The Convolution Operation and Its Role in Solving ODEs

The primary goal of a course in Differential Equations is to arm you with tools (aka techniques) to be able to solve ODEs of various types. Hopefully you have seen at this point in your study of solving ODEs that we can use several different methods to solve the same type of equation. We see that some methods that are used to solve an ODE are much quicker than other methods ALL THE TIME! Other ODEs can be solved by different methods more efficiently depending upon how complex the functions that appear in the equation tend to be. For example, $y'' + y = \sec(x)$ would be easier to solve by Variation of Parameters than by Method of Undetermined Coefficients or Laplace Transforms.

Note that in our study of Laplace Transforms and their use in solving ODEs, we saw that equations like...

- $y'' + 4y = \sin(2t)$; $y(0) = 3$, $y'(0) = 5$
- $y'' - 4y' + 13y = e^{2t}\sin(3t)$; $y(0) = 4$ and $y'(0) = 3$
- $y''' - 27y = e^{-3t}$; $y(0) = 2$, $y'(0) = 3$, and $y''(0) = 4$

took a major effort to solve using just Laplace Transforms, but these linear, nonhomogeneous equations would also take some effort to solve by Method of Undetermined Coefficients! With having to put in so much effort to solve these types of equations using either aforementioned method, the internal question you probably ask yourself is...

THERE HAS GOT TO BE A BETTER, FASTER, MORE EFFICIENT WAY THAT WE COULD POSSIBLY SOLVE SOME OF THESE ODEs THAT HAVE MORE COMPLEX FUNCTIONS IN THEM ???

Fortunately, there is a technique that we can use to possibly accomplish this goal in some situations!

Consider the following ODE with initial conditions below.

$$y' - 5y = t^2 \; ; \; y(0) = 0$$

If we choose to solve this ODE by using Laplace Transforms, we would see that...

$$\mathcal{L}\{y' - 5y\} = \mathcal{L}\{t^2\} \Rightarrow \mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = \mathcal{L}\{t^2\}$$

$$\therefore \; sY(s) - \cancel{y(0)}^{0} - 5Y(s) = \frac{2!}{s^3} \Rightarrow Y(s)[s-5] = \frac{2}{s^3}$$

$$\therefore \; Y(s) = \frac{2}{s^3(s-5)} = \left(\frac{2}{s^3}\right)\left(\frac{1}{s-5}\right)$$

At this point we would probably revert to doing Partial Fraction Expansion/Decomposition (PFE/PFD) by rewriting $Y(s)$ as...

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-5}$$

, and find $A, B, C, + D$ by doing the following steps.

(1) Use "cover-up" method to find constants $C + D$.

(2) With $C + D$ known, rewrite our $Y(s)$ in PFE form as

$$\frac{2}{s^3(s-5)} - \frac{C}{s^3} - \frac{D}{s-5} = \frac{A}{s} + \frac{B}{s^2}$$

and algebraically manipulate the left side of this equation to match the right side of this equation

(3) Determine constants A + B are step (2) is completed.

NOTE! what we are proposing might take a long time to do!!

what if there was a way to directly solve for the solution of y(t) vs. going through steps (1) - (3) and still having to take the Inverse Laplace Transforms of all 4 terms to finally get to y(t)? What would that look like?

well, since $Y(s) = \left(\frac{2}{s^3}\right)\left(\frac{1}{s-5}\right)$, let us make $X(s) = \frac{2}{s^3}$ and $H(s) = \frac{1}{s-5}$ so that $Y(s) = X(s) \cdot H(s)$. Naturally, we would think that $Y(s) = X(s) \cdot H(s) \Rightarrow y(t) = x(t) \cdot h(t)$ (i.e. if we just take the Inverse Laplace Transform of Y(s), X(s), and H(s), that should conveniently give us the answer), right?

Unfortunately, this method would not work because if we did what we just suggested, we would get...

$Y(s) = \left(\frac{2}{s^3}\right)\left(\frac{1}{s-5}\right) \Rightarrow y(t) = t^2 \cdot e^{5t} = x(t) \cdot h(t)$, BUT IF WE SUBSTITUTED y(t) BACK INTO OUR ODE, WE SEE THAT...

$$y' - 5y = t^2$$
$$\Rightarrow [t^2 e^{5t}]' - 5[t^2 e^{5t}] = t^2$$
$$\Rightarrow [2te^{5t} + t^2 \cdot 5e^{5t}] - 5t^2 e^{5t} = t^2$$
$$\Rightarrow 2te^{5t} + \cancel{5t^2 e^{5t}} - \cancel{5t^2 e^{5t}} = t^2$$
$$\Rightarrow 2te^{5t} = t^2 \quad \text{(FALSE)}$$

This is showing us that $y(t) = x(t) \cdot h(t)$ is not a valid solution to this ODE !!

BIG NOTE: The reason why this approach did not work is because we have (mistakenly) assumed that...

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{x(t)\,h(t)\} = \underbrace{\int_0^\infty e^{-st}[x(t)\,h(t)]\,dt = \int_0^\infty e^{-st}x(t)\,dt \cdot \int_0^\infty e^{-st}h(t)\,dt}_{\text{These two statements do not equal each other as the properties of integrals don't work this way!}}$$

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* **BIG IDEA** * : What if there was a way to REDEFINE MULTIPLICATION in the t-domain in such a way to where the product of $x(t)$ + $h(t)$ using this alternative multiplication operation would actually yield $X(s) \cdot H(s)$ (i.e. the product of $X(s)$ and $H(s)$ using what we know multiplication to be normally) in the s-domain?

If we could find a way to redefine multiplication in this fashion, we would (possibly) be able to find $y(t)$ directly from just knowing what $x(t)$ and $h(t)$ are from performing the Inverse Laplace Transform operation on $X(s)$ and $H(s)$!

GOOD NEWS!! It turns out that we do have such an operation, and this operation is called Convolution!! The secret here is that convolution is defined as an integral transformation somewhat like the Laplace Transform is defined as an integral transformation!

We will now state the definition of convolution, work through a few simple examples of using this new (binary) operation, + discuss some basic properties we need to be aware of as we use this new operation. Finally, we will show how this operation could be used to solve ODEs by revisiting some of the examples we did when learning how to solve ODEs using Laplace Transforms.

<u>Convolution</u> (Def'n): Let $f(t)$ and $g(t)$ be two functions that are defined on the interval $t \geq 0$. Then, the "convolution of f and g", denoted by $f * g = (f * g)(t) = f(t) * g(t)$ is defined as...

$$\boxed{f * g = \int_0^t f(x) \cdot g(t-x)\,dx}.$$

<u>Ex. 1</u>: Evaluate $(f*g)(x)$ if $f(x) = e^{3x}$ and $g(x) = e^{5x}$

$$\therefore\ e^{3x} * e^{5x} = \int_0^x f(t)\cdot g(x-t)\,dt = \int_0^x e^{3t}\cdot e^{5(x-t)}\,dt =$$

$$\int_0^x \left(e^{3t}\cdot e^{5x}\cdot e^{-5t}\right)dt.$$ Note that since x is considered to be a constant to our integral, we can factor out this expression!

$$\therefore \int_0^x \left(e^{3t}\cdot e^{5x}\cdot e^{-5t}\right)dt = e^{5x}\int_0^x e^{3t}\cdot e^{-5t}\,dt = e^{5x}\int_0^x e^{-2t}\,dt =$$

$$e^{5x}\left[\frac{e^{-2t}}{-2}\right]_0^x = \frac{e^{5x}}{-2}\left[e^{-2x} - \cancel{e^{-2(0)}}^{\,1}\right] = -\frac{1}{2}e^{5x}\left[e^{-2x} - 1\right] = \frac{-e^{3x}}{2} + \frac{e^{5x}}{2}$$

$$\therefore\ \boxed{e^{3x} * e^{5x} = \frac{1}{2}\left[e^{5x} - e^{3x}\right]}$$

Ex. 2: Find the convolution $f*g$ for each of the pair of functions.

a) $f(t)=\sin(t)$; $g(t)=e^{-2t}$

$$(f*g)(t)=\int_0^t f(x)\cdot g(t-x)\,dx=\int_0^t \sin(x)\cdot e^{-2(t-x)}\,dx=$$

$$\int_0^t \sin(x)\cdot e^{-2t}\cdot e^{2x}\cdot dx=e^{-2t}\int_0^t\left(\sin(x)\cdot e^{2x}\right)dx=e^{-2t}\int_0^t u\,dv\text{ , where}$$

$u=e^{2x}$ and $dv=\sin(x)\,dx$. Now we will perform (augmented) tabular integration by parts.

$\Rightarrow$

	u		dv	
$u_1=$	e^{2x}	$+$	$\sin(x)\,dx$	$=dv_1$
$u_2=$	$2e^{2x}$	$-$	$-\cos(x)$	$=dv_2$
$u_3=$	$4e^{2x}$	$+$	$-\sin(x)$	$=dv_3$

$\therefore \int u\,dv=u_1dv_2-u_2dv_3+\int u_3\,dv_3$

Augmented Tabular Integration (By Parts)

$$\therefore \int_0^t e^{2x}\sin(x)\,dx=\left[-e^{2x}\cos(x)+2e^{2x}\sin(x)\right]_0^t+\int_0^t -4e^{2x}\sin(x)\,dx$$

$$\Rightarrow \overset{5}{(\cancel{1}+4)}\int_0^t e^{2x}\sin(x)\,dx=\left(-e^{2t}\cos(t)+2e^{2t}\sin(t)\right)-\left(-\overset{1}{\cancel{e^{2(0)}}}\overset{1}{\cancel{\cos(0)}}+2\overset{1}{\cancel{e^{2(0)}}}\overset{0}{\cancel{\sin(0)}}\right)$$

$$\therefore \int_0^t e^{2x}\sin(x)\,dx=-\tfrac{1}{5}e^{2t}\cos(t)+\tfrac{2}{5}e^{2t}\sin(t)+\tfrac{1}{5}$$

$$\therefore e^{-2t}\int_0^t e^{2x}\sin(x)\,dx=-\tfrac{1}{5}\overset{1}{\cancel{e^{2t-2t}}}\cos(t)+\tfrac{2}{5}\overset{1}{\cancel{e^{2t-2t}}}\sin(t)+\tfrac{1}{5}e^{-2t}$$

$$\therefore \boxed{(f*g)(t)=e^{-2t}\int_0^t e^{2x}\sin(x)\,dx=-\tfrac{1}{5}\cos(t)+\tfrac{2}{5}\sin(t)+\tfrac{1}{5}e^{-2t}}$$

Ex. 2 : (cont'd)

b) $f(t) = \frac{1}{\sqrt{t}}$; $g(t) = t^2$

$$f(t) * g(t) = \int_0^t f(z) \cdot g(t-z)\, dz = \int_0^t z^{-1/2} \cdot (t-z)^2\, dz =$$

$$\int_0^t z^{-1/2}(t^2 - 2zt + z^2)\, dz = \int_0^t \left(t^2 z^{-1/2} - 2t z^{1/2} + z^{3/2}\right) dz$$

$$= \int_0^t t^2 z^{-1/2}\, dz - \int_0^t 2t z^{\frac{1}{2}}\, dz + \int_0^t z^{3/2}\, dz$$

$$= t^2 \int_0^t z^{-1/2}\, dz - 2t \int_0^t z^{1/2}\, dz + \int_0^t z^{3/2}\, dz$$

$$= t^2 \left[2z^{1/2} \right]_0^t - 2t \left[\tfrac{2}{3} z^{3/2} \right]_0^t + \left[\tfrac{2}{5} z^{5/2} \right]_0^t$$

$$= t^2 \cdot (2t^{1/2} - \cancelto{0}{2(0)^{1/2}}) - \tfrac{4}{3} t \left(t^{3/2} - \cancelto{0}{(0)^{3/2}} \right) + \tfrac{2}{5} \left(t^{5/2} - \cancelto{0}{(0)^{5/2}} \right)$$

$$= 2t^{5/2} - \tfrac{4}{3} t^{5/2} + \tfrac{2}{5} t^{5/2}$$

$$= \left(2 - \tfrac{4}{3} + \tfrac{2}{5}\right) t^{5/2} = \left(\frac{30 - 20 + 6}{15}\right) t^{5/2} = \frac{16}{15} t^{5/2}$$

$$\therefore \boxed{f(t) * g(t) = \frac{16}{15} t^{5/2}}$$

Ex. 2 (cont'd - 2)

c) $f(x) = \cos(x)$; $g(x) = \cos(x)$

$$f*g = \int_0^x f(t)\cdot g(x-t)\,dt = \int_0^x \cos(t)\cdot\cos(x-t)\,dt$$

NOTE! $\cos(A)\cdot\cos(B) = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$, where $A = t$ and $B = x-t$

$$\therefore \cos(t)\cdot\cos(x-t) = \frac{1}{2}\left[\cos(t+(x-t)) + \cos(t-(x-t))\right]$$

$$\Rightarrow \cos(t)\cdot\cos(x-t) = \frac{1}{2}\left[\cos(x) + \cos(2t-x)\right]$$

This is a const. to the integral

$$\therefore f*g = \int_0^x \left(\frac{1}{2}\left[\cos(x) + \cos(2t-x)\right]\right)dt = \frac{1}{2}\int_0^x \left[\cos(x) + \cos(2t-x)\right]dt$$

$$= \frac{1}{2}\left[\cos(x)\cdot t + \frac{\sin(2t-x)}{2}\right]_0^x = \frac{1}{2}\left[\left(\cos(x)\cdot x + \frac{\sin(2x-x)}{2}\right) - \left(0 + \frac{\overset{-\sin(x)}{\cancel{\sin(-x)}}}{2}\right)\right]$$

$$= \frac{1}{2}\left[x\cos(x) + \tfrac{1}{2}\sin(x) + \sin(x)\right] = \frac{1}{2}\left[x\cos(x) + \tfrac{3}{2}\sin(x)\right]$$

$$\therefore \boxed{f*g = \frac{x\cos(x)}{2} + \frac{3\sin(x)}{4}}$$

Basic Properties of the Convolution (Binary) operation

Now that we have defined the binary operation called convolution, we need to know its flexibility in performing calculations. Most of us identify this "flexibility" as knowing the properties of an operation. The basic properties we want to know to see if "$*$" is...

- Communtative
- Associative
- Distributive
- What element(s) "f" used with "$*$" will always yield 0.
- What element(s) "f" used with "$*$" will always yield f.

It turns out that the convolution operation (where the "elements" that are used with this operation are functions or constant, real numbers) possesses all of the properties stated above. We will state each property (but without proof). Let $\alpha, \beta \in \mathbb{R}$ and $f, g, + h$ be functions.

- (Factor out constants): $[\alpha f] * g = f * [\alpha g] = \alpha [f * g]$
- (Communitive): $f * g = g * f$
- (Associative): $[f * g] * h = f * [g * h]$
- (Distributive from right to left): $[f + g] * h = [f * h] + [g * h]$
- (Distributive from left to right): $f * [g + h] = [f * g] + [f * h]$

- $0 * g = g * 0 = \int_0^t 0 \cdot g(t-x)\,dx = 0$

- $f * 1 = 1 * f = \int_0^t f(x) \cdot 1\,dx = \int_0^t f(x)\,dx$

- $(f * g)(0) = \int_0^0 f(x) \cdot g(0-x)\,dx = 0$

A Note on Evaluating $(f*g)(t)$ at Specific Values of t

When using the convolution operation, make sure to complete the computation for $*$ <u>1st</u> before evaluating $(f*g)(t)$ at a specific value of t.

For example, we found out earlier that $e^{3x} * e^{5x} = \frac{1}{2}\left[e^{5x} - e^{3x}\right]$.

Therefore, note the correct <u>and</u> incorrect way to evaluate $(e^{3x} * e^{5x})(-2)$ (i.e. $(e^{3x} * e^{5x})(x)$ evaluated at $x=-2$).

- <u>Incorrect Way</u>: $(e^{3x} * e^{5x})(-2) = e^{3(-2)} * e^{5(-2)} = e^{-6} * e^{-10} = \int_0^x \overset{e^{-6}}{\cancel{f(t)}}\, \overset{e^{-10}}{\cancel{g(x-t)}}\,dt$

$\int_0^x e^{-16}\,dt = \left[e^{-16}t\right]_0^x = e^{-16}x - e^{-16}(0) = e^{-16}x \longrightarrow$ INCORRECT ANSWER !!!

- <u>Correct Way</u>: $(e^{3x} * e^{5x})(-2) = \left[\int_0^x e^{3t} \cdot e^{5(x-t)} \cdot dt\right]_{x=-2} = \frac{1}{2}\left[e^{5x} - e^{3x}\right]_{x=-2} =$

$\frac{1}{2}\left[e^{5(-2)} - e^{3(-2)}\right] = \frac{1}{2}\left[e^{-10} - e^{-6}\right] \longrightarrow$ CORRECT ANSWER !!!

Using Laplace Transform and Convolution to Solve ODEs

Recall earlier that we wanted to find a solution to the initial value problem

$$y' - 5y = t^2 \quad ; \quad y(0) = 0$$

by saying that $Y(s) = X(s) \cdot H(s) \Rightarrow y(t) = x(t)\, h(t) = t^2 e^{5t}$, but we saw that this assumption was incorrect. It turns out that if we would have said that $y(t) = x(t) * h(t) = t^2 * e^{5t}$, we would have arrived at a solution to our IVP ODE! Let's check it out for ourselves.

NOTE: If you were to solve the 1st-order Linear ODE IVP by other methods (e.g. using the formula $y = e^{-h}\left[\int e^{h} \cdot r \cdot dx + C\right]$, where $h = \int p\, dx$ and $y' - 5y = t^2 \Rightarrow y' + py = r \Rightarrow p = -5$ and $r = t^2$, or by allowing $y = y_h + y_p$ (and use Method of Undetermined Coefficients to find y_p)), then you can verify that the solution to this IVP ODE would be

$$y(t) = -\frac{1}{5}t^2 - \frac{2}{25}t - \frac{2}{125} + \frac{2}{125}e^{5t}.$$

viewed as const. to integral

$$\therefore\ t^2 * e^{5t} = \int_0^t f(x) \cdot g(t-x)\, dx = \int_0^t x^2 \cdot e^{5(t-x)}\, dx = \int_0^t x^2 \cdot e^{5t} \cdot e^{-5x}\, dx$$

$$= e^{5t}\int_0^t x^2 e^{-5x}\, dx = e^{5t}\int_0^t u\, dv, \text{ where } u = x^2 \text{ and } dv = e^{-5x}\, dx.$$

(Tabular Integration by Parts)

$\underline{u}$		$\underline{dv}$
x^2	$+$	$e^{-5x}\,dx$
$2x$	$-$	$-\frac{1}{5}e^{-5x}$
2	$+$	$\frac{1}{25}e^{-5x}$
0		$\frac{-1}{125}e^{-5x}$

$$\therefore \int u\,dv = -\frac{1}{5}x^2e^{-5x} - \frac{2}{25}xe^{-5x} - \frac{2}{125}e^{-5x}$$

$$\therefore t^2 * e^{5t} = e^{5t}\int_0^t u\,dv = e^{5t}\left[-\frac{1}{5}x^2e^{-5x} - \frac{2}{25}xe^{-5x} - \frac{2}{125}e^{-5x}\right]_0^t$$

$$= e^{5t}\left[\left(-\frac{1}{5}t^2e^{-5t} - \frac{2}{25}te^{-5t} - \frac{2}{125}e^{-5t}\right) - \left(0 - 0 - \frac{2}{125}(1)\right)\right]$$

$$= -\frac{1}{5}t^2\overset{1}{\cancel{e^{5t-5t}}} - \frac{2}{25}t\overset{1}{\cancel{e^{5t-5t}}} - \frac{2}{125}\overset{1}{\cancel{e^{5t-5t}}} + \frac{2}{125}e^{5t}$$

$$= -\frac{1}{5}t^2 - \frac{2}{25}t - \frac{2}{125} + \frac{2}{125}e^{5t}$$

$$\therefore y(t) = t^2 * e^{5t} = -\frac{1}{5}t^2 - \frac{2}{25}t - \frac{2}{125} + \frac{2}{125}e^{5t}$$ (SAME ANSWER !!!)

From seeing this example and noting that the "t^2" and "e^{5t}" functions came from $t^2 = \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}$ and $e^{5t} = \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}$, respectively, the following observations can be made (which we will state them as theorems).

Thm 1 (Convolution) : Let $f(t)$ and $g(t)$ be functions such that they both satisfy the following criteria.

- $f(t)$ and $g(t)$ are both continuous (or at least piecewise continuous on $[0, \infty)$.
- $\lim_{t \to \infty} [f(t) \cdot e^{-st}] = 0$ and $\lim_{t \to \infty} [g(t) \cdot e^{-st}] = 0$ (i.e. both $f(t)$ and $g(t)$ are of exponential order s_K, where $s > s_K$).

Then; it follows that ...

(1) $\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\} = F(s) \cdot G(s)$

and

(2) $\mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\} = f(t) * g(t) = f * g$

(3) Let $y(t)$, $x(t)$, and $h(t)$ be functions such that all are Laplace-Transformable. Also, let $Y(s)$, $X(s)$, + $H(s)$ be the corresponding Laplace Transform of $y(t)$, $x(t)$, and $h(t)$. Then, if we can express $Y(s) = X(s) \cdot H(s)$, then it follows that...

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{X(s)\} * \mathcal{L}^{-1}\{H(s)\} = x(t) * h(t)}$$

Thm 2 (Duhamel's Principle)

Let $a_0, a_1, \ldots, a_N$ be any real number where N is any natural number. Also, let $f(t)$ be a function such that $\mathcal{L}\{f(t)\} = F(s)$ exists. Then, the solution, $y(t)$, to the N^{th}-order Linear ODE ...

$$(\star) \quad \boxed{a_0 y^{(N)} + a_1 y^{(N-1)} + a_2 y^{(N-2)} + \ldots + a_{N-2} y'' + a_{N-1} y' + a_N y = f(t)}$$

with the following "zero" initial conditions ...

$$y(0) = 0, \; y'(0) = 0, \; y''(0) = 0, \; \ldots, \text{ and } y^{(N-1)}(0) = 0$$

can be found by ...

$$y(t) = (h * f)(t) = h * f = \int_0^t h(x) f(t-x)\, dx,$$

where ...

$$h(x) = \mathcal{L}^{-1}\{H(s)\}\Big|_x \quad \text{and} \quad H(s) = \frac{1}{a_0 s^n + a_1 s^{n-1} + \ldots a_{n-1} s + a_n}.$$

Finally, $h(x)$ is known as the impulse response function (aka weight function) of equation $(\star)$.

A Few Tips, Tricks, and Observations About Duhamel's Principle

- Recall that $\int_0^t h(x)\, f(t-x)\, dx = \int_0^t h(t-x)\, f(x)\, dx = (h * f)(t)$.

  In practice, use whichever integral form is easiest to work with.

- Note that if $y(t) = (h * f)(t)$, then both "y" and "h" will need to be Laplace-Transformable, but $f(t)$ does not have to be Laplace-Transformable. (Similarly, if $y(t) = (f * h)(t)$, then "y" and "f" will need to be Laplace-Transformable, but $h(t)$ will not have to meet this condition). This is true (in the case of considering $y(t) = (h * f)(t)$) because it can be shown that if ...

$$f_T(t) = \begin{cases} f(t), & \text{if } t < T \\ 0 ; & \text{if } t \geq T \end{cases}, \text{ where } T \to \infty, \text{ then } y = h * f$$

  is well-defined and satisfies the IVP ODE stated in Thm 2 regardless if $f(t)$ is a continuous function or piecewise-cont. function on $(0, \infty)$.

- If we consider the ODE in equation (☆) for Thm 2, but have initial conditions for $y(0), y'(0), \ldots, y^{(N-1)}(0)$ where at least one of these conditions does not equal zero, it can be shown that the solution to this (new) IVP ODE can be obtained by...

$$y(t) = \left[\begin{array}{l}\text{Solution obtained by} \\ \text{Duhamel's Principle} \\ \text{(with all ``zero''} \\ \text{initial conditions)}\end{array}\right] + \left[\begin{array}{l}\text{solution to ODE...} \\ a_0 y^{(N)} + a_1 y^{(N-1)} + \ldots + a_{N-1} y + a_N y = 0 \\ \text{with desired initial conditions (i.e.} \\ \text{initial conditions where at least 1} \\ \text{of them is not equal to zero).}\end{array}\right]$$

Now we will revisit a few examples we did when we learned how to solve ODEs via Laplace Transform techniques. We will now use convolution to help us solve these same ODEs!!! Specifically, we will revisit the following ODEs below.

- $y' - 5y = \text{step}_4(t)$; $y(0) = 0$
- $y' + 7y = 10$; $y(0) = -3$
- $y'' + 4y = \sin(2t)$; $y(0) = 3$ and $y'(0) = 5$

Ex. 1: Solve the following IVP ODE by Convolution.

$$y' - 5y = \text{step}_4(t) \; ; \; y(0) = 0 \quad \Rightarrow Y(s) = \frac{e^{-4s}}{s(s-5)}$$

$$\text{Solution: } y(t) = \frac{1}{5}\left[e^{5(t-4)} - \text{step}_4(t)\right]; \quad \text{step}_4(t) = \begin{cases} 0 \; ; t < 4 \\ 1 \; ; t \geq 4 \end{cases}$$

NOTE: Since we know what $y(t)$ is, we are just verifying that we can arrive at the same solution by using Convolution.

Since we know that taking the Laplace Transform of our ODE yields $Y(s) = \frac{e^{-4s}}{s(s-5)}$, we will $X(s) = \frac{e^{-4s}}{s}$ and $H(s) = \frac{1}{s-5}$ so that $Y(s) = X(s) \cdot H(s)$. Now we will solve for $y(t)$ using convolution.

$$\therefore y(t) = \mathcal{L}^{-1}\{X(s)\} * \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} = \text{step}_4(t) * e^{5t}$$

$$\therefore y(t) = \begin{cases} 0 * e^{5t} \; ; \; t < 4 \\ 1 * e^{5t} \; ; \; t \geq 4 \end{cases} = \begin{cases} 0 \; ; \; t < 4 \\ 1 * e^{5t} \; ; \; t \geq 4 \end{cases}, \text{ where } 1 * e^{5t} = \int_0^{t-4} e^{5x}\, dx$$

$$\therefore \int_0^{t-4} e^{5x}\, dx = \left[\frac{e^{5x}}{5}\right]_0^{t-4} = \frac{1}{5}\left[e^{5(t-4)} - e^{5(0)}\right] = \frac{1}{5}\left[e^{5(t-4)} - 1\right] \; ; \; t \geq 4$$

Note that $1 \; ; \; t \geq 4 = \text{step}(t-4) = \text{step}_4(t)$.

$$\therefore \boxed{y(t) = \text{step}_4(t) * e^{5t} = \int_0^{t-4} e^{5x}\, dx = \frac{1}{5}\left[e^{5(t-4)} - \text{step}_4(t)\right], \text{ where } \text{step}_4(t) = \begin{cases} 0 \; ; \; t < 4 \\ 1 \; ; \; t \geq 4 \end{cases}}$$

Ex. 2 : Verify the solution for the following IVP ODE by using Convolution.

$$y' + 7y = 10 \; ; \; y(0) = -3 \quad \Rightarrow Y(s) = \frac{10}{s(s+7)} - \frac{3}{s+7}$$

$$\text{solution : } y(t) = \frac{10}{7} - \frac{31}{7} e^{-7t}$$

$Y(s) = \frac{10}{s(s+7)} - \frac{3}{s+7}$. Since $Y(s)$ has more than 1 term, this lets us know that $Y(s)$ came from an ODE where at least 1 initial condition did not equal zero. Therefore, we will let $y(t) = y_D + y_{HI}$, where

y_D = solution to ODE using Duhamel's Principle (i.e. $y(0) = 0$) and

y_{HI} = solution to corresponding homogeneous ODE with actual initial conditions of ODE (i.e. solution to $y' + 7y = 0 \; ; \; y(0) = -3$).

Find y_D : Note that $\mathcal{L}\{y' + 7y\} = \mathcal{L}\{10\} \Rightarrow sY(s) - \cancel{y(0)}^{0} + 7Y(s) = \frac{10}{s}$.

Since we assume for y_D that $y(0) = 0 \Rightarrow sY(s) + 7Y(s) = \frac{10}{s}$

$\therefore Y(s)[s+7] = \frac{10}{s} \Rightarrow Y(s) = \frac{10}{s} \cdot \frac{1}{s+7} = X(s) \cdot H(s)$, where $X(s) = \frac{10}{s}$ and $H(s) = \frac{1}{s+7}$. (NOTE: This is exactly the same as the 1st term of $Y(s)$ that was given to us)!

$$\therefore y_D = \mathcal{L}^{-1}\{X(s)\} * \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{10}{s}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+7}\right\} = 10(1) * e^{-7t} = 10 * e^{-7t}$$

$$\therefore y_D = 10 * e^{-7t} = \int_0^t x(z) \cdot h(t-z)\, dz = \int_0^t 10 \cdot e^{-7(t-z)}\, dz = 10 e^{-7t} \int_0^t e^{7z}\, dz$$

Ex. 2 : (cont'd)

$$\therefore y_\Delta = 10e^{-7t}\left[\frac{e^{7z}}{7}\right]_0^t = \frac{10e^{-7t}}{7}\left[e^{7t} - \cancel{e^{7(0)}}^{1}\right] = \frac{10}{7}\left[\cancel{e^{-7t+7t}}^{1} - e^{-7t}\right]$$

$$\therefore \boxed{y_\Delta = \frac{10}{7} - \frac{10}{7}e^{-7t}}$$

[*] NOTE: $y' + 7y = 0$ is a separable equation. I could have solved this ODE using separable equation solving techniques as well!

Find y_{HI} : We need to solve the follow IVP ODE to solve for y_{HI}.

- $y_h' + 7y_h = 0$; $y_h(0) = -3$ (1st order Linear IVP) [*]

$\therefore$ $y_h = e^{-h}\left[\int e^{h} \cdot r \cdot dt + C\right]$, where C = integration constant, $h = \int p \cdot dt$,

$p = 7$, and $r = 0$.

$\therefore$ $h = \int p\, dt = \int 7\, dt = 7t \Rightarrow e^{h} = e^{7t}$ and $e^{-h} = e^{-7t}$.

$\therefore$ $y_h = e^{-7t}\left[\int e^{7t} \cdot 0 \cdot dt + C\right] = e^{-7t}\left[\int 0\, dt + C\right] = e^{-7t}\left[A + C\right]$,

where A and C are both constants. Let K_1 = constant such that $K_1 = A + C$.

$\therefore$ $y_h = K_1 e^{-7t}$. Now we need to find K_1. So, $y(0) = -3 \Rightarrow -3 = K_1 \cancel{e^{-7(0)}}^{1}$

$\therefore K_1 = -3 \Rightarrow \boxed{y_h = y_{HI} = -3e^{-7t}}$

Find $y(t) = y_\Delta(t) + y_{HI}(t)$

$\therefore$ $y(t) = y_\Delta(t) + y_{HI}(t) = \frac{10}{7} - \frac{10}{7}e^{-7t} - 3e^{-7t} = \frac{10}{7} - \frac{31}{7}e^{-7t}$

$$\therefore \boxed{y(t) = \frac{10}{7} - \frac{31}{7}e^{-7t}}$$

Ex. 3 : Verify the solution for the following IVP ODE by using Convolution.

$$y'' + 4y = \sin(2t) \ ; \ y(0) = 3 \text{ and } y'(0) = 5 \Rightarrow Y(s) = \frac{2}{(s^2+4)^2} + \frac{3s+5}{s^2+4}$$

solution : $y(t) = \frac{21}{8}\sin(2t) + 3\cos(2t) - \frac{1}{4}t\cos(2t)$

Similar to Ex. 2, if we are going to employ Convolution in solving this IVP ODE, we need to let $y(t) = y_D(t) + y_{HI}(t)$, where...

- $y_D(t)$ = solution to ODE using Duhamel's Principle (i.e. $y(0) = y'(0) = 0$)
- $y_{HI}(t)$ = solution to corresponding homogeneous ODE with actual initial conditions of the ODE (i.e. solution to $y'' + 4y = 0$; $y(0) = 3$ and $y'(0) = 5$).

Find $y_D(t)$: Note that $y'' + 4y = \sin(2t)$, where $y(0) = 0$ and $y'(0) = 0$ will have the Laplace Transform equivalent of $s^2 Y(s) + 4Y(s) = \frac{2}{s^2+4}$ $\Rightarrow$

$Y(s) = \left(\frac{2}{s^2+4}\right)\left(\frac{1}{s^2+4}\right) = \frac{2}{(s^2+4)^2}$. Let $Y(s) = X(s) \cdot H(s) = \left(\frac{2}{s^2+4}\right)\left(\frac{1}{s^2+4}\right)$,

where $X(s) = \frac{2}{s^2+4}$ and $H(s) = \frac{1}{s^2+4}$. Therefore, $y_D = x(t) * h(t)$, where

$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$ and $h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$.

$$\therefore y_D(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^2+(2)^2}\right\} * \mathcal{L}^{-1}\left\{\frac{1 \cdot 2 \cdot \frac{1}{2}}{s^2+(2)^2}\right\} = \sin(2t) * \tfrac{1}{2}\sin(2t) = \tfrac{1}{2}\left[\sin(2t) * \sin(2t)\right]$$

$$\therefore y_D(t) = \tfrac{1}{2}\int_0^t x(\tau)\cdot h(t-\tau)\, d\tau = \tfrac{1}{2}\int_0^t \sin(2\tau)\cdot \sin(2(t-\tau))\, d\tau$$

NOTE 0: $\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$ (Product-to-Sum Formula)

If we let $\alpha = 2z$ and $\beta = 2(t-z) = 2t - 2z$, then $\alpha - \beta = 2z - 2t + 2z = 4z - 2t$

and $\alpha + \beta = \cancel{2z} + 2t - \cancel{2z} = 2t$.

$\therefore \sin(2z)\cdot\sin(2(t-z)) = \frac{1}{2}\left[\cos(4z-2t) - \cos(2t)\right]$

$$\therefore y_D(t) = \frac{1}{2}\int_0^t \frac{1}{2}\left[\cos(4z-2t) - \cos(2t)\right]dz = \frac{1}{4}\int_0^t \left[\cos(4z-2t) - \cos(2t)\right]dz$$

these expressions are viewed as constants to the integral

$$= \frac{1}{4}\left[\frac{\sin(4z-2t)}{4} - z\cdot\cos(2t)\right]_0^t$$

$$= \frac{1}{4}\left[\left(\frac{\sin(\overset{2t}{\cancel{4t-2t}})}{4} - t\cos(2t)\right) - \left(\frac{\overset{-\sin(2t)}{\cancel{\sin(-2t)}}}{4} - \overset{0}{\cancel{0\cdot\cos(2t)}}\right)\right]$$

$$\therefore y_D(t) = \frac{\sin(2t)}{16} - \frac{t\cos(2t)}{4} + \frac{\sin(2t)}{16} = \boxed{\frac{\sin(2t)}{8} - \frac{t\cos(2t)}{4} = y_D(t)}$$

Find $y_{HI}(t)$: We need to solve the IVP ODE: $y'' + 4y = 0$; $y(0) = 3$ and $y'(0) = 5$

$$\therefore \mathcal{L}\{y''+4y\} = \mathcal{L}\{0\} \Rightarrow s^2Y(s) - s\overset{3}{\cancel{y(0)}} - \overset{5}{\cancel{y'(0)}} + 4Y(s) = 0$$

$$\therefore (s^2+4)Y(s) = 5 + 3s \Rightarrow Y(s) = \frac{5+3s}{s^2+4} = \frac{5}{s^2+4} + \frac{3s}{s^2+4}$$

NOTE 1: We will not rewrite $Y(s) = X(s)\cdot H(s)$ and find $y(t)$ by the convolution of $x(t)$ and $h(t)$ (i.e. $x(t) * h(t)$) because the Inverse Laplace Transform of both terms is relatively easy to find!!

NOTE 2 : We could actually use the characteristic equation for this ODE to quickly find what y_{HI} would be because our IVP ODE is just a 2^{nd}-order Linear ODE with constant coefficients. This would end up being a Case III situation (i.e. $y_h(t) = e^{\lambda t}[\cos(\omega t) + \sin(\omega t)]$). However, we are purposely choosing to solve this ODE using Laplace Transforms since the point of convolution is to use it with Laplace + Inverse Laplace Transforms !

$$\therefore \mathcal{L}^{-1}\left\{\frac{5}{s^2+4}\right\} = 5\mathcal{L}^{-1}\left\{\frac{2\cdot\frac{1}{2}}{s^2+(2)^2}\right\} = \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+(2)^2}\right\} = \frac{5}{2}\sin(2t)$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{3s}{s^2+4}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+(2)^2}\right\} = 3\cos(2t) \rightarrow \therefore \boxed{y_{HI}(t) = \left\{\frac{5}{2}\sin(2t) + 3\cos(2t)\right\}}$$

Find $y(t) = y_D(t) + y_{HI}(t)$

$$\therefore y(t) = y_D(t) + y_{HI}(t) = \left[\frac{1}{8}\sin(2t) - \frac{1}{4}t\cos(2t)\right] + \left[\frac{5}{2}\sin(2t) + 3\cos(2t)\right]$$

$$\Rightarrow y(t) = \left(\cancel{\frac{1}{8} + \frac{5}{2}}\right)^{21/8}\sin(2t) - \frac{1}{4}t\cos(2t) + 3\cos(2t)$$

$$\boxed{y(t) = \frac{21}{8}\sin(2t) - \frac{1}{4}t\cos(2t) + 3\cos(2t)}$$