Solving 1^{st}-order ODEs via Substitution Techniques

- The Basics on Solving 1^{st}-order ODEs via substitution
- Linear substitution, $u = Ax + By + C$, technique
- Rational/Fractional substitution, $u = y/x$, technique for Homogeneous ODEs
- Radical/Fractional Exponent substitution, $u = y^r \iff \sqrt[r]{u} = y$, technique for Bernoulli Equations

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Up to this point in our study of this subject, we have studied equation solving techniques for the following types of equations.

- Directly Integrable : $\frac{dy}{dx} = f(x)$
- Separable : $\frac{dy}{dx} = f(x) \cdot g(y)$
- Autonomous : $\frac{dy}{dx} = f(y)$ (which is a special case of a Separable equation)
- Linear (1^{st}-order) : $\frac{dy}{dx} + p(x)y = r(x) \iff \frac{dy}{dx} = \underbrace{r(x) - p(x)y}_{\text{i.e. specific } f(x,y)}$

<u>Our goal now is to develop techniques that would help us work with ODEs that would not be categorized as any of equation types above, but, with a little "help" (i.e. clever substitution and algebraic manipulation), get these ODEs in a form that would either be separable or linear 1^{st}-order</u>

The Basics

A first-order ODE of the type $\frac{dy}{dx} = f(x,y)$, where $f(x,y) = f(g(x,y))$ (i.e. $f(x,y)$ is a composition function of "x" and "y"), often times are not able to be written in separable or linear (1^{st}-order ODE) form directly. (Note that since $\frac{dy}{dx} = f(x,y) \neq f(x)$, the case of being directly integrable or autonomous (i.e. $\frac{dy}{dx} = f(y)$) is not considered as a possible alternative form for these equations). In general, in order to use the technique of solving ODEs that are not separable or linear via substitution, the key to using this method is to get "creative" with your substitution!

The goal of the creativity that you (ideally) would use to solve equations of this ilk would be to eventually end up with the ODE in another form that is more managable to solve after the substitution is complete (i.e. our ODE turns into either a separable or linear equation). Unfortunately, there is no single, bullet-proof method or suggestion for substitution into these problems. We will look at special cases though.

Linear Substitution Technique ($u = Ax + By + C$)

1^{st}-order ODEs that can be rewritten in a way where...

$$\frac{dy}{dx} = f(Ax + By + C), \text{ where } A, B, C \text{ are constants,}$$

are ODEs that should be solved using substitution where we let $u = Ax + By + C$! We say that $u = Ax + By + C$ is a "linear" substitution because (by definition) if we let $Ax + By + C = 0$, then this is an equation of a line. (Consider that we can rewrite $Ax + By + C = 0$ as $Ax + By = -C$, where $C \leq 0$ or $C \geq 0$ depending on the situation).

Steps for Linear Substitution

1. Verify that ODE is not separable or linear.
2. Verify that ODE is of type $\frac{dy}{dx} = f(Ax + By + C)$
3. Let $u = Ax + By + C$. Solve for $y \Rightarrow y = \frac{u - C - Ax}{B}$
4. Find $\frac{dy}{dx}$. (Note: $\frac{dy}{dx} = \frac{1}{B}\frac{du}{dx} - \frac{A}{B}$)
5. Substitute the new representations for both "y" and "$\frac{dy}{dx}$" into the original ODE
6. Resulting equation is separable. Solve via separable techniques.

Ex: Verify that each ODE is a candidate for linear substitution. (4)
If it is, solve the ODE using suggested $u = Ax + By + C$ substitution
Write final answer as an explicit function of one variable if possible.

a) $\frac{dy}{dx} = \tan^2(x+y)$

Sol'n: Note that no constant solutions for this ODE exist. Also, we see that $\frac{dy}{dx} = f(Ax+By+C)$, where $f(Ax+By+C) = \tan^2(x+y)$ since $x+y = Ax+By+C$ if $A=B=1$ and $C=0$. So, let $u = x+y$!

$\therefore u = x+y \Rightarrow y = u-x \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$

$\therefore \frac{dy}{dx} = \tan^2(x+y) \Rightarrow \frac{du}{dx} - 1 = \tan^2(u) \Rightarrow \frac{du}{dx} = \cancel{\tan^2(u)+1} \; \sec^2(u)$

$\therefore \frac{du}{dx} = \sec^2(u)$ is separable $\Rightarrow \frac{du}{\sec^2(u)} = dx \Rightarrow \int \cos^2(u)\,du = \int dx.$

NOTE: $\cos(2u) = 2\cos^2(u) - 1 \Rightarrow \cos^2(u) = \frac{1}{2}[\cos(2u)+1].$

$\therefore \int \cos^2(u)\,du = \frac{1}{2}\int [\cos(2u)+1]\,du = \frac{1}{2}\cdot\frac{\sin(2u)}{2} + u + C = \frac{\sin(2u)}{4} + u + C$

$\therefore \int \cos^2(u)\,du = \int dx \Rightarrow \frac{\sin(2u)}{4} + u = x + C \Rightarrow \frac{\sin(2(x+y))}{4} + \cancel{x} + y = \cancel{x} + C$

$\therefore \boxed{\frac{1}{4}\sin(2x+2y) + y = C}$

Ex: (cont'd)

$= f(Ax+By+C) \Rightarrow$ candidate for lin. sub.

b) $\frac{dy}{dx} = 1+e^{y-x+5}$. Let $u = y-x+5 \Rightarrow u+x-5 = y$

$\therefore \frac{dy}{dx} = \frac{du}{dx}+1$. Therefore, $\frac{dy}{dx} = 1+e^{y-x+5} \Rightarrow \frac{du}{dx}+1 = 1+e^{u}$

$\therefore \frac{du}{dx}+1 = 1+e^{u} \Rightarrow \frac{du}{dx} = e^{u} \Rightarrow e^{-u}\,du = dx$

$\therefore \int e^{-u}\,du = \int dx \Rightarrow \frac{e^{-u}}{-1} = x+C \Rightarrow -e^{-u} = x+C$

Recall that $u = y-x+5 \Rightarrow -u = -y+x-5 = x-y-5$

$\therefore -e^{-u} = x+C \Rightarrow -e^{x-y-5} = x+C \Rightarrow e^{x-y-5} = -x-C$

$\therefore \ln\left[e^{x-y-5}\right] = \ln[-x-C] \Rightarrow x-y-5 = \ln(-x-C)$

$$\therefore \boxed{x-5-\ln(-x-C) = y = y(x)}$$

NOTE! $-x-C > 0 \Rightarrow x+C < 0 \Rightarrow x < -C$ in order for us to use $y(x)$ to produce viable solutions.

Ex: (cont'd_2)

c) $\dfrac{dy}{dx} = \dfrac{3x+2y}{3x+2y+2}$; $y(-1) = -1$

Sol'n: Let $u = 3x+2y+2$. Then $u-2 = 3x+2y$. So, we see that

$\dfrac{3x+2y}{3x+2y+2} = \dfrac{u-2}{u} = f(u) = f(Ax+By+C)$ if $A=3$, $B=2$, $C=2$.

Therefore, our ODE is now in the form $\dfrac{dy}{dx} = f(Ax+By+C)$, so

we can employ linear substitution.

$\therefore$ Let $u = 3x+2y-2$. Then, $\dfrac{u-3x+2}{2} = y$. It follows that

$\dfrac{dy}{dx} = \frac{1}{2}\dfrac{du}{dx} - \frac{3}{2}$.

$\therefore \dfrac{dy}{dx} = \dfrac{3x+2y}{3x+2y+2} \Rightarrow \frac{1}{2}\dfrac{du}{dx} - \frac{3}{2} = \dfrac{u-2}{u} \Rightarrow \dfrac{du}{dx} - 3 = 2\left[\dfrac{u-2}{u}\right]$

$\therefore \dfrac{du}{dx} = 2\left[\dfrac{u-2}{u}\right] + 3 = \dfrac{2u-4+3u}{u} = \dfrac{5u-4}{u} \Rightarrow \dfrac{du}{dx} = \dfrac{5u-4}{u}$

$\therefore \dfrac{u\,du}{5u-4} = dx$. Let $w = 5u-4$. Then, $\frac{1}{5}(w+4) = u$ and

$\dfrac{dw}{du} = 5 \Rightarrow \frac{1}{5}dw = du$. So, $\dfrac{u\,du}{5u-4} = \dfrac{\frac{1}{5}(w+4)\cdot\frac{1}{5}dw}{w}$

$\therefore \dfrac{u\,du}{5u-4} = \frac{1}{25}\left[1 + 4\cdot\frac{1}{w}\right]dw$

Ex: (Cont'd_3)

c) (cont'd)

$$\therefore \frac{u\,du}{5u-4} = dx \Rightarrow \tfrac{1}{25}\left[1+4\cdot\tfrac{1}{w}\right]dw = dx$$

$$\therefore \int \tfrac{1}{25}\left[1+4\cdot\tfrac{1}{w}\right]dw = \int dx$$

$$\Rightarrow \tfrac{1}{25}\left[w+4\ln|w|\right] = x + C \quad . \text{ But } w = 5u-4$$

$$\therefore \tfrac{1}{25}\left[5u-4+4\ln|5u-4|\right] = x + C \text{ . But } u = 3x+2y-2$$

$$\therefore \tfrac{1}{25}\left[5(3x+2y-2)+4\ln|5(3x+2y-2)-4|\right] = x+C$$

$$\Rightarrow \tfrac{1}{5}(3x+2y-2)+4\ln|15x+10y-10-4| = x+C$$

$$\Rightarrow (3x+2y-2)+20\ln|15x+10y-14| = 5x+D \;;\; D = 5C$$

Applying initial cond'n $y(-1)=-1$ ...

$$3(-1)+2(-1)-2+20\ln|15(-1)+10(-1)-14| = 5(-1)+D$$

$$\Rightarrow -7+20\ln|-39| = -5+D \quad \Rightarrow 20\ln(39) = \cancel{-5+7}^{\,2}+D$$

$$\therefore 20\ln(39)-2 = D \quad \Rightarrow D = \ln(39^{20})-2$$

$\therefore$ $\boxed{\text{Final answer: } (3x+2y-2)+20\ln|15x+10y-14| = 5x+\ln(39^{20})-2}$

NOTE: Because $\frac{dy}{dx} = \frac{3x+2y}{3x+2y+2} \Rightarrow 3x+2y+2 \neq 0 \Rightarrow y \neq -\frac{3}{2}x-1$.

Thus x, y can be any combo of real #s except when $y = -\frac{3}{2}x-1$!!

Ex : (cont'd – 4)

d) $\dfrac{dy}{2+\sqrt{y-2x+3}} = dx$; $y(-1) = -1$

$\therefore \dfrac{dy}{dx} = 2+\sqrt{y-2x+3} = f(Ax+By+C)$, where $A=-2$, $B=1$, $+ C=3$.

Let $u = y-2x+3$. Then, $u+2x-3 = y \Rightarrow \dfrac{dy}{dx} = \dfrac{du}{dx}+2$

$\therefore \dfrac{dy}{dx} = 2+\sqrt{y-2x+3} \Rightarrow \dfrac{du}{dx}+2 = 2+\sqrt{u} \Rightarrow \dfrac{du}{dx} = \sqrt{u}$

$\therefore \dfrac{du}{\sqrt{u}} = dx \Rightarrow u^{-1/2}du = dx \Rightarrow \int u^{-1/2}du = \int dx$

$\therefore 2u^{1/2} = x+C \Rightarrow \boxed{2\sqrt{y-2x+3} = x+C}$ (Implicit, General solution)

Apply initial condition $y(-1) = -1$ yields...

$2\sqrt{-1-2(-1)+3} = -1+C \Rightarrow 2\cancel{\sqrt{4}}^{\,2} = -1+C \Rightarrow 4 = -1+C \Rightarrow \boxed{5 = C}$

$\therefore$ (Implicit, Particular Solution): $2\sqrt{y-2x+3} = x+5$

Now, attempting to solve for "y" yields...

$\left(2\sqrt{y-2x+3}\right)^2 = (x+5)^2 \Rightarrow 4(y-2x+3) = (x+5)^2 \Rightarrow 4y-8x+12 = (x+5)^2$

$\therefore 4y = (x+5)^2+8x-12 \Rightarrow \boxed{y = y(x) = \frac{1}{4}(x+5)^2+2x-3}$

NOTE: There exist no constant solutions for this ODE because $\sqrt{y-2x+3} \neq -2$!

Particular, Explicit solution!

Reason why you don't do $\left(2\sqrt{y-2x+3}\right)^2=(x+c)^2$ before finding "c" (7b)

If we did this step before finding "c" in this IVP, note that...

$$\left(2\sqrt{y-2x+3}\right)^2=(x+c)^2$$

$$\Rightarrow \quad 4(y-2x+3)=(x+c)^2$$

Applying initial cond'n $y(-1)=-1$ yields...

$$4(-1-2(-1)+3)=(-1+c)^2$$

$$4(4)=(c-1)^2 \Rightarrow 16=(c-1)^2 \Rightarrow \pm\sqrt{\cancel{16}}^{\,4}=c-1 \Rightarrow c=1\pm4$$

$\therefore c=5$ or -3

THIS IS A PROBLEM BECAUSE IVP. ODEs MUST HAVE EXACTLY 1 SOLUTION!! ONE OF THESE SOLUTIONS IS EXTRANEOUS!

It turns out that if you let $c=-3$ and apply $y(-1)=-1$ to your general solution $2\sqrt{y-2x+3}=x+c$, it turns out to be...

$$2\sqrt{-1-2(-1)+3}=-1+(-3) \Rightarrow 2\sqrt{\cancel{4}}^{\,2}=-4 \Rightarrow 4=-4 \text{ (FALSE)!}$$

Thus, doing the same thing but letting $c=5$ will yield $4=4$ (TRUE)!

To avoid all this extra steps; we don't square both sides until after finding "c" because when you do, you're inadvertently stating that both $-2\sqrt{y-2x+3}=x-3$ and $2\sqrt{y-2x+3}=x+5$ are solutions to this ODE for $y(-1)=-1$ which can't be true!

Ex: (contd - 4)

e) $\frac{dy}{dx} = \cos(x+y)$; $y(0) = \pi/4$ $= f(Ax+By+C) \Rightarrow$ candidate for linear sub!

Let $u = x+y \Rightarrow u - x = y \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1$

$\therefore \frac{dy}{dx} = \cos(x+y) \Rightarrow \frac{du}{dx} - 1 = \cos(u) \Rightarrow \frac{du}{dx} = \cos(u) + 1$

$\therefore \frac{du}{\cos(u)+1} = dx$. NOTE: $\frac{1}{\cos(u)+1} = \frac{1}{\cos(u)+1} \cdot \frac{\cos(u)-1}{\cos(u)-1} =$

$\frac{\cos(u)-1}{\cos^2(u)-1} = \frac{\cos(u)-1}{-\sin^2(u)} = \frac{\cos(u)\cdot 1}{-\sin(u)\cdot\sin(u)} + \frac{-1}{-\sin^2(u)} =$

$-\cot(u)\cdot\csc(u) + \csc^2(u)$. Therefore, $\int \frac{du}{\cos(u)+1} = \int[-\cot(u)\cdot\csc(u) + \csc^2(u)]\,du$

Thus, $\int[-\cot(u)\cdot\csc(u) + \csc^2(u)]\,du = \csc(u) - \cot(u) + C$

$\therefore \int \frac{du}{\cos(u)+1} = \int dx \Rightarrow \csc(u) - \cot(u) = x + C$

But $u = x+y$, so our general solution is $\csc(x+y) - \cot(x+y) = x + C$

Applying in initial condition $y(0) = \pi/4$ yields...

$\csc(0+\pi/4) - \cot(0+\pi/4) = 0 + C \Rightarrow \frac{1}{\cancel{\sin(\pi/4)}}^{\sqrt{2}} - \frac{1}{\cancel{\tan(\pi/4)}}^{1} = C \Rightarrow \boxed{C = \sqrt{2} - 1}$

$\therefore$ $\boxed{\text{Final answer: } \csc(x+y) - \cot(x+y) = x + (\sqrt{2} - 1)}$

Homogeneous ODEs and Rational / Fractional substitution technique

If you have a 1^{st}-order ODE that is not separable, linear, and is not of the form $\frac{dy}{dx} = f(Ax+By+C)$, where $A, B, C \in \mathbb{R}$, then the next type of "equation type" you want to test for is to see if its homogeneous or not.

Homogeneous equations (of 1^{st} order) are ODE that satisfy the following criterion:

$$\frac{dy}{dx} = f(x,y) = f(tx, ty), \text{ where } t \in \mathbb{R}.$$

TRANSLATION: If we let $x = tx$ and $y = ty$ for our function $f(x,y)$, then our ODE will be homogeneous if after doing this substitution (and some algebraic manipulation) the result just ends up being $f(x,y)$ again!

WHY IS THIS IMPORTANT?: If the function $f(x,y)$ in our ODE is equal to $f(tx, ty)$, then this means can ultimately rewrite our homogeneous ODE into separable form!

In order to make this happen, it is suggested to use the substitution

$$u = y/x \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = \frac{d}{dx}[ux] = \frac{du}{dx}x + \cancel{\frac{dx}{dx}}^{1} \cdot u = \frac{du}{dx}x + u.$$

★) Using $y = ux$ and $\frac{dy}{dx} = \frac{du}{dx}x + u$ will force our homogeneous ODE to be separable!! (★)

It is important to know when a 1^{st}-order ODE is homogeneous as well as what it looks like when a 1^{st}-order ODE is not homogeneous. The next example will help us get practice in verifying such a condition for an ODE.

Ex: Verify whether the following ODEs are homogeneous or not.

a) $\dfrac{dy}{dx} = \dfrac{2x+y^2}{xy}$

WMS that $f(tx,ty) = f(x,y)$ if $f(x,y) = \dfrac{2x+y^2}{xy}$ and we let $x = tx$ and $y = ty$.

$$\therefore f(tx,ty) = \frac{2(tx)+(ty)^2}{(tx)(ty)} = \frac{2tx+t^2y^2}{t^2xy} = \frac{t[2x+ty^2]}{t^2xy} = \frac{2x+ty^2}{txy}$$

But $\dfrac{2x+ty^2}{txy} \neq \dfrac{2x+y^2}{xy} \Rightarrow f(tx,ty) \neq f(x,y) \Rightarrow$ not homogeneous!

b) $\dfrac{dy}{dx} = \dfrac{y-x}{x}$

WMS that $f(tx,ty) = f(x,y)$ if $f(x,y) = \dfrac{y-x}{x}$ and we let $x = tx$ and $y = ty$.

$$\therefore f(tx,ty) = \frac{ty-tx}{tx} = \frac{\cancel{t}(y-x)}{\cancel{t}x} = \frac{y-x}{x} = f(x,y)$$

$\Rightarrow$ our ODE is homogeneous!!

Ex: (cont'd)

c) $y' = \dfrac{x^2 + 2y^2}{xy}$. Let $x = tx$ and $y = ty$ and $f(x,y) = \dfrac{x^2+2y^2}{xy}$.

WMS that $f(tx, ty) = f(x,y)$ if this ODE is homogeneous.

$$\therefore f(tx,ty) = \frac{(tx)^2 + 2(ty)^2}{(tx)(ty)} = \frac{t^2x^2 + 2(t^2y^2)}{t^2 xy} = \frac{\cancel{t^2}[x^2+2y^2]}{\cancel{t^2}[xy]} = \frac{x^2+2y^2}{xy} = f(x,y)$$

$\therefore$ Our ODE is homogeneous!

d) $y' = \dfrac{y^2}{xy + (xy^2)^{1/3}}$. Let $x = tx$ and $y = ty$ and $f(x,y) = \dfrac{y^2}{xy + (xy^2)^{1/3}}$

WMS that $f(tx,ty) = f(x,y)$ if this ODE is homogeneous.

$$\therefore f(tx,ty) = \frac{(ty)^2}{(tx)(ty) + [(tx)(ty)^2]^{1/3}} = \frac{t^2y^2}{t^2xy + [t^3x\,y^2]^{1/3}} =$$

$$\frac{t^2y^2}{t^2xy + tx^{1/3}y^{2/3}} = \frac{\cancel{t}\cdot t\cdot y^2}{\cancel{t}[txy + x^{1/3}y^{2/3}]} = \frac{ty^2}{txy + (xy^2)^{1/3}} \neq f(x,y)$$

$\therefore$ Our ODE is not homogeneous!!

Now, we will do some examples of solving homogeneous ODEs.

Ex: All of the following ODEs are homogeneous. Find all solutions (including any constant solutions) to the given ODEs. Please express your final answer explicitly (as a function of x) when possible.

a) $\frac{dy}{dx} = \frac{y-x}{y+x}$. Since our (1st-order) ODE is homogeneous, let $u = y/x$

$\Rightarrow y = ux$. Therefore, $\frac{dy}{dx} = \frac{du}{dx}x + u$

$\therefore \frac{dy}{dx} = \frac{y-x}{y+x} \Rightarrow \frac{du}{dx}x + u = \frac{ux-x}{ux+x} = \frac{x(u-1)}{x(u+1)} = \frac{u-1}{u+1}$

$\therefore \frac{du}{dx}x + u = \frac{u-1}{u+1} \Rightarrow \frac{du}{dx}x = \frac{u-1}{u+1} - \frac{u(u+1)}{u+1} = \frac{\cancel{u}-1-u^2-\cancel{u}}{u+1}$

$\therefore \frac{du}{dx}x = \frac{-(1+u^2)}{u+1} \Rightarrow \frac{(u+1)\,du}{-(1+u^2)} = \frac{dx}{x} \Rightarrow \int \frac{-(u+1)}{1+u^2}du = \int \frac{dx}{x}$

For $\int \frac{-(u+1)}{1+u^2}du$, let $w = 1+u^2 \Rightarrow dw = 2u \cdot du \Rightarrow \frac{1}{2}dw = u \cdot du$

$\therefore \int \frac{-(u+1)}{1+u^2}du = \int \frac{-u}{1+u^2}du + \int \frac{-1}{1+u^2}du = \int \frac{-\frac{1}{2}dw}{w} - \int \frac{du}{1+u^2} = -\frac{1}{2}\ln|w| - \tan^{-1}(u)$

$\therefore \int \frac{-(u+1)}{1+u^2}du = \int \frac{dx}{x} \Rightarrow -\frac{1}{2}\ln|w| - \tan^{-1}(u) = \ln|x| + C$. Recall that $w = 1+u^2$.

$\therefore -\frac{1}{2}\ln|1+u^2| - \tan^{-1}(u) = \ln|x| + C \Rightarrow -\frac{1}{2}\ln\left|1+\left(\frac{y}{x}\right)^2\right| - \tan^{-1}\left(\frac{y}{x}\right) = \ln|x| + C$

$\therefore -\frac{1}{2}\ln\left|\frac{x^2+y^2}{x^2}\right| - \tan^{-1}\left(\frac{y}{x}\right) = \ln|x| + C \Rightarrow \ln\left|\frac{x^2+y^2}{x^2}\right| + 2\tan^{-1}\left(\frac{y}{x}\right) = -2\ln|x| + \cancel{D}^{\,-2C}$

$\therefore \ln|x^2+y^2| \cancel{-\ln|x^2|} + 2\tan^{-1}\left(\frac{y}{x}\right) = \cancel{-\ln|x^2|} + D \Rightarrow \boxed{\ln|x^2+y^2| + 2\tan^{-1}\left(\frac{y}{x}\right) = D,\ D = -2C}$

b) $-y\,dx + (x+\sqrt{xy})\,dy = 0 \Rightarrow (x+\sqrt{xy})\,dy = y\,dx$

Let $y=ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx}x + u$. ⊛

$\therefore (x+\sqrt{xy})\,dy = y\,dx \Rightarrow \boxed{\frac{dy}{dx} = \frac{y}{(x+\sqrt{xy})}} \Rightarrow \frac{du}{dx}x + u = \frac{ux}{x+\sqrt{x(ux)}}$

$\therefore \frac{du}{dx}x + u = \frac{u\cancel{x}^{1}}{\cancel{x}_{1}(1+\sqrt{u})} \Rightarrow \frac{du}{dx}x = \frac{u}{1+\sqrt{u}} - \frac{u(1+\sqrt{u})}{1+\sqrt{u}} = \frac{\cancel{u}-\cancel{u}-u\sqrt{u}}{1+\sqrt{u}}$

$\therefore \frac{du}{dx}x = \frac{-u\sqrt{u}}{1+\sqrt{u}} \Rightarrow \frac{1+\sqrt{u}}{-u\sqrt{u}} = \frac{dx}{x} \Rightarrow \frac{1+u^{1/2}}{-(u^{3/2})}du = \frac{dx}{x}$

$\therefore -u^{-3/2} - u^{-1} = \frac{dx}{x} \Rightarrow -\int u^{-3/2}du - \int \frac{1}{u}du = \int \frac{dx}{x}$

$\therefore -\frac{u^{-\frac{3}{2}+\frac{2}{2}}}{-\frac{3}{2}+\frac{2}{2}} - \ln|u| = \ln|x| + C \Rightarrow 2u^{-1/2} - \ln|u| = \ln|x| + C$. Recall $y=ux$.

$\therefore \frac{2}{\sqrt{y/x}} - \ln|y/x| = \ln|x| + C \Rightarrow \frac{2\sqrt{x}}{\sqrt{y}} - \ln|y| + \cancel{\ln|x|} = \cancel{\ln|x|} + C$; $y>0$

$\therefore \left[\frac{2\sqrt{x}}{\sqrt{y}}\right]^2 = [C + \ln(y)]^2 \Rightarrow \frac{4x}{y} = [\ln(y)+C]^2 \Rightarrow 4x = y[\ln(y)+C]^2$

⊛ NOTE: $\frac{dy}{dx} = 0$ when $y=0$, so this ODE has a constant solution $y=0$!

$\therefore$ $\boxed{\text{Final Answer: } y=0;\ 4x = y[\ln(y)+C]^2}$

c) $x \frac{dy}{dx} - y = \sqrt{x^2+y^2}$; $x < 0$

So, $x \frac{dy}{dx} - y = \sqrt{x^2+y^2} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2+y^2}+y}{x}$

Constant Solutions?

If this ODE has constant solution, then it must be true that there exists at lea 1 "y" such that $\frac{dy}{dx} = 0$. In this case, this must mean that $\sqrt{x^2+y^2}+y = 0 \Rightarrow \sqrt{x^2+y^2} = -y \Rightarrow y \leq 0$ since $\sqrt{\square} \geq 0$ always! Therefore, $\sqrt{x^2+y^2} = -y \Rightarrow x^2+y^2 = y^2$ (if we square both sides of this equation) $\Rightarrow x^2 = 0 \Rightarrow x = 0$!

So, $x=0$ and $y<0$. But if $x=0$, then $\frac{dy}{dx}$ = DNE (because of 0 in denominator when $x=0$). Thus, there are no constant solutions for this ODE!

$\boxed{\text{Const. Solutions: NONE}}$

Non-constant solutions? : Since we know our ODE is homogeneous, let $u = y/x \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx}x + u$ and substitute these expressions into $\frac{dy}{dx} = \frac{\sqrt{x^2+y^2}+y}{x}$.

$$\therefore \frac{dy}{dx} = \frac{\sqrt{x^2+y^2}+y}{x} \Rightarrow \frac{du}{dx}x + u = \frac{\sqrt{x^2+(ux)^2}+(ux)}{x}$$

<u>Ex</u>: (c) - cont'd - 5

$$\therefore \frac{du}{dx}x + u = \frac{\sqrt{x^2+u^2x^2} + ux}{x} = \frac{\overset{|x|}{\cancel{\sqrt{x^2}}}\sqrt{1+u^2} + ux}{x} = \frac{-x\sqrt{1+u^2}+ux}{x}$$

since $x < 0$.

$$\therefore \frac{du}{dx}x + u = \frac{\overset{1}{\cancel{x}}\left[-\sqrt{1+u^2} + u\right]}{\underset{1}{\cancel{x}}} = u - \sqrt{1+u^2}$$

$$\Rightarrow \frac{du}{dx}x = -u + u - \sqrt{1+u^2} = -\sqrt{1+u^2}$$

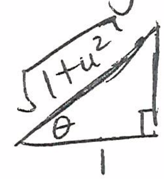
$$\therefore \frac{du}{dx} = -\sqrt{1+u^2} \Rightarrow \frac{du}{\sqrt{1+u^2}} = \frac{-dx}{x} \Rightarrow \boxed{\int \frac{du}{\sqrt{1+u^2}}} = \int -\frac{dx}{x}$$

will need to do trigonometric substitution on this one since don't know anti-derivative straight-up and u-sub does work!

$\therefore$ For $\int \frac{du}{\sqrt{1+u^2}}$, note that $\sin^2(\theta) + \cos^2(\theta) = 1 \Rightarrow \tan^2(\theta) + 1 = \sec^2(\theta)$

Since $1+u^2 = 1+\tan^2(\theta)$ if $u = \tan(\theta)$, let $u = \tan(\theta)$. Then, $\frac{du}{d\theta} = \sec^2(\theta) \Rightarrow du = \sec^2(\theta)\,d\theta$. Also, $\sqrt{1+u^2} = \sqrt{1+\tan^2(\theta)} = \overset{|\sec(\theta)|}{\cancel{\sqrt{\sec^2(\theta)}}}$

$$\therefore \int \frac{du}{\sqrt{1+u^2}} = \int \frac{\sec^2(\theta)\,d\theta}{|\sec(\theta)|} = \int \frac{\overset{1}{\cancel{|\sec(\theta)|}} \cdot |\sec(\theta)|}{\underset{1}{\cancel{|\sec(\theta)|}}}\,d\theta = \int |\sec(\theta)|\,d\theta$$

Recall that we let $u = \tan(\theta) \Rightarrow \tan(\theta) = \frac{u}{1} \Rightarrow$ 

$\Rightarrow u, 1, \sqrt{1+u^2}$, and $1+u^2 > 0$ (since this expressions represent various side (lengths) of a right triangle. It follows that $\sec(\theta) = \frac{\sqrt{1+u^2}}{u} > 0$, so $|\sec(\theta)| = \sec(\theta) > 0$. So, $\int |\sec(\theta)|\,d\theta = \int \sec(\theta)\,d\theta$

$$= \ln|\sec(\theta) + \tan(\theta)|$$

$$\therefore \int \frac{du}{\sqrt{1+u^2}} = \int \sec(\theta)\,d\theta = \ln\left|\sec(\theta)+\tan(\theta)\right| + D$$

But $\sec(\theta) = \frac{\sqrt{1+u^2}}{u}$ and $\tan(\theta) = \frac{u}{1} = u$. So, it follows that

$$\ln\left|\sec(\theta)+\tan(\theta)\right| + D = \ln\left|\frac{\sqrt{1+u^2}}{u} + u\right|$$

$$\therefore \int \frac{du}{\sqrt{1+u^2}} = \int -\frac{dx}{x} \Rightarrow \ln\left|\frac{\sqrt{1+u^2}}{u} + u\right| = -\ln|x| + C \ ; \ C = -D$$

Recall that $u = y/x \Rightarrow y = ux$. Substituting back in $u = y/x$ yields...

$$\ln\left|\frac{\sqrt{1+(y/x)^2}}{(y/x)} + (y/x)\right| = -\ln|x| + C \Rightarrow \ln\left|\sqrt{\frac{x^2+y^2}{x^2}} \cdot \frac{x}{y} + \frac{y}{x}\right| = -\ln|x| + C$$

$$\Rightarrow \ln\left|\frac{\sqrt{x^2+y^2}}{\cancel{\sqrt{x^2}}\ \cancel{|x|}} \cdot \frac{\cancel{x}}{y} + \frac{y}{x}\right| = -\ln|x| + C$$

$$\Rightarrow \ln\left|\frac{\sqrt{x^2+y^2}}{y} + \frac{y}{x}\right| = -\ln|x| + C$$

$$\Rightarrow \ln\left|\frac{\sqrt{x^2+y^2} + y^2}{yx}\right| = -\ln|x| + C$$

$$\Rightarrow \ln\left|\sqrt{x^2+y^2} + y^2\right| - \ln|y| \cancel{- \ln|x|} = \cancel{-\ln|x|} + C$$

$$\ln\left|\sqrt{x^2+y^2} + y^2\right| = \ln|y| + C$$

$$\Rightarrow e^{\ln\left|\sqrt{x^2+y^2} + y^2\right|} = e^{\ln|y| + C}$$

$$\Rightarrow \left|\sqrt{x^2+y^2} + y^2\right| = |y| \cdot A$$

where $A = e^C$

$$\Rightarrow \boxed{\sqrt{x^2+y^2} + y^2 = Ay}$$

Final Answer

Ex: cont'd _ 7

$$(x+y)\,dx + x\cdot dy = 0 \Rightarrow (x+y)\,dx = -x\cdot dy$$
$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{-x}$$

Let $u = y/x \Rightarrow y = ux \Rightarrow \frac{dy}{dx} = \frac{du}{dx}x + u$

$$\therefore \frac{dy}{dx} = \frac{x+y}{-x} \Rightarrow \frac{du}{dx}x + u = \frac{x+ux}{-x} = -1-u$$

$$\therefore \frac{du}{dx}x + u = -1-u \Rightarrow \frac{du}{dx}x = -1-2u \Rightarrow \frac{-du}{1+2u} = \frac{dx}{x}$$

Let $w = 1+2u \Rightarrow dw = 2\,du \Rightarrow \frac{1}{2}dw = du$. So, $\frac{-du}{1+2u} = \frac{-\frac{1}{2}dw}{w}$

$$\therefore \frac{-du}{1+2u} = \frac{dx}{x} \Rightarrow \int \frac{-du}{1+2u} = \int \frac{dx}{x} \Rightarrow \int \frac{-\frac{1}{2}dw}{w} = \int \frac{dx}{x}$$

$$\therefore -\frac{1}{2}\ln|w| = \ln|x| + c \Rightarrow \ln|w| = -2\ln|x| - 2c \Rightarrow \ln|w| = -\ln|x^2| + D$$

where $D = -2c$

$\therefore \ln|w| = -\ln|x^2| + D \Rightarrow \ln|w| + \ln|x^2| = D$. Recall that $w = 1+2u$ and that $y = ux \Rightarrow u = y/x$.

$$\therefore \ln|1+2u| + \ln|x^2| = D \Rightarrow \ln\left|1+2\left(\frac{y}{x}\right)\right| + \ln|x^2| = D \Rightarrow \ln\left|\frac{x+2y}{x}\right| + \ln|x^2| = D$$

$$\therefore \ln|x+2y| - \ln|x| + 2\ln|x| = D \Rightarrow \ln|x+2y| + \ln|x| = \ln|x^2+2xy| = D$$

$$\therefore e^{\ln|x^2+2xy|} = e^{D} \Rightarrow A = |x^2+2xy| \Rightarrow x^2+2xy = A \text{ or } x^2+2xy = -A$$

$$\therefore y = \frac{A-x^2}{2x} \text{ or } y = \frac{-A-x^2}{2x} \Longrightarrow \boxed{y = \frac{\pm A - x^2}{2x} = \frac{\pm A}{2x} - \frac{1}{2}x}$$

Bernoulli Equations and Radical / Fractional Exponent substitution

Recall that a (1^{st}-order) Linear ODE has the form ...

$$\frac{dy}{dx} + p(x)y = r(x) \quad \text{, where } p(x) \text{ and } r(x) \text{ are known.}$$

But what if we had an equation that had an addition factor y^n multiplied to $r(x)$. How would we solve that? Probably the 1^{st} thing you may have thought of is how you could possible convert this ODE into one that actually is linear (again) since its form looks so close to a (1^{st}-order) Linear ODE, right? Well, it turns out that if we make a good choice for a substitution for y, what we desire can actually be achieved.

It turns out that because Bernoulli Equations (i.e. ODEs of the type... $\boxed{\frac{dy}{dx} + p(x)y = r(x)\cdot y^n \;,\; \begin{matrix} n\in\mathbb{R} \\ n>1 \end{matrix}}$) have the "y^n" factor that is the issue with it not being truly (1^{st}-order) Linear, a good substitution to counteract this issue is to sub. $u = y^{1-n} \Rightarrow \sqrt[1-n]{u} = y \Rightarrow u^{\frac{1}{1-n}} = y$, where $n \in \mathbb{R}$ is the same "n" from the y^n factor multiplied by $r(x)$ earlier.

Once we know what "n" is and use it to form $u = y^{1-n} \Rightarrow u^{\frac{1}{1-n}} = y$ we find the derivative of y with respect to u (i.e. find $\frac{dy}{dx}$) and substitute both "y" and "$\frac{dy}{du}$" into the given (Bernoulli) equation.

By doing these substitutions, the Bernoulli equation you are working to solve will relegate back to a linear (1st-order) ODE where you can use the formula...

$$\boxed{u = e^{-h}\left[\int e^{h} \cdot r \cdot dx + C\right] \; ; \; h = \int p \cdot dx = \text{integrating factor}}$$

Lastly, to recover our (function) y, we recall that $y = u^{\frac{1}{1-n}}$.

NOW, LET'S DO SOME PRACTICE PROBLEMS !!!

SUMMARY

0. Note that for $n > 0$, $y(x) = 0$ for all x, is a constant solution
1. For ODE $\frac{dy}{dx} + p(x)y = r(x)y^n$, identify value on "n".
2. Let $u = y^{1-n} \Rightarrow u^{\frac{1}{1-n}} = y$. Find $\frac{dy}{dx}$ as well.
3. Substitute "y" + "$\frac{dy}{dx}$" back into (Bernoulli) ODE.
4. Since resulting equation will now be (1st-order) Linear, use corresponding formula: $u = e^{-h}\left[\int e^{h} \cdot r \cdot dx\right]$; $h = \int p \cdot dx$; $p = p(x)$ and $r = r(x)$.
5. Recover "y" by noting that $y = u^{\frac{1}{1-n}}$.

Ex: Solve the following Bernoulli equations given the steps listed in the "Summary" on the previous page.

a) $y' + xy = xy^2$

$y' + xy = xy^2 \Rightarrow \frac{dy}{dx} + \overline{p(x)}\,y = \overline{r(x)}\,y^n$, where $\bar{p} = \overline{p(x)} = x$, $\bar{r} = \overline{r(x)} = x$, and $n = 2$. Therefore, let $u = y^{1-n} = y^{1-2} = y^{-1} \Rightarrow u = \frac{1}{y} \Rightarrow y = \frac{1}{u}$

$\therefore y = \frac{1}{u} = u^{-1} \Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[u^{-1}\right] = -u^{-2} \cdot \frac{du}{dx} = \frac{-1}{u^2} \cdot \frac{du}{dx}$

$u \neq 0 \Rightarrow y \neq 0$

$\therefore y' + xy = xy^2 \Rightarrow \frac{-1}{u^2} \cdot \frac{du}{dx} + x\frac{1}{u} = x\left[\frac{1}{u}\right]^2 \Rightarrow \frac{-1}{u^2}\frac{du}{dx} + \frac{x}{u} = \frac{x}{u^2}$

$\therefore u^2\left[\frac{-1}{u^2}\frac{du}{dx} + \frac{x}{u}\right] = \left[\frac{x}{u^2}\right]u^2 \Rightarrow -\frac{du}{dx} + \underbrace{-x}_{p = p(x)}u = \underbrace{-x}_{r = r(x)} \Rightarrow$ Linear ODE

$\therefore u = e^{-h}\left[\int e^h \cdot r \cdot dx + C\right]$ where $h = \int p \cdot dx$

So, $h = \int p \cdot dx = \int -x \cdot dx = -\frac{x^2}{2} \Rightarrow u = e^{\frac{1}{2}x^2}\left[\int \left(e^{-\frac{1}{2}x^2} \cdot -x\right) dx + C\right]$,

$\therefore$ Let $z = -\frac{1}{2}x^2 \Rightarrow dz = -x \cdot dx$

$\therefore u = e^{x^2/2}\left[\int e^z \cdot dz + C\right] = e^{x^2/2}\left[e^z + C\right] = e^{x^2/2}\left[e^{-\frac{1}{2}x^2} + C\right]$

$\therefore u = 1 + Ce^{x^2/2}$. Therefore, recall that $y = \frac{1}{u} \Rightarrow y = \frac{1}{1 + Ce^{x^2/2}}$

NOTE: Since $n = 2 > 0 \Rightarrow y(x) = 0$ is a constant sol'n $\forall\, x \in \mathbb{R}$ $\Rightarrow$ $\boxed{y(x) = \frac{1}{1 + Ce^{x^2/2}} \text{ and } y(x) = 0 \text{ are solutions}}$

Ex : cont'd - 1

Since $n = \frac{1}{3} > 0 \Rightarrow$ Const. sol'n $y(x) = 0$

b) $y' - \frac{3}{x} y = x^4 y^{\frac{1}{3}}$; $n = \frac{1}{3}$ $\Rightarrow$ let $u = y^{1-n} = y^{1-\frac{1}{3}} = y^{\frac{2}{3}}$

$\therefore u = y^{\frac{2}{3}} \Rightarrow \boxed{u^{\frac{3}{2}} = y} \Rightarrow \sqrt{u^3} = y \Rightarrow u^3 \geq 0 \Rightarrow u, y \geq 0$

$\therefore u^{\frac{3}{2}} = y \Rightarrow \frac{dy}{dx} = \frac{3}{2} u^{\frac{1}{2}} \cdot \frac{du}{dx}$ and $u = y^{\frac{2}{3}} \Rightarrow u^{\frac{1}{2}} = \sqrt{u} = (y^{\frac{2}{3}})^{\frac{1}{2}} = y^{\frac{1}{3}}$

$\Rightarrow \boxed{u^{\frac{1}{2}} = y^{\frac{1}{3}}}$

$\therefore y' - \frac{3}{x} y = x^4 y^{\frac{1}{3}} \Rightarrow \frac{3}{2} u^{\frac{1}{2}} \cdot \frac{du}{dx} - \frac{3}{x} u^{\frac{3}{2}} = x^4 u^{\frac{1}{2}}$

$\therefore \frac{du}{dx} - \frac{3}{x} \cdot \frac{2}{3u^{\frac{1}{2}}} \cdot u^{\frac{3}{2}} = x^4 \cdot u^{\frac{1}{2}} \cdot \frac{2}{3u^{\frac{1}{2}}}$

$\Rightarrow \frac{du}{dx} \underbrace{- \frac{2}{x}}_{p} u = \underbrace{\frac{2x^4}{3}}_{r}$ $\Rightarrow$ Linear (1st-order) ODE

$\therefore u = e^{-h} \left[\int e^{h} \cdot r \cdot dx + C \right]$, where $h = \int p \cdot dx = -\int \frac{2}{x} dx = -2 \ln|x|$

$= \ln\left|\frac{1}{x^2}\right|$

$\therefore h = -\ln|x^2|$

$\therefore u = e^{\ln|x^2|} \left[\int e^{-\ln|x^2|} \cdot \frac{2}{3} x^4 \cdot dx + C \right] = \cancel{|x^2|}\, x^2 \left[\int x^{-2} \cdot \frac{2}{3} \cdot x^4 \cdot dx + C \right]$

$\Rightarrow u = x^2 \left[\frac{2}{3} \int x^2 \cdot dx + C \right] = x^2 \left[\frac{2}{3} \cdot \frac{x^3}{3} + C \right] = \frac{2}{9} x^5 + Cx^2$

$\therefore u = \frac{2}{9} x^5 + Cx^2 \Rightarrow y^{\frac{2}{3}} = \frac{2}{9} x^5 + Cx^2 \Rightarrow y = \left[\frac{2}{9} x^5 + Cx^2 \right]^{\frac{3}{2}}$

Final answers : $\boxed{y(x) = 0 \text{ (for all } x) \; ; \; y = \left[\frac{2}{9} x^5 + Cx^2 \right]^{\frac{3}{2}}}$

c) $y' + \frac{2}{x} y = -x^9 y^5$; $y(-1) = 2$

Note: $n = 5 \Rightarrow$ ODE is a Bernoulli equation $\Rightarrow y(x) = 0$ for all $x \in \mathbb{R}$ is a constant solution since $n = 5 > 0$!

Let $u = y^{1-n} = y^{1-5} = y^{-4} \Rightarrow u = y^{-4} \Rightarrow \boxed{u^{-1/4} = (y^{-4})^{-1/4} = y \;;\; y \geq 0}$

$\therefore u^{-1/4} = y \Rightarrow \boxed{\frac{dy}{dx} = -\frac{1}{4} u^{-5/4} \cdot \frac{du}{dx} = y'}$

$\therefore y' + \frac{2}{x} y = -x^9 y^5 \Rightarrow -\frac{1}{4} u^{-5/4} \cdot \frac{du}{dx} + \frac{2}{x} \cdot u^{-1/4} = -x^9 (u^{-1/4})^5$

$\therefore -4u^{5/4}\left[-\frac{1}{4} u^{-5/4} \cdot \frac{du}{dx} + \frac{2}{x} \cdot u^{-1/4}\right] = \left[-x^9 u^{-5/4}\right] \cdot \left(-4u^{5/4}\right)$

$\Rightarrow \frac{du}{dx} + \underbrace{\left(\frac{-8}{x}\right)}_{p} u = \underbrace{4x^9}_{r}$ (which is a linear (1st-order) ODE w/ $p = \frac{-8}{x}$ + $r = 4x^9$)

$\therefore u = e^{-h}\left[\int e^{h} \cdot r \cdot dx + C\right]$; $h = \int p \cdot dx = \int \frac{-8}{x} dx = -8 \ln|x| = \ln|x^{-8}|$

$\therefore u = e^{-\ln|x^{-8}|}\left[\int e^{\ln|x^{-8}|} \cdot 4x^9 \cdot dx + C\right] = \cancel{|x^8|}\left[\int \cancel{\left|\frac{1}{x^8}\right|} \cdot 4x^9 \cdot dx + C\right] =$

$x^8\left[\int 4x \cdot dx + C\right] = x^8\left[\frac{4x^2}{2} + C\right] = 2x^{10} + Cx^8$. Recall that $u = \frac{1}{y^4}$.

$\therefore \frac{1}{y^4} = 2x^{10} + Cx^8 \Rightarrow \boxed{\sqrt[4]{\frac{1}{2x^{10} + Cx^8}} = y = y(x)}$

Now we will apply the initial condition $y(-1) = 2$!

Ex :

c) cont'd_2

$$\therefore\ y(-1)=2 \Rightarrow \sqrt[4]{\frac{1}{2(-1)^{10}+C(-1)^{8}}}=2 \Rightarrow \sqrt[4]{\frac{1}{2+C}}=2$$

$$\Rightarrow \frac{1}{2+C}=2^4 \Rightarrow \frac{1}{2^4}=2+C \Rightarrow \frac{1}{16}-\cancel{2}^{\,32/16}=C \Rightarrow \boxed{C=\frac{-31}{16}}$$

$\therefore$ Final Answer(s): $y(x)=0\ (x\in\mathbb{R})$; $y(x)=\sqrt[4]{\dfrac{1}{2x^{10}-\frac{31}{16}x^8}}$

NOTE: $y(x)=\sqrt[4]{\dfrac{1}{2x^{10}-\frac{31}{16}x^8}}=\dfrac{1}{\sqrt[4]{x^8\left(2x^2-\frac{31}{16}\right)}}=\dfrac{1}{x^2\sqrt[4]{2x^2-\frac{31}{16}}}$

Alternative Form for $y(x)$

d) $y' + xy = 6x\sqrt{y}$

$y' + xy = 6xy^{1/2} \Rightarrow n = 1/2$ (Bernoulli) w/ $\boxed{y(x) = 0 \text{ const. sol'n for all } x \text{ !!}}$

$\therefore$ Let $u = y^{1-n} = y^{1-1/2} = y^{1/2}$. Then $u^2 = y$ and $\frac{dy}{dx} = 2u \cdot \frac{du}{dx}$

$\therefore y' + xy = 6xy^{1/2} \Rightarrow 2u \cdot \frac{du}{dx} + x[u^2] = 6x[u]$

$\therefore \frac{du}{dx} + \left(\frac{1}{2}x\right)u = (3x) \Rightarrow$ Linear ODE

($p = \frac{1}{2}x$, $r = 3x$)

$\therefore u = e^{-h}\left[\int e^{h} \cdot r \cdot dx + C\right]$; $h = \int p \cdot dx = \int \frac{1}{2}x\,dx = \frac{1}{2}\frac{x^2}{2} = \frac{x^2}{4}$

$\therefore u = e^{-x^2/4}\left[\int \left(e^{x^2/4} \cdot 3x\right) dx + C\right]$. Let $w = \frac{1}{4}x^2$. Then, $dw = \frac{1}{2}x \cdot dx \Rightarrow 2\,dw = x \cdot dx \Rightarrow 6\,dw = 3x \cdot dx$

$\therefore u = e^{-x^2/4}\left[\int e^{w} \cdot 6 \cdot dw + C\right] = e^{-x^2/4}\left[6e^{w} + C\right] = e^{-x^2/4}\left[6e^{\frac{1}{4}x^2} + C\right]$

$\therefore u = 6 + Ce^{-x^2/4} \Rightarrow \boxed{u^2 = y = \left(6 + Ce^{-x^2/4}\right)^2}$

$\therefore$ Final answer(s): $y(x) = 0$ for all $x \in \mathbb{R}$; $y(x) = \left(6 + Ce^{-x^2/4}\right)^2$