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MA238 **Final exam work.**

I took pictures with my phone and uploaded it to a word document. Last exam you told me it was hard to see my work, so I am trying a different method. I hope this works better.

Number 1

1) $x^2 \frac{dy}{dx} - xy = y^2$

$y' + p(x)y = g(x)$

$x^2 y' - xy = y^2 \Rightarrow \dfrac{y' - \frac{1}{x}y = \frac{1}{x^2}y^2}{y^2}$ $\quad p(x) = -\frac{1}{x}$ $\quad g(x) = \frac{1}{x^2}$ $\quad n = 2$

$\frac{y'}{y^2} - \frac{1}{xy} = \frac{1}{x^2} \Rightarrow v = y^{1-n}$ then $v' = (1-n)\frac{y'}{y^n}$

$\frac{y'}{y^2} - \frac{1}{x}v = \frac{1}{x^2}$, find $v' = (y^{-1})' = (u^{-1})' = -\frac{1}{u^2}$ $\quad u = y$.

$= \left(-\frac{1}{u^2}\right)y' = \left(-\frac{1}{y^2}\right)y' = -\frac{y'}{y^2} = v'$

$-v' - \frac{1}{x}v = \frac{1}{x^2} \Rightarrow v' + \frac{1}{x}v = -\frac{1}{x^2}$ $\quad P(x) = \frac{1}{x}$, $q(x) = -\frac{1}{x^2}$

Integrating factor $= x$.

$\cancel{x}v' + \frac{1}{x}v = -\frac{1}{x^2} \Rightarrow xv' + v = -\frac{1}{x}$ w/ integrating factor.

Product rule: $(xv)' = -\frac{1}{x} \Rightarrow xv = \int -\frac{1}{x}\,dx = xv = -\ln(x) + c_1$

$\therefore\ v = \dfrac{-\ln(x) + c_1}{x}$, then sub back y^{-1} for v:

$y^{-1} = \dfrac{-\ln(x) + c_1}{x} = \dfrac{1}{y} = \dfrac{-\ln(x) + c_1}{x} \Rightarrow x = y(-\ln(x) + c_1)$

$$\boxed{y = \frac{x}{-\ln(x) + c_1}}$$

Number 2

2) $\frac{dy}{dx} + \frac{2}{x}y = 4\sqrt{y} \Rightarrow y' + \frac{2}{x}y = 4\sqrt{y}$; $p(x) = \frac{2}{x}$; $q(x) = 4$ $n = \frac{1}{2}$

$y' + \frac{2}{x}y = 4\sqrt{y} = \frac{y'}{\sqrt{y}} + \frac{2\sqrt{y}}{x} = 4$ $\quad v = \sqrt{y}$

$\frac{y'}{\sqrt{y}} + \frac{2v}{x}$ $\quad v' = \frac{y'}{2\sqrt{y}} = \frac{1}{\frac{1}{2}} v' + \frac{2}{x}v = 4$

$\Rightarrow v' + \frac{v}{x} = 2$ $\quad p(x) = \frac{1}{x}$; $q(x) = 2$

Integrating factor: x = $xv' + \frac{vx}{x} = 2x$

$(xv') = 2x \Rightarrow xv = \int 2x\,dx = xv = x^2 + c_1$

$v = \frac{x^2+c_1}{x}$, but $v = \sqrt{y}$

$\sqrt{y} = \frac{x^2+c_1}{x} = y = \left(\frac{x^2+c_1}{x}\right)^2 \Rightarrow y = x^2 + 2c_1 + \frac{c_1^2}{x^2}$

Simplify constants:.. $\boxed{y = x^2 + cx^{-2}}$

Number 3

3) $y^4 \frac{dy}{dx} = x+1 - \frac{dy}{dx}$

y' for $\frac{dy}{dx}$: $y^4 y' = x+1 - y' \Rightarrow y^4 y' - y' = x+1 \Rightarrow (y^4+1)y' = x+1$

$(y^4+1)dy = (x+1)\,dx \Rightarrow \int y^4+1\,dy = \int x+1\,dx$

$\int x+1\,dx = \frac{x^2}{2}+x+C_1$ $\quad \int y^4+1 = \frac{y^5}{5}+y+C_2$

$\Rightarrow \frac{y^5}{5}+y+C_2 = x^2+x+C_1$

$$\boxed{\frac{1}{5}y^5+y = \frac{1}{2}x^2+x+C_1}$$

Number 4

4) $y''' - 8y'' + 37y' - 50y = 0$

Assume a solution of e^{yt}.

$(e^{yt})''' - 8(e^{yt})'' + 37(e^{yt})' - 50(e^{yt}) = 0$

$y^3 e^{yt} - 8y^2 e^{yt} + 37e^{yt} y - 50e^{yt} = 0$

$e^{yt}(y^3 - 8y^2 + 37y - 50) = 0$, since $e^{yt} \neq 0$...

$y^3 - 8y^2 + 37y - 50 = 0$

$(y-2)(y^2 - 6y + 25) = 0 \Rightarrow y_1 = 2$

$\frac{6 \pm \sqrt{36 - 4(1)(25)}}{2} \Rightarrow \frac{6 \pm \sqrt{64}\,i}{2} = \frac{6 \pm 8i}{2} \Rightarrow$

$y_2 = 3 + 4i \quad y_3 = 3 - 4i$

Solution for $y = 2$: $C_1 e^{2t}$

Now, for imaginary.

$y = \alpha + i\beta$; Solution: $y = e^{\alpha t}(C_1 \cos(\beta t) + C_2 \sin(\beta t))$

$y = e^{3t}(C_1 \cos(4t) + C_2 \sin(4t))$

Foil out & combine:

$$\boxed{y = C_1 e^{2t} + C_2 e^{3t} \cos(4t) + C_3 e^{3t} \sin(4t)}$$

Number 5

5) $2x^2y'' + 5xy' + y = 0$

2nd order Euler cauchy: $ax^2y'' + bxy' + cy = 0$

Assume a solution of x^r

$2x^2(x^r)'' + 5x(x^r)' + (x^r) = 0$

$(x^r)'' = \underbrace{(x^r)' = rx^{r-1}}_{(x^r)'},\ \underbrace{(rx^{r-1})' = rx^{r-2}(r-1)}_{(x^r)''}$

$2x^2\, rx^{r-2}(r-1) + 5x\, rx^{r-1} + x^r = 0$

$= 2r^2x^r + 3rx^r + x^r = 0 = x^r(2r^2 + 3r + 1) = 0$

$2r^2 + 3r + 1 = 0 \quad \frac{-3 \pm \sqrt{9 - 4(2)(1)}}{4} \quad \frac{-3 \pm \sqrt{9 - 8}}{4} = \frac{-3 \pm \sqrt{1}}{4}$

$\frac{-3 \pm 1}{4} = \underline{r_1 = -\frac{1}{2} \quad r_2 = -1}$

$y = c_1x^{r_1} + c_2x^{r_2}$

$\boxed{y = c_1x^{-1/2} + c_2x^{-1}}$

Number 6

6) $y'' + y = \tan(x)$

General form: $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x) = R(x)$

$(A^2+1)y = \tan(x)$, where $A = \frac{d}{dx}$

$\therefore m^2+1 \Rightarrow m^2 = -1,\ m = \pm\sqrt{-1} = \pm i$

$m = a \pm ib$, the C.F. $= e^{ax}(C_1\cos bx + C_2 \sin bx)$, here $a = 0,\ b = 1$

$\therefore$ Function $= C_1\cos(x) + C_2\sin(x)$

Let $u = \cos(x)$ $\quad v = \sin(x)$

$\therefore$ P.I. $= A(x)u(x) + B(x)v(x)$

$uv' - vu' = \cos^2(x) + \sin^2(x) = 1$

$A(x) = \int \frac{vR}{uv'-vu'} = -\int \frac{\sin x \tan x}{1}dx = -\int \frac{\sin^2(x)}{\cos(x)}dx$

$= -\int \frac{1-\cos^2(x)}{\cos(x)}dx = -\int \sec(x)dx + \int \cos(x)dx$

$= -\ln|\sec x + \tan x| + \sin x$

$B(x) = \int \frac{uR}{uv'-vu'} = \int \frac{\cos x \tan x}{1}dx = \int \sin x\, dx = -\cos(x)$

P.I. $= -\ln|\sec x + \tan x|\cos x + \sin x \cos x - \cos x \sin x$

$= -\ln|\sec x + \tan x|\cos x$

General sol'n: $y(x) = CF + PI =$

$$\boxed{y(x) = C_1\cos x + C_2 \sin x - (\cos x)\ln|\sec x + \tan x|}$$

Number 7

7) $y'' - 9y = 24e^{-3t}$ with $y(0)=6$ and $y'(0)=2$

General sol'n for $ay''+by'+cy=g(x)$ can be written as: $y = y_h + y_p$

$y''-9y=0 \Rightarrow$ assume e^{yt}

$(e^{yt})''-9(e^{yt})=0 \Rightarrow y^2e^{yt}-9e^{yt} \Rightarrow e^{yt}(y^2-9)=0$

$y^2-9=0$; $y=3$; $y=-3$ $\therefore$ $\boxed{y_h = C_1e^{3t}+C_2e^{-3t}}$

$\underline{y_p}$: assume a solution of $y = a_0te^{-3t}$

$(a_0te^{-3t})'' - 9a_0te^{-3t} = -24e^{-3t}$

$\Rightarrow a_0(9e^{-3t}t - 6e^{-3t}) - 9a_0te^{-3t} = -24e^{-3t}$

$$\frac{-6a_0e^{-3t}}{-6e^{-3t}} = \frac{24e^{-3t}}{-6e^{-3t}} \Rightarrow a_0 = -4$$

$y = -4te^{-3t} \Rightarrow y = C_1e^{3t}+C_2e^{-3t}-4te^{-3t}$

$6 = C_1e^0 + C_2e^0 - 4te^0 = \; 6 = C_1+C_2 \Rightarrow \underline{C_1 = 6-C_2}$

$y' = 3C_1e^{3t} - 3C_2e^{-3t} + 12te^{-3t} - 4e^{-3t}$

$2 = 3(6-C_2)e^{3t} - 3C_2e^{-3t} + 12te^{-3t} - 4e^{-3t}$

$2 = 3(6-C_2)e^0 - 3(C_2e^{-3(0)}) + 0 - 4e^{-3(0)}$

$2 = 18-3C_2-6-4 \Rightarrow 18-6C_2=6 \quad -6C_2=-12 \quad \underline{C_2=2}$

$y = (6-2)e^{3t} - 3(2)e^{-3t} - 4e^{-3t}$

$$\boxed{y = 4e^{3t}+2e^{-3t}-4e^{-3t}t}$$

Number 8

8) $y'' + 4y = 2\cos(2t);\ y(0) = y'(0) = 0$

$y'' + 4y = 2\cos(2t)$

$L\{y''\} + 4L\{y\} = 2L\{\cos 2t\}$

$s^2 L\{y\} - s y(0) - y'(0) + 4L(y) = \frac{2s}{s^2+4}$

$(s^2+4)L\{y\} - 0 - 0 = \frac{2s}{s^2+4}$

$y(s)[s^2+4] = \frac{2s}{s^2+4}$

$y(s) = \left(\frac{1}{s^2+4}\right)\left(\frac{2s}{s^2+4}\right)$

$y(s) = F(s)\,H(s)$

$= L^{-1}\left\{\frac{2s}{(s^2+4)^2}\right\} = \frac{t}{2}\sin(2t) \to$ From # 9

$n(s) = \frac{t}{2}\sin(2t)$

$F(s) = \frac{1}{s^2+4}$ $H(s) = \frac{2s}{s^2+4}$ $Y(s) = \left(\frac{1}{s^2+4}\right)\left(\frac{2s}{s^2+4}\right)$

$n(s) = \frac{t}{2}\sin 2t$

None of the answers match...

Number 9

9) $y'' + 4y = 2\cos(2t)$; $y(0) = y'(0) = 0$

$L\{y''\} + 4L\{y\} = L\{2\cos 2t\}$

$\therefore L\{y''\} + 4L\{y\} = \frac{2p}{p^2+4}$

$\therefore L\{y''\} = p^2 L\{y\} - py(0) - y'(0)$, given $y(0)=0$, $y'(0)$

$\therefore L\{y''\} = p^2 L\{y\} - p\times 0 - 0 \Rightarrow L\{y''\} = p^2 L\{y\}$

$L\{y\}(p^2+4) = \frac{2p}{(p^2+4)} = L\{y\} = \frac{2p}{(p^2+4)^2} \Rightarrow y = L^{-1}\left\{\frac{2p}{(p^2+4)^2}\right\}$

$L^{-1}\left\{\frac{p}{(p^2+4)}\right\} = \cos(2t) \Rightarrow L^{-1}\left\{\frac{1}{p^2+2^2}\right\} = \frac{1}{2}\sin 2t$

$\therefore$ by convolution: $L^{-1}\left\{\frac{p}{p^2+2^2}\cdot\frac{1}{p^2+4}\right\} = \frac{1}{2}\int_0^t \cos 2u \sin 2(t-u)\,du$

$= \frac{1}{2}\int_0^t \cos 2u[\sin 2t\cos 2u - \sin 2u\cos 2t]\,du$

$= \frac{1}{2}\int_0^t \sin 2t\cos^2 2u\,du - \frac{1}{2}\int_0^t \cos 2t\sin 2u\cos 2u\,du$

$= \frac{1}{2}\sin 2t\int_0^t \cos^2 2u\,du - \frac{1}{4}\int_0^t 2\sin 2u\cos 2u\cos 2t\,du$

$= \frac{1}{2}\int_0^t \sin 2t\left(\frac{1+\cos 4u}{2}\right)du - \frac{1}{4}\int_0^t \cos 2t\sin 4u\,du$

$= \frac{1}{4}\sin 2t\left[u + \frac{\sin 4u}{4}\right]_0^t + \frac{1}{4}\cos 2t\left(\frac{\cos 4u}{4}\right)_0^t$

$= \frac{1}{4}\sin 2t\left[t + \frac{\sin 4t}{4}\right] + \frac{1}{4}\cos 2t\left[\frac{\cos 4t}{4} - \frac{1}{4}\right]$

$= \frac{1}{4}t\sin 2t + \frac{1}{16}\sin 2t\sin 4t + \frac{1}{16}\cos 2t\cos 4t - \frac{1}{16}\cos 2t$

$= \frac{t\sin 2t}{4} - \frac{1}{16}\cancel{\cos 2t + \frac{1}{16}\cos(2t)}$

$= \frac{t\sin 2t}{4}$ $\therefore L^{-1}\left\{\frac{p}{p^2+4}\cdot\frac{1}{p^2+4}\right\} = \frac{t\sin 2t}{4}$

$y = \frac{2t\sin 2t}{4} = \boxed{y = \frac{t}{2}\sin 2t}$

Number 10

10) i) This is under damped response. It is a sinusoidal wave that increases in amplitude.

ii) Yes, it can achieve resonance. The natural frequency of oscillation : $\omega_n = (\text{coeff of } y'')^{1/2}$, made l it can reach 2 rad/s. Input frequency is also 2 rad/s, Thus the system can achieve resonance.

iii) Let $D = d/dt$: $D^2 y + 4y = 0$, or $y(D^2+4) = 0$

or $D^2 = -4 \Rightarrow D = 2i$ and $D = -2i$,

Therefore the system is somewhat stable.

iv) The given input; $2\cos(2t)$, therefore the response is unbounded. And therefore it approaches infinity as t approaches infinity.