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On closed-form expressions to Gompertz–Makeham life expectancy

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ABSTRACT

Missov and Lenart (2013) derived closed-form solutions to the life expectancy and remaining life expectancy at age x when the mortality is governed by a Gompertz–Makeham hazard, which is a parametric model commonly applied to human mortality data at adult and old ages. However, the closed-form expressions provided by these authors are not correct. We provide, therefore, valid and correct expressions using elementary calculus. We also consider numerical studies using real data to show that the formulas we provide deliver satisfactory results.

1. Introduction

It is well-known that Gompertz-related models are quite useful to describe the pattern of adult human deaths. In a recent and important paper, Missov and Lenart (2013) derived closed-form expressions for the remaining life expectancy for individuals aged x, denoted as e_x, assuming that they follow the Gompertz ('G' for short) or Gompertz–Makeham ('GM' for short) mortality laws. The remaining life expectancy at age x is useful for describing the pattern of adult human deaths and, hence, is extremely important to have a valid expression to compute correctly e_x. Missov and Lenart (2013) considered the following expression to compute e_x:

$$e_x = \int_x^\infty S(t)\,dt,$$

where $S(\cdot)$ denotes the survival function (see, Missov and Lenart, 2013, Eq. (4)). It is worth stressing that the above expression does not correspond to the correct formula to compute the remaining life expectancy. To verify it, we start with some notation. Let (x) denote a life aged x, where $x \geq 0$. The death of (x) can occur at any age greater than x, and we model the future lifetime of (x) by a continuous random variable which we denote by T_x. This means that $x+T_x$ represents the age-at-death random variable for (x). For $x = 0$, let $T_0 = X$ be a random variable with cumulative distribution function $F(\cdot)$. In addition, define $T_x := X - x | X > x$. The cumulative distribution function of T_x is given by $F_{T_x}(t) := \Pr(T_x \leq t)$, that is,

$$F_{T_x}(t) = \frac{F(t+x) - F(x)}{1 - F(x)} = \frac{S(x) - S(x+t)}{S(x)},$$

where $S(x) = 1 - F(x)$ is a survival function of X. The mean residual life corresponds to the expected value of T_x, and it is given by

$$e_x := \mathbb{E}(T_x) = \int_0^\infty t\, f_{T_x}(t)dt = \int_0^\infty t\, S_{T_x}(t)\, \mu_{T_x}(t)dt,$$

where $f_{T_x}(\cdot)$ is the probability density function of T_x, and $\mu_{T_x}(\cdot)$ denotes the mortality function of T_x. After some algebra, it is possible to show that the remaining life expectancy at age x reduces to

$$e_x = \frac{1}{S(x)} \int_0^\infty S(t+x)dt = \frac{1}{S(x)} \int_x^\infty S(t)dt;$$

see, for example, Elandt-Johnson and Johnson (1980). In particular, the remaining life expectancy at birth is $e_0 = \int_0^\infty S(t)dt$. Depending on the algebraic form of $S(\cdot)$, the above integral may not have an algebraic solution. Consequently, numerical integration methods have to be considered. Evidently, some numerical problems can eventually occur in solving integral numerically. Therefore, it is quite important to provide a simple closed-form expression to compute e_x.

Let Z be a positive random variable, called frailty, that modulates individual hazards. It is common to assume that the individual frailty follows a gamma distribution (Vaupel et al., 1979) with probability density function in the form

$$\pi(z) = \frac{\lambda^k}{\Gamma(k)} z^{k-1} e^{-\lambda z}, \quad z > 0,$$

where $k > 0$ (shape parameter), $\lambda > 0$ (scale parameter), and $\Gamma(\cdot)$ denotes the complete gamma function. Assuming that mortality is considered constant, given frailty $Z = z$ at age x, the force of mortality is given by $\mu(x|z) = zae^{bx} + c$, and the survival function is

$$S(x|z) = \exp\left[-z\frac{a}{b}(e^{bx} - 1) - cx\right],$$

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where $a > 0$ and $b > 0$ are parameters associated with the Gompertz model, and $c \geq 0$ stands for the level of age-independent extrinsic mortality (Missov and Lenart, 2013). Under this setup, the survival function of the gamma Gompertz–Makeham ('GGM' for short) model is

$$S(x) = \int_0^\infty S(x|z)\pi(z)dz = \mathrm{e}^{-cx}\left[1 + \frac{a}{b\lambda}(\mathrm{e}^{bx} - 1)\right]^{-k}.$$

Hence, the remaining life expectancy at age x for the GMM model is computed numerically from

$$e_x = \left[1 + \frac{a}{b\lambda}(\mathrm{e}^{bx} - 1)\right]^{k} \int_0^\infty \mathrm{e}^{-ct}\left[1 + \frac{a}{b\lambda}(\mathrm{e}^{b(x+t)} - 1)\right]^{-k} dt. \quad (1)$$

Evidently, the above integral has no algebraic solution and hence numerical integration has to be considered. When $c = 0$ in Eq. (1), we obtain the remaining life expectancy at age x for the gamma Gompertz ('GG' for short) model. According to Missov and Lenart (2013), the GG and GGM models are the simplest models (in terms of frailty distribution choice) that capture observed mortality dynamics at adult, old, and oldest-old ages. Unfortunately, the closed-form solutions to the remaining life expectancy at age x in (1) reported in Missov and Lenart (2013) for these models do not appear correct. In this short note, we shall provide valid and correct closed-form expressions for e_x. We shall also provide numerical studies.

2. Gompertz life expectancy

The closed-form expression to the remaining life expectancy at age x derived by Missov and Lenart (2013, eq. (5)) in the Gompertz case is given by

$$e_x := e_{\mathrm{G}}(x) = \frac{\mathrm{e}^{a/b}}{b} E_1\left(\frac{a\mathrm{e}^{bx}}{b}\right), \quad (2)$$

where $E_1(z) = \int_1^\infty t^{-1}\mathrm{e}^{-zt}dt = \int_z^\infty t^{-1}\mathrm{e}^{-t}dt$ is the integro-exponential function for $z \in \mathbb{C}$ (see, for instance, Milgram, 1985). Expression (2) is not correct. We provide the correct result in the following proposition.

Proposition 1. *The remaining life expectancy at age x for the Gompertz model is*

$$e_x := \mathcal{M}_{\mathrm{G}}(x) = \frac{1}{b}\exp\left(\frac{a\mathrm{e}^{bx}}{b}\right) E_1\left(\frac{a\mathrm{e}^{bx}}{b}\right). \quad (3)$$

Proof. The cumulative distribution function of the Gompertz model is given by

$$F(x) = 1 - \exp\left[-\frac{a}{b}(\mathrm{e}^{bx} - 1)\right], \quad x \geq 0,$$

where $a > 0$ and $b > 0$. Consequently, the survival function of the Gompertz model is

$$S(x) = 1 - F(x) = \exp\left[-\frac{a}{b}(\mathrm{e}^{bx} - 1)\right], \quad x \geq 0.$$

We can express

$$\frac{S(x+t)}{S(x)} = \exp\left[-\frac{a\mathrm{e}^{bx}}{b}(\mathrm{e}^{bt} - 1)\right].$$

Hence,

$$e_x = \int_0^\infty \exp\left[-\frac{a\mathrm{e}^{bx}}{b}(\mathrm{e}^{bt} - 1)\right] dt.$$

Note that

$$\int_0^\infty \exp\left[-\frac{a\mathrm{e}^{bx}}{b}(\mathrm{e}^{bt} - 1)\right] dt = \exp\left(\frac{a\mathrm{e}^{bx}}{b}\right)\int_0^\infty \exp\left(-\frac{a\mathrm{e}^{bx}}{b}\mathrm{e}^{bt}\right) dt.$$

Letting $u = \mathrm{e}^{bt}$, it follows that

$$\exp\left(\frac{a\mathrm{e}^{bx}}{b}\right)\int_0^\infty \exp\left(-\frac{a\mathrm{e}^{bx}}{b}\mathrm{e}^{bt}\right) dt = \frac{1}{b}\exp\left(\frac{a\mathrm{e}^{bx}}{b}\right)\int_1^\infty \exp\left(-\frac{a\mathrm{e}^{bx}}{b}u\right) du.$$

Finally, we obtain

$$e_x = \frac{1}{b}\exp\left(\frac{a\mathrm{e}^{bx}}{b}\right)\int_1^\infty \exp\left(-\frac{a\mathrm{e}^{bx}}{b}u\right) du = \frac{1}{b}\exp\left(\frac{a\mathrm{e}^{bx}}{b}\right) E_1\left(\frac{a\mathrm{e}^{bx}}{b}\right). \quad \square$$

Remark 1. Note that $e_{\mathrm{G}}(0) = \mathcal{M}_{\mathrm{G}}(0)$ and, hence, the life expectancy at birth obtained from Eqs. (2) and (3) is the same for the Gompertz model. Recall that life expectancy at birth actually refers to the average number of years a newborn is expected to live if mortality patterns at the time of its birth remain constant in the future. In a probabilistic mortality model, it is simply the expected value of the underlying distribution.

3. Gompertz-Makeham life expectancy

The closed-form expression to the remaining life expectancy at age x derived by Missov and Lenart (2013, eq. (7)) in the Gompertz–Makeham case is given by

$$e_x := e_{\mathrm{GM}}(x) = \frac{\mathrm{e}^{a/b}}{b}\left(\frac{a}{b}\right)^{c/b} \Gamma\left(-\frac{c}{b}, \frac{a\mathrm{e}^{bx}}{b}\right), \quad (4)$$

where

$$\Gamma(u, x) = \int_x^\infty t^{u-1}\mathrm{e}^{-t}dt, \quad x > 0, \quad u \in \mathbb{R};$$

that is, $\Gamma(\cdot, \cdot)$ is the complementary incomplete gamma function. In particular, we have $\Gamma(0, z) = E_1(z)$. Expression (4) is not correct. We provide the correct result in the following proposition.

Proposition 2. *The remaining life expectancy at age x for the Gompertz–Makeham model is*

$$e_x := \mathcal{M}_{\mathrm{GM}}(x) = \frac{1}{b}\exp\left(\frac{a\mathrm{e}^{bx}}{b}\right)\left(\frac{a\mathrm{e}^{bx}}{b}\right)^{c/b} \Gamma\left(-\frac{c}{b}, \frac{a\mathrm{e}^{bx}}{b}\right). \quad (5)$$

Proof. The cumulative distribution function of the Gompertz–Makeham model is given by

$$F(x) = 1 - \exp\left[-cx - \frac{a}{b}(\mathrm{e}^{bx} - 1)\right], \quad x \geq 0,$$

where $a > 0$, $b > 0$ and $c \geq 0$, and the corresponding survival function is

$$S(x) = \exp\left[-cx - \frac{a}{b}(\mathrm{e}^{bx} - 1)\right], \quad x \geq 0.$$

We have that

$$\frac{S(x+t)}{S(x)} = \exp\left[-ct - \frac{a\mathrm{e}^{bx}}{b}(\mathrm{e}^{bt} - 1)\right].$$

Hence,

$$e_x = \int_0^\infty \exp\left[-ct - \frac{a\mathrm{e}^{bx}}{b}(\mathrm{e}^{bt} - 1)\right] dt.$$

From a simple computation, it then follows that

$$e_x = \frac{1}{b}\exp\left(\frac{a\mathrm{e}^{bx}}{b}\right)\left(\frac{a\mathrm{e}^{bx}}{b}\right)^{c/b} \Gamma\left(-\frac{c}{b}, \frac{a\mathrm{e}^{bx}}{b}\right). \quad \square$$

Corollary 1. *When $c = 0$ in Eq. (5), we obtain the remaining life expectancy at age x for the Gompertz model provided in Eq. (3). This result is expected since the Gompertz–Makeham model reduces to the Gompertz model when $c = 0$.*

Remark 2. Note that $e_{\mathrm{GM}}(0) = \mathcal{M}_{\mathrm{GM}}(0)$. Hence, the life expectancy at birth obtained from Eqs. (4) and (5) is the same for the Gompertz–Makeham model.

4. Gamma Gompertz–Makeham life expectancy

The closed-form expression to the remaining life expectancy at age x derived by Missov and Lenart (2013, eq. (11)) in the gamma Gompertz–Makeham case is given by

$$e_x := e_{\mathrm{GGM}}(x) = \frac{\mathrm{e}^{-(bk+c)x}}{bk+c}\left(\frac{b\lambda}{a}\right)^k \times {}_2F_1\left(k+\frac{c}{b}, k, k+\frac{c}{b}+1; \left(1-\frac{b\lambda}{a}\right)\mathrm{e}^{-bx}\right), \tag{6}$$

where ${}_2F_1(p, q, m; z)$ is the Gaussian hypergeometric function (see, for instance, Rainville, 1960), which is given by

$${}_2F_1(p, q, m; z) = \sum_{n=0}^{\infty} \frac{(p)_n (q)_n}{(m)_n} \frac{z^n}{n!}, \quad |z| < 1,$$

where $(q)_n$ is the (rising) Pochhammer symbol defined by $(q)_0 = 1$, and $(q)_n = q(q+1)\cdots(q+n-1)$ for $n \geq 1$. The Gaussian hypergeometric function is undefined (or infinite) if m equals a non-positive integer. Missov and Lenart (2013, Eq. (13)) also computed a closed-form expression to the life expectancy at birth in the gamma Gompertz–Makeham case, which is given by

$$\frac{1}{bk+c}\,{}_2F_1\left(k, 1, k+\frac{c}{b}+1; 1-\frac{a}{b\lambda}\right). \tag{7}$$

Note that is not possible to obtain the life expectancy at birth in Eq. (7) directly from Eq. (6) when $x = 0$.

Unfortunately, the remaining life expectancy at age x in Eq. (6) does not appear correct. According to Missov (2013, p. 264) and Missov and Lenart (2013, p. 30), we have that $a \propto 10^{-6}$, $b \approx 0.14$ and $k = \lambda > 1$ for human populations corresponding to mortality patterns in modern societies. However, for these parameter values, the Gaussian hypergeometric function in (6) can be divergent since there is no guarantee that

$$\left(1-\frac{b\lambda}{a}\right)\mathrm{e}^{-bx}$$

belongs to $(-1, 1)$. Therefore, the computation of e_x from Eq. (6) is not reliable. In fact, numerical results using real data in the next section reveal that expression (6) is not convergent and, therefore, cannot be used for practical purposes.

The following theorem will be used to compute a valid and correct closed-form expression for e_x in the gamma Gompertz–Makeham case.

Theorem 1. *If $|z| < 1$ and $Re(m) > Re(q) > 0$, then*

$${}_2F_1(p, q, m; z) = \frac{\Gamma(m)}{\Gamma(q)\Gamma(m-q)} \int_0^1 u^{q-1}(1-u)^{m-q-1}(1-zu)^{-p}\, du,$$

where $\Gamma(\cdot)$ denotes the complete gamma function.

Proof. The proof can be found in Rainville (1960, p. 47). □

Remark 3. Theorem 1 is enunciated, in general, for complex variables, where $|\cdot|$ means module in the field of complex functions, and $Re(z)$ is the real part of the complex variable z. If z is a real variable, then $Re(z) = z$, and $|z|$ is simply the absolute value of z.

The following proposition provides a correct and valid closed-form expression for computing e_x in the gamma Gompertz–Makeham case.

Proposition 3. *The remaining life expectancy at age x for the gamma Gompertz–Makeham model is*

$$e_x := \mathcal{M}_{\mathrm{GGM}}(x) = \frac{1}{bk+c}\,{}_2F_1\left(k, 1, k+1+\frac{c}{b}; \frac{(1-\frac{a}{b\lambda})}{1-\frac{a}{b\lambda}+\frac{a}{b\lambda}\mathrm{e}^{bx}}\right). \tag{8}$$

Proof. We have that

$$e_x = \frac{1}{S(x)}\int_0^{\infty} S(t+x)dt,$$

where

$$S(x) = \mathrm{e}^{-cx}\left[1+\frac{a}{b\lambda}(\mathrm{e}^{bx}-1)\right]^{-k} = \mathrm{e}^{-cx}\left[1-\frac{a}{b\lambda}+\frac{a}{b\lambda}\mathrm{e}^{bx}\right]^{-k},$$

and

$$S(x+t) = \mathrm{e}^{-cx}\mathrm{e}^{-ct}\left[1+\frac{a}{b\lambda}(\mathrm{e}^{bx+bt}-1)\right]^{-k}.$$

Using the transformation $u = 1-\mathrm{e}^{-bt}$, we obtain

$$\begin{aligned} e_x &= \frac{\mathrm{e}^{-cx}}{S(x)}\int_0^1 (1-u)^{\frac{c}{b}}\left[1-\frac{a}{b\lambda}+\frac{a}{b\lambda}\mathrm{e}^{bx}\frac{1}{(1-u)}\right]^{-k}\frac{du}{b(1-u)} \\ &= \frac{\mathrm{e}^{-cx}}{b\,S(x)}\int_0^1 (1-u)^{\frac{c}{b}+k-1}\left[1-\frac{a}{b\lambda}+\frac{a}{b\lambda}\mathrm{e}^{bx}-\left(1-\frac{a}{b\lambda}\right)u\right]^{-k}du. \end{aligned}$$

After some algebra, it follows that

$$e_x = \frac{\mathrm{e}^{-cx}}{bS(x)}\left[1-\frac{a}{b\lambda}+\frac{a}{b\lambda}\mathrm{e}^{bx}\right]^{-k} \times \int_0^1 (1-u)^{\frac{c}{b}+k-1}\left[1-\frac{(1-\frac{a}{b\lambda})u}{1-\frac{a}{b\lambda}+\frac{a}{b\lambda}\mathrm{e}^{bx}}\right]^{-k}du.$$

Thus, we obtain

$$e_x = \frac{1}{b}\int_0^1 u^{1-1}(1-u)^{\frac{c}{b}+k-1}(1-zu)^{-k}\,du,$$

where

$$z = \frac{\left(1-\frac{a}{b\lambda}\right)}{1-\frac{a}{b\lambda}+\frac{a}{b\lambda}\mathrm{e}^{bx}}.$$

From Theorem 1, we have that

$$\int_0^1 u^{1-1}(1-u)^{(\frac{c}{b}+k+1)-1-1}(1-zu)^{-k} = {}_2F_1\left(k, 1, k+1+\frac{c}{b}; z\right)\frac{b}{bk+c}.$$

Therefore, it follows that

$$e_x = \frac{1}{bk+c}\,{}_2F_1\left(k, 1, k+1+\frac{c}{b}; \frac{(1-\frac{a}{b\lambda})}{1-\frac{a}{b\lambda}+\frac{a}{b\lambda}\mathrm{e}^{bx}}\right). \quad \square$$

Corollary 2. *When $c = 0$ in Eq. (8), we obtain the remaining life expectancy at age x in the gamma Gompertz case, which is given by*

$$e_x := \mathcal{M}_{\mathrm{GG}}(x) = \frac{1}{bk}\,{}_2F_1\left(k, 1, k+1; \frac{(1-\frac{a}{b\lambda})}{1-\frac{a}{b\lambda}+\frac{a}{b\lambda}\mathrm{e}^{bx}}\right).$$

Remark 4. For human populations corresponding to mortality patterns in modern societies, we have that $a \propto 10^{-6}$, $b \approx 0.14$ and $k = \lambda > 1$ (Missov, 2013; Missov and Lenart, 2013).

From Eq. (8), note that

$$\frac{\left(1-\frac{a}{b\lambda}\right)}{1-\frac{a}{b\lambda}+\frac{a}{b\lambda}\,\mathrm{e}^{bx}}$$

always belongs to $(-1, 1)$, which ensures the convergence of the Gaussian hypergeometric function.

Remark 5. The life expectancy at birth in the gamma Gompertz and gamma Gompertz–Makeham cases reduce to

$$\mathcal{M}_{\mathrm{GG}}(0)=\frac{1}{bk}\,{}_2F_1\left(k,1,k+1;1-\frac{a}{b\lambda}\right)$$

and

$$\mathcal{M}_{\mathrm{GGM}}(0)=\frac{1}{bk+c}\,{}_2F_1\left(k,1,k+1+\frac{c}{b};1-\frac{a}{b\lambda}\right),$$

respectively.

Remark 6. Note that $e_{\mathrm{GGM}}(0)\neq\mathcal{M}_{\mathrm{GGM}}(0)$. Additionally, $\mathcal{M}_{\mathrm{GGM}}(0)$ coincides with the life expectancy at birth in Eq. (7).

5. Numerical illustration

In this section, we discuss the estimation of the unknown parameters of the Gompertz-related models discussed in the previous sections by the maximum likelihood (ML) method. Additionally, we consider real data to confirm that the closed-form expressions derived in Missov and Lenart (2013) to compute e_x are not correct, while the closed-form expressions we provide to compute e_x work properly. The numerical illustration will be performed on the basis of mortality datasets available in HMD (2012).

5.1. Parameter estimation

Let D_x be the number of deaths in a given age interval $[x, x+1)$ for $x = 0, 1, \ldots, m$. If the number of deaths and the number of person-years exposed to the risk of dying can be observed, then we can assume that D_x follows a Poisson distribution. Under a Poisson distribution, we have that $\mathbb{E}(D_x) = \mu(x,\boldsymbol{\theta})E_x$ and $\mathbb{VAR}(D_x)=\mu(x,\boldsymbol{\theta})E_x$, where $\mu(x,\boldsymbol{\theta})$ represents the mortality rate at age x, E_x denotes the number of person-years with age x exposed to the risk of dying (see, for example, Brillinger, 1986), and $\boldsymbol{\theta}=(\theta_1,\theta_2,\ldots,\theta_d)'$ is the parameter vector of dimension d that characterizes the mortality rate. Let $\boldsymbol{D}=(D_0,D_1,\ldots,D_m)'$ and $\boldsymbol{E}=(E_0,E_1,\ldots,E_m)'$. The likelihood function for the parameter vector $\boldsymbol{\theta}=(\theta_1,\theta_2,\ldots,\theta_d)'$ is given by

$$L(\boldsymbol{\theta}|\boldsymbol{D},\boldsymbol{E})=\prod_x\frac{\lambda(x,\boldsymbol{\theta})^{D_x}\mathrm{e}^{-\lambda(x,\boldsymbol{\theta})}}{D_x!},$$

where $\lambda(x,\boldsymbol{\theta})=\mu(x,\boldsymbol{\theta})E_x$. The log-likelihood function becomes

$$\ell(\boldsymbol{\theta}|\boldsymbol{D},\boldsymbol{E})=\log L(\boldsymbol{\theta}|\boldsymbol{D},\boldsymbol{E})=\sum_x\left[D_x\log\lambda(x,\boldsymbol{\theta})-\log D_x!-\lambda(x,\boldsymbol{\theta})\right].$$

The ML estimate $\widehat{\boldsymbol{\theta}}=(\widehat{\theta}_1,\widehat{\theta}_2,\ldots,\widehat{\theta}_d)'$ of $\boldsymbol{\theta}=(\theta_1,\theta_2,\ldots,\theta_d)'$ is obtained by maximizing the log-likelihood function with respect to parameter vector $\boldsymbol{\theta}=(\theta_1,\theta_2,\ldots,\theta_d)'$. By taking the partial derivatives of the log-likelihood function with respect to the elements of $\boldsymbol{\theta}$, we obtain the likelihood equations

$$\frac{\partial\ell(\boldsymbol{\theta}|\boldsymbol{D},\boldsymbol{E})}{\partial\theta_j}=\sum_x\left[\frac{D_x}{\mu(x,\boldsymbol{\theta})}-E_x\right]\frac{\partial\mu(x,\boldsymbol{\theta})}{\partial\theta_j},\quad j=1,2,\ldots,d.$$

The ML estimate can also be obtained by equating the likelihood equations to zero and solving simultaneously the resulting system of equations. The complexity of the likelihood equations depends mainly on the algebraic form of the mortality rate $\mu(x,\boldsymbol{\theta})$ and, in general, the resulting system of equations has no closed-form solution. Hence, the ML estimates need to be obtained through a numerical maximization of the log-likelihood function using nonlinear optimization algorithms. In particular, we shall consider the well-developed statistical software R (R Core Team, 2019) to compute the ML estimate of $\boldsymbol{\theta}=(\theta_1,\theta_2,\ldots,\theta_d)'$ by using the *genetic algorithm* (Goldberg, 1989) through the library GA (Scrucca, 2013, 2017). This algorithm ensures the global maximum of the estimation process by the ML method. It is worth stressing that the R program is the most used and reliable statistical software for numerical purposes in practice. In the numerical illustrations, we obtain the ML estimates of the Gompertz-related model parameters, and then we compute the life expectancies at ages 30, 60 and 90.

5.2. Gompertz model

In the Gompertz case, the force of mortality is given by $\mu(x,\boldsymbol{\theta})=a\mathrm{e}^{bx}$, where $\boldsymbol{\theta}=(a,b)'$, and $a>0$ and $b>0$. The likelihood equations can be expressed as

$$\frac{\partial\ell(\boldsymbol{\theta}|\boldsymbol{D},\boldsymbol{E})}{\partial a}=\sum_x\left(\frac{D_x}{a\mathrm{e}^{bx}}-E_x\right)\mathrm{e}^{bx},$$

$$\frac{\partial\ell(\boldsymbol{\theta}|\boldsymbol{D},\boldsymbol{E})}{\partial b}=ab\sum_x\left(\frac{D_x}{a\mathrm{e}^{bx}}-E_x\right)x\mathrm{e}^{bx}.$$

Hence, the ML estimate $\widehat{\boldsymbol{\theta}}=(\widehat{a},\widehat{b})'$ of $\boldsymbol{\theta}=(a,b)'$ is obtained by equating these likelihood equations to zero and solving simultaneously the resulting system of equations. Table 1 lists the ML estimates, and the life expectancies (at ages 30, 60 and 90) computed from e_{G} given in (2), and $\mathcal{M}_{\mathrm{G}}$ provided in (3). For comparison, we also compute the life expectancy numerically from

$$e_x=\int_0^\infty\exp\left[-\frac{a\mathrm{e}^{bx}}{b}(\mathrm{e}^{bt}-1)\right]dt.$$

From Table 1, note that the life expectancies computed from expression (3) are the same as that computed numerically. Also, for age under 30 years, the life expectancies computed from expression (2) are the same as that computed numerically. However, when the age increases, it is evident that they differ greatly, going over 3 years (at ages like 90) in some countries, demonstrating that the closed-form expression provided by Missov and Lenart (2013) in the Gompertz case is not correct. Obviously this can severely impact any projection that involves calculating life expectancy.

5.3. Gompertz-Makeham model

In the Gompertz–Makeham case, the force of mortality is given by $\mu(x,\boldsymbol{\theta})=a\mathrm{e}^{bx}+c$, where $\boldsymbol{\theta}=(a,b,c)'$, and $a>0$, $b>0$ and $c\geq 0$. The likelihood equations can be expressed as

$$\frac{\partial\ell(\boldsymbol{\theta}|\boldsymbol{D},\boldsymbol{E})}{\partial a}=\sum_x\left(\frac{D_x}{a\mathrm{e}^{bx}+c}-E_x\right)\mathrm{e}^{bx},$$

$$\frac{\partial\ell(\boldsymbol{\theta}|\boldsymbol{D},\boldsymbol{E})}{\partial b}=ab\sum_x\left(\frac{D_x}{a\mathrm{e}^{bx}+c}-E_x\right)x\mathrm{e}^{bx},$$

$$\frac{\partial\ell(\boldsymbol{\theta}|\boldsymbol{D},\boldsymbol{E})}{\partial c}=\sum_x\left(\frac{D_x}{a\mathrm{e}^{bx}+c}-E_x\right).$$

By equating these likelihood equations to zero and solving simultaneously the resulting system of equations, we obtain the ML estimate $\widehat{\boldsymbol{\theta}}=(\widehat{a},\widehat{b},\widehat{c})'$ of $\boldsymbol{\theta}=(a,b,c)'$. Table 2 lists the numerical

Table 1
Life expectancy: Gompertz model.

Country	Year	$\widehat{a}$	$\widehat{b}$	Age	e_G	$\mathcal{M}_G$	Integration
Sweden	2010	0.00018	0.11120	30	52.70204	52.70204	52.70204
				60	23.04392	24.07710	24.07710
				90	1.26220	4.52385	4.52385
Germany	2009	0.00035	0.10077	30	50.80376	50.80376	50.80376
				60	21.35607	22.83666	22.83666
				90	1.07576	4.56308	4.56308
Japan	2009	0.00024	0.10114	30	54.38963	54.38963	54.38963
				60	24.77029	25.93740	25.93740
				90	2.15471	5.87486	5.87486
USA	2007	0.00057	0.09023	30	50.06454	50.06454	50.06454
				60	20.83892	22.77322	22.77322
				90	1.24814	5.15682	5.15682

Table 2
Life expectancy: Gompertz–Makeham model.

Country	Year	$\widehat{a}$	$\widehat{b}$	$\widehat{c}$	Age	e_{GM}	$\mathcal{M}_{GM}$	Integration
Sweden	2010	0.00014	0.11521	0.00033	30	52.65278	52.65278	52.65278
					60	23.08803	24.22260	24.22260
					90	1.24628	4.43064	4.43064
Germany	2009	0.00023	0.10779	0.00075	30	50.60062	50.60062	50.60062
					60	21.34957	23.02619	23.02619
					90	1.02544	4.35396	4.35396
Japan	2009	0.00016	0.10825	0.00056	30	54.22616	54.22616	54.22616
					60	24.75870	26.10025	26.10025
					90	2.08145	5.63451	5.63451
USA	2007	0.00037	0.09820	0.00088	30	50.01326	50.01326	50.01326
					60	20.95964	23.02648	23.02648
					90	1.19111	4.88020	4.88020

results regarding the life expectancies computed from e_{GM} given in (4), $\mathcal{M}_{GM}$ provided in (5), and numerically from

$$e_x = \int_0^\infty \exp\left[-ct - \frac{ae^{bx}}{b}(e^{bt}-1)\right]dt.$$

First, note that the life expectancies computed from expression (5) are the same as that computed numerically (see Table 2). Once again, we can see some differences between the calculations proposed by Missov and Lenart (2013) and the one proposed in this paper. Similar to the Gompertz case, the differences can reach more than 3 years (at ages like 90), which leads to an underestimation of life expectancy for older individuals (90 years). Consequently, this can impact any projection that involves calculating life expectancy.

5.4. Gamma Gompertz–Makeham model

In the gamma Gompertz–Makeham case, the frailty Z follows a gamma distribution with parameters $\lambda > 0$ and $k > 0$, where $\mathbb{E}(Z) = \lambda/k$ and $\mathbb{VAR}(Z) = \lambda/k^2$. It is common to assume that $k = \lambda$ (see, for example, Canudas-Romo et al., 2018), and so $\mathbb{E}(Z) = 1$ and $\mathbb{VAR}(Z) = 1/k := \sigma^2$. Hence, in the gamma Gompertz–Makeham setup, the mortality rate can be expressed as

$$\mu(x, \boldsymbol{\theta}) = \frac{ae^{bx}}{1 + \sigma^2 \frac{a}{b}\left(e^{bx}-1\right)} + c,$$

where $\boldsymbol{\theta} = (a, b, c, \sigma^2)'$, and $a > 0$, $b > 0$, $c \geq 0$ and $\sigma^2 > 0$. According to Böhnstedt and Gampe (2019) and Böhnstedt et al. (2019), the parameter σ^2 describes the heterogeneity of frailty in the gamma Gompertz–Makeham model, as well as in the gamma Gompertz model ($c = 0$). If σ^2 is not close to zero, then there is heterogeneity in the risk of death and selection of the most robust individual will occur. On the other hand, $\sigma^2 \approx 0$ (i.e. very close to zero) may indicate that there is no heterogeneity and the force of mortality is increasing exponentially, such that $\mu(x, \boldsymbol{\theta}) = ae^{bx}$. Böhnstedt and Gampe (2019) have discussed about this issue and, in addition, some statistical properties of the ML estimators are derived.

The likelihood equations are

$$\frac{\partial \ell(\boldsymbol{\theta}|\boldsymbol{D}, \boldsymbol{E})}{\partial a} = b^2 \sum_x \left[D_x \left(\frac{ae^{bx}}{1+\sigma^2\frac{a}{b}\left(e^{bx}-1\right)} + c\right)^{-1} - E_x\right] \times \frac{e^{bx}}{[a\sigma^2(e^{bx}-1)+b]^2},$$

$$\frac{\partial \ell(\boldsymbol{\theta}|\boldsymbol{D}, \boldsymbol{E})}{\partial b} = a \sum_x \left[D_x \left(\frac{ae^{bx}}{1+\sigma^2\frac{a}{b}\left(e^{bx}-1\right)} + c\right)^{-1} - E_x\right] \times \left\{\frac{xe^{bx}}{1+\sigma^2\frac{a}{b}(e^{bx}-1)} - \frac{a\sigma^2 e^{bx}\left[\frac{xe^{bx}}{b} - \frac{(e^{bx}-1)}{b^2}\right]}{\left[1+\sigma^2\frac{a}{b}(e^{bx}-1)\right]^2}\right\},$$

$$\frac{\partial \ell(\boldsymbol{\theta}|\boldsymbol{D}, \boldsymbol{E})}{\partial c} = \sum_x \left[D_x \left(\frac{ae^{bx}}{1+\sigma^2\frac{a}{b}\left(e^{bx}-1\right)} + c\right)^{-1} - E_x\right],$$

$$\frac{\partial \ell(\boldsymbol{\theta}|\boldsymbol{D}, \boldsymbol{E})}{\partial \sigma^2} = \frac{a^2}{b} \sum_x \left[D_x \left(\frac{ae^{bx}}{1+\sigma^2\frac{a}{b}\left(e^{bx}-1\right)} + c\right)^{-1} - E_x\right] \times \frac{e^{bx}(e^{bx}-1)}{\left[\sigma^2\frac{a}{b}(e^{bx}-1)+1\right]^2}.$$

Hence, the ML estimate $\widehat{\boldsymbol{\theta}} = (\widehat{a}, \widehat{b}, \widehat{c}, \widehat{\sigma}^2)'$ of $\boldsymbol{\theta} = (a, b, c, \sigma^2)'$ is obtained by equating these likelihood equations to zero and solving simultaneously the resulting system of equations. Table 3 lists the ML estimates, the life expectancies (at ages 30, 60 and 90) computed from e_{GGM} given in (6), and $\mathcal{M}_{GGM}$ provided in

Table 3
Life expectancy: gamma Gompertz–Makeham model.

Country	year	$\widehat{a}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\sigma}^2$	Age	e_{GGM}	$\mathcal{M}_{GGM}$	Integration
Sweden	2010	0.00016	0.11107	0.00050	0.00291	30	∞	53.25777	53.06439
						60	∞	25.06624	24.89456
						90	∞	5.02367	4.94986
Germany	2009	0.00045	0.09706	0.00007	0.06863	30	∞	49.97458	49.95937
						60	∞	22.45431	22.44277
						90	∞	5.00831	5.00401
Japan	2009	0.00009	0.11691	0.00025	0.02974	30	∞	56.32456	56.15136
						60	∞	27.48690	27.32225
						90	∞	5.90270	5.82392
USA	2007	0.00047	0.09324	0.00005	0.00157	30	∞	50.72140	50.79600
						60	∞	23.19501	23.25364
						90	∞	5.16066	5.18469

Table 4
Convergence radius.

Country	Year	Age	$z_{\text{Missov-Lenart}}$	z_{proposed}
Sweden	2010	30	−243505.341901	0.999996
		60	−8696.340013	0.999885
		90	−310.573596	0.996790
Germany	2009	30	−3145.853395	0.999682
		60	−171.064218	0.994188
		90	−9.302076	0.902932
Japan	2009	30	−44603.014326	0.999978
		60	−1337.280889	0.999253
		90	−40.094155	0.975666
USA	2007	30	−125299.950049	0.999992
		60	−7641.208934	0.999869
		90	−465.986411	0.997859

(8), and life expectancy computed numerically from (1) assuming $k = \lambda$. Note that there is heterogeneity in the deaths in the four countries since the values of $\widehat{\sigma}^2$ are not very close to zero. Now, let us take a look at the life expectancies at ages 30, 60 and 90 provided in Table 3. By using expression (6) derived in Missov and Lenart (2013) to compute such quantities, we obtain the life expectancies equal to ∞, which is unacceptable, of course. On the other hand, the expression in (8) derived in this paper to compute e_x works properly, delivering plausible results; that is, the values obtained from $\mathcal{M}_{GGM}$ are quite close to the results obtained from numerical integration.

It is worth stressing that the wrong (absurd) values obtained from expression (6) in Table 3 are due to the non-convergence of the Gaussian hypergeometric function. As noted earlier, there is no guarantee that

$$z_{\text{Missov-Lenart}} := \left(1 - \frac{b\lambda}{a}\right) e^{-bx}$$

belongs to $(-1, 1)$. On the other hand, we have that

$$z_{\text{proposed}} := \frac{\left(1 - \frac{a}{b\lambda}\right)}{1 - \frac{a}{b\lambda} + \frac{a}{b\lambda} e^{bx}}$$

always belongs to $(-1, 1)$ and, consequently, the convergence of the Gaussian hypergeometric function is guaranteed (see Remark 4). On the basis of the ML estimates in Table 3, we present in Table 4 the estimated values of $z_{\text{Missov-Lenart}}$ and z_{proposed}, that is, the values of these quantities evaluated at the ML estimates. Note that $z_{\text{Missov-Lenart}}$ does not belong to $(-1, 1)$, while z_{proposed} is always smaller than 1, as expected, which ensures the convergence of the Gaussian hypergeometric function.

We also compute the remaining life expectancy to other historical population data (Sweden and Japan) under the gamma Gompertz–Makeham model. We obtain the remaining life expectancy (at ages 30 and 90) for these population data in the years 1950, 1960, 1970, 1980, 1990, 2000 and 2010. The ML estimates are listed in Table 5, and the remaining life expectancies are presented in Table 6. The estimated values of $z_{\text{Missov-Lenart}}$ and z_{proposed} are listed in Table 7. Again, the remaining life expectancies computed using expression (6) derived in Missov and Lenart (2013) are all equal to ∞, whereas the expression (8) derived in this paper to compute these quantities delivery plausible results in all cases (see Table 6). Finally, Fig. 1 displays the actual (calculated by life table methods) and model-predicted (based on the estimates of the gamma Gompertz–Makeham parameters) remaining life expectancies (at age 30 and at age 90) for the Sweden

Table 5
ML estimates: Sweden and Japan.

Sweden

Year	$\widehat{a}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\sigma}^2$
1950	0.00041	0.11188	0.00085	0.04394
1960	0.00037	0.11120	0.00027	0.02867
1970	0.00028	0.11137	0.00044	0.02346
1980	0.00031	0.10658	0.00028	0.00070
1990	0.00025	0.10855	0.00024	0.00220
2000	0.00012	0.12063	0.00044	0.01554
2010	0.00016	0.11107	0.00050	0.00291

Japan

Year	$\widehat{a}$	$\widehat{b}$	$\widehat{c}$	$\widehat{\sigma}^2$
1950	0.00074	0.10196	0.00373	0.03493
1960	0.00046	0.11022	0.00125	0.05349
1970	0.00027	0.11834	0.00084	0.08036
1980	0.00019	0.11782	0.00049	0.00929
1990	0.00012	0.12093	0.00047	0.00027
2000	0.00014	0.11103	0.00039	0.00002
2010	0.00004	0.13206	0.00072	0.02978

Table 6
Life expectancy: Sweden and Japan.

Sweden

Year	$e_{GGM}(30)$	$\mathcal{M}_{GGM}(30)$	$e_{GGM}(90)$	$\mathcal{M}_{GGM}(90)$
1950	∞	44.45237	∞	2.65107
1960	∞	46.18740	∞	2.84602
1970	∞	48.15765	∞	3.36722
1980	∞	49.10522	∞	3.73948
1990	∞	50.50775	∞	4.05584
2000	∞	52.13494	∞	4.06903
2010	∞	53.25777	∞	5.02367

Japan

Year	$e_{GGM}(30)$	$\mathcal{M}_{GGM}(30)$	$e_{GGM}(90)$	$\mathcal{M}_{GGM}(90)$
1950	∞	39.690799	∞	2.647626
1960	∞	43.576301	∞	2.694148
1970	∞	46.078373	∞	2.925495
1980	∞	49.358731	∞	3.307441
1990	∞	52.132095	∞	3.998831
2000	∞	54.512493	∞	5.464440
2010	∞	56.338899	∞	5.412301

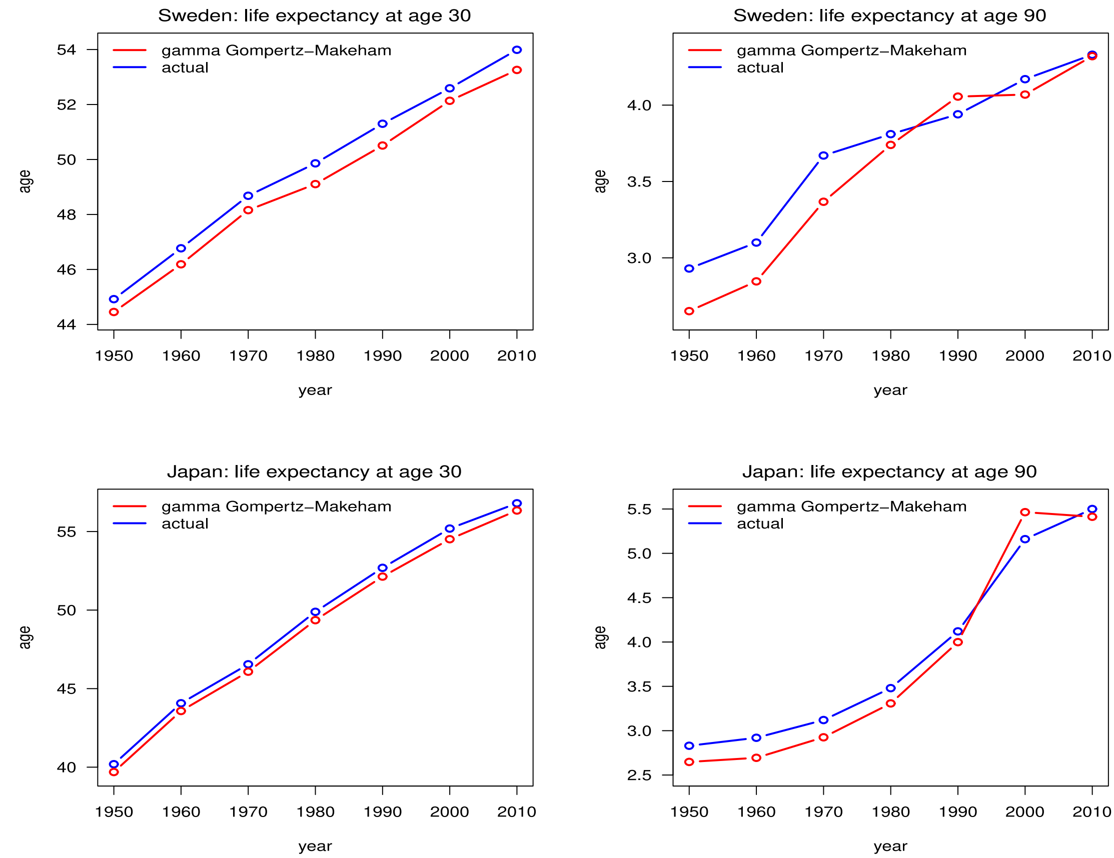

Fig. 1. Actual and gamma-Gompertz life expectancies.

Table 7
Convergence radius: Sweden and Japan.

Year	age 30		age 90	
Sweden				
	$z_{\text{Missov-Lenart}}$	z_{proposed}	$z_{\text{Missov-Lenart}}$	z_{proposed}
1950	−6207.629547	0.999839	−7.543557	0.882953
1960	−10562.920054	0.999905	−13.372692	0.930424
1970	−16751.012472	0.999940	−20.995128	0.954535
1980	−488535.183312	0.999998	−816.194454	0.998776
1990	−198476.812227	0.999995	−294.572870	0.996617
2000	−65582.163561	0.999985	−47.150565	0.979232
2010	−110767.625726	0.999991	−66.295668	0.985140
Japan				
Year	age 30		age 90	
	$z_{\text{Missov-Lenart}}$	z_{proposed}	$z_{\text{Missov-Lenart}}$	z_{proposed}
1950	−6207.629547	0.999839	−7.543557	0.882953
1960	−10562.920054	0.999905	−13.372692	0.930424
1970	−16751.012472	0.999940	−20.995128	0.954535
1980	−488535.183312	0.999998	−816.194454	0.998776
1990	−198476.812227	0.999995	−294.572870	0.996617
2000	−65582.163561	0.999985	−47.150565	0.979232
2010	−110767.625726	0.999991	−66.295668	0.985140

and Japan countries in the specified years. It is noteworthy that the gamma Gompertz–Makeham model performs satisfactorily for estimating life expectancies for both countries.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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