SIGNALS AND SYSTEMS USING MATLAB
Chapter 3 — The Laplace Transform

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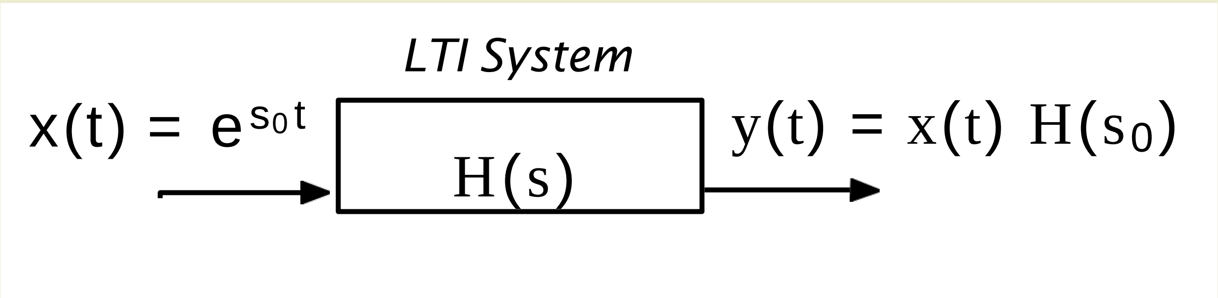
Eigenfunction property of LTI systems

LTI system with h(t) as impulse response:

$$
\begin{aligned}
\text{input} \quad x(t) &= e^{s_0 t}, \quad s_0 = \sigma_0 + j\Omega_0, \quad -\infty < t < \infty \\
\text{convolution} \quad y(t) &= \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau \\
&= e^{s_0 t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-\tau s_0} d\tau}_{H(s_0)} = x(t) H(s_0)
\end{aligned}
$$



Two–sided Laplace transform

The two-sided Laplace transform of $f(t)$ is

$$\boxed{\begin{array}{c} F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-st}dt \qquad s \in \mathrm{ROC} \\ s = \sigma + j\Omega, \;\; \text{damping } \sigma, \; \text{frequency } \Omega \end{array}}$$

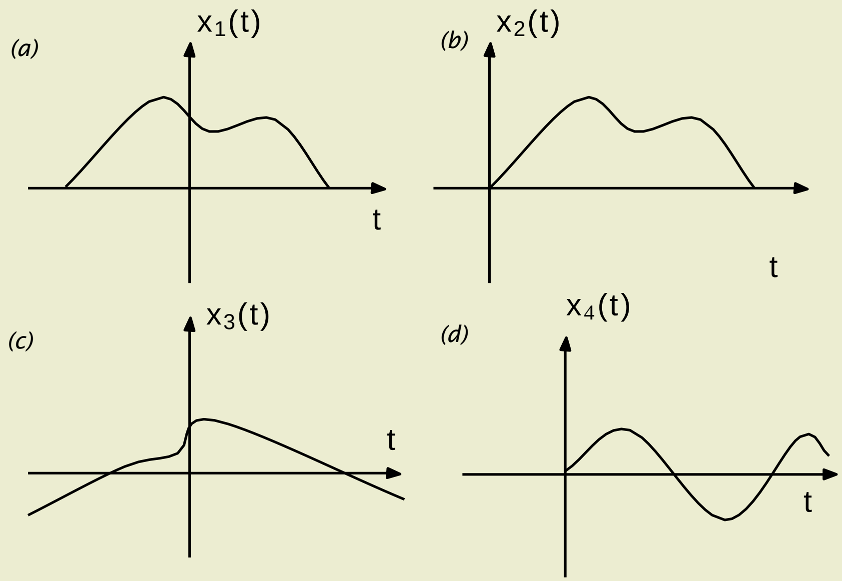
The inverse Laplace transform is

$$\boxed{f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds \qquad \sigma \in \mathrm{ROC}}$$

Functions :Finite support functions:

$f(t) = 0$, for t not in a finite segment $t_1 \leq t \leq t_2$

- Infinite support functions: $f(t)$ defined in infinite support, $t_1<t<t_2$ where either t_1 or t_2 or both are infinite



Examples of (a) non-causal finite support signal $x_1(t)$*, (b) causal finite support signal* $x_2(t)$*, (c) non-causal infinite support signal* $x_3(t)$*, and (d) causal infinite-support* $x_4(t)$

Rational function $F(s) = \mathcal{L}[f(t)] = N(s)/D(s)$

- zeros: values of s such that $F(s) = 0$
- poles: values of s such that $F(s) \to \infty$

ROC: where $F(s)$ is defined (integral converges) where $\{\sigma_i\} = \{Re(p_i)\}$

- Causal $f(t)$, $f(t) = 0$ for $t < 0$,

$$R_c = \{(\sigma, \Omega) : \sigma > \max\{\sigma_i\}, -\infty < \Omega < \infty\},$$

right of poles

- Anti-causal $f(t)$, $f(t) = 0$ for $t > 0$,

$$R_{ac} = \{(\sigma, \Omega) : \sigma < \min\{\sigma_i\}, -\infty < \Omega < \infty\},$$

left of poles

- Non-causal $f(t)$ defined for $-\infty < t < \infty$,

$$R_c \cap R_{ac},$$ poles in middle

Example:

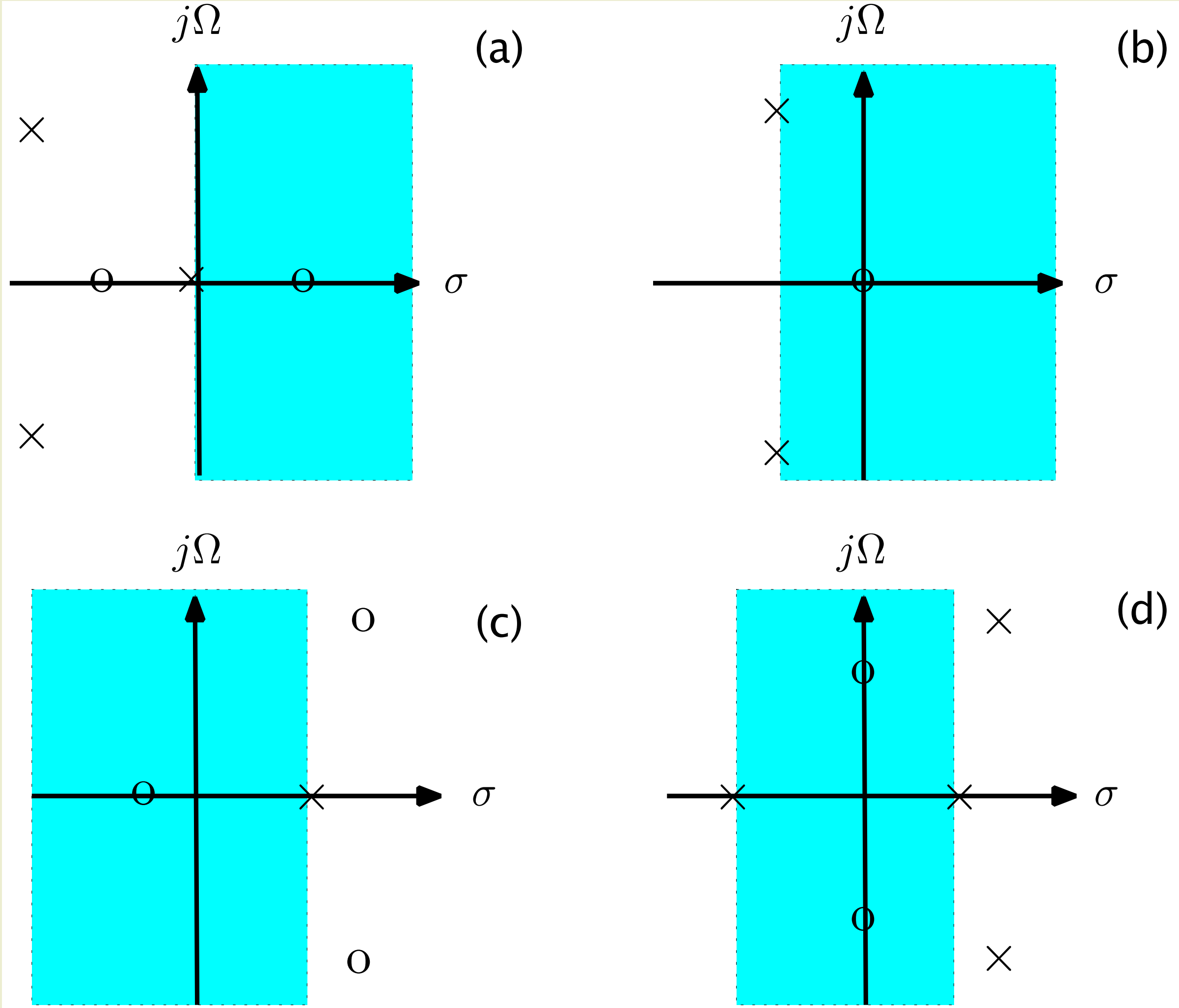
- $\delta(t)$ and $u(t)$

$$\mathcal{L}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-st} dt = \int_{-\infty}^{\infty} \delta(t) e^{-s0} dt = 1, \; ROC \text{ whole s-plane}$$

$$\begin{aligned} U(s) &= \mathcal{L}[u(t)] = \int_{-\infty}^{\infty} u(t) e^{-st} dt = \int_{0}^{\infty} e^{-st} dt = \int_{0}^{\infty} e^{-\sigma t} e^{-j\Omega t} dt \\ &= \frac{1}{s}, \quad ROC = \{(\sigma, \Omega) : \sigma > 0, -\infty < \Omega < \infty\} \end{aligned}$$

- Pulse $p(t) = u(t) - u(t-1)$

$$\begin{aligned} P(s) &= \mathcal{L}[u(t) - u(t-1)] = \int_{0}^{1} e^{-st} dt = \frac{-e^{-st}}{s}\Big|_{t=0}^{1} \\ &= \frac{1}{s}[1 - e^{-s}] \quad ROC = \text{whole s–plane} \end{aligned}$$



ROC for (a) causal signal with poles with $\sigma_{max} = 0$; (b) causal signal with poles with $\sigma_{max} < 0$; (c) anti-causal signal with poles with $\sigma_{min} > 0$; (d) two-sided or noncausal signal where ROC is bounded by poles. The ROCs do not contain poles, but they can contain zeros

For function $f(t), -\infty < t < \infty$, its one-sided Laplace transform is

$$\boxed{F(s) = \mathcal{L}[f(t)u(t)] = \int_{0-}^{\infty} f(t)e^{-st}dt, \quad \text{ROC}}$$

- Finite support $f(t)$, i.e., $f(t) = 0$ for $t<t_1$ and $t>t_2$, $t_1<t_2$,

$$F(s) = \mathcal{L}\left[f(t)[u(t-t_1) - u(t-t_2)]\right] \qquad \text{ROC: whole s-plane}$$

- Causal $g(t)$, i.e., $g(t)=0$ for $t<0$, is

$$G(s) = \mathcal{L}[g(t)u(t)] \qquad \mathcal{R}_c = \{\sigma > \max\{\sigma_i\}\}$$

- Anti–causal $h(t)$, i.e., $h(t)=0$ for $t>0$, is

$$H(s) = \mathcal{L}[h(-t)u(t)]_{(-s)} \qquad \mathcal{R}_{ac} = \{\sigma < \min\{\sigma_i\}\}$$

- Non-causal $p(t)$, i.e., $p(t) = p_{ac}(t) + p_c(t) = p(t)u(-t) + p(t)u(t)$, is

$$P(s) = \mathcal{L}[p_{ac}(-t)u(t)]_{(-s)} + \mathcal{L}[p_c(t)u(t)] \qquad \mathcal{R}_c \bigcap \mathcal{R}_{ac}$$

Example:

$$\mathcal{L}[e^{j(\Omega_0 t+\theta)}u(t)] = \frac{e^{j\theta}}{s - j\Omega_0} \qquad \text{ROC: } \sigma > 0.$$

Laplace transform of $x(t) = \cos(\Omega_0 t + \theta)u(t)$

$$\begin{aligned} X(s) &= 0.5\mathcal{L}[e^{j(\Omega_0 t+\theta)}u(t)] + 0.5\mathcal{L}[e^{-j(\Omega_0 t+\theta)}u(t)] \\ &= \frac{s\cos(\theta) - \Omega_0 \sin(\theta)}{s^2 + \Omega_0^2}, \quad ROC : \sigma > 0 \end{aligned}$$

For $\theta = 0, -\pi/2$

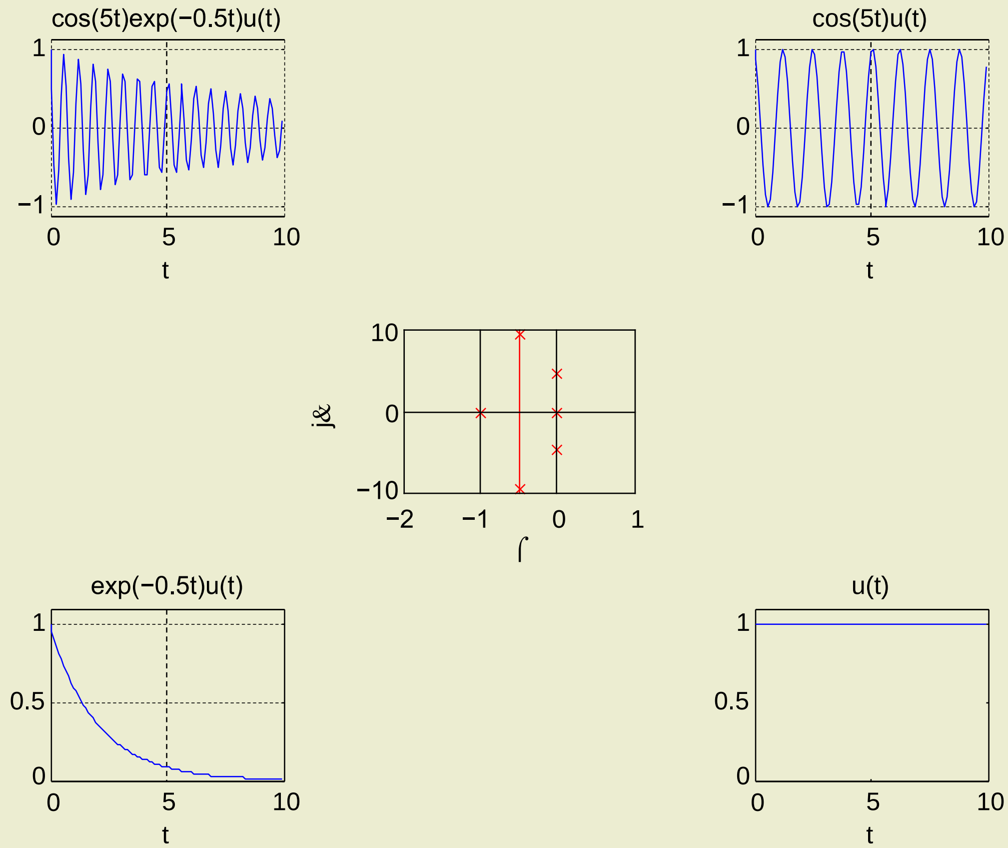
$$\mathcal{L}[\cos(\Omega_0 t)u(t)] = \frac{s}{s^2 + \Omega_0^2},$$

$$\mathcal{L}[\sin(\Omega_0 t)u(t)] = \frac{\Omega_0}{s^2 + \Omega_0^2}, \quad ROC : \sigma > 0$$

Basic Properties of One-sided Laplace Transforms

Causal functions and constants

Linearity	$\alpha f(t),\ \beta g(t)$	$\alpha F(s),\ \beta G(s)$
	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Time shifting	$f(t-\alpha)u(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$t\ f(t)$	$-\dfrac{dF(s)}{ds}$
Derivative	$\dfrac{df(t)}{dt}$	$sF(s) - f(0-)$
Second derivative	$\dfrac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0-) - f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt'$	$\dfrac{F(s)}{s}$
Expansion/contraction	$f(\alpha t),\ \alpha \neq 0$	$\dfrac{1}{\lvert\alpha\rvert}F\left(\dfrac{s}{\alpha}\right)$
Initial value	$f(0-) = \lim_{s\to\infty} sF(s)$	



For poles in middle plot:

pole s = 0 corresponds to u(t);

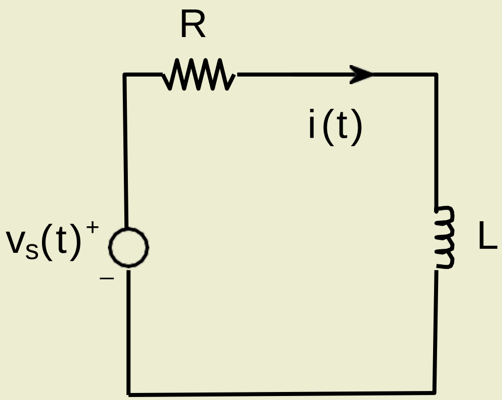
complex conjugate poles on jΩ-axis correspond to sinusoid;

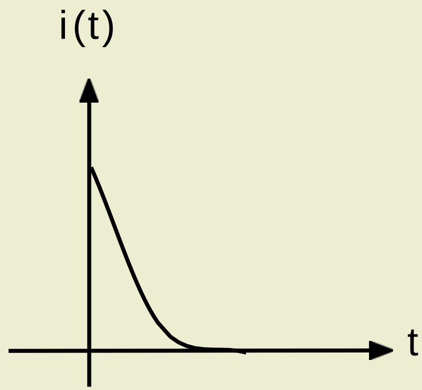
complex conjugate poles with negative real part corresponds to sinusoid multiplied by an exponential;

the pole in negative real axis gives decaying exponential

Derivative property – solution of o.d.e.

Example: Impulse response of RL circuit





$$v_s(t) = L\frac{di(t)}{dt} + Ri(t), \qquad i(0-) = 0$$

impulse response:

$$\mathcal{L}[\delta(t)] = \mathcal{L}[L\frac{di(t)}{dt} + Ri(t)]$$

$$1 = sLI(s) + RI(s)$$

$$I(s) = \frac{1/L}{s + R/L} \quad \Rightarrow i(t) = \frac{1}{L}e^{-(R/L)t}u(t)$$

Integral property

Example: Find $y(t)$ for

$$\int_0^t y(\tau)d\tau = 3u(t) - 2y(t)$$

Method 1 Using integration property

$$\frac{Y(s)}{s} = \frac{3}{s} - 2Y(s)$$

$$Y(s) = \frac{3}{2(s+0.5)} \quad \Rightarrow y(t) = 1.5e^{-0.5t}u(t)$$

Method 2 Using derivative property

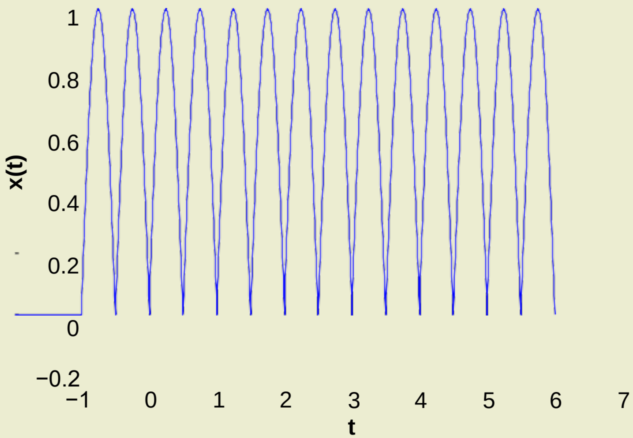
$$y(t) = 3\delta(t) - 2\frac{dy(t)}{dt}, \quad \text{assume} \quad y(0) = 0$$

$$Y(s) = 3 - 2sY(s)$$

$$Y(s) = \frac{3}{2(s+0.5)} \quad \Rightarrow y(t) = 1.5e^{-0.5t}u(t)$$

Time–shifting property

Example: Causal full-wave rectified signal



first period: $x_1(t) = \sin(2\pi t)u(t) + \sin(2\pi(t-0.5))u(t-0.5)$

$$X_1(s) = \frac{2\pi(1+e^{-0.5s})}{s^2+(2\pi)^2}$$

train of sinusoidal pulses $x(t) = \sum_{k=0}^{\infty} x_1(t-0.5k)$

$$X(s) = \frac{X_1(s)}{1-e^{-s/2}} = \frac{2\pi(1+e^{-s/2})}{(1-e^{-s/2})(s^2+4\pi^2)}$$

One-sided Laplace Transforms

(1)	$\delta(t)$	1, whole s-plane
(2)	$u(t)$	$\frac{1}{s}$, $Re[s] > 0$
(3)	$r(t)$	$\frac{1}{s^2}$ $Re[s] > 0$
(4)	$e^{-at}u(t),\ a > 0$	$\frac{1}{s+a}$ $Re[s] > -a$
(5)	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $Re[s] > 0$
(6)	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $Re[s] > 0$
(7)	$e^{-at}\cos(\Omega_0 t)u(t),\ a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $Re[s] > -a$
(8)	$e^{-at}\sin(\Omega_0 t)u(t),\ a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $Re[s] > -a$
(9)	$2Ae^{-at}\cos(\Omega_0 t+\theta)u(t),\ a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0}+\frac{A\angle-\theta}{s+a+j\Omega_0}$, $Re[s] > -a$
(10)	$\frac{1}{(N-1)!}\ t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $Re[s] > 0$

Inverse Laplace transform – PFE

One–sided inverse Laplace transform

$$\text{Given } F(s) = \frac{N(s)}{D(s)},\ ROC\text{, find causal } f(t)u(t)$$

- Basic idea: decompose proper rational functions (order $N(s)$ < order $D(s)$) into proper rational components with inverse in tables
- Poles of $X(s)$ provide basic characteristics of $x(t)$
- For $N(s)$ and $D(s)$ polynomials with real coefficients — zeros and poles of $X(s)$ are real and/or complex conjugate pairs, and can be simple or multiple,
- $u(t)$ is integral part of the one–sided inverse
- Avoid errors using generic inverse from poles and *initial–value theorem*

Simple real poles

$$X(s) = \frac{N(s)}{(s+p_1)(s+p_2)} \quad ,\{-p_i \; , i = 1, 2\} \text{ real poles}$$

partial fraction expansion and inverse

$$\boxed{\begin{aligned} & X(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} \quad \Rightarrow \quad x(t) = [A_1 e^{-p_1 t} + A_2 e^{-p_2 t}]u(t) \\ & A_k = X(s)(s+p_k)\,|_{s=-p_k} \quad k = 1, 2 \end{aligned}}$$

Simple complex conjugate poles

$$X(s) = \frac{N(s)}{(s+\alpha)^2 + \Omega_0^2} = \frac{N(s)}{(s+\alpha-j\Omega_0)(s+\alpha+j\Omega_0)}, \quad \text{poles: } \{-\alpha \pm j\Omega_0\}$$

partial fraction expansion and inverse

$$\boxed{\begin{aligned} & X(s) = \frac{A}{s+\alpha-j\Omega_0} + \frac{A^*}{s+\alpha+j\Omega_0} \quad \Rightarrow \quad x(t) = 2|A|e^{-\alpha t}\cos(\Omega_0 t + \theta)u(t) \\ & A = X(s)(s+\alpha-j\Omega_0)|_{s=-\alpha+j\Omega_0} = |A|e^{j\theta} \end{aligned}}$$

Example: Causal inverse of

$$X(s) = \frac{3s+5}{s^2+3s+2} = \frac{3s+5}{(s+1)(s+2)}$$

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

generic solution $x(t) = [A_1 e^{-t} + A_2 e^{-t}]u(t)$

$$A_1 = X(s)(s+1)|_{s=-1} = \frac{3s+5}{s+2}|_{s=-1} = 2 \qquad \text{and}$$

$$A_2 = X(s)(s+2)|_{s=-2} = \frac{3s+5}{s+1}|_{s=-2} = 1$$

$$X(s) = \frac{2}{s+1} + \frac{1}{s+2} \quad \Rightarrow \quad x(t) = [2e^{-t} + e^{-2t}]u(t)$$

Example: Causal inverse

$$X(s) = \frac{4}{s((s+1)^2+3)}, \text{ poles: } s=0, \; s=-1\pm j\sqrt{3}$$

$$X(s) = \frac{A}{s+1-j\sqrt{3}} + \frac{A^*}{s+1+j\sqrt{3}} + \frac{B}{s}$$

$$B = sX(s)|_{s=0} = 1$$

$$A = X(s)(s+1-j\sqrt{3})|_{s=-1+j\sqrt{3}} = 0.5(-1+\frac{j}{\sqrt{3}}) = \frac{1}{\sqrt{3}}\angle 150^o$$

$$\begin{aligned} x(t) &= \frac{2}{\sqrt{3}}e^{-t}\cos(\sqrt{3}t+150^o)u(t)+u(t) \\ &= -[\cos(\sqrt{3}t)+0.577\sin(\sqrt{3}t)]e^{-t}u(t)+u(t) \end{aligned}$$

Double real poles

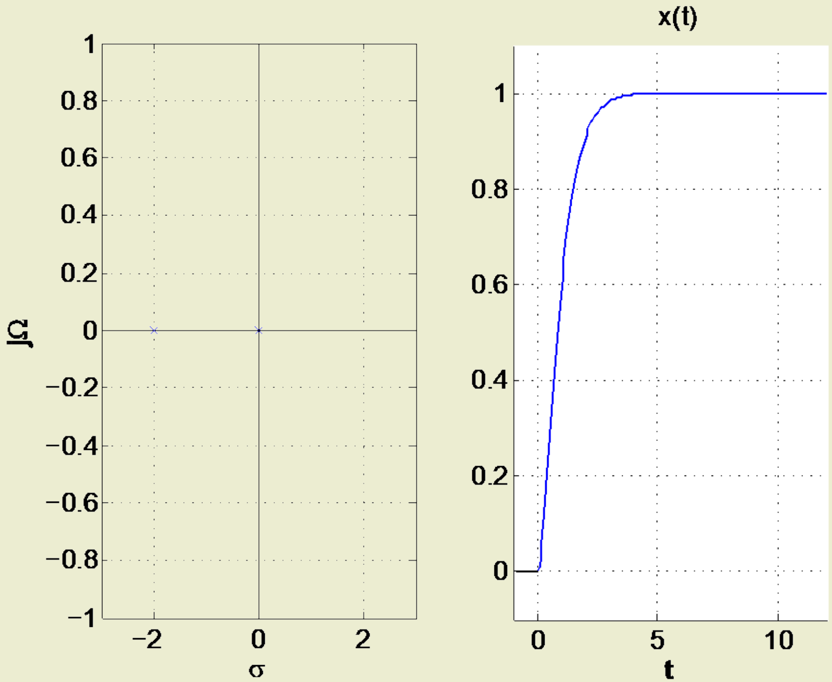
$$X(s) = \frac{N(s)}{(s+\alpha)^2}$$ proper rational, poles $s_{1,2} = -\alpha$

partial fraction expansion and inverse

$$\boxed{\begin{aligned} X(s) &= \frac{a + b(s+\alpha)}{(s+\alpha)^2} = \frac{a}{(s+\alpha)^2} + \frac{b}{s+\alpha} \\ x(t) &= [ate^{-\alpha t} + be^{-\alpha t}]u(t) \\ a &= X(s)(s+\alpha)^2 |_{s=-\alpha} \end{aligned}}$$

b found by computing $X(s_0)$ for $s_0 \neq -\alpha$

$$
\begin{aligned}
X(s) &= \frac{4}{s(s+2)^2}, \text{ poles: } s=0, \quad -2 \text{ double} \\
&= \frac{A}{s} + \frac{B}{(s+2)^2} + \frac{C}{s+2} \\
& \quad A = X(s)s|_{s=0} = 1 \\
& \quad B = X(s)(s+2)^2|_{s=-2} = -2 \\
& \quad X(1) = \frac{4}{9} = \frac{A}{1} + \frac{B}{9} + \frac{C}{3} = 1 - \frac{2}{9} + \frac{C}{3} \Rightarrow C = -1 \\
x(t) &= [1 - 2te^{-2t} - e^{-2t}]u(t)
\end{aligned}
$$



Complete response $y(t)$ of system represented by

$$y^{(N)}(t) + \sum_{k=0}^{N-1} a_k y^{(k)}(t) = \sum_{\ell=0}^{M} b_\ell x^{(\ell)}(t) \qquad N > M$$

$$x(t)\ y(t)\ \text{input, output,}\ \{y^{(k)}(t),\ 0 \le k \le N-1\}\ \text{ IC}$$

$$y(t) = \mathcal{L}^{-1}\left[Y(s) = \frac{B(s)}{A(s)}X(s) + \frac{1}{A(s)}I(s)\right]$$

$$\begin{aligned}
Y(s) &= \mathcal{L}[y(t)],\ X(s) = \mathcal{L}[x(t)] \\
A(s) &= \sum_{k=0}^{N} a_k s^k,\ a_N = 1, \quad B(s) = \sum_{\ell=0}^{M} b_\ell s^\ell \\
I(s) &= \sum_{k=1}^{N} a_k \left(\sum_{m=0}^{k-1} s^{k-m-1} y^{(m)}(0) \right)
\end{aligned}$$

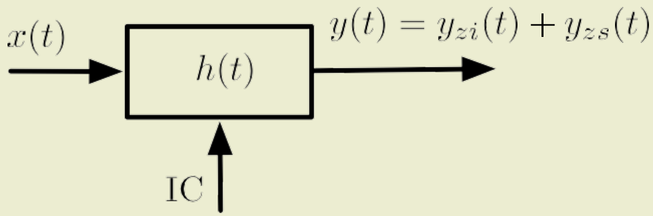
Zero–input, zero–state responses

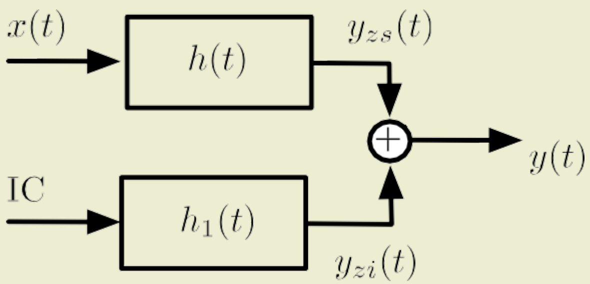
$$Y(s) = H(s)X(s) + H_1(s)I(s), \quad H(s) = \frac{B(s)}{A(s)}, \quad H_1(s) = \frac{1}{A(s)}$$

$$y(t) = y_{zs}(t) + y_{zi}(t)$$

$y_{zs}(t) = \mathcal{L}^{-1}[H(s)X(s)]$ system's zero-state response

$y_{zi}(t) = \mathcal{L}^{-1}[H_1(s)I(s)]$ system's zero-input response





Transient and steady–state responses

$$\boxed{\text{LTI, BIBO system} \quad y(t) = \underbrace{y_t(t)}_{transient} + \underbrace{y_{ss}(t)}_{steady-state}}$$

1. Steady state is due to simple real or complex conjugate pairs poles of $Y(s)$ in $j\Omega$-axis
2. Transient is due to poles of $Y(s)$ in the left-hand s-plane
3. Multiple poles in the $j\Omega$-axis and poles in the right-hand s-plane give unbounded responses

Example: Impulse response of system represented by o.d.e.

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t), \quad \text{input, output}: \; x(t), y(t)$$

$$Y(s)[s^2 + 3s + 2] = X(s) \quad \Rightarrow \quad H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{s+1} + \frac{-1}{s+2}$$

$$h(t) = \left[e^{-t} - e^{-2t}\right] u(t) \qquad \text{(transient only)}$$

Example: Unit–step response

$$S(s)[s^2+3s+2] = X(s) \quad \Rightarrow \quad S(s) = \frac{H(s)}{s} = \frac{1}{s(s^2+3s+2)}$$

$$S(s) = \frac{0.5}{s} + \frac{-1}{s+1} + \frac{0.5}{s+2}$$

$$s(t) = 0.5u(t) - e^{-t}u(t) + 0.5e^{-2t}u(t)$$

$$s_t(t) = -e^{-t}u(t) + 0.5e^{-2t}u(t), \quad \text{(transient)}$$

$$s_{ss}(t) = \lim_{t\to\infty} = 0.5, \quad \text{(steady–state)}$$

Unit–step $s(t)$ and impulse $h(t)$ responses

$$sS(s) = H(s) \quad \Rightarrow \quad \frac{ds(t)}{dt} = [e^{-t} - e^{-2t}]u(t) = h(t)$$

Computation of convolution integral

$$y(t) = [x * h](t) \ \text{convolution} \quad \Rightarrow \quad Y(s) = X(s)H(s)$$

$$H(s) = \mathcal{L}[h(t)] = \frac{Y(s)}{X(s)} \quad \text{trasnfer function of system}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)]$$

Example: Convolution $y(t) = [x * h](t)$ when

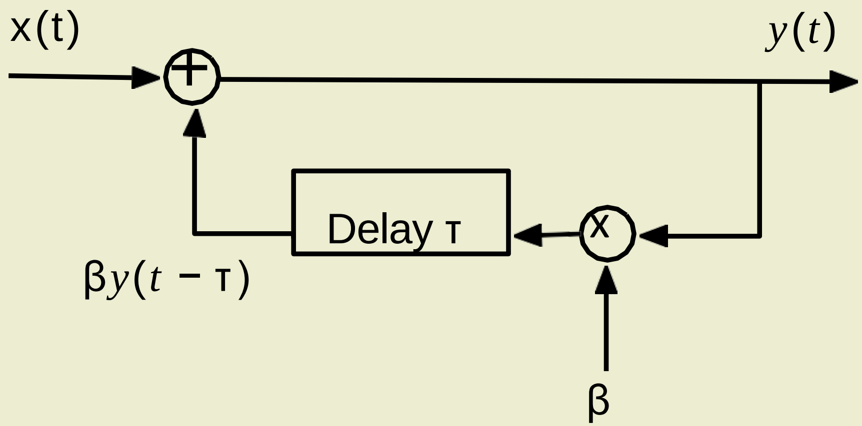
$x(t) = u(t)$, $h(t) = u(t) - u(t-1)$

$$X(s) = \mathcal{L}[u(t)] = \frac{1}{s}, \quad H(s) = \mathcal{L}[h(t)] = \frac{1 - e^{-s}}{s}$$

$$Y(s) = H(s)X(s) = \frac{1 - e^{-s}}{s^2}$$

$$y(t) = r(t) - r(t-1)$$

Example: Positive feedback created by closeness of a microphone to a set of speakers



- Impulse response $x(t) = \delta(t)$, IC$= 0$, $y(t) = h(t)$

$$y(t) = x(t) + y(t-1) \quad \Rightarrow \quad h(t) = \delta(t) + \beta h(t-1)$$
$$H(s) = 1 + H(s)e^{-s}$$
$$H(s) = \frac{1}{1-\beta e^{-s\tau}} = \frac{1}{1-e^{-s}} = \sum_{k=0}^{\infty} e^{-sk} = 1 + e^{-s} + e^{-2s} + e^{-3s} + \cdots$$
$$h(t) = \delta(t) + \delta(t-1) + \delta(t-2) + \cdots = \sum_{k=0}^{\infty} \delta(t-k)$$

- BIBO stability of positive feedback system

  absolute integrability

$$\begin{aligned}\int_{-\infty}^{\infty} |h(t)|dt &= \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \delta(t-k)dt \\ &= \sum_{k=0}^{\infty} \int_{-\infty}^{\infty} \delta(t-k)dt \\ &= \sum_{k=0}^{\infty} 1 \to \infty\end{aligned}$$

  pole location

  poles: roots of $1 - e^{-s} = 0$, or $e^{-s_k} = 1 = e^{j2\pi k} \quad \Rightarrow s_k = \pm j2\pi k$

  System is not BIBO stable ($h(t)$ is not absolutely integrable, or poles of $H(s)$ are not in open left–hand s–plane)