Dealing With Differentiation in Using LaPlace Transform

Our primary goal of using the LaPlace Transform of functions is to give us an alternative way of solving ODEs. Inherently, this causes us to deal with basic calculus operators like the differentiation and integration. We will learn how the LaPlace Transform deals with derivatives of a function $f(t)$ as well as how to deal with the derivative of the LaPlace Transform $F(s)$ itself. From here, we will demonstrate how to use this information to find the LaPlace Transform of a solution $y(t)$ (i.e. $Y(s)$). The final step in this process is to take the LaPlace Transform, $Y(s)$, and translate it back to $y(t)$. We call this performing the Inverse LaPlace Transform. There is a long, complicated (integral) formula for finding Inverse LaPlace Transforms, but we avoid using it. What we will do is (1) refer to our LaPlace Transform tables and (2) use algebra manipulation techniques (e.g. Partial Fraction Decomposition) to get to our desired results.

Ex 1: Find the LaPlace Transform of $f'(t)$, $f''(t)$, and $f'''(t)$. From this info, see if you can find a formula for the LaPlace Transform of $f^{(n)}(t)$.

a) $\mathcal{L}\{f'(t)\} = ?$

$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} \cdot f'(t)\,dt$. Using Integration by Parts, we allow $u = e^{-st}$ and $dv = f'(t)\,dt$.

$\therefore$ $u = e^{-st}$ $\quad dv = f'(t)\,dt$ $\quad \Rightarrow \int_0^\infty e^{-st} \cdot f'(t)\,dt = \int_0^\infty u\,dv = \lim_{b\to\infty}\left(\left[uv\right]_0^b\right) - \int_0^\infty v\,du.$

$du = -se^{-st}$ $\quad v = f(t)$

$$\therefore \lim_{b\to\infty}\left(\left[uv\right]_0^b\right) = \lim_{b\to\infty}\left(\left[e^{-st}f(t)\right]_0^b\right) = \lim_{b\to\infty}\left(\left[\frac{f(t)}{e^{st}}\right]_0^b\right) =$$

$$\lim_{b\to\infty}\left(\frac{f(b)}{e^{sb}} - \frac{f(0)}{e^{\cancel{s(0)}}_{1}}\right) = \lim_{b\to\infty}\left(\frac{f(b)}{e^{sb}} - f(0)\right) = \underbrace{\cancel{\lim_{b\to\infty}\left(\frac{f(b)}{e^{sb}}\right)}}_{0\ (*)} - \underbrace{\cancel{\lim_{b\to\infty}}\left(f(0)\right)}_{f(0)}$$

(*) NOTE: $f(t)$ decreases slower than e^{-st} (for any given value of t) as this is a condition for $f(t)$ to be a "suitable" function, and, thus LaPlace-Transformable. Thus, $\lim_{b\to\infty}\left(\frac{f(b)}{e^{sb}}\right) \le \lim_{b\to\infty}\left(\frac{Me^{s_0 b}}{e^{sb}}\right) = \lim_{b\to\infty}\left(\frac{M}{e^{(s-s_0)b}}\right) = \frac{M}{\infty} \to 0$ if $s - s_0 > 0 \Rightarrow s > s_0$. Since $f(t)$ is assumed to be a suitable function for a LaPlace Transform, then $|f(t)| \le Me^{s_0 t}$ where $s > s_0$ (i.e. $f(t)$ is of exponential order s_0 (which also means that $f(t)$ decreases slower than e^{-st}). Therefore, $\lim_{b\to\infty}\left[\frac{f(t)}{e^{st}}\right] = 0$ no matter what $f(t)$ is defined to be!

Ex 1: (cont'd)

a) $\therefore \int_0^\infty v\,du = \int_0^\infty f(t)\cdot(-se^{-st})\,dt = -s\cdot\mathcal{L}\{f(t)\} = -sF(s)$ (by definition).

$\therefore \int_0^\infty e^{-st} f'(t)\,dt = \lim_{b\to\infty}\left([uv]_0^b\right) - \int_0^\infty v\,du = -f(0) - (-sF(s)) = sF(s) - f(0)$

$\therefore \boxed{\mathcal{L}\{f'(t)\} = sF(s) - f(0)}$

⊛NOTE! For part b, if $f(t)$ decreases slower than e^{-st}, then so does $f^{n}(t)$ where $n = 1, 2, \ldots$

b) $\mathcal{L}\{f''(t)\} = ?$

$\mathcal{L}\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$. Let $u = e^{-st}$ and $dv = f''(t)\,dt$.

$\therefore \int_0^\infty e^{-st}\cdot f''(t)\,dt = \int_0^\infty u\,dv = \lim_{b\to\infty}\left([uv]_0^b\right) - \int_0^\infty v\,du$, where $du = -se^{-st}\cdot dt$ + $v = f'(t)\,dt$.

$\therefore \lim_{b\to\infty}\left([uv]_0^b\right) = \lim_{b\to\infty}\left(\left[e^{-st} f'(t)\,dt\right]_0^b\right) = \lim_{b\to\infty}\left(\left[\frac{f'(t)}{e^{st}}\right]_0^b\right) = \lim_{b\to\infty}\left[\frac{f'(b)}{e^{sb}} - \frac{f'(0)}{e^{s(0)}}\right]$

$= \cancel{\lim_{b\to\infty}\left(\frac{f'(b)}{e^{sb}}\right)}^{\,0\ ⊛} - \lim_{b\to\infty}\left(f'(0)\right) = 0 - f'(0) = -f'(0)$

$\therefore \int_0^\infty v\,du = \int_0^\infty f'(t)\cdot(-se^{-st})\,dt = -s\int_0^\infty e^{-st}\cdot f(t)\,dt = -s\cdot\cancel{\mathcal{L}\{f(t)\}}^{\,sF(s)-f(0)}$
$= -s^2F(s) + sf(0)$

$\therefore \mathcal{L}\{f''(t)\} = \lim_{b\to\infty}\left([uv]_0^b\right) - \int_0^\infty v\,du = -f'(0) - \left[-s^2F(s) + sf(0)\right] = s^2F(s) - sf(0) - f'(0) \Rightarrow \boxed{\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)}$

Ex. 1 : (cont'd_2)

c) $\mathcal{L}\{f'''(t)\} = ?$

Sol'n : $\mathcal{L}\{f'''(t)\} = \int_0^\infty e^{-st} f'''(t)\,dt$. Let $u = e^{-st}$ and $dv = f'''(t)\,dt$.

∴ $u = e^{-st}$; $dv = f'''(t)\,dt$ $\Rightarrow \int_0^\infty e^{-st} f'''(t)\,dt = \int_0^\infty u\,dv = \lim_{b\to\infty}\left[[uv]_0^b\right] - \int_0^\infty v\,du$.

$du = -se^{st}dt$; $v = f''(t)$

$$\therefore \lim_{b\to\infty}\left([uv]_0^b\right) = \lim_{b\to\infty}\left(\left[e^{-st}f''(t)\right]_0^b\right) = \lim_{b\to\infty}\left(\frac{f''(b)}{e^{sb}} - \frac{f''(0)}{\underset{1}{\cancel{e^{s(0)}}}}\right) =$$

$$\lim_{b\to\infty}\left(\cancel{\frac{f''(b)}{e^{sb}}}^{\circledast \to 0}\right) - \lim_{b\to\infty}\left(f''(0)\right) = 0 - f''(0) = -f''(0)$$

$$\therefore \int_0^\infty v\,du = \int_0^\infty f''(t)\cdot(-se^{-st})\,dt = -s\int_0^\infty f''(t)\,e^{-st}\,dt = -s\cdot\cancel{\mathcal{L}\{f''(t)\}}^{s^2F(s) - sf(0) - f'(0)}$$

$$= -s^3F(s) + s^2 f(0) + s f'(0)$$

$$\therefore \mathcal{L}\{f'''(t)\} = \lim_{b\to\infty}\left([uv]_0^b\right) - \int_0^\infty v\,du = -f''(0) - \left[-s^3F(s) + s^2f(0) + sf'(0)\right]$$

$$= s^3F(s) - s^2 f(0) - s f'(0) - f''(0)$$

$$\therefore \boxed{\mathcal{L}\{f'''(t)\} = s^3 - s^2 f(0) - s f'(0) - f''(0)}$$

Ex. 1 : (cont'd _ 3)

d) $\mathcal{L}\{f^{(n)}(t)\} = ?$

Recall that... $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$$\mathcal{L}\{f''(t)\} = s^2F(s) - s f(0) - f'(0)$$

$$\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2 f(0) - s f'(0) - f''(0)$$

From this info, we can surmise that the pattern for $\mathcal{L}\{f^{(n)}(t)\}$ will be...

$$\boxed{\mathcal{L}\{f^{(n)}(t)\} = \left\{\begin{array}{l} s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \dots \\ - s^{n-(n-1)} f^{(n-2)}(0) - f^{(n-1)}(0) \end{array}\right\}}$$

Ex. 2 : Using the identity above for $\mathcal{L}\{f^{(n)}(t)\}$, find $\mathcal{L}\{f^{(6)}(t)\}$

Sol'n : $\mathcal{L}\{f^{(6)}(t)\} = \left\{\begin{array}{l} s^6F(s) - s^5 f(0) - s^4 f'(0) - s^3 f''(0) - s^2 f'''(0) \\ - s f^{(4)}(0) - f^{(5)}(0) \end{array}\right\}$

Ex. 3 : What would $\mathcal{L}\{f^{(6)}(t)\}$ be if $f^{(z)}(0) = 0$ for $z = 0, 1, 2, 3, 4, + 5$

Sol'n : $f^{(0)}(0) = f(0) = 0$; $f'(0) = f''(0) = f'''(0) = f^{(4)}(0) = f^{(5)}(0) = 0$

$$\therefore \mathcal{L}\{f^{(6)}(t)\} = s^6F(s) - s^5 \cancelto{0}{f(0)} - s^4 \cancelto{0}{f'(0)} - s^3 \cancelto{0}{f''(0)} - s^2 \cancelto{0}{f'''(0)} - s \cancelto{0}{f^{(4)}(0)} - \cancelto{0}{f^{(5)}(0)}$$

$$\boxed{\mathcal{L}\{f^{(6)}(t)\} = s^6 F(s)}$$

Ex. 4 : Find $F'(s)$, $F''(s)$, $F'''(s)$, and $F^{(4)}(s)$. Use this info to come up with a general formula for $F^{(n)}(s)$.

a) $F'(s) = ?$; $F'(s) = \frac{dF}{ds} = \frac{d}{ds}\left[F(s)\right] = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t)\, dt\right] =$

$\int_0^\infty \left[\frac{d}{ds}\left(e^{-st} f(t)\right)\right] dt$. Note that since "$\frac{d}{ds}$" operator views anything in terms of "t" (or any other variable for that matter) as a constant since "t" is not assumed to be a function of (the constant) "s" and vice versa. Thus, we can use the "$\frac{d}{ds}$" operator to complete this derivation!

$$\therefore \int_0^\infty \left[\frac{d}{ds}\left(e^{-st} f(t)\right)\right] dt = \int_0^\infty \left[\frac{d}{ds}\left(e^{-st} f(t)\right)\right] dt = \int_0^\infty f(t)\left[\frac{d}{ds}\left(e^{-st}\right)\right] dt =$$

$$\int_0^\infty f(t)\cdot\left(-te^{-st}\right) dt = \int_0^\infty \left(-t\cdot f(t)\right)\cdot e^{-st}\, dt = \mathcal{L}\{-t\, f(t)\}$$

$$\therefore \boxed{F'(s) = \mathcal{L}\{-t\, f(t)\}}$$

b) $F''(s) = ?$; $F''(s) = \frac{d^2F}{ds^2} = \frac{d^2}{ds}\left[F(s)\right] = \frac{d^2}{ds^2}\left[\int_0^\infty e^{-st} f(t)\, dt\right] =$

$$\int_0^\infty \left[\frac{d^2}{ds^2}\left(e^{-st}\cdot f(t)\right)\right] dt = \int_0^\infty \left[\frac{d^2}{ds}\left(e^{-st}\cdot f(t)\right)\right] dt = \int_0^\infty f(t)\cdot\left[\frac{d^2}{ds^2}\left(e^{-st}\right)\right] dt =$$

$$\int_0^\infty f(t)\cdot\left((-t)^2\cdot e^{-st}\right) dt = \int_0^\infty \left(t^2 f(t)\right)\cdot e^{-st}\, dt = \mathcal{L}\{t^2 f(t)\}$$

$$\therefore \boxed{F''(s) = \mathcal{L}\{t^2 f(t)\}}$$

Ex. 4 ; (cont'd)

c) $F'''(s) = ?$; $F'''(s) = \frac{d^3}{ds^3}\left[F(s)\right] = \frac{d^3}{ds^3}\left[\int_0^\infty e^{-st}\cdot f(t)\,dt\right] =$

$\int_0^\infty \frac{d^3}{ds^3}\left(e^{-st}\cdot f(t)\right)dt = \int_0^\infty f(t)\cdot\left[\frac{d^3}{ds^3}\left(e^{-st}\right)\right]dt = \int_0^\infty f(t)\cdot\left[(-t)^3 e^{-st}\right]dt =$

$\int_0^\infty \left(-t^3 f(t)\right)\cdot e^{-st}\,dt = \mathcal{L}\{-t^3 f(t)\}$.

$$\therefore \boxed{F'''(s) = \mathcal{L}\{-t^3 f(t)\}}$$

d) $F^{(4)}(s) = ?$; $F^{(4)}(s) = \frac{d^4}{ds^4}\left[F(s)\right] = \frac{d^4}{ds^4}\left[\int_0^\infty e^{-st}\cdot f(t)\,dt\right] =$

$\int_0^\infty \frac{d^4}{ds^4}\left(e^{-st}\cdot f(t)\right)dt = \int_0^\infty f(t)\cdot\frac{d^4}{ds^4}\left(e^{-st}\right)dt = \int_0^\infty f(t)\left[(-t)^4\cdot e^{-st}\right]dt =$

$\int_0^\infty t^4 e^{-st} f(t)\,dt = \int_0^\infty \left(t^4 f(t)\right)\cdot e^{-st}\,dt = \mathcal{L}\{t^4 f(t)\}$

$$\therefore \boxed{F^{(4)}(s) = \mathcal{L}\{t^4 f(t)\}}$$

NOTE:

$$\left.\begin{aligned} F'(s) &= \mathcal{L}\{-t f(t)\} = \mathcal{L}\{(-1)^1 t^1 f(t)\} \\ F''(s) &= \mathcal{L}\{t^2 f(t)\} = \mathcal{L}\{(-1)^2 t^2 f(t)\} \\ F'''(s) &= \mathcal{L}\{-t^3 f(t)\} = \mathcal{L}\{(-1)^3 t^3 f(t)\} \\ F^{(4)}(s) &= \mathcal{L}\{t^4 f(t)\} = \mathcal{L}\{(-1)^4 t^4 f(t)\} \end{aligned}\right\} \ldots \boxed{\begin{aligned} F^{(n)}(s) &= \mathcal{L}\{(-1)^n t^n f(t)\} \\ &= (-1)^n \mathcal{L}\{t^n f(t)\} \end{aligned}}$$

To summarize what we have done so far, we have found the following LaPlace Transform identities:

- $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$ [*]

(*) <u>Note</u>: $s = \text{const}$
$n = 1, 2, 3, \ldots$

- $\mathcal{L}\{f''(t)\} = s^2 F(s) - s f(0) - f'(0)$
- $\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s^{n-(n-1)} f^{(n-2)} - f^{(n-1)}(0)$
- $F'(s) = \dfrac{dF}{ds} = \mathcal{L}\{-t f(t)\}$
- $F''(s) = \dfrac{d^2F}{ds} = \mathcal{L}\{t^2 f(t)\}$
- $F^{(n)}(s) = (-1)^n \mathcal{L}\{t^n f(t)\}$ (☆)

Note that if we were to take identity (☆) and rearrange it, we will see that ... $\boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)}$ (☆☆)

So identities [*], (☆), + (☆☆) address how we ① find the LaPlace Transform of a derivative (of a function $f(t)$) and ② find the derivative of the LaPlace Transform $\mathcal{L}\{f(t)\} = F(s)$ without having to find these functions by brute force! Although these identities are good to know, we will see that these identities help tremendously when we are called to reverse this process (i.e. apply Inverse LaPlace Transforms)!!

(More on that later as we will tackle how to do that towards the end of this set of notes).

Just like we have identities [*] and ($\bar{\text{A}}$A) that address derivatives and LaPlace Transforms, we also have similiar identities that address ① how to find the LaPlace Transform of (a function defined as) an integral and ② how to find the integral of the LaPlace Transform of a function $f(t)$ (i.e. $\int F(s)\,ds$). We will now state these 2 analogous identities. Afterwards, we will get some practice finding the LaPlace Transform of $y(t)$ (i.e. $Y(s)$) for several ODEs.

<u>Thm (Find LaPlace Transform of Integral)</u>: Let f be a continuous (or piecewise continuous) function on $[0,\infty)$ (which will include $(0,\infty)$) such that $\lim_{t\to\infty}\left[e^{-st}f(t)\right]=0$ (i.e. $f(t)$ is of exponential order s_0, where $s>s_0$). Also, let $g(t)=\int_0^t f(\tau)\,d\tau$ be another function. Then $g(t)$ is (also) a continuous (or piecewise-continuous) function on $[0,\infty)$ of exponential order s_1, where $s>s_1$ and ...

$$\boxed{\mathcal{L}\{g(t)\}=\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}\Big|_s=\frac{F(s)}{s} \Rightarrow G(s)=\frac{F(s)}{s}}.$$

Thm (Find the Integral of the LaPlace Transform of a Function $f(t)$):

Let f be a continuous (or piecewise-continuous) function on $(0,\infty)$ (or possibly $[0,\infty)$ if $f(0)$ is defined) such that $\lim_{t\to\infty}\left[e^{-st}f(t)\right]=0$ (i.e. $f(t)$ is of exponential order s_0, where $s>s_0$. Also, let $g(t)$ be a function defined as...

$$g(t)=\frac{f(t)}{t}, \text{ for } t=(0,\infty), \text{ such that...}$$

$$\lim_{t\to 0^+}\left[\frac{f(t)}{t}\right]=L\in\mathbb{R} \quad \left(\begin{array}{l}\text{i.e. this limit exists AND its}\\ \text{converges to a finite value}\end{array}\right).$$

Then, $g(t)$ is (also) a continuous (or piecewise-continuous) function on $(0,\infty)$ that is of exponential order s_0 and ...

$$\boxed{\mathcal{L}\{g(t)\}=\mathcal{L}\left\{\frac{f(t)}{t}\right\}\bigg|_s=\int_s^\infty F(\sigma)\,d\sigma}.$$

Ex. 5: Let $f(t)=e^{3t}$. Find $F'''(s)$ by using the identity $\mathcal{L}\{t^n f(t)\}=(-1)^n F^{(n)}(s)$.

Sol'n: $\mathcal{L}\{t^n f(t)\}=\mathcal{L}\{t^n e^{3t}\}$. But $\mathcal{L}\{t^n e^{at}\}=\dfrac{n!}{(s-a)^{n+1}}$.

$\therefore\ \mathcal{L}\{t^n e^{3t}\}=\dfrac{n!}{(s-3)^{n+1}}$ and $\mathcal{L}\{t^n e^{3t}\}=(-1)^n F^{(n)}(s)$.

$$\therefore\ \frac{n!}{(s-3)^{n+1}}=(-1)^n F^{(n)}(s) \Rightarrow F^{(n)}(s)=(-1)^n\cdot\frac{n!}{(s-3)^{n+1}}$$

Finally, $F'''(s)=F^{(3)}(s) \Rightarrow n=3$.

$$\therefore\ F'''(s)=\cancelto{-1}{(-1)^3}\cdot\frac{\cancelto{6}{3!}}{(s-3)^{3+1}} \Rightarrow \boxed{F'''(s)=\frac{-6}{(s-3)^4}}$$

Laplace Transforms of Common Functions

In the following, α and ω are real-valued constants, and, unless otherwise noted, $s > 0$

$f(t)$	$F(s) = \mathcal{L}[f(t)]\|_s$	Restrictions
1	$\frac{1}{s}$	
t	$\frac{1}{s^2}$	
t^n	$\frac{n!}{s^{n+1}}$	$n = 1,\ 2,\ 3,\ \ldots$
$\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{s}}$	
t^α	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$	$-1 < \alpha$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$	$\alpha < s$
$e^{i\alpha t}$	$\frac{1}{s - i\alpha}$	
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	
$\text{step}_\alpha(t)$, $\text{step}(t-\alpha)$	$\frac{e^{-\alpha s}}{s}$	$0 \le \alpha$

Commonly Used LaPlace Tranforms Identities

In the following, $F(s) = \mathcal{L}[f(t)]|_s$.

$h(t)$	$H(s) = \mathcal{L}[h(t)]|_s$	Restrictions
$f(t)$	$\int_0^\infty f(t)e^{-st}\,dt$	
$e^{\alpha t} f(t)$	$F(s-\alpha)$	α is real
$\frac{df}{dt}$	$sF(s) - f(0)$	
$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0) - f'(0)$	
$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \cdots - f^{(n-1)}(0)$	$n = 1, 2, 3, \ldots$
$t\ f(t)$	$-\frac{dF}{ds}$	
$t^n\ f(t)$	$(-1)^n \frac{d^n F}{ds^n}$	$n = 1, 2, 3, \ldots$
$e^{i\alpha t} f(t)$	$F(s - i\alpha)$	α is real

Ex. 6: Find the LaPlace Transform, $Y(s)$, for the following initial-value ODEs. DO NOT SOLVE THESE ODEs for $y(t)$ using methods previously learned in this course. Use the LaPlace Transform tables given to you to help you with this process.

a) $y' + 7y = 10$; $y(0) = -3$

sol'n: $y' + 7y = 10 \Rightarrow \mathcal{L}\{y' + 7y\} = \mathcal{L}\{10\}$

$$\Rightarrow \mathcal{L}\{y'\} + \mathcal{L}\{7y\} = \mathcal{L}\{10\}$$

$$\Rightarrow \mathcal{L}\{y'\} + 7\mathcal{L}\{y\} = 10\mathcal{L}\{1\}$$

Note that from using our LaPlace Transform tables that ...

- $\mathcal{L}\{y'\} = \mathcal{L}\{y'(t)\} = sY(s) - \cancelto{-3}{y(0)} = sY(s) + 3$
- $\mathcal{L}\{y\} = \mathcal{L}\{y(t)\} = Y(s) \Rightarrow 7\mathcal{L}\{y\} = 7Y(s)$
- $\mathcal{L}\{1\} = \frac{1}{s} \Rightarrow 10\mathcal{L}\{1\} = \frac{10}{s}$

$\therefore \mathcal{L}\{y'(t)\} + 7\mathcal{L}\{y(t)\} = 10\mathcal{L}\{1\}$

$$\Rightarrow sY(s) + 3 + 7Y(s) = \frac{10}{s}$$

$$\Rightarrow Y(s)[s+7] + 3 = \frac{10}{s}$$

$$\Rightarrow Y(s)[s+7] = \frac{10}{s} - 3$$

$$\Rightarrow \boxed{Y(s) = \frac{10}{s(s+7)} - \frac{3}{s+7} = \frac{10-3s}{s(s+7)}}$$

Ex. 6 : (cont'd)

b) $y' - 5y = \text{step}_4(t)$; $y(0) = 0$

sol'n : $\mathcal{L}\{y' - 5y\} = \mathcal{L}\{\text{step}_4(t)\} \Rightarrow \mathcal{L}\{y'(t)\} - 5\mathcal{L}\{y(t)\} = \mathcal{L}\{\text{step}_4(t)\}$

$\therefore \mathcal{L}\{y'(t)\} = sY(s) - \cancel{y(0)}^{0} = sY(s)$

$\therefore \mathcal{L}\{y(t)\} = Y(s) \Rightarrow -5\mathcal{L}\{y(t)\} = -5Y(s)$

$\therefore \mathcal{L}\{\text{step}_4(t)\} = \dfrac{e^{-4s}}{s}$

This info is found from using LaPlace Transform tables!

$\therefore \mathcal{L}\{y'(t)\} - 5\mathcal{L}\{y(t)\} = \mathcal{L}\{\text{step}_4(t)\} \Rightarrow sY(s) - 5Y(s) = \dfrac{e^{-4s}}{s}$

$\therefore Y(s)[s-5] = \dfrac{e^{-4s}}{s} \Rightarrow \boxed{Y(s) = \dfrac{e^{-4s}}{s(s-5)}}$

c) $y'' + 4y = \sin(2t)$; $y(0) = 3$ and $y'(0) = 5$

sol'n : $\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{\sin(2t)\} \Rightarrow \mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{\sin(2t)\}$

$\therefore \mathcal{L}\{y''(t)\} = s^2Y(s) - s\cancel{y(0)}^{3} - \cancel{y'(0)}^{5} = s^2Y(s) - 3s - 5$

$\therefore 4\mathcal{L}\{y(t)\} = 4Y(s)$

$\therefore \mathcal{L}\{\sin(2t)\} = \dfrac{2}{s^2 + (2)^2} = \dfrac{2}{s^2+4}$

This info is found from using LaPlace Transform tables

So, $\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{\sin(2t)\} \Rightarrow s^2Y(s) - 3s - 5 + 4Y(s) = \dfrac{2}{s^2+4}$

$\therefore (s^2+4)Y(s) = \dfrac{2}{s^2+4} + 3s + 5 \Rightarrow$

$$\boxed{Y(s) = \frac{2}{(s^2+4)^2} + \frac{3s+5}{s^2+4} \quad \text{OR} \quad Y(s) = \frac{2 + (3s+5)(s^2+4)}{(s^2+4)^2}}$$

d) $y'' - 4y' + 13y = e^{2t}\sin(3t)$; $y(0)=4$ and $y'(0)=3$

sol'n : $\mathcal{L}\{y''-4y'+13y\} = \mathcal{L}\{e^{2t}\sin(3t)\}$

$\Rightarrow \mathcal{L}\{y''(t)\} - 4\mathcal{L}\{y'(t)\} + 13\mathcal{L}\{y(t)\} = \mathcal{L}\{e^{2t}\sin(3t)\}$

$\Rightarrow \left[s^2Y(s) - s\overset{4}{\cancel{y(0)}} - \overset{3}{\cancel{y'(0)}}\right] - 4\left[sY(s) - \overset{4}{\cancel{y(0)}}\right] + 13Y(s) = \mathcal{L}\{e^{\alpha t}f(t)\}\Big|_{\substack{\alpha=2 \\ f(t)=\sin(3t)}}$

$\Rightarrow s^2Y(s) - 4s - 3 - 4sY(s) + 16 + 13Y(s) = F(s-2)$, where $F(s) = \frac{3}{s^2+(3)^2}$

$\Rightarrow Y(s)[s^2-4s+13] - 4s + 13 = \frac{3}{(s-2)^2+9} \Rightarrow Y(s)[s^2-4s+13] = \frac{3}{(s-2)^2+9} + 4s - 13$

NOTE: $(s-2)^2+9 = (s^2-4s+4)+9 = s^2-4s+13$

$\therefore Y(s)[s^2-4s+13] = \frac{3}{(s-2)^2+9} + 4s - 13 \Rightarrow \boxed{Y(s) = \frac{3}{(s^2-4s+13)^2} + \frac{4s-13}{s^2-4s+13}}$

e) $y''' - 27y = e^{-3t}$; $y(0)=2$, $y'(0)=3$, and $y''(0)=4$

sol'n : $\mathcal{L}\{y'''-27y\} = \mathcal{L}\{e^{-3t}\}$

$\Rightarrow \mathcal{L}\{y'''(t)\} - 27\mathcal{L}\{y\} = \mathcal{L}\{e^{-3t}\}$

$\Rightarrow \left[s^3Y(s) - s^2\overset{2}{\cancel{y(0)}} - s\overset{3}{\cancel{y'(0)}} - \overset{4}{\cancel{y''(0)}}\right] - 27Y(s) = \frac{1}{s-(-3)}$

$\Rightarrow s^3Y(s) - 2s^2 - 3s - 4 - 27Y(s) = \frac{1}{s+3}$

$\Rightarrow Y(s)[s^3-27] = \frac{1}{s+3} + 2s^2 + 3s + 4 \Rightarrow Y(s) = \frac{1}{(s+3)(s^3-27)} + \frac{2s^2+3s+4}{s^3-27}$

$\therefore Y(s) = \frac{1+(2s^2+3s+4)(s+3)}{(s+3)(s^3-27)} = \frac{1+2s^3+3s^2+4s+6s^2+9s+12}{(s+3)(s+3)(s^2+3s+9)}$

$\therefore \boxed{Y(s) = \frac{2s^3+9s^2+13s+13}{(s+3)^2(s^2+3s+9)} = \frac{1}{(s+3)(s^3-27)} + \frac{2s^2+3s+4}{s^3-27}}$

Inverse LaPlace Transform and Solving ODEs

In short, the LaPlace Transform has an inverse operation called the Inverse LaPlace Transform, denoted as $\mathcal{L}^{-1}\{F(s)\}$, that effectively undoes the LaPlace Transform, $F(s)$, to return back to function in the t-domain, $f(t)$. The actual integral transform that performs the operation is...

$$f(t) = \frac{1}{2\pi i} \int_{\bar{s}}^{s} F(s)\, e^{st}\, dt \quad \text{, where} \ldots$$

↳ $s = \lambda + wi$

↳ $\bar{s} = \lambda - wi$

↳ λ, w are real #s ,

but we do not use this formula. In practice, what we do to a function F(s) is refer to the LaPlace Transform tables to see what the corresponding f(t) will be. In the cases where there is not a direct correlation with F(s) (or a term of F(s)) in the LaPlace Transform table, we must algebraically manipulate what F(s) looks like to force this correlation between F(s) and f(t) (for each term of F(s)) before using the LaPlace Transform tables in reverse. This usually involves heavy use Partial Fraction Decomposition!

There are a lot of properties we could state about the Inverse LaPlace Transform, but we will not do that here in order to avoid any over-complication of this concept. The main concept to take away here is that the "$\mathcal{L}^{-1}$" operator will possess most of the same properties as the "$\mathcal{L}$" operator. Our efforts will be focused on showing you how to get from $F(s)$ to $f(t)$ by using algebraic manipulation which will (usually) include heavy use of Partial Fraction Expansion / Decomposition (PFE/PFD) to solve the ODEs in Exs. 6a - 6e for $y(t)$!

Ex. 7: Recall the ODEs given in Exs. 6a - 6e. These equations as well as their corresponding LaPlace Transform $Y(s)$ is given. Use this info along with the LaPlace Transform tables to find $\mathcal{L}^{-1}\{Y(s)\} = y(t)$.

a) $y' + 7y = 10$; $y(0) = -3$ $\Rightarrow$ $Y(s) = \dfrac{10}{s(s+7)} - \dfrac{3}{s+7}$

$$\therefore \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{10}{s(s+7)}\right\} - \mathcal{L}^{-1}\left\{\frac{3}{s+7}\right\}$$

We can use our tables and properties for LaPlace Transform to find its inverse directly. We will wait to do this until later!

We need to find the PFE equivalent of $\frac{10}{s(s+7)}$. Using the (shortcut) "cover up" method...

$$\frac{10}{s(s+7)} = \frac{A}{s} + \frac{B}{s+7}, \text{ where } A = \left[\frac{10\cancel{s}}{\cancel{s}(s+7)}\right]_{s=0} \text{ and } B = \left[\frac{10\cancel{(s+7)}}{s\cancel{(s+7)}}\right]_{s=-7}$$

Ex. 7 :

a) (cont'd)

$$\therefore A = \left[\frac{10}{s+7}\right]_{s=0} = \frac{10}{0+7} = \frac{10}{7} \; ; \; B = \left[\frac{10}{s}\right]_{s=-7} = \frac{-10}{7}$$

$$\therefore \frac{10}{s(s+7)} = \frac{A}{s} + \frac{B}{s+7} = \frac{10/7}{s} + \frac{-10/7}{s+7}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{10}{s(s+7)}\right\} = \mathcal{L}^{-1}\left\{\frac{10/7}{s} - \frac{10/7}{s+7}\right\} = 10/7\left[\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+7}\right\}\right]$$

Using our LaPlace Transform Tables, we can find the Inverse LaPlace Transform of both $\frac{10}{s(s+7)}$ and $\frac{3}{s+7}$

$$\therefore \mathcal{L}^{-1}\left\{\frac{10}{s(s+7)}\right\} = \frac{10}{7}\left[\cancelto{1}{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}} - \cancelto{e^{-7t}}{\mathcal{L}^{-1}\left\{\frac{1}{s+7}\right\}}\right] = \frac{10}{7}\left[1 - e^{-7t}\right] = \frac{10}{7} - \frac{10}{7}e^{-7t}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{3}{s+7}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s+7}\right\} = 3e^{-7t}$$

So, $\mathcal{L}^{-1}\{Y(s)\} = y(t) = \mathcal{L}^{-1}\left\{\frac{10}{s(s+7)}\right\} - \mathcal{L}^{-1}\left\{\frac{3}{s+7}\right\}$

$$\Rightarrow y(t) = \left(\frac{10}{7} - \frac{10}{7}e^{-7t}\right) - \left(3e^{-7t}\right) = \frac{10}{7} - \frac{31}{7}e^{-7t}$$

$$\therefore \boxed{y(t) = \frac{10}{7} - \frac{31}{7}e^{-7t}}$$

Ex. 7 : (cont'd – 2)

b) $y' - 5y = \text{step}_4(t)$; $y(0) = 0 \Rightarrow Y(s) = \dfrac{e^{-4s}}{s(s-5)}$

NOTE: We will perform PFE on $\dfrac{1}{s(s-5)}$ only. The "e^{-4s}" factor just let us know that there will be a horizontal shift of 4 units to the right in the t-domain (i.e. $f(t-4)$). We will apply this horizontal shift (aka time shift if we interpret t as time) after finding out what the inverse LaPlace Transform of $\dfrac{1}{s(s-5)}$ turns out to be.

$$\therefore \frac{1}{s(s-5)} = \frac{A}{s} + \frac{B}{s-5}, \text{ where } A = \left[\frac{1s}{s(s-5)}\right]_{s=0} \text{ and } B = \left[\frac{1(s-5)}{s(s-5)}\right]_{s=5}$$

by using the "cover-up" method (shortcut) for PFE!

$$\therefore A = \left[\frac{1}{s-5}\right]_{s=0} = \frac{1}{-5} \quad \text{and } B = \left[\frac{1}{s}\right]_{s=5} = \frac{1}{5}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s(s-5)}\right\} = \mathcal{L}^{-1}\left\{\frac{-1/5}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1/5}{s-5}\right\} = -\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}$$

$$= -\frac{1}{5}(1) + \frac{1}{5}e^{5t} = \frac{1}{5}e^{5t} - \frac{1}{5}, \quad t \geq 0.$$

$$\therefore y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s(s-5)}\right\} = \left[\frac{1}{5}e^{5x} - \frac{1}{5}\right]_{x=t-4} = \frac{1}{5}e^{5(t-4)} - \frac{1}{5} \; ; \; t-4 \geq 0$$

$$\therefore \boxed{y(t) = \frac{1}{5}\left[e^{5(t-4)} - \text{step}_4(t)\right], \text{ where } \text{step}_4(t) = \text{step}(t-4) = \begin{cases} 0 \; ; \; t < 4 \\ 1 \; ; \; t \geq 4 \end{cases}}$$

Ex. 7: (cont'd – 3)

c) $y'' + 4y = \sin(2t)$; $y(0) = 3$ and $y'(0) = 5 \Rightarrow Y(s) = \dfrac{2}{(s^2+4)^2} + \dfrac{3s+5}{s^2+4}$

NOTE! To perform PFE on $Y(s)$ and maximize the efficiency in using our LaPlace Transform tables, we will need to express $Y(s)$ differently than in its current form. Noting that $s^2+4 = s^2+(2)^2 = (s+2i)(s-2i)$, the term $\dfrac{2}{(s^2+4)^2} = \dfrac{2}{[(s+2i)(s-2i)]^2} = \dfrac{2}{(s+2i)^2(s-2i)^2}$.

$\therefore Y(s) = \dfrac{A}{s+2i} + \dfrac{B}{(s+2i)^2} + \dfrac{C}{s-2i} + \dfrac{D}{(s-2i)^2} + \dfrac{3s}{s^2+4} + \dfrac{5}{s^2+4}$,

where $\dfrac{A}{s+2i} + \dfrac{B}{(s+2i)^2} + \dfrac{C}{s-2i} + \dfrac{D}{(s-2i)^2} = \dfrac{2}{(s+2i)^2(s-2i)^2}$. Therefore, we will be performing PFE (the traditional way) to find A, B, C, + D. Note that we could employ the use of the "cover-up" (shortcut) method. We will choose to stay with the traditional way of doing PFE in this example + use this alternative method in the next example!

$$\therefore \frac{A}{s+2i} + \frac{B}{(s+2i)^2} + \frac{C}{s-2i} + \frac{D}{(s-2i)^2} = \frac{2}{(s+2i)^2(s-2i)^2}$$

$$\Rightarrow \frac{A(s+2i)(s-2i)^2 + B(s-2i)^2 + C(s-2i)(s+2i)^2 + D(s+2i)^2}{(s+2i)^2(s-2i)^2} = \frac{2}{(s+2i)^2(s-2i)^2}$$

$$\therefore A(s+2i)(s-2i)^2 + B(s-2i)^2 + C(s-2i)(s+2i)^2 + D(s+2i)^2 = 2$$

c) (cont'd)

$$\therefore\ A(s+2i)(s-2i)^2 = A(s+2i)(s^2-4is-4) = A[s^3-4is^2-4s+2is^2 \overset{+8s}{\cancel{-8i^2s}} -8i] =$$

$$A[s^3-2is^2+4s-8i] = \boxed{As^3-2As^2i+4As-8Ai}$$

$$\therefore\ B(s-2i)^2 = B[s^2-4is-4] = \boxed{Bs^2-4Bsi-4B}$$

$$\therefore\ C(s-2i)(s+2i)^2 = C(s-2i)(s^2+4is-4) = C[s^3+4is^2-4s-2is^2 \overset{+8s}{\cancel{-8i^2s}}+8i] =$$

$$C[s^3+2is^2+4s+8i] = \boxed{Cs^3+2Cs^2i+4Cs+8Ci}$$

$$\therefore\ D(s+2i)^2 = D[s^2+4is-4] = \boxed{Ds^2+4Dsi-4D}$$

So, $A(s+2i)(s-2i)^2+B(s-2i)^2+C(s-2i)(s+2i)^2+D(s+2i)^2 = 2$

$$\Rightarrow \left\{\begin{array}{l} As^3-2As^2i+4As-8Ai+Bs^2-4Bsi-4B \\ +Cs^3+2Cs^2i+4Cs+8Ci+Ds^2+4Dsi-4D \end{array}\right\} = 2$$

$$\Rightarrow \left\{\begin{array}{l} (A+C)s^3+(-2Ai+2Ci+B+D)s^2+(4A+4C-4Bi+4Di)s \\ \qquad +(-8Ai+8Ci-4B-4D) \end{array}\right\} = 2$$

s^3	s^2	s
$A+C=0$	$2(C-A)i+B+D=0$	$4\overset{0}{\cancel{(A+C)}}+4(D-B)i=0$

constants

$$8(C-A)i-4(B+D)=2$$

c) (cont'd – 2)

$\therefore A + C = 0 \Rightarrow \boxed{C = -A}$ (1)

$\therefore 2(\overset{-A}{\cancel{C}} - A)i + (B + D) = 0 \Rightarrow \boxed{+4Ai = +(B+D)}$ (2)

$\therefore 4(\overset{0}{\cancel{A+C}}) + 4(D-B)i = 0 \Rightarrow 4(D-B)i = 0 \Rightarrow D - B = 0 \Rightarrow \boxed{D = B}$ (3)

$\therefore 8(\overset{-A}{\cancel{C}} - A)i - 4(\overset{4Ai}{\cancel{B+D}}) = 2 \Rightarrow -16Ai - 16Ai = 2 \Rightarrow -32Ai = 2 \Rightarrow A = \frac{2}{-32i}$

$\Rightarrow \boxed{A = \frac{-1}{16i}}$

So, from (1), we see that $C = -A = -\left(-\frac{1}{16i}\right) = \frac{1}{16i} \Rightarrow \boxed{C = \frac{1}{16i}}$

Also, from (2) + (3), we see that... $4Ai = B + \overset{B}{\cancel{D}} \Rightarrow 4Ai = 2B$,

$2Ai = B = 2\left(\frac{-1}{16i}\right)i = \frac{-2}{16} = \frac{-1}{8} \Rightarrow \boxed{B = D = \frac{-1}{8}}$

FINALLY !!! WE HAVE A, B, C, + D, so now we can use this info to substitute back into Y(s) so that we can find the Inverse LaPlace Transform of each term!

$$\therefore Y(s) = \frac{A}{s+2i} + \frac{B}{(s+2i)^2} + \frac{C}{s-2i} + \frac{D}{(s-2i)^2} + \frac{3s}{s^2+4} + \frac{5}{s^2+4}$$

$$Y(s) = \frac{-\frac{1}{16i}}{s+2i} + \frac{-\frac{1}{8}}{(s+2i)^2} + \frac{\frac{1}{16i}}{s-2i} + \frac{-\frac{1}{8}}{(s-2i)^2} + \frac{3s}{s^2+4} + \frac{5}{s^2+4}$$

Now we will take the Inverse LaPlace Transform of each term of Y(s)!

Ex. 7 : (cont'd_6)

(*) NOTE: $\mathcal{L}\{e^{i\alpha t} f(t)\} = F(s-i\alpha)$

$\therefore \mathcal{L}^{-1}\{F(s-i\alpha)\} = e^{i\alpha t} \cdot \mathcal{L}^{-1}\{F(s)\} = e^{i\alpha t} \cdot f(t)$

c) (cont'd_3)

$$\therefore \mathcal{L}^{-1}\left\{\frac{-\frac{1}{16i}}{s+2i}\right\} = -\frac{1}{16i}\mathcal{L}^{-1}\left\{\frac{1}{s-(-2i)}\right\} = -\frac{1}{16i} \cdot e^{-2it} = \boxed{-\frac{e^{i(-2t)}}{16i}}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{-\frac{1}{8}}{(s+2i)^2}\right\} = -\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{(s-(-2i))^2}\right\}^{(*)} = -\frac{1}{8} e^{i(-2i)t}\, \cancel{\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}}^{(*)\,\to\, t} =$$

$$\boxed{-\frac{1}{8} e^{(-2t)i} \cdot t}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{\frac{1}{16i}}{s-2i}\right\} = \frac{1}{16i}\mathcal{L}^{-1}\left\{\frac{1}{s-2i}\right\} = \frac{1}{16i} \cdot e^{2it} = \boxed{\frac{e^{i(2t)}}{16i}}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{-\frac{1}{8}}{(s-2i)^2}\right\} = -\frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{(s-2i)^2}\right\}^{(*)} = -\frac{1}{8} e^{2it}\, \cancel{\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}}^{(*)\,\to\, t} =$$

$$-\frac{1}{8} e^{(2t)i} \cdot t$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{3s}{s^2+4}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+(2)^2}\right\} = 3\cos(2t)$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{5}{s^2+4}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{s^2+(2)^2}\right\} = 5\mathcal{L}^{-1}\left\{\frac{1}{2}\cdot\frac{2}{s^2+(2)^2}\right\} = \frac{5}{2}\sin(2t)$$

So, $y(t) = \mathcal{L}^{-1}\{Y(s)\} = -\dfrac{e^{i(-2t)}}{16i} - \dfrac{t}{8}e^{(-2t)i} + \dfrac{e^{i(2t)}}{16i} - \dfrac{t}{8}e^{(2t)i} + 3\cos(2t) + \dfrac{5}{2}\sin(2t)$

$$\therefore y(t) = \frac{e^{i(2t)} - e^{-i(2t)}}{8(2i)} - \frac{t}{4}\left[\frac{e^{(2t)i} + e^{-(2t)i}}{2}\right] + 3\cos(2t) + \frac{5}{2}\sin(2t).$$

Ex. 7 : (cont'd-7)

c) (cont'd_4)

Recall that $\sin(\theta) = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$ and $\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$. So, if we let $\theta = 2t$, then...

$$* \ \frac{e^{i(2t)} - e^{-i(2t)}}{2i} = \sin(2t) \quad ; \quad * \ \frac{e^{i(2t)} + e^{-i(2t)}}{2} = \cos(2t)$$

$$\therefore \ y(t) = \frac{1}{8}\sin(2t) - \frac{t}{4}\cos(2t) + 3\cos(2t) + \frac{5}{2}\sin(2t)$$

$$\Rightarrow y(t) = \left(\frac{1}{8} + \frac{5}{2}\right)\sin(2t) + 3\cos(2t) - \frac{1}{4}t\cos(2t)$$

$$\boxed{y(t) = \frac{21}{8}\sin(2t) + 3\cos(2t) - \frac{1}{4}t\cos(2t)} \rightarrow \text{Final Answer!!}$$

NOTE: Our IVP ODE $y'' + 4y = \sin(2t)$; $y(0) = 3$ and $y'(0) = 5$ was solved using Laplace Transform technique, but notice that this process actually involved more work we would have encountered if we had done this problem using the method of solving Non-homogeneous ODEs using the Method of Undetermined Coefficients or Variation of Parameters!

d) $y'' - 4y' + 13y = e^{2t}\sin(3t)$; $y(0) = 4$ and $y'(0) = 3$

$$\therefore Y(s) = \frac{3}{(s^2-4s+13)^2} + \frac{4s-13}{s^2-4s+13}$$

NOTE: $s^2 - 4s + 13 = (s-2)^2 + 9$. Also note that if $(s-2)^2+9=0$, then solving for s yields ...

$$(s-2)^2 + 9 = 0 \Rightarrow (s-2)^2 = -9 \Rightarrow s-2 = \pm\sqrt{-9} \Rightarrow s = 2 \pm 3i$$

$$\therefore (s-2)^2 + 9 = (s-2-3i)(s-2+3i) = [(s-2)-3i][(s-2)+3i].$$

$$\therefore (s^2-4s+13)^2 = [(s-2)+9]^2 = [(s-2)-3i]^2 \cdot [(s-2)+3i]^2$$

So, we can rewrite Y(s) using PFE techniques as...

$$Y(s) = \frac{3}{[(s-2)-3i]^2 \cdot [(s-2)+3i]^2} + \frac{4s}{(s-2)^2+9} - \frac{13}{(s-2)^2+9}$$

$$\Rightarrow Y(s) = \frac{A}{(s-2)-3i} + \frac{B}{((s-2)-3i)^2} + \frac{C}{(s-2)+3i} + \frac{D}{((s-2)+3i)^2}$$

$$+ \frac{4s}{(s-2)^2+9} - \frac{13}{(s-2)^2+9}$$

We will 1st find A, B, C, + D using the "cover up" method to start doing PFE. Afterwards, we will find the Inverse LaPlace Transform of each term of Y(s)!!

d) (cont'd)

$$\therefore B = \left[\frac{3\,\cancel{[(s-2)-3i]^2}}{\cancel{[(s-2)-3i]^2}\cdot[(s-2)+3i]^2} \right]_{s=2+3i} = \left[\frac{3}{[(s-2)+3i]^2} \right]_{s=2+3i} =$$

$$\frac{3}{[(\cancel{2}+3i)-\cancel{2}+3i]^2} = \frac{3}{(6i)^2} = \frac{3}{36(-1)} = \boxed{\frac{-1}{12} = B}$$

$$\therefore D = \left[\frac{3\,\cancel{[(s-2)+3i]^2}}{[(s-2)-3i]^2\,\cancel{[(s-2)+3i]^2}} \right]_{s=2-3i} = \left[\frac{3}{[(s-2)-3i]^2} \right]_{s=2-3i} =$$

$$\frac{3}{[(2-3i)-2-3i]^2} = \frac{3}{(-6i)^2} = \frac{3}{36i^2} = \frac{3}{36(-1)} = \boxed{\frac{-1}{12} = D}$$

$$\therefore \frac{3}{[(s-2)-3i]^2\cdot[(s-2)+3i]^2} = \left\{ \begin{array}{l} \dfrac{A}{(s-2)-3i} + \dfrac{-\frac{1}{12}}{[(s-2)-3i]^2} \\ + \dfrac{C}{(s-2)+3i} + \dfrac{-\frac{1}{12}}{[(s-2)+3i]^2} \end{array} \right\}$$

Now we will solve for $\dfrac{A}{(s-2)-3i} + \dfrac{C}{(s-2)+3i}$ by adding the other 2 terms to both sides of our equation above.

$$\therefore \frac{3}{[(s-2)-3i]^2\cdot[(s-2)+3i]^2} + \frac{\frac{1}{12}}{[(s-2)-3i]^2} + \frac{\frac{1}{12}}{[(s-2)+3i]^2} = \left\{ \begin{array}{l} \dfrac{A}{(s-2)-3i} \\ + \dfrac{C}{(s-2)+3i} \end{array} \right\}$$

d) (cont'd _2)

$$\therefore \frac{3+\frac{1}{12}\left[(s-2)+3i\right]^2+\frac{1}{12}\left[(s-2)-3i\right]^2}{\left[(s-2)-3i\right]^2\cdot\left[(s-2)+3i\right]^2} = \frac{A}{(s-2)-3i}+\frac{C}{(s-2)+3i}$$

$$\Rightarrow \frac{3+\frac{1}{12}\left[s^2-4s+4+\cancel{6(s-2)}^{6si-12i}i-9\right]+\frac{1}{12}\left[s^2-4s+4-\cancel{6(s-2)}^{-6si+12i}i-9\right]}{\left[(s-2)-3i\right]^2\cdot\left[(s-2)+3i\right]^2} = \left\{\frac{A}{(s-2)-3i}+\frac{C}{(s-2)+3i}\right\}$$

$$\Rightarrow \frac{36+s^2-4s+4+\cancel{6si}-\cancel{12i}-9+s^2-4s+4-\cancel{6si}+\cancel{12i}-9}{12\left[(s-2)-3i\right]^2\cdot\left[(s-2)+3i\right]^2}$$

$$\Rightarrow \frac{2(s^2-4s+4)+\cancel{36}^{18}-18}{12\left[(s-2)-3i\right]^2\cdot\left[(s-2)+3i\right]^2} = \frac{\cancel{2}^{1}\left[s^2-4s+4+9\right]}{\cancel{12}_{6}\left[(s-2)-3i\right]^2\left[(s-2)+3i\right]^2} = \left\{\frac{A}{(s-2)-3i}+\frac{C}{(s-2)+3i}\right\}$$

$$\Rightarrow \frac{(s-2)^2+9}{6\left[(s-2)-3i\right]^2\left[(s-2)+3i\right]^2} = \frac{\cancel{\left[(s-2)+3i\right]}^{1}\cancel{\left[(s-2)-3i\right]}^{1}}{6\left[(s-2)-3i\right]^{\cancel{2}\,1}\left[(s-2)+3i\right]^{\cancel{2}\,1}} = \left\{\frac{A}{(s-2)-3i}+\frac{C}{(s-2)+3i}\right\}$$

$$\Rightarrow \frac{1/6}{\left[(s-2)-3i\right]\left[(s-2)+3i\right]} = \frac{A}{(s-2)-3i}+\frac{C}{(s-2)+3i}$$

Now we will use the "cover up" method again to solve for A + C !

$$\therefore A = \left[\frac{1/6\,\cancel{\left[(s-2)-3i\right]}^{1}}{\cancel{\left[(s-2)-3i\right]}_{1}\left[(s-2)+3i\right]}\right]_{s=2+3i} = \left[\frac{1/6}{\left[(s-2)+3i\right]}\right]_{s=2+3i} = \frac{1/6}{(\cancel{2}+3i)\cancel{-2}+3i} = \frac{\frac{1}{6}}{6i} = \frac{1}{36i}$$

d) (cont'd – 3)

$$\therefore C = \left[\frac{\frac{1}{6}\cancel{[(s-2)+3i]}}{[(s-2)-3i]\cancel{[(s-2)+3i]}} \right]_{s=2-3i} = \left[\frac{\frac{1}{6}}{[(s-2)-3i]} \right]_{s=2-3i} = \frac{\frac{1}{6}}{(\cancel{2}-3i)-\cancel{2}-3i} = \frac{\frac{1}{6}}{-6i} = \frac{-1}{36i}$$

So $\boxed{A = \frac{1}{36i}, \ C = \frac{-1}{36i}, \text{ and } B = D = \frac{-1}{12}}$.

$$\therefore Y(s) = \frac{\frac{1}{36i}}{(s-2)-3i} + \frac{-\frac{1}{12}}{[(s-2)-3i]^2} + \frac{-\frac{1}{36i}}{(s-2)+3i} + \frac{-\frac{1}{12}}{[(s-2)+3i]^2} + \frac{4s}{(s-2)^2+9} + \frac{-13}{(s-2)^2+9}$$

Now we will find the Inverse LaPlace Transform of each term of $Y(s)$!

$$\therefore \mathcal{L}^{-1}\left\{ \frac{\frac{1}{36i}}{(s-2)-3i} \right\} = \frac{1}{36i}\mathcal{L}^{-1}\left\{ \frac{1}{(s-2)-3i} \right\} = \frac{e^{2t}}{36i}\mathcal{L}^{-1}\left\{ \frac{1}{s-3i} \right\} = \boxed{\frac{e^{2t}}{36i} \cdot e^{i(3t)}}$$

$$\therefore \mathcal{L}^{-1}\left\{ \frac{-\frac{1}{12}}{[(s-2)-3i]^2} \right\} = -\frac{1}{12}\mathcal{L}^{-1}\left\{ \frac{1}{[(s-2)-3i]^2} \right\} = -\frac{e^{2t}}{12}\mathcal{L}^{-1}\left\{ \frac{1}{(s-3i)^2} \right\}$$

$$= -\frac{e^{2t}}{12} \cdot e^{i(3t)} \overset{t}{\cancel{\mathcal{L}^{-1}\left\{ \frac{1}{s^2} \right\}}} = \boxed{-\frac{e^{2t}}{12} \cdot e^{i(3t)} \cdot t}$$

$$\therefore \mathcal{L}^{-1}\left\{ \frac{-\frac{1}{36i}}{(s-2)+3i} \right\} = -\frac{1}{36i}\mathcal{L}^{-1}\left\{ \frac{1}{(s-2)+3i} \right\} = -\frac{e^{2t}}{36i} \overset{e^{i(-3t)}}{\cancel{\mathcal{L}^{-1}\left\{ \frac{1}{s+3i} \right\}}} = \frac{-e^{2t}}{36i} \cdot e^{i(-3t)}$$

$$= \boxed{-\frac{e^{2t}}{36i} \cdot e^{-i(3t)}}$$

d) (cont'd_4)

$$\therefore \mathcal{L}^{-1}\left\{\frac{-\frac{1}{12}}{[(s-2)+3i]^2}\right\} = -\frac{1}{12}\mathcal{L}^{-1}\left\{\frac{1}{[(s-2)+3i]^2}\right\} = -\frac{e^{2t}}{12}\mathcal{L}^{-1}\left\{\frac{1}{(s+3i)^2}\right\} =$$

$$-\frac{e^{2t}}{12}\cdot e^{i(-3t)}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}}_{t} = \boxed{-\frac{e^{2t}}{12}\cdot e^{-i(3t)}\cdot t}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{4s}{(s-2)^2+9}\right\} = 4\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2+(3)^2}\right\} = 4\mathcal{L}^{-1}\left\{\frac{s-2+2}{(s-2)^2+(3)^2}\right\} =$$

$$= 4\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+(3)^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+(3)^2}\right\}$$

$$= 4e^{2t}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{s}{s^2+(3)^2}\right\}}_{\cos(3t)} + 4e^{2t}\mathcal{L}^{-1}\left\{\frac{2\cdot\frac{1}{3}\cdot 3}{s^2+(3)^2}\right\}$$

$$= 4e^{2t}\left[\cos(3t) + \frac{2}{3}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{3}{s^2+(3)^2}\right\}}_{\sin(3t)}\right] = \boxed{4e^{2t}\cos(3t) + \frac{8}{3}e^{2t}\sin(3t)}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{-13}{(s-2)^2+9}\right\} = -13\mathcal{L}^{-1}\left\{\frac{3\cdot\frac{1}{3}}{(s-2)^2+(3)^2}\right\} = -\frac{13}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+(3)^2}\right\} =$$

$$-\frac{13}{3}e^{2t}\,\underbrace{\mathcal{L}^{-1}\left\{\frac{3}{s^2+(3)^2}\right\}}_{\sin(3t)} = \boxed{-\frac{13}{3}e^{2t}\cdot\sin(3t)}$$

d) (cont'd - 5)

$$\therefore y(t) = \mathcal{L}^{-1}\{Y(s)\} = \begin{cases} \dfrac{e^{2t}}{36i} \cdot e^{i(3t)} + \dfrac{-e^{2t}}{12} \cdot e^{i(3t)} \cdot t + 4e^{2t}\cos(3t) \\ -\dfrac{e^{2t}}{36i} \cdot e^{-i(3t)} + \dfrac{-e^{2t}}{12} \cdot e^{-i(3t)} \cdot t + \tfrac{8}{3} e^{2t}\sin(3t) - \tfrac{13}{3} e^{2t}\sin(3t) \end{cases}$$

$$\therefore y(t) = \frac{e^{2t}}{18}\left[\cancelto{\sin(3t)}{\frac{e^{i(3t)} - e^{-i(3t)}}{2i}}\right] - \frac{e^{2t}}{6}\left[\cancelto{\cos(3t)}{\frac{e^{i(3t)} + e^{-i(3t)}}{2}}\right] + 4e^{2t}\cos(3t) - \frac{5}{3}e^{2t}\sin(3t)$$

$$y(t) = \frac{e^{2t}}{18}\sin(3t) - \frac{te^{2t}}{6}\cos(3t) + 4e^{2t}\cos(3t) - \frac{5}{3}e^{2t}\sin(3t)$$

$$y(t) = e^{2t}\left[\left(\frac{1}{18} - \frac{5}{3}\right)\sin(3t) + \left(4 - \frac{t}{6}\right)\cos(3t)\right]$$

$$\boxed{y(t) = e^{2t}\left[-\frac{29}{18}\sin(3t) + \left(4 - \frac{t}{6}\right)\cos(3t)\right]}$$

Ex. 7: (cont'd - 14)

e) $y''' - 27y = e^{-3t}$; $y(0) = 2$, $y'(0) = 3$, and $y''(0) = 4$

$$\therefore Y(s) = \frac{1}{(s+3)(s^3-27)} + \frac{2s^2+3s+4}{s^3-27}$$

NOTE: $s^3 - 27 = (s-3)(s^2+3s+9) = (s-3)\left[s^2+3s+\frac{9}{4}+9-\frac{9}{4}\right]$

$$= (s-3)\left[\left(s+\tfrac{3}{2}\right)^2 + \frac{27}{4}\right]$$

$$\therefore Y(s) = \frac{1}{(s+3)(s^3-27)} + \frac{2s^2+3s+4}{s^3-27} = \frac{1+(2s^2+3s+4)(s+3)}{(s+3)(s^3-27)} =$$

$$\frac{1+2s^3+3s^2+4s+6s^2+9s+12}{(s+3)(s^3-27)} = \frac{2s^3+9s^2+13s+13}{(s+3)(s^3-27)}$$

$$\therefore \frac{2s^3+9s^2+13s+13}{(s+3)(s-3)\left[\left(s+\frac{3}{2}\right)^2+\frac{27}{4}\right]} = \frac{A}{s+3} + \frac{B}{s-3} + \frac{Cs+D}{\left(s+\frac{3}{2}\right)^2+\frac{27}{4}}$$

So we will use the "cover-up" method to find A + B. Afterwards, we will substitute A + B back into Y(s) and rearrange this equation to find out what $\dfrac{Cs+D}{\left(s+\frac{3}{2}\right)^2+\frac{27}{4}}$ is, and, thus, the values of C + D!

$$\therefore A = \left[\frac{(2s^3+9s^2+13s+13)\cancel{(s+3)}}{\cancel{(s+3)}(s-3)(s^2+3s+9)}\right]_{s=-3} = \left[\frac{2s^3+9s^2+13s+13}{(s-3)(s^2+3s+9)}\right]_{s=-3} =$$

$$\frac{2(-3)^3+9(-3)^2+13(-3)+13}{(-3-3)\left[(-3)^2+3(-3)+9\right]} = \frac{-54+81-39+13}{(-6)[9-9+9]} = \frac{1}{-54} \Rightarrow \boxed{A = -\frac{1}{54}}$$

e) (cont'd)

$$\therefore B = \left[\frac{(2s^3+9s^2+13s+13)\cancel{(s-3)}^{1}}{(s+3)\cancel{(s-3)}_{1}(s^2+3s+9)}\right]_{s=3} = \left[\frac{2s^3+9s^2+13s+13}{(s+3)(s^2+3s+9)}\right]_{s=3} =$$

$$\frac{2(3)^3+9(3)^2+13(3)+13}{(3+3)((3)^2+3(3)+9)} = \frac{54+81+39+13}{(6)(9+9+9)} = \frac{187}{162} \Rightarrow \boxed{B = \frac{187}{162}}$$

$$\therefore \frac{2s^3+9s^2+13s+13}{(s+3)(s-3)\left[\left(s+\frac{3}{2}\right)^2+\frac{27}{4}\right]} = \frac{-\frac{1}{54}}{s+3} + \frac{\frac{187}{162}}{s-3} + \frac{Cs+D}{\left(s+\frac{3}{2}\right)^2+\frac{27}{4}}$$

$$\Rightarrow \frac{2s^3+9s^2+13s+13}{(s+3)(s-3)\left[\left(s+\frac{3}{2}\right)^2+\frac{27}{4}\right]} + \frac{\frac{1}{54}}{s+3} - \frac{\frac{187}{162}}{s-3} = \frac{Cs+D}{\left(s+\frac{3}{2}\right)^2+\frac{27}{4}}$$

$$\Rightarrow \frac{2s^3+9s^2+13s+13+\frac{1}{54}\underbrace{\cancel{(s-3)\left[\left(s+\frac{3}{2}\right)^2+\frac{27}{4}\right]}}_{s^3-27} - \frac{187}{162}(s+3)\underbrace{\cancel{\left[\left(s+\frac{3}{2}\right)^2+\frac{27}{4}\right]}}_{s^2+3s+9}}{(s+3)(s-3)\left[\left(s+\frac{3}{2}\right)^2+\frac{27}{4}\right]} = \frac{Cs+D}{\left(s+\frac{3}{2}\right)^2+\frac{27}{4}}$$

$$\Rightarrow \frac{2s^3+9s^2+13s+13+\cancel{\frac{1}{54}}^{\frac{3}{162}}(s^3-27) - \frac{187}{162}\underbrace{\cancel{\left(s^3+3s^2+9s+3s^2+9s+27\right)}}_{s^3+6s^2+18s+27}}{(s+3)(s-3)\left[\left(s+\frac{3}{2}\right)^2+\frac{27}{4}\right]} = \frac{Cs+D}{\left(s+\frac{3}{2}\right)^2+\frac{27}{4}}$$

$$\Rightarrow \frac{2s^3+9s^2+13s+13+\frac{3}{162}s^3-\frac{81}{162}-\frac{187}{162}s^3-\frac{1122}{162}s^2-\frac{3366}{162}s-\frac{5049}{162}}{(s+3)(s-3)\left[\left(s+\frac{3}{2}\right)^2+\frac{27}{4}\right]} = \frac{Cs+D}{\left(s+\frac{3}{2}\right)^2+\frac{27}{4}}$$

$$\Rightarrow \frac{s^3\underbrace{\cancel{\left(2+\frac{3}{162}-\frac{187}{162}\right)}}_{\frac{140}{162}}+s^2\underbrace{\cancel{\left(9-\frac{1122}{162}\right)}}_{\frac{336}{162}}+s\underbrace{\cancel{\left(13-\frac{3366}{162}\right)}}_{-\frac{1260}{162}}+\underbrace{\cancel{\left(13-\frac{81}{162}-\frac{5049}{162}\right)}}_{-\frac{3024}{162}}}{(s+3)(s-3)\left[\left(s+\frac{3}{2}\right)^2+\frac{27}{4}\right]} = \frac{Cs+D}{\left(s+\frac{3}{2}\right)^2+\frac{27}{4}}$$

Now, we will multiply the numerator and denominator of the left side by $\frac{162}{162}$!

Ex. 7: (Cont'd_16)

e) (cont'd_2)

$$\therefore \frac{140s^3 + 336s^2 - 1260s - 3024}{162(s+3)(s-3)\left[\left(s+\frac{3}{2}\right)^2 + \frac{27}{4}\right]} = \frac{Cs + D}{\left(s+\frac{3}{2}\right)^2 + \frac{27}{4}}$$

$$\Rightarrow \frac{\cancel{4}^{2}(35s^3 + 84s^2 - 315s - 756)}{\cancel{162}_{81}(s+3)(s-3)\left[\left(s+\frac{3}{2}\right)^2 + \frac{27}{4}\right]} = \frac{Cs + D}{\left(s+\frac{3}{2}\right)^2 + \frac{27}{4}}$$

$$\Rightarrow \frac{\frac{2}{81}(35s^3 + 84s^2 - 315s - 756)}{(s+3)(s-3)\left[\left(s+\frac{3}{2}\right)^2 + \frac{27}{4}\right]} = \frac{Cs + D}{\left(s+\frac{3}{2}\right)^2 + \frac{27}{4}}$$

NOTE: We are expecting that our numerator should be able to be rewritten as $(s+3)(s-3)(\ ?\ )$ since the right side of our equation above does not have "$s+3$" and "$s-3$" in its denominator. (In other words, the left-side of our equation should be divisible by $(s-3)(s+3)$). We will use synthetic division to factor down $35s^3+84s^2-315s-756$ into a product of linear factors. Our result will indirectly tell us what constants C + D turn out to be!

Since $(s+3)(s-3) = 0 \Rightarrow s = \pm 3$, we will use these values to factor the numerator of the left side of our equation above.

$s=3$:

$$\begin{array}{r|rrrr} 3 & 35 & 84 & -315 & -756 \\ & \downarrow & 105 & 567 & 756 \\ \hline & 35 & 189 & 252 & \boxed{0} \end{array} \Rightarrow \begin{array}{r|rrr} -3 & 35 & 189 & 252 \\ & \downarrow & -105 & -252 \\ \hline & 35 & 84 & \boxed{0} \\ & s & \# & \end{array}$$

$$\therefore \frac{2}{81}(35s^3+84s^2-315s-756) = \frac{2}{81}(s+3)(s-3)(35s+84)$$

e) (cont'd-3)

$$\therefore \frac{\frac{2}{81}\cancel{(s+3)}\cancel{(s-3)}(35s+84)}{\cancel{(s+3)}\cancel{(s-3)}\left[(s+\frac{3}{2})^2+\frac{27}{4}\right]} = \frac{Cs+D}{(s+\frac{3}{2})^2+\frac{27}{4}} \Rightarrow \frac{\frac{70}{81}s+\frac{168}{81}}{(s+\frac{3}{2})^2+\frac{27}{4}} = \frac{Cs+D}{(s+\frac{3}{2})^2+\frac{27}{4}}$$

$$\therefore \boxed{C = \frac{70}{81} \quad \text{and} \quad D = \frac{168}{81} = \frac{56}{27}}$$

So, $A = -\frac{1}{54}$, $B = \frac{187}{162}$, $C = \frac{70}{81}$, and $D = \frac{56}{27}$

$$\therefore Y(s) = \frac{-\frac{1}{54}}{s+3} + \frac{\frac{187}{162}}{s-3} + \frac{\frac{70}{81}s+\frac{56}{27}}{(s+\frac{3}{2})^2+\frac{27}{4}}$$

$$\therefore y(t) = \mathcal{L}^{-1}\{Y(s)\} = \left\{ \begin{array}{l} -\frac{1}{54}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + \frac{187}{162}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{70}{81}\mathcal{L}^{-1}\left\{\frac{s+\frac{3}{2}-\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{27}{4}}\right\} \\ + \frac{56}{27}\mathcal{L}^{-1}\left\{\frac{\frac{3\sqrt{3}}{2}\cdot\frac{2}{3\sqrt{3}}}{(s+\frac{3}{2})^2+(\frac{3\sqrt{3}}{2})^2}\right\} \end{array} \right.$$

$$\therefore y(t) = \left\{ \begin{array}{l} -\frac{1}{54}e^{-3t} + \frac{187}{162}e^{3t} + \frac{70}{81}\mathcal{L}^{-1}\left\{\frac{s+\frac{3}{2}}{(s+\frac{3}{2})^2+\frac{27}{4}}\right\} + \frac{70}{81}\mathcal{L}^{-1}\left\{\frac{-\frac{3}{\cancel{2}}\cdot\frac{3\sqrt{3}}{2}\cdot\frac{\cancel{2}}{3\sqrt{3}}}{(s+\frac{3}{2})^2+(\frac{3\sqrt{3}}{2})^2}\right\} \\ + \frac{56}{27}\cdot\frac{2}{3\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\frac{3\sqrt{3}}{2}}{(s+\frac{3}{2})^2+(\frac{3\sqrt{3}}{2})^2}\right\} \end{array} \right.$$

$$\therefore y(t) = \left\{ \begin{array}{l} -\frac{1}{54}e^{-3t} + \frac{187}{162}e^{3t} + \frac{70}{81}e^{-\frac{3}{2}t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+(\frac{3\sqrt{3}}{2})^2}\right\} + \frac{70}{81\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\frac{3\sqrt{3}}{2}}{(s+\frac{3}{2})^2+(\frac{3\sqrt{3}}{2})^2}\right\} \\ + \frac{112\sqrt{3}}{243}\mathcal{L}^{-1}\left\{\frac{\frac{3\sqrt{3}}{2}}{(s+\frac{3}{2})^2+(\frac{3\sqrt{3}}{2})^2}\right\} \end{array} \right.$$

$$\therefore y(t) = -\frac{1}{54}e^{-3t} + \frac{187}{162}e^{3t} + \frac{70}{81}e^{-\frac{3}{2}t}\cos\left(\frac{3\sqrt{3}}{2}t\right) + \frac{70\sqrt{3}}{243}\sin\left(\frac{3\sqrt{3}}{2}t\right) + \frac{112\sqrt{3}}{243}\sin\left(\frac{3\sqrt{3}}{2}t\right)$$

$$\boxed{y(t) = -\frac{1}{54}e^{-3t} + \frac{187}{162}e^{3t} + \frac{70}{81}e^{-\frac{3}{2}t}\cos\left(\frac{3\sqrt{3}}{2}t\right) + \frac{182\sqrt{3}}{243}\sin\left(\frac{3\sqrt{3}}{2}t\right)}$$