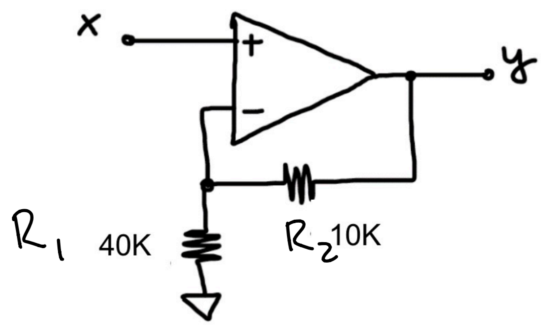
1. (10 points) What is the transfer function of the following circuit:



$$y(t) \cdot \frac{R_1}{R_1+R_2} = x(t)$$

$$y(t) = \frac{R_1+R_2}{R_1} \cdot x(t)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[x(t)]}$$

$$y(t) = \frac{40k + 10K}{40k} \cdot x(t)$$

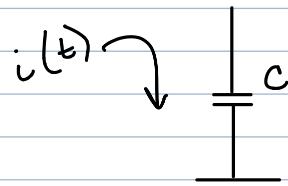
$$y(t) = \frac{5}{4} x(t)$$

$$\mathcal{L}[y(t)] = \frac{5}{4} \mathcal{L}[x(t)]$$

$$H(s) = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[x(t)]} = \boxed{\frac{5}{4}}$$

3. (10 points)
Find impulse response of capacitor and its unit step response. How step response depends on the capacitance of the capacitor?

Impulse Response of a Capacitor



$$v_C(t) = \frac{1}{C}\int_0^t i(t)\,dt$$

$$h(t) = \frac{1}{C}\int_0^t \cancel{\delta(\tau)}\,d\tau$$ 1 impulse

$$\boxed{h(t) = \frac{1}{C}} u(t)$$ impulse

Unit step:

$$\cancel{u(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}$$

$$\cancel{u(t) = \int_0^t 1\cdot\frac{1}{C}\,d\tau = \frac{1}{C}\int_0^{-\infty} d\tau}$$

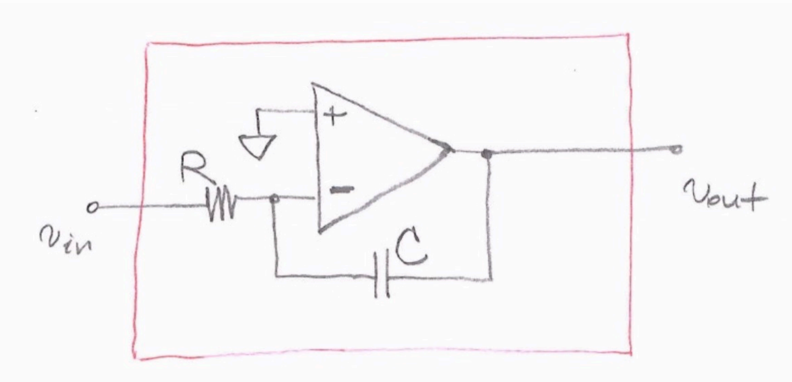
$$\cancel{u(t) = \frac{1}{C}t = \boxed{\frac{t}{C}}}$$ ~~Unit step response~~

$$v_C(t) = \int_{-\infty}^{\infty} h(t-\tau)\,i(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} \frac{1}{C}\,u(t-\tau)\,u(t)\,d\tau$$

$$= \frac{1}{C}\int_0^t d\tau = \boxed{\frac{1}{C}\,r(t)}$$

4. (15 points) Find transfer function of the following circuit. What is the output voltage at t=0.4s if the input is step function u(t), R=10KΩ, and C=10 μF?



$$v_c(t) = \frac{1}{C}\int_0^t i(t)\, d\tau \qquad v_{out} = -v_{in}$$

$$V = IR \rightarrow I = \frac{V}{R}$$

$$\Rightarrow v_{out}(t) = -\frac{1}{C}\int_0^t \frac{v_i(t)}{R}\, d\tau$$

$$v_{out}(t) = -\frac{1}{RC}\int_0^t v_i(t)\, d\tau$$

$$v_{out}(t) = \frac{-1}{RC}\left[v_i t\right]_0^t = \frac{-v_i}{RC}\left[t\right]_0^t$$

$$\frac{v_o(t)}{v_i(t)} = \frac{-1}{RC}\left[t\right]_0^t$$

$$\frac{v_o(t)}{v_i(t)} = \frac{-1}{RC}\left[t - \cancel{0}\right] = \frac{-t}{RC}$$

$$H(s) = \frac{\mathcal{L}[v_o(t)]}{\mathcal{L}[v_i(t)]} = \mathcal{L}\left[\frac{-t}{RC}\right] = \frac{-1}{RC}\mathcal{L}[t] \qquad \mathcal{L}[t] = \frac{1}{s}$$

Transfer function: w/ $R = 10\,k\Omega$ and $C = 10\,\mu F$

$$H(s) = \frac{-1}{RCs} \quad \text{or} \quad h(t) = \frac{-t}{RC}$$

$$h(0.4) = \frac{v_o(0.4)}{v_i(0.4)} = \frac{-(0.4)}{(10k)(10\mu)}$$

given no v_i

$$v_o(t) = \frac{-(0.4)}{(10k)(10\mu)} \cdot v_i(t) = \boxed{-4\ V} \quad \text{or} \quad -4v_i$$

6. (10 points)

Describe the basic properties of the one sided Laplace transform.

1. It is linear, so $\mathcal{L}[af(t) + bg(t)] = aF(s) + bF(s)$
2. It can be shifted by time or freq. so:

$$\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$$

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

3. For multiplication by t:

$$\mathcal{L}[t\,f(t)] = -\frac{dF(s)}{ds}$$

4. For the derivative + second derivative:

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0-)$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = s^2F(s) - sf(0-) - f$$

5. For integrals:

$$\mathcal{L}\left[\int_{0-}^{t} f(t')\,dt'\right] = \frac{F(s)}{s}$$

6. For expantion / contraction:

$$\mathcal{L}[f(at)] = \frac{1}{|a|}F\left(\frac{s}{a}\right) \quad \text{where } a \neq 0$$

7. For initial value:

$$\mathcal{L}[f(0-)] = \lim_{s\to\infty} sF(s)$$

7. (15 points)

Find and use the Laplace transform of $e^{j(\Omega_0 t+\Theta)}u(t)$ to obtain the Laplace transform of

$$x(t) = cos(\Omega_0 t + \Theta) \cdot u(t)$$

Consider the special cases for Θ =0, Θ = -π/2, and Θ = π/4.

$$\mathcal{L}\left[e^{j(\Omega_0 t + \theta)} u(t)\right] = \int_0^\infty e^{j(\Omega_0 t + \theta)} e^{-st} dt$$

$$= e^{j\theta} \int_0^\infty e^{-(s - j\Omega_0)t} dt = \frac{-e^{j\theta}}{s - j\Omega_0} e^{-\sigma t - j(\Omega - \Omega_0)t} \Big]_0^\infty$$

$$= \frac{e^{j\theta}}{s - j\Omega_0} \rightarrow \text{complex causal side}$$

By Euler's Identity:

$$\cos(\Omega_0 t + \theta) = \frac{e^{j(\Omega_0 t + \theta)} + e^{-j(\Omega_0 t + \theta)}}{2}$$

Knowing ↑ and linearity of the integral:

$$X(s) = \mathcal{L}\left[\cos(\Omega_0 t + \theta) u(t)\right]$$

$$= \underbrace{0.5 \, \mathcal{L}\left[e^{j(\Omega_0 t + \theta)} u(t)\right] + 0.5 \, \mathcal{L}\left[e^{-j(\Omega_0 t + \theta)} u(t)\right]}_{\text{solved for in first step}}$$

$$= 0.5 \left[\frac{e^{j\theta}(s + j\Omega_0) + e^{-j\theta}(s - j\Omega_0)}{s^2 + \Omega_0^2}\right]$$

$$= \boxed{0.5 \left[\frac{s\cos(\theta) - \Omega_0 \sin(\theta)}{s^2 + \Omega_0^2}\right]}$$

<u>Case θ=0</u>

$$\mathcal{L}\left[\cos(\Omega_0 t) u(t)\right] = \boxed{\frac{s}{s^2 + \Omega_0^2}}$$

Case $\theta = \frac{-\pi}{2}$

$\cos(\Omega_o t - \pi/2) = \sin(\Omega_o t)$ so:

$$\mathcal{L}[\sin(\Omega_o t)] = \boxed{\frac{\Omega_o}{s^2 + \Omega_o^2}}$$

Case $\theta = \frac{\pi}{4}$

$\cos(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

$$\mathcal{L}[\cos(\Omega_o t + \frac{\pi}{4})] = \boxed{\frac{\frac{\sqrt{2}}{2}(s + \Omega_o)}{s^2 + \Omega_o^2}}$$