Solving 2^{nd}-order Linear Homogeneous ODEs w/ Constant Coefficients

The purpose of this set of notes is to apply the "Big Theorem for 2^{nd}-order Linear Homogeneous ODEs" to the sub-class of ODEs of the type...

$$ay'' + by' + cy = 0 \quad , \text{where } a,b,c \in \mathbb{R} \text{ and } a \neq 0.$$

In words, this type of ODE is a 2^{nd}-order Linear Homogeneous ODE w/ constant coefficients. This type of ODE inherently meets all the conditions of the "Big Theorem..." and solutions will be valid on $x = (-\infty, \infty)$.

Just as we did in the previous set of notes, we will find suitable solutions for this class of ODEs by making inferences (i.e. good, educated guesses) on what the solution (or solutions) might be and testing them out to verify our guesses.

It turns out that for these type of ODEs, one such function that is a robust (starting-point for a) solution is $y = y(x) = e^{rx}$, where r can be a real and/or complex number! We will verify this solution in order to discern when r is real, when r is complex, and discern what type of general solutions we will end up with.

After going through this verification process, we will resort to a summarized list of possible general solutions based upon what our constant "r" turns out to be!

Verification of $y = e^{rx}$ as a solution to $ay'' + by' + cy = 0$

Let $y = e^{rx}$. Then, $y' = re^{rx}$ and $y'' = r^2 e^{rx}$

$\therefore\ ay'' + by' + cy = 0 \Rightarrow a[r^2 e^{rx}] + b[re^{rx}] + c[e^{rx}] = 0$

$\therefore\ e^{rx}[ar^2 + br + c] = 0 \Rightarrow ar^2 + br + c = 0$ since $e^{rx} \neq 0$

$\therefore\ ar^2 + br + c = 0 \Rightarrow \boxed{r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

NOTE: At this point our derivation is communicating to us that the way our general solutions will look shall depend upon what our roots to our quadratic equation $ar^2 + br + c = 0$ look like. In particular, it is the discriminant, $b^2 - 4ac$, that really determines what our solution to our ODE will look like. We will need to find what $y(x)$ will look like considering 3 different cases:

(1) $b^2 - 4ac > 0$ (i.e. r_1 + r_2 are 2 real, distinct numbers)
(2) $b^2 - 4ac = 0$ (i.e. $r_1 = r_2 \Rightarrow r_1$ is a double root)
(3) $b^2 - 4ac < 0$ (i.e. r_1 + r_2 will be complex conjugate solutions)

Case (1): $b^2-4ac>0$

If $b^2-4ac>0$, then r_1 and r_2 will be 2 real, distinct numbers (i.e. $r_1, r_2 \in \mathbb{R}$ such that $r_1 \neq r_2$). In this case, $y=y(x)=c_1y_1+c_2y_2$, where $y_1=e^{r_1x}$ and $y_2=e^{r_2x}$. Thus, ...

$$\boxed{\text{General sol'n}: \; y=y(x)=c_1e^{r_1x}+c_2e^{r_2x}}$$

Case (2): $b^2-4ac=0$

If $b^2-4ac=0$, then $r_1=r_2=\frac{-b}{2a}$. Therefore, this method of verifying our (proposed) solution only yields one solution, $y_1=e^{r_1x}=e^{r_2x}$.

CONNECTION: Recall that for 2nd-order Linear Homogeneous ODEs, we expect to have at least 1 pair of solutions that will form a fundamental set of solutions (i.e. this pair will form a basis of solutions). Since we don't have the second function in this expected pair, we can use the method of reduction of order to find out our possible second function to complete our pair.

So, let $y_1=e^{r_1x}$ and $y_2=u\cdot e^{r_1x}$, where $u=u(x)$ (i.e. u is a function of x). We need to find what $u=u(x)$ will be so that we can (eventually) find $y_2=y_2(x)$.

Case (2): $b^2-4ac=0$ (cont'd)

$$\therefore \boxed{y_2 = ue^{r_1x}} \Rightarrow y_2' = u'e^{r_1x} + u[r_1e^{r_1x}] = \boxed{e^{r_1x}[u'+r_1u] = y_2'}$$

$$\Rightarrow y_2'' = [r_1e^{r_1x}][u'+ur_1] + [e^{r_1x}][u''+r_1u']$$

$$\Rightarrow y_2'' = e^{r_1x}[u'r_1 + ur_1^2] + e^{r_1x}[u''+r_1u']$$

$$\Rightarrow \boxed{y_2'' = e^{r_1x}[u'' + 2r_1u' + ur_1^2]}$$

$$\therefore ay_2'' + by_2' + cy_2 = 0$$

$$\Rightarrow a[e^{r_1x}(u''+2r_1u'+ur_1^2)] + b[e^{r_1x}(u'+r_1u)] + c[ue^{r_1x}] = 0$$

$$\Rightarrow e^{r_1x}[au''+2ar_1u'+ar_1^2u] + e^{r_1x}[bu'+br_1u] + e^{r_1x}[cu] = 0$$

$$\Rightarrow e^{r_1x}[(a)u'' + (2ar_1+b)u' + (ar_1^2+br_1+c)u] = 0$$

$$\Rightarrow (a)u'' + (2ar_1+b)u' + (ar_1^2+br_1+c)u = 0 \quad \text{since } e^{r_1x} \neq 0$$

<u>NOTE 1:</u> Recall that $ar^2+br+c=0 \Rightarrow r_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\Rightarrow r_1 = r_2 = -\frac{b}{2a}$ when $b^2-4ac=0$. Thus, $ar_1^2+br_1+c=0$!!!

So, just as we expected with this method, the terms with the "u"'s in it would cancel to 0 <u>and</u> our resulting equation will be separable since we are dealing with a 2nd-order homogeneous eqn!

$\therefore (a)u'' + (2ar_1 + b)u' + \underbrace{(ar_1^2 + br_1 + c)}_{0}u = 0 \quad [*]$

$\Rightarrow (a)u'' + (2ar_1 + b)u' = 0$

NOTE 2: Recall that $r_1 = \frac{-b}{2a}$. Thus, $2ar_1 + b = 2a\left(\frac{-b}{2a}\right) + b = 0$!

So, it turns out for this case that both the "u" and "u'" terms cancel to zero for equation [*] above!

$\therefore (a)u'' + \underbrace{(2ar_1 + b)}_{0}u' = 0 \Rightarrow (a)u'' = 0 \Rightarrow \underbrace{u'' = 0}_{\text{directly-integrable!!}}$

$\therefore u'' = 0 \Rightarrow u' = K_1 \Rightarrow u = K_1 x + K_2$, where $K_1, K_2 \in \mathbb{R}$.

For simplicity, we let $u = u(x) = x$ because we know that K_1 & K_2 are constants that will be "recovered" within finding constants c_1 and c_2 in our general solution $y = y(x) = c_1 y_1 + c_2 y_2$.

So, $y_2 = u \cdot y_1 = x \cdot e^{r_1 x} = xe^{r_1 x}$

$\therefore$ A basis of solutions for our ODE $ay'' + by' + cy = 0$ is $\{e^{r_1 x}, xe^{r_1 x}\}$ and our general solution for this ODE when $ar^2 + br + c = 0 \Rightarrow b^2 - 4ac = 0$ is $y = y(x) = c_1 e^{r_1 x} + c_2 xe^{r_1 x}$

$$\therefore \boxed{y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x} = (c_1 + c_2 x)e^{r_1 x}}$$

NOTE 3 : In practice, we will NOT go through the process of Reduction of Order to find a second function to complete the pair for a basis of solutions when $b^2-4ac=0$. We will just note that when $b^2-4ac=0$ (for what will later call our auxillary or characteristic equation $ar^2+br+c=0$), we will automatically know that the set $\{e^{r_1 x}, xe^{r_1 x}\}$ will be a basis of solutions for the ODE and our general solution will be $y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{r_1 x} + c_2 (xe^{r_1 x})$!

Case (3) : $b^2-4ac<0$

If $b^2-4ac<0$, then r_1 and r_2 will be complex conjugates of each other. Therefore $r = r_{1,2} = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = -\dfrac{b}{2a} \pm \dfrac{\sqrt{4ac-b^2}}{2a} i =$ $\lambda \pm \omega i$, where $\lambda = -\dfrac{b}{2a}$ and $\omega = \dfrac{\sqrt{4ac-b^2}}{2a}$, where $4ac-b^2>0$

Thus, $r = \lambda \pm \omega i \Rightarrow y(x) = e^{rx} = e^{(\lambda \pm \omega i)x} = e^{\lambda x}\left[e^{\pm(\omega x)i}\right] =$ $e^{\lambda x}\left[e^{i\theta}\right]$, where $\theta = \pm \omega x$.

ATTENTION ! : Recall that Euler's Formula states that

$e^{i\theta} = \cos(\theta) + i\sin(\theta) \Rightarrow e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos(\theta) - i\sin(\theta)$

Case(3): $b^2-4ac<0$ (cont'd)

$\therefore$ If $r=r_1=\lambda+wi$, then one solution to our ODE for this case will be
$y_1=e^{r_1x}=e^{(\lambda+wi)x}=e^{\lambda x}\left[e^{(wx)i}\right]=e^{\lambda x}\left[\cos(wx)+i\sin(wx)\right]$. Similarly,

if $r=r_2=\lambda-wi$, then another solution to our ODE for this case will be
$y_2=e^{r_2x}=e^{(\lambda-wi)x}=e^{\lambda x}\left[e^{(-wx)i}\right]=e^{\lambda x}\left[\cos(-wx)+i\sin(-wx)\right]=$
$e^{\lambda x}\left[\cos(wx)-i\sin(wx)\right]$.

$\therefore$ $y_1=e^{\lambda x}\left[\cos(wx)+i\sin(wx)\right]$ and $y_2=e^{\lambda x}\left[\cos(wx)-i\sin(wx)\right]$ are both "suitable" solutions.

NOTE: What we have is great, but it is not that useful to us since we will be using general solution in the case of when $b^2-4ac<0$ to model real applications using real numbers, not complex numbers! To get around this dilemma, we note that we can use the superposition principle to (hopefully) come up with a solution that will consist of only terms that are real only. Thus, we need to use the principle of superposition with our solutions y_1 and y_2 to eliminate the imaginary part of both of these solutions, and, thus come up with a new general solution!)

If we add y_1 and y_2, this yields...

$$y_1 + y_2 = e^{\lambda x}\left[\cos(wx) + i\sin(wx)\right] + e^{\lambda x}\left[\cos(wx) - i\sin(wx)\right]$$

$$'' \quad = e^{\lambda x}\left[\cos(wx) + i\sin(wx) + \cos(wx) - i\sin(wx)\right]$$

$$'' \quad = e^{\lambda x}\left[\left(\cos(wx) + \cos(wx)\right) + \left(\underbrace{\cancel{\sin(wx)} - \sin(wx)}_{0}\right) i\right]$$

$$= e^{\lambda x}\left[2\cos(wx)\right] = e^{\lambda x} c_1 \cos(wx), \text{ where } c_1 = 2.$$

So, let $\boxed{\bar{y}_1 = e^{\lambda x} c_1 \cos(wx)}$.

It also turns out that we can find another "suitable" solution by using the linear combination of y_1 and y_2 in the form $i[y_1 - y_2]$!

$$\therefore \; i[y_1 - y_2] = i\left[e^{\lambda x}\left(\cos(wx) + i\sin(wx)\right) - e^{\lambda x}\left(\cos(wx) - i\sin(wx)\right)\right]$$

$$= e^{\lambda x}\left[\cancel{i\cos(wx)} + \overset{-1}{i^2}\sin(wx) - \cancel{i\cos(wx)} + \overset{-1}{i^2}\sin(wx)\right]$$

$$= e^{\lambda x}\left[(-1)\sin(wx)\right] = e^{\lambda x} c_2 \sin(wx), \text{ where } c_2 = -1.$$

So, let $\boxed{\bar{y}_2 = e^{\lambda x} c_2 \sin(wx)}$.

It follows from the superposition principle (again) that $\bar{y}_1 + \bar{y}_2$ is also a "suitable" solution of our ODE <u>AND</u> $\bar{y}_1 + \bar{y}_2$ will only have real terms!

$\therefore$ For $ar^2 + br + c = 0 \Rightarrow b^2 - 4ac < 0 \Rightarrow$ we have a (real-valued) general solution $\boxed{y(x) = c_1 e^{\lambda x}\cos(wx) + c_2 e^{\lambda x}\sin(wx)}$.

Summary of Cases (1) – (3) for the ODE $ay''+by'+cy=0$

The ODE $ay''+by'+cy=0 \Rightarrow y(x)=e^{rx}$ will be a solution and the auxillary / characteristic equation $ar^2+br+c=0$ will decide on what type of (root) values "r" we have as well as what our general solution to this ODE will look like based on the following 3 cases:

Case (1): $b^2-4ac>0$

$r_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ yields 2 real, distinct solutions.

$\therefore \boxed{y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} \quad \text{(General solution)}}$

Case (2): $b^2-4ac=0$

$r_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ yields only 1 real solution $r_1=r_2=\frac{-b}{2a}=r$.

$\therefore \boxed{y(x) = c_1 e^{rx} + c_2 x e^{rx} = e^{rx}(c_1+c_2x) \quad \text{(General solution)}}$

Case (3): $b^2-4ac<0$

$r_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ yields complex conjugate solutions $r=\lambda\pm wi$.

$\therefore \boxed{y(x) = e^{\lambda x}\left[c_1\cos(wx)+c_2\sin(wx)\right] \quad \text{(General solution)}}$

Now we will do a few examples in order to get practice on what our general solutions (and particular solutions for IVP-type of problems) for each of Cases (1)-(3)!

Ex (Case 1): Find the general solution of each ODE given. If initial conditions are given, find the unique solution for the ODE.

a) $3y'' + 7y' - 6y = 0$

Characteristic eqn: $3r^2 + 7r - 6 = 0 \Rightarrow (3r-2)(r+3) = 0$

$\therefore 3r-2=0$ or $r+3=0 \Rightarrow r = 2/3$ or $r = -3$

$\therefore$ General solution: $y(x) = c_1 e^{2/3 x} + c_2 e^{-3x}$

b) $y'' + 3y' = 0$

Characteristic eqn: $r^2 + 3r = 0 \Rightarrow r(r+3) = 0 \Rightarrow r = 0, -3$

$\therefore$ General solution: $y(x) = c_1 e^{0x} + c_2 e^{-3x} \Rightarrow \boxed{y(x) = c_1 + c_2 e^{-3x}}$

Case (1) examples : cont'd

(c) $y'' - 9y = 0$ with $\boxed{y(0) = 1 \text{ and } y'(0) = 0}$ (*)

Characteristic Eqn: $r^2 - 9r = 0 \Rightarrow r(r-9) = 0 \Rightarrow r = 0, 9$

$\therefore$ General solution: $y(x) = c_1 e^{0x} + c_2 e^{9x} \Rightarrow \boxed{y(x) = c_1 + c_2 e^{9x}}$

Applying I.C. $y(0) = 1$

$y(0) = 1 \Rightarrow c_1 + c_2 \cancel{e^{9(0)}}^{1} = 1 \Rightarrow c_1 + c_2 = 1 \Rightarrow \boxed{c_1 = 1 - c_2}$

Applying I.C. $y'(0) = 0$

NOTE: $y(x) = c_1 + c_2 e^{9x} \Rightarrow y'(x) = 9c_2 e^{9x}$

$\therefore y'(0) = 0 \Rightarrow 9c_2 \cancel{e^{9(0)}}^{1} = 0 \Rightarrow 9c_2 = 0 \Rightarrow c_2 = \frac{0}{9} = 0$

$\therefore c_2 = 0 \Rightarrow c_1 = 1 - c_2 = 1 - 0 = 1$. So, $\boxed{c_1 = 1 \text{ and } c_2 = 0}$!

$\therefore y(x) = 1 + 0e^{9x} \Rightarrow$ * $\boxed{y(x) = 1}$ * (Final Answer)

(*) NOTE: This answer may seem strange, but it actually makes sense if you observe the combination of the initial conditions state that $y(x)$ will have a horizontal tangent line (i.e. $y'(0) = 0$) at the point $(0,1)$ (i.e. $y(0) = 1$)

Case (1) examples: cont'd - 2

(d) $y'' - 8y' + 15y = 0$ with $y(0) = 5$ and $y'(0) = 19$

Characteristic eqn: $r^2 - 8r + 15 = 0 \Rightarrow (r-3)(r-5) = 0 \Rightarrow r_{1,2} = 3, 5$

$$\therefore \boxed{y(x) = c_1 e^{3x} + c_2 e^{5x} \quad \text{(General solution)}}$$

Applying I.C. $y(0) = 5$

$$y(0) = 5 \Rightarrow c_1 e^{3(0)} + c_2 e^{5(0)} = 5 \Rightarrow c_1 + c_2 = 5 \Rightarrow \boxed{c_1 = -5c_2}$$

Applying I.C. $y'(0) = 19$

$$y'(x) = 3c_1 e^{3x} + 5c_2 e^{5x}$$

$$\therefore y'(0) = 19 \Rightarrow 3c_1 e^{3(0)} + 5c_2 e^{5(0)} = 19 \Rightarrow 3c_1 + 5c_2 = 19$$

$$\Rightarrow 3[-5c_2] + 5c_2 = 19$$

$$\Rightarrow -15c_2 + 5c_2 = 19$$

$$\Rightarrow -10c_2 = 19$$

$$\Rightarrow \boxed{c_2 = \frac{-19}{10}}$$

$$\therefore c_1 = -5c_2 = -5\left[\frac{-19}{10}\right] = \boxed{\frac{19}{2} = c_1}$$

$$\therefore \boxed{y(x) = \frac{19}{2} e^{3x} - \frac{19}{10} e^{5x}}$$

Final answer

Case (1) examples : cont'd_3

(e) $4y'' - 4y' - 3y = 0$ with $y(-2) = e$ and $y'(-2) = -e/2$

Characteristic eqn : $4r^2 - 4r - 3 = 0 \Rightarrow (2r+1)(2r-3) = 0$

$\therefore r_{1,2} = -1/2,\ 3/2 \Rightarrow$ $\boxed{\text{General solution} : y(x) = c_1 e^{-\frac{1}{2}x} + c_2 e^{\frac{3}{2}x}}$

Applying I.C. $y(-2) = e$

$y(-2) = e \Rightarrow c_1 e^{-\frac{1}{2}(-2)} + c_2 e^{\frac{3}{2}(-2)} = e \Rightarrow c_1(e) + c_2(e^{-3}) = e$

$\therefore [c_1(e^1) + c_2(e^{-3})]e^{-1} = [e^1]e^{-1} \Rightarrow \boxed{c_1 + c_2(e^{-4}) = 1}$

Applying I.C. $y'(-2) = -e/2$

$y'(x) = -\frac{1}{2} c_1 e^{-\frac{1}{2}x} + \frac{3}{2} c_2 e^{\frac{3}{2}x}$

$\therefore y'(-2) = -e/2 \Rightarrow 2\left[-\frac{1}{2} c_1 e^{-\frac{1}{2}(-2)} + \frac{3}{2} c_2 e^{\frac{3}{2}(-2)}\right] = \left[-\frac{e}{2}\right]2$

$\therefore -c_1 e^1 + 3c_2 e^{-3} = -e^1 \Rightarrow e^{-1}\left[-c_1 e^1 + 3c_2 e^{-3}\right] = \left[-e^1\right]e^{-1}$

$\Rightarrow \boxed{-c_1 + 3c_2 e^{-4} = -1}$

$\therefore \begin{cases} c_1 + c_2(e^{-4}) = 1 \\ -c_1 + 3c_2(e^{-4}) = -1 \end{cases} \Rightarrow 4c_2 e^{-4} = 0 \Rightarrow c_2 = \frac{0}{4e^{-4}} = 0 \Rightarrow \boxed{c_2 = 0}$

$\therefore c_1 + \cancel{c_2}^{0}(e^{-4}) = 1 \Rightarrow \boxed{c_1 = 1}$ ★ $\boxed{\text{Final answer} : y(x) = e^{-\frac{1}{2}x}}$ ★

Ex (Case 2): Same directions as for Ex (Case 1).

a) $4y'' - 4y' + y = 0$

Characteristic eqn: $4r^2 - 4r + 1 = 0 \Rightarrow (2r-1)(2r-1) = 0$

$\Rightarrow (2r-1)^2 = 0$

$\Rightarrow 2r - 1 = \pm\sqrt{0} = 0$

$\Rightarrow r = \frac{1}{2}$ (double root)

$\therefore y(x) = (c_1 + c_2 x)e^{rx}$

$\Rightarrow \boxed{y(x) = (c_1 + c_2 x)e^{\frac{1}{2}x} = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}}$

b) $y'' + 16y' + 64y = 0$

$\therefore$ Characteristic eqn: $r^2 + 16r + 64 = 0 \Rightarrow (r+8)(r+8) = 0$

$\Rightarrow (r+8)^2 = 0$

$\Rightarrow r + 8 = \pm\sqrt{0} = 0$

$\Rightarrow r = -8$ (double root)

$\therefore y(x) = (c_1 + c_2 x)e^{rx}$

$\Rightarrow \boxed{y(x) = (c_1 + c_2 x)e^{-8x}}$

Case (2) examples: cont'd

c) $y'' - 8y' + 16y = 0$ with $y(0) = 3$ and $y'(0) = 14$

Characteristic eqn: $r^2 - 8r + 16 = 0 \Rightarrow (r-4)(r-4) = 0 \Rightarrow r - 4 = 0$
$\Rightarrow r = 4$
(double root)

$\therefore y(x) = (c_1 + c_2 x)e^{rx} \Rightarrow \boxed{y(x) = (c_1 + c_2 x)e^{4x}}$

Applying I.C. $y(0) = 3$

$y(0) = 3 \Rightarrow (c_1 + \cancel{c_2 (0)}^{0}) \cancel{e^{4(0)}}^{1} = 3 \Rightarrow \boxed{c_1 = 3}$

Applying I.C. $y'(0) = 14$

$y'(x) = c_2 e^{4x} + (c_1 + c_2 x)(4e^{4x})$

$\therefore y'(0) = 14 \Rightarrow c_2 \cancel{e^{4(0)}}^{1} + (\cancel{c_1}^{3} + \cancel{c_2(0)}^{0})(4\cancel{e^{4(0)}}^{1}) = 14 \Rightarrow c_2 + \cancel{(3)(4)}^{12} = 14$
$\Rightarrow c_2 = 14 - 12 = 2$
$\Rightarrow \boxed{c_2 = 2}$

$\therefore$ Final answer: $y(x) = (3 + 2x)e^{4x}$

or

$y(x) = 3e^{4x} + 2xe^{4x}$

Case (2) examples : cont'd _ 2

d) $4y'' + 4y' + y = 0$ with $y(0) = 0$ and $y'(0) = 1$

Characteristic eqn : $4r^2 + 4r + 1 = 0 \Rightarrow (2r+1)(2r+1) = 0$

$\Rightarrow 2r + 1 = 0$

$\Rightarrow r = -\frac{1}{2}$ (double root)

$\therefore y(x) = (c_1 + c_2 x) e^{rx}$

$\Rightarrow \boxed{y(x) = (c_1 + c_2 x) e^{-\frac{1}{2}x}}$

Applying I.C. $y(0) = 0$

$y(0) = 0 \Rightarrow (c_1 + \cancel{c_2(0)}^{0}) \cancel{e^{-\frac{1}{2}(0)}}^{1} = 0 \Rightarrow \boxed{c_1 = 0}$

Applying I.C. $y'(0) = 1$

$y'(x) = c_2 e^{-\frac{1}{2}x} + (c_1 + c_2 x)(-\frac{1}{2} e^{-\frac{1}{2}x})$

$\therefore y'(0) = 1 \Rightarrow c_2 \cancel{e^{-\frac{1}{2}(0)}}^{1} + (\cancel{c_1}^{0} + \cancel{c_2(0)}^{0})(-\frac{1}{2} \cancel{e^{-\frac{1}{2}(0)}}^{1}) = 1 \Rightarrow c_2 + 0 = 1$

$\Rightarrow \boxed{c_2 = 1}$

$\therefore$ $\boxed{\text{Final answer : } y(x) = x e^{-\frac{1}{2}x}}$

Ex (Case 3): Same directions as for Ex (Case 1)

a) $y'' + 36y = 0$

Characteristic eqn: $r^2 + 36 = 0 \Rightarrow r^2 = -36 \Rightarrow r = \pm\sqrt{-36} = \pm\sqrt{36}\,i$

$\Rightarrow r = \pm 6i = 0 \pm 6i$

$\therefore y(x) = e^{\lambda x}\left[c_1 \cos(\omega x) + c_2 \sin(\omega x)\right]$, where $r = \lambda \pm \omega i \Rightarrow \lambda = 0 + \omega = 6.$

$\therefore y(x) = \cancel{e^{0x}}^{\,1}\left[c_1 \cos(6x) + c_2 \sin(6x)\right]$

$\Rightarrow \boxed{y(x) = c_1 \cos(6x) + c_2 \sin(6x)}$ Final answer

b) $y'' - 4y' + 13y = 0$

Characteristic eqn: $r^2 - 4r + 13 = 0 \Rightarrow r_{1,2} = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)}$

$= \dfrac{4 \pm \sqrt{-36}}{2} = \dfrac{4 \pm 6i}{2} = 2 \pm 3i = \lambda + \omega i \Rightarrow \lambda = 2;\ \omega = 3$

$\therefore y(x) = e^{\lambda x}\left[c_1 \cos(\omega x) + c_2 \sin(\omega x)\right]$

$\Rightarrow \boxed{y(x) = e^{2x}\left[c_1 \cos(3x) + c_2 \sin(3x)\right]}$

Ex (Case 3): cont'd

c) $y'' - 4y' + 13y = 0$ with $y(0) = 1$ and $y'(0) = 0$

NOTE: We found the generic, general solution to this ODE in the previous example. Therefore, all we need to do is find the constants c_1 + c_2 for $y(x) = e^{2x}\left[c_1 \cos(3x) + c_2 \sin(3x)\right]$!

So, since we know a general (generic) solution for $y(x)$, we can find the corresponding derivative, $y'(x)$, for $y(x)$.

$$\therefore y'(x) = 2e^{2x}\left[c_1 \cos(3x) + c_2 \sin(3x)\right] + e^{2x}\left[-3c_1 \sin(3x) + 3c_2 \cos(3x)\right]$$

$$= e^{2x}\left[2c_1 \cos(3x) + 2c_2 \sin(3x) - 3c_1 \sin(3x) + 3c_2 \cos(3x)\right]$$

$$= e^{2x}\left[\cos(3x)(2c_1 + 3c_2) + \sin(3x)(2c_2 - 3c_1)\right]$$

Applying I.C. $y(0) = 1$

$$y(0) = 1 \Rightarrow \cancel{e^{2(0)}}^{1}\left[c_1 \cancel{\cos(0)}^{1} + c_2 \cancel{\sin(0)}^{0}\right] = 1 \Rightarrow c_1 + 0 = 1 \Rightarrow \boxed{c_1 = 1}$$

Applying I.C. $y'(0) = 0$

$$y'(0) = 0 \Rightarrow \cancel{e^{2(0)}}^{1}\left[\cancel{\cos(0)}^{1}(2\cancel{c_1}^{1} + 3c_2) + \cancel{\sin(0)}^{0}(2c_2 - 3c_1)\right] = 0 \Rightarrow 2 + 3c_2 = 0$$

$$\Rightarrow \boxed{c_2 = -2/3}$$

$\therefore$ $\boxed{\text{Final answer: } y(x) = e^{2x}\left[\cos(3x) - \tfrac{2}{3}\sin(3x)\right]}$

Ex (Case 3) : cont'd _ 2

d) $y'' + y' + 3.25y = 0$ with $y\left(\frac{\pi}{\sqrt{3}}\right) = -2$ and $y'\left(\frac{\pi}{\sqrt{3}}\right) = 6$

Characteristic eqn: $r^2 + r + 3.25 = 0 \Rightarrow r_{1,2} = \frac{-(1) \pm \sqrt{(1)^2 - 4(1)(3.25)}}{2(1)} =$

$$\frac{-1 \pm \sqrt{1-13}}{2} = \frac{-1 \pm \sqrt{-12}}{2} = \frac{-1 \pm \sqrt{4}^{\,2}\sqrt{3}\,i}{2} = -\tfrac{1}{2} \pm \sqrt{3}\,i = \lambda \pm wi$$

$\therefore \lambda = -\tfrac{1}{2}$ and $w = \sqrt{3}$ $\Rightarrow y(x) = e^{\lambda x}\left[c_1 \cos(wx) + c_2 \sin(wx)\right]$

$$\Rightarrow \boxed{y(x) = e^{-\frac{1}{2}x}\left[c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\right]}$$

Applying $y\left(\frac{\pi}{\sqrt{3}}\right) = -2$

$$y\left(\tfrac{\pi}{\sqrt{3}}\right) = -2 \Rightarrow e^{-\frac{1}{2}\left(\frac{\pi}{\sqrt{3}}\right)}\left[c_1 \overset{-1}{\cancel{\cos\left(\sqrt{3}\cdot\tfrac{\pi}{\sqrt{3}}\right)}} + c_2 \overset{0}{\cancel{\sin\left(\sqrt{3}\cdot\tfrac{\pi}{\sqrt{3}}\right)}}\right] = -2$$

$$\Rightarrow e^{-\frac{\sqrt{3}\pi}{6}}\left[-c_1\right] = -2 \Rightarrow c_1 = \frac{-2}{-e^{-\frac{\sqrt{3}\pi}{6}}} = 2e^{\frac{\sqrt{3}\pi}{6}}$$

$$\Rightarrow \boxed{c_1 = 2e^{\frac{\sqrt{3}\pi}{6}}}$$

Applying $y'\left(\frac{\pi}{\sqrt{3}}\right) = 6$

$$y'(x) = -\tfrac{1}{2}e^{-\frac{1}{2}x}\left[c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\right] + e^{-\frac{1}{2}x}\left[-\sqrt{3}c_1 \sin(\sqrt{3}x) + \sqrt{3}c_2 \cos(\sqrt{3}x)\right]$$

$$\therefore y'\left(\tfrac{\pi}{\sqrt{3}}\right) = 6 \Rightarrow \left(-\tfrac{1}{2}e^{-\frac{1}{2}\cdot\frac{\pi}{\sqrt{3}}}\left[c_1 \overset{-1}{\cancel{\cos\left(\sqrt{3}\cdot\tfrac{\pi}{\sqrt{3}}\right)}} + c_2 \overset{0}{\cancel{\sin\left(\sqrt{3}\cdot\tfrac{\pi}{\sqrt{3}}\right)}}\right] + e^{-\frac{1}{2}\cdot\frac{\pi}{\sqrt{3}}}\left[-\sqrt{3}c_1 \overset{0}{\cancel{\sin\left(\sqrt{3}\cdot\tfrac{\pi}{\sqrt{3}}\right)}} + \sqrt{3}c_2 \overset{-1}{\cancel{\cos\left(\sqrt{3}\cdot\tfrac{\pi}{\sqrt{3}}\right)}}\right]\right) = 6$$

d) cont'd

$$\therefore -\tfrac{1}{2} e^{-\frac{\sqrt{3}\pi}{6}} [-c_1] + e^{-\frac{\sqrt{3}\pi}{6}} [-\sqrt{3}\, c_2] = 6$$

But $\boxed{c_1 = 2e^{\frac{\sqrt{3}\pi}{6}}}$.

$$\therefore \cancel{-\tfrac{1}{2} e^{-\frac{\sqrt{3}\pi}{6}} \left[-2e^{\frac{\sqrt{3}\pi}{6}}\right]}^{\,1} + e^{-\frac{\sqrt{3}\pi}{6}} [-\sqrt{3}\, c_2] = 6$$

$$\Rightarrow 1 + e^{-\frac{\sqrt{3}\pi}{6}} [-\sqrt{3}\, c_2] = 6$$

$$\Rightarrow e^{-\frac{\sqrt{3}\pi}{6}} [-\sqrt{3}\, c_2] = 5$$

$$\Rightarrow c_2 = \frac{5}{-\sqrt{3}\, e^{-\frac{\sqrt{3}\pi}{6}}} = -\frac{5}{\sqrt{3}} e^{\frac{\sqrt{3}\pi}{6}} \Rightarrow \boxed{c_2 = -\frac{5}{\sqrt{3}} e^{\frac{\sqrt{3}\pi}{6}}}$$

$$\therefore \boxed{y(x) = e^{-\frac{1}{2}x} \left[2e^{\frac{\sqrt{3}\pi}{6}} \cos(\sqrt{3}\, x) - \frac{5}{\sqrt{3}} \sin(\sqrt{3}\, x) \right]}$$

Final answer