2^{nd}-ORDER ODEs and Reduction of Order Method

In previous chapters in this course, we have spent a good amount of time learning how to solve 1^{st} and 2^{nd}-order ODEs using various substitution methods. Recall the following ODE types.

ODE Name	ODE Profile	Recommended Substitution
1^{st}-order Linear Substitution	$\frac{dy}{dx} = f(Ax+By+C)$	$u = Ax+By+C$
Homogenous (1^{st}-order)	$\frac{dy}{dx} = f(tx,ty) = f(x,y)$	$y = ux \Rightarrow u = y/x$
Bernoulli	$\frac{dy}{dx} + p(x)\,y = r(x)\cdot y^n$	$u = y^{1-n} \Rightarrow y = u^{\frac{1}{1-n}}$

Unfortunately, for ODEs of order 2 and higher, there are a limited number of ODEs of this ilk that are able to be solved via (traditional) substitution methods. On a good note, we will develop other methods for solving these type of ODEs that will prove to be very useful to us in the near future.

For now, our primary focus will be on developing methods for solving higher-order ODEs that arise a lot in applications.

It turns out that 2^{nd}-order linear ODEs arise a bunch in application problems, so we will spend a lot of time developing methods to solve these type of ODEs in the next few chapters. One somewhat primitive, but very important (from a foundational point of view) method (and skill) we will need to know is the Reduction of Order method of solving an ODE. The Reduction of Order method is essentially a method that helps one solve for an additional solution to an ODE given that at least one prior solution to the ODE is known. (The need and use for such method occur a lot in solving 2^{nd}-order linear ODEs since solutions to these ODE are often linearly independent of each other (i.e. you have more than 1 function $y(x)$ that will satisfy the ODE and each function is not a constant multiple of any of the other functions that satisfy the ODE as well). Thus, the primary importance of reduction of order is to help to find a basis of solutions to an ODE (i.e. a fancy way to say that we want all linearly independent functions that will satisfy the given ODE).

Definition of Higher-Order Linear ODEs

Recall that $\frac{dy}{dx} + py = r$, where p and r are functions of x is defined as a 1^{st}-order linear ODE. Note that an equivalent way of expressing this ODE could be...

$$a\frac{dy}{dx} + by = f \quad \text{, if } a=1,\ b=p \text{, and } r=f$$

Therefore, we could express as 2^{nd}-order linear ODE as...

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = g$$

, where a_0, a_1, a_2, and g are all functions of x (or constants). Similiarly, a 3^{rd}-order linear ODE could be expressed as...

$$a_0\frac{d^3y}{dx^2} + a_1\frac{d^2y}{dx^2} + a_2\frac{dy}{dx} + a_3 y = g$$

, where a_0, a_1, a_2, a_3, and g are functions of x (or constants).

In general, an N^{th}-order linear ODE would look like...

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \dots + a_{N-2}y'' + a_{N-1}y' + a_N y = g$$

, where $a_0, a_1, \dots, a_{N-1}, a_N$, and g are functions of x (or constants)

Examples of $2^{nd} + 3^{rd}$ order Linear ODEs

2^{nd}-order

- $\frac{d^2y}{dx^2} + x^3\frac{dy}{dx} + 5x^5y = \sqrt[3]{x-2}$
- $-4\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 3y = 0$

3^{rd}-order

- $x^5\frac{d^3y}{dx^3} + x^3\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 15y = 2e^{3x}$
- $\frac{d^3y}{dx^3} - \frac{dy}{dx} + ky = 0$

Examples of ODEs that are not 2^{nd}- + 3^{rd} order ODEs

- $\frac{d^2y}{dx^2} + y\frac{dy}{dx} = \sqrt[5]{2x-3}$: This is not in terms of just "x". We don't know if $y = y(x)$ or $y = y(x, y', \ldots)$.
- $3\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 - 1$: Can't have a $y^{(N)}$ term raised to any power
- $\frac{d^4y}{dx^4} - y^3\frac{d^2y}{dx^2} - y = 0$: This is not in terms of just "x". We don't know if $y = y(x)$ or $y = y(x, y', \ldots)$
- $4\frac{d^3y}{dx^3} + y^2\frac{dy}{dx} = 0$: Same reason as above!

Homogeneous vs. Non-homogeneous Higher-Order ODEs

For our general form of our N^{th}-order ODE, if $g = 0 \Rightarrow$ the N^{th}-order ODE is homogeneous. Otherwise, the N^{th}-order ODE is non-homogeneous.

Basic Process of Applying Reduction of Order Technique to 2nd-order Linear ODEs.

1. Given that $y_1(x) = y_1$ is a known solution to your ODE, let $y_2(x) = u(x) \cdot y_1(x) = uy_1$.

2. Find y_2' and y_2''. Substitute y_2, y_2', and y_2'' into original ODE. Note that...

   * $y_2' = (uy_1)' = u'y_1 + uy_1'$
   * $y_2'' = (u''y_1 + u'y_1') + (u'y_1' + uy_1'') = u''y_1 + 2u'y_1' + uy_1''$

3. Simplify the ODE with all the substitutions from step 2 above as much as possible.

NOTE: If given ODE was 2nd-order Linear Homogeneous, the result of simplifying after our substitutions in step 2 will be a 1st-order separable ODE provided that we let $v = u'$ and $v' = u''$. If the given ODE was 2nd-order Linear Non-homogeneous the result of simplifying after our substitutions in step 2 will be a 1st-order Linear ODE provided that we let $v = u'$ and $v' = u''$.

4. Solve the resulting ODE using either separable or 1^{st}-order Linear Techniques previously learned for a solution to $v = v(x) = u'(x) = u'$.

5. Since $v = u'$ $\Rightarrow \int v \, dx = u$. Evaluate the integral to solve for $u = u(x)$.

6. Now $y_1(x)$ and $y_2(x)$ are known. Verify that these solutions are linearly independent of each other (i.e. they are not non-zero constant multiples of each other).

7. Because each linearly independent solution of our ODE can satisfy the ODE on its own, we can use the superposition principle to conclude that $y = y(x) = c_1 y_1(x) + c_2 y_2(x)$, where $c_1, c_2 \in \mathbb{R}$. Note that the superposition principle simply states that if y_1 + y_2 are 2 linearly independent solutions of our ODE, then any linear combination of y_1 + y_2 are also solutions to our ODE!

NOW WE WILL DO SEVERAL EXAMPLES TO PRACTICE THIS TECHNIQUE!

Ex.1: For each ODE stated below, identify the following properties:

(i) order of ODE

(ii) Is it a (2nd-order +) Linear ODE ?

(iii) If it is a (2nd-order +) Linear ODE, is the ODE homogeneous ?

NOTE! An ODE of the type ...

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = g,$$

where $a_0, a_1, \ldots, a_N$, and g are functions of x, $y = y(x)$, and $a_0 = a_0(x) \neq 0$ for any x-value in interval of interest of our ODE ...

is considered to be a linear ODE of order N!

ATTENTION! If $g = g(x) = 0$, then our N^{th}-order ODE is considered to be homogeneous!

ODE	ORDER?	LINEAR?	HOMOGENEOUS?
a) $y'' + x^2 y' - 4y = x^3$	2	YES	NO
b) $y'' + x^2 y' = 4y$	2	yes	YES
c) $y''' + y = 0$	3	YES	YES
d) $(y+1) y'' = (y')^3$	2	NO	NO
e) $y^{(iv)} + 6y'' + 3y' - 83y - 25 = 0$	4	YES	YES
f) $y^{(55)} = \sin(x)$	55	YES	NO

Ex. 2: For the following 2^{nd}-order linear ODE's, verify that the solution $y_1(x)$ is indeed a solution to the ODE. Then use the method of reduction of order to find another solution, $y_2(x)$. Finally, state a general solution for the 2^{nd}-order linear ODE using the superposition principle.

a) $y'' - 4y' + 4y = 0 \quad ; \; y_1 = e^{2x}$

Sol'n: Since $y_1 = e^{2x} \Rightarrow y_1' = 2e^{2x} \Rightarrow y_2'' = 4e^{2x}$

$$\therefore \; y'' - 4y' + 4y = 0 \Rightarrow 4e^{2x} - 4(2e^{2x}) + 4(e^{2x}) = 0$$
$$\Rightarrow 8e^{2x} - 8e^{2x} = 0$$
$$\Rightarrow 0 = 0 \; \checkmark$$

So, $y_1(x) = e^{2x}$ is a sol'n to our ODE.

Find $y_2(x)$: Let $y_2(x) = u(x) \cdot y_1(x) = u y_1$. Then, it follows that

$$y_2' = u' \cdot y_1 + u y_1' = u'(e^{2x}) + u(2e^{2x}) = e^{2x}[u' + 2u] \text{, and}$$
$$y_2'' = (u'' y_1 + u' y_1') + (u' y_1' + u y_1'') = u'' y_1 + 2u' y_1' + u y_1'' =$$
$$u''(e^{2x}) + 2u'(2e^{2x}) + u(4e^{2x}) = e^{2x}[u'' + 2u' \cdot 2 + 4u]$$
$$= e^{2x}[u'' + 4u' + 4u]$$

So, $y_2 = u y_1 = e^{2x}[u]$, $y_2' = e^{2x}[u' + 2u]$, and $y_2'' = e^{2x}[u'' + 4u' + 4u]$

Ex. 2:

a) (cont'd)

$\therefore\ y_2'' - 4y_2' + 4y_2 = 0$

$\Rightarrow e^{2x}[u'' + 4u' + 4u] - 4[e^{2x}(u' + 2u)] + 4[e^{2x}(u)] = 0$

$\Rightarrow e^{2x}[u'' + 4u' + 4u - 4u' - 8u + 4u] = 0$

$\Rightarrow u'' + (\cancel{4u' - 4u'}^{\,0}) + (\cancel{4u - 8u + 4u}^{\,0}) = 0$, since $e^{2x} \neq 0$ and $\frac{0}{e^{2x}} = 0$.

$\Rightarrow u'' = 0$

$\Rightarrow u' = C \quad \Rightarrow u = Cx + D$, where $C, D \in \mathbb{R}$.

For simplicity, we let $u = u(x) = x \quad \Rightarrow y_2 = u y_1 = x e^{2x}$

$\therefore$ A general solution of our 2^{nd}-order Linear ODE is...

$y = y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{2x} + c_2 x e^{2x}$ via the superposition principle.

NOTE: If we would have let $u(x) = Cx + D$, thus, making $y(x) = c_1(e^{2x}) + c_2[(Cx + D)e^{2x}] = c_1 e^{2x} + c_2 C x e^{2x} + c_2 D e^{2x} = (c_1 + c_2 D)e^{2x} + c_2 C x e^{2x} = K_1 e^{2x} + K_2 x e^{2x}$, where $K_1 = c_1 + c_2 D$ and $K_2 = c_2 C$. So, we see that it is convenient + efficient to let $y = c_1 y_1 + c_2 y_2$!!.

Ex. 2 : (cont'd_2)

b) $y'' - y = 0$; $y_1 = \cosh(x)$

Sol'n: Since $y_1 = \cosh(x) \Rightarrow y_1' = \sinh(x) \Rightarrow y_1'' = \cosh(x)$

$\therefore\ y'' - y = 0 \Rightarrow \cosh(x) - \cosh(x) = 0 \Rightarrow 0 = 0$ ✓

So $y_1 = \cosh(x)$ is indeed a sol'n to $y'' - y = 0$

Find $y_2(x)$: Let $y_2(x) = y_2 = u(x) \cdot y_1 = u y_1 = u[\cosh(x)]$

$\therefore\ y_2' = u'[\cosh(x)] + u[\sinh(x)]$

$\Rightarrow y_2'' = \left(u''[\cosh(x)] + u'[\sinh(x)]\right) + \left(u'[\sinh(x)] + u[\cosh(x)]\right)$

$y_2'' = u'' \cdot \cosh(x) + 2u' \cdot \sinh(x) + u \cdot \cosh(x)$

$\therefore\ y_2'' - y_2' = 0 \Rightarrow u'' \cdot \cosh(x) + 2u' \cdot \sinh(x) \cancel{+ u \cdot \cosh(x)} \cancel{- u \cdot \cosh(x)} = 0$

$\Rightarrow u'' \cdot \cosh(x) + 2u' \cdot \sinh(x) = 0$. Let $v = u' \Rightarrow v' = u''$,

Then, $u'' \cdot \cosh(x) + 2u' \cdot \sinh(x) = 0 \Rightarrow v' \cdot \cosh(x) + 2v \cdot \sinh(x) = 0$

$\therefore\ v' \cdot \cosh(x) = -2v \cdot \sinh(x) \Rightarrow \dfrac{v'}{v} = -\dfrac{2\sinh(x)}{\cosh(x)} \Rightarrow \dfrac{dv}{v} = -2\dfrac{\sinh(x)}{\cosh(x)}\,dx$

$\therefore\ \displaystyle\int \frac{dv}{v} = -2\int \frac{\sinh(x)}{\cosh(x)}\,dx$. Let $w = \cosh(x) \Rightarrow dw = \sinh(x)\,dx$

$\therefore\ \displaystyle\int \frac{dv}{v} = -2\int \frac{dw}{w} \Rightarrow \ln|v| = -2\ln|w| + C \Rightarrow \ln|v| = \ln|w^{-2}| + C$

Ex. 2 : (cont'd – 3)

b) $\therefore \; e^{\ln|v|} = e^{\ln|w^{-2}| + c}$

$$\Rightarrow |v| = e^c \cdot w^{-2} = \frac{e^c}{w^2} = \frac{e^c}{\cosh^2(x)},$$

$$\Rightarrow v = \pm e^c \cdot \frac{1}{\cosh^2(x)} = A \cdot \operatorname{sech}^2(x), \text{ where } A = \pm e^c$$

But $v = u' \Rightarrow \int v \, dx = u \Rightarrow A \int \operatorname{sech}^2(x)\,dx = u$

Recall that $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} \Rightarrow [\tanh(x)]' =$

$$\frac{\cosh(x)\cdot\cosh(x) - \sinh(x)\cdot\sinh(x)}{\cosh^2(x)} = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)} =$$

$\operatorname{sech}^2(x)$!

$$\therefore A \int \operatorname{sech}^2(x)\,dx = u \Rightarrow A\tanh(x) + B = u$$

For simplicity, we let $u = \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$

$$\therefore y_2 = u y_1 = \frac{\sinh(x)}{\cosh(x)} \cdot \cosh(x) = \sinh(x) \Rightarrow y_2(x) = \sinh(x)$$

$\therefore$ Our general sol'n : $y(x) = c_1 y_1 + c_2 y_2 = c_1 \cosh(x) + c_2 \sinh(x)$

NOTE: Our ODE has a constant solution of $y = 0$. This can be achieved in the general solution if $c_1 = 0$ and $x = 0$. So, we don't need to state $y = 0$ as a solution explicitly.

Ex. 3 : The following ODEs are 2^{nd}-order Linear Homogeneous ODEs. Use the method of reduction of order to find another solution $y_2(x)$ for the ODE given that $y_1(x)$ is a known solution. Finally, state a general solution for the ODE using $y_1(x)$ and $y_2(x)$.

a) $x^2y'' - 7xy' + 16y = 0$; $y_1(x) = x^4$

Let $y_2(x) = u(x) \cdot y_1(x) = u \cdot y_1 = u[x^4]$

$\therefore y_2' = u'[x^4] + u[x^4]' = x^4u' + 4x^3u$

$\Rightarrow y_2'' = \left[(x^4)'u' + (x^4)u''\right] + \left[(4x^3)'u + (4x^3)u'\right]$

$y_2'' = 4x^3u' + x^4u'' + 12x^2u + 4x^3u'$

$y_2'' = x^4u'' + 8x^3u' + 12x^2u$

Now we will sub. y_2, y_2', and y_2'' into our 2^{nd}-order Linear ODE.

$\therefore x^2y'' - 7xy' + 16y = 0$

$\Rightarrow x^2\left[x^4u'' + 8x^3u' + 12x^2u\right] - 7x\left[x^4u' + 4x^3u\right] + 16\left[ux^4\right] = 0$

$\Rightarrow x^6u'' + 8x^5u' + 12x^4u - 7x^5u' - 28x^4u + 16x^4u = 0$

$\Rightarrow u''[x^6] + u'[8x^5 - 7x^5] + u[\underbrace{12x^4 - 28x^4 + 16x^4}_{0}] = 0$

$\Rightarrow u''[x^6] + u'[x^5] = 0$

NOTE: $x^2 \neq 0 \Rightarrow x \neq 0$. So, we can divide by x if we want!

Ex. 3:

a) cont'd

$$\therefore \frac{u''[x^6] + u'[x^5]}{x^5} = \frac{0}{x^5} \Rightarrow u'' \cdot x + u' = 0$$

Let $v = u'$. Then, $v' = u''$. Therefore, $v' \cdot x + v = 0$

$$\therefore \frac{dv}{dx} \cdot x + v = 0 \Rightarrow \frac{dv}{dx} x = -v \Rightarrow \frac{dv}{v} = \frac{-dx}{x} \Rightarrow \int \frac{dv}{v} = -\int \frac{dx}{x}$$

$$\therefore \ln|v| = -\ln|x| + C \Rightarrow \ln|v| + \ln|x| = C \Rightarrow \ln|vx| = C$$

$$\therefore e^{\ln|vx|} = e^{C} \Rightarrow |vx| = e^{C} \Rightarrow vx = \pm e^{C} = A \text{ ; } A = e^{C}$$

$$\therefore vx = A \Rightarrow \boxed{v = \frac{A}{x}} \text{. But } v = u' \Rightarrow \int v \, dx = u$$

$$\therefore \int \frac{A}{x} \, dx = u \Rightarrow A \cdot \ln|x| + C = u$$. For simplicity, we let $\boxed{u = \ln|x|}$.

$$\therefore y_2(x) = u \cdot y_1 = \ln|x| \cdot x^4 = x^4 \ln|x|$$

Our general solution (which includes our constant solution of $y=0$) will be...

$$y(x) = c_1 y_1 + c_2 y_2 = c_1[x^4] + c_2[x^4 \ln|x|]$$ since when $x = 0 \Rightarrow y(x) = ($

$$\therefore \boxed{y(x) = c_1 x^4 + c_2 x^4 \ln|x| = x^4[c_1 + c_2 \ln|x|]}$$

Ex. 3 : cont'd

b) $x^2y'' - xy' + 2y = 0$; $y_1 = x \cdot \sin(\ln(x))$

Let $y_2(x) = y_2 = u(x) \cdot y_1 = uy_1 = u[x \cdot \sin(\ln(x))]$

$\therefore y_2' = u'[x \cdot \sin(\ln(x))] + u\left[\sin(\ln(x)) + x \cdot \cos(\ln(x)) \cdot \frac{1}{x}\right]$

$y_2' = u'[x \cdot \sin(\ln(x))] + u[\sin(\ln(x)) + \cos(\ln(x))]$

$\therefore y_2'' = u''[x \cdot \sin(\ln(x))] + u'\left[\sin(\ln(x)) + \cancel{x} \cdot \cos(\ln(x)) \cdot \frac{1}{\cancel{x}}\right]$

$+ u'[\sin(\ln(x)) + \cos(\ln(x))] + u\left[\cos(\ln(x)) \cdot \frac{1}{x} - \sin(\ln(x)) \cdot \frac{1}{x}\right]$

$y_2'' = u''[x \cdot \sin(\ln(x))] + 2u'[\sin(\ln(x)) + \cos(\ln(x))]$

$+ u\left[\cos(\ln(x)) \cdot \frac{1}{x} - \sin(\ln(x)) \cdot \frac{1}{x}\right]$

Now we will sub. in y_2, y_2', and y_2'' into our 2^{nd}-order Linear ODE.

$\therefore x^2 y_2'' - xy_2' + 2y_2 = 0$

$$\Rightarrow x^2\left[u''(x \cdot \sin(\ln(x))) + 2u'(\sin(\ln(x)) + \cos(\ln(x))) + \frac{u}{x}(\cos(\ln(x)) - \sin(\ln(x)))\right]$$

$$-x\left[\qquad + u'(x \cdot \sin(\ln(x))) + u(\sin(\ln(x)) + \cos(\ln(x)))\right]$$

$$+2\left[\qquad + ux(\sin(\ln(x))) \qquad\right] = 0$$

Ex. 3 :

b) cont'd_1

$$\therefore \Big[u''\cdot x^3 \cdot \sin(\ln(x)) + 2x^2u'(\sin(\ln(x)) + \cos(\ln(x))) \overset{-ux(-\cos\ln(x) + \sin(\ln(x)))}{\cancel{+ux(\cos\ln(x) - \sin(\ln(x)))}}$$

$$- x^2u'(\sin(\ln(x))) \qquad - ux(\sin(\ln(x)) + \cos(\ln(x)))$$

$$+ 2ux(\sin(\ln(x))) \Big] = 0$$

$$\Rightarrow u''\cdot x^3 \cdot \sin(\ln(x)) + x^2u'\cdot \sin(\ln(x)) + 2x^2u'\cdot\cos(\ln(x))$$

$$-2ux(\cancel{\sin(\ln(x))}) + 2ux(\cancel{\sin(\ln(x))}) = 0 \quad ; \text{ note that } x \neq 0.$$

$$\Rightarrow \frac{u''\cdot x^3 \cdot \sin(\ln(x)) + x^2u'\cdot\sin(\ln(x)) + 2x^2u'\cdot\cos(\ln(x))}{x^2} = \frac{0}{x^2}$$

$$\Rightarrow u''x\cdot\sin(\ln(x)) + u'\Big[\sin(\ln(x)) + 2\cdot\cos(\ln(x))\Big] = 0$$

Let $v = u'$. Then, $v' = u''$.

$$\therefore v'x\cdot\sin(\ln(x)) = -v\Big[\sin(\ln(x)) + 2\cos(\ln(x))\Big]$$

$$\Rightarrow \frac{v'}{v} = -\frac{1}{x}\left[\frac{\sin(\ln(x) + 2\cos(\ln(x))}{\sin(\ln(x))}\right]$$

$$\Rightarrow \frac{dv}{v} = -\frac{1}{x}\Big[1 + 2\coth(\ln(x))\Big]\,dx$$

$$\Rightarrow \int \frac{dv}{v} = \int -\frac{1}{x}\big(1 + 2\coth(\ln(x))\big)\,dx$$

Ex. 3 :

b) cont'd _ 2

let $\theta = \ln(x) \Rightarrow d\theta = \frac{1}{x}\,dx$. Also, recall that $\coth(\theta) = \frac{\cosh(\theta)}{\sinh(\theta)}$

$\therefore \int \coth(\theta)\,d\theta = \int \frac{\cosh(\theta)}{\sinh(\theta)}\,dx$. let $z = \sinh(\theta) \Rightarrow dz = \cosh(\theta)\,d\theta$

$$\therefore \int \coth(\theta)\,dx = \int \frac{dz}{z} = \ln|z| = \ln|\sinh(\theta)|$$

$$\therefore \int -\frac{1}{x}\Big(1+2\coth\big(\ln(x)\big)\Big)\,dx = \int -\frac{1}{x}\,dx - 2\int \frac{\coth(\ln(x))}{x}\,dx =$$

$$-\ln|x| - 2\ln\big|\sinh(\ln(x))\big|$$

$$\therefore \int \frac{dv}{v} = \int -\frac{1}{x}\Big(1+2\coth\big(\ln(x)\big)\Big)\,dx$$

$$\Rightarrow \ln|v| = -\ln|x| - 2\ln\big|\sinh(\ln(x))\big| + C$$

$$\Rightarrow e^{\ln|v|} = e^{\ln\left|\frac{1}{x}\right| + \ln\left|\frac{1}{\sinh^2(\ln(x))}\right| + C}$$

$$\Rightarrow |v| = \left|\frac{1}{x}\right| \cdot \frac{1}{\sinh^2(\ln(x))} \cdot e^{C}$$

$$\Rightarrow v = \pm e^{C} \cdot \pm\frac{1}{x} \cdot \operatorname{csch}^2(\ln(x)) = \frac{A}{x}\operatorname{csch}^2(\ln(x)) \;; \quad A = \pm e^{C}$$

But $v = u' \Rightarrow \int v\,dx = u$. Also, note that $\left[\coth(\theta)\right]' = -\operatorname{csch}^2(\theta)$

which implies that $\int -\operatorname{csch}^2(\theta)\,d\theta = \coth(\theta)$

Ex. 3

b) cont'd – 3

$$\therefore \int v\,dx = u \quad \Rightarrow \quad \int \frac{A}{x} \cdot \operatorname{csch}^2(\ln(x))\,dx = u$$

let $\beta = \ln(x) \quad \Rightarrow \quad d\beta = \frac{1}{x}\,dx$

$$\therefore \int \frac{A}{x} \operatorname{csch}^2(\ln(x))\,dx = \int A \operatorname{csch}^2(\beta)\,d\beta = A\left[-\coth(\beta)\right] + C_1 =$$

$$-A\left[\coth(\ln(x))\right] + C_1 \quad \Rightarrow \quad u = -A\coth(\ln(x)) + C_1$$

For simplicity, we let $u = \coth(\ln(x))$

$$\therefore y_2(x) = y_2 = u \cdot y_1 = \coth(\ln(x)) \cdot \left[x \cdot \sin(\ln(x))\right]$$

$$\Rightarrow y_2(x) = \frac{\cosh(\ln(x))}{\cancel{\sinh(\ln(x))}_{\,1}} \cdot \left[x \cdot \cancel{\sin(\ln(x))}^{\,1}\right]$$

$$\Rightarrow \boxed{y_2(x) = x \cdot \cosh(\ln(x))}$$

$\therefore$ Our general sol'n for this ODE (which includes the constant solution $y=0$) is $y(x) = c_1 y_1 + c_2 y_2 = c_1\left[x \cdot \sinh(\ln(x))\right] + c_2\left[x \cdot \cosh(\ln(x))\right]$

$$\therefore y(x) = c_1\left[x \cdot \sinh(\ln(x))\right] + c_2\left[x \cdot \cosh(\ln(x))\right]$$

Ex. 3: cont'd

c) $(1-2x-x^2)y'' + 2(1+x)y' - 2y = 0$; $y_1(x) = x+1$

Since $y_1(x) = x+1 \Rightarrow y_1' = 1 \Rightarrow y_1'' = 0$

$\therefore (1-2x-x^2)y_1'' + 2(1+x)y_1' - 2y_1 = 0$

$\Rightarrow (1-2x-x^2)(0) + 2(1+x)(1) - 2(x+1) = 0$

$\Rightarrow 0 + \cancel{2} + \cancel{2x} - \cancel{2x} - \cancel{2} = 0 \qquad \Rightarrow 0 = 0$ ✓

So, $y_1(x) = x+1$ is indeed a solution to the given ODE.

Find $y_2(x)$: Let $y_2(x) = u(x) \cdot y_1(x) = uy_1 = u[x+1] = ux + u$

$\therefore y_2' = u'x + u(1) + u' = u'x + u' + u$

$\therefore y_2'' = u''x + u'(1) + u'' + u' = u''x + u'' + 2u'$

Now we will sub. in y_2, y_2', and y_2'' into our 2^{nd}-order linear ODE...

$\therefore (1-2x-x^2)y_2'' + 2(1+x)y_2' - 2y_2 = 0$

$\Rightarrow (1-2x-x^2)(u''x + u'' + 2u') + (2+2x)(u'x + u' + u) - 2(ux + u) = 0$

$\Rightarrow u''x + u'' + 2u' - 2x^2u'' - 2xu'' - 4xu' - x^3u'' - x^2u'' - 2x^2u'$
$+ 2u'x + 2u' + \cancel{2u} + 2x^2u' + 2xu' + \cancel{2xu} - \cancel{2ux} - \cancel{2u} = 0$

$\Rightarrow u''[x + 1 - 2x^2 - 2x - x^3 - x^2] + u'[2 - \cancel{4x} - \cancel{2x^2} + \cancel{2x} + 2 + \cancel{2x^2} + \cancel{2x}] = 0$

$\Rightarrow u''[-x^3 - 3x^2 - x + 1] + u'[4] = 0$

Ex. 3 :

c) cont'd_1

Let $v = u'$. Then, $v' = u''$. Therefore, $v'[-x^3 - 3x^2 - x + 1] + 4v = 0$

$\therefore -v'[x^3 + 3x^2 + x - 1] = -4v$

$\Rightarrow v'[x^3 + 3x^2 + x - 1] = 4v \Rightarrow \dfrac{v'}{v} = \dfrac{4}{x^3 + 3x^2 + x - 1}$

NOTE: Since $x^3 + 3x^2 + x - 1$ is not factorable via factor-by-grouping technique, we will use the Rational Zeros Theorem and synthetic division to see if this expression has any rational zeros. If it does, we will use this fact to "factor down" $x^3 + 3x^2 + x - 1$.

List of possible rational zeros: $x = \dfrac{\text{factors of } \pm a_0}{\text{factors of } \pm a_n}$, where $x^3 + 3x^2 - x - 1 = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$, where $n = 3$.

$\therefore a_0 = 1$ and $a_n = 1$. So, $x = \dfrac{\{\pm 1\}}{\{\pm 1\}} = \pm 1$. Using synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 1 & -1 \\ & \downarrow & -1 & -2 & 1 \\ \hline & 1 & 2 & -1 & \boxed{0} \end{array} \qquad ; \qquad \begin{array}{r|rrrr} 1 & 1 & 3 & 1 & -1 \\ & \downarrow & 1 & 4 & 5 \\ \hline & 1 & 4 & 5 & \boxed{4} \end{array}$$

$\therefore x = -1$ is a factor of $x^3 + 3x^2 - x - 1$

$x = 1$ is not a factor of $x^3 + 3x^2 - x - 1$

$\therefore x^3 + 3x^2 - x - 1 = (x+1)(x^2 + 2x - 1)$

Ex. 3 :

c) cont'd – 2

So, $\frac{v'}{v} = \frac{4}{x^3+3x^2+x-1} \Rightarrow \frac{dv}{v} = \frac{4}{(x+1)(x^2+2x-1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x-1}$

Performing (traditional) partial fraction expansion (PFE)...

$$\frac{A(x^2+2x-1)+(Bx+C)(x+1)}{(x+1)(x^2+2x-1)} = \frac{4}{(x+1)(x^2+2x-1)}$$

$$\Rightarrow A(x^2+2x-1)+(Bx+C)(x+1) = 4$$

$$\Rightarrow Ax^2+2Ax-A+Bx^2+Bx+Cx+C = 4+0x+0x^2$$

$$\Rightarrow x^2(A+B)+x(2A+B+C)+(-A+C) = 4+0x+0x^2$$

x^2	x	#
$A+B=0$	$2A+B+C=0$	$-A+C=4$
$\Rightarrow A=-B$	$-2B+B+C=0$	$-A-A=4$
or	$-B+C=0$	$-2A=4$
$B=-A$	$C=B$	$\boxed{A=-2} \Rightarrow \boxed{B=2=C}$
	$\Rightarrow C=-A$	

$$\therefore \frac{dv}{v} = \frac{-2}{x+1} + \frac{2x+2}{x^2+2x-1} = \frac{-2}{x+1} + \frac{2(x+1)}{x^2+2x-1}$$

$$\Rightarrow \int \frac{dv}{v} = \int\left(\frac{-2}{x+1}\right)dx + 2\int\left(\frac{x+1}{x^2+2x-1}\right)dx$$

Let $w = x^2+2x-1$

$\Rightarrow dw = (2x+2)\,dx$

$\Rightarrow dw = 2(x+1)\,dx$

$\Rightarrow \frac{1}{2}dw = (x+1)\,dx$

Ex. 3 :

c) cont'd _ 3

$$\therefore \int \frac{dv}{v} = \int \left(\frac{-2}{x+1}\right) dx + 2\int \frac{\frac{1}{2}\,dw}{w}$$

$$\Rightarrow \ln|v| = -2\ln|x+1| + \ln|w| + C$$

$$\Rightarrow \ln|v| = \ln\left|\frac{1}{(x+1)^2}\right| + \ln|w| + C$$

$$\Rightarrow \ln|v| = \ln\left|\frac{x^2+2x-1}{(x+1)^2}\right| + C \text{, since } w = x^2+2x-1$$

$$\Rightarrow e^{\ln|v|} = e^{\ln\left|\frac{x^2+2x-1}{(x+1)^2}\right| + C}$$

$$\Rightarrow |v| = e^C \cdot \left|\frac{x^2+2x-1}{(x+1)^2}\right|$$

$$\Rightarrow v = \pm e^C \cdot \pm\left(\frac{x^2+2x-1}{(x+1)^2}\right) \Rightarrow v = A\left(\frac{x^2+2x-1}{(x+1)^2}\right);\ A = \pm e^C$$

$\boxed{\nabla}$: Alternatively, we can just divide this rational function !

$$\frac{x^2+2x-1}{(x+1)^2} = \frac{x^2+2x-1}{x^2+2x+1}$$

$$\begin{array}{r|l} & 1 \\ \hline x^2+2x+1 & \cancel{x^2}+\cancel{2x}-1 \\ & -\cancel{x^2}-\cancel{2x}-1 \\ \hline & -2 \end{array}$$

$$\therefore \frac{x^2+2x-1}{(x+1)^2} = 1 - \frac{2}{x^2+2x+1}$$

But $v = u' \Rightarrow \int v\,dx = u \Rightarrow \int A\left(\frac{x^2+2x-1}{(x+1)^2}\right)dx = u$

NOTE : Substitution method for integration won't work directly here. We need to rewrite our integrand $\boxed{\nabla}$ in a way that will allow us to use substitution rule.

$$\therefore \frac{x^2+2x-1}{(x+1)^2} = \frac{x^2+2x-1+1-1}{(x+1)^2} = \frac{x^2+2x+1-1-1}{(x+1)^2} = \frac{\overbrace{x^2+2x+1}^{(x+1)^2}}{(x+1)^2} - \frac{2}{(x+1)^2} =$$

$1 - \frac{2}{(x+1)^2}$. Therefore, $\int A\left(\frac{x^2+2x-1}{(x+1)^2}\right)dx = A\int\left(1 - \frac{2}{(x+1)^2}\right)dx$

Ex. 3 :

c) cont'd – 4

$$\therefore \quad u = A\int\left(\frac{x^2+2x-1}{(x+1)^2}\right)dx = A\int\left(1-\frac{2}{(x+1)^2}\right)dx = A\left[x - 2\cdot\frac{-1}{x+1}\right] + C_1$$

$z = x+1$
$dz = dx$

$$\Rightarrow u = A\left[x + \frac{2}{x+1}\right] + C_1$$

$$\Rightarrow u = A\left[\frac{x(x+1)+2}{x+1}\right] + C_1$$

$$\Rightarrow u = A\left[\frac{x^2+x+2}{x+1}\right] + C_1$$

For simplicity, we let $u = u(x) = \dfrac{x^2+x+2}{x+1}$

$$\therefore \quad y_2(x) = u \cdot y_1 = \left(\frac{x^2+x+2}{x+1}\right)(x+1) = x^2+x+2 \quad \Rightarrow \boxed{y_2(x) = x^2+x+2}$$

$\therefore$ Our general solution is $y(x) = c_1 y_1 + c_2 y_2 = c_1(x+1) + c_2(x^2+x+2)$ and our constant solution is $y = 0$.

$\therefore$ General solutions: $y = 0$; $y(x) = c_1(x+1) + c_2(x^2+x+2)$; $c_1, c_2 \in \mathbb{R}$

NOTE: There is not a value of x with combinations of c_1 + c_2 that will make $y(x) = 0$ except for when $c_1 = c_2 = 0$. Thus, the constant solution has to be stated explicitly.