SIGNALS AND SYSTEMS USING MATLAB
Chapter 4 — Frequency Analysis: The Fourier Series

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Eigenfunctions

$x(t) = e^{j\Omega_0 t}, \quad -\infty < t < \infty$, input to a causal and stable LTI system

$$\text{steady state output} \quad y(t) = e^{j\Omega_0 t} H(j\Omega_0)$$

$$H(j\Omega_0) = \int_0^{\infty} h(\tau) e^{-j\Omega_0 \tau} d\tau = H(s)|_{s=j\Omega_0}$$

$s=\cancel{\sigma}+j\Omega_0$

frequency response at Ω_0

$x(t) = e^{j\Omega_0 t}$ is eigenfunction of LTI system

Example: RC circuit, voltage source be $v_s(t) = 4\cos(t + \pi/4)$, $R = 1\ \Omega$, $C = 1\text{F}$

$$\text{transfer function} \quad H(s) = \frac{V_c(s)}{V_s(s)} = \frac{1}{s+1}$$

$$H(j1) = \frac{\sqrt{2}}{2}\angle{-\pi/4} \quad \text{frequency response at}\ \Omega_0 = 1$$

$$\text{steady-state output} \quad v_c(t) = 4|H(j1)|\cos(t + \pi/4 + \angle H(j1)) = 2\sqrt{2}\cos(t)$$

Complex exponential Fourier series

Fourier Series of periodic signal $x(t)$, of fundamental period T_0, is infinite sum of ortho–normal complex exponentials of frequencies multiples of fundamental frequency $\Omega_0 = 2\pi/T_0$ (rad/sec) of $x(t)$:

$$\boxed{\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \\ \text{FS coefficients} \quad X_k &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t) e^{-jk\Omega_0 t}\, dt \end{aligned}}$$

$$\underbrace{\cos(k\Omega_0 t)}_{Re} + j\,\underbrace{\sin(k\Omega_0 t)}_{Im}$$

$\{e^{jk\Omega_0 t}\}$ are ortho-normal Fourier basis

$$\frac{1}{T_0}\int_{t_0}^{t_0+T_0} e^{jk\Omega_0 t}[e^{j\ell\Omega_0 t}]^* dt = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} e^{j(k-\ell)\Omega_0 t} dt$$

$$= \begin{cases} 0 & k \neq \ell \quad \text{orthogonal} \\ 1 & k = \ell \quad \text{normal} \end{cases}$$

Line spectrum

- Parseval's power relation

$$P_x: \text{ power of periodic signal } x(t) \text{ of fundamental period} T_0$$

$$P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2, \qquad \text{for any } \; t_0$$

- Periodic $x(t)$ is represented in frequency by
  - Magnitude line spectrum $|X_k|$ vs $k\Omega_0$
  - Phase line spectrum $\angle X_k$ vs $k\Omega_0$
  - Power line spectrum $|X_k|^2$ vs $k\Omega_0$

- Real-valued periodic signal $x(t)$, of fundamental period T_0,

  $X_k = X^*_{-k}$ or equivalently

  (i) $|X_k| = |X_{-k}|$, i.e., magnitude $|X_k|$ is even function of $k\Omega_0$.

  (ii) $\angle X_k = -\angle X_{-k}$, i.e., phase $\angle X_k$ is odd function of $k\Omega_0$

Trigonometric Fourier series

Real-valued, periodic signal $x(t)$, of fundamental period T_0,

$$\boxed{\begin{aligned} x(t) &= \underbrace{X_0}_{dc-component} + 2\sum_{k=1}^{\infty} \underbrace{|X_k|\cos(k\Omega_0 t + \theta_k)}_{k^{th}\,harmonic} \\ &= c_0 + 2\sum_{k=1}^{\infty}[c_k\cos(k\Omega_0 t) + d_k\sin(k\Omega_0 t)] \qquad \Omega_0 = \frac{2\pi}{T_0} \end{aligned}}$$

Fourier coefficients $\{c_k,\ d_k\}$

$$c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(k\Omega_0 t)\,dt \qquad k = 0, 1, \cdots$$

$$d_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(k\Omega_0 t)\,dt \qquad k = 1, 2, \cdots$$

Sinusoidal basis functions $\{\sqrt{2}\cos(k\Omega_0 t),\ \sqrt{2}\sin(k\Omega_0 t)\}$, $k = 0, \pm1, \cdots,$ are orthonormal in $[0,\ T_0]$

Example: $x(t) = B + A\cos(\Omega_0 t + \theta)$ periodic of fundamental period T_0

trigonometric Fourier series: $X_0 = B$; $|X_1| = A/2$, $\angle X_1 = \theta$

exponential Fourier series:

$$x(t) = B + \frac{A}{2}\left[e^{j(\Omega_0 t+\theta)} + e^{-j(\Omega_0 t+\theta)}\right]$$

$$X_0 = B, \quad X_1 = \frac{Ae^{j\theta}}{2}, \quad X_{-1} = X_1^* = \frac{Ae^{-j\theta}}{2}$$

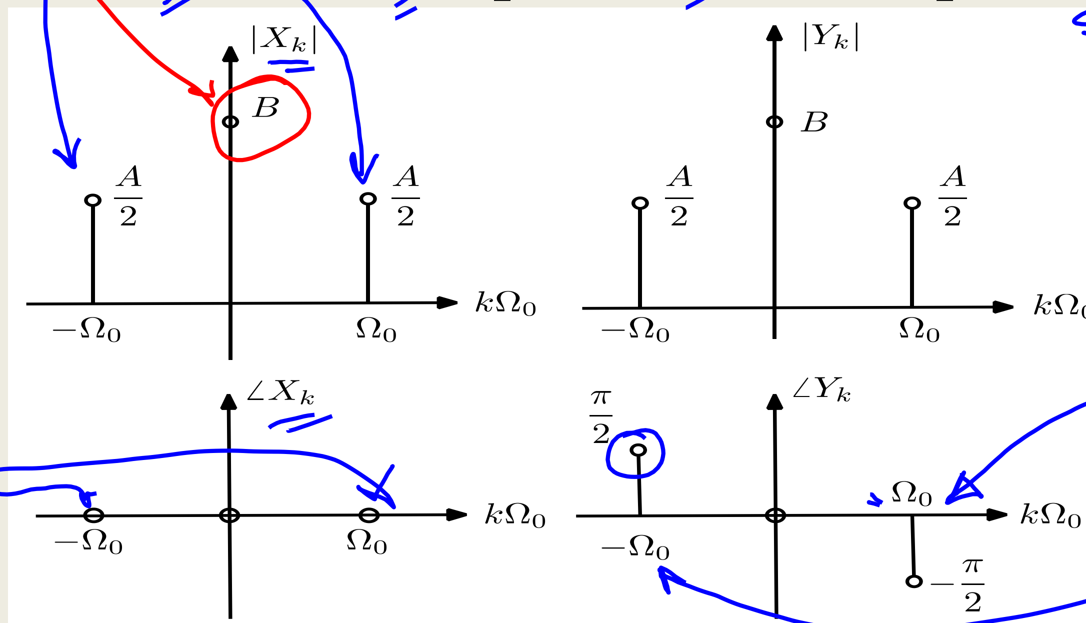
sin: $\theta = -\frac{\pi}{2}$

$-\Omega_0$: $-\theta = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$

cos

$\theta = 0$

$-\theta = 0$



Line spectrum of $x(t) = B + A\cos(\Omega_0 t)$ and of $y(t) = B + A\sin(\Omega_0 t)$ (right).

Fourier coefficients from Laplace

$x(t)$, periodic of fundamental period T_0

$$\boxed{\begin{array}{l} \text{period:} \quad x_1(t) = x(t)[u(t-t_0) - u(t-t_0-T_0)], \quad \text{any } t_0 \\ X_k = \dfrac{1}{T_0}\mathcal{L}\left[x_1(t)\right]_{s=jk\Omega_0} \quad \Omega_0 = \dfrac{2\pi}{T_0} \text{ (fundamental frequency)}, \; k = 0, \pm 1, \cdots \end{array}}$$

Example: $x(t)$ periodic, $T_0 = 2$, $x_1(t) = u(t) - u(t-1)$

$$x(t) = \sum_{m=-\infty}^{\infty} x_1(t-2m) = \sum_{k=-\infty}^{\infty} X_k e^{jk\pi t}$$

$$X_k = \frac{1}{2}\mathcal{L}\left[x_1(t)\right]_{s=jk\pi} = \frac{1-e^{-jk\pi}}{jk\pi} = e^{-jk\pi/2} \; \frac{\sin(k\pi/2)}{k\pi/2}$$

Time and frequency shifting

Periodic signal $x(t)$

- Time-shifting: $x(\pm t_0)$ remains periodic of the same fundamental period

$$x(t) \;\leftrightarrow\; \{X_k\} \;\Rightarrow\; x(t \mp t_0) \;\leftrightarrow\; X_k e^{\mp jk\Omega_0 t_0} = |X_k| e^{j(\angle X_k \mp k\Omega_0 t_0)}$$

  only change in phase

- Frequency-shifting:
  - $x(t)e^{j\Omega_1 t}$ is periodic of fundamental period T_0 if $\Omega_1 = M\Omega_0$, for an integer $M \geq 1$,
  - for $\Omega_1 = M\Omega_0$, $M \geq 1$, the Fourier coefficients X_k are shifted to frequencies $k\Omega_0 + \Omega_1 = (k+M)\Omega_0$
  - the modulated signal is real-valued by multiplying $x(t)$ by $\cos(\Omega_1 t)$.

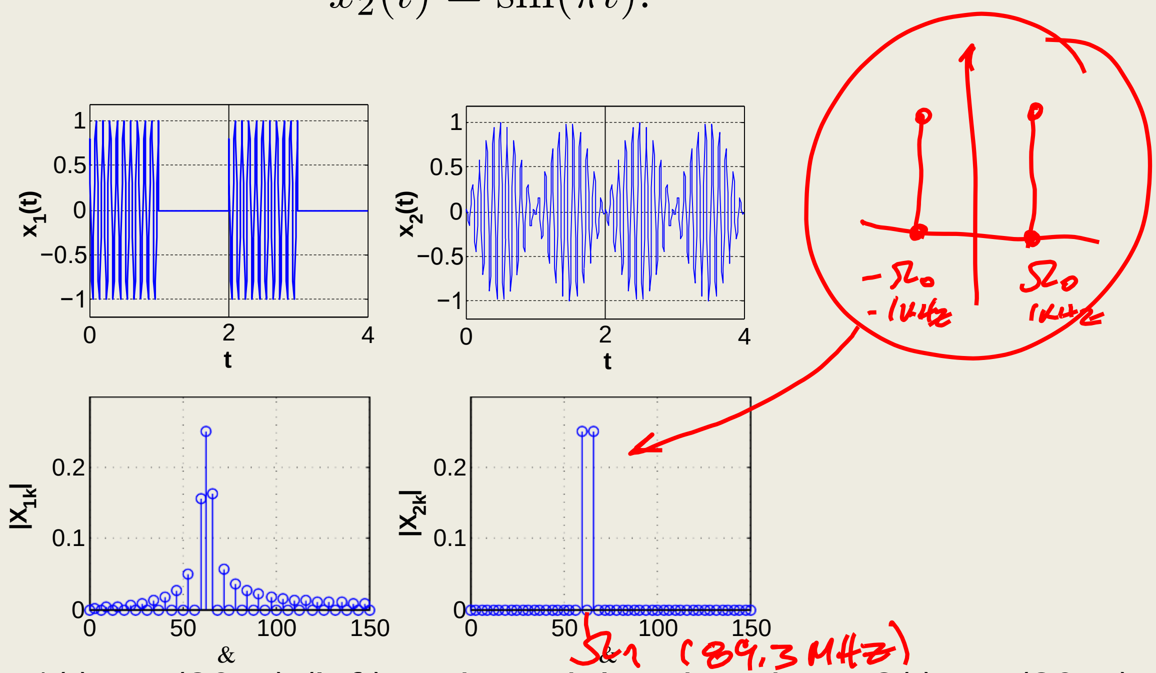
Example: Modulating $\cos(20\pi t)$ with

- a periodic train of square pulses

$$x_1(t) = 0.5[1 + \mathrm{sign}(\sin(\pi t))] = \begin{cases} 1 & \sin(\pi t) \geq 0 \\ 0 & \sin(\pi t) < 0 \end{cases}$$

- with a sinusoid

$$x_2(t) = \sin(\pi t).$$



Modulated square-wave x1(t) cos(20πt) (left) and modulated cosine x2(t) cos(20πt)

Response of LTI systems to periodic signals

Periodic input $x(t)$ of causal and stable LTI system, with impulse response $h(t)$, by eigenfunction property of LTI systems

Fourier series $\quad x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k| \cos(k\Omega_0 t + \angle X_k) \qquad \Omega_0 = \frac{2\pi}{T_0}$

$$y_{ss}(t) = X_0 \underline{|H(j0)|} + 2\sum_{k=1}^{\infty} |X_k| \underline{|H(jk\Omega_0)|} \cos(k\Omega_0 t + \angle X_k + \angle H(jk\Omega_0))$$

where $\underline{H}(jk\Omega_0) = |H(\underline{jk\Omega_0})| e^{j\angle \underline{H(jk\Omega_0)}} H(s)|_{\underline{s=jk\Omega_0}}$

frequency response of the system at $k\Omega_0$