Solving N^{th}-order Homogeneous ODEs w/ Constant Coefficients

The purpose of this set of notes is to extend our knowledge of solving 2^{nd}-order linear homogeneous ODEs to N^{th}-order linear homogeneous ODEs that we can solve. Many of these types of ODEs are difficult to solve mainly because of the difficulty in successfully factoring out the characteristic equation for these ODEs. We will confined our examples to N^{th}-order Linear Homogeneous ODEs that have characteristic (polynomial) equations that can be factored out by using any one (or a combination) of the following techniques traditionally taught (or reviewed) in a Precalculus course

- Traditional Factoring (i.e. Trial-and Error, By "Grouping", or Divide + Squeeze
- Completing the Square (to factor irreducible expression using complex #s)
- Rational Zeros Theorem

Note that it is well-known that all polynomials can be factored completely although it may be very tedious to do it! Recalling some of the factoring techniques above and employing them for these type of problems is really the motivation for this lesson.

Thm 1: Let r be a root of multiplicity "m" to the characteristic polynomial for ...

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \dots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0,$$

where $a_0, a_1, \dots, a_N$ are all constants. Then the set ...

$$\{e^{rx}, xe^{rx}, x^2 e^{rx}, \dots, x^{m-1} e^{rx}\}$$

forms a basis of "m" solutions (that are all linearly independent of each other)

Thm 2: Consider the same ODE, characteristic (polynomial) equation, root r of multiplicity "m", and the basis of solutions $\{e^{rx}, xe^{rx}, \dots, x^{m-1} e^{rx}\}$.

(a) If $r = \lambda + \omega i$ (i.e. r is a complex root), then it must be true that its conjugate $\bar{r} = \lambda - \omega i$ is also a root of our ODE in question

(b) If $r = \lambda \pm \omega i$ is a complex root of multiplicity "m" to the corresponding characteristic equation of this ODE, then the sets $\{e^{(\lambda+\omega i)x}, xe^{(\lambda+\omega i)x}, \dots, x^{m-1} e^{(\lambda+\omega i)x}\}$ and $\{e^{(\lambda-\omega i)x}, xe^{(\lambda-\omega i)x}, \dots, x^{m-1} e^{(\lambda-\omega i)x}\}$ form a basis of "m" linearly independent solutions corresponding to $r = \lambda \pm \omega i$!

Thm 3: Considering the same situation as in Thm 2, the sets of linearly independent solutions of the form $e^{(\lambda \pm \omega i)x}$ can be alternatively expressed in the form $e^{\lambda x}[\cos(\omega x) + \sin(\omega x)]$. Thus, our ODE in Thm 2 ensures that ...

$$\{e^{\lambda x}\cos(\omega x),\ xe^{\lambda x}\cos(\omega x),\ x^2e^{\lambda x}\cos(\omega x),\ \ldots,\ x^{m-1}e^{\lambda x}\cos(\omega x)\}$$

and

$$\{e^{\lambda x}\sin(\omega x),\ xe^{\lambda x}\sin(\omega x),\ x^2e^{\lambda x}\sin(\omega x),\ \ldots,\ x^{m-1}e^{\lambda x}\sin(\omega x)\}$$

are sets of "m" linearly independent solutions to our N^{th}-order Linear ODE with constant coefficients

NOW WE SHALL DO A FEW EXAMPLES TO GET PRACTICE USING THESE FACTORING TECHNIQUES AND FACTS TO SOLVE N^{th}-ORDER LINEAR ODEs w/ CONSTANT COEFFICIENTS!

Ex. 1: Solve the following homogeneous ODE via high-degree factor techniques.

$$y^{(5)} + 10y^{(4)} + 40y''' + 80y'' + 80y' + 32y = 0$$

Char eqn: $r^5 + 10r^4 + 40r^3 + 80y^2 + 80y + 32 = 0$

Using Rational Zeros Thm, the potential zeros for this char (poly.) equation are...

$$r = \frac{\{\pm \text{ all factors of } 32\}}{\{\pm \text{ all factors of } 1\}} = \frac{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32}{\pm 1}$$

$$\therefore r = \pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32$$

NOTE: To find which of these "potential zeros" of our char. (poly.) are actually zeros, you can substitute each of them for r into this polynomial to test it out. It turns out $r=-2$ ends up being the only rational zero we have because if we let $f(r)$ = our char. poly. equation, then, ...

$$f(-2) = (-2)^5 + 10(-2)^4 + 40(-2)^3 + 80(-2)^2 + 80(-2) + 32$$
$$" = -32 + 160 - 320 + 320 - 160 + 32$$
$$f(-2) = 0$$

Trying $r=-2$ and using synthetic division to factor down our char. poly. equation, we get...

⊛ (Notice that we can evaluate poly. by doing synthetic division. See that $f(r)$ = remainder from synthetic div.)

-2	1	10	40	80	80	32
	↓	-2	-16	-48	-64	-32
	1	8	24	32	16	0 $= f(-2)$ *
	r^4	r^3	r^2	r	#	

$$\Rightarrow f(\;) = (r+2)\underbrace{(r^4 + 8r^3 + 24r^2 + 32r + 16)}_{\text{Need to factor this down now}}$$

Since we only have $r=-2$ as a rational zero, we check to see if this zero's multiplicity is larger than 1. Therefore, employing synthetic division again ...

$$\begin{array}{r|rrrrr} -2 & 1 & 8 & 24 & 32 & 16 \\ & \downarrow & -2 & -12 & -24 & -16 \\ \hline & 1 & 6 & 12 & 8 & \boxed{0} \\ & r^3 & r^2 & r & \# & \end{array}$$

Notice that we are using the coefficients from the quotient of our 1st round of synthetic division. This is how we use this tool to factor down a polynomial of high degree

$$\therefore f(r) = (r+2)(r+2)\underbrace{(r^3 + 6r^2 + 12r + 8)}_{\text{Need to factor this down now}}$$

Repeating again ...

$$\begin{array}{r|rrrr} -2 & 1 & 6 & 12 & 8 \\ & \downarrow & -2 & -8 & -8 \\ \hline & 1 & 4 & 4 & \boxed{0} \\ & x^2 & x & \# & \end{array}$$

$$\Rightarrow f(r) = (r+2)(r+2)(r+2)\overbrace{(r^2+4r+4)}^{\text{Can factor this with traditional method of factoring}}$$

$$f(\ ) = (r+2)(r+2)(r+2)(r+2)(r+2) = (r+2)^5$$

$\therefore$ $r = -2$ is a zero of multiplicity 5

$\Rightarrow \{e^{-2x}, xe^{-2x}, x^2e^{-2x}, x^3e^{-2x}, x^4e^{-2x}\}$ is the set of linearly independent fundamental solutions for our original 5^{th}-order Linear Homogeneous ODE!

$\therefore$ (General) Homogeneous sol'n

$$\boxed{y(x) = c_1 e^{-2x} + c_2 xe^{-2x} + c_3 x^2 e^{-2x} + c_4 x^3 e^{-2x} + c_5 x^4 e^{-2x}}$$

Ex. 2 : Solve the following homogeneous ODE via factoring techniques.

$$y''' - 27y = 0 \qquad \text{Assume } y = y(x).$$

Char eqn: $r^3 - 27 = 0 \Rightarrow r^3 - (3)^3 = 0 \Rightarrow (r-3)(r^2+3r+9) = 0$

$\therefore\ r - 3 = 0$ or $r^2 + 3r + 9 = 0$

$\boxed{r = 3}$ or $r_{1,2} = \dfrac{-3 \pm \sqrt{(3)^2 - 4(1)(9)}}{2(1)} = \dfrac{-3 \pm \sqrt{-27}}{2} = \dfrac{-3 \pm 3\sqrt{3}\,i}{2}$

$$\therefore (r-3)(r^2+3r+9) = (r-3)\left[r - \left(-\tfrac{3}{2} + \tfrac{3\sqrt{3}}{2}i\right)\right]\left[r - \left(-\tfrac{3}{2} - \tfrac{3\sqrt{3}}{2}i\right)\right] = 0$$

$\Rightarrow r = 3,\ -\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i \Rightarrow r_1 = 3,\ r_2 = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i,\ r_3 = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$

$\Rightarrow r_2 = \lambda_2 + \omega_2 i \qquad \Rightarrow r_3 = \lambda_3 - \omega_3 i$

NOTE: $r_1, r_2, + r_3$ all are roots of multiplicity 1

$\therefore$ Homogeneous Sol'n

From $r_1 = 3$: $y_1 = c_1 e^{3x}$

From $r_{2,3} = -\frac{3}{2} \pm \frac{3\sqrt{3}}{2} i = \lambda_2 \pm \omega_2 i$: $y_2 = e^{\lambda_2 x}\left[c_2 \cos(\omega_2 x) + c_3 \sin(\omega_2 x)\right]$

$$\therefore \boxed{y = y_1(x) + y_2(x) = c_1 e^{3x} + e^{-\frac{3}{2}x}\left[c_2 \cos\left(\frac{3\sqrt{3}}{2}x\right) + c_3 \sin\left(\frac{3\sqrt{3}}{2}x\right)\right]}$$

Ex. 3 : Solve the following homogeneous ODE using factoring techniques.

$$y^{(4)} - 5y''' - 9y'' + 81y' - 108y = 0$$

Char eqn : $r^4 - 5r^3 - 9r^2 + 81r - 108 = 0 = f(r)$.

Using Rational Zeros Thm (RZT), we will find a list of potential (rational #) zeros that f(r) could have. We will test out these zeros to see which ones are actually zeros. Any rational zero we find will be tested to see if its multiplicity is higher than 1 if we do not find 4 distinct zeros from this list.

$\therefore$ RZT potential zeros : $r = \dfrac{\{\pm \text{ factors of } 108\}}{\{\pm \text{ factors of } 1\}} = \dfrac{\pm1, \pm2, \pm3, \pm4, \pm6, \pm9, \pm12, \pm18, \pm27, \pm36, \pm54, \pm108}{\pm1}$

$\therefore$ potential zeros $r = \pm1, \pm2, \pm3, \pm4, \pm6, \pm9, \pm12, \pm18, \pm27, \pm36, \pm54, \pm108$

Using Technology to "quickly" find out which of these potential zeros are actual zeros ; we see that our list is $r = 3, -4$

Since $r=3$ and $r=-4$ do not total up to 4 zeros (as this is needed to match the degree of our characteristic polynomial), we will test both of these rational zeros to see if there multiplicity is higher than 1.

$\therefore$ Testing $r = 3$:

3	1	−5	−9	81	−108
	↓	3	−6	−45	108
	1	−2	−15	36	0
	r^3	r^2	r	#	

This just shows that $r=3$ is an actual zero of $f(r)$. The actual testing for multiplicity >1 comes next

$\therefore f(r) = (r-3)(r^3 - 2r^2 - 15r + 36)$

Testing $r = 3$ for multiplicity > 1

Can be factored by traditional techniques

$$\begin{array}{r|rrrr|r} 3 & 1 & -2 & -15 & 36 \\ & \downarrow & 3 & 3 & -36 \\ \hline & 1 & 1 & -12 & 0 \\ & r^2 & r & \# & \end{array} \Rightarrow f(r) = (r-3)(r-3)(r^2 + r - 12)$$

$\therefore r = 3$ has a multiplicity of at least 2

$$\therefore f(r) = (r-3)^2(r^2 + r - 12) = (r-3)^2(r+4)(r-3) = (r-3)^3(r+4) = 0$$

$\therefore r = 3, -4$

So, $r = 3$ has a multiplicity of 3 and $r = -4$ has a multiplicity of 1!

- From $r = 3$ (multiplicity 3): $\{e^{3x}, xe^{3x}, x^2e^{3x}\}$ is the set of linearly independent functions that form a basis of solutions for our 4^{th}-order Linear Homogeneous ODE corresponding to this root!

- From $r = -4$ (multiplicity 1): $\{e^{-4x}\}$ is set of linearly independent function(s) that form of a basis of solutions for our aforementioned ODE corresponding to this root!

$\therefore$ Final answer: $\boxed{y(x) = c_1e^{3x} + c_2xe^{3x} + c_3x^2e^{3x} + c_4e^{-4x}}$

Ex. 4: Using the fact that $r^4-4r^3+14r^2-20r+25$ can be factored into $[(r-1)-2i]^2 \cdot [(r-1)+2i]^2$, find a general solution to the ODE: $y^{(4)}-4y'''+14y''-20y'+25y=0$, where $y=y(t)$. (7)

For our ODE, the char. (poly.) eqn. is $r^4-4r^3+14r^2-20r+25=0$

$\therefore r^4-4r^3+14r^2-20r+25=[(r-1)-2i]^2[(r-1)+2i]^2=0$

$\therefore [(r-1)-2i]^2=0$ or $[(r-1)+2i]^2=0$

$\Rightarrow [r-(1+2i)]^2=0$ or $[r-(1-2i)]^2=0$

$\Rightarrow r-(1+2i)=0$ or $r-(1-2i)=0$

$\Rightarrow r=1+2i$ (multiplicity 2) or $r=1-2i$ (multiplicity 2)

For $r_1=1+2i$ (mult. 2): $\{e^t\cos(2t), e^t\sin(2t), te^t\cos(2t), te^t\sin(2t)\}$ is the set of linearly independent fundamental solutions for our given ODE.

For $r_2=1-2i$ (mult. 2): same set of fundamental, linearly independent solutions as $r_1=1+2i$ since r_1 and r_2 are (complex) conjugates of each other.

$\therefore \boxed{y(t)=c_1e^t\cos(2t)+c_2e^t\sin(2t)+c_3te^t\cos(2t)+c_4te^t\sin(2t)}$

$\Rightarrow \boxed{y(t)=(c_1+c_3t)e^t\cos(2t)+(c_2+c_4t)e^t\sin(2t)}$