Properties of 2^{nd}-Order (Linear) Homogeneous Equations

In our study of reduction of order for a 2^{nd}-order Linear ODE, we noticed that this method required us to already know a possible solution to our ODE so that we could find another solution that would satisfy the ODE. We also observed that by using the substitution $y_2 = u y_1$; where y_1, y_2, and u are all functions of x ; that finding the other solution we were looking for could end up being a very laborious task!

Our goal in this set of notes is to set the stage to give us some foundational tools and observations that will help us in our goal of (eventually) solving 2^{nd}-order Linear Homogeneous ODEs! Again, these type of ODEs are very important to us because they arise in so many applications. Our focus for this set of notes will be on the following topics:

- Recognizing "suitable" choices of solutions for 2^{nd}-order Linear ODEs and forming a basis of solutions for one of these ODEs.
- Discerning between Linearly Independent vs. Linearly Dependent solutions to a 2^{nd}-order Linear ODE.
- Applying the principles of a Linear Combination of solutions and superposition to finding general solutions to a 2^{nd}-order Linear ODE.
- The existence and uniqueness of initial-value problems (IVPs) when our ODE is 2^{nd}-order Linear.

We will begin with some (relatively) simple 2^{nd}-order Linear ODEs to help demonstrate the main concepts that we wish to illuminate in our aforementioned list of topics we want to focus on. From here, we will state a theorem that will summarize all of these observations (we will call it the "Big Theorem for 2^{nd}-order (Linear) Homogeneous Equations") and close out with a few examples applying this theorem.

"Suitable" solutions and Basis of Solutions

Let us consider the following (2nd-order Linear Homogeneous) ODE.

$$y'' + y = 0 \quad \Rightarrow y'' = -y$$

Note that our ODE here is basically implying that any (known) function that we recognize that has the property that the sum of the original function and its second derivative equals zero would satisfy this ODE! (Alternatively, we could also say that any function whose 2nd derivative is equal to the negative of the original function would also satisfy this ODE).

It turns that our well-known (periodic) functions $\sin(x)$ and $\cos(x)$ have this property!

$y = \sin(x)$

$\Rightarrow y' = \cos(x)$

$\Rightarrow y'' = -\sin(x)$

$\therefore y'' + y = -\sin(x) + \sin(x) = 0$

$y = \cos(x)$

$\Rightarrow y' = -\sin(x)$

$\Rightarrow y'' = -\cos(x)$

$\therefore y'' + y = -\cos(x) + \cos(x) = 0$

Therefore, we see that the functions $\sin(x)$ and $\cos(x)$ are indeed "suitable" choices for solutions to this ODE!

Also, if we wanted to identify a set of corresponding functions that would satisfy our ODE $y'' + y = 0$, we see that the set $\{\sin(x), \cos(x)\}$ could be a "suitable" set of functions

Thus, we make these observations:

* The functions $\sin(x)$ and $\cos(x)$ are (independently) solutions to our given ODE on the interval $x = (-\infty, \infty)$. This is why they are "suitable" choices of solutions,
* The set $\{\sin(x), \cos(x)\}$ is called a basis of solutions for our given ODE! Note that if we found any other combinations of solutions for this ODE $y'' + y = 0$, the functions would also need to have similar periodic properties to $\sin(x)$ + $\cos(x)$!

In general, any function $y(x)$ that is a solution to an ODE can be classified as a "suitable" solution to that ODE. Any collection of suitable solutions to an ODE will form a basis of solutions for the ODE!!

Linearly Independent vs. Linearly Dependent Solutions

Note that sin(x) and cos(x) are considered to be linearly independent solutions to our ODE because there does not exist a way to rewrite sin(x) as a non-zero constant multiple of cos(x) and vice versa!

NOTE: $\cos(x) = \sin(x + \frac{\pi}{2}) \neq c \cdot \sin(x)$, where c = constant.

because...

$$\therefore \cos(x) = \sin(x + \tfrac{\pi}{2}) = \sin(x)\cancelto{0}{\cos(\tfrac{\pi}{2})} + \cos(x)\cancelto{1}{\sin(\tfrac{\pi}{2})}$$

$$= 0 \cdot \sin(x) + 1 \cdot \cos(x)$$

$$= 1 \cdot \cos(x)$$

$$\neq c \cdot \sin(x)$$

The same would hold true if we used $\sin(x) = \cos(x - \frac{\pi}{2})$ as well!

On the other hand, for example, $y_1 = \sin(x)$ and $y_2 = -2\sin(x)$ are linearly dependent solutions since $y_2 = -2\sin(x) = -2 \cdot y_1 \Rightarrow y_2 = c \cdot y_1$ where $c = -2$!

NOTE: When identifying a basis of solutions for a given ODE, we are only concerned in identifying linearly independent functions!

General Solutions to ODEs using Linear Independence + Superposition

Since the set $\{\sin(x), \cos(x)\}$ forms of a basis of solutions for our ODE $y'' + y = 0$ AND our elements / functions in the set are linearly independent of each other, it turns out that any linear combination of the basis of these solutions (i.e. if we let $y = c_1 \cdot \sin(x) + c_2 \cdot \cos(x)$) will also be a "suitable" solution to the ODE $y'' + y = 0$! (*)

(*) See the theorem about superposition on the next page.

This fact is easy to see if $c_1 = 0$ and $c_2 = 1$ (i.e. $y = \cos(x)$) OR if $c_1 = 1$ and $c_2 = 0$ (i.e. $y = \sin(x)$), but it can also be shown that this is true for $y = c_1 \cdot \sin(x) + c_2 \cdot \cos(x)$.

Let $y = c_1 \cdot \sin(x) + c_2 \cdot \cos(x)$. Then, $y' = c_1 \cdot \cos(x) - c_2 \sin(x)$ and $y'' = -c_1 \cdot \sin(x) - c_2 \cdot \cos(x)$. Therefore, ...

$$y'' + y = [-c_1 \cdot \sin(x) - c_2 \cdot \cos(x)] + [c_1 \cdot \sin(x) + c_2 \cdot \cos(x)]$$

$$= (\cancel{-c_1 + c_1}^{\,0}) \cdot \sin(x) + (\cancel{-c_2 + c_2}^{\,0}) \cdot \cos(x) = 0$$

$\therefore$ $y'' + y = 0$ for $y = c_1 \cdot \sin(x) + c_2 \cdot \cos(x)$, where $c_1, c_2 \in \mathbb{R}$!!

In demonstrating that $y = c_1 \cdot \sin(x) + c_2 \cdot \cos(x)$, where $c_1, c_2 \in \mathbb{R}$, is a solution to $y'' + y = 0$ is showing the principle of superposition at work!

★ Thm (Principle of Superposition [for 2nd-order ODEs]): Any linear combination of solutions to a 2nd-order homogeneous linear ODE is also a solution to that 2nd-order linear homogeneous ODE!

2nd-order ODEs and Existence + Uniqueness for IVP ODE solutions

Recall that when we studied 1st-order IVPs and we concluded that a 1st-order ODE with an initial condition $y(x_0) = A$, where $x_0, A \in \mathbb{R}$ such that x_0 is within the interval of interest for the ODE, has a solution to it AND that solution was unique as well! This concept of an IVP having a solution AND the uniqueness of these solutions also extend over to 2nd-order (homogeneous linear) ODEs as well! (Recall that the concept of IVPs for 1st-order ODEs was mentioned in the notes "Differential Equations: Basic Definitions and Classifications" on pages 6 + 8 of these notes!

Using this concept in an example, consider the following 2 IVPs:

(a) $y'' + y = 0$; $y(0) = 1$ and $y'(0) = 2$

(b) $y'' + y = 0$; $y(\frac{\pi}{2}) = 1$ and $y'(\frac{\pi}{2}) = 2$

Since we stated earlier that the 2^{nd}-order IVP should have solutions that are unique, it follows that these 2 IVPs should have different, unique solutions (i.e. the values of c_1 and c_2 for (a) and (b) will be different). We will find c_1 and c_2 for (a) and (b) to show that this is indeed the case.

<u>sol'n for (a)</u>

Let $y = c_1 \sin(x) + c_2 \cos(x)$.

$\Rightarrow y' = c_1 \cos(x) - c_2 \sin(x)$

$\therefore y(0) = 1 \Rightarrow c_1 \underbrace{\sin(0)}_{0} + c_2 \underbrace{\cos(0)}_{1} = 1$

$\Rightarrow 0 - c_2(1) = 1 \Rightarrow \boxed{c_2 = 1}$

$\therefore y'(0) = 2 \Rightarrow c_1 \underbrace{\cos(0)}_{1} - c_2 \underbrace{\sin(0)}_{0} = 2$

$\Rightarrow c_1(1) - 0 = 2 \Rightarrow \boxed{c_1 = 2}$

<u>sol'n for (b)</u>

Let $y = c_1 \sin(x) + c_2 \cos(x)$

$\Rightarrow y' = c_1 \cos(x) - c_2 \sin(x)$

$\therefore y(\frac{\pi}{2}) = 1 \Rightarrow c_1 \underbrace{\sin(\frac{\pi}{2})}_{1} + c_2 \underbrace{\cos(\frac{\pi}{2})}_{0} = 1$

$\Rightarrow c_1(1) + 0 = 1 \Rightarrow \boxed{c_1 = 1}$

$\therefore y'(\frac{\pi}{2}) = 2 \Rightarrow c_1 \underbrace{\cos(\frac{\pi}{2})}_{0} - c_2 \underbrace{\sin(\frac{\pi}{2})}_{1} = 2$

$\Rightarrow 0 - c_2(1) = 2 \Rightarrow \boxed{c_2 = -2}$

So, IVPs (a) + (b) have different, unique solutions since the values for c_1 and c_2 for each case are different!!!

In summary, we conclude that for $y''+y=0$:

(1) The functions $\sin(x)$ + $\cos(x)$ are "suitable" solutions for our ODE and the set $\{\sin(x), \cos(x)\}$ form a basis of solutions for the ODE $y''+y=0$

(2) The functions $\sin(x)$ + $\cos(x)$ in our basis of solutions are linearly independent of each other since $\sin(x) \neq c \cdot \cos(x)$ where $c \in \mathbb{R}$ (constant) and $\cos(x) \neq c \cdot \sin(x)$!

(3) Since $\sin(x)$ and $\cos(x)$ are linearly independent of each other, it follows that any linear combination of $\sin(x)$ + $\cos(x)$ (i.e. $y = c_1 \cos(x) + c_2 \sin(x)$, where $c_1, c_2 \in \mathbb{R}$) can also be a general solution to $y''+y=0$ via the superposition principle

(4) If $y''+y=0$ had initial conditions $y(x_1)=A_1$ and $y'(x_2)=B_2$, the (particular) solution of this IVP would be unique! Thus, if we considered initial conditions $y(x_3)=A_3$ and $y'(x_4)=B_4$, where $x_1 \neq x_3$, $A_1 \neq A_3$, $x_2 \neq x_4$, and $B_2 \neq B_4$, then this implies that c_1 + c_2 will be different for each case!

Now we shall generalize what we concluded for our example using the ODE $y''+y=0$ for any 2^{nd}-order homogeneous (linear) ODE before working out a few extra examples for practice

Thm (Big Theorem for 2^{nd}-order Linear Homogeneous ODEs)

Let $x=(\alpha,\beta)$ and $y=y(x)$ be a solution for a 2^{nd}-order Linear Homogeneous ODE ...

$$[\star]\quad ay''+by'+cy=0$$, where a,b,c can either be constants or functions of x such that $a\neq 0$ and a,b,c are all continuous (functions of x if not constants). If all previously stated is true, then the following is also true:

(i) At least 1 basis of solutions $\{y_1, y_2\}$ exists for equation $[\star]$.

(ii) Every basis of solutions consists of a pair of (function) solutions. (Note that constants can be considered as at least 1 of the 2 functions in each pair. Also, note that $y=0$ is a trivial, constant solution for all 2^{nd}-order Linear Homogeneous ODEs!)

(iii) If a fundamental sets of solutions (aka basis of a solution) $\{y_1, y_2\}$ contain functions y_1 & y_2 that are linearly independent, then a general solution for the ODE can be expressed as $y=c_1y_1+c_2y_2$, where c_1, c_2 are constants (i.e. real numbers).

Thm (Big Theorem for 2^{nd}-order Linear Homogeneous ODEs) : cont'd

(iv) For any x-value $x_0 \in (\alpha, \beta)$ and any 2 fixed values A and B (that are not required to be within (α, β) but could be), there exists exactly 1 ordered pair of constants $\{c_1, c_2\}$ such that ...

$$y = y(x) = c_1 y_1(x) + c_2 y_2(x)$$

also satisfies the initial conditions ...

$$y(x_0) = A \quad \text{and} \quad y'(x_0) = B .$$

(In other words, the initial conditions $y(x_0) = A$ and $y'(x_0) = B$ guarantee a unique solution (i.e. unique values of c_1 + c_2) for our ODE $ay'' + by' + cy = 0$)

NOW WE WILL DO A FEW EXAMPLES TO CLOSE OUT THIS SET OF NOTES !!

For the given 2^{nd}-order Linear Homogeneous ODEs, do the following: (12)

(i) Verify that each pair $\{y_1, y_2\}$ is a fundamental sets of solutions (i.e. forms a basis of solutions).

(ii) Find a linear combination of y_1 + y_2 that satisfy the given initial conditions

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(a) ODE: $y'' - y = 0$ with $y(\ln(2)) = 1$ and $y(\ln(2)) = 6$

FUNCTIONS: $y_1(x) = \sinh(x)$ and $y_2(x) = \cosh(x)$

(i): $y_1 = \sinh(x) \Rightarrow y_1' = \cosh(x) \Rightarrow y_1'' = \sinh(x)$

$\therefore\ y_1'' - y_1 = 0 \Rightarrow \sinh(x) - \sinh(x) = 0 \Rightarrow 0 = 0$ ✓

$\boxed{\therefore\ y_1 = \sinh(x) \text{ is a suitable solution}}$

$y_2 = \cosh(x) \Rightarrow y_2' = \sinh(x) \Rightarrow y_2'' = \cosh(x)$

$\therefore\ y_2'' - y_2 = 0 \Rightarrow \cosh(x) - \cosh(x) = 0 \Rightarrow 0 = 0$ ✓

$\boxed{\therefore\ y_2 = \cosh(x) \text{ is a suitable solution}}$

(ii) Let $y = y(x) = c_1 \sinh(x) + c_2 \cosh(x)$

$\therefore\ y' = c_1 \cosh(x) + c_2 \sinh(x)$

(a) : cont'd

(ii) cont'd

NOTE! $y_1 = \sinh(x) = \frac{e^x - e^{-x}}{2} = \frac{1}{2}[e^x - e^{-x}]$ and

$y_2 = \cosh(x) = \frac{e^x + e^{-x}}{2} = \frac{1}{2}[e^x + e^{-x}]$

$\therefore \; y = y(x) = c_1\left[\frac{1}{2}(e^x - e^{-x})\right] + c_2\left[\frac{1}{2}(e^x + e^{-x})\right]$

$\Rightarrow y' = y'(x) = c_1\left[\frac{1}{2}(e^x + e^{-x})\right] + c_2\left[\frac{1}{2}(e^x - e^{-x})\right]$

I.C. : $y(\ln(2)) = 1$

$y(\ln(2)) = 1 \Rightarrow c_1\left[\frac{1}{2}(e^{\ln(2)} - e^{\cancel{-\ln(2)}\,\ln(\frac{1}{2})})\right] + c_2\left[\frac{1}{2}(e^{\ln(2)} + e^{\cancel{-\ln(2)}\,\ln(\frac{1}{2})}\right] = 1$

$\Rightarrow c_1\left[\frac{1}{2}(2 - \frac{1}{2})\right] + c_2\left[\frac{1}{2}(2 + \frac{1}{2})\right] = 1$

$\Rightarrow c_1(\frac{3}{4}) + c_2(\frac{5}{4}) = 1 \quad \Rightarrow \boxed{3c_1 + 5c_2 = 4}$

$y'(\ln(2)) = 6 \Rightarrow c_1\left[\frac{1}{2}(e^{\ln(2)} + e^{-\ln(2)})\right] + c_2\left[\frac{1}{2}(e^{\ln(2)} - e^{-\ln(2)})\right] = 6$

$\Rightarrow c_1\left[\frac{1}{2}(2 + \frac{1}{2})\right] + c_2\left[\frac{1}{2}(2 - \frac{1}{2})\right] = 6$

$\Rightarrow c_1(\frac{5}{4}) + c_2(\frac{3}{4}) = 6 \quad \Rightarrow \boxed{5c_1 + 3c_2 = 24}$

$\therefore \begin{cases} 3c_1 + 5c_2 = 4 \\ 5c_1 + 3c_2 = 24 \end{cases} \Rightarrow \begin{cases} -15c_1 - 25c_2 = -20 \\ 15c_1 + 9c_2 = 72 \end{cases} \Rightarrow -16c_2 = 52 \Rightarrow c_2 = \frac{52}{-16} = \frac{-13}{4}$

$\therefore \boxed{c_2 = -\frac{13}{4}} \Rightarrow c_1 = \frac{4 - 5c_2}{3} = \frac{4 - 5(\frac{-13}{4})}{3} = \frac{27}{4}$

$\therefore \boxed{c_1 = \frac{27}{4}}$

$\therefore \boxed{y = \frac{27}{4}\sinh(x) - \frac{13}{4}\cosh(x)}$

(b) ODE : $y'' + 4y = 0$ with $y(0) = 2$ and $y'(0) = 6$ 14

FUNCTIONS : $y_1(x) = \cos(2x)$ and $y_2(x) = \sin(2x)$

(i) $y_1 = \cos(2x) \Rightarrow y_1' = -2\sin(2x) \Rightarrow y_1'' = -4\cos(2x)$

$\therefore y_1'' + 4y_1 = 0 \Rightarrow \cancel{-4\cos(2x)} + 4[\cancel{\cos(2x)}] = 0 \Rightarrow 0 = 0$ ✓

$y_2 = \sin(2x) \Rightarrow y_2' = -2\cos(2x) \Rightarrow y_2'' = -4\sin(2x)$

$\therefore y_2'' + 4y_2 = 0 \Rightarrow \cancel{-4\sin(2x)} + 4[\cancel{\sin(2x)}] = 0 \Rightarrow 0 = 0$ ✓

(ii) Let $y = y(x) = c_1\cos(2x) + c_2\sin(2x)$. Then $y' = -2c_1\sin(2x) + 2c_2\cos(2x)$

Applying I.C. : $y(0) = 2$

$y(0) = 2 \Rightarrow c_1 \cancelto{1}{\cos(0)} + c_2 \cancelto{0}{\sin(0)} = 2 \Rightarrow c_1(1) + 0 = 2 \Rightarrow \boxed{c_1 = 2}$

Applying I.C. : $y'(0) = 6$

$y'(0) = 6 \Rightarrow -2c_1 \cancelto{0}{\sin(0)} + 2c_2 \cancelto{1}{\cos(0)} = 6 \Rightarrow 0 + 2c_2 = 6 \Rightarrow \boxed{c_2 = 3}$

$\therefore \boxed{y = y(x) = 2\cos(2x) + 3\sin(2x)}$

(c) ODE: $(x+1)^2 y'' - 2(x+1)y' + 2y = 0$ with $y(0)=0$ and $y'(0)=4$ 15

FUNCTIONS: $y_1(x) = x^2 - 1$ and $y_2(x) = x+1$

(i) $y_1 = x^2 - 1 \Rightarrow y_1' = 2x \Rightarrow y_1'' = 2$

$\therefore (x+1)^2 y_1'' - 2(x+1)y_1' + 2y_1 = 0 \Rightarrow (x+1)^2(2) - (2x+2)(2x) + 2(x^2-1) = 0$

$\therefore (x^2+2x+1)(2) - (4x^2+4x) + 2x^2 - 2 = 0$

$\Rightarrow 2x^2 + 4x + 2 - 4x^2 - 4x + 2x^2 - 2 = 0$

$\Rightarrow x^2[\cancel{2-4+2}]^{0} + x[\cancel{4-4}]^{0} + [\cancel{2-2}]^{0} = 0 \Rightarrow 0 = 0$ ✓

$y_2 = x+1 \Rightarrow y_2' = 1 \Rightarrow y_2'' = 0$

$\therefore (x+1)^2 y_2'' - 2(x+1)y_2' + 2y_2 = 0 \Rightarrow (x+1)^2(0) - (2x+2)(1) + 2(x+1) = 0$

$\therefore 0 - 2x - 2 + 2x + 2 = 0 \Rightarrow (\cancel{-2x+2x})^{0} + (\cancel{2-2})^{0} = 0 \Rightarrow 0 = 0$ ✓

(ii) Let $y = y(x) = c_1(x^2-1) + c_2(x+1)$. Then $y' = c_1(2x) + c_2$

<u>Applying I.C. $y(0)=0$</u>

$y(0) = 0 \Rightarrow 0 = c_1((0)^2 - 1) + c_2(0+1) \Rightarrow 0 = -c_1 + c_2 \Rightarrow c_1 = c_2$

<u>Applying I.C. $y'(0)=4$</u>

$y'(0) = 4 \Rightarrow 4 = c_1\cancel{(2(0))}^{0} + c_2 \Rightarrow 4 = c_2 \Rightarrow \boxed{c_1 = 4 = c_2}$

$\therefore y = y(x) = 4(x^2-1) + 4(x+1) = 4x^2 \cancel{-4} + 4x \cancel{+4} \Rightarrow \boxed{y = 4x^2 + 4x}$

(d) ODE: $xy'' - y' + 4x^3y = 0$ with $y(\sqrt{\pi}) = 3$ and $y'(\sqrt{\pi}) = 4$

FUNCTIONS: $y_1(x) = \cos(x^2)$ and $y_2(x) = \sin(x^2)$

(i) $y_1 = \cos(x^2) \Rightarrow y_1' = -2x\sin(x^2) \Rightarrow y_1'' = -2\sin(x^2) - 4x^2\cos(x^2)$

$\therefore\ xy_1'' - y_1' + 4x^3y_1 = 0$

$\Rightarrow x[-2\sin(x^2) - 4x^2\cos(x^2)] - (-2x\sin(x^2)) + 4x^3[\cos(x^2)] = 0$

$\Rightarrow -2x\sin(x^2) - 4x^3\cos(x^2) + 2x\sin(x^2) + 4x^3\cos(x^2) = 0$

$\Rightarrow [\underbrace{\cancel{-2x\sin(x^2) + 2x\sin(x^2)}}_{0}] + [\underbrace{\cancel{-4x^3\cos(x^2) + 4x^3\cos(x^2)}}_{0}] = 0 \Rightarrow 0 = 0$ ✓

$y_2 = \sin(x^2) \Rightarrow y_2' = 2x\cos(x^2) \Rightarrow y_2'' = 2\cos(x^2) - 4x^2\sin(x^2)$

$\therefore\ xy_2'' - y_2' + 4x^3y_2 = 0$

$\Rightarrow x[2\cos(x^2) - 4x^2\sin(x^2)] - 2x\cos(x^2) + 4x^3[\sin(x^2)] = 0$

$\Rightarrow 2x\cos(x^2) - 4x^3\sin(x^2) - 2x\cos(x^2) + 4x^3\sin(x^2) = 0$

$\Rightarrow [\underbrace{\cancel{2x\cos(x^2) - 2x\cos(x^2)}}_{0}] + [\underbrace{\cancel{-4x^3\sin(x^2) + 4x^3\sin(x^2)}}_{0}] = 0 \Rightarrow 0=0$ ✓

(ii) Let $y = y(x) = c_1\cos(x^2) + c_2\sin(x^2) \Rightarrow y' = c_1[-2x\sin(x^2)] + c_2[2x\cos(x^2)]$

Applying I.C. $y'(\sqrt{\pi}) = 4$

$y(\sqrt{\pi}) = 4 \Rightarrow 4 = c_1[-2\sqrt{\pi}\,\underbrace{\cancel{\sin((\sqrt{\pi})^2)}}_{0}] + c_2[2\sqrt{\pi}\,\underbrace{\cancel{\cos((\sqrt{\pi})^2)}}_{-1}]$

$\Rightarrow 4 = c_2(-2\sqrt{\pi})$

$\Rightarrow \frac{-4}{2\sqrt{\pi}} = c_2 \Rightarrow \boxed{-\frac{2}{\sqrt{\pi}} = c_2}$

(d) cont'd

(ii) cont'd

Applying I.C. $y(\sqrt{\pi}) = 3$

$$y(\sqrt{\pi}) = 3 \Rightarrow c_1 \cancelto{-1}{\cos\left((\sqrt{\pi})^2\right)} + c_2 \cancelto{0}{\sin\left((\sqrt{\pi})^2\right)} = 3$$

$$\Rightarrow c_1(-1) = 3$$

$$\Rightarrow \boxed{c_1 = -3}$$

$$\therefore \boxed{y = y(x) = -3\cos(x^2) - \frac{2}{\sqrt{\pi}}\sin(x^2)}$$