CPE 381 Equation Sheet

Chapter 0:

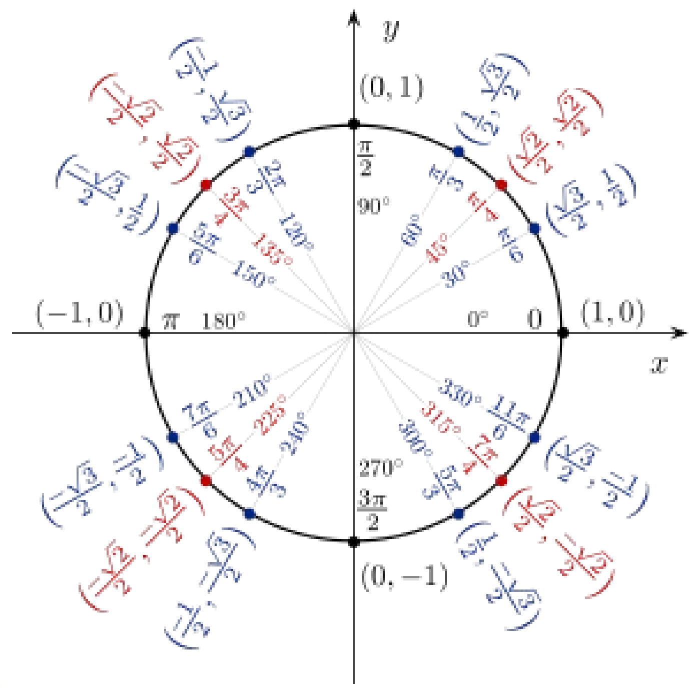
n	2^n	Significance
0	1	
1	2	
2	4	
4	16	
8	256	
10	1024	
16	65536	

Range of numbers:

0 to $2^n - 1$ for unsigned

-2^{n-1} to $2^{n-1}-1$

Unit Circle:



Arithmetic Series:

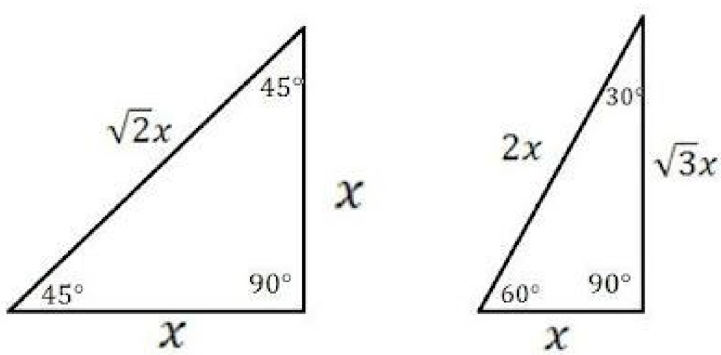
$a_n = a_1 + (n-1)d$

$$\sum_{i=1}^{n} a_i = n \cdot \frac{a_1 + a_n}{2}$$

Geometric Series:

$$\sum_{k=1}^{n} a_k r^k = a \cdot \frac{1-r^n}{1-r}$$

Triangles to Know:



Integrals to Know:

$$\int x^n \, dx = \frac{1}{n+1} \cdot x^{n+1}$$

$$\int \frac{1}{x} dx = \ln|x|$$

$$\int u\, dv = uv - \int v\, du$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}$$

$$\int \sin(ax)\, dx = \frac{-1}{a} \cos(ax)$$

$$\int \cos(ax)\, dx = \frac{1}{a} \sin(ax)$$

$$\int \sin^2(ax)\, dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \cos^2(ax)\, dx = \frac{x}{2} + \frac{\sin(2ax)}{4a}$$

<u>Euler's Identity:</u>

$$e^{j\theta} = cos\theta + jsin\theta$$

<u>Other forms:</u>

$$e^{-j\theta} = \cos(-\theta) + jsin(-\theta)$$

$$e^{-j\theta} = cos\theta - jsin\theta$$

$$e^{j\theta} + e^{-j\theta} = 2\,cos\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

<u>Sampling continuous time signal x(t) into discrete time signal sequence x[n]:</u>

$$x[n] = x(nT_s) = x(t) \vee t = nT_s$$

<u>Derivative and Forward Difference:</u>

$$D[x(t)] = \frac{dx(t)}{dt} = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$

$$\Delta[x(nT_s)] = x((n+1)T_s) - x(nT_s)$$

<u>Integral and Summation:</u>

$$I(t)=\int_{t_0}^{t} x(\tau)\,d\tau \; x(t)=\frac{dI(t)}{dt}$$

$$I(t)\approx\sum_{n} x(nT_s)\,p(n)$$ p(n) pulses of width T_s

<u>DE for RC circuit with constant voltage:</u>

$$v_i(t)=v_c(t)+\frac{d\,v_c(t)}{dt}\; t\geq 0$$

<u>Approximate Integral for a Trapezoid & DE:</u>

$$v_c(t)=\int_{0}^{t}\left[v_i(\tau)-v_c(\tau)\right]d\tau+v_c(0)$$

$$v_c(nT)=\frac{T}{2+T}\left[v_i(nT)+v_i((n-1)T)\right]+\frac{2-T}{2+T}v_c((n-1)T),\, v_c(0)=0,\, n\geq 1$$

<u>Converting between Polar and Rectangular:</u>

$z=x+jy=|z|e^{j\angle(z)}$

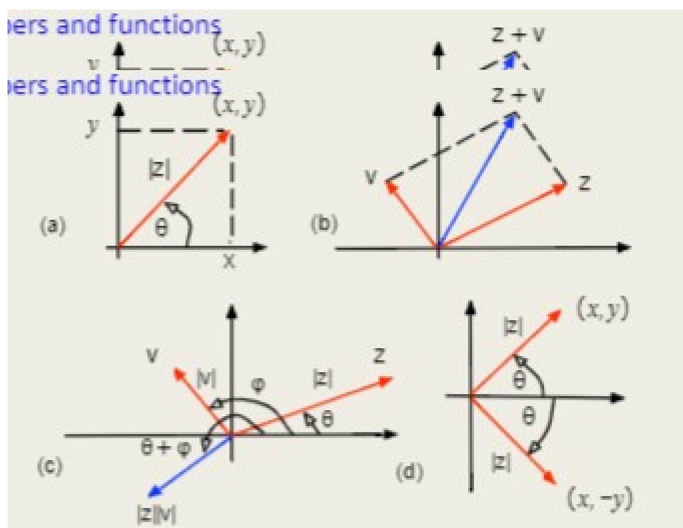
$v=u+jw=|v|e^{j\angle(v)}$

$z+v=(x+u)+j(y+w)$

$zv=|z||v|e^{j(\angle(z)+\angle(v))}$

$z^{¿}=x-jy=|z|e^{-j\angle z}$

- Be able to convert between the two in any quadrant



<u>Euler's Identity:</u>

$e^{j\theta}=\cos(\theta)+jsin(\theta)$

$\cos(\theta)=\Re\left[e^{j\theta}\right]=\dfrac{e^{j\theta}+e^{-j\theta}}{2}$

$\sin(\theta)=lm\left[e^{j\theta}\right]=\dfrac{e^{j\theta}-e^{-j\theta}}{2j}$

<u>Trig. Identities:</u>

$$\sin(-\theta)=\frac{e^{-j\theta}-e^{j\theta}}{2j}=-\sin(\theta)$$

$$\cos(\pi+\theta)=e^{j\pi}\frac{e^{j\theta}+e^{-j\theta}}{2}=-\cos(\theta)$$

$$\cos^2(\theta)=\left[\frac{e^{j\theta}+e^{-j\theta}}{2}\right]^2=\frac{1}{2}+\frac{1}{2}\cos(2\theta)$$

$$\sin(\theta)\cos(\theta)=\frac{e^{j\theta}-e^{-j\theta}}{2j}\cdot\frac{e^{j\theta}+e^{-j\theta}}{2}=\frac{1}{2}\sin(2\theta)$$

Sinusoids and phasors

$$x(t) = A\cos(\Omega_0 t + \psi) \qquad -\infty < t < \infty$$

A amplitude, $\Omega_0 = 2\pi f_0$ frequency (rad/sec), ψ phase (rad)

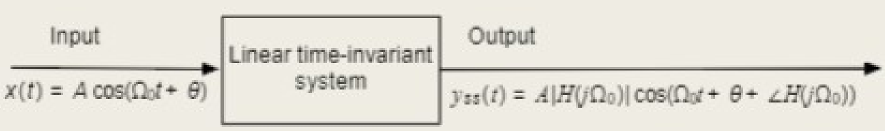
$$\boxed{\text{Phasor}: X = Ae^{j\psi},\ x(t) = \mathrm{Re}[Xe^{j\Omega_0 t}]}$$

Eigenfunction property of LTI systems

Input: $x(t) = \mathrm{Re}[Xe^{j\Omega_0 t}]$, input phasor $X = Ae^{j\theta}$

Output: $y(t) = \mathrm{Re}[Ye^{j\Omega_0 t}]$, output phasor $Y = XH(j\Omega_0)$

Steady-state response



Frequency response of system

$$H(j\Omega_0) = |H(j\Omega_0)|e^{\angle H(j\Omega_0)}$$

Chapter 1:

$$\boxed{\begin{array}{l} x(.) : \mathcal{R} \to \mathcal{R} \;\; (\mathcal{C}) \\ \quad\quad t \to \;\; x(t) \end{array}}$$

Example: complex signal $y(t) = (1+j)e^{j\pi t/2}, \quad 0 \leq t \leq 10, \quad 0$ otherwise

$$y(t) = \begin{cases} \sqrt{2}\left[\cos(\pi t/2 + \pi/4) + j\sin(\pi t/2 + \pi/4)\right], & 0 \leq t \leq 10, \\ 0, & \text{otherwise} \end{cases}$$

If $\quad x(t) = \sqrt{2}\cos(\pi t/2 + \pi/4), -\infty < t < \infty$

$p(t) = 1, \quad 0 \leq t \leq 10, \quad 0$ otherwise

then

$y(t) = [x(t) + jx(t-1)]p(t)$

Given signals x (t), y (t), constants α and τ , and function w(t):

- Signal addition/subtraction: x(t) + y (t), x (t) − y (t)
- Constant multiplication: αx (t)
- Time shifting
  - x (t − τ) is x (t) delayed by τ
  - x (t + τ) is x (t) advanced by τ
- Time scaling x(αt)
  - $\alpha = -1$, x (−t) reversed in time or reflected
  - $\alpha > 1$, x (αt) is x (t) compressed
  - $\alpha < 1$, x (αt) is x (t) expanded
- Time windowing x(t)w(t), w(t) window
- Integration

$$y(t) = \int_{t_0}^{t} x(\tau)d\tau + y(t_0)$$

Example

$$x(t) = \begin{cases} t & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

delayed by 1: $x(t-1) = \begin{cases} t-1 & 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$

advanced by 1: $x(t+1) = \begin{cases} t+1 & -2 \le t \le 0 \\ 0 & \text{otherwise} \end{cases}$

reflected: $x(-t) = \begin{cases} -t & -1 < t < 1 \\ 0 & \text{otherwise} \end{cases}$

reflected and delayed by 1: $x(-t+1) = \begin{cases} -t+1 & 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$

reflected and advanced by 1: $x(-t-1) = \begin{cases} -t-1 & -2 \le t \le 0 \\ 0 & \text{otherwise} \end{cases}$

compressed by 2: $x(2t) = \begin{cases} 2t & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$

expanded by 2: $x(t/2) = \begin{cases} t/2 & -2 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$

x(t) even: x (t) = x (−t)
odd : x (t) = −x (−t)

$x(t)$ is periodic if
(i) $x(t)$ defined in $-\infty < t < \infty$. and
(ii) there is $T_0 > 0$, the fundamental period of $x(t)$,
such that $x(t + kT_0) = x(t)$, integer k

$$\text{Energy of } x(t): \quad E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt,$$

$$\text{Power of } x(t): \quad P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

- *x (t) is finite-energy, or square integrable, if Ex < ∞*
- *x (t) is finite-power if Px < ∞*

x (t) period of fundamental period T_0 is

$$\boxed{P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^2(t)dt}$$

- Complex exponential

$$\boxed{\begin{aligned} x(t) &= Ae^{at} = |A|e^{j\theta}e^{(r+j\Omega_0)t} \\ &= |A|e^{rt}[\cos(\Omega_0 t + \theta) + j\sin(\Omega_0 t + \theta)] \quad -\infty < t < \infty \end{aligned}}$$

Sinusoid

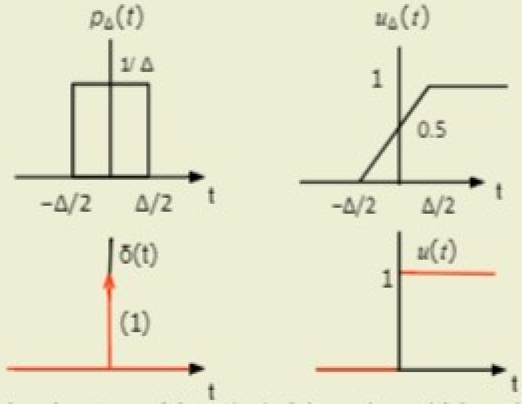
$$\boxed{A\cos(\Omega 0t + \theta) = A\sin(\Omega 0t + \theta + \pi/2) \qquad -\infty < t < \infty}$$

Modulation systems

$$A(t)\cos(\Omega(t)t + \theta(t))$$

- Amplitude modulation or AM: A(t) changes according to the message, frequency and phase constant,
- Frequency modulation or FM: Ω(t) changes according to the message, amplitude and phase constant,
- Phase modulation or PM: θ(t) changes according to the message, amplitude and frequency constant

Unit-impulse signal



Unit-impulse δ(t) and unit–step u(t) as Δ → 0 in pulse pΔ(t) and its integral uΔ(t).

Unit-impulse

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0 \end{cases}$$

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 1 & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

Unit–step signal

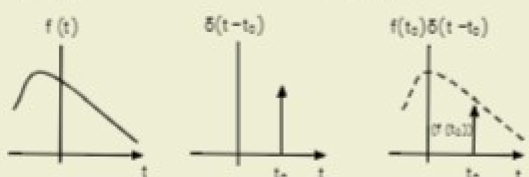
$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Ramp signal

$$r(t) = tu(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Sifting property of $\delta(t)$

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)dt = f(\tau)\int_{-\infty}^{\infty} \delta(t-\tau)dt = f(\tau), \quad \text{for any } \tau$$



Generic representation

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

$x(t) \Rightarrow y(t) = S[x(t)]$

Input Output

Continuous time system

A system S is linear if for inputs $x(t)$ and $v(t)$, and constants α and β, superposition holds, i.e.,

$$\begin{aligned} S[\alpha x(t) + \beta v(t)] &= S[\alpha x(t)] + S[\beta v(t)] \\ &= \alpha S[x(t)] + \beta S[v(t)] \end{aligned}$$

· System represented by linear, constant coefficient differential equation: System S, with input x (t) and output y (t), represented by

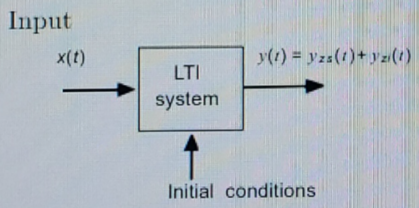
$$a_0 y(t) + a_1 \frac{dy(t)}{dt} + \cdots + a_N \frac{d^N y(t)}{dt^N} = b_0 x(t) + b_1 \frac{dx(t)}{dt} + \cdots + b_M \frac{d^M x(t)}{dt^M} \qquad t \geq 0$$

is linear time-invariant (LTI) if

- IC are zero
- input x (t) is causal (i.e., zero for $t < 0$)

i.e., the system is not initially energized

If IC$\neq 0$, $x(t)$ causal consider superposition



LTI system with x (t) and IC as inputs

Impulse response of LTI system, $h(t)$, is output of the system corresponding to an impulse $\delta(t)$, and initial conditions of zero

Convolution integral

$$\begin{aligned} \delta(t) &\rightarrow h(t) \quad \text{(definition)} \\ \delta(t-\tau) &\rightarrow h(t-\tau) \quad \text{(TI)} \\ x(\tau)h(t-\tau) &\rightarrow x(\tau)h(t-\tau) \quad \text{(L)} \\ x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau &\rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad \text{(L)} \end{aligned}$$

$$\boxed{\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \\ &= [x * h](t) = [h * x](t) \end{aligned}}$$

Impulse response $h(t)$, unit-step response $s(t)$, and ramp response $\rho(t)$ are related by

$$h(t) = \begin{cases} ds(t)/dt \\ d^2\rho(t)/dt^2 \end{cases}$$

- LTI system S represented by its impulse response $h(t)$ is causal if

$h(t) = 0 \qquad \text{for } t < 0$

output of causal LTI system for causal input $x(t) = 0,\ t < 0$

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau$$

Graphical computation of convolution

S is LTI and causal, $h(t) = 0, t < 0$, input is causal, $x(t) = 0, \quad t < 0$, output

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t h(\tau)x(t-\tau)d\tau$$

Graphical procedure

- Choose time t_0 to compute $y(t_0)$,
- Plot as functions of τ , $x(\tau)$ and the reflected and delayed $h(t_0 - \tau)$,
- Obtain $x(\tau)h(t_0-\tau)$ and integrate it from 0 to t_0 to obtain $y(t_0)$.
- Increase t_0, move from $-\infty$ to ∞

Equal results obtained if $x(t-\tau)$ and $h(\tau)$ used

BIBO stability

- Bounded-input-bounded-output (BIBO) stability: for a bounded $x(t)$ the output $y(t)$ *is also bounded*
- LTI S is BIBO stable if

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty, \quad \text{(absolutely integrable)}$$

Indeed

$$|y(t)| = \left|\int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau\right| \le M\int_{-\infty}^{\infty} |h(\tau)|d\tau \le MK < \infty$$

Example: RL circuit (R=L=1)

$$v_s(t) = i(t) + \frac{di(t)}{dt}$$

$$v_s(t) = \delta(t), i(0) = 0, \quad i(t) = h(t) = e^{-t}u(t)$$

$$\int_{-\infty}^{\infty} |h(t)|dt = -e^{-t}|_{t=0}^{\infty} = 1$$

Eigenfunction Property of LTI systems

Eigenfunction property of LTI systems

LTI system with h(t) as impulse response:

$$
\begin{aligned}
\text{input} \quad x(t) &= e^{s_0 t}, \quad s_0 = \sigma_0 + j\Omega_0, \quad -\infty < t < \infty \\
\text{convolution} \quad y(t) &= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \\
&= e^{s_0 t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-\tau s_0} d\tau}_{H(s_0)} = x(t) H(s_0)
\end{aligned}
$$

LTI System

$$x(t) = e^{s_0 t} \longrightarrow \boxed{H(s)} \longrightarrow y(t) = x(t)\, H(s_0)$$

Two–sided Laplace transform

The two-sided Laplace transform of $f(t)$ is

$$\boxed{\begin{array}{c} F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-st} dt \qquad s \in \text{ROC} \\ s = \sigma + j\Omega, \quad \text{damping } \sigma, \ \text{frequency } \Omega \end{array}}$$

The inverse Laplace transform is

$$\boxed{f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds \qquad \sigma \in \text{ROC}}$$

Functions :Finite support functions:

$f(t) = 0$, for t not in a finite segment $t_1 \le t \le t_2$

- Infinite support functions: $f(t)$ defined in infinite support, $t_1 < t < t_2$ where either t_1 or t_2 or both are infinite

Poles/zeros and ROC

Rational function $F(s) = \mathcal{L}[f(t)] = N(s)/D(s)$

- zeros: values of s such that $F(s) = 0$
- poles: values of s such that $F(s) \to \infty$

For function $f(t), -\infty < t < \infty$, its one-sided Laplace transform is

$$\boxed{F(s) = \mathcal{L}[f(t)u(t)] = \int_{0-}^{\infty} f(t)e^{-st}dt, \quad \text{ROC}}$$

- Finite support $f(t)$, i.e., $f(t) = 0$ for $t<t_1$ and $t>t_2$, $t_1<t_2$,

$$F(s) = \mathcal{L}\left[f(t)[u(t-t_1) - u(t-t_2)]\right] \qquad \text{ROC: whole s-plane}$$

- Causal $g(t)$, i.e., $g(t)=0$ for $t<0$, is

$$G(s) = \mathcal{L}[g(t)u(t)] \qquad \mathcal{R}_c = \{\sigma > \max\{\sigma_i\}\}$$

- Anti–causal $h(t)$, i.e., $h(t)=0$ for $t>0$, is

$$H(s) = \mathcal{L}[h(-t)u(t)]_{(-s)} \qquad \mathcal{R}_{ac} = \{\sigma < \min\{\sigma_i\}\}$$

- Non-causal $p(t)$, i.e., $p(t) = p_{ac}(t) + p_c(t) = p(t)u(-t) + p(t)u(t)$, is

$$P(s) = \mathcal{L}[p_{ac}(-t)u(t)]_{(-s)} + \mathcal{L}[p_c(t)u(t)] \qquad \mathcal{R}_c \bigcap \mathcal{R}_{ac}$$

Example

$$\mathcal{L}[e^{j(\Omega_0 t+\theta)}u(t)] = \frac{e^{j\theta}}{s - j\Omega_0} \qquad \text{ROC:} \quad \sigma > 0.$$

Laplace transform of $x(t) = \cos(\Omega_0 t + \theta)u(t)$

$$\begin{aligned} X(s) &= 0.5\mathcal{L}[e^{j(\Omega_0 t+\theta)}u(t)] + 0.5\mathcal{L}[e^{-j(\Omega_0 t+\theta)}u(t)] \\ &= \frac{s\cos(\theta) - \Omega_0 \sin(\theta)}{s^2 + \Omega_0^2}, \quad ROC: \sigma > 0 \end{aligned}$$

For $\theta = 0, -\pi/2$

$$\begin{aligned} \mathcal{L}[\cos(\Omega_0 t)u(t)] &= \frac{s}{s^2 + \Omega_0^2}, \\ \mathcal{L}[\sin(\Omega_0 t)u(t)] &= \frac{\Omega_0}{s^2 + \Omega_0^2}, \quad ROC: \sigma > 0 \end{aligned}$$

Basic Properties of One-Sided Laplace Transforms

Causal functions and constants		
Linearity	$\alpha f(t),\ \beta g(t)$	$\alpha F(s),\ \beta G(s)$
	$\alpha f(t)+\beta g(t)$	$\alpha F(s)+\beta G(s)$
Time shifting	$f(t-\alpha)u(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$t\,f(t)$	$-\dfrac{dF(s)}{ds}$
Derivative	$\dfrac{df(t)}{dt}$	$sF(s)-f(0-)$
Second derivative	$\dfrac{d^2f(t)}{dt^2}$	$s^2F(s)-sf(0-)-f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt'$	$\dfrac{F(s)}{s}$
Expansion/contraction	$f(\alpha t),\ \alpha \neq 0$	$\dfrac{1}{\|\alpha\|}F\left(\dfrac{s}{\alpha}\right)$
Initial value	$f(0-)=\lim_{s\to\infty} sF(s)$	

One-sided Laplace Transforms

(1)	$\delta(t)$	1, whole s-plane
(2)	$u(t)$	$\dfrac{1}{s}$, $Re[s]>0$
(3)	$r(t)$	$\dfrac{1}{s^2}$ $Re[s]>0$
(4)	$e^{-at}u(t),\ a>0$	$\dfrac{1}{s+a}$ $Re[s]>-a$
(5)	$\cos(\Omega_0 t)u(t)$	$\dfrac{s}{s^2+\Omega_0^2}$, $Re[s]>0$
(6)	$\sin(\Omega_0 t)u(t)$	$\dfrac{\Omega_0}{s^2+\Omega_0^2}$, $Re[s]>0$
(7)	$e^{-at}\cos(\Omega_0 t)u(t),\ a>0$	$\dfrac{s+a}{(s+a)^2+\Omega_0^2}$, $Re[s]>-a$
(8)	$e^{-at}\sin(\Omega_0 t)u(t),\ a>0$	$\dfrac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $Re[s]>-a$
(9)	$2Ae^{-at}\cos(\Omega_0 t+\theta)u(t),\ a>0$	$\dfrac{A\angle\theta}{s+a-j\Omega_0}+\dfrac{A\angle-\theta}{s+a+j\Omega_0}$, $Re[s]>-a$
(10)	$\dfrac{1}{(N-1)!}t^{N-1}u(t)$	$\dfrac{1}{s^N}$ N an integer, $Re[s]>0$

Random Formulas

1

$\dfrac{1}{s}$

e^{-at} $\dfrac{1}{s+a}$

$$\mathcal{L}\left[e^{j(\Omega_0+\theta)}\right]=\frac{e^{j\theta}}{s-j\Omega_0}$$

$$\mathcal{L}\left[\cos(\Omega_0 + \theta)\cdot u(t)\right] = \frac{s\cdot\cos(\theta) - \Omega_0 \sin(\theta)}{s^2 + \Omega_0^2}$$

$$\mathcal{L}\left[\cos(\Omega_0 t)\cdot u(t)\right] = \frac{s}{s^2 + \Omega_0^2}$$

$$\mathcal{L}\left[\sin(\Omega_0 t)\cdot u(t)\right] = \frac{\Omega_0}{s^2 + \Omega_0^2}$$

$$\mathcal{L}\left\{\frac{dt(t)}{dt}\right\} = s\cdot F(s) - f(0-)$$

$$\mathcal{L}\left[\frac{d^2 d(t)}{dt^2}\right] = s^2\cdot F(s) - s\cdot f(0-) - \left.\frac{df(t)}{dt}\right|_{t=0}$$

$$\mathcal{L}\left[\int_0^t y(\tau)\,d\tau\right] = \frac{Y(s)}{s}$$