider our spring-mass system if $m = 12$ kg, $\kappa = 12\ \text{kg}/\text{sec}^2$, $F_0 = 12\sin(t)$, and $\gamma = 18$. Find the general solution for $y(t)$.

ODE: $my'' + \gamma y' + \kappa y = 12\sin(t) \Rightarrow 12y'' + 18y' + 12y = 12\sin(t)$.

Find $y_h(t)$: From Ex. 1c, we found that $\boxed{y_h(t) = e^{-\frac{3}{4}t}\left[c_1\cos\left(\frac{\sqrt{7}}{4}t\right) + c_2\sin\left(\frac{\sqrt{7}}{4}t\right)\right]}$.

Find $y_p(t)$: Since $y_h(t)$ and $F_0 = 12\sin(t)$ do not have any like terms, we can let $\boxed{y_p(t) = A\sin(t) + B\cos(t)}$ via Method of Undetermined Coefficients.

$\therefore \boxed{y_p' = A\cos(t) - B\sin(t)}$ and $\boxed{y_p'' = -A\sin(t) - B\cos(t)}$.

$\therefore 12y_p'' + 18y_p' + 12y_p = 12\sin(t) \Rightarrow 2y_p'' + 3y_p' + 2y_p = 2\sin(t)$

$\therefore 2\left[-A\sin(t) - B\cos(t)\right] + 3\left[A\cos(t) - B\sin(t)\right] + 2\left[A\sin(t) + B\cos(t)\right] = 2\sin(t)$

$\Rightarrow \cancel{-2A\sin(t)} - \cancel{2B\cos(t)} + 3A\cos(t) - 3B\sin(t) + \cancel{2A\sin(t)} + \cancel{2B\cos(t)} = 2\sin(t)$

$\Rightarrow 3A\cos(t) - 3B\sin(t) = 2\sin(t) + 0\cdot\cos(t)$

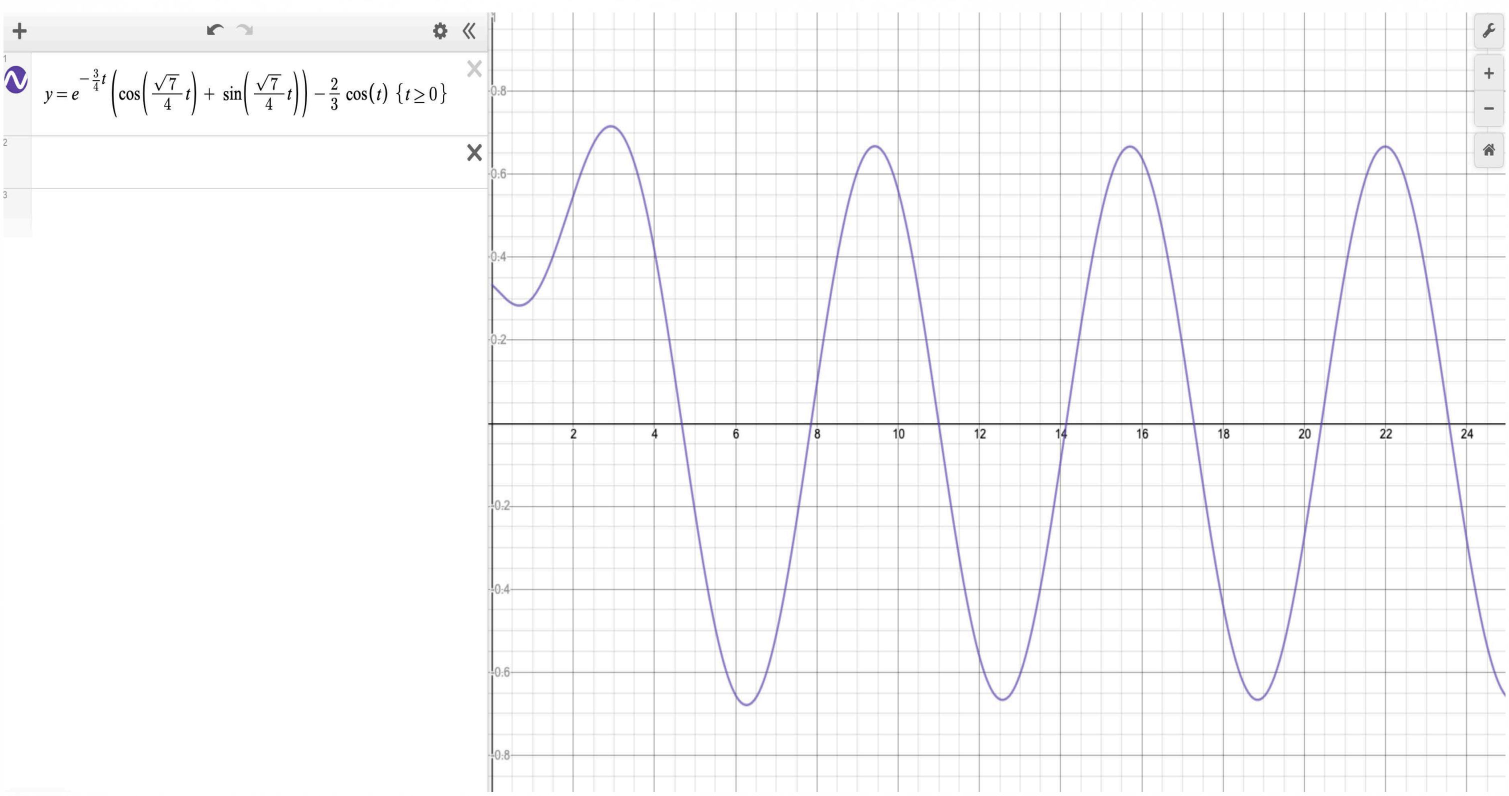
$\cos(t)$	$\sin(t)$
$3A = 0$	$-3B = 2$
$\Rightarrow A = 0$	$\Rightarrow B = -\frac{2}{3}$

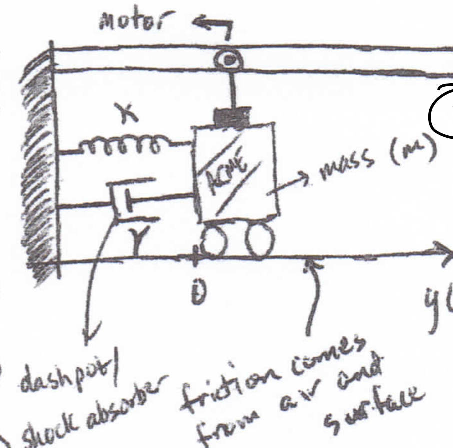
$\therefore \boxed{y_p(t) = -\frac{2}{3}\cos(t)}$

$\therefore$ General sol'n for $\boxed{y(t) = y_h(t) + y_p(t) = e^{-\frac{3}{4}t}\left[c_1\cos\left(\frac{\sqrt{7}}{4}t\right) + c_2\sin\left(\frac{\sqrt{7}}{4}t\right)\right] - \frac{2}{3}\cos(t)}$.

h) Is the spring-mass system under the conditions of Ex. 5h stable? Why or why not? Yes, the system is stable. Similar to this strategy we used to find the behavior of $y(t)$ in Ex. 4f, we see that because of the Squeeze Theorem (from Calc A) that $\lim_{t\to\infty}[y_h(t)] = 0$ since $e^{-\frac{3}{4}t} \to 0$ as $t \to \infty$. Thus, as $t \to \infty$, $y(t) \to -\frac{2}{3}\cos(t) = y_p(t)$ (i.e. our steady-state solution for this system). Since $-\frac{2}{3} \le y_p(t) \le \frac{2}{3}$ for all values of $t \ge 0$ and $-C \le y_h(t) \le C$, where $C = \sqrt{c_1^2 + c_2^2}$, this means that $-C - \frac{2}{3} \le y(t) \le C + \frac{2}{3}$. So $y(t)$ does not oscillate out of control!

i) Graph the function you arrived at for $y(t)$ in Ex. 5g. Does this graph verify your answers above in Ex. 5h? (Let $c_1 = c_2 = 1$).





Ex. 6: Consider the spring-mass system shown where $m = 12$ kg, $\kappa = 12\ \text{kg}/\text{sec}^2$, and $F_0 = 12\sin(\frac{t}{2})$ is a continuous force function applied to our spring-mass system via the motor moving the mass "m".

- m = mass = 12 kg
- κ = spring constant (kg/sec^2)
- γ = damping coefficient ($\frac{\text{kg}}{\text{sec}}$) of dashpot
- $m, \kappa > 0$
- $\gamma \geq 0$
- F_0 = force acting on the spring-mass system with magnitude $|F_0|$

a) If $\gamma = 0$ (i.e. no dampening occurs due to the dashpot and surface the mass "m" moves on), find the general solution for $y(t)$.

ODE: $my'' + \gamma y' + \kappa y = F_0 \Rightarrow 12y'' + 12y = 12\sin(\frac{t}{2})$

Find $y_h(t)$: From Ex. 1a: $\boxed{y_h(t) = c_1\cos(t) + c_2\sin(t)}$.

Find $y_p(t)$: Since $y_h(t)$ and $F_0 = g(t) = 12\sin(\frac{t}{2})$ do not have any like terms, we can let $\boxed{y_p(t) = A\sin(\frac{t}{2}) + B\cos(\frac{t}{2})}$ via Method of Undetermined Coefficients.

$\therefore \boxed{y_p' = \frac{A}{2}\cos(\frac{t}{2}) - \frac{B}{2}\sin(\frac{t}{2})}$ and $\boxed{y_p'' = -\frac{A}{4}\sin(\frac{t}{2}) - \frac{B}{4}\cos(\frac{t}{2})}$.

$\therefore 12y_p'' + 12y_p = 12\sin(\frac{t}{2}) \Rightarrow y_p'' + y_p = \sin(\frac{t}{2})$

$\therefore \left[-\frac{A}{4}\sin(\frac{t}{2}) - \frac{B}{4}\cos(\frac{t}{2})\right] + \left[A\sin(\frac{t}{2}) + B\cos(\frac{t}{2})\right] = \sin(\frac{t}{2})$

$\Rightarrow -A\sin(\frac{t}{2}) - B\cos(\frac{t}{2}) + 4A\sin(\frac{t}{2}) + 4B\cos(\frac{t}{2}) = 4\sin(\frac{t}{2})$

$\Rightarrow 3A\sin(\frac{t}{2}) + 3B\cos(\frac{t}{2}) = 4\sin(\frac{t}{2}) + 0\cdot\cos(\frac{t}{2})$

$\therefore$

$\sin(\frac{t}{2})$	$\cos(\frac{t}{2})$
$3A = 4$	$3B = 0$
$\Rightarrow A = \frac{4}{3}$	$\Rightarrow B = 0$

$\Rightarrow y_p(t) = \frac{4}{3}\sin(\frac{t}{2})$

$\Rightarrow \boxed{y(t) = y_h(t) + y_p(t) = c_1\cos(t) + c_2\sin(t) + \frac{4}{3}\sin(\frac{t}{2})}$

b) Find the (natural) resonance frequency of the spring-mass system in Ex. 6a. $w_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{12}{12}} = 1$ rad/sec

NOTE: $w_0 = 2\pi f_0 \Rightarrow f_0 = \frac{w_0}{2\pi} = \frac{1}{2\pi}$ cycle/sec $= \frac{1}{2\pi}$ Hz.

c) What is the frequency of the force function $F_0 = 12\sin(t/2)$? Is this frequency the same as the frequency you found in Ex. 6b?

$F_0 = 12\sin(\frac{1}{2}t) = 12\sin(w_z t) \Rightarrow w_z = \frac{1}{2}$ rad/sec

NOTE: $w_z = 2\pi f_z \Rightarrow f_z = \frac{w_z}{2\pi} = \frac{1/2}{2\pi}$ cycle/sec $= \frac{1}{4\pi}$ Hz.

d) What is the frequency of the homogeneous solution y_h to our spring-mass system in Ex. 5a? Is this frequency the same as the frequency you found in Ex. 6b? The frequency of y_h is the same as the (natural) resonance frequency of this spring-mass system.

$\therefore w_0 =$ freq. of $y_h = 1$ rad/sec

e) What frequency would F_0 have to have in order to achieve resonance for this spring-mass system? If we achieve resonance, will this system be stable? What do you think would happen if the system was unstable? The frequency of F_0, w_z, would have to equal $w_0 = 1$ rad/sec in order for the spring-mass system to achieve resonance. If this system achieved resonance, it would be unstable! (Recall Ex. 5f).

If this system was unstable, at some point either the dashpot or spring would break!!

f) Graph the function you arrived at for $y(t)$ in Ex. 5g. Does this graph reflect that the spring-mass system is stable or unstable?

STABLE