Natalie Taggart

1. (20 points) What is the inverse Laplace transform of the function

$$X(s) = \frac{1}{(s+1)^2+4}$$

Plot the function in time domain.

$$X(s) = \frac{1}{(s+1)^2+4} = \frac{1}{(s+1)^2+2^2}$$

Property: $\mathcal{L}^{-1}\left[\frac{b}{(s-a)^2+b^2}\right] = e^{at}\sin(bt)$

$$X(s) = \frac{2}{2}\cdot\frac{1}{(s+1)^2+2^2} = \frac{1}{2}\left(\frac{2}{(s+1)^2+2^2}\right)$$

$$\mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+2^2}\right] = \boxed{\frac{1}{2}e^{-t}\sin(2t)}$$

$\star$ notes from class review.

$\star$ Page 202 formulas!

$\star$ PLOT

2. (20 points) A system with input x(t) and output y(t) is defined by the following differential equation:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

Find the impulse response $h(t)$ and the unit-step response $s(t)$.

$$X(s) = \mathcal{L}[x(t)] = \mathcal{L}\left[\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t)\right]$$

$$X(s) = \left[s^2Y(s) - \cancel{sy(0^-)} - \cancel{y'(0^-)}\right] + 3\left[sY(s) - \cancel{y(0^-)}\right] + 2Y(s)$$

$$X(s) = s^2Y(s) + 3sY(s) + 2Y(s)$$

$$Y(s) = \underbrace{\frac{1}{s^2+3s+2}}_{H(s)} \cdot X(s)$$

<u>Find $h(t)$</u> $\quad x(t) = \delta(t) \Rightarrow X(s) = 1$

$$H(s) = \frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$A = H(s)(s+2)\Big|_{s=-2} = \frac{1}{s+1}\Big|_{s=-2} = -1$$

$$B = H(s)(s+1)\Big|_{s=-1} = \frac{1}{s+2}\Big|_{s=-1} = 1$$

$$H(s) = \frac{-1}{s+2} + \frac{1}{s+1}$$

$$\Rightarrow \boxed{h(t) = \left[e^{-t} - e^{-2t}\right]u(t)}$$

Find $s(t)$ (convolution property) → $X(s) = u(t)$

$$S(s) = X(s) \cdot H(s) \qquad \mathcal{L}[u(t)] = \frac{1}{s}$$

$$X(s) = \frac{1}{s}$$
$$H(s) = \frac{1}{s+1} \cdot \frac{1}{s+2}$$

$$\mathcal{L}^{-1}[S(s)] = \mathcal{L}^{-1}[X(s) \cdot H(s)]$$

$$s(t) = \mathcal{L}^{-1}\left[\frac{1}{s} \cdot \frac{1}{s^2+3s+2}\right] = \mathcal{L}^{-1}\left[\frac{1}{s(s+2)(s+1)}\right]$$

$$s(t) = \mathcal{L}^{-1}\left[\frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s}\right]$$

$$A = S(s)(s+2)\Big|_{s=-2} = \frac{1}{(-2)(-1)} = \frac{1}{2}$$

$$B = S(s)(s+1)\Big|_{s=-1} = \frac{1}{(1)(-1)} = -1$$

$$C = S(s)\, s\Big|_{s=0} = \frac{1}{(2)(1)} = \frac{1}{2}$$

$$S(s) = \frac{1/2}{s+2} + \frac{-1}{s+1} + \frac{1/2}{s} = \frac{1}{2(s+2)} - \frac{1}{s+1} + \frac{1}{2s}$$

$$\mathcal{L}^{-1}[S(s)] = s(t)$$

$$\boxed{s(t) = 0.5e^{-2t}u(t) - e^{-t}u(t) + 0.5u(t)}$$

☆ know how to find steady + transient state/response

Partial Fraction Decomp./Exp.

$$\frac{1}{s(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s}$$

$$A = X(s)(s+2)\Big|_{s=-2}$$

$$s+2=0$$
$$s=-2$$

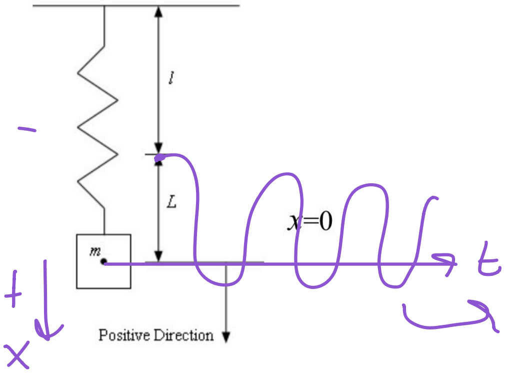
$$\frac{1}{s\cancel{(s+2)}(s+1)} \cdot \cancel{s+2} = \frac{1}{s(s+1)} \quad \frac{1}{(-2)(\underbrace{\cancel{-2+1}}_{-1})} = \frac{1}{2}$$

$$B = X(s)(s+1)\Big|_{s=-1} = \frac{1}{(s+2)(s)}\Big|_{s=-1} = \frac{1}{1(-1)} = -1$$

base unit

3. (20 points) A 1 kg weight is hung on the end of a vertically suspended spring, thereby stretching the spring L= 20 cm. The weight is raised 10 cm above its equilibrium position and released from rest at time t=0. Find the displacement x of the weight from its equilibrium position at time t=2.5s. Use g=10m/s^2.

All forces, velocities, and displacements in the upward direction will be negative, according to the Figure below.



$$k = \frac{mg}{l}$$

would have to invert graph in MATLAB to match

$$m\frac{d^2x}{dt^2} + kx + \cancel{c\frac{dx}{dt}}^{\,0} = 0 \rightarrow \Sigma F = 0$$

★ Know how to do w/ C value for resistance

$$k = \frac{mg}{l} = \frac{(1)(10)}{0.2} = 50 \text{ N/m}$$

$$1\frac{d^2x}{dt^2} + 50x = 0$$

$$\mathcal{L}\left[\frac{d^2x}{dt^2} + 50x\right] = 0$$

$$s^2X(s) - sf(0-) - \cancel{f'(0-)}^{\,0} + 50X(s) = 0$$

↳ $f(0-) = -10$ cm

~~★ need u(t)?~~ no

$$s^2X(s) + 0.1s + 50X(s) = 0$$

$$X(s)(s^2+50) + 0.1s = 0$$

prop. on pg. 202

$$\mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{-0.1s}{s^2+50}\right] \rightarrow \text{add of. sqs.}$$

$$x(t) = \mathcal{L}^{-1}\left[-0.1\left(\frac{s}{s^2+50}\right)\right] = -0.1\cos(\sqrt{50}\,t) \quad t \geq 0$$

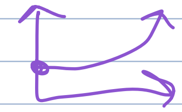
<u>at $t = 2.5$</u>

$$x(2.5) = -0.1\cos(\sqrt{50}(2.5)) = \boxed{-0.039 \text{ m}}$$

★ rads

↳ above equilibrium.

Unstable



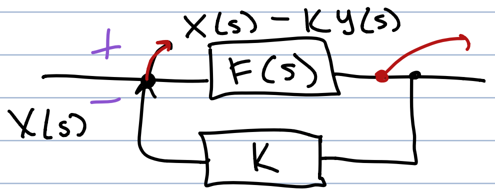
4. (20 points) An unstable system can be stabilized by using negative feedback with gain K in the feedback loop. For the given unstable system with pole in the right-hand *s*-plane:

$$F(s) = \frac{1}{2 \cdot s - 3}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

what should be the value of the gain K (K is integer greater than zero) with *F(s)* in the forward loop that will make the system BIBO stable.

Draw block diagram of the system, write overall transfer function of the system *H(s)*, and impulse response $h(t)$.



$$F(s)\left(X(s) - K \cdot Y(s)\right) = Y(s)$$

$$F(s)X(s) - F(s) \cdot K \cdot Y(s) = Y(s)$$

$$F(s)X(s) = (1 + F(s)K)Y(s)$$

$$Y(s) = \frac{F(s)}{1 + KF(s)} \cdot X(s)$$

unstable

correction

↳ modify K to make it stable

$$H(s) = \frac{F(s)}{1 + G(s)F(s)}$$

let $G(s) = K$

$$H(s) = \frac{\frac{1}{2s-3}}{1 + \frac{K}{2s-3}} = \frac{1}{2s-3+K} = \frac{1}{2}\left(\frac{1}{s - \frac{3}{2} + \frac{K}{2}}\right)$$

$$-\frac{3}{2} + \frac{K}{2} > 0 \quad \text{for stable system}$$

$$\frac{K}{2} > \frac{3}{2}$$

$$K > 3 \Rightarrow K = 4$$

if $K = 3 \rightarrow a = 0$ so $\frac{1}{s}$ & response would be step function (unstable)

$$\boxed{H(s) = \frac{1}{2}\left(\frac{1}{s + \frac{1}{2}}\right)}$$

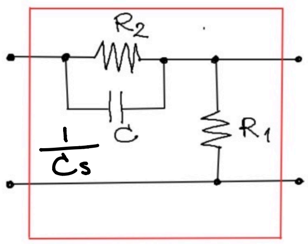
$$h(t) = \mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}\left[\frac{1}{2}\left(\frac{1}{s + \frac{1}{2}}\right)\right]$$

$$\boxed{h(t) = \frac{1}{2} e^{-t/2} u(t)}$$

5. a) (4 points) What is the transfer function of the following circuit:

$Z_R = R$

$Z_C = \frac{1}{Cs}$



a.) $Y(s) = \frac{Z_{R_1}}{Z_{R_1} + Z_{R_2C}} \cdot X(s)$

$Z_{R_2C} = \frac{1}{R_2} + \frac{Cs\,R_2}{R_2} = \frac{R_2}{CsR_2+1}$

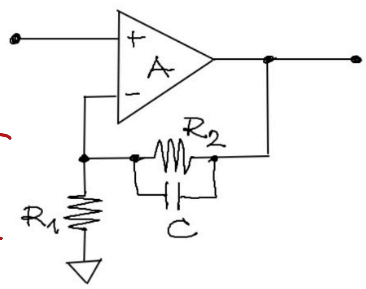
$Y(s) = \frac{R_1}{R_1 + \underbrace{\frac{R_2}{CsR_2+1}}_{H(s)}} \cdot X(s)$

$Y(s) = \underbrace{\frac{R_1CsR_2 + R_1}{R_1CsR_2 + R_1 + R_2}}_{H(s)} \cdot X(s)$

$$\boxed{H(s) = \frac{R_1}{R_1 + \frac{R_2}{CsR_2+1}}}$$

b) (6 points) What is the transfer function of the following circuit?
Hints:

- you can use solutions of problems #4 and #5a
- to simplify the result you can assume that A → ∞



from a

$\frac{A}{1 + AG(s)} = \frac{1}{\cancel{\frac{1}{A}} + G(s)}$ 0

$A \to \infty$ $\frac{1}{G(s)}$

c) (10 points) Find and plot the unit-step response $s(t)$ of the system. Assume R_1=1Ω, R_2=2Ω, C=1F.

b.) $H(s) = \frac{F(s)}{1 + F(s)G(s)}$ as $A \to \infty$ $\frac{\cancel{F(s)}}{\cancel{1+F(s)}G(s)} = \frac{1}{G(s)}$

$G(s) = H(s)$ from part a $\Rightarrow H(s) = \frac{1}{\text{part a}}$

$$\boxed{H(s) = \frac{R_1 + \frac{R_2}{CsR_2+1}}{R_1} = \frac{R_1CsR_2 + R_1 + R_2}{R_1CsR_2 + R_1}}$$

c.) $S(s) = X(s)H(s)$ $\quad X(s) = \frac{1}{s}$

$S(s) = \frac{1}{s} \cdot \frac{2s + 1 + 2}{2s + 1} = \frac{2s+3}{s(2s+1)} = \frac{A}{s} + \frac{B}{2s+1}$

$A = X(s)s\Big|_{s=0} = \frac{2(0)+3}{2(0)+1} = 3$

$B = X(s)(2s+1)\Big|_{s=-\frac{1}{2}} = \frac{2(-\frac{1}{2})+3}{(-1/2)} = \frac{2}{-1/2} = -4$

$S(s) = \frac{3}{s} + \frac{-4}{2s+1} = \frac{3}{s} - \frac{1}{2}\left(\frac{4}{s+\frac{1}{2}}\right)$

$$\mathcal{L}^{-1}[S(s)] = \boxed{s(t) = 3u(t) - 2e^{-1/2t}u(t)}$$

★ PLOT

u(t)?

★ Work a positive feedback example

★ look at 3.18, 3.19, 3.21 for MATLAB Examples

shows answers in lec. 3/8/21

3.2, 3.4, 3.24

★ Keep splane w/ you