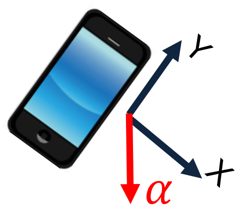
1. The output of a causal LTI system with the impulse response $h(t)$ to a causal input $x(t)$ is

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau$$

2. Accelerometer (± 4g) with analog output and power supply of +4V is used in smartphone to determine orientation of the smartphone according to the figure below. What are the values of X and Y components of the accelerometer for α = 45°.



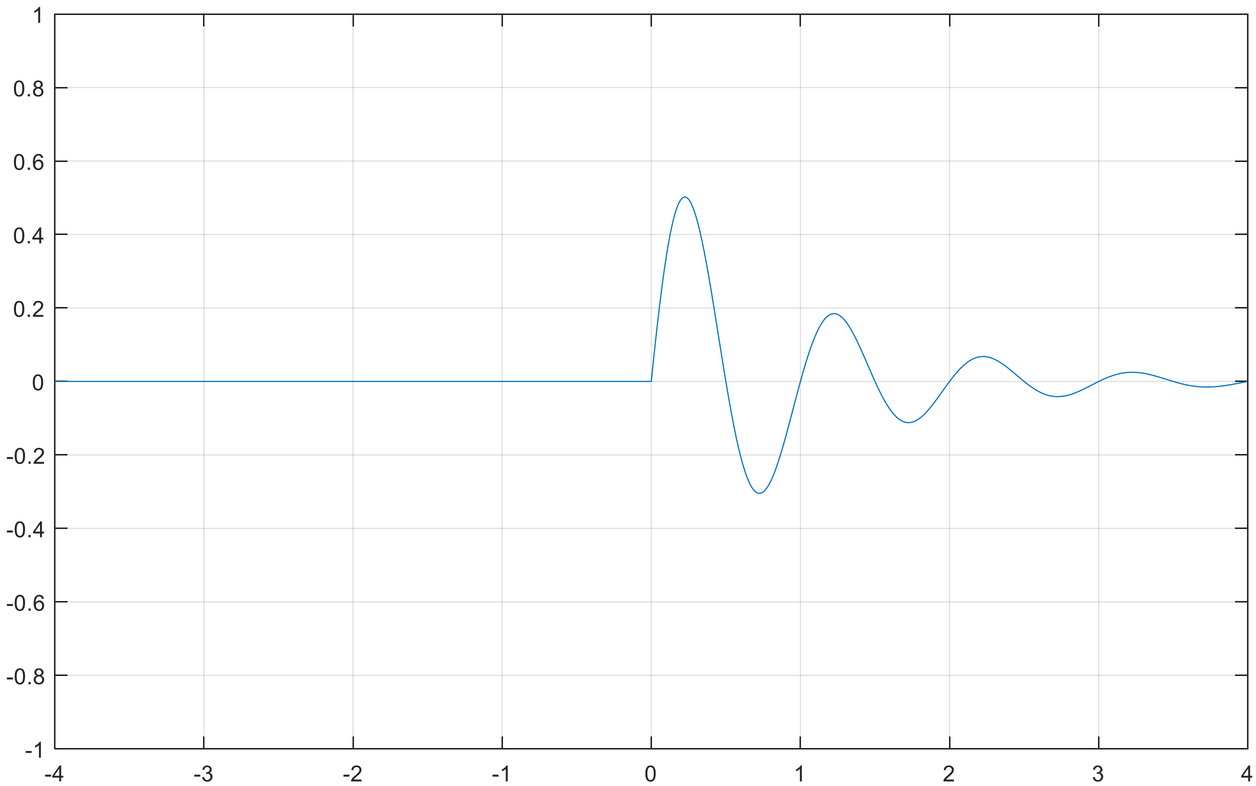
Sensitivity of the accelerometer 1g → s = 4V / 8g = 0.5 [V / g]

A_0 (0 g) = 2V

$A_X = A_0 +$ 1g * cos(π/4) * s = 2.35 V

$A_Y = A_0 +$ -(1g * sin(π/4)) * s = 1.65 V

3. $x(t) = \frac{2}{\pi} e^{-t} \cdot \sin(2\pi t) \cdot u(t)$



4. Consider the periodic signal x(t) = cos(0.4π·*t*) + 3·cos(2 π·*t/7*), −∞ < t < ∞..

Is x(t) periodic? If it is, what is the period T_0 of $x(t)$? T_0 = 35 s

$x(t) = x_1(t) + x_2(t)$

$T_1 = 2\pi / (0.2\pi) = 5$ s

$T_2 = 2\pi / (2\pi/7) = 7$ s

$T_0 = N*T_1 = M*T_2$ → The least common multiple of 5 and 7 is 35, therefore 7N = 5M → $T_0 = 7*5 = 35$ s

What is the average power of $x(t)$?

$$\int_0^x \cos^2(x)dx = \int_0^x \frac{1}{2}(1+\cos(2x)) = \frac{1}{2}\int_0^x dx + \frac{1}{4}\int_0^x \cos(y)dy = \left(\frac{x}{2} + \frac{1}{4}\sin(2x)\right)\Bigg|_0^x$$

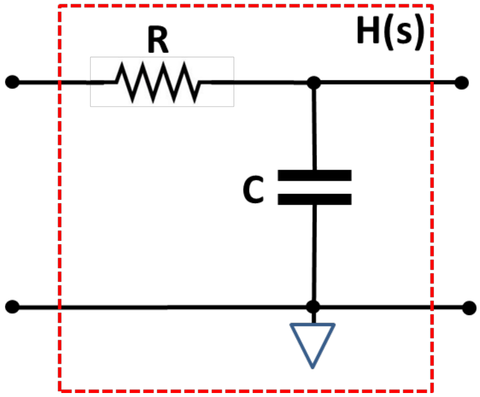
$$\int_0^t \cos^2(x)dx = \left(\frac{x}{2} + \frac{1}{4}\sin(2x)\right)\Bigg|_0^t = \frac{t}{2} + \frac{1}{4}\sin(2t) \Rightarrow \textit{for } t = T \int_0^T \cos^2(x)dx = \frac{1}{2}$$

$$P_{x1} = \frac{1}{T_1}\int_0^{T_1} x_1^{\,2}(t)dt = \frac{1}{0.5}\cdot\left(\frac{x}{2} + \frac{1}{4}\sin(2x)\right)\Bigg|_0^{T_1} = \frac{1}{T_1}\left(\frac{T_1}{2} + \frac{1}{4}\sin\left(12\pi\cdot\frac{1}{6}\right)\right) = 1\cdot\frac{1}{2} = 0.5$$

$$P_{x2} = \frac{1}{T_2}\int_0^{T_2} x_2^{\,2}(t)dt = \frac{1}{T_2}\int_0^{T_2} (3\cos(16\pi t))^2 dt = 9\cdot\frac{1}{T_2}\int_0^{T_2} \cos^2(16\pi\, t)dt = 9\cdot\frac{1}{2} = 4.5$$

$P = P_{x1} + P_{x2} = 0.5 + 4.5 = 5$

5. (4 points)



$$H(s)=\frac{Y(s)}{X(s)}=\frac{\frac{1}{Cs}}{R+\frac{1}{Cs}}=\frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

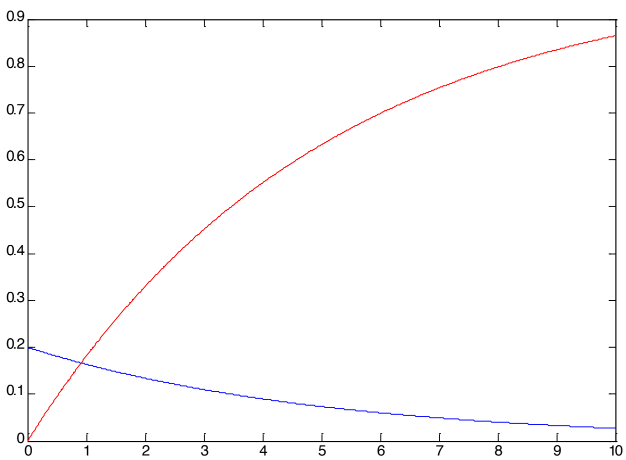
$$h(t)=\frac{1}{RC}\cdot e^{-\frac{t}{RC}}$$

b) (6 points) Step response:

$$\frac{1}{RC}=\frac{1}{5M\Omega\cdot 1\mu F}=0.2$$

$$S(s)=\frac{1}{s}H(s)=\frac{1}{s}\frac{0.2}{s+0.2}=\frac{A}{s}+\frac{B}{s+0.2}=\frac{1}{s}-\frac{1}{s+0.2}$$

$$s(t)=\left(1-e^{-0.2\cdot t}\right)\cdot u(t)$$



6. A system with input x(t) and output y(t) is defined by the following differential equation:

$$\ddot{y}(t) + 2\dot{y}(t) + 1 - y(t) = \dot{x}(t)$$

Initial conditions are $y(0\text{-})$=0, $dy/dt\,|_{t=0}$ =1, and $x(t)=u(t)$. Find the response $y(t)$ and identify steady state response and transient response.

Find Laplace transform of the equation:

$$s^2Y(s) - sy(0-) - \dot{y}(0) \quad + 2\big(sY(s) - y(0-)\big) \quad + \frac{1}{s} \quad - Y(s) \ = 1$$

Since $\mathcal{L}[1] = \int_0^\infty 1e^{-st}dt = \frac{e^{-st}}{-s}\Big|_0^\infty = \frac{1}{s}$;

derivative of the step function $\dot{u}(t)$ is impulse, and Laplace transform of impulse is 1.

$$s^2Y(s) - 1 + 2sY(s) + \frac{1}{s} - Y(s) \ = 1$$

$$Y(s)(s^2 + 2s - 1) = 2 - \frac{1}{s} = \frac{2s-1}{s}$$

$$Y(s) = \frac{2s-1}{s(s^2+2s-1)} = \frac{2s-1}{s(s-p_2)(s-p_3)}$$

Roots of $(s^2 + 2s - 1)$ are $p_{2,3} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2\pm\sqrt{4+4}}{2} = -1 \pm \sqrt{2} = \{-2.4, 0.4\}$ and $p_1 = 0$.

$$Y(s) = \frac{A}{s} + \frac{B}{s-p_2} + \frac{C}{s-p_3}$$

we find

$$A = \frac{2s-1}{(s-p_2)(s-p_3)}\Big|_{s=0} = \frac{-1}{p_2p_3} = 1$$

$$B = \frac{2s-1}{s(s-p_3)}\Big|_{s=p_2} = \frac{2*(-2.4)-1}{-2.4*(-2.4-0.4)} = -0.85$$

$$C = \frac{2s-1}{s(s-p_2)}\Big|_{s=p_3} = \frac{2*0.4-1}{0.4*(0.4+2.4)} = -0.15$$

therefore:

$$y(t) = \mathcal{L}^{-1}\left(\frac{A}{s} + \frac{B}{s-p_2} + \frac{C}{s-p_3}\right) = (1 - 0.85e^{-2.4t} - 0.15e^{0.4t})u(t)$$

steady state response is $\qquad y_{ss}(t) = 1 \cdot u(t)$

and transient response is $\qquad y_{tr}(t) = (-0.85e^{-2.4t} - 0.15e^{0.4t})u(t)$

Alternative problem

$$\ddot{y}(t) + 2\dot{y}(t) + 10y(t) = \dot{x}(t)$$

Initial conditions are $y(0\text{-})=0$, $dy/dt\,|_{t=0} = 1$, and $x(t)=u(t)$. Find the response $y(t)$ and identify steady state response and transient response.

Find Laplace transform of the equation:

$$s^2Y(s) - sy(0-) - \dot{y}(0) \quad + 2\big(sY(s) - y(0-)\big) + 10Y(s) \; = 1$$

derivative of the step function $\dot{u}(t)$ is impulse, and Laplace transform of impulse is 1.

$$s^2Y(s) - 1 + 2sY(s) + 10Y(s) \; = 1$$

$$Y(s)(s^2 + 2s + 10) = 2$$

$$Y(s) = \frac{2}{s^2 + 2s + 10} = \frac{2}{(s+1)^2 + 9}$$

therefore:

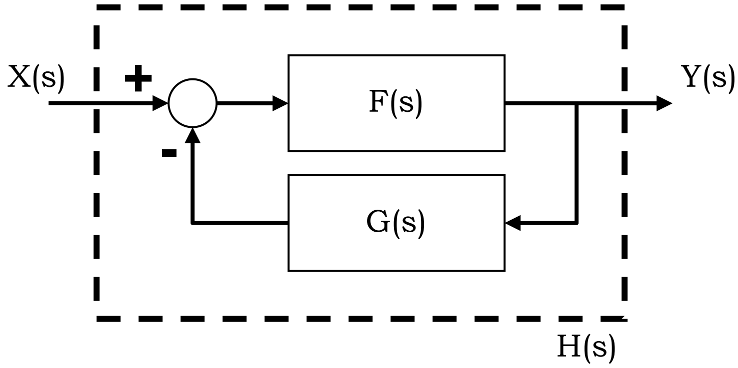
$$y(t) = \mathcal{L}^{-1}\left(\frac{2}{3}\frac{3}{(s+1)^2 + 9}\right) = \frac{2}{3}e^{-t}\sin\,(3t)u(t)$$

steady state response is $y_{ss}(t) = 0$

and transient response is

$$y_{tr}(t) = \frac{2}{3}e^{-t}\sin\,(3t)u(t)$$

7. (5 points) What is the transfer function H(s) of the system represented below?



Y(s) = F(s)*(X(s) - G(s)*Y(s)) = F(s)*X(s) - F(s)*G(s)*Y(s)

$$H(s) = \frac{Y(s)}{X(s)} = \frac{F(s)}{1 + F(s) \cdot G(s)}$$

b) (10 points) Transfer function of the feedback block is

$$G(s) = \frac{Z_{R||C}}{Ls + Z_{R||C}}$$

$$Z_{R||C} = \frac{R\frac{1}{Cs}}{R + \frac{1}{Cs}} = \frac{R}{RCs + 1}$$

$$G(s) = \frac{R}{R + Ls(RCs + 1)} = \frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

Transfer function of the system is:

$$H(s) = \frac{F(s)}{1 + F(s) \cdot G(s)}$$

$F(s) = A$ and $G(s) = \frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$

$$H(s) = \frac{A}{1 + A\left(\frac{\frac{1}{LC}}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}\right)} \; for\ A \to \infty\ H(s) = LCs^2 + \frac{L}{R}s + 1$$

8.