First-Order ODEs (The Basics)

- Directly Integrable vs. Autonomous vs. Separable ODE forms
- Solving 1^{st}-order ODEs with Constant/Equilibrium Solutions (IVPs
- Existence and Uniqueness of IVPs

Up to this point in our study of solving ODEs, we have just encountered solving for general and particular solutions of directly integrable ODEs (i.e. $\frac{dy}{dx} = f(x)$). In dealing with 1^{st}-order ODEs, we will often encounter ODEs where $\frac{dy}{dx} = f(x,y)$ (i.e. $\frac{dy}{dx}$ is a function of both "x" and/or "y"). In summary, we will see these 3 types of 1^{st}-order ODEs a lot in our study of solving ODEs!

- Directly Integrable: $\frac{dy}{dx} = f(x)$
- Autonomous: $\frac{dy}{dx} = f(y)$
- Separable: $\frac{dy}{dx} = f(x) \cdot g(y)$

NOTE: All of these equati are considered to be of the form $\frac{dy}{dx} = f(x,y)$ where $f(x,y)$ means that f is a function of "x" and/or "y"

It is important to know and recognize the different forms of ODEs in this course as recognizing the ODE form will guide you in letting you know what technique(s) and/or other special considerations you need to think about in the process of finding solutions to the ODEs!

Ex: Determine if each of the following ODEs of the form $\frac{dy}{dx} = f(x,y)$ is directly integrable, autonomous, or neither.

a) $x \frac{dy}{dx} = -\frac{x}{y}$; $y(4) = -3$ Ans: autonomous

b) $(x^2-1)\frac{dy}{dx} = \sqrt{1-y}$; $x \neq 1$; $y \leq 1$ Ans: neither

c) $x^3 \frac{dy}{dx} - 2x = 6$; $x \neq 0$ Ans: directly integrable

d) $(y^2 - \pi)\frac{dx}{dy} + 5y - 8 = 0$; $y \neq \sqrt{\pi}$ Ans: directly integrable

e) $\frac{du}{dy} - 6u = 10u^2 - 3$ Ans: autonomous

f) $\frac{dz}{dy} = -2z + 3y^2$ Ans: neither

g) $y^3 - 16y - \frac{dy}{dx} = 0$ Ans: autonomous

h) $x^2 \frac{dy}{dx} = -\frac{yx}{2}$; $x \neq 0$ Ans: neither

i) $|x| \cdot \frac{dy}{dx} = -xy^4$; $x > 0$ Ans: autonomous

Constant / Equilibrium Solutions to ODEs

A constant solution to an ODE is just a constant function $f(x,y) = c$ where c is a constant (real number). For example, the ODE ...

$$\frac{dy}{dx} = 3x^2y - 4x^2y^2$$

has a trivial solution $y_1(x) = 0$ and non-trivial solution $y_2(x) = \frac{3}{4}$ because $y_1'(x) = y_2'(x) = 0$ and $3x^2y - 4x^2y^2 = 0$ when $y = 0$ or $\frac{3}{4}$ for all $x =$ domain of ODE $= (-\infty, \infty) = \mathbb{R}$ (in this case). We verify this fact below

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For $y_1(x) = 0$

$$\frac{dy_1}{dx} = [0]' = 0$$

Let $y_1 = y$. Then,

$$3x^2(0) - 4x^2(0)^2 = 0 - 0 = 0$$

$$\therefore \frac{dy_1}{dx} = 3x^2y - 4x^2y^2 \text{ if}$$

$$y_1(x) = 0$$

For $y_2(x) = \frac{3}{4}$

$$\frac{dy_2}{dx} = \left[\frac{3}{4}\right]' = 0$$

Let $y_2 = y$. Then,

$$3x^2\left(\frac{3}{4}\right) - 4x^2\left(\frac{3}{4}\right)^2$$

$$= \frac{9}{4}x^2 - \cancel{4}x^2 \cdot \frac{9}{\cancel{16}\,4}$$

$$= \frac{9}{4}x^2 - \frac{9}{4}x^2$$

$$= 0$$

Therefore,

$$\frac{dy_2}{dx} = 3x^2 - 4x^2y^2$$

if $y_2(x) = \frac{3}{4}$.

However, we see that if $y(x) = 2$, $y(x)$ will not be a solution to the ODE $\frac{dy}{dx} = 3x^2y - 4x^2y^2$. We leave it to the reader to verify this fact.

NOTE! It is possible for ODEs to have either (1) only constant solutions, (2) only non-constant solutions, or (3) a combination of constant and non-constant solutions! Many of the ODEs that we will deal with in our study of this body of knowledge will b of the type that does not contain constant solutions. However, we will now show an example of an ODE that has both constant and non-constant solutions (since we have already seen an example where an ODE had multiple constant solutions).

Ex: Consider the ODE $\frac{dy}{dx} - xy^2 + 5y^3 = 20y - 4x$.
Suppose that $y = y(x) = c$, where c = constant (real number), is a solution to this ODE. Find the value(s) of c that will satisfy this ODE.

Sol'n: First, let's rewrite our ODE in the form of $\frac{dy}{dx} = f(x,y)$ to see how we could possibly find suitable value(s) of c for $y(x)$.

$$\therefore \frac{dy}{dx} - xy^2 + 5y^3 = 20y - 4x$$
$$\Rightarrow \frac{dy}{dx} = xy^2 - 5y^3 + 20y - 4x$$
$$\Rightarrow \frac{dy}{dx} = y^2(x - 5y) + (-4)(-5y + x)$$
$$\Rightarrow \frac{dy}{dx} = (y^2 - 4)(x - 5y)$$

Now, we note that $\frac{dy}{dx} = [c]' = 0$ and $y = y(x) = c$. We can substitute these values into our ODE and work to solve for suitable values of c!

$$\therefore \overbrace{\frac{dy}{dx}}^{0} = (\overbrace{y^2}^{c^2} - 4)(x - 5\overbrace{y}^{c})$$
$$\Rightarrow 0 = (c^2 - 4)(x - 5c)$$

$\therefore\ 0 = (c^2-4)(x-5c) \Rightarrow 0 = c^2-4$ or $x-5c=0$

$\underline{0 = c^2-4}$

$0 = (c-2)(c+2)$

$\Rightarrow \boxed{c = \pm 2}$

constant solutions

$\underline{x-5c=0}$

$x = 5c$

$\Rightarrow \boxed{\frac{1}{5}x = c}$

non-constant solution

Thus, $y(x) = \pm 2$ and $y(x) = \frac{1}{5}x$ are the 3 solutions to this ODE. However, $y(x) = \pm 2$ are the only 2 solutions that are constant solutions!

$\therefore$ $\boxed{\text{Constant sol'ns: } y(x) = 2 \text{ and } y(x) = -2}$

NOTE: We could have simply substituted $\frac{dy}{dx} = 0$ and $y = c$ into our ODE $\frac{dy}{dx} - xy^2 + 5y^3 = 20y - 4x$ directly to find these same solutions. However, part of the purpose of the example was to showcase how algebraic manipulation techniques need to be remembered as you will need them as we get deeper in our study of solving various types of ODEs.

ATTENTION: In our last example, the ODE $\frac{dy}{dx} - xy^2 + 5y^3 = 20y - 4x$ with the constant solutions $y(x) = c = \pm 2$ is technically an IVP ODE because $y(x) = \pm 2$ can be interpreted as (separate) initial conditions for our given ODE.

So how can we determine if an IVP-type of 1st-order ODE actual has the potential to have at least 1 solution without having to actually go through the process of solving for these solutions first? Is there a way to know exactly how many solutions a 1st-order IVP ODE will yield? It turns out that we do have a way of answering these questions. The Existence + Uniqueness Theorem for 1st-order IVP ODEs addresses how to do it!

<u>Thm (Existence & Uniqueness of 1st-order IVP ODEs)</u>: The IVP problem of type ...

$$\frac{dy}{dx} = F(x,y), \text{ where } y(x_0) = y_0,$$

has exactly 1 solution on the (x-value) interval $x = (a, b)$ where $a < x_0 < b$ if the following 2 conditions are met:

(1) $F = F(x,y)$ is a continuous function on the open region R within the x-y plane where $R = \{(x,y) \mid x = (a,b) \text{ and } y = (-\infty, \infty)\}$ *

(2) $\frac{\partial F}{\partial y}$ is a continuous function on the open region R within the x-y plane where $R = \{(x,y) \mid x = (a,b) \text{ and } y = (-\infty, \infty)\}$ *

* point (x_0, y_0) is in this region

shaded region of points (x, y)

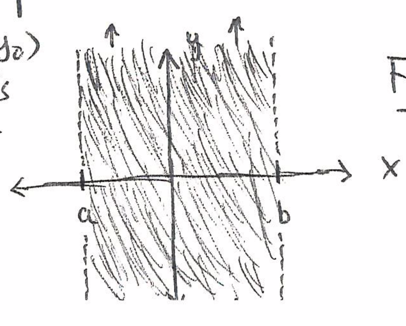

Figure 3.1

Figure 3.1 is a physical representation of what the region R looks like.

In summary, the translation of this theorem is that if $F(x,y)$ can be expressed as $\frac{dy}{dx}$, knowing that $y = y(x)$ is a constant function, then
① $F(x,y)$ will be (reasonably) "well-behaved" within the region R and
② $y(x)$ is the only solution for this IVP ODE if $y(x) = y_0$ is a distinct constant function corresponding to the point (x_0, y_0) in the region R.

We will look at our last example to see how we could practically apply this theorem.

Ex: Consider The ODE $\frac{dy}{dx} - xy^2 + 5y^3 = 20y - 4x$ which can be rewritten as $\frac{dy}{dx} = xy^2 - 5y^3 + 20y - 4x = F(x,y)$. We must show that
① $F(x,y)$ is continuous on the region of interest for our given ODE and ② $\frac{\partial F}{\partial y}$ is continuous on the region of interest for our given ODE.

Region of Interest: Our region of interest in the example is the entire x-y plane (i.e. $R = \{(x,y) \mid x = (-\infty, \infty)$ and $y = (-\infty, \infty)\}$ since their is no combo of x and/or y values that will make $\frac{dy}{dx}$ = DNE (i.e. $\frac{dy}{dx}$ will not be undefined). (✱) It follows from (1st-year) calculus that if $\frac{dy}{dx}$ is defined, then by definition of continuity, $\frac{dy}{dx} = F(x,y)$ is continuous.

F(x,y) continuous on R: Sentence (✱) above addresses verification of this fact.

$\frac{\partial F}{\partial y}$ is continuous on R : $\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[xy^2 - 5y^3 + 20y - 4x\right]$

$\therefore \frac{\partial F}{\partial y} = 2xy - 15y^2 + 20 - 0 = 2xy - 15y^2 + 20$. Again, there are no combos of x and/or y values that will make $\frac{\partial F}{\partial y}$ = DNE. Thus, from the definition of continuity, $\frac{\partial F}{\partial y}$ is a continuous function on R (as well).

NOTE: The phrase "well-behaved" for $F(x,y)$ in the context of this theorem means that $F(x,y)$ will behave as the derivative of a function $y(x)$ = constant function should for a distinct $y(x) = y_0$ as long as $x = x_0$ where x_0 is within the (x-value) interval of interest. The function $F(x,y)$ will not act like anything other than $\frac{dy}{dx} = 0$ provided that $y(x)$ meets the parameters/conditions of the aforementioned Existence & Uniqueness Theorem.

Now we will do some examples where we will find constant solutions of some ODEs as well as determine if the ODEs are directly integrable, autonomous, or neither.

Ex: Rewrite each given ODE in the form $\frac{dy}{dx} = F(x,y)$, and then find all constant solutions of the ODE, if any. Also, determine if the ODE is directly integrable, autonomous, or neither

a) $\frac{dy}{dx} + y^3 = 8$

$\therefore \frac{dy}{dx} = 8 - y^3 = f(y)$ (autonomous)

Let $y = y(x) = c$. Then $\frac{dy}{dx} = 0 \Rightarrow 0 = 8 - c^3 \Rightarrow 0 = -(c^3 - 8)$

$\Rightarrow c^3 - 8 = 0 \Rightarrow c^3 = 8 \Rightarrow c = \sqrt[3]{8} = 2$

$\therefore \boxed{y(x) = 2 \rightarrow \text{constant sol'n}}$

b) $\sin(x+y) - y\frac{dy}{dx} = 0$

$\therefore \sin(x+y) = y\frac{dy}{dx} \Rightarrow \underbrace{\frac{\sin(x+y)}{y}}_{= f(x,y)} = \frac{dy}{dx}$; $y \neq 0$

$\Rightarrow$ neither directly integrable or autonomous!

Let $y = y(x) = c$. Then $\frac{dy}{dx} = 0 \Rightarrow 0 = \frac{\sin(x+c)}{c}$, $c \neq 0$.

$\therefore 0 = \sin(x+c) \Rightarrow x + c = n\pi$, n = integer $\Rightarrow c = n\pi - x$; n = integer

But $n\pi - x$ is not a constant function. Therefore, our ODE does not have any constant function solutions!

$\therefore \boxed{y(x) = \text{none}}$

Ex: (cont'd)

c) $\frac{dy}{dx} + (8-x)y = y^2 - 8x \Rightarrow \frac{dy}{dx} = -(8-x)y + y^2 - 8x$

$\Rightarrow \frac{dy}{dx} = (-8+x)y + y^2 - 8x = -8y + xy + y^2 - 8x = (xy - 8x) + (y^2 - 8y)$

$\Rightarrow \frac{dy}{dx} = x(y-8) + y(y-8) \Rightarrow \frac{dy}{dx} = (x+y)(y-8) \Rightarrow$ neither directly integrable nor autonomous. It is actually separable!

$\therefore \frac{dy}{dx} = 0$ iff $x+y=0$ or $y-8=0$. Since $x+y=0 \Rightarrow y=-x \neq$ constant,

then the only constant solution occurs when $y-8=0 \Rightarrow \boxed{y=8}$ → only constant sol'n!

d) $(x-2)\cdot\frac{dy}{dx} = y+3 \Rightarrow \frac{dy}{dx} = \frac{y+3}{x-2} \Rightarrow$ neither directly integrable nor autonomous. Again, this ODE is actually separable

$\therefore \frac{dy}{dx} = 0$ iff $y+3=0 \Rightarrow \boxed{y=-3}$

→ this is our only constant solution!

Separable Equations

A 1^{st}-order ODE is called separable if $\frac{dy}{dx} = F(x,y)$ such that $F(x,y) = f(x) \cdot g(y)$

NOTE: If $F(x,y) = f(x) \cdot g(y)$, then $\frac{dy}{dx} = f(x) \cdot g(y)$ which implies that $h(y) \cdot dy = f(x) \cdot dx$, where $h(y) = \frac{1}{g(y)}$.

The steps for solving separable ODE's are as follows:

(1) Rewrite your ODE in the form of $\frac{dy}{dx} = f(x) \cdot g(y)$

(2) Identify all constant solutions y_n, where $n = 0, 1, 2, \dots$, such that $g(y_n) = 0$.

(3) Divide both sides of $\frac{dy}{dx} = f(x) \cdot g(y)$ by $g(y)$ so that it results in an equation of the form $h(y) \cdot y'(x) = f(x)$, where $h(y) = \frac{1}{g(y)}$, $g(y) \neq 0$

NOTE! You have already considered the cases where $g(y) = 0$ in step 2, so you can ignore this case now.

(4) Integrate both sides of the equation $h(y) \cdot y'(x) = f(x)$ with respect to x and note that $y'(x) = \frac{dy}{dx} \Rightarrow y'(x)\, dx = dy$. Therefore,

$$h(y) \cdot y'(x) = f(x) \Rightarrow h(y) \cdot y'(x) \cdot dx = f(x)\, dx \Rightarrow h(y)\, dy = f(x) \cdot dx$$

$$\therefore \underbrace{\int h(y)\, dy = \int f(x) \cdot dx}_{\text{no initial condition given}} \quad \underline{\text{OR}} \quad \int_{y(0)}^{y(x)} h(y) \cdot dy = \int_0^x f(s)\, ds, \; x = s.$$

(5) Solve $\int h(y)\cdot dy = \int f(x)\cdot dx$ for "y". If y is explicitly know, sub in y (in terms of x) into final answer for $y = y(x)$. Otherwise, leave the solution as an implicit one (i.e. express $y = f(x,y)$)

(6) See if constant solutions in step (2) can be combined with solution(s) in step (5) so that the solution(s) "y" can be expressed as one single function. If this is not possible, make sure to list all solutions of "y".

Ex: Determine if each equation is separable or not. If it is separable, rewrite each equation in separable form (i.e. $\frac{dy}{dx} = f(x)\cdot g(y)$)

a) $\frac{dy}{dx} = 3x - y\sin(x)$

$\frac{dy}{dx} = f(x,y) \neq f(x)\cdot g(y)$

NOT SEPARABLE

b) $\frac{dy}{dx} = \sin(x+y) = f(x,y) \neq f(x)\cdot g(y)$

NOT separable!

c) $\frac{dy}{dx} = xy - 3x - 2y + 6$

$\frac{dy}{dx} = x(y-3) - 2(y-3)$

$\frac{dy}{dx} = (x-2)(y-3) = f(x)\cdot g(y)$

SEPARABLE

d) $y\,\frac{dy}{dx} = e^{x-3y^2} = \frac{e^x}{e^{3y^2}}$

$\Rightarrow \frac{dy}{dx} = \frac{e^x}{e^{3y^2}}\cdot\frac{1}{y} = e^x\cdot\frac{1}{ye^{3y^2}} = f(x)\cdot g(y)$

SEPARABLE!

Ex: Find the general solution, including any constant solutions, if possible. Write the final answer as an explicit solution where possible as well. ③

a) $y\ln(x)\cdot\frac{dx}{dy}=\left(\frac{y+1}{x}\right)^2$

$\therefore\ y\cdot\ln(x)\cdot\frac{dx}{dy}=\frac{(y+1)^2}{x^2}\Rightarrow x^2\cdot\ln(x)\,\frac{dx}{dy}=\frac{(y+1)^2}{y}$

Let $x'(y)=\frac{dx}{dy}\Rightarrow x'(y)\,dy=dx$

$\therefore\ x^2\cdot\ln(x)\cdot x'(y)=\frac{(y+1)^2}{y}$. Multiply both sides by dy yields...

$x^2\cdot\ln(x)\cdot\underbrace{\cancel{x'(y)}\cdot dy}_{dx}=\frac{(y+1)^2}{y}\cdot dy\Rightarrow x^2\cdot\ln(x)\cdot dx=\frac{y^2+2y+1}{y}\,dy;\ y\neq 0$

$\therefore\ x^2\cdot\ln(x)\cdot dx=\left(y+2-\frac{1}{y}\right)dy$

NOTE: To identify the constant solutions, note that

$x^2\cdot\ln(x)\cdot dx=\left(y+2-\frac{1}{y}\right)dy\Rightarrow\frac{dy}{dx}=\frac{x^2\ln(x)}{y+2-\frac{1}{y}}=\left[x^2\ln(x)\right]\left[\frac{y}{y^2-2y-1}\right]$

$\therefore\ \frac{dy}{dx}=f(x)\cdot g(y)$, where $f(x)=x^2\ln(x)$ and $g(y)=\frac{y}{y^2-2y-1}$. It follows that $g(y)=0$ only when $y=0$. Thus, $y=y(x)=0$ is the only (trivial) constant solution to this equation.

So, $x^2\cdot\ln(x)\,dx=\left(y+2-\frac{1}{y}\right)dy\Rightarrow\underbrace{\int x^2\cdot\ln(x)\,dx}_{\text{integrate via integration by parts}}=\underbrace{\int\left(y+2-\frac{1}{y}\right)dy}_{\text{integrate normally}}$

Ex !

a) (Cont'd). Now we will evaluate the integral $\int x^2 \cdot \ln(x)\,dx$.

$$u = x^2 \qquad dv = \ln(x)\,dx \qquad \Rightarrow \int x^2 \cdot \ln(x)\,dx = \int u \cdot dv = uv - \int v\,du$$

$$du = 2x \cdot dx \qquad v = x\ln(x) - x$$

$$\therefore\ uv - \int v\,du = (x^2)(x\ln(x) - x) - \int \left[x\ln(x) - x\right] \cdot 2x \cdot dx =$$

$$x^3\ln(x) - x^3 - 2\int (x^2\ln(x) - x^2)\,dx = x^3\ln(x) - x^3 - 2\int x^2\ln(x)\,dx + 2\int x^2 d$$

$$x^3\ln(x) - x^3 - 2\int x^2\ln(x)\,dx + 2 \cdot \tfrac{1}{3}x^3$$

$$\therefore\ uv - \int v\,du = x^3\ln(x) - x^3 - 2\int x^2\ln(x)\,dx + \tfrac{2}{3}x^3 + C$$

$$\Rightarrow \int u\,dv = uv - \int v\,du \Rightarrow \int x^2\ln(x)\,dx = x^3\ln(x) - x^3 - 2\int x^2\ln(x)\,dx + \tfrac{2}{3}x^2 +$$

$$\therefore\ \int x^2\ln(x)\,dx + 2\int x^2\ln(x)\,dx = x^3\ln(x) - x^3 + \tfrac{2}{3}x^2 + C$$

$$\overset{3}{\cancel{(1+2)}}\int x^2\ln(x)\,dx = x^3\ln(x) - x^3 + \tfrac{2}{3}x^2 + C$$

$$\therefore\ \int x^2\ln(x)\,dx = \tfrac{1}{3}x^3\ln(x) - \tfrac{1}{3}x^3 + \tfrac{2}{9}x^2 + D, \text{ where } D = \tfrac{C}{3}$$

Now we will evaluate the integral $\int \left(y + 2 - \frac{1}{y}\right) dy = \frac{1}{2}y^2 + 2y - \ln|y| + F$

Finally, $\int x^2\ln(x)\,dx = \int \left(y + 2 - \frac{1}{y}\right) dy \Rightarrow \boxed{\tfrac{1}{3}x^3\ln(x) - \tfrac{1}{3}x^3 + \tfrac{2}{9}x^2 = \tfrac{1}{2}y^2 + 2y - \ln|y| + G \text{, where } G = F - D}$

b) $\frac{dy}{dx} = e^{3x+2y} = e^{3x} \cdot e^{2y} = f(x) \cdot g(y) \Rightarrow$ separable equation

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NOTE: Since $e^{3x} \neq 0$ for any $x \in \mathbb{R}$ and $e^{2y} \neq 0$ for any $y \in \mathbb{R}$, it follows that this ODE has no constant solutions!

Looking for non-constant solutions

$$\frac{dy}{dx} = e^{3x} \cdot e^{2y} \Rightarrow y'(x) = e^{3x} \cdot e^{2y} \Rightarrow y'(x)\,dx = e^{3x} \cdot e^{2y} \cdot dx$$

Since $\frac{dy}{dx} = y'(x) \Rightarrow y'(x)\,dx = dy$, we can say that $dy = e^{3x} \cdot e^{2y} \cdot dx$

$\therefore e^{-2y} \cdot dy = e^{3x} \cdot dx$. Now we will integrate both sides and solve for y.

$$\therefore \int e^{-2y}\,dy = \int e^{3x} \cdot dx \Rightarrow \frac{e^{-2y}}{-2} = \frac{e^{3x}}{3} + C \Rightarrow -6\left[\frac{e^{-2y}}{-2}\right] = -6\left[\frac{e^{3x}}{3} + C\right]$$

$$\Rightarrow 3e^{-2y} = -2e^{3x} - 6C \Rightarrow e^{-2y} = \frac{-2e^{3x} - 6C}{3} = -\tfrac{2}{3}e^{3x} - 2C$$

$$\therefore \ln(e^{-2y}) = \ln\left[-\tfrac{2}{3}e^{3x} - 2C\right] \Rightarrow -2y = \ln\left[-\tfrac{2}{3}e^{3x} + D\right]; \quad D = -2C = \text{const.}$$

Finally, $\boxed{y = y(x) = -\tfrac{1}{2}\ln\left[-\tfrac{2}{3}e^{3x} + D\right]}$

c) $\sec^2(x)\,dy + \csc(y)\,dx = 0$ (6)

$\therefore \sec^2(x)\,dy = -\csc(y)\,dx \Rightarrow \dfrac{dy}{-\csc(y)} = \dfrac{dx}{\sec^2(x)}$, where $y \neq 0$ and $x \neq \frac{(2n+1)\pi}{2}$; n = integer

$\therefore -\sin(y)\,dy = \cos^2(x)\,dx$

NOTE: $\cos(2x) = \cos^2(x) - \sin^2(x) = \cos^2(x) - (1-\cos^2(x)) = 2\cos^2(x) - 1$

$\therefore \cos(2x) = 2\cos^2(x) - 1 \Rightarrow \cos^2(x) = \dfrac{\cos(2x)+1}{2} = \frac{1}{2}[\cos(2x)+1]$.

$\therefore -\sin(y)\,dy = \frac{1}{2}[\cos(2x)+1]\,dx$. Therefore, integrating both with respect to x yields...

$\int -\sin(y)\,dy = \frac{1}{2}\int(\cos(2x)+1)\,dx \Rightarrow \cos(y) = \frac{1}{2}\left[\frac{1}{2}\sin(2x) + x\right] + C$

$\therefore \cos(y) = \frac{1}{4}\sin(2x) + \frac{1}{2}x + C \Rightarrow y = \arccos\left[\frac{1}{4}\sin(2x) + \frac{1}{2}x + C\right]$

So, we can express our answer in one of the following 3 ways

(1): $\cos(y) = \frac{1}{4}\sin(2x) + \frac{1}{2}x + C$

or

(2) $4\cos(y) = \sin(2x) + 2x + C$

or

(3) $\boxed{y = \arccos\left[\frac{1}{4}\sin(2x) + \frac{1}{2}x + C\right]}$ → this is y written explicitly in terms of x

Ex: Solve each IVP ODE. If possible, express each solution as an explicit solution.

a) $\frac{dx}{dt} = 4(x^2+1)$; $x(\frac{\pi}{4}) = 1$

let $x'(t) = \frac{dx}{dt}$. Then $x'(t) \cdot dt = dx$. Thus, $\overset{x'(t)}{\frac{dx}{\cancel{dt}}} \cdot \cancel{dt} = 4(x^2+1) \cdot dt \Rightarrow$

$x'\overset{dx}{\cancel{(t)\,dt}} = 4(x^2+1)\,dt \Rightarrow dx = 4(x^2+1)\,dt \Rightarrow \frac{dx}{4(x^2+1)} = dt$

$\therefore \int \frac{dx}{4(x^2+1)} = \int dt \Rightarrow \frac{1}{4}\tan^{-1}(x) = t + C \Rightarrow \tan^{-1}(x) = 4t + 4C$

$\therefore \tan[\tan^{-1}(x)] = \tan[4t+4C] \Rightarrow x = x(t) = \tan(4t + D)$; $D = 4C$

Apply Initial Condition $x(\frac{\pi}{4}) = 1$ (*)

$x(\frac{\pi}{4}) = 1 \Rightarrow \tan(4(\frac{\pi}{4}) + D) = 1 \Rightarrow \tan(\pi + D) = 1 \Rightarrow \pi + D = \frac{\pi}{4}$

(*) NOTE! The range of $\tan^{-1}(x) = \tan^{-1}(x(t))$ is $-\frac{\pi}{2} < \tan^{-1}(x(t)) < \frac{\pi}{2}$. Therefore, only (radian) angles in the range can be considered. This is why we do not consider ALL possibilities when $\tan(\pi + D) = 1$ (i.e. $\pi + D = \frac{\pi}{4} + n\pi$; n = integer).

$\therefore \pi + D = \frac{\pi}{4} \Rightarrow D = \frac{\pi}{4} - \pi = \frac{\pi}{4} - \frac{4\pi}{4} = \frac{-3\pi}{4}$

$\therefore$ $\boxed{x(t) = \tan(4t - \frac{3\pi}{4}) \text{, where } x(t) \text{ has no constant solutions.}}$

NOTE! Since $4(x^2+1) \neq 0$ for all $x \in \mathbb{R} \Rightarrow \frac{dx}{dt} \neq 0$ for $x \in \mathbb{R} \Rightarrow x = x(t)$ will never be constant.

d) $(x^2+1)\frac{dy}{dx} = y^2+1$. Let $y'(x) = \frac{dy}{dx} \Rightarrow y'(x)\,dx = dy$

$\therefore (x^2+1)\,y'(x) = y^2+1 \Rightarrow (x^2+1)\cdot \underbrace{\cancel{y'(x)\cdot dx}}_{dy} = (y^2+1)\,dx$

$\therefore (x^2+1)\cdot dy = (y^2+1)\,dx \Rightarrow \frac{dy}{y^2+1} = \frac{dx}{x^2+1} \Rightarrow \int \frac{dy}{y^2+1} = \int \frac{dx}{x^2+1}$

$\therefore \arctan(y) = \arctan(x) + C \Rightarrow \tan\left[\tan^{-1}(y)\right] = \tan\left[\tan^{-1}(x)+C\right]$

$\therefore y = y(x) = \tan\left[\tan^{-1}(x)+C\right] \Rightarrow$ non-constant solution!

NOTE: $\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$ is an alternative way our ODE can be written. Thus, $\frac{dy}{dx} = 0$ iff $y^2+1=0$. But $y^2+1>0$ for all $y \in \mathbb{R} \Rightarrow$ no constant solution!

e) $\frac{dy}{dx} = 3y^2 - y^2\sin(x) = [3-\sin(x)]\cdot y^2 \Rightarrow$ separable eqn

Let $\frac{dy}{dx} = y'(x) \Rightarrow dy = y'(x)\,dx$. Therefore, it follows that

$\frac{dy}{dx} = (3-\sin(x))\cdot y^2 \Rightarrow \underbrace{\cancel{y'(x)\cdot dx}}_{dy} = (3-\sin(x))\cdot y^2 \cdot dx \Rightarrow y^{-2}\cdot dy = (3-\sin(x))\cdot dx$

$\therefore \int y^{-2}\cdot dy = \int (3-\sin(x))\cdot dx \Rightarrow -\frac{1}{y} = 3x + \cos(x) + C \Rightarrow \boxed{y = y(x) = \frac{-1}{3x+\cos(x)+C}}$

NOTE: $\frac{dy}{dx} = (3-\sin(x))\cdot y^2 = 0$ iff $y^2 = 0 \Rightarrow y = 0$. Thus $y=0$ is the only constant solution for our ODE. Recall that $-1 \le \sin(x) \le 1$. Therefore, $(-1)(-1) \ge (-1)\cdot\sin(x) \ge (-1)(1) \Rightarrow 1 \ge -\sin(x) \ge -1 \Rightarrow 1+3 \ge 3-\sin(x) \ge 3-1$. Thus, $2 \le 3-\sin(x) \le 4$. So, $3-\sin(x) > 0$ for all $x \in \mathbb{R}$. We can't get any constant solutions from $3-\sin(x)$ factor of $\frac{dy}{dx}$!

b) $\sqrt{1-y^2}\,dx - \sqrt{1-x^2}\,dy = 0$; $y(0) = \frac{\sqrt{3}}{2}$ (9)

$\sqrt{1-y^2}\,dx = \sqrt{1-x^2}\,dy \Rightarrow \frac{dx}{\sqrt{1-x^2}} = \frac{dy}{\sqrt{1-y^2}} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ *

NOTE: $\frac{dy}{dx} = 0$ iff $\sqrt{1-y^2} = 0 \Rightarrow 1-y^2 = 0 \Rightarrow 1 = y^2 \Rightarrow y = \pm 1$

$\therefore$ $\boxed{y = \pm 1 \text{ are the 2 constant solutions}}$

* Breaking notation rules here for simplicity and to save space

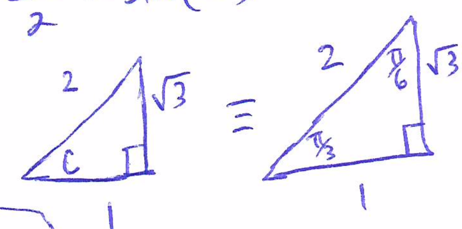
Looking for non-constant solutions

$\frac{dx}{\sqrt{1-x^2}} = \frac{dy}{\sqrt{1-y^2}} \Rightarrow \int \frac{dx}{\sqrt{1-x^2}} = \int \frac{dy}{\sqrt{1-y^2}} \Rightarrow \arcsin(x) = \arcsin(y) - C$

$\therefore \sin[\arcsin(x) + C] = \sin[\arcsin(y)] \Rightarrow y = y(x) = \sin[\arcsin(x) + C]$

Applying initial condition $y(0) = \frac{\sqrt{3}}{2}$

$y(0) = \frac{\sqrt{3}}{2} \Rightarrow \frac{\sqrt{3}}{2} = \sin[\underbrace{\arcsin(0)}_{0} + C] = \sin(C) \Rightarrow \frac{\sqrt{3}}{2} = \sin(C) \Rightarrow C = \frac{\pi}{3}$



$\therefore$ $\boxed{y = y(x) = \sin\left[\arcsin(x) + \frac{\pi}{3}\right] \text{ is the non-constant solution}}$

NOTE: $\arcsin(x) + \frac{\pi}{3}$ would need to equal $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ in order for $y(x) = \pm 1$

Specifically, $\arcsin(x) + \frac{\pi}{3} = \frac{\pi}{2} \Rightarrow \arcsin(x) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$ and $\arcsin(x) + \frac{\pi}{3} = \frac{3\pi}{2}$

$\Rightarrow \arcsin(x) = \frac{3\pi}{2} - \frac{\pi}{3} = \frac{9\pi - 2\pi}{6} = \frac{7\pi}{6}$ in order for $y(x) = 1$ and $y(x) = -1$, respectively

Thus, our non-constant solution includes the constant solutions specifically when $x = \frac{1}{2}$ or $\frac{-1}{2}$!!

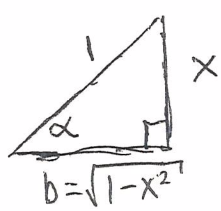
Ex:

b) (cont'd). Show that the solution for example (b) on the previous page can also be expressed as $y(x) = \frac{1}{2}x + \frac{\sqrt{3}}{2}\sqrt{1-x^2}$; $x = (-1, 1)$

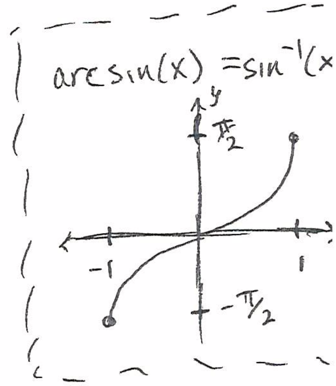
Sol'n: From example (b) on the previous page, we see that the form of the solution $y(x)$ is $y(x) = \sin\left[\arcsin(x) + \frac{\pi}{3}\right]$ (in explicit terms of x

Let $\alpha = \arcsin(x) \Rightarrow \sin(\alpha) = x$; $x = [-1, 1]$ and $\alpha = [-\frac{\pi}{2}, \frac{\pi}{2}]$

Using (visual) right angle trigonometry, we can express $\sin(\alpha) = x$ in terms of sides of a right triangle.



$\therefore b^2 + x^2 = (1)^2 \Rightarrow b = \sqrt{1-x^2}$



Since we let $\alpha = \arcsin(x) \Rightarrow y(x) = \sin\left[\alpha + \frac{\pi}{3}\right]$. Using the Addition formula $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$, we see that we can express our (particular) solution as $y(x) = \sin(\alpha)\cdot\cos(\frac{\pi}{3}) + \cos(\alpha)\cdot\sin(\frac{\pi}{3})$

But $\sin(\alpha) = x$, $\cos(\frac{\pi}{3}) = \frac{1}{2}$, $\cos(\alpha) = \sqrt{1-x^2}$, and $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

$\therefore y(x) = x\cdot\frac{1}{2} + \sqrt{1-x^2}\cdot\frac{\sqrt{3}}{2} \Rightarrow \boxed{y(x) = \frac{1}{2}x + \frac{\sqrt{3}}{2}\sqrt{1-x^2}}$

Finally, note that the graph of $\arcsin(x)$ has endpoints at $x = \pm 1$. From calculus, we know that $\frac{d}{dx}[\arcsin(x)] = \text{DNE}$ @ $x = \pm 1$. Thus, the interval of interest will be $x = (-1, 1)$, not $x = [-1, 1]$. So, our final answer is

$$\boxed{y(x) = \frac{1}{2}x + \frac{\sqrt{3}}{2}\sqrt{1-x^2} \; ; \; x = (-1, 1)}$$

c) $x\,\frac{dy}{dx} = y^2 - y$; $y(2) = 1$

Let $y'(x) = \frac{dy}{dx} \Rightarrow y'(x)\cdot dx = dy$. Also, let $y^2 - y = y(y-1)$.

Thus, $\frac{dy}{dx} = \frac{y^2-y}{x} = \frac{y(y-1)}{x}$. So, $\frac{dy}{dx} = 0$ when $y=0$ or $y=1$

$\therefore$ $\boxed{y = 0, 1 \text{ are the constant solutions.}}$ (*) We will see if any of these solutions are included in the non-constant solution results later.

Looking for non-constant solutions

$x\cdot\frac{dy}{dx} = y(y-1) \Rightarrow x\cdot y'(x) = y(y-1) \Rightarrow x\cdot \cancel{y'(x)}^{dy}\cdot dx = y(y-1)\cdot dx$

$\therefore$ $x\cdot dy = y(y-1)\cdot dx \Rightarrow \frac{dy}{y(y-1)} = \frac{dx}{x} \Rightarrow \left(\frac{A}{y} + \frac{B}{y-1}\right)dy = \frac{1}{x}\,dx$,

where $A + B$ are constants we will find via Partial Fraction Expansion.

$$\therefore\ A = \frac{1}{y(y-1)}\cdot y\Big|_{y=0} = \frac{1}{y-1}\Big|_{y=0} = \frac{1}{0-1} = -1$$

$$B = \frac{1}{y(y-1)}\cdot(y-1)\Big|_{y=1} = \frac{1}{y}\Big|_{y=1} = \frac{1}{1} = 1$$

$$\therefore\ \frac{A}{y} + \frac{B}{y-1} = \frac{-1}{y} + \frac{1}{y-1}$$

$\therefore \left(\frac{1}{y-1} - \frac{1}{y}\right)dy = \frac{1}{x}\,dx \Rightarrow \int\left(\frac{1}{y-1} - \frac{1}{y}\right)dy = \int\frac{1}{x}\,dx \Rightarrow \ln|y-1| - \ln|y| = \ln|x| + c$

$\Rightarrow \ln\left|\frac{y-1}{y}\right| = \ln|x| + c \Rightarrow e^{\ln\left|\frac{y-1}{y}\right|} = e^{\ln|x|}\cdot e^{c} \Rightarrow \left|\frac{y-1}{y}\right| = |x|\cdot A$, $A = e^c = \text{const.}$

Since $\frac{dy}{dx} = \frac{y^2-y}{x} = f(x)\cdot g(y) \Rightarrow g(y) = y^2 - y = x$ is a function $\Rightarrow y \geq \frac{1}{2}$ + $x \geq \frac{-1}{4}$ since allowing $y < \frac{1}{2}$ as well would violate the vertical line test for the graph of $y^2 - y = x$

$\therefore \left|\frac{y-1}{y}\right| = |x|\cdot A \Rightarrow \frac{y-1}{y} = A|x| \Rightarrow y - 1 = A|x|y \Rightarrow y - A|x|y = 1 \Rightarrow y[1 - A|x|] = 1$

d) $\dfrac{dy}{dx} = \dfrac{y^2-1}{xy}$; $y(1) = -2$

$\therefore$ Let $y'(x) = \frac{dy}{dx}$. Then, $dy = y'(x) \cdot dx$.

$\therefore$ $y'(x) = \dfrac{y^2-1}{xy} \Rightarrow \cancel{y'(x)}\,dx = \dfrac{y^2-1}{xy}\,dx \Rightarrow dy = \dfrac{y^2-1}{xy}\,dx$

$\Rightarrow \dfrac{y \cdot dy}{y^2-1} = \dfrac{1}{x}\,dx$. We choose to perform integration by using definite integrals since we have initial conditions. In order to set this up properly, we need to do the following:

- Let $x = s$ and $w = y = y(x)$ $\Rightarrow dx = ds$ and $dw = y'(x)\,dx$
- Integrate both sides of equation from $s=1$ to $s=x$ (which is equivalent to $w = y(1) = -2$ to $w = y(x)$ for the side $w = y(x)$ substitution is used)

$\therefore$ $\dfrac{y\,dy}{y^2-1} = \dfrac{1}{x}\,dx \Rightarrow \displaystyle\int_1^x \frac{y\,dy}{y^2-1} = \int_1^x \frac{1}{x}\,dx \Rightarrow \int_{\cancel{y(1)}=-2}^{y(x)} \frac{w\,dw}{w^2-1} = \int_1^x \frac{1}{s}\,ds$

NOTE: Using substitution rule, let $u = w^2-1$. Then, $du = 2w \cdot dw \Rightarrow$ $\frac{1}{2}du = w \cdot dw$. So, $\displaystyle\int_{-2}^{y(x)} \frac{w\,dw}{w^2-1} = \int_{u(-2)}^{u(y(x))} \frac{\frac{1}{2}du}{u} = \frac{1}{2}\int_3^{[y(x)]^2-1} \frac{1}{u}\,du = \frac{1}{2}\Big[\ln|u|\Big]_3^{[y(x)]^2-1} =$ $\frac{1}{2}\Big[\ln|y^2(x)-1| - \ln|3|\Big] = \frac{1}{2}\ln\left|\dfrac{y^2(x)-1}{3}\right| = \ln\left|\sqrt{\dfrac{y^2(x)-1}{3}}\right|$

$\therefore$ $\displaystyle\int_{-2}^{y(x)} \frac{w\,dw}{w^2-1} = \int_1^x \frac{1}{s}\,ds \Rightarrow \ln\left|\sqrt{\frac{y^2(x)-1}{3}}\right| = \Big[\ln|s|\Big]_1^x \Rightarrow \ln\left|\sqrt{\frac{y^2(x)-1}{3}}\right| = \ln|x| - \cancelto{0}{\ln|1|}$

$\Rightarrow \sqrt{\dfrac{y^2(x)-1}{3}} = x$, where $x \neq 0$.

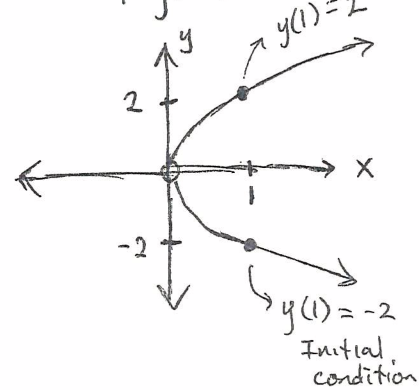
Ex: (d) cont'd

$$\therefore \sqrt{\frac{y^2(x)-1}{3}} = x \Rightarrow \frac{y^2(x)-1}{3} = x^2 \Rightarrow y^2(x) - 1 = 3x^2$$

$$\therefore y^2(x) = 3x^2 + 1 \Rightarrow y(x) = \pm\sqrt{3x^2+1}, \quad x \neq 0$$

NOTE: Since $y(x)$ has to be a function, we have to determine if $y(x)$ should be positive or negative because $y(x)$ in its current form violates the vertical line test for functions. Please see Fig. 2 below of the graph of $y(x) = \pm\sqrt{3x^2+1}$, $x \neq 0$. Since our initial condition implies that our y-values for this function can be negative $\Rightarrow y(x) = -\sqrt{3x^2+1}$, $x \neq 0$ will satisfy all conditions our original ODE $\frac{dy}{dx} = \frac{y^2-1}{xy}$ with initial condition that $y(1) = -2$

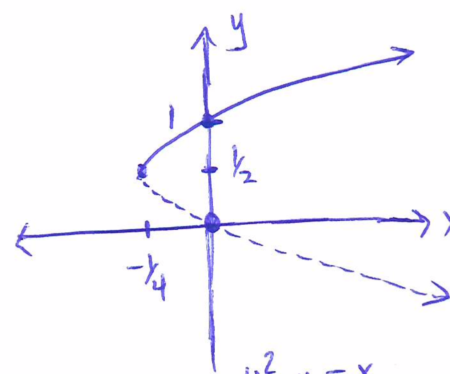
Fig. 2



Final answer: $\boxed{y(x) = -\sqrt{3x^2+1}\ ;\ x \neq 0}$

c) cont'd

$\therefore y(x) = \dfrac{1}{1 - A|x|}$, where $x \geq -\frac{1}{4}$ and $y \geq \frac{1}{2}$



$y^2 - y = x$
$\hookrightarrow y \geq \frac{1}{2}$; $x \geq -\frac{1}{4}$

Apply initial condition $y(2) = 1$

$y(2) = 1 \Rightarrow \dfrac{1}{1 - A|2|} = 1$

$\therefore \dfrac{1}{1-2A} = 1 \Rightarrow 1 = 1 - 2A \Rightarrow 0 = -2A \Rightarrow \boxed{A = 0}$

$\therefore y(x) = \dfrac{1}{1 - 0|x|} = \frac{1}{1} \equiv 1$ (constant solution)

(*) Note that we found that $y = y(x) = 0, 1$ were constant solutions of this ODE. We now see that $y(x) = 0, 1$ are the <u>ONLY</u> solutions to the IVP ODE !!