SIGNALS AND SYSTEMS USING MATLAB
Chapter 10 — The Z–transform

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Laplace Transform of Sampled Signals

$$x(t) = \sum_n x(nT_s)\delta(t - nT_s) \quad \text{(sampled signal)}$$

$$X(s) = \sum_n x(nT_s)\mathcal{L}[\delta(t - nT_s)] = \sum_n x(nT_s)e^{-nsT_s}$$

Letting $z = e^{sT_s}$

$$\mathcal{Z}[x(nT_s)] = \mathcal{L}[x_s(t)]|_{z=e^{sT_s}} = \sum_n x(nT_s)z^{-n} \quad \text{Z–transform}$$

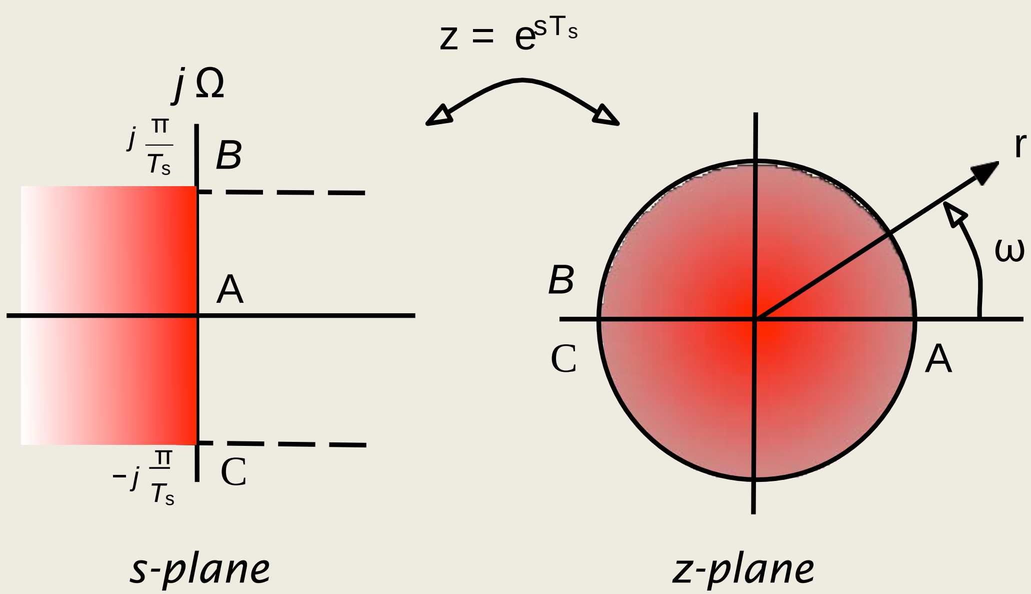


Figure: Mapping of the Laplace plane into the Z–plane

Two-sided/ One-sided Z-transforms

- Two-sided Z-transform

$$\boxed{\begin{array}{l}\text{discrete-time signal } \; x[n], -\infty < n < \infty \\ \qquad X(z) = \displaystyle\sum_{n=-\infty}^{\infty} x[n] z^{-n}, \quad ROC: \; \mathcal{R}\end{array}}$$

- One–sided Z–transform

$$\boxed{\begin{array}{l}\text{causal signal } \; x[n]u[n] \\ X_1(z) = \mathcal{Z}(x[n]u[n]) = \displaystyle\sum_{n=0}^{\infty} x[n]u[n] z^{-n}, \quad ROC: \; \mathcal{R}_1\end{array}}$$

- Two–sided in terms of one–sided Z–transform

$$x[n] = x[n]u[n] + x[n]u[-n] - x[0]$$

$$X(z) = \mathcal{Z}\left(x[n]u[n]\right) + \mathcal{Z}\left(x[-n]u[n]\right)|_{z} - x[0], \quad \mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$$

$$\mathcal{R}_1 = ROC[\mathcal{Z}\left(x[n]u[n]\right)], \quad \mathcal{R}_2 = ROC[\mathcal{Z}\left(x[-n]u[n]\right)|_{z}]$$

Poles/Zeros, ROC

- Z-transform $X(z)$
  - pole p_k such that $X(p_k) \to \infty$
  - zero z_k such that $X(z_k) = 0$
- ROC of finite-support signal

$$x[n], \quad \text{finite support} \; -\infty < N_0 \le n \le N_1 < \infty$$

$$X(z) = \sum_{n=N_0}^{N_1} x[n] z^{-n}$$

$ROC:$ whole Z–plane, excluding 0 and/or $\pm\infty$ depending on $N_0, \; N_1$

Examples:

$$(i) \quad X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} = \frac{z^3 + 2z^2 + 3z + 4}{z^3} = \frac{N_1(z)}{D_1(z)}$$

zeros: roots of $N_1(z) = 0, \; z_1 = -1.65, \; z_2 = -0.175 \pm j1.547$

poles: roots of $D_1(z) = 0 \quad z = 0$ triple

$$(ii) \quad X_2(z) = \frac{(z^{-1} - 1)(z^{-1} + 2)^2}{z^{-1}(z^{-2} + \sqrt{2}z^{-1} + 1)} = \frac{(1 - z)(1 + 2z)^2}{1 + \sqrt{2}z + z^2} = \frac{N_2(z)}{D_2(z)}$$

zeros: roots of $N_2(z) = 0, \quad z_1 = 1, \; z_{2,3} = -0.5$

poles: roots of $D_2(z) = 0, \quad p_{1,2} = -0.707 \pm j0.707$

Example: Discrete-time pulse $x[n] = u[n] - u[n-10]$

$$X(z) = \sum_{n=0}^{9} 1 \; z^{-n} = \frac{1 - z^{-10}}{1 - z^{-1}} = \frac{z^{10} - 1}{z^9(z-1)}$$

zeros: roots of $z^{10} - 1 = 0$, or $z_k = e^{j2\pi k/10}$, $k = 0 \cdots 9$

$z_0 = 1$ cancels pole $p = 1 \quad \Rightarrow X(z) = \dfrac{\prod_{k=1}^{9}(z - e^{j\pi k/5})}{z^9}$,

ROC whole z–plane excluding the origin

$$X(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} + z^{-7} + z^{-8} + z^{-9}$$

only tends to infinity when $z = 0$

ROC of Z-transform of infinite–support signals

- causal signal $x[n]$, ROC: $|z| > R_1$, R_1 the largest radius of poles of $X(z)$
- anti-causal signal $x[n]$, ROC: $|z| < R_2$, R_2 smallest radius of poles of $X(z)$
- non-causal signal $x[n]$, ROC: $R_1 < |z| < R_2$, or inside a torus of inside radius R_1 and outside radius R_2

Example: Possible regions of convergence of $X(z)$ with poles $z = 0.5$ and $z = 2$

- $\{\mathcal{R}_1 : |z| > 2\}$, outside of circle of radius 2, $X(z)$ associated with causal signal $x_1[n]$
- $\{\mathcal{R}_2 : |z| < 0.5\}$, inside of circle of radius 0.5, $X(z)$ associated with anti-causal signal $x_2[n]$
- $\{\mathcal{R}_3 : 0.5 < |z| < 2\}$, torus of radii 0.5 and 2, $X(z)$ associated with non-causal signal $x_3[n]$

Example: Noncausal $c[n] = \alpha^{|n|}$, $0 < \alpha < 1$, (autocorrelation function related to the power spectrum of a random signal)

$$\mathcal{Z}(c[n]u[n]) = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}}, \quad ROC: \ |z| > \alpha$$

$$\mathcal{Z}(c[-n]u[n])_z = \sum_{n=0}^{\infty} \alpha^n z^n = \frac{1}{1-\alpha z}, \quad ROC: \ |z| < 1/\alpha$$

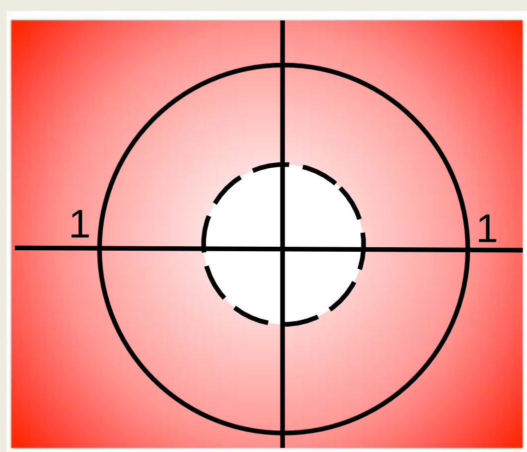
$$C(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\alpha z} - 1 = \frac{(\alpha - 1/\alpha)z}{(z-\alpha)(z-1/\alpha)}$$

$$ROC: \ \alpha < |z| < \frac{1}{\alpha}$$

Example: Causal $x[n] = \alpha^n u[n]$

$$X(z) \quad = \quad \sum_{n=0}^{\infty} \alpha^n z^{-n} = \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha} \qquad \text{ROC: } \ |z| > |\alpha|$$

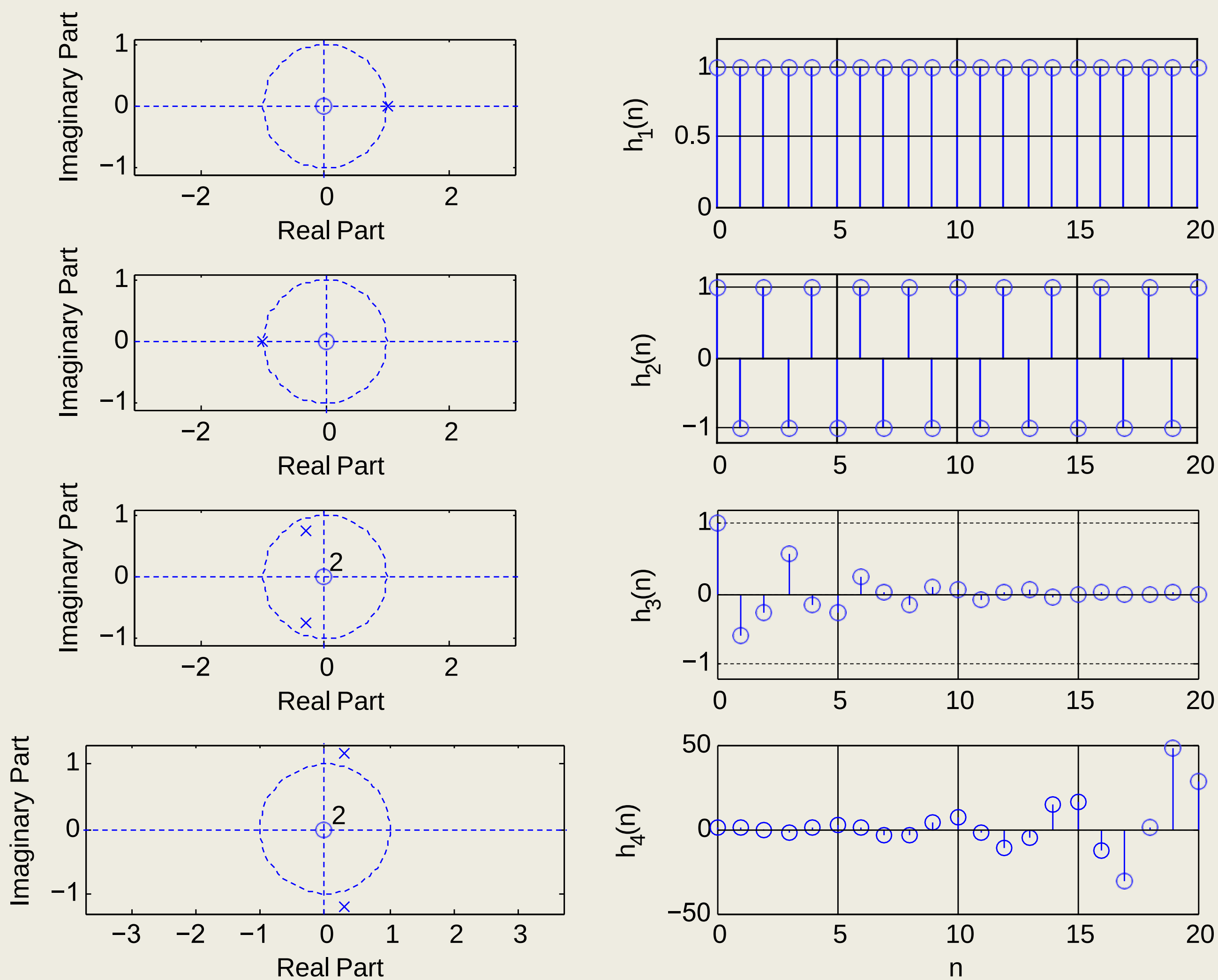
z-plane



Region of convergence (shaded area) of X (z) with a pole at z = α, α < 0

Table 10.1 One-sided Z-transforms

$\delta[n]$	$1, \quad \text{whole z-plane}$
$u[n]$	$\dfrac{1}{1-z^{-1}}, \quad \lvert z\rvert > 1$
$nu[n]$	$\dfrac{z^{-1}}{(1-z^{-1})^2}, \quad \lvert z\rvert > 1$
$n^2u[n]$	$\dfrac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}, \quad \lvert z\rvert > 1$
$\alpha^n u[n],\ \lvert\alpha\rvert < 1$	$\dfrac{1}{1-\alpha z^{-1}}, \quad \lvert z\rvert > \lvert\alpha\rvert$
$n\alpha^n u[n],\ \lvert\alpha\rvert < 1$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}, \quad \lvert z\rvert > \lvert\alpha\rvert$
$\cos(\omega_0 n)u[n]$	$\dfrac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}, \quad \lvert z\rvert > 1$
$\sin(\omega_0 n)u[n]$	$\dfrac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}, \quad \lvert z\rvert > 1$
$\alpha^n\cos(\omega_0 n)u[n],\ \lvert\alpha\rvert < 1$	$\dfrac{1-\alpha\cos(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2 z^{-2}}, \quad \lvert z\rvert > \lvert\alpha\rvert$
$\alpha^n\sin(\omega_0 n)u[n],\ \lvert\alpha\rvert < 1$	$\dfrac{\alpha\sin(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2 z^{-2}}, \quad \lvert z\rvert > \lvert\alpha\rvert$



Effect of pole location on the inverse Z-transform (from top to bottom): if pole is at $z = 1$ the signal is $u(n)$, constant for $n \geq 0$; if pole is at $z = -1$ the signal is a cosine of frequency π continuously changing, constant amplitude; when poles are complex, if inside the unit circle the signal is a decaying modulated exponential, and if outside the unit circle the signal is a growing modulated exponential

Table 10.2 Basic Properties of One-sided Z-transform

Causal signals	$\alpha x[n],\ \beta y[n]$	$\alpha X(z),\ \beta Y(z)$
Linearity	$\alpha x[n] + \beta y[n]$	$\alpha X(z) + \beta Y(z)$
Convolution sum	$\sum_k x[n]y[n-k]$	$X(z)Y(z)$
Time shifting	$x[n-N],\ N > 0$	$z^{-N}X(z) + x[-1]z^{-N+1} + x[-2]z^{-N+2} + \cdots + x[-N]$
Time reversal	$x[-n]$	$X(z^{-1})$
Multiplication	$n\ x[n]$	$-z\dfrac{dX(z)}{dz}$
	$n^2\ x[n]$	$z^2\dfrac{d^2X(z)}{dz^2} + z\dfrac{dX(z)}{dz}$
Finite difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z) - x[-1]$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\dfrac{X(z)}{1 - z^{-1}}$
Initial value	$x[0]$	$\lim_{z\to\infty} X(z)$
Final value	$\lim_{n\to\infty} x[n]$	$\lim_{z\to 1}(z-1)X(z)$

Convolution sum and transfer Function

output of causal LTI system

$$y[n] = [x * h][n] = \sum_{k=0}^{n} x[k]h[n-k] = \sum_{k=0}^{n} h[k]x[n-k]$$

$x[n]$ causal input, $h[n]$ impulse response of system

$$Y(z) = \mathcal{Z}\{[x * h][n]\} = \mathcal{Z}\{x[n]\}\mathcal{Z}\{h[n]\} = X(z)H(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\mathcal{Z}[\text{ output } y[n]]}{\mathcal{Z}[\text{ input } x[n]]} \quad \text{transfer function}$$

- Convolution gives coefficients of multiplication of polynomials
- FIR systems implemented using convolution
- Length of convolution of two sequences of lengths M and N is $M + N - 1$

Example: FIR filter

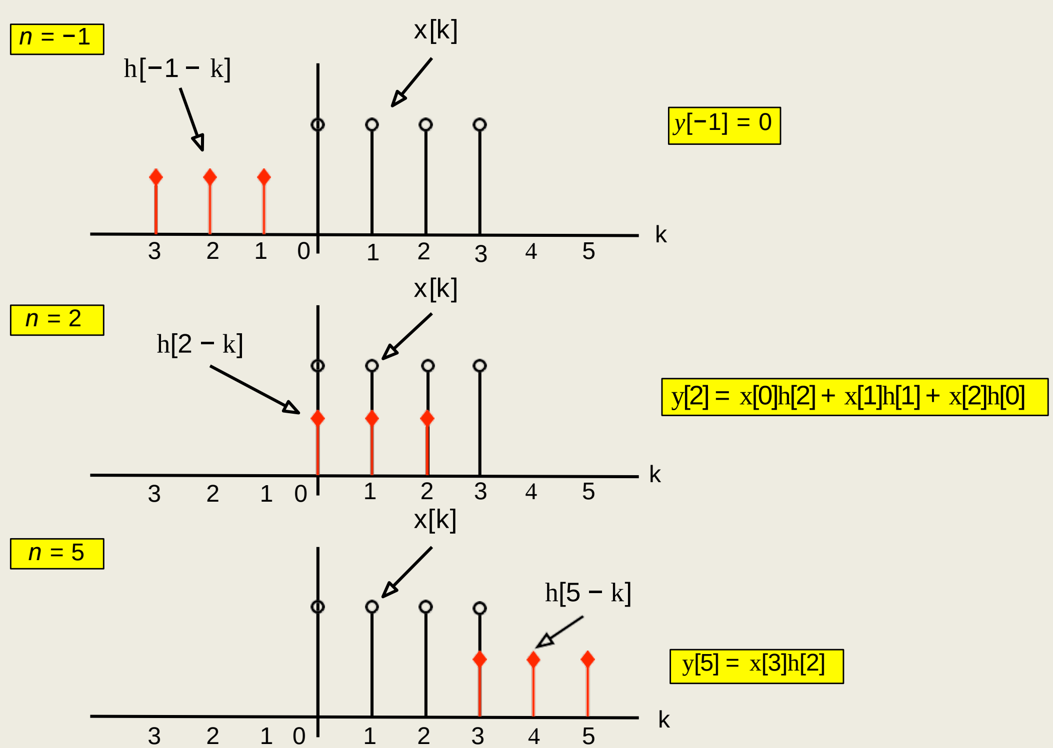
$$y[n] = \frac{1}{2}\left(x[n] + x[n-1] + x[n-2]\right)$$

$$x[n] = u[n] - u[n-4], \quad h[n] = 0.5(\delta[n] + \delta[n-1] + \delta[n-2])$$

$$X(z) = 1 + z^{-1} + z^{-2} + z^{-3}, \quad H(z) = \frac{1}{2}[1 + z^{-1} + z^{-2}]$$

$$Y(z) = X(z)H(z) = \frac{1}{2}(1 + 2z^{-1} + 3z^{-2} + 3z^{-3} + 2z^{-4} + z^{-5})$$

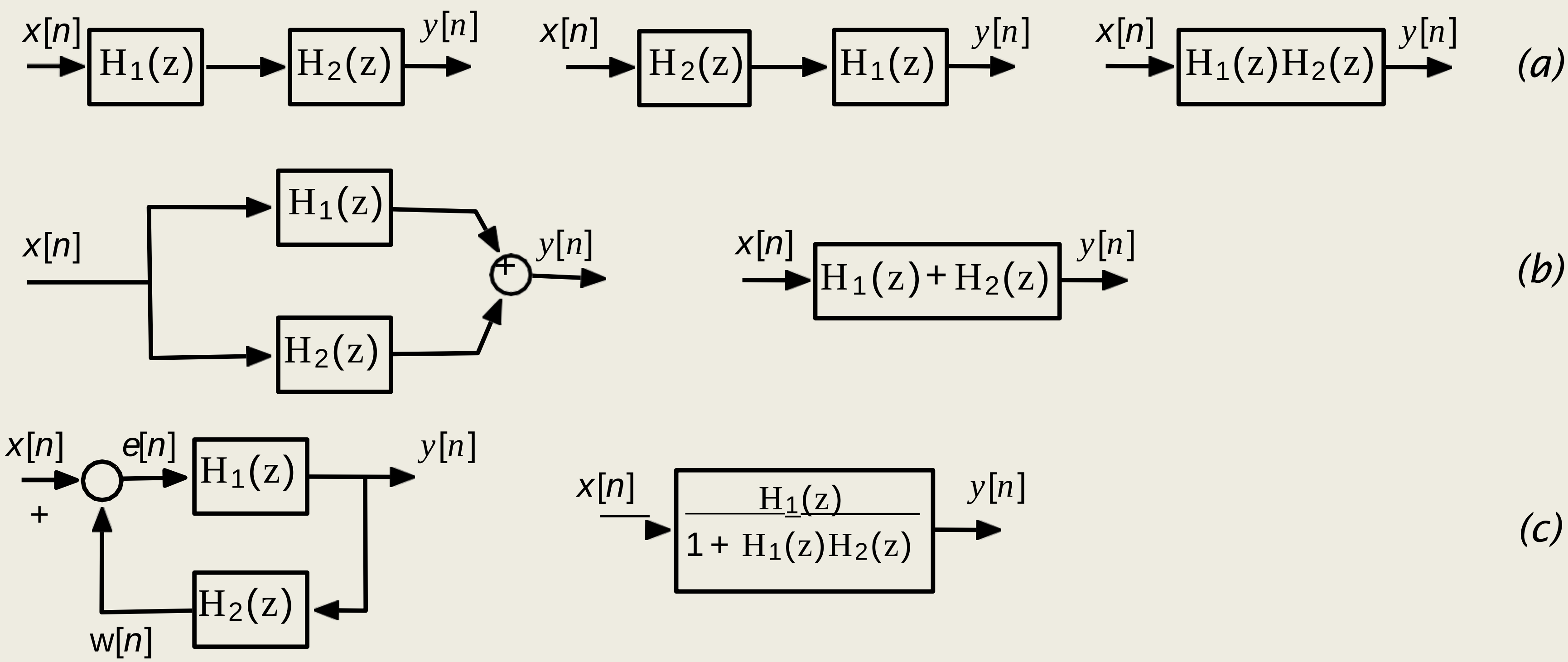
$$y[0] = 0.5,\ y[1] = 1,\ y[2] = 1.5,\ y[3] = 1.5,\ y[4] = 1,\ y[5] = 0.5, \cdots$$



Graphical approach: x [k] and h[n − k] are plotted as functions of k for a given value of n.

The signal x [k] remains stationary, while h[n − k] moves linearly from left to right

Interconnection of discrete-time systems



Connections of LTI systems: (a) cascade, (b) parallel, and (c) negative feedback.

One-sided Z-transform inverse

- Long–division Rational function $X(z) = \mathcal{Z}[x[n]] = B(z)/A(z)$, $x[n]$ causal. By division

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$
$$\text{inverse} \;\; x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

- Partial fraction expansion

$$X(z) = \mathcal{Z}[x[n]] = B(z)/A(z), \quad x[n] \text{ causal}$$

  – proper rational $X(z)$: degree $N(z) <$ degree $D(z)$
  – $N(z)$, $D(z)$ polynomials with real coefficients poles/zeros are
    (i) real
    (ii) complex conjugate pairs
    (iii) simple
    (iv) multiple

Example: Non-proper rational function

$$X(z) = \frac{2 + z^{-2}}{1 + 2z^{-1} + z^{-2}}$$

By division

$$X(z) = 1 + \frac{1 - 2z^{-1}}{1 + 2z^{-1} + z^{-2}} \quad \Rightarrow \quad x[n] = \delta[n] + \mathcal{Z}^{-1}\left[\frac{1 - 2z^{-1}}{1 + 2z^{-1} + z^{-2}}\right]$$

Example:

$$X(z) = \frac{1 + z^{-1}}{(1 + 0.5z^{-1})(1 - 0.5z^{-1})} = \frac{z(z+1)}{(z+0.5)(z-0.5)} \qquad |z| > 0.5$$

Partial fraction expansion in z^{-1} terms

$$\begin{aligned}
X(z) &= \frac{1 + z^{-1}}{(1 + 0.5z^{-1})(1 - 0.5z^{-1})} = \frac{A}{1 + 0.5z^{-1}} + \frac{B}{1 - 0.5z^{-1}} \\
A &= X(z)(1 + 0.5z^{-1})|_{z^{-1}=-2} = -0.5 \\
B &= X(z)(1 - 0.5z^{-1})|_{z^{-1}=2} = 1.5
\end{aligned}$$

Partial fraction expansion in positive powers of z

$$\begin{aligned}
\frac{X(z)}{z} &= \frac{z+1}{(z+0.5)(z-0.5)} = \frac{C}{z+0.5} + \frac{D}{z-0.5} \\
C &= \frac{X(z)}{z}(z+0.5)|_{z=-0.5} = -0.5 \\
D &= \frac{X(z)}{z}(z-0.5)|_{z=0.5} = 1.
\end{aligned}$$

Either gives $x[n] = [-0.5(-0.5)^n + 1.5(0.5)^n]u[n]$

Solution of difference equations

$$\boxed{\begin{array}{l} x[n] \;\leftrightarrow\; X(z) \\ \mathcal{Z}[x[n-N]] = z^{-N}X(z) + x[-1]z^{-N+1} + x[-2]z^{-N+2} + \cdots + x[-N] \end{array}}$$

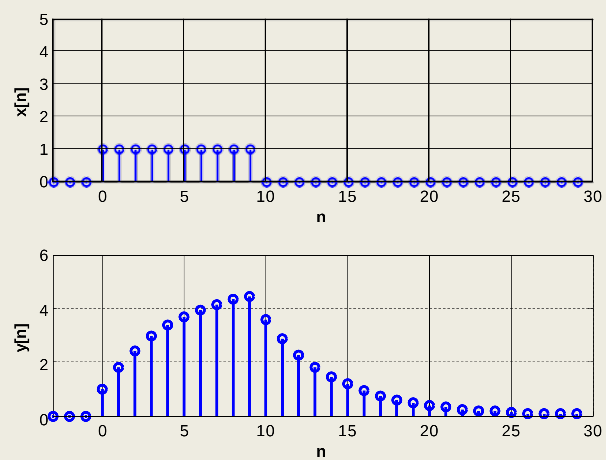
Example: IIR system with input $x[n]$, $y[n]$ output, is represented by

$$y[n] = 0.8y[n-1] + x[n] \qquad n \geq 0, \quad IC: \; y[-1]$$

Closed-form solution

$$\begin{aligned} \mathcal{Z}(y[n]) &= \mathcal{Z}(0.8y[n-1]) + \mathcal{Z}[x[n]) \\ Y(z) &= 0.8(z^{-1}Y(z) + y[-1]) + X(z) \\ Y(z) &= \underbrace{\frac{X(z)}{1-0.8z^{-1}}}_{y_{zs}[n]} + \underbrace{\frac{0.8y[-1]}{1-0.8z^{-1}}}_{y_{zi}[n]} \end{aligned}$$

$y_{zs}[n]$ $y_{zi}[n]$



Solution of difference equation (bottom) with input x [n] = u[n] − u[n − 11], y [−1] = 0

Example: Steady–state response

$$y[n] + y[n-1] - 4y[n-2] - 4y[n-3] = 3x[n], \quad n \geq 0,$$
$$y[-1] = 1,\ y[-2] = y[-3] = 0, \quad x[n] = u[n]$$

$$Y(z) = 3\frac{X(z)}{A(z)} + \frac{-1 + 4z^{-1} + 4z^{-2}}{A(z)}, \quad |z| > 2,\ A(z) = (1 + z^{-1})(1 + 2z^{-1})(1 - 2z^{-1})$$

BIBO stability: transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{3}{A(z)},\ \text{poles } z = -1,\ z = -2,\ z = 2 \text{ (on and outside UC)}$$

$h[n] = \mathcal{Z}^{-1}[H(z)]$ not absolutely summable, so system is not BIBO stable

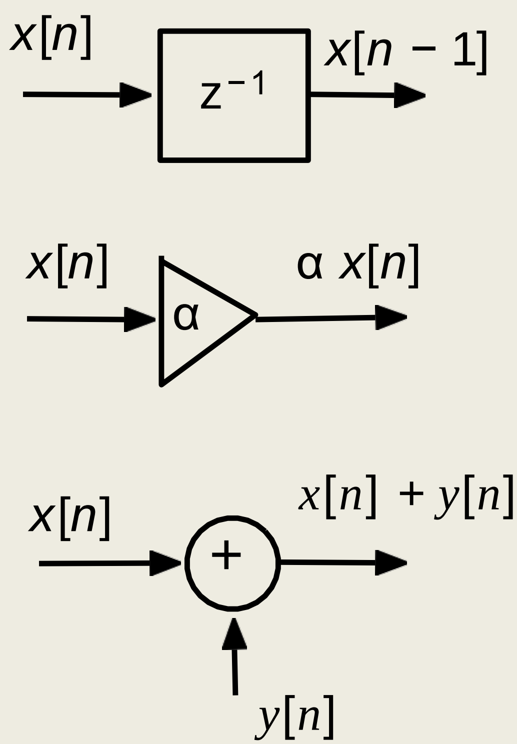
$$Y(z) = \frac{2 + 5z^{-1} - 4z^{-3}}{(1 - z^{-1})(1 + z^{-1})(1 + 2z^{-1})(1 - 2z^{-1})}$$
$$= \frac{B_1}{1 - z^{-1}} + \frac{B_2}{1 + z^{-1}} + \frac{B_3}{1 + 2z^{-1}} + \frac{B_4}{1 - 2z^{-1}}$$

$$B_1 = Y(z)(1 - z^{-1})|_{z^{-1}=1} = -\tfrac{1}{2}, \qquad B_2 = Y(z)(1 + z^{-1})|_{z^{-1}=-1} = -\tfrac{1}{6},$$
$$B_3 = Y(z)(1 + 2z^{-1})|_{z^{-1}=-1/2} = 0, \quad B_4 = Y(z)(1 - 2z^{-1})|_{z^{-1}=1/2} = \tfrac{8}{3},$$

$$y[n] = \left(-0.5 - \frac{1}{6}(-1)^n + \frac{8}{3}2^n\right) u[n] \ \to \infty \text{ as } n \to \infty, \quad \text{no steady-state}$$

State variable representation

- Used in modern control theory
- State variables are memory of a system
- State variable representation is non–unique internal representation



Different components used to represent discrete–time systems (top to bottom): delay, constant multiplier and adder.