Linear 1^{st}-order Equations

- Basic Notation / Form of 1^{st}-order linear ODEs
- Detailed Procedure on Solving Linear 1^{st}-order ODEs / Integrating Factor
- Abbreviated Summary of Solving Linear 1^{st}-order ODEs
- Practice Problems

What is a Linear 1^{st}-order ODE?

A 1^{st}-order ODE of the form ...

$$\frac{dy}{dx} = r(x) - p(x)y \iff \underbrace{\frac{dy}{dx} + p(x)y = r(x)}_{\text{standard form of Linear } 1^{st}\text{-order ODE}},$$

where $p(x)$ and $r(x)$ are known functions of <u>only</u> x (or they can be a constant value).

<u>NOTE</u>: We will rearrange an ODE that is classified as Linear-1^{st}-order in the standard form as it will be easier to recognize and use in this form

Now we will investigate how we actually solve equations like these. Note that we are purposely going through the "long way" of solving this type of equation so that you gain the intuition of knowing the twists, turns, and tricks involved in solving these equations.

Procedure for Solving Linear 1st-order ODEs (Long Way)

The procedure that follows will take you through the derivation process to see how we solve linear, 1st-order ODEs. Although the end result will be a nice, concise formula we can use to solve these type of equations quickly, it is important to actually go through this process at least once as by doing it, we will gain intuition that will (1) help us to know when actually need to use the resulting formula we will come up with vs. figuring out the key "thing" we need by inspection and (2) help us to understand the motivation for other ODE solving techniques for other types of ODEs whose foundation is rooted in the procedure we will learn here. Two such examples of other equation types that will have its foundation in 1st-order, linear ODEs are Bernoulli equations and (1st-order) Homogeneous equations.

Goal of Derivation: To transform a linear, 1st-order ODE (i.e. $\frac{dy}{dx} = g(x) - p(x)\,y = f(x,y)$) into an ODE that is just a function of x on both sides. Consequently, this will change our 1st-order, linear ODE into a directly integrable ODE!

Here is how we do it!

(1) Given that our 1^{st}-order ODE is of the form $\frac{dy}{dx} + py = r$ (whe
$p = p(x)$, $y = y(x) =$ solution to ODE that is not known yet, and $r = r(x)$ and p and q
are known functions or constants from our given ODE), let $w = w(x)$ be a ne
(unknown) function of x such that $wy = w(x) \cdot y(x) = F(x)$ (i.e. the produ
of functions "w" + "y" create a new function only in terms of x).

NOTE: The function $w = w(x)$ will be called our __________ factor because it is this function we will use to manipulate our Linear, 1^{st}-order ODE into a directly integrable equation!

(2) Multiply 1^{st}-order ODE on both sides by "w".

$$\therefore \frac{dy}{dx} + py = r \quad \Rightarrow \quad w\frac{dy}{dx} + wpy = wr \quad [1]$$

(3) Note that $\frac{dF}{dx} = \frac{d}{dx}[wy] = \frac{dw}{dx}y + w\frac{dy}{dx} = w\frac{dy}{dx} + \frac{dw}{dx}y$ [2].

(4) Set equations [1] and [2] equal to each other and determine what conditions have to be met for these equations to actually be equal to each other.

Condition #2

$$\therefore [1] = [2] \Rightarrow \underbrace{w\frac{dy}{dx} + wpy = wr}_{[1]} = \underbrace{\frac{d}{dx}[wy] = w\frac{dy}{dx} + \frac{dw}{dx}y}_{[2]}$$

Condition #1

(4) cont'd

$\therefore \frac{d}{dx}[wy] = wr$ and $\frac{dw}{dx} = wp$ must both be true in order for equations [1] and [2] to actually be equal to each other!

NOTE: The functions $r = r(x)$ and $p = p(x)$ are already known. We don't know what $w = w(x)$ is (yet) and we are ultimately looking for $y = y(x)$ (since $y = y(x)$ will be the solution to this linear, 1^{st}-order ODE). However, we can use the equation $\frac{dw}{dx} = wp$ to find w. Afterwards, we can use what we found w to be (i.e. once we find the integrating factor) in the equation $\frac{d}{dx}[wy] = wr$ in order to find out what $y = y(x)$ ultimately is going to be!!

(5) Use $\frac{dw}{dx} = wp$ to find the integrating factor, $w = w(x)$.

$$\frac{dw}{dx} = wp \Rightarrow w'(x) = wp \Rightarrow w'(x)\,dx = wp\cdot dx.$$

$$\text{Since } \frac{dw}{dx} = w'(x) \Rightarrow dw = w'(x)\cdot dx$$

$$\therefore \overset{dw}{\cancel{w'(x)\cdot dx}} = wp\cdot dx \Rightarrow dw = wp\cdot dx \Rightarrow \frac{dw}{w} = p\cdot dx$$

$$\therefore \int \frac{dw}{w} = \int p\cdot dx \Rightarrow \ln|w| = \int p\cdot dx + C \Rightarrow \overset{|w|}{\cancel{e^{\ln|w|}}} = e^{\left(\int p\cdot dx\right)+C}$$

$$\therefore |w| = e^{\int p\cdot dx}\cdot e^{C} \Rightarrow |w| = Ae^{\int p\cdot dx};\ A = e^{C} \Rightarrow \boxed{w = \pm Ae^{\int p\cdot dx} = w(x)}$$

(*) This is the actual result, but in practice, we will just use $w = e^{\int p\cdot}$

(6) Use integrating factor $w = w(x) = e^{\int p \cdot dx}$ ⊛ in the equation $\frac{d}{dx}[wy] = wr$ to solve for $y = y(x)$.

Recall that $F(x) = w(x) \cdot y(x)$. Thus, $\frac{dF}{dx} = \frac{d}{dx}[wy] = wr \Rightarrow \frac{dF}{dx} = wr$.

Since $\frac{dF}{dx} = F'(x)$, it follows that $F'(x) = wr \Rightarrow F'(x) \cdot dx = wr \cdot dx$

$\therefore \int F'(x)\,dx = \int wr \cdot dx \Rightarrow F(x) = \left(\int wr \cdot dx\right) + C$. But $F(x) = wy$.

$$\therefore \quad wy = \left(\int wr \cdot dx\right) + C \quad \Rightarrow \quad y = \frac{1}{w}\left[\left(\int wr \cdot dx\right) + C\right]$$

$\therefore y = w^{-1}\int wr \cdot dx + Cw^{-1}$. Substituting $w = e^{\int p \cdot dx}$ yields...

$$y = e^{-\int p \cdot dx}\left(\int e^{\int p \cdot dx} \cdot r \cdot dx\right) + Ce^{-\int p \cdot dx}$$

(7) Let $h = \int p \cdot dx$ and rewrite formula for $y = y(x)$ in concise form.

Letting $h = \int p \cdot dx \Rightarrow y = y(x) = e^{-h}\left(\int e^{h} \cdot r\,dx\right) + Ce^{-h}$; $h = \int p \cdot dx = h($

$$\therefore \boxed{\begin{gathered} y(x) = e^{-h(x)}\left[\int e^{h(x)} \cdot r(x) \cdot dx\right] + Ce^{-h(x)} \; ; \; h(x) = \int p(x) \cdot dx \\ \text{or} \\ y = e^{-h}\left[\int e^{-h} \cdot r \cdot dx\right] + Ce^{-h} \; ; \; h = \int p \cdot dx \end{gathered}}$$

↳ Formula to solve 1st-order, linear ODEs

Abbreviated Summary of Solving Linear 1st order ODEs

From our derivation process, we found that $y = y(x)$ can be easily found by the formula...

$$\boxed{y = e^{-h}\left[\int e^{-h} \cdot r \cdot dx\right] + Ce^{-h}, \text{ where } h = \int p \cdot dx \text{ and } e^{h} = \text{integrating factor!}}$$

This formula is nice, but it causes us to remember more formulas!! (UGH)

So is there a way for us to solve these kinds of ODE without having to (1) do the derivation process every time and/or (2) remember the formula above? It turns out that the answer is YES!! The key to solving these Linear, 1st-order ODEs in an alternative manner is to remember the "key thing" which is what is the integrating factor $w(x)$ and how to find it.

Recall that $\frac{dy}{dx} + py = r \Rightarrow w\frac{dy}{dx} + wpy = wr$ (in step 2 on pg. 3) and that in step 4 that $\boxed{\frac{dw}{dx} = wp}$ was one of the 2 conditions that had to be satisfied to have equations [1] and [2] equal to each other. Remembering this fact along with the fact that $\boxed{\frac{d}{dx}[wy] = wr}$ (i.e. the other condition for equations [1] and [2] to be equal can be used without doing any of the other steps.

Therefore, to solve linear, 1^{st}-order ODEs, we can summarize this process in a few steps.

(1) Consider and rewrite $\frac{dy}{dx} = r(x) - p(x)y = f(x,y)$ into the (standard) fo

$\frac{dy}{dx} + p(x)y = r(x)$

(2) Identify $p(x)$ and $r(x)$.

(3) Find integrating factor $w = w(x)$ by solving equation $\frac{dw}{dx} = wp$.

(4) Note that $\frac{dy}{dx} + p(x)y = q(x)$ will reduce to $\frac{d}{dx}[wy] = wr$

(5) Use known functions/values $w = w(x)$ and $r = r(x)$ to find $y = y(x)$ by solving the equation $\frac{d}{dx}[wy] = wr$

WE WILL NOW DO A FEW EXAMPLES TO GET FAMILIAR WITH BOTH METHODS OF SOLVING LINEAR, 1^{ST}-ORDER ODEs

Ex: Verify that each ODE is linear. Afterwards, solve the ODE for a general solution and determine the largest interval of interest. Use steps (1) thru (5) on page 7 of these notes (i.e. don't use formula $y = e^{-h}\left[\int e^{h} \cdot r\, dx + C\right]$)

a) $\frac{dy}{dx} + y = e^{3x}$

$\therefore \frac{dy}{dx} + y = e^{3x} \Rightarrow \frac{dy}{dx} + py = r$, where $p = p(x) = 1$ and $r = r(x) = e^{3x}$

So, our ODE is a 1^{st}-order Linear ODE!

To find our integrating factor, note that $\frac{dw}{dx} = wp \Rightarrow \int \frac{dw}{w} = \int p \cdot dx \Rightarrow$ $\ln|w| = \int p \cdot dx \Rightarrow w = \pm A e^{\int p \cdot dx}$. For simplicity, we use $w = e^{\int p \cdot dx}$

$\therefore w = w(x) = e^{\int 1 \cdot dx} = e^{x + c} = Ae^{x}$; $A = e^{c}$. For simplicity, let $A = 1$.

$\therefore w = w(x) = e^{x}$.

So, $\frac{dy}{dx} + y = e^{3x} \Rightarrow \frac{d}{dx}[wy] = wr \Rightarrow \frac{d}{dx}[e^{x}y] = e^{x} \cdot e^{3x} = e^{4x}$.

$\therefore e^{x}y = \int e^{4x}\, dx \Rightarrow y = e^{-x}\int e^{4x}\, dx$. Using u-substitution, let

$u = 4x \Rightarrow du = 4 \cdot dx \Rightarrow \frac{1}{4}\, du = dx$.

$\therefore y = e^{-x}\int e^{4x}\, dx = e^{-x}\int \frac{1}{4} e^{u}\, du = e^{-x}\left[\frac{1}{4} e^{u} + C\right] = e^{-x}\left[\frac{1}{4} \cdot e^{4x} + C\right]$

$\therefore y = y(x) = \frac{1}{4} e^{3x} + Ce^{-x}$. Since e^{-x} and e^{3x} are continuous for all $x \in \mathbb{R}$, it follows that the interval of interest is $x = (-\infty, \infty)$

$\therefore \boxed{y(x) = \frac{1}{4} e^{3x} + Ce^{-x}\ ;\ x = (-\infty, \infty)}$

Ex:

b) $x^2y' + xy = 1 \Rightarrow y' + \frac{xy}{x^2} = \frac{1}{x^2} \Rightarrow y' + \frac{1}{x}\cdot y = \frac{1}{x^2}$

$\Rightarrow y' + p(x)\cdot y = r(x)$, where $p(x) = \frac{1}{x}$ and $r(x) = \frac{1}{x^2}$. So, our ODE is a 1st-order linear ODE!

If our ODE is (1st-order) linear $\Rightarrow \frac{dw}{dx} = wp$ and $\frac{d}{dx}[wy] = wr$, where $w = w(x)$. We need to find $w(x) = w$ (i.e. the integrating factor).

$\therefore \frac{dw}{dx} = w\left[\frac{1}{x}\right] \Rightarrow \frac{dw}{w} = \frac{dx}{x} \Rightarrow \int \frac{dw}{w} = \int \frac{dx}{x} \Rightarrow \ln|w| = \ln|x| + C$

$\therefore e^{\ln|w|} = e^{\ln|x|}\cdot e^{C} \Rightarrow |w| = |x|\cdot A;\ A = e^C \Rightarrow |w| = |x| \Rightarrow w = |x|, x \neq$

Note that we drop the constant A for simplicity since (1) we have no initial conditions and (2) it can be recovered once we find out $y = y(x)$.

$\therefore \frac{d}{dx}[wy] = wr \Rightarrow \int \frac{d}{dx}[wy]\cdot dx = \int (wr)\cdot dx$

$\Rightarrow wy = \int (wr)\cdot dx \Rightarrow y = y(x) = \frac{\int (wr)\cdot dx}{w}$. Recall that $w = |x|$ & $r = \frac{1}{x^2}$

$\therefore y(x) = \frac{\int \left(|x|\cdot\frac{1}{x^2}\right)\cdot dx}{|x|} = \frac{\int \left|\frac{x}{x^2}\right| dx}{|x|} = \frac{\int \left|\frac{1}{x}\right|\cdot dx}{|x|} = \frac{\int \frac{1}{x}\cdot dx}{|x|};\ x > 0$ (*)

$\therefore y(x) = \frac{\ln|x| + C}{|x|};\ x > 0 \Rightarrow y(x) = \frac{\ln(x) + C}{x} = \frac{\ln(x)}{x} + Cx^{-1};\ x > 0$

Sol'n: $\boxed{y(x) = \frac{\ln(x)}{x} + Cx^{-1};\ x > 0}$

(*) $\left|\frac{1}{x}\right|$ can be rewritten as $\frac{1}{x}$ only if $x > 0$. Also, the expression $\frac{\ln|x|}{|x|}$ has to be positive & it can only happen if $\frac{\ln(x)}{x}$ where $x > 0$!

Ex: (cont'd)

c) $\frac{dz}{d\theta} + z\sec(\theta) = \cos(\theta)$

$\therefore \frac{dz}{d\theta} + pz = r$, where $p = p(\theta) = \sec(\theta)$ and $r = r(\theta) = \cos(\theta)$

$\therefore$ ODE is 1^{st}-order, linear!

$$\therefore \frac{dw}{d\theta} = wp \Rightarrow w = e^{\int p \cdot d\theta} = e^{\int \sec(\theta) \cdot d\theta} = e^{\ln|\sec(\theta) + \tan(\theta)| + c}$$
$$= A e^{\ln|\sec(\theta) + \tan(\theta)|}$$

where $A = e^c$

For simplicity, let $A = 1 \Rightarrow w = w(x) = e^{\ln|\sec(\theta) + \tan(\theta)|} = |\sec(\theta) + \tan(\theta)|$

$$\therefore \frac{d}{d\theta}[wz] = wr \Rightarrow wz = \int wr \cdot d\theta \Rightarrow z = w^{-1}\left[\int wr \cdot d\theta\right]$$

$$\therefore z = \frac{1}{|\sec(\theta) + \tan(\theta)|}\left[\int |\sec(\theta) + \tan(\theta)| \cdot \cos(\theta) \cdot d\theta\right]$$

see next page ↑ ⊛

$$z = \frac{1}{\sec(\theta) + \tan(\theta)}\left[\int (\sec(\theta) + \tan(\theta)) \cdot \cos(\theta) \cdot d\theta\right] ; \; x = (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

$$z = \frac{1}{\sec(\theta) + \tan(\theta)}\left[\int (1 + \sin(\theta)) \cdot d\theta\right] = \frac{\theta - \cos(\theta) + c}{\sec(\theta) + \tan(\theta)}$$

$$\therefore \boxed{z = z(\theta) = \frac{\theta - \cos(\theta) + C}{\sec(\theta) + \tan(\theta)} \Leftrightarrow (\sec(\theta) + \tan(\theta)) \cdot z = \theta - \cos(\theta) + c , \text{ where } x = (-\tfrac{\pi}{2}, \tfrac{\pi}{2})}$$

(*) Note that $\dfrac{1}{\sec(\theta)+\tan(\theta)} = \left[\sec(\theta)+\tan(\theta)\right]^{-1} =$

$$\left[\frac{1}{\cos(\theta)} + \frac{\sin(\theta)}{\cos(\theta)}\right]^{-1} = \left[\frac{1+\sin(\theta)}{\cos(\theta)}\right]^{-1} = \frac{\cos(\theta)}{1+\sin(\theta)}.$$

Also, note that both $\cos(\theta) > 0$ and $1+\sin(\theta) > 0$ for $\theta = (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\theta = (-\pi, \pi)$, respectively, within the interval $\theta = [-\pi, \pi]$ (i.e. for 1 full period of both $\cos(\theta)$ and $1+\sin(\theta)$). The intersection of these 2 sets are $(-\frac{\pi}{2}, \frac{\pi}{2}) \cap (-\pi, \pi) = (-\frac{\pi}{2}, \frac{\pi}{2})$. This is (an) explanatio why the interval of interest that we picked here is $\theta = (-\frac{\pi}{2}, \frac{\pi}{2})$

Also, note that the graphs of both $\sec(\theta)$ and $\tan(\theta)$ are continuous on the interval $\theta = (-\frac{\pi}{2}, \frac{\pi}{2})$ and the interval falls within the interval $\theta = (-\pi, \pi)$ as well. That makes the interval of interest $\theta = (-\frac{\pi}{2}, \frac{\pi}{2})$ an easy choice.

Ex :

d) $\cos(x)\cdot\frac{dy}{dx}+[\sin(x)]\cdot y=1 \Rightarrow \frac{dy}{dx}+\frac{\sin(x)}{\cos(x)}\cdot y=\frac{1}{\cos(x)}$

$\therefore \frac{dy}{dx}+\tan(x)\cdot y=\sec(x) \Rightarrow \frac{dy}{dx}+p(x)\cdot y=r(x)$; $p(x)=\tan(x)$ and $r(x)=\sec(x)$. Thus, our ODE is (1^{st}-order) linear!

$\therefore$ It is true that $\frac{dw}{dx}=wp$ and $\frac{d}{dx}[wy]=wr$ if our ODE is linear.

Finding integrating factor w

$\therefore \frac{dw}{dx}=wp \Rightarrow \frac{dw}{w}=p\cdot dx=p(x)\cdot dx \Rightarrow \int\frac{1}{w}\,dw=\int p\cdot dx$

$\therefore \ln|w|=\int\tan(x)\,dx=-\ln|\cos(x)|+C=\ln|\sec(x)|+C$

$\therefore e^{\ln|w|}=e^{\ln|\sec(x)|+C} \Rightarrow |w|=\cancel{e^{C}}^{A=\text{const}}\cdot|\sec(x)| \Rightarrow w=|\sec(x)|$ (*)

(*) constant A dropped for simplicity

Finding y, using w and r

$\frac{d}{dx}[wy]=wr \Rightarrow \int\frac{d}{dx}(wy)\cdot dx=\int(wr)\cdot dx \Rightarrow wy=\int(wr)\cdot dx$

$\therefore y=y(x)=\frac{\int(wr)\cdot dx}{w}=\frac{\int\left(|\sec(x)|\cdot\sec(x)\right)dx}{|\sec(x)|}=\frac{\int\sec^2(x)\,dx}{|\sec(x)|}=\frac{\tan(x)+C}{|\sec(x)|}$

$\therefore y=y(x)=\frac{\tan(x)+C}{\sec(x)}$; $x=(-\frac{\pi}{2},\frac{\pi}{2})$ $=\cancel{\cos(x)}\cdot\frac{\sin(x)}{\cancel{\cos(x)}}+C\cos(x)$; $x=(-\frac{\pi}{2},\frac{\pi}{2})$

$\therefore \boxed{y=y(x)=\sin(x)+C\cos(x)\ ;\ x=(-\frac{\pi}{2},\frac{\pi}{2})}$

Ex: Solve the Linear, 1st-order IVP by using the formula

$$y = e^{-h}\left[\int e^{h} \cdot r \cdot dx + C\right] \; ; \; h = \int p \cdot dx$$

a) $xy' + y = e^{x}$; $y(1) = 2$

$\therefore y' + \frac{1}{x} \cdot y = \frac{e^{x}}{x}$ $\Rightarrow$ Linear ODE where $p = 1/x$ and $r = \frac{e^{x}}{x}$

$\therefore h = \int p \cdot dx = \int \frac{1}{x} \, dx = \ln|x| + c$ $\Rightarrow$ $h = \ln|x|$ (for simplicity)

$$\therefore y = e^{-h}\left[\int e^{h} \cdot r \cdot dx + C\right] = e^{-\ln|x|}\left[\int e^{\ln|x|} \cdot \frac{e^{x}}{x} \cdot dx + C\right] =$$

$$e^{\ln\left|\frac{1}{x}\right|}\left[\int \cancel{|x|}^{1} \cdot \frac{e^{x}}{\cancel{x}_{1}} \cdot dx + C\right] = \left|\frac{1}{x}\right|\left[\int e^{x} \cdot dx + C\right] \text{ if } x > 0!$$

if $x > 0$

$$\therefore y = \frac{1}{x}\left[e^{x} + C\right] \; ; \; x > 0 \quad \Rightarrow y = y(x) = \frac{e^{x}}{x} + \frac{C}{x} \; ; \; x > 0$$

Apply initial condition $y(1) = 2$

$\therefore y(1) = 2 \Rightarrow \frac{e^{1}}{1} + \frac{C}{1} = 2 \Rightarrow e + C = 2 \Rightarrow \boxed{C = 2 - e}$

$\therefore$ Final (particular) solution : $\boxed{y(x) = \frac{e^{x} + (2-e)}{x} \; ; \; x > 0}$

Ex: (cont'd)

b) $L\frac{di}{dt} + Ri = E$; $i(0) = i_0$; i_0, L, R, E are constants

$\therefore \frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$ $\Rightarrow$ Linear ODE where $p(t) = p = \frac{R}{L}$ and $r(t) = r = \frac{E}{L}$.

So, $i = e^{-h}\left[\int e^{h} \cdot r \cdot dt + C\right]$, where $h = \int p \cdot dt$

$\therefore h = \int p \cdot dt = \int \frac{R}{L} \cdot dt = \frac{R}{L} \cdot t$ (integration constant dropped for simplicity).

$$\therefore i = e^{-\frac{R}{L}t}\left[\int e^{\frac{R}{L}t} \cdot \frac{E}{L} \cdot dt + C\right] = e^{-\frac{R}{L}t}\left[\frac{E}{\cancel{L}} \cdot \frac{\cancel{L}}{R} \cdot e^{\frac{R}{L}t} + C\right] =$$

$$e^{-R/L\,t}\left[\frac{E}{R} e^{R/L\,t} + C\right] = \frac{E}{R} \cdot e^{-R/L\,t} \cdot e^{R/L\,t} + Ce^{-R/L\,t} = \frac{E}{R} \cdot 1 + Ce^{-R/L\,t}$$

$\therefore i = \boxed{i(t) = \frac{E}{R} + Ce^{-R/L\,t}}$. Applying the initial condition $i(0) = i_0$ yields..

$$i(0) = i_0 \Rightarrow \frac{E}{R} + C\cancelto{1}{e^{-R/L(0)}} = i_0 \Rightarrow \frac{E}{R} + C = i_0 \Rightarrow i_0 - \frac{E}{R} = C$$

$$\therefore \boxed{i(t) = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right)e^{-R/L\,t}}$$

Ex: (cont'd)

c) $(x+1)\cdot\frac{dy}{dx}+y=\ln(x)$; $y(1)=10$

$\therefore \frac{dy}{dx}+\frac{1}{x+1}\cdot y=\frac{\ln(x)}{x+1} \Rightarrow$ (1^{st}-order) linear ODE with $p=p(x)=\frac{1}{x+1}$ and $r=r(x)=\frac{\ln(x)}{x+1}$.

constant dropped for simplicity

$$\therefore y=e^{-h}\left[\int e^{h}\cdot r\cdot dx+C\right], \text{ where } h=\int p\cdot dx=\int\frac{1}{x+1}\cdot dx=\ln|x+1| \circledast$$

$$\therefore y=e^{-\ln|x+1|}\left[\int e^{\ln|x+1|}\cdot\frac{\ln(x)}{x+1}\cdot dx+C\right],$$ where $x>0$ since $\ln(x)$ is defined only on $x>0$. Consequently, $\ln|x+1|=\ln(x+1)$

$$\therefore y=e^{-\ln(x+1)}\left[\int e^{\ln(x+1)}\cdot\frac{\ln(x)}{x+1}\cdot dx+C\right]=\frac{1}{x+1}\left[\int\frac{\cancel{x+1}}{1}\cdot\frac{\ln(x)}{\cancel{x+1}}\cdot dx+C\right]$$

$$\therefore y=\frac{1}{x+1}\left[\int\ln(x)\cdot dx+C\right]=\frac{1}{x+1}\left[x\ln(x)-x+C\right]=\frac{x\ln(x)-x}{x+1}+\frac{C}{x+1}$$

$\therefore y=y(x)=\frac{x\ln(x)-x+C}{x+1}$; $x>0$. Now, applying the initial condition $y(1)=10$ yields...

$$y(1)=10\Rightarrow\frac{1\cdot\cancelto{0}{\ln(1)}-1+C}{1+1}=10\Rightarrow\frac{C-1}{2}=10\Rightarrow C-1=20\Rightarrow\boxed{C=21}$$

$$\therefore \boxed{y(x)=\frac{x\ln(x)-x+21}{x+1}}$$

Ex: (cont'd)

d) $\dfrac{dy}{dx} + 2xy = f(x)$; $f(x) = \begin{cases} x; & 0 \le x < 1 \\ 0; & x \ge 1 \end{cases}$; $y(0) = 2$

$2x \to p = p(x)$ $\quad f(x) \to r = r(x)$ Linear (1^{st} order) ODE!

$\therefore \dfrac{dy}{dx} + 2xy = x$; $x = [0, 1)$ or $\dfrac{dy}{dx} + 2xy = 0$, $x = [1, \infty)$

Finding solutions to $\dfrac{dy}{dx} + 2xy = x$; $x = [0,1)$; $y(0) = 2$

NOTE: $\dfrac{dy}{dx} + 2xy = x \Rightarrow \dfrac{dy}{dx} = x - 2xy = x(1-2y) \Rightarrow \dfrac{dy}{dx} = 0$ if

$1 - 2y = 0 \Rightarrow 1 = 2y \Rightarrow y = \frac{1}{2}$. Therefore, $\boxed{y = \frac{1}{2} \text{ is a constant solution}}$

Now, looking for non-constant solutions, we will use our formula...

$$y = e^{-h}\left[\int e^{h} \cdot r \cdot dx + C\right]; \quad h = \int p \cdot dx = \int 2x \cdot dx = 2\frac{x^2}{2} = x^2$$

$$\therefore y = e^{-x^2}\left[\int e^{x^2} \cdot x \cdot dx + C\right] = e^{-x^2}\left[\int \tfrac{1}{2} e^{u} \cdot du + C\right], \text{ where } \begin{array}{l} u = x^2 \\ du = 2x \cdot dx \\ \frac{1}{2} du = x \cdot dx \end{array}.$$

$$\therefore y = e^{-x^2}\left[\tfrac{1}{2} e^{u} + C\right] = e^{-x^2}\left[\tfrac{1}{2} e^{x^2} + C\right] = \tfrac{1}{2} + Ce^{-x^2}$$

Applying the initial condition $y(0) = 2 \Rightarrow \frac{1}{2} + C\cancel{e^{-0}}^{\,1} = 2 \Rightarrow C = 2 - \frac{1}{2} \Rightarrow \boxed{\frac{3}{2} = C}$

$$\therefore \boxed{y_1 = y_1(x) = \tfrac{1}{2} + \tfrac{3}{2} e^{-x^2} \; ; \; x = [0, 1)}$$

Finding solutions to $\frac{dy}{dx} + 2xy = 0$; $x = [1, \infty)$; $y(0) = 2$

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NOTE: Since $x = [1, \infty)$, the initial condition $y(0) = 2$ does not apply to this "piece" of the piecewise function $y = y(x)$ that will result in the solution to our ODE. However, it is required that $y(x)$ be continuous for all $x = [0, \infty)$. Thus, if $y_1(x)$ is the solution for $y(x)$ on $x = [0, 1)$ and $y_2(x)$ is the solution for $y(x)$ on $x = [1, \infty)$, then we will have to ensure that $\lim_{x \to 1^-} [y_1(x)] = \lim_{x \to 1^+} [y_2(x)]$ in order to ensure continuity at $x = 1$!

$$\therefore \frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{dx} = -2xy \Rightarrow \frac{dy}{y} = -2x \cdot dx$$

$$\therefore \int \frac{dy}{y} = \int -2x \cdot dx \Rightarrow \ln|y| = -\frac{2x^2}{2} + D \Rightarrow e^{\ln|y|} = e^{-x^2} \cdot \cancel{e^D} F \text{ ; } F = \text{const.}$$

$$\therefore |y| = Fe^{-x^2} \Rightarrow y = y(x) = Fe^{-x^2} \text{ since } e^{-x^2} > 0 \text{ for all } x \in \mathbb{R}.$$

$$\therefore \lim_{x \to 1^-} [y_1(x)] = \lim_{x \to 1^+} [y_2(x)] \Rightarrow \lim_{x \to 1^-} \left[\frac{1}{2} + \frac{3}{2} e^{-x^2}\right] = \lim_{x \to 1^+} \left[Fe^{-x^2}\right]$$

$$\therefore \frac{1}{2} + \frac{3}{2} e^{-(1)^2} = Fe^{-(1)^2} \Rightarrow \frac{\frac{1}{2} + \frac{3}{2} e^{-1}}{e^{-1}} = F \Rightarrow \boxed{F = \frac{e}{2} + \frac{3}{2} = \frac{e+3}{2}}$$

$$\therefore y_2 = y_2(x) = \left(\frac{e+3}{2}\right) e^{-x^2} \Rightarrow \boxed{y = y(x) = \begin{cases} \frac{1}{2} + \frac{3}{2} e^{-x^2} & ; x = [0, 1) \\ \left(\frac{e+3}{2}\right) e^{-x^2} & ; x = [1, \infty) \end{cases} \text{ and } y = \frac{1}{2}}$$

Final solutions to Linear ODE!