Using Indefinite and Definite Integrals in Solving Differential Equations

- Definition: Directly-Integrable Equations (1^{st}-order ODE)
- Solving Directly-Integrable Equations using Indefinite Integrals
- " " " " " Definite Integrals
- " " " " Containing Piecewise Functions

Definition (Directly Integrable 1^{st}-order ODE): A 1^{st}-order ODE is considered to be directly integrable iff the ODE can be (re)written in the form...

$$\frac{dy}{dx} = f(x)$$

, where $f(x)$ is interpreted as some function of "x" values (and no other variables).

NOTE: This includes variables like $y', y'', \ldots, y^N$ as these can't be variables either!

It follows that if an equation can be written in the form...

$$\frac{d^N y}{dx^N} = f(x)$$

, then the N^{th}-order ODE is considered to be directly integrable.

NOTE! Knowing that an ODE/PDE is directly integrable means that you can use indefinite and definite integration techniques from calculus to find solutions to these equations!!

Ex: Determine which of the following DEs are directly integrable. Circle "y" or "N".

a) $\frac{dy}{dx} = -4 + \tan(x) = f(x)$

y or N

b) $\frac{dy}{dx} = 14 - \sec(y) = f(y) \neq f(x)$

y or N

c) $y\frac{dy}{dx} = \frac{3}{x} \Rightarrow \frac{dy}{dx} = \frac{3}{xy} = f(x,y) \neq f(x)$

y or N

d) $x\frac{dy}{dx} = \operatorname{arccot}(x^2)$

$\frac{dy}{dx} = \frac{\operatorname{arccot}(x^2)}{x} = f(x)$

y or N

e) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 8y = e^{-x^2}$

$\therefore y'' = e^{-x^2} - 8y - 3y' = f(x, y, y') \neq f(x)$

y or N

f) $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} = 0$

$\therefore y'' = \frac{-3xy'}{x^2} = \frac{-3y'}{x} = f(x, y') \neq f(x)$

y or N

Solving Directly-Integrable Equations using Indefinite Integrals

We will solve directly-integrable equations similar to the way you learned how to solve IVP problem in calculus when you 1st learned about antiderivatives (and, eventually, basic integration). Note that when using indefinite integrals, we always end up with a family of functions for a solution since we must add the arbitrary constant of integration "c" to our final answers.

If we have initial conditions $y(x_0)=a$, $y'(x_0)=b$, $\ldots$, $y^{N-1}(x_0)=z$, where $a, b, \ldots, z$ are constants and (x_0, y_0) is a point on the graph of $y(x)$, then we can go use these conditions to go and find all the arbitrary constants, including the constant of integration. However, in doing these problems, error may arise from misuse (i.e. double-use) of certain parameters. For example, if you have constants C, C_1, + C_2 in your problem, you might forget to put the subscript "2" on C_2 and mistakenly substitute the value you might have found for C in for this constant. It turns out that using definite integrals is a way efficiently solve IVP problems and avoid (some) notation errors simultaneously.

Solving Directly - Integrable Equations Using Definite Integrals

Let $y(x)$ be a (general) solution to the ODE $\frac{dy}{dx}=f(x)$. If $\frac{dy}{dx}=y'(x)$, then it is also true that...

$$\frac{dy}{dx}=y'(x) \Rightarrow dy = y'(x)\,dx \Rightarrow \underline{\int dy} = \int y'(x)\,dx = \underline{y(x)+C}$$

Note that we could come to the same conclusion by using definite integrals if we did the following:

- Let $x=s$, where s will be the variable of integration
- Allow upper + lower limits of integration to be "a" and "x", respectively, where "a" is a fixed constant.

$\therefore \frac{dy}{dx} = y'(x) \Rightarrow \frac{dy}{ds} = y'(s) \Rightarrow dy = y'(s)\,ds$

Now integrating both from $s=a$ to $s=x$ yields...

$$\int_a^x dy = \int_a^x y'(s)\,ds$$

$$\Rightarrow \int_a^x dy = \Big[y(s)\Big]_{s=a}^{s=x} = y(x) - y(a) \Rightarrow \int_a^x dy = y(x) - \overset{C}{y(a)}$$

where $y(a) = C$ = constant of integration. By knowing what $y(a) = \underline{\ ?\ }$ directly, you can substitute this value into your final answer without fear of confusing it with something else in your problem!

$$\therefore \int_a^x y'(x)\,dx = y(x) - y(a) \Rightarrow \boxed{y(x) = \int_a^x y'(x)\,dx + y(a)}$$

<u>So why is the ability to use definite integrals in solving ODEs so important?</u>

(1) Using definite integrals allows you to often times avoid dealing with arbitrary constants, and, thus, end up with answers (solutions) involving the initial values directly.

(2) Answers/Solutions to ODEs are in compact formulas that can be computed, if necessary, regardless of if the indefinite integral (i.e. most general antiderivative) can't be found via by-hand methods!!

Special Functions to Know

The following functions are special functions that pop up alot in applications that are modeled by ODEs. Note that the functions below do not have elementary antiderivatives (i.e. none of the integration techniques learned in a 1^{st} or 2^{nd} year calculus course wi help you find an exact answer to the antiderivatives by just using algebraic, trigonometric, or transcendental function (i.e. a^x and $\log_a(x)$

Often times, either power series expansion and/or software is needed to use these functions to compute actual values for these functions.

- Error Function: (aka Gauss Error Function) $\boxed{\operatorname{erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-s^2}\, ds}$

This function is used in probability, statistics, and engineering for such things as finding (more simple or compact) ways to express the error that a random variable lies within a certain range $(-s, s)$, where $s>0$, bit rate error in digital communications, and solutions to the heat/diffusion equation (with set boundary conditions) just to name a few.

- Sine-Integral Function: $\boxed{\operatorname{Si}(x) = \int_0^x \frac{\sin(s)}{s}\, ds = \int_0^x \operatorname{sinc}(s)\, ds}$

This function is used to measure area under the curve of the $\operatorname{sinc}(x) = \frac{\sin(x)}{x}$ function, which is seen a lot in signal processing among other uses.

To illustrate how these functions can be used, lets consider the following example.

Ex: Find the solution to the IVP stated below.

$$x \cdot \frac{dy}{dx} = \sin(x^2) \quad ; \; y(0) = 0$$

Sol'n: $x \cdot \frac{dy}{dx} = \sin(x^2) \Rightarrow \frac{dy}{dx} = \frac{\sin(x^2)}{x}$

NOTE: $\frac{\sin(x^2)}{x} = \frac{\sin(x^2)}{x} \cdot \frac{x^2}{x^2} = \frac{\sin(x^2)}{x^2} \cdot \frac{x^2}{x} = \frac{\sin(x^2)}{x^2} \cdot x$

$\therefore \frac{dy}{dx} = \frac{\sin(x^2)}{x^2} \cdot x \Rightarrow dy = \frac{\sin(x^2) \cdot x}{x^2}\, dx$

So, let's integrate both sides of our last equation from 0 to x.

$\therefore \int_0^x dy = \int_0^x \frac{x \cdot \sin(x^2)}{x^2}\, dx$. Let $s = x^2$. Then, $ds = 2x \cdot dx \Rightarrow$

$\frac{1}{2}\, ds = x \cdot dx$. Also, $s(0) = 0^2 = 0$ and $s(x) = x^2 = s$

Thus, $\int_0^x \frac{x \cdot \sin(x^2)}{x^2}\, dx = \int_0^{x^2 \to s} \frac{\sin(s)}{s} \cdot \frac{1}{2}\, ds = \frac{1}{2} \int_0^s \frac{\sin(s)}{s}\, ds$

But, note that $Si(s) = \int_0^s \frac{\sin(s)}{s}\, ds$. So, $\frac{1}{2} \int_0^s \frac{\sin(s)}{s}\, ds = \frac{1}{2} Si(s)$
$= \frac{1}{2} Si(x^2)$

initial condition

$\therefore \int_0^x dy = y(x) - \overset{0}{y(0)} = \frac{1}{2} Si(x^2) \Rightarrow \boxed{y(x) = \frac{1}{2} Si(x^2)}$

Solving Directly - Integrable Equations w/ Piecewise Functions

When solving directly-integrable ODE (or any type of ODE for that matter) using piecewise functions, we just need to remember that when we perform the integration, we need to do the following:

(1) Integrate each "piece" of the piecewise function, making sure that each "piece" we integrate is a function that is continuous over a closed interval.

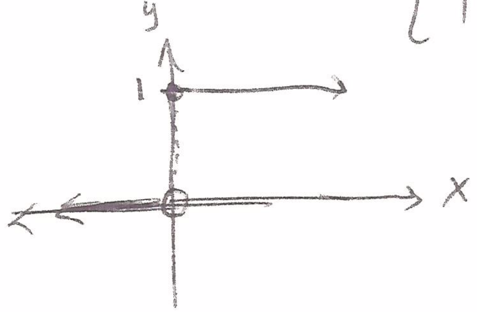
(2) If the piecewise function has true discontinuities in it, the solution will not be valid at these discontinuities

(3) At places of possible non-continuity, we must make sure that the behavior of the (piecewise) function at this place, say @ $x = x_0$, is the same on both the left and right of x_0. In other words, $\lim_{x \to x_0^-} \left[f(x) \right] = \lim_{x \to x_0^+} \left[f(x) \right] = L$, where $L \in \mathbb{R}$ (i.e. L is a real # constant).

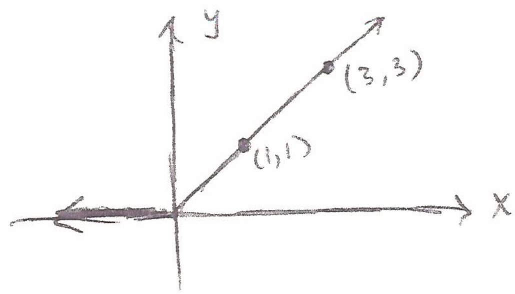
<u>NOTE</u>! Investigating (3) will ensure that we have the same arbitrary constant for each "piece" of the piecewise function.

Special Piecewise Functions You Need to Know

- $\text{step}(x) = u(t) = \text{Heaviside function} = \begin{cases} 0 \; ; & \text{if } x < 0 \\ 1 \; ; & \text{if } x \geq 0 \end{cases}$

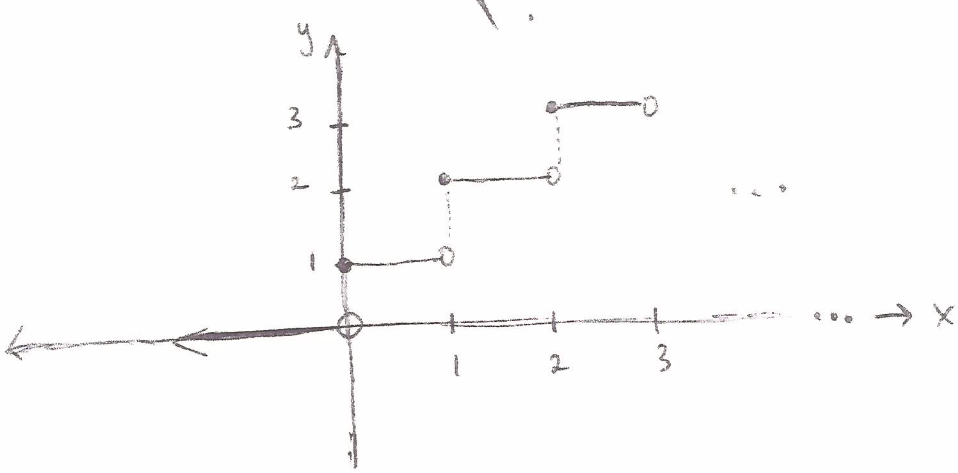


- $\text{ramp}(x) = \begin{cases} 0 \; ; & \text{if } x < 0 \\ x \; ; & \text{if } x \geq 0 \end{cases}$



- $\text{stair}(x) = \text{int}(x) + 1 \quad = \begin{cases} 0 \; ; & \text{if } x < 0 \\ 1 \; ; & \text{if } 0 \leq x < 1 \\ 2 \; ; & \text{if } 2 \leq x < 3 \\ \vdots & \end{cases}$

  (for $x \geq 0$)



Ex. 1: Solve each IVP using an indefinite integral

a) $\cos(x) \cdot \frac{dy}{dx} - \sin(x) = 0$; $y(0) = 3$ ($\frac{dy}{dx} \to y'$)

$\therefore \cos(x) \cdot y' - \sin(x) = 0 \Rightarrow y' = \frac{\sin(x)}{\cos(x)} = \tan(x) \Rightarrow y(x)\,dx = \tan(x)$

$\therefore \int y(x)\,dx = \int \tan(x)\,dx \Rightarrow y(x) = -\ln|\cos(x)| + C = \ln|\sec(x)| + C$ (*)

$\therefore$ when $y(0) = 3$: $-\ln|\cos(0)| + C = 3 \Rightarrow -\cancel{\ln|1|} + C = 3 \Rightarrow \boxed{C = 3}$ ($\ln|1| \to 0$)

$$\boxed{\therefore y(x) = -\ln|\cos(x)| + 3 = \ln|\sec(x)| + 3}$$

(*) $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$. Let $u = \cos(x)$. Then, $du = -\sin(x)\,dx$.

$\Rightarrow -du = \sin(x)\,dx$.

$\therefore \int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{-du}{u} = -\ln|u| + C$

But $u = \cos(x) \Rightarrow \int \tan(x)\,dx = -\ln|\cos(x)| + C = \ln\left|\frac{1}{\cos(x)}\right| + C$

$= \ln|\sec(x)| + C$ //

b) $x\dfrac{d^2y}{dx^2} + 2 = \sqrt{x}$; $y(1) = 8$; $y'(1) = 6$

Note: $\sqrt{x} \Rightarrow x \geq 0$

$\therefore\ xy'' + 2 = x^{1/2} \Rightarrow xy'' = x^{1/2} - 2 \Rightarrow y'' = \dfrac{x^{1/2} - 2}{x} = x^{-1/2} - \dfrac{2}{x}$

$\therefore\ y''(x)\,dx = \left(x^{-1/2} - \dfrac{2}{x}\right)dx$

$\Rightarrow \int y''(x)\,dx = \int \left(x^{-1/2} - \frac{2}{x}\right)dx \Rightarrow y'(x) = \dfrac{x^{-\frac{1}{2}+\frac{2}{2}}}{-\frac{1}{2}+\frac{2}{2}} - 2\ln|x| + C$

$\Rightarrow y'(x) = 2x^{1/2} - 2\ln(x) + C$; $x > 0$

$\therefore\ y'(x)\,dx = \left(2x^{1/2} - 2\ln(x) + C\right)dx$; $x > 0$

$\Rightarrow \int y'(x)\,dx = \int \left(2x^{1/2} - 2\ln(x) + C\right)dx$; $x > 0$

$\Rightarrow y(x) = \dfrac{2x^{\frac{1}{2}+\frac{2}{2}}}{\frac{1}{2}+\frac{2}{2}} - 2\left[x\ln(x) - x\right]^{(*)} + Cx + D$; $x > 0$

$\Rightarrow y(x) = \frac{2}{3}\cdot 2x^{3/2} - 2x\ln(x) + 2x + Cx + D$

$\Rightarrow y(x) = \frac{4}{3}x\sqrt{x} - 2x\cdot\ln(x) + (C+2)x + D$ → general sol'n

(*) NOTE: $\int \ln(x)\,dx$ can be solved by using integratio[n] by parts! Let $u = \ln(x)$ a[nd] $dv = dx$.

$u = \ln(x)$ $\quad dv = dx$

$u' = \frac{1}{x}\,dx$ $\quad v = x$

$\therefore\ uv - \int v\,du$

$= \ln(x)\cdot x - \int x\cdot\frac{1}{x}\,d[x]$

$= \ln(x)\cdot x - \int dx$

$= x\ln(x) - x + [C]$

when $y(1) = 8$: $\frac{4}{3}(1)\sqrt{1} - 2(1)\cdot\cancel{\ln(1)}^{0} + (C+2)(1) + D = 8 \Rightarrow \frac{4}{3} - \cancel{2} + C + \cancel{2} + D = 8$

$\Rightarrow \frac{4}{3} + C + D = 8$

$\Rightarrow C + D = 8 - \frac{4}{3} = \frac{20}{3}$

$\Rightarrow \boxed{3C + 3D = 20}$

when $y'(1) = 6$: $y'(x) = 2\sqrt{x} + 2\ln(x) + C$

$\therefore\ y'(1) = 6 \Rightarrow 2\sqrt{1} + 2\cancel{\ln(1)}^{0} + C = 6 \Rightarrow 2 + C = 6 \Rightarrow \boxed{C = 4}$

Thus, $3\cancel{(4)}^{12} + 3D = 20 \Rightarrow 3D = 20 - 12 = 8 \Rightarrow \boxed{D = 8/3}$

$\therefore\ \boxed{y(x) = \frac{4}{3}x\sqrt{x} - 2x\cdot\ln(x) + 6x + \frac{8}{3}\ ;\ x > 0}$

Ex. 2: Solve the follow IVPs using definite integrals.

a) $\frac{dy}{dx} = \frac{1}{x^2+1}$; $y(1) = 0$

$\therefore dy = \frac{1}{x^2+1}\,dx \Rightarrow \int dy = \int \frac{1}{x^2+1}\,dx = \arctan(x) + C$

$\therefore y(x) = \arctan(x) + C$

when $y(1)=0$: $\cancel{\arctan(1)}^{\pi/4} + C = 0 \Rightarrow \pi/4 + C = 0 \Rightarrow C = -\pi/4$

$$\therefore \boxed{y(x) = \arctan(x) - \pi/4 = \tan^{-1}(x) - \pi/4}$$

b) $x\,\frac{dy}{dx} = \sin(x)$; $y(0) = -7$

$x \cdot \frac{dy}{dx} = \sin(x) \Rightarrow \frac{dy}{dx} = \frac{\sin(x)}{x} \Rightarrow dy = \frac{\sin(x)}{x}\,dx$

NOTE: Since $\sin(x)/x$ does not have an elementary (function) antiderivative, we can use the technique of integrating via definite integral by (1) Letting $x=s$ and (2) integrating both side of our equation $dy = \frac{\sin(x)}{x}\,dx = \frac{\sin(s)}{s}$ from $s=0$ to $s=x$.

$$\therefore \int_0^x dy = \int_0^x \frac{\sin(s)}{s}\,ds \Rightarrow y(x) - \cancel{y(0)}^{-7} = \int_0^x \frac{\sin(s)}{s}\,ds \Rightarrow y(x) = \int_0^x \frac{\sin(s)}{s}\,ds + 7$$

Recall that $Si(s) = \int_0^x \frac{\sin(s)}{s}\,ds$. So, $y(x) = Si(s) + 7 \Rightarrow y(x) = Si(x) + 7$

since $x = s$.

$$\boxed{y(x) = Si(x) + 7}$$

Ex. 3: Solve the IVP containing piecewise function for a solution.

a) $\frac{dy}{dx} = \text{stair}(x)$ for $x<4$; $y(0)=0$

$\therefore \frac{dy}{dx} = \text{stair}(x) \Rightarrow dy = \text{stair}(x)\cdot dx \Rightarrow \int dy = \int \text{stair}(x)\,dx$; $x=(-\infty,4)$.

NOTE: $\text{stair}(x) = \begin{cases} 0; & \text{if } x<0 \\ 1; & \text{if } 0\le x<1 \\ 2; & \text{if } 1\le x<2 \\ 3; & \text{if } 2\le x<3 \\ 4; & \text{if } 3\le x<4 \end{cases} \Rightarrow \int \text{stair}(x)\,dx = \begin{cases} C; & \text{if } x<0 \\ x+D; & \text{if } 0\le x<1 \\ 2x+E; & \text{if } 1\le x<2 \\ 3x+F; & \text{if } 2\le x<3 \\ 4x+G, & \text{if } 3\le x<4 \end{cases}$;

where $C, D, E, F, + G$ are constants! Since $\int dy = y(x) + \bar{C}$, $\bar{C}\in\mathbb{R}$, this implies that $y(x)$ must be continuous on $x=(-\infty,4)$. Therefore, $\int dy = \int \text{stair}(x)\,d$
$\Rightarrow y(x) = \int \text{stair}(x)\,dx$ and $y(x)$ will be continuous at all values of $x=(-\infty,4)$, where there will be corners in the graph of $y(x)$ at $x=0,1,2,$ and 3. From 1st year calculus we know that the behavior (i.e. limit) of $y(x)$ at $x=0,1,2,+3$ on left and right of each of these numbers should be equal to $y(0), y(1), y(2), +$ $y(3)$, respectively if $y(x)$ is to be continuous at all these values of x. Therefore it must be true that...

- $\lim_{x\to0^-}[C] = \lim_{x\to0^+}[x+D] = y(0) \Rightarrow C = 0+D = 0 \Rightarrow \boxed{C=D=0}$

- $\lim_{x\to1^-}[x+D] = \lim_{x\to1^+}[2x+E] = y(1) \Rightarrow 1+0 = 2(1)+E = y(1) \Rightarrow 1 = 2+E = y(1)$
$\Rightarrow \boxed{-1 = E = y(1)}$

- $\lim_{x\to2^-}[2x+E] = \lim_{x\to2^+}[3x+F] = y(2) \Rightarrow 2(2)-1 = 3(2)+F = y(2) \Rightarrow 3 = 6+F = y(2)$
$\Rightarrow \boxed{-3 = F = y(2)}$

- $\lim_{x\to3^-}[3x+F] = \lim_{x\to3^+}[4x+G] = y(3) \Rightarrow 3(3)-3 = 4(3)+G = y(3) \Rightarrow 6 = 12+G = y(3)$
$\Rightarrow \boxed{-6 = G = y(3)}$

Ex. 3

a) cont'd

$\therefore C = D = 0\ ;\ E = -1\ ;\ F = -3\ ;\ G = -6$

$$\therefore y(x) = \int \text{stair}(x)\,dx = \begin{cases} C\ ; & \text{if } x < 0 \\ x + D\ ; & \text{if } 0 \le x < 1 \\ 2x + E\ ; & \text{if } 1 \le x < 2 \\ 3x + F\ ; & \text{if } 2 \le x < 3 \\ 4x + G\ ; & \text{if } 3 \le x < 4 \end{cases} = \begin{cases} 0\ ; & \text{if } x < 0 \\ x\ ; & \text{if } 0 \le x < 1 \\ 2x - 1\ ; & \text{if } 1 \le x < 2 \\ 3x - 3\ ; & \text{if } 2 \le x < 3 \\ 4x - 6\ ; & \text{if } 3 \le x < 4 \end{cases}$$

Ex. 3

b) $\frac{dy}{dx} = \text{step}(x)$ with $y(0) = 0$

$\therefore\ dy = \text{step}(x)\,dx \Rightarrow \int dy = \int \text{step}(x)\,dx = \int \left[\begin{cases} 0, & \text{if } x<0 \\ 1, & \text{if } x \geq 0 \end{cases} \right] dx$

$$= \begin{cases} \int 0\,dx; & \text{if } x<0 \\ \int 1\,dx; & \text{if } x \geq 0 \end{cases}$$

$$= \begin{cases} C; & \text{if } x<0 \\ x+D; & \text{if } x \geq 0 \end{cases}$$

where C, D are constants. Since $y(x) = \int dy$ is a solution to this ODE, it follow that $y(x)$ must be continuous on $x = (-\infty, \infty) \Rightarrow y(x)$ must be continuous @ $x=1$. Similiar to Ex 3 (a), we know that the behavior (i.e. limit) of $y(x)$ as $x \to 0$ from both the left and right will have equal the same value $y(0)$ @ $x=0$

$\therefore\ \lim_{x\to 0^-}[C] = \lim_{x \to 0^+}[x+D] = y(0) \Rightarrow C = 0 + D = \overset{0}{\cancel{y(0)}} \Rightarrow \boxed{C = D = 0}$

$$\therefore\ y(x) = \int \text{step}(x)\,dx = \begin{cases} 0; & \text{if } x<0 \\ x; & \text{if } x \geq 0 \end{cases}$$