Contraction-Free Sequent Calculus for BI

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***Abstract*—We study LBI backward proof searches, presenting a precise analysis on how its structural rules interact with logical rules. In particular we study two sources of non-termination, one inherent in unary structural connectives and one inherent in contraction, and develop a new BI sequent calculus LBI3, resolving the first problem altogether and restricting the second problem by absorbing contraction into logical inference rules. We then adopt a contraction elimination technique well-known for the intuitionistic logic, and show a promising result towards a conclusion of purely syntactical BI decidability analysis.**

I. Introduction

BI is a substructural logic which arises, roughly speaking, in a free combination of the intuitionistic logic taking a role as the BI additive, and the multiplicative intuitionistic linear logic (without exponentials) as the BI multiplicative[1]. Categorically it originates in a doubly-closed single category, and BI, instead of a single implication, facilitates two implications: $\rightarrow$ drawn from the intuitionistic logic and $-\!\!*$ from the linear logic, giving rise to two adjoint relations: $[F \wedge G \vdash H] \simeq [F \vdash G {\rightarrow} H]$ and $[F * G \vdash H] \simeq [F \vdash G {-\!\!*} H]$ [6]. BI proof systems define two structural connectives ";" and "," in response, the former reserved for additive structure formation and the latter for multiplicative structure formation. This differentiation helps insulate additive structures from multiplicative ones and vice versa in a syntactically unambiguous way, as is observed for instance in the pair of LBI (BI sequent calculus developed in [7]) rules below:

$$\frac{\Gamma; F \vdash G}{\Gamma \vdash F {\rightarrow} G} \rightarrow R \quad \frac{\Gamma, F \vdash G}{\Gamma \vdash F {-\!\!*} G} -\!\!*R$$

The same differentiation is convenient also for structural rules, weakening and contraction in particular, which are valid only in the context of BI additive structures connected by ";":

$$\frac{\Gamma(\Delta; \Delta) \vdash F}{\Gamma(\Delta) \vdash F} \text{Contraction} \quad \frac{\Gamma(\Delta) \vdash F}{\Gamma(\Delta; \Delta') \vdash F} \text{Weakening}$$

That the intuitionistic logic (respectively the multiplicative intuitionistic linear logic) faithfully encodes into BI additive (respectively multiplicative) fragments [6] is not hard to see. If we remove the multiplicative connectives out of LBI, the result is the intuitionistic logic. Similarly, the removal of the additive connectives reduces BI into the multiplicative intuitionistic linear logic.

As is well known, however, BI encodes into neither of the logics. There emerges in fact somewhat a curious phenomenon around the interactions between additives and multiplicatives for the logic as a whole. From the viewpoint of backward theorem proving, one exemplar derives from contraction, and the issue is characterised in the following problem statement: "Given a LBI sequent $\Gamma(F, G) \vdash H$, describe circumstances in which Ctr_3 must be applied instead of Ctr_1 or Ctr_2".

$$\frac{\Gamma((F; F), G) \vdash H}{\Gamma(F, G) \vdash H} Ctr_1 \quad \frac{\Gamma(F, (G; G)) \vdash H}{\Gamma(F, G) \vdash H} Ctr_2$$

$$\frac{\Gamma((F, G); (F, G)) \vdash H}{\Gamma(F, G) \vdash H} Ctr_3$$

Interestingly, BI contraction, though certainly must connect a duplicating structure with a ";" on the premise sequent, does not actually impose any restriction on what it could be. That is, not only a structure in the form "$\Delta_1; \Delta_2$", but also that in the form "Δ_1, Δ_2" possibly duplicates. Is it possible to ascertain that Ctr_3 is always admissible, or should there be any situations where it must, as the statement seems to insinuate, take place? And if it is not admissible, what exactly is demanding the presence of Ctr_3? To endeavour contraction restriction in sequent calculus, one must first answer these questions.

Another issue in backward BI theorem proving is the equivalence of BI structures: $\Gamma, \varnothing \equiv \Gamma \equiv \Gamma; \emptyset$ (where "$\emptyset$" denotes the additive structural unit and "$\varnothing$" the multiplicative structural unit), which is by nature bidirectional, and by abusing the use of which a non-terminating LBI derivation results.

It is worth mentioning that a resolution of these two problems could effortlessly extend to BI decidability proof within sequent calculus purely syntactically, and hence in a manner perspicuous to all, as cut elimination in LBI is known to hold [1], [7]. But these problems are rather obnoxious ones, in particular the first problem; virtually no sensible analysis is known to date around the behaviour of contraction within BI sequent calculus. Currently known treatment is conservative enough to deal with *a symptom rather than a cause* as is nicely put by O'Hearn [5]. The established practice is in fact to *not face* it, compiling away the complexity of contraction in semantics [4]. While it is largely thanks to this judicious swerving that we are aware of the indication of BI decidability [4], its practical merit has been less significant than the theoretical implication itself, as attested by the long absence of automated BI decision procedures.

The present work we deliver here is intended to be a pathway for resolution of the dilemma through a study of the *cause*. It lays down a foundation of analysis by scrutinizing the effect of structural rules upon logical rules. Out of the

[1]We only consider propositional logics in this work

$$\dfrac{}{F \vdash F}\ \text{id} \qquad \dfrac{\Delta \vdash G \qquad \Gamma(G) \vdash H}{\Gamma(\Delta) \vdash H}\ \text{Cut} \qquad \dfrac{}{\Gamma(\bot) \vdash H}\ \bot L \qquad \dfrac{}{\Gamma \vdash \top}\ \top R$$

$$\dfrac{\Gamma(\emptyset) \vdash H}{\Gamma(\top) \vdash H}\ \top L \qquad \dfrac{\Gamma(\varnothing) \vdash H}{\Gamma(\top^*) \vdash H}\ \top^* L \qquad \dfrac{}{\varnothing \vdash \top^*}\ \top^* R \qquad \dfrac{\Gamma(F; G) \vdash H}{\Gamma(F \wedge G) \vdash H}\ \wedge L$$

$$\dfrac{\Gamma(F) \vdash H \qquad \Gamma(G) \vdash H}{\Gamma(F \vee G) \vdash H}\ \vee L \qquad \dfrac{\Delta \vdash F \qquad \Gamma(\Delta; G) \vdash H}{\Gamma(\Delta; F \to G) \vdash H} \to L \qquad \dfrac{\Gamma(F, G) \vdash H}{\Gamma(F * G) \vdash H}\ *L \qquad \dfrac{\Delta \vdash F \qquad \Gamma(G) \vdash H}{\Gamma(\Delta, F \mathrel{-\!\!*} G) \vdash H}\ \mathrel{-\!\!*} L$$

$$\dfrac{\Gamma \vdash F \qquad \Gamma \vdash G}{\Gamma \vdash F \wedge G}\ \wedge R \qquad \dfrac{\Gamma \vdash F_i}{\Gamma \vdash F_1 \vee F_2}\ \vee R \qquad \dfrac{\Gamma; F \vdash G}{\Gamma \vdash F \to G} \to R \qquad \dfrac{\Gamma \vdash F \qquad \Delta \vdash G}{\Gamma, \Delta \vdash F * G}\ *R$$

$$\dfrac{\Gamma, F \vdash G}{\Gamma \vdash F \mathrel{-\!\!*} G}\ \mathrel{-\!\!*} R \qquad \dfrac{\Gamma(\Delta) \vdash H}{\Gamma(\Delta; \Delta') \vdash H}\ \text{Wk L} \qquad \dfrac{\Gamma(\Delta; \Delta) \vdash H}{\Gamma(\Delta) \vdash H}\ \text{Ctr L} \qquad \dfrac{\Gamma(\Delta; \emptyset) \vdash H}{\Gamma(\Delta) \vdash H}\ EqAnt_1$$

$$\dfrac{\Gamma(\Delta, \varnothing) \vdash H}{\Gamma(\Delta) \vdash H}\ EqAnt_2$$

Fig. 1: LBI: a BI sequent calculus. Associativity and commutativity of ";" and "," are assumed. Inference rules with a double line are bidirectional.

said two sources of non-termination in LBI backward theorem proving, the bidirectionality of the structural equivalence is resolved altogether. The common perception about it is that the presence of the two structural units poses a non-trivial issue [1], [2]. We find, however, that by a close examination of the role of each connective and each LBI inference rule it becomes fairly straightforward to solve the puzzle.

For the other source of non-termination arising from contraction, we make the following discovery: contraction on "Δ_1, Δ_2" is required only for multiplicative implications - a hitherto unknown result which implies that contraction interferes with multiplicative structures in a limited fashion. The consequence is a new contraction-free BI sequent calculus LBI3 in which no trace of contraction is found other than in the two BI implications. The greater locality that LBI3 offers in fact allows us to embark on the last step of contraction analysis, namely implicit contraction elimination, succeeding which the purely syntactical BI decidability proof follows. As a promising guide towards the ultimate goal, we examine a well-known technique [3] in the intuitionistic logic and investigate how it may adapt to LBI3. The result is its extension to multiplicative connectives: "$\top^*$", *i.e.* the multiplicative top, and "$*$", *i.e.* the multiplicative conjunction.

We close the present introductory note with an overview of the remaining sections. Section II presents technical preliminaries of LBI. Section III is dedicated to LBI3 and its main properties which include admissibility of weakening, of bidirectionality of the rules for the structural equivalence, and of contraction, along with the equivalence of LBI3 to LBI. In the same section is found a discussion on the consequence of weakening absorption. Section IV then meditates implicit contraction elimination through a study of a renowned technique for the intuitionistic logic [3], and successfully extends the framework to multiplicative connectives. Section V concludes.

II. Preliminaries

A BI formula $F(, G, H)$ is defined as:

$F := p \mid \top \mid \bot \mid \top^* \mid F \wedge F \mid F \vee F \mid F {\to} F \mid F * F \mid F \mathrel{-\!\!*} F$
where "p" denotes a propositional variable; "$\top$" the additive top unit; "$\bot$" the additive bottom unit; "$\top^*$" the multiplicative top unit; "$\wedge$", "$\vee$", "$\to$" additive (logical) connectives; "$*$" and "$\mathrel{-\!\!*}$" multiplicative (logical) connectives; "$F \wedge F$", "$F \vee F$" and "$F {\to} F$" additive formulas; "$F * F$" and "$F \mathrel{-\!\!*} F$" multiplicative formulas. A BI single structure α is defined as:

$\alpha := F \mid \emptyset \mid \varnothing$
where "$\emptyset$" (resp. "$\varnothing$") denotes a unary additive (resp. multiplicative) structural connective which acts as a proxy for "$\top$" (resp. for "$\top^*$"). A BI structure $\Gamma(, \Delta, Re)$, which is commonly referred to a bunch [6], is then defined as:

$\Gamma := \alpha \mid \Gamma; \Gamma \mid \Gamma, \Gamma$
where ";" denotes a binary additive structural connective as a proxy for "$\wedge$", and "," a binary multiplicative structural connective as a proxy for "$*$". The three connectives "$\wedge$", "$\vee$", and "$*$" bind the strongest, followed by "$\to$" and "$\mathrel{-\!\!*}$", followed by ";" and ",". LBI [7], the first BI sequent calculus, is shown in Figure 1.

The structural connectives on the antecedent part of sequents can nest and form clusters of additive/multiplicative structures, and it is usually a tree that is employed to represent the antecedent structure. The notation "$\Gamma(\Delta)$" is used to specify which part of a BI antecedent structure is currently being accessed by an inference rule, stating that Δ occurs as a substructure of $\Gamma(\Delta)$. That is, $\Gamma(-)$ represents a BI structure with a "hole" which is filled with Δ as in $\Gamma(\Delta)$. Equivalence of two antecedent structures is identified through coherent equivalence [6].

Definition 1 (BI Coherent equivalence [6])**.** *"$\equiv$" is the least equivalence relation on antecedent structures satisfying*

1) *(Additive commutativity)* $\Gamma_1; \Gamma_2 \equiv \Gamma_2; \Gamma_1$

$$\frac{}{\Gamma; p \vdash p}\ id \qquad \frac{}{\Gamma; \emptyset \vdash \top^*}\ \top^* R \qquad \frac{\Delta; F \to G \vdash F \qquad \Gamma(\Delta; G) \vdash H}{\Gamma(\Delta; F \to G) \vdash H}\ \to L$$

$$\frac{Re_1 \vdash F \qquad \Gamma((Re_2, G); (\Delta, (\Gamma_1; F \mathrel{-\!\!*} G))) \vdash H}{\Gamma(\Delta, (\Gamma_1; F \mathrel{-\!\!*} G)) \vdash H}\ \mathrel{-\!\!*} L_1 \qquad \frac{\Delta \vdash F \qquad \Gamma(G; (\Delta, (\Gamma_1; F \mathrel{-\!\!*} G))) \vdash H}{\Gamma(\Delta, (\Gamma_1; F \mathrel{-\!\!*} G)) \vdash H}\ \mathrel{-\!\!*} L_2$$

$$\frac{\emptyset \vdash F \qquad \Gamma((\Delta, G); (\Delta, (\Gamma_1; F \mathrel{-\!\!*} G)) \vdash H}{\Gamma(\Delta, (\Gamma_1; F \mathrel{-\!\!*} G)) \vdash H}\ \mathrel{-\!\!*} L_3 \qquad \frac{\emptyset \vdash F \qquad \Gamma(G; \Gamma_1; F \mathrel{-\!\!*} G) \vdash H}{\Gamma(\Gamma_1; F \mathrel{-\!\!*} G) \vdash H}\ \mathrel{-\!\!*} L_4$$

$$\frac{Re_1 \vdash F_1 \qquad Re_2 \vdash F_2}{\Delta \vdash F_1 * F_2}\ *R_1 \qquad \frac{\Delta \vdash F_1 \qquad \emptyset \vdash F_2}{\Delta \vdash F_1 * F_2}\ *R_2 \qquad \frac{\Gamma(\Delta) \vdash H}{\Gamma(\Delta, \emptyset) \vdash H}\ EqAnt_2'' \qquad \frac{\Gamma(\emptyset) \vdash H}{\Gamma(\emptyset; \emptyset) \vdash H}\ \emptyset L$$

Fig. 2: A set of LBI3 inference rules to replace corresponding LBI inference rules. The associativity and commutativity of binary connectives except for $\to$ and $\mathrel{-\!\!*}$ are assumed. A finite number of weakening ($WkL_{\texttt{LBI}}$) applications is internalised in id, $\top^* R$, $\mathrel{-\!\!*} L_{1,2,3,4}$ and $*R_1$. Contraction ($CtrL_{\texttt{LBI}}$) is internalised in $\to L$ and $\mathrel{-\!\!*} L_{1,2,3,4}$. $\Gamma \equiv \Gamma, \emptyset$ ($EqAnt_{2\,\texttt{LBI}}$) is internalised in $\mathrel{-\!\!*} L_{3,4}$ and $*R_2$. $\emptyset L$ is a specific instance of $WkL_{\texttt{LBI}}$. The Re_1/Re_2 pair appearing in $*R_1$ and $\mathrel{-\!\!*} L_1$ results from Δ through the internalised weakening.

2) *(Additive unit)* $\Gamma_1; \emptyset \equiv \Gamma_1$
3) *(Multiplicative commutativity)* $\Gamma_1, \Gamma_2 \equiv \Gamma_2, \Gamma_1$
4) *(Multiplicative unit)* $\Gamma_1, \emptyset \equiv \Gamma_1$
5) *(Congruence) If* $\Gamma_1 \equiv \Gamma_2$ *then* $\Gamma(\Gamma_1) \equiv \Gamma(\Gamma_2)$

The associativity and commutativity of each BI binary connective except for $\to$ and $\mathrel{-\!\!*}$ are assumed throughout the text.

While additive inference rules share antecedent structures, *e.g.* in $\vee L$ the same structure in the conclusion propagates onto both premises, multiplicative inference rules are context-free [9] or resource aware. A good example to illustrate this is $*R$: in the antecedent structure in the conclusion sequent, both Γ and Δ are viewed as resources for the inference rule and are split into the premises of the rule. In backward derivation, a decision has to be made when $*R$ applies as to how the antecedent resource is to split, which is referred to as the resource distribution problem. For the two structural rules, *i.e.* WkL and $CtrL$, a resemblance to the contraction/weakening in G1i [9] is easily observed in that they are valid only in the context of additive structures connected by ";". The rules that would result if it were replaced with "," are not generally valid. Cut is admissible in LBI [1], [7], and hence is dropped from farther consideration.

Alternating nested sequents

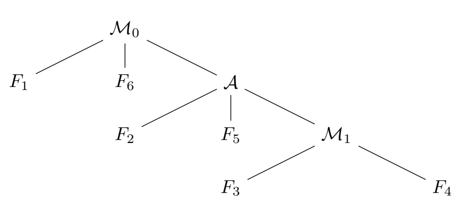

Fig. 3: $F_1, ((F_3, F_4); F_2; F_5), F_6$ as represented in alternating nested sequents.

Among many possible representations of a BI antecedent structure, it is one which we term alternating nested sequents that we base our reasoning on in the present work.

Definition 2 (A bunch in alternating nested sequents)**.** *An antecedent structure* Γ *in alternating nested sequents is defined as follows:*

$$\begin{array}{rcl} \Gamma & := & \alpha \mid \mathcal{M} \mid \mathcal{A} \\ \mathcal{M} & := & \alpha, \mathcal{M}' \mid \mathcal{A}, \mathcal{M}' \\ \mathcal{M}' & := & \alpha \mid \mathcal{A} \mid \alpha, \mathcal{M}' \mid \mathcal{A}, \mathcal{M}' \\ \mathcal{A} & := & \alpha; \mathcal{A}' \mid \mathcal{M}; \mathcal{A}' \\ \mathcal{A}' & := & \alpha \mid \mathcal{M} \mid \alpha; \mathcal{A}' \mid \mathcal{M}; \mathcal{A}' \end{array}$$

Each of the $\mathcal{A}$ *(resp.* $\mathcal{M}$*) substructures of* Γ *is termed an additive (resp. multiplicative) structural layer.*

An example of a BI structure in alternating nested sequents is shown in Figure 3 in which are found two multiplicative structural layers: "$F_1, F_6, \mathcal{A}$" and "F_3, F_4"; and one additive structural layer "$F_2; F_5; \mathcal{M}_1$", with $\mathcal{A}$ denoting "$F_2; F_5; \mathcal{M}_1$" and $\mathcal{M}_1$ denoting "F_3, F_4". For any structure in which two structural layers nest, the structural layer holding the other structural layer within is described as the outer structural layer of the two, while that enclosed in the other is described as the inner structural layer.

III. LBI3: a Contraction-Free BI Sequent Calculus

This section presents a new BI sequent calculus LBI3 in which no explicit contraction appears. Changes are made to the following LBI inference rules: id, $\top^* R$, $\to L$, $\mathrel{-\!\!*} L$, $*R$ and $EqAnt_2$, which are replaced with the corresponding inference rules shown in Figure 2. A remnant of $WkL_{\texttt{LBI}}$ is visible only in $\emptyset L$. $EqAnt_{1\,\texttt{LBI}}$ is absent.

Definition 3 (LBI3)**.** LBI3 *comprises the following inference rules*[2]*:*

Axioms: $\underline{id}$ $\underline{\top^* R}$ $\bot L$ $\top R$

[2] A correction to an error in the submitted version here: $\top L$ and $\top^* L$ have been included in "Other logical rules".

Other logical rules: $\wedge L \quad \wedge R \quad \vee L \quad \vee R \quad \underline{\to L} \quad \to R$
$*L \quad \underline{*R_1} \quad \underline{*R_2} \quad \underline{-\!\!*L_1} \quad \underline{-\!\!*L_2} \quad \underline{-\!\!*L_3} \quad \underline{-\!\!*L_4} \quad -\!\!*R$
$\top L \quad \top^* L$
Structural rules: $\underline{EqAnt_2''} \quad \underline{\emptyset L}$
each of which is identical to a corresponding inference rule in LBI *unless otherwise stated in Figure 2 (underlined in this definition for clarity).*

In the rest of Section III, we focus on proving the main properties of LBI3, *i.e.* admissibility of weakening, of $EqAnt_1$ and one direction of $EqAnt_2$, contraction admissibility, and its equivalence to LBI to conclude. Before we proceed any farther, however, it seems appropriate that the correspondence between Re_1/Re_2 and Δ in $*R_1$ and $-\!\!*L_1$ be elaborated.

A. Correspondence between Re_1/Re_2 and Δ

We define the said correspondence by binding $*R_1$ and $-\!\!*L_1$ to a corresponding LBI-derivation which in LBI3-space is implicit. The following notation of sequents' transition appears recurrently in the present work.

Definition 4 (Transitions of sequents)**.** *"$\rightsquigarrow$" is defined for two sequents D_1 and D_2 such that $D_1 \rightsquigarrow D_2$ is a one-step transition (in backward derivation) via an inference rule, satisfying 1) that D_2 is the premise sequent (or one of the premise sequents) of the inference rule and 2) that D_1 is the conclusion sequent of the same inference rule. Specialisation of the one-step transition from D_1 to D_2 is defined as $D_1 \rightsquigarrow_{\textbf{\textit{Inf}}} D_2$, explicitly stating that it is **Inf** which has effected the sequent transition. A transition from D_1 to D_2 in zero (*i.e. *no transition) or more applications of inference rule(s) is defined by $D_1 \rightsquigarrow^* D_2$.*

Definition 5 (Re_1/Re_2 in $*R_1/-\!\!*L_1$)**.** *In* LBI3*, correspondence of premise/conclusion sequents in $*R_1$ and $-\!\!*L_1$ are defined with respect to $*R$/$-\!\!*L$/WkL/$CtrL$ in* LBI*:*

For $*R_{1\,\text{LBI3}}$:

Suppose a sequent $D_1 : \Delta \vdash F$ as the conclusion sequent of the inference rule. Then the corresponding derivation of $*R_{1\,\text{LBI3}}$ within LBI *is defined to be*

- $D_1 \rightsquigarrow^*_{WkL_{\text{LBI}}} [D_1' : Re_1, Re_2 \vdash F * G]$
- $D_1' \rightsquigarrow_{*R_{\text{LBI}}} [D_2 : Re_1 \vdash F]$
- $D_1' \rightsquigarrow_{*R_{\text{LBI}}} [D_3 : Re_2 \vdash G]$

in which D_2 and D_3 correspond to the premise sequents of $*R_{1\,\text{LBI3}}$ (with D_1 as its conclusion sequent).

For $-\!\!*L_{1\,\text{LBI3}}$:

Suppose a sequent $D_1 : \Gamma(\Delta, (\Gamma_1; F-\!\!*G)) \vdash H$ as the conclusion sequent. Then the corresponding derivation of $-\!\!*L_{1\,\text{LBI3}}$ within LBI *is defined as below. "$\Gamma'(-)$" denotes "$\Gamma(-; (\Delta, (\Gamma_1; F-\!\!*G)))$" which is used for simplification.*

- $D_1 \rightsquigarrow_{CtrL_{\text{LBI}}} [D_1' : \Gamma'(\Delta, (\Gamma_1; F-\!\!*G)) \vdash H]$
- $D_1' \rightsquigarrow_{WkL_{\text{LBI}}} [D_1'' : \Gamma'(\Delta, F-\!\!*G) \vdash H]$
- $D_1'' \rightsquigarrow^*_{WkL_{\text{LBI}}} [D_1''' : \Gamma'(Re_1, Re_2, F-\!\!*G) \vdash H]$
- $D_1''' \rightsquigarrow_{-\!\!*L_{\text{LBI}}} [D_2 : Re_1 \vdash F]$
- $D_1''' \rightsquigarrow_{-\!\!*L_{\text{LBI}}} [D_3 : \Gamma'(Re_2, G) \vdash H]$

in which D_2 and D_3 correspond to the premise sequents of $-\!\!*L_{1\,\text{LBI3}}$ (with D_1 as its conclusion sequent).

In effect, these rules internalise general weakening (WkL_{LBI}) and, in case of $-\!\!*L_1$, also contraction ($CtrL_{\text{LBI}}$), which would be otherwise explicit in LBI. Since WkL_{LBI} is general and can extend its reach to several additive structural layers of the antecedent structure, there naturally are many Re_1/Re_2 pairs to result through the internalised weakening process (WkL_{LBI}).

The following proposition indicates that the use of weakening rules which only act for the outermost additive structural layer of Δ: WkL_1 for $*R_1$; $WkL_{1,2}'$ for $-\!\!*L_1$, is not always sufficient.

$$\frac{\Delta_1 \vdash H}{\Delta_1; \Delta_2 \vdash H}\ WkL_1$$

$$\frac{\Gamma(\Delta_1, F-\!\!*G) \vdash H}{\Gamma(\Delta_1, (\Gamma_1; F-\!\!*G)) \vdash H}\ WkL_1'$$

$$\frac{\Gamma(\Delta_1, F-\!\!*G) \vdash H}{\Gamma(\Delta_1; \Delta_2, F-\!\!*G) \vdash H}\ WkL_2'$$

Proposition 1. *There are sequents $D : \Gamma \vdash F$ which are derivable in* LBI3 *and* LBI *but not derivable in* LBI3*' in which the internalised weakening for $*R_1$ is restricted to WkL_1 and for $-\!\!*L_1$ to $WkL_{1,2}'$.*

Proof: With $p_0; (p_1, ((p_2, p_3); p_4)) \vdash (p_5 \to (p_1 * p_2)) * p_3$ for $*R_1$, and $p_1, ((p_2, p_3); p_5), (p_1 * p_2) -\!\!* (p_3 -\!\!* p_4) \vdash p_4$ for $-\!\!*L_1$. ■

To complete a picture, we mention comprehensively those rules that internalise LBI rules.

- LBI3 inference rules internalising
  WkL_{LBI}: $\quad id \quad \top^* R \quad *R_1 \quad -\!\!*L_{1,2,3,4}$
  $CtrL_{\text{LBI}}$: $\quad \to L \quad -\!\!*L_{1,2,3,4}$
  $EqAnt_{2\,\text{LBI}}$: $\quad -\!\!*L_{3,4} \quad *R_2$

Their derivations within LBI, apart from $*R_1$ and $-\!\!*L_{1,4}$, are straightforward. Only Γ_1 in the conclusion sequent is weakened away (in backward derivation) in $-\!\!*L_{2,3}$. For the remaining $-\!\!*L_4$, $[D : \Gamma(\Gamma_1; F-\!\!*G) \vdash H] \rightsquigarrow_{CtrL} [D' : \Gamma(F-\!\!*G; \Gamma_1; F-\!\!*G) \vdash H] \rightsquigarrow_{EqAnt_2} [D'' : \Gamma((\emptyset, F-\!\!*G); \Gamma_1; F-\!\!*G) \vdash H]$ to take place first internally, followed by $-\!\!*L$.

B. Inversion lemma and weakening admissibility

In this subsection, we first of all introduce LBI3 inversion lemma which is crucial in simplification of the subsequent discussion.

Definition 6 (Derivation depth)**.** *Given a derivation of a* LBI3 *sequent D, denoted as $\Pi(D)$, its derivation depth* `der_depth`$(\Pi(D))$ *is defined inductively as follows:*

- *if D is the conclusion sequent of an axiom, then* `der_depth`$(\Pi(D)) = 1$.
- *if D is the conclusion sequent of a single-premised inference rule whose premise sequent is D_1, then* `der_depth`$(\Pi(D)) =$ `der_depth`$(\Pi(D_1)) + 1$.

- *if D is the conclusion sequent of a two-premised inference rule whose premise sequents are D_1 and D_2, then* $\texttt{der\_depth}(\Pi(D)) = max(\texttt{der\_depth}(\Pi(D_1)), \texttt{der\_depth}(\Pi(D_2))) + 1$.

1) `LBI3` *inversion lemma:*

Lemma 1 (Inversion lemma for LBI3)**.** *For the following sequent pairs, if the sequent shown on the left is* `LBI3`*-derivable at most with the derivation depth of k, then so is (are) the sequent(s) shown on the right.*

$$\begin{array}{rl}
\Gamma(F \wedge G) \vdash H, & \Gamma(F; G) \vdash H \\
\Gamma(F_1 \vee F_2) \vdash H, & \textit{both } \Gamma(F_1) \vdash H \textit{ and } \Gamma(F_2) \vdash H \\
\Gamma(\Gamma_1; F \to G) \vdash H, & \Gamma(\Gamma_1; G) \vdash H \\
\Gamma(F * G) \vdash H, & \Gamma(F, G) \vdash H \\
\Gamma(\top) \vdash H, & \Gamma(\emptyset) \vdash H \\
\Gamma(\top^*) \vdash H, & \Gamma(\varnothing) \vdash H \\
\Gamma(\varnothing; \varnothing) \vdash H, & \Gamma(\varnothing) \vdash H \\
\Gamma(\Delta, \varnothing) \vdash H, & \Gamma(\Delta) \vdash H \\
\Gamma \vdash F \wedge G, & \textit{both } \Gamma \vdash F \textit{ and } \Gamma \vdash G \\
\Gamma \vdash F \to G, & \Gamma; F \vdash G \\
\Gamma \vdash F \mathrel{-\!\!*} G, & \Gamma, F \vdash G
\end{array}$$

Proof: By induction on k. Details are in Appendix A. ■

2) Weakening admissibility:

We are now ready to prove the admissibility of weakening in `LBI3`, depth-preserving admissibility as is customary.

Proposition 2 (LBI3 weakening admissibility)**.** *If a sequent $D : \Gamma(\Delta) \vdash F$ is* `LBI3`*-derivable, then so is $D' : \Gamma(\Delta; \Delta') \vdash F$, preserving the derivation depth.*

Proof: By induction on the derivation depth of $\Pi(D)$, $\Pi(D)$ denoting a derivation of D. If it is one, *i.e.* D is the conclusion sequent of an axiom, then so is D'. For inductive cases, assume that the current proposition holds for all the derivations of depth up to k. It must be now demonstrated that it still holds for derivations of depth $k+1$. Consider what the last inference rule is in $\Pi(D)$.

1) $\to L$: $\Pi(D)$ looks like:

$$\frac{\begin{array}{c}\vdots \\ \Delta; F \to G \vdash F\end{array} \qquad \begin{array}{c}\vdots \\ \Gamma(\Delta; G) \vdash H\end{array}}{\Gamma(\Delta; F \to G) \vdash H} \to L$$

By induction hypothesis on both of the premises, $\Delta; F \to G; \Delta' \vdash F$ and $\Gamma(\Delta; G; \Delta') \vdash H$. Then, $\Gamma(\Delta; F \to G; \Delta') \vdash H$ via $\to L$.

2) $\mathrel{-\!\!*} L_1$: $\Pi(D)$ looks like:

$$\frac{\begin{array}{c}\vdots \\ Re_1 \vdash F\end{array} \qquad \begin{array}{c}\vdots \\ \Gamma((Re_2, G); (\Delta, (\Gamma_1; F \mathrel{-\!\!*} G))) \vdash H\end{array}}{\Gamma(\Delta, (\Gamma_1; F \mathrel{-\!\!*} G)) \vdash H} \mathrel{-\!\!*} L_1$$

Note, for this case and also for $*R_1$, we must additionally cater for the internalised weakening. Suppose that Δ' results from Δ through a sequence of induction hypothesis applications at additive structural layers in Δ, then by induction hypothesis twice more on the right premise, $\Gamma((Re_2, G); (\Delta', (\Gamma_1; F \mathrel{-\!\!*} G; \Delta_2)); \Delta_1) \vdash H$. Then $\Gamma((\Delta', (\Gamma_1; F \mathrel{-\!\!*} G; \Delta_2)); \Delta_1) \vdash H$ via $\mathrel{-\!\!*} L_1$.

3) Other cases are similar or trivial. For $\varnothing L$, the inversion lemma (Lemma 1) on the conclusion sequent in the end. ■

C. Admissibility of bidirectional $EqAnt_{1,2}$ in `LBI3`

Replacement of `LBI` inference rules with those stated in Figure 2 admits $EqAnt_{1\,\texttt{LBI}}$ altogether and also $EqAnt_2'$:

$$\frac{\Gamma(\Delta, \varnothing) \vdash F}{\Gamma(\Delta) \vdash F} EqAnt_2'$$

Proposition 3 (Admissibility of $EqAnt_{1,2}$ bidirectionality)**.** *$EqAnt_{1\,\texttt{LBI}}$ and $EqAnt_2'$ are admissible in* `LBI3`*, preserving the derivation depth.*

Proof: We divide $EqAnt_{1\,\texttt{LBI}}$ into two rules, one for each direction:

$$\frac{\Gamma(\Delta; \emptyset) \vdash F}{\Gamma(\Delta) \vdash F} EqAnt_1' \quad \frac{\Gamma(\Delta) \vdash F}{\Gamma(\Delta; \emptyset) \vdash F} EqAnt_1''$$

Proof is by induction on the derivation depth routinely. Note the internalisation of $EqAnt_2'$ in $\mathrel{-\!\!*} L_{3,4}$ and $*R_2$ (*c.f.* subsection A in Section III). Proposition 2 for $EqAnt_1''$. ■

D. Preparation for contraction admissibility in $*R_1$/$\mathrel{-\!\!*} L_1$

We dedicate one subsection here to fortify ourselves with a further observation about the generation process of Re_1/Re_2, preparing for the main proof of contraction admissibility.

1) Maximal Re_1/Re_2:

Having arrived at Proposition 2 and Proposition 3, an observation can be made concerning the internalised weakening ($WkL_{\texttt{LBI}}$) within $*R_1$ and $\mathrel{-\!\!*} L_1$ that there is no need to consider an arbitrary $WkL_{\texttt{LBI}}$ application in the process.

Lemma 2 (Sufficiency of incremental weakening)**.** *In an application of $*R_1$ (in backward derivation) on a* `LBI3`*-derivable sequent $D : \Delta \vdash F * G$, if there is a* `LBI3`*-derivable pair of D_1 and D_2 such that $D \rightsquigarrow_{*R_1} D_1$ and $D \rightsquigarrow_{*R_1} D_2$, then there exists a* `LBI3`*-derivable pair of D_1' and D_2' such that $D \rightsquigarrow_{*R_1'} D_1'$ and $D \rightsquigarrow_{*R_1'} D_2'$ where $*R_1'$ is defined here to be $*R$ except that its internalised weakening is carried out only with WkL_1 and WkL_2 as stated below instead of the general weakening ($WkL_{\texttt{LBI}}$):*

$$\frac{\Delta_1 \vdash F}{\Delta_1; \Delta_2 \vdash F} WkL_1 \quad \frac{\Delta_1, \Delta_2 \vdash F}{\Delta_1, (\Delta_2; \Delta_3) \vdash F} WkL_2$$

Similarly, in an application of $\mathrel{-\!\!*} L_1$ (in backward derivation) on a sequent $D : \Gamma(\Delta, (\Gamma_1; F \mathrel{-\!\!*} G)) \vdash H$, it suffices to apply the following restricted weakening rules in the internalised weakening process:

$$\frac{\Gamma(\Delta_1, F \mathrel{-\!\!*} G) \vdash F}{\Gamma(\Delta_1, (\Gamma_1; F \mathrel{-\!\!*} G)) \vdash F} WkL_1'$$

$$\frac{\Gamma(\Delta_1, F \mathrel{-\!\!*} G) \vdash F}{\Gamma((\Delta_1; \Delta_2), F \mathrel{-\!\!*} G) \vdash F} \; WkL'_2$$

$$\frac{\Gamma(\Delta_1, \Delta_2, F \mathrel{-\!\!*} G) \vdash F}{\Gamma(\Delta_1, (\Delta_2; \Delta_3), F \mathrel{-\!\!*} G) \vdash F} \; WkL'_3$$

Proof: Found in Appendix B. ■

Corollary 1 (Maximal Re_1/Re_2). *For a* LBI3*-derivable sequent $D : \Delta \vdash F * G$, if there exists a pair of* LBI3*-derivable sequents $D'_1 : Re'_1 \vdash F$ and $D'_2 : Re'_2 \vdash G$ such that $D \rightsquigarrow_{*R_1} D'_1$ and $D \rightsquigarrow_{*R_1} D'_2$, then there exists a pair of* LBI3*-derivable sequents $D_1 : Re_1 \vdash F$ and $D_2 : Re_2 \vdash G$ such that all the following conditions satisfy.*

- *$D \rightsquigarrow_{*R_1} D_1$ (resp. $D \rightsquigarrow_{*R_1} D_2$).*
- *D_1 (resp. D_2) is a sequent that can be derived by weakening D'_1 (resp. D'_2)* (c.f. *Proposition 2).*[3]
- *there exists no $D^*_1 : Re^*_1 \vdash F$ (resp. $D^*_2 : Re^*_2 \vdash G$) such that all the following conditions satisfy.*
  - *D^*_1 (resp. D^*_2) is a sequent that can be derived by weakening D_1 (resp. D_2).*
  - *$D^*_1 \not\equiv D_1$ (resp. $D^*_2 \not\equiv D_2$).*
  - *$D \rightsquigarrow_{*R_1} D^*_1$ (resp. $D \rightsquigarrow_{*R_1} D^*_2$).*

Such a Re_1/Re_2 pair is called a maximal Re_1/Re_2 pair. Likewise, with an abbreviation $\Gamma'(-)$ denoting $\Gamma(-; (\Delta, (\Gamma_1; F \mathrel{-\!\!*} G)))$, for a LBI3*-derivable sequent $D : \Gamma(\Delta, (\Gamma_1; F \mathrel{-\!\!*} G)) \vdash H$, if there exists a pair of* LBI3*-derivable sequents $D'_1 : Re'_1 \vdash F$ and $D'_2 : \Gamma'(Re'_2, G) \vdash H$ such that $D \rightsquigarrow_{-*L_1} D'_1$ and $D \rightsquigarrow_{-*L_1} D'_2$, then there exists a pair of* LBI3*-derivable sequents $D_1 : Re_1 \vdash F$ and $D_2 : \Gamma'(Re_2, G) \vdash H$ such that the following conditions all satisfy.*

- *$D \rightsquigarrow_{-*L_1} D_1$ (resp. $D \rightsquigarrow_{-*L_1} D_2$).*
- *D_1 (resp. D_2) is a sequent that can be derived by weakening D'_1 (resp. D'_2)* (c.f. *Proposition 2).*
- *there exists no $D^*_1 : Re^*_1 \vdash F$ (resp. $D^*_2 : \Gamma'(Re^*_2, G) \vdash H$) such that the following conditions all satisfy.*
  - *D^*_1 (resp. D^*_2) is a sequent that can be derived by weakening D_1 (resp. D_2).*
  - *$D^*_1 \not\equiv D_1$ (resp. $D^*_2 \not\equiv D_2$).*
  - *$D \rightsquigarrow_{-*L_1} D^*_1$ (resp. $D \rightsquigarrow_{-*L_1} D^*_2$).*

The reason why, in the presence of Re_1/Re_2 in $*R_1$ or $-\!\!*L_1$, neither $*R_2$, $-\!\!*L_2$ nor $-\!\!*L_3$ exhibits a similar notation is inferrable from Corollary 1. Hereafter, we assume only some maximal Re_1/Re_2 pair for $*R_1$ and $-\!\!*L_1$.

E. Admissibility of contraction in LBI3

Contraction admissibility in LBI3 follows.

Theorem 1 (Contraction admissibility in LBI3). *If $D : \Gamma(\Delta; \Delta) \vdash F$ is* LBI3*-derivable, then so is $D' : \Gamma(\Delta) \vdash F$, preserving the derivation depth.*

[3]An application of Proposition 2 on a LBI3-derivable sequent D is reflexive if it only introduces "$\emptyset$"s on D due to Proposition 3.

Proof: By induction on der_depth($\Pi(D)$) where $\Pi(D)$ denotes a LBI3 derivation of D. Appendix C for details. The base cases are when it is 1. Trivial if the last rule applied is $\top R$ or $\bot L$. For both id and $\top^* R$, $\Pi(D)$ looks like:

$$\frac{}{\Gamma; \alpha \vdash F}$$

where (α, F) is (p, p) for id, and $(\emptyset, \top^*)$ for $\top^* R$. As the antecedent is either a single structure or an additive structural layer, irrespective of where Δ in D is, if D is LBI3-derivable, then so is D'.

For inductive cases, consider what the LBI3 inference rule applied last is, and, in case of a left inference rule, consider where the active structure Γ_1 of the inference rule is in $\Gamma(\Delta; \Delta)$.

1) $\wedge L$, and Γ_1 is $F_1 \wedge F_2$: if it does not appear in Δ, induction hypothesis on the premise sequent concludes. Otherwise, $\Pi(D)$ looks like:

$$\frac{\begin{matrix}\vdots \\ D_1 : \Gamma(\Delta'(F_1; F_2); \Delta'(F_1 \wedge F_2)) \vdash H\end{matrix}}{D : \Gamma(\Delta'(F_1 \wedge F_2); \Delta'(F_1 \wedge F_2)) \vdash H} \; \wedge L$$

$D'_1 : \Gamma(\Delta'(F_1; F_2); \Delta'(F_1; F_2)) \vdash H$ is LBI3-derivable (inversion lemma); $D''_1 : \Gamma(\Delta'(F_1; F_2)) \vdash H$ is also LBI3-derivable (induction hypothesis); then a forward (as opposed to backward; assumed such for the rest of the cases) application of $\wedge L$ on D''_1 concludes.

2) $\emptyset L$, and Γ_1 is $\emptyset; \emptyset$: if it does not appear in Δ, the induction hypothesis on the premise sequent concludes. Otherwise, if it is entirely in Δ, then $\Pi(D)$ looks like:

$$\frac{\begin{matrix}\vdots \\ D_1 : \Gamma(\Delta'(\emptyset); \Delta'(\emptyset; \emptyset)) \vdash H\end{matrix}}{D : \Gamma(\Delta'(\emptyset; \emptyset); \Delta'(\emptyset; \emptyset)) \vdash H} \; \emptyset L$$

$D'_1 : \Gamma(\Delta'(\emptyset); \Delta'(\emptyset)) \vdash H$ is LBI3-derivable (inversion lemma); so is $D''_1 : \Gamma(\Delta'(\emptyset)) \vdash H$ (induction hypothesis); then $\emptyset L$ on D''_1 concludes. If only one of the $\emptyset$s appears in Δ, then $\Pi(D)$ looks like:

$$\frac{\begin{matrix}\vdots \\ D_1 : \Gamma'(\emptyset; \Delta'; \emptyset; \Delta') \vdash H\end{matrix}}{D : \Gamma'(\emptyset; \emptyset; \Delta'; \emptyset; \Delta') \vdash H} \; \emptyset L$$

By induction hypothesis on D_1 twice, $D'_1 : \Gamma'(\emptyset; \Delta') \vdash H$ upon which $\emptyset L$ applies for a conclusion.

3) $\rightarrow L$, and Γ_1 is $\Gamma'; F \rightarrow G$: if it does not appear in Δ, then the induction hypothesis on both of the premises concludes. If it is entirely in Δ, then $\Pi(D)$ looks either like:

$$\frac{\begin{matrix}\vdots \\ D_1 : \Gamma'; F \rightarrow G \vdash F\end{matrix} \qquad \begin{matrix}\vdots \\ D_2\end{matrix}}{D : \Gamma(\Delta'(\Gamma'; F \rightarrow G); \Delta'(\Gamma'; F \rightarrow G)) \vdash H} \; \rightarrow L$$

where $D_2 : \Gamma(\Delta'(\Gamma'; G); \Delta'(\Gamma'; F \rightarrow G)) \vdash H$, or, in case Δ is $\Delta'; \Gamma_1; F \rightarrow G$, like:

$$\frac{\begin{matrix}\vdots \\ D_1 : \Delta'; \Gamma'; F \rightarrow G; \Delta'; \Gamma'; F \rightarrow G \vdash F\end{matrix} \qquad \begin{matrix}\vdots \\ D_2\end{matrix}}{D : \Gamma(\Delta'; \Gamma'; F \rightarrow G; \Delta'; \Gamma'; F \rightarrow G) \vdash H} \; \rightarrow L$$

where $D_2 : \Gamma(\Delta'; \Gamma'; G; \Delta'; \Gamma'; F \to G) \vdash H$. In the former, $D_2' : \Gamma(\Delta'(\Gamma'; G); \Delta'(\Gamma'; G)) \vdash H$ (inversion lemma); $D_2'' : \Gamma(\Delta'(\Gamma'; G)) \vdash H$ (induction hypothesis); then $\to L$ on D_1 and D_2'' concludes. In the latter, induction hypothesis on D_1, meanwhile the inversion lemma and the induction hypothesis on D_2; then via $\to L$ to conclude. Finally, if only a substructure of Γ_1 is in Δ with the rest spilling out of Δ, then similar to the latter case with appropriate applications of the induction hypothesis.

4) $*R_1$: $\Pi(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots \\ D_1 : Re_1 \vdash F_1\end{matrix} \qquad \begin{matrix}\vdots \\ D_2 : Re_2 \vdash F_2\end{matrix}}{D : \Gamma(\Delta; \Delta) \vdash F_1 * F_2} *R_1$$

We show that the internalised weakening process to generate a maximal Re_1/Re_2 pair must either weaken away one Δ completely or preserve $\Delta; \Delta$ as a substructure of Re_1[4], and that no other possibilities exist. But due to the formulation of a maximal Re_1/Re_2 pair (*c.f.* Corollary 1), such must be the case. If $\Delta; \Delta$ is preserved in Re_1, then induction hypothesis on D_1 concludes; otherwise, it is trivial to see that only a single Δ needs to be present in D.

5) $-\!*L_1$, and Γ_1 is $\Delta_1, (\Gamma'; F -\!* G)$: if Γ_1 is not in Δ, then induction hypothesis on the right premise sequent concludes. If it is in Δ, $\Pi(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots \\ D_1 : Re_1 \vdash F\end{matrix} \qquad \begin{matrix}\vdots \\ D_2\end{matrix}}{D : \Gamma(\Delta'(\Delta_1, (\Gamma'; F -\!* G)); \Delta'(\Delta_1, (\Gamma'; F -\!* G))) \vdash H} -\!*L_1$$

where D_2 is: $\Gamma(\Delta'((Re_2, G); (\Delta_1, (\Gamma'; F -\!* G))); \Delta'(\Delta_1, (\Gamma'; F -\!* G))) \vdash H$. $D_2' : \Gamma(\Delta'((Re_2, G); (\Delta_1, (\Gamma'; F -\!* G))); \Delta'((Re_2, G); (\Delta_1, (\Gamma'; F -\!* G)))) \vdash H$ due to Proposition 2 is also LBI3-derivable. $D_2'' : \Gamma(\Delta'((Re_2, G); (\Delta_1, (\Gamma'; F -\!* G)))) \vdash H$ via induction hypothesis. Then $-\!*L_1$ on D_1 and D_2'' concludes. If, on the other hand, Δ is in Γ_1, then it is either in Γ' or in Δ_1. But if it is in Γ', then it must be weakened away, and if it is in Δ_1, similar to the $*R_1$ case.

6) The rest: similar.

■

Hence, no structural contraction is required other than for $-\!*L_{1,2,3}$.

Proposition 4 (Non-admissible structural contraction)**.** *There exist sequents $\Gamma \vdash F$ which are* LBI*-derivable but not* LBI*-derivable without structural contraction.*

Proof: With a sequent $p_5; (\top -\!* p_1, \top -\!* (p_1 \to p_2), p_4) \vdash p_2$. Details are in Appendix D. ■

F. Equivalence of LBI3 *to* LBI

The following equivalence theorem of LBI3 to LBI concludes Section III.

[4] Re_1 or Re_2 if associativity and commutativity of "," is not assumed.

Theorem 2 (Equivalence between LBI3 and LBI)**.** *$D : \Gamma \vdash F$ is* LBI3*-derivable if and only if it is* LBI*-derivable.*

Proof: Details are found in Appendix E. Into the *only if* direction, we mention that each LBI3 inference rule is derivable in LBI. Into the *if* direction, assume that D is LBI-derivable, and then show that there is a corresponding LBI3-derivation to each LBI derivation by induction on the derivation depth of $\Pi_{\text{LBI}}(D)$ ($\Pi_{\text{LBI}}(D)$ denotes a LBI-derivation of D).

If it is 1, *i.e.* if D is the conclusion sequent of an axiom, then $\bot L_{\text{LBI}}$ is identical to $\bot L_{\text{LBI3}}$; id_{LBI} and resp. $\top^* R_{\text{LBI}}$ via id_{LBI3} and resp. $\top^* R_{\text{LBI3}}$ with Proposition 2; and $\top R_{\text{LBI}}$ identical to $\top R_{\text{LBI3}}$. For inductive cases, consider what the LBI rule applied last is:

1) $\to L_{\text{LBI}}$: $\Pi_{\text{LBI}}(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots \\ D_1 : \Delta \vdash F\end{matrix} \qquad \begin{matrix}\vdots \\ D_2 : \Gamma(\Delta; G) \vdash H\end{matrix}}{D : \Gamma(\Delta; F \to G) \vdash H} \to L_{\text{LBI}}$$

By induction hypothesis, both D_1 and D_2 are also LBI3-derivable. Proposition 2 on D_1 in LBI3-space results in $D_1' : \Delta; F \to G \vdash F$. Then an application of $\to L_{\text{LBI3}}$ on D_1' and D_2 concludes in LBI3-space.

2) $-\!*L_{\text{LBI}}$: $\Pi_{\text{LBI}}(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots \\ D_1 : \Delta \vdash F\end{matrix} \qquad \begin{matrix}\vdots \\ D_2 : \Gamma(G) \vdash H\end{matrix}}{D : \Gamma(\Delta, F -\!* G) \vdash H} -\!*L_{\text{LBI}}$$

By induction hypothesis, D_1 and D_2 are also LBI3-derivable. We consider (1) if Δ is $\emptyset$ and (2) what $\Gamma(-)$ is which surrounds G, to conclude.

3) WkL_{LBI}: Proposition 2.
4) $CtrL_{\text{LBI}}$: Theorem 1.
5) $EqAnt_{1\,\text{LBI}}$: Proposition 3.
6) $EqAnt_{2\,\text{LBI}}$: Proposition 3 and $EqAnt''_{2\,\text{LBI3}}$.
7) Others: straightforward.

■

IV. Towards Truly Contraction-Free BI Sequent Calculus - A Preliminary Investigation

In this section, an improvement over LBI3 is pondered upon based on a well-known implicit contraction elimination in the standard intuitionistic logic [3]. LBI3 is contraction-free in a sense that an explicit structural rule of contraction does not appear within, but the termination property of the calculus is not immediately apparent. This is due to, though only implicit, contraction that is internalised in the two implications. We pave a way towards the complete BI decidability analysis within sequent calculus by demonstrating an extension of the said technique to multplicative connectives "$\top^*$" and "$*$". The following sub-calculus LBI3_1 is specifically examined.

Definition 7 (LBI3_1)**.** LBI3_1 *comprises the following inference rules:*

Axioms: id_{LBI3} $\top^* R_{\text{LBI3}}$ $\bot L_{\text{LBI3}}$ $\top R_{\text{LBI3}}$

Other logical rules: $\wedge L_{\text{LBI3}}$ $\wedge R_{\text{LBI3}}$ $\vee L_{\text{LBI3}}$ $\vee R_{\text{LBI3}}$ $\to L_{\text{LBI3}}$ $\to R_{\text{LBI3}}$ $* L_{\text{LBI3}}$ $* R_{1,2\,\text{LBI3}}$ $\top L$ $\top^* L$

$$\frac{\Gamma(p; G) \vdash H}{\Gamma(p; p{\to}G) \vdash H} \to L_p \qquad \frac{\Gamma(\emptyset; G) \vdash H}{\Gamma(\emptyset; \top^*{\to}G) \vdash H} \to L_{\top^*}$$

$$\frac{\Gamma(F_1{\to}G; F_2{\to}G) \vdash H}{\Gamma((F_1 \vee F_2){\to}G) \vdash H} \to L_\vee \qquad \frac{\Gamma(F_1{\to}(F_2{\to}G)) \vdash H}{\Gamma((F_1 \wedge F_2){\to}G) \vdash H} \to L_\wedge$$

$$\frac{\Delta; F_2{\to}G \vdash F_1 \to F_2 \qquad \Gamma(\Delta; G) \vdash H}{\Gamma(\Delta; (F_1{\to}F_2){\to}G) \vdash H} \to L_\to \qquad \frac{\Delta \vdash F_1^\star * F_2^\star \qquad \Gamma(\Delta; G) \vdash H}{\Gamma(\Delta; (F_1^\star * F_2^\star) \to G) \vdash H} \to L_{*1}$$

$$\frac{\Delta; (F_1 * F_2){\to}G \vdash F_1 * F_2 \qquad \Gamma(\Delta; G) \vdash H}{\Gamma(\Delta; (F_1 * F_2){\to}G) \vdash H} \to L_{*2} \qquad \frac{\Gamma(G) \vdash H}{\Gamma(\top{\to}G) \vdash H} \to L_\top$$

Restriction: $\emptyset \nvdash F^\star$

Fig. 4: A new set of $\to L$ rules. Subformula property holds for all but $\to L_{*2}$. $\to L_{*1}$ applies only when both F_1 and F_2 satisfy $\emptyset \nvdash F_i$, $i \in \{1, 2\}$.

Structural rules: $EqAnt''_{2\,\mathtt{LBI3}}$ $\quad \emptyset L_{\mathtt{LBI3}}$

In accordance with the restriction, we assume the availability of only those connectives valid in $\mathtt{LBI3}_1$ to all the sequents appearing in a $\mathtt{LBI3}_1$ derivation.

A. Contraction elimination in $\mathtt{LBI3}_1$

Our intention here is to prove that replacement of $\to L_{\mathtt{LBI3}_1}$ with those in Figure 4 is sound and complete, to render $\mathtt{LBI3}_1$ nearly almost contraction-free even implicitly.

1) Preparation:
We define three concepts for the main proofs: (1) non-theorem formulas; (2) the derivation length; and (3) irreducible $\mathtt{LBI3}_1$ sequents.

Definition 8 (Non-theorem formulas)**.** *We call a* $\mathtt{LBI3}_1$ *formula F a non-theorem formula if it satisfies $\emptyset \nvdash F$, and define it by $F^\star$ to mean such.*

Definition 9 (Derivation length)**.** *Given a derivation of a* $\mathtt{LBI3}_1$ *sequent D, denoted as $\Pi(D)$, the derivation length between D and some sequent D' that occurs in $\Pi(D)$ $(\mathtt{der\_len}(D, D'))$ is inductively defined as follows:*

- $\mathtt{der\_len}(D, D') = 0$ *if D' coincides with D.*
- $\mathtt{der\_len}(D, D') = \mathtt{der\_len}(D'', D') + 1$ *if there exists a sequent transition from D into D'' and from D'' into D' such that $D \rightsquigarrow D'' \rightsquigarrow^* D'$.*

Definition 10 (Irreducible $\mathtt{LBI3}_1$ sequents)**.**
An antecedent structure Γ in $\mathtt{LBI3}_1$ *is said to be irreducible if it contains as its substructure none of the following: (1) $p; p{\to}G$ (2) $\emptyset; \top^*{\to}G$ (3) $\top{\to}G$ (4) $\emptyset; \emptyset$ (5) $\Delta, \emptyset$ (6) $\bot$ (7) $\top$ (8) $\top^*$ (9) $H_1 \wedge H_2$ (10) $H_1 \vee H_2$ (11) $H_1 * H_2$. A* $\mathtt{LBI3}_1$ *sequent $D : \Gamma \vdash F$ is said to be irreducible if Γ is irreducible.*

Notice that a $\mathtt{LBI3}_1$ sequent D that contains any one of the 11 structures can be reduced via the inversion lemma (Lemma 1) such that D is $\mathtt{LBI3}_1$-derivable if and only if the reduced sequent(s) is (are).

2) A subformula property in $\to L_p$, $\to L_{\top^*}$ and $\to L_{*1}$:
We now show that any $\to L_{\mathtt{LBI3}_1}$ application on "$p{\to}G$" or "$\top^*{\to}G$" can be deferred until "$p; p{\to}G$" or resp. "$\emptyset; \top^*{\to}G$" appears on the antecedent part. Such also is the case for $(F_1^\star * F_2^\star){\to}F_3$ under a set of conditions.

Lemma 3. *Any* $\mathtt{LBI3}_1$*-derivable irreducible sequent $D : \Gamma \vdash H$ has a closed derivation in which the principal of the last rule applied is neither $p{\to}G$ nor $\top^*{\to}G$ (on the antecedent part of D), nor $(F_1^\star * F_2^\star){\to}G$ if not all of the following conditions satisfy:*

- $[D : \Gamma(\Delta'; (F_1^\star * F_2^\star){\to}G) \vdash H] \rightsquigarrow_{\to L}$ $[D_1 : \Delta'; (F_1^\star * F_2^\star){\to}G \vdash F_1^\star * F_2^\star]$
- $D_1 \rightsquigarrow_{*R_1} [D_2 : Re_1 \vdash F_1^\star]$
- $D_1 \rightsquigarrow_{*R_1} [D_3 : Re_2 \vdash F_2^\star]$
- *D_2 and D_3 (and hence also D_1) are both* $\mathtt{LBI3}_1$*-derivable.*

Proof: By contradiction. As in [3], we assume without loss of generality that inference rules to apply in the leftmost branch were cleverly chosen such that the derivation length between D and the conclusion sequent of an axiom in the leftmost branch be shortest. Suppose, by way of showing contradiction, that there cannot exist any other shorter derivations of D than the ones ending in $\to L$ for which a formula in the form either $p{\to}G$, $\top^*{\to}G$ or $(F_1^\star * F_2^\star){\to}G$ (under the given set of conditions) is its principal, then $\Pi(D)$, a derivation of D, looks like:

$$\dfrac{\dfrac{\begin{matrix}\vdots & \vdots \\ D_3 & D_4\end{matrix}}{D_1 : \Delta; B{\to}G \vdash B}\,\textbf{Inf} \qquad \begin{matrix}\vdots \\ D_2 : \Gamma(\Delta; G) \vdash H\end{matrix}}{D : \Gamma(\Delta; B{\to}G) \vdash H} \to L$$

where B is p if the principal is "$p{\to}G$"; is $\top^*$ if "$\top^*{\to}G$"; or is "$F_1^\star * F_2^\star$" if "$(F_1^\star * F_2^\star){\to}G$". As D is irreducible, so is D_1 which, therefore, cannot be the conclusion sequent of an axiom. If B is either p or $\top^*$, then the consequent formula of D_1 can be active only for an axiom. Likewise, due to the given set of conditions, if B is $F_1^\star * F_2^\star$ in D_1, its consequent part cannot be active for **Inf**. Therefore, **Inf** is known to be $\to L$. Moreover, as the leftmost branch is supposed shortest, the principal for **Inf** must be from among those constituents residing in the same additive structural layer as the $B{\to}G$. Furthermore, that the leftmost branch is shortest has to dictate that the principal for **Inf** is in neither of the following: "$p_i{\to}G_i$", "$\top_j^*{\to}G_j$" or "$(F_{k1}^\star * F_{k2}^\star){\to}G_k$" for some propositional variable p_i, some multiplicative logical unit $\top_j^*$, some $F_{k1}^\star * F_{k2}^\star$ (satisfying the same set of the conditions as stated) and some formula G_i, G_j or G_k.

These points taken into account, D, D_1, D_2, D_3 and D_4 are actually seen taking the following forms for some other formula F:

- $D : \Gamma(\Delta'; F{\to}G'; B{\to}G) \vdash H$
- $D_1 : \Delta'; F{\to}G'; B{\to}G \vdash B$
- $D_2 : \Gamma(\Delta'; F{\to}G'; G) \vdash H$
- $D_3 : \Delta'; F{\to}G'; B{\to}G \vdash F$
- $D_4 : \Delta'; G'; B{\to}G \vdash B$

But, then, this perforce implies the existence of an alternative derivation $\Pi'(D)$ which results by permuting $\Pi(D)$:

$$\dfrac{\begin{matrix}\vdots\\ D_3\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ D_4\end{matrix} \qquad \begin{matrix}\vdots\\ D_2' : \Gamma(\Delta'; G'; G) \vdash H\end{matrix}}{\Gamma(\Delta'; G'; B{\to}G) \vdash H} \to L}{D} \to L$$

D_2' can be shown derivable from D_2 via the inversion lemma (Lemma 1). A direct contradiction to the supposition has been drawn, for the leftmost branch in $\Pi'(D)$ is shorter. ■

From Lemma 3 follows an observation.

Lemma 4. *In* LBI3$_1$, $\to L_*'$ *as below is admissible in* $\to L_{*1}$.

$$\dfrac{D_1 : \Delta; (F_1^\star * F_2^\star){\to}G \vdash F_1^\star * F_2^\star \qquad D_2 : \Gamma(\Delta; G) \vdash H}{D : \Gamma(\Delta; (F_1^\star * F_2^\star){\to}G) \vdash H} \to L_*'$$

Proof: Any application of $\to L$ with $(F_1^\star * F_2^\star){\to}G$ as its principal can be deferred until all the four conditions hoisted in Lemma 3 are satisfied. Under the assumption, there exists a pair of sequent transitions via $*R_1$ from the left premise sequent D_1 of $\to L_*'$ into D_2 and D_3 such that (1) $D_1 \rightsquigarrow_{*R_1} D_2$; (2) $D_1 \rightsquigarrow_{*R_1} D_3$; and (3) both D_2 and D_3 are LBI3$_1$-derivable. Then, because the antecedent part of D_1 is not a multiplicative structural layer (checked by eye inspection on $\to L_*'$) nor can it be a single structure $(F_1^\star * F_2^\star){\to}G$ (otherwise D_1 is not LBI3$_1$-derivable), it must be an additive structural layer, and moreover, it must be such that there exists at least one multiplicative structural layer as its constituent (because the four conditions in Lemma 3 are assumed satisfied). By the process of a maximal Re_1/Re_2 generation (*c.f.* Lemma 2 and Corollary 1), it cannot be the case that two constituents of the outermost additive structural layer be retained simultaneously. And so there could be only one from among the $\mathcal{M}$ constituents which is to remain after a sequence of WkL_1 (*c.f.* Lemma 2) so that the result be a multiplicative structural layer to appear at the outermost structural layer. But $(F_1 * F_2){\to}F_3$ is not a multiplicative structural layer. ■

Proposition 5. *Replacement of* $\to L_{\text{LBI3}_1}$ *with those in Figure 4 is sound and complete.*

Proof: One direction: to assume inference rules in Figure 4 and to show corresponding derivations with $\to L_{\text{LBI3}_1}$, is trivial. Into the other direction, we consider what the actual instance F is in the principal $F{\to}G$, and turn to Lemma 3 and Lemma 4, for $\to L_{*1}$, $\to L_p$ and $\to L_{\top^*}$. $\to L_\top$ is straightforward. For the other cases, Dyckhoff92 [3]. ■

3) Remaining issues:
We have extended [3] to multiplicative connectives in the previous subsection, rendering LBI3$_1$ nearly almost contraction-free even implicitly. Nevertheless, there still is a certain challenge in establishing a subformula property for $\to L_{*2}$. The difficulty we see is as follows[5]:

1) Suppose a partial derivation tree comprising the following sequents:
   - $[D : \Gamma(\Delta; p{\to}(F_4 * F_5); (F_1 * (p{\to}F_2)){\to}G) \vdash H]$
   - $[D_1 : \Delta; p{\to}(F_4 * F_5); (F_1 * (p{\to}F_2)){\to}G) \vdash F_1 * (p{\to}F_2)]$
   - $[D_2 : \Gamma(\Delta; p{\to}(F_4 * F_5); G) \vdash H]$
   - $[D_3 : \emptyset \vdash F_1]$
   - $[D_4 : \Delta; p{\to}(F_4 * F_5); (F_1 * (p{\to}F_2)){\to}G \vdash p \to F_2]$
   - $[D_4' : \Delta; p; p{\to}(F_4 * F_5); (F_1 * (p{\to}F_2)){\to}G \vdash F_2]$
   - $[D_4'' : \Delta; p; F_4 * F_5; (F_1 * (p{\to}F_2)){\to}G \vdash F_2]$
   - $[D_4''' : \Delta; p; (F_4, F_5); (F_1 * (p{\to}F_2)){\to}G \vdash F_2]$
   - $[D_5 : \Delta; p; (F_4, F_5); (F_1 * (p{\to}F_2)){\to}G \vdash F_1 * (p{\to}F_2)]$
   - $[D_6 : \Delta; p; (F_4, F_5); G \vdash F_2]$
   - $[D_7 : F_4 \vdash F_1]$
   - $[D_8 : F_5 \vdash p{\to}F_2]$

   such that

$$\dfrac{\dfrac{}{\overline{D_3}} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\D_7\end{matrix}\quad\begin{matrix}\vdots\\D_8\end{matrix}}{D_5} *R_1 \qquad \begin{matrix}\vdots\\D_6\end{matrix}}{D_4'''} \to L_{*2}}{\begin{matrix}\vdots\ \text{Inversion}\\ D_4'' \\ \vdots\ \text{Inversion}\\ D_4' \\ \vdots\ \text{Inversion}\\ D_4\end{matrix}}}{D_1} *R_2 \qquad \begin{matrix}\vdots\\D_2\end{matrix}}{D} \to L_{*2}$$

2) In the above derivation, the formula "$(F_1 * (p{\to}F_2)) \to G$" becomes the principal for $\to L_{*2}$ twice without a redundancy in the given partial derivation tree.
3) If it were eliminated on D_1, then $\to L_{*2}$ could not apply on D_4''' for the same formula, and could then affect the derivability of D.

[5] A correction to an error in the submitted paper here to make the example effective according to the intention of the authors.

Indeed, the problem is more immanent than just within $\to L_{*2}$: although $-\!\!*L_{*1}$ works as intended, it may not be known whether some formula F is a non-theorem formula. The last analysis step towards the syntactical conclusion of BI decidability analysis should first concentrate on gaining a finer understanding of the proof-theoretical behaviour of $\mathtt{LBI3}_1$. By solving the simpler problem for $\mathtt{LBI3}_1$, an extension of the analysis framework to LBI3 may well be in scope.

V. Conclusion and Related Work

We have investigated the effect of LBI structural rules upon logical rules and presented as a consequence a contraction-free BI sequent calculus, LBI3. The well-known implicit contraction elimination technique in the intuitionistic logic [3] was then applied which was extended successfully to BI multiplicative connectives: "$\top^*$" and "$*$".

Comparisons between our work and related works may be of some interest here. For contraction absorption, albeit not in sequent calculus, we recognize nonetheless an attempt in [5] which suggests contraction absorption for $*$. Within LBI3, this idea may translate into the following inference rule as an alternative to $*L_{\mathtt{LBI3}}$:

$$\frac{\Gamma((F,G);(F*G)) \vdash H}{\Gamma(F*G) \vdash H} *L'_{\mathtt{LBI3}}$$

Although this conservative change does not affect the equivalence of LBI3 to LBI, it also does not augment our knowledge about the way contraction interacts with logical rules in BI sequent calculus. In fact, the only information we could gather would be that the structural contraction is generally required whenever multiplicative structures are involved. With the vague understanding, however, it could not be easily seen how the technique we studied in Section IV might be extended beyond BI additive fragments, *i.e.* the intuitionistic logic, for which we have nothing else to do but to quote the original work by Dyckhoff [3]. Fortunately our analysis speaks the contrary: structural contraction is admissible unless $-\!\!*L$ applies, which enabled us the said extension to "$\top^*$" and "$*$".

For the elimination of the bidirectionality of the structural equivalence, we are not aware of any preceding works. The equivalence manifests as a source of non-termination in $\mathrm{DL}_{\mathtt{BI}}$ [1], and also in [2] which for this reason opted for excluding the structural units altogether as a remedy. What we illustrated on the contrary was that the role that the unary structural connectives undertake in BI sequent calculus might be much simpler than commonly believed. The best way of reasoning about them we feel is by viewing a BI structure in alternating nested sequents, which helps one see the interactions between the additives/multiplicatives more lucidly than as some esoteric construct.

Lastly for weakening absorption, the work that comes closest is that by Donnelly *et al* [2] which looks into the forward BI (for its subset without units) theorem proving. As a proof search commences from axiom sequents, it faces a different set of non-termination problems arising in weakening than in contraction and, though not included in their analysis, in $EqAnt''_2$. Regarding the omitted units, first of all, our own analysis seems to suggest a straightforward extension of their calculus to also handle those. Absorption of our $\emptyset L$ into their internal weakening, of $EqAnt''_{1,2}$ (not $EqAnt'_{1,2}$) in their logical inference rules in a similar manner to our approach, along with an addition of $EqAnt'_2$, seem sufficient. To talk of calculus design issues, their approach differs in philosophy from ours in that the effect of weakening is absorbed into contraction, *i.e.* another structural rule, than purely into logical inference rules. One advantage the solution has is that the sequent calculus to result remains relatively conventional, unlike in our case where Re_1/Re_2 pairs appear. What must be lost on a downside is the flexibility of what conclusion sequents can be. For instance, it is not possible, using their sequent calculus, to derive the sequent $p_0;(p_1,((p_2,p_3);p_4)) \vdash (p_5 \to (p_1 * p_2)) * p_3$, although $p_0 \wedge (p_1 * ((p_2 * p_3) \wedge p_4) \vdash (p_5 \to (p_1 * p_2)) * p_3$ is derivable. This can result in a derivation becoming longer than truly necessary. Nevertheless, this seems to be a delicate decision point as far as the forward BI theorem proving is concerned, and it is too early to judge how weakening may be best handled. In retrospect, it might have paid off better, incurring possibly much less technical difficulties in the present work, in [2], and also in [1], if, in LBI, an alternative definition of weakening had been invented which would have extended its reach to one antecedent structure rather than one additive structural layer. Such weakening formulation seems to be more in keeping with BI semantics [4], [8]. Perhaps it is time, in view of the still remaining issues, that the base BI sequent calculus were revamped for a better usability; and the idiosyncrasy as apparent in $*R_{1\,\mathtt{LBI3}}$ and $-\!\!*L_{1\,\mathtt{LBI3}}$ were expelled once and for all.

In a broader picture, we mention a work by Galmiche *et al* [4] which indicated BI decidability in semantic tableaux. Automated decision procedures for BI have not yet been developed, however. The present work laid down an analysis foundation towards the purely syntactical conclusion of BI decidability analysis.

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APPENDIX A

Proof of Lemma 1 (Inversion lemma for LBI3)

Proof: By induction on the derivation depth k.

1) For a `LBI3` sequent $\Gamma(F \wedge G) \vdash H$, the base case is when it is an axiom, and the proof is trivial. For the inductive cases, assume that the statement holds true for all the derivation depths up to k, and show that it still holds true at $k+1$. Consider what the last inference rule applied is.

   a) $\top L$[6]: The derivation ends in:
   $$\frac{\Gamma(F \wedge G)(\emptyset) \vdash F}{\Gamma(F \wedge G)(\top) \vdash F}\ \top L$$
   where the representation $\Gamma(\Gamma_1)(\Gamma_2)$ is an abbreviation of $(\Gamma(\Gamma_1))(\Gamma_2)$ which indicates that Γ_1 is not a subbunch of Γ_2 nor is Γ_2 a subbunch of Γ_1. By induction hypothesis, $\Gamma(F; G)(\emptyset) \vdash F$. Then, $\Gamma(F; G)(\top) \vdash F$ as required by $\top L$.

   b) $\top^* L$: Similar.

   c) $\wedge L$: Similar, or trivial when the principal should coincide with $F \wedge G$.

   d) $\vee L$: The derivation ends in:
   $$\frac{\Gamma(F \wedge G)(F_1) \vdash H \qquad \Gamma(F \wedge G)(F_2) \vdash H}{\Gamma(F \wedge G)(F_1 \vee F_2) \vdash H}\ \vee L$$
   By induction hypothesis, both $\Gamma(F; G)(F_1) \vdash H$ and $\Gamma(F; G)(F_2) \vdash H$. Then $\Gamma(F; G)(F_1 \vee F_2) \vdash H$ as required via $\vee L$.

   e) $\to L$: The derivation ends in one of the following:
   $$\frac{\Gamma_1(F \wedge G); F_1 \to G_1 \vdash F_1 \qquad \Gamma(\Gamma_1(F \wedge G); G_1) \vdash H}{\Gamma(\Gamma_1(F \wedge G); F_1 \to G_1) \vdash H}\ \to L$$
   $$\frac{\Gamma_1'; F_1 \to G_1 \vdash F_1 \qquad \Gamma'(F \wedge G)(\Gamma_1'; G_1) \vdash H}{\Gamma'(F \wedge G)(\Gamma_1'; F_1 \to G_1) \vdash H}\ \to L$$
   By induction hypothesis, both $\Gamma_1(F; G); F_1 \to G_1 \vdash F_1$ and $\Gamma(\Gamma_1(F; G); G_1) \vdash H$ in case the former, or $\Gamma'(F; G)(\Gamma_1'; G_1) \vdash H$ in case the latter. Then $\to L$ (with the untouched left premise if the latter) produces the required result.

   f) $* L$: The derivation ends in:
   $$\frac{\Gamma(F \wedge G)(F_1, G_1) \vdash H}{\Gamma(F \wedge G)(F_1 * G_1) \vdash H}\ * L$$
   By induction hypothesis, $\Gamma(F; G)(F_1, G_1) \vdash H$. Then, $\Gamma(F; G)(F_1 * G_1) \vdash H$ as required via $* L$.

   g) $-\!\!* L_1$: The derivation ends as one of the follows, depending on the location at which $F \wedge G$ appears.
   $$\frac{Re_1 \vdash F_1 \qquad \Gamma((Re_2, G_1); (\Delta, \Gamma_1(F \wedge G); F_1 -\!\!* G_1)) \vdash H}{\Gamma(\Delta, \Gamma_1(F \wedge G); F_1 -\!\!* G_1) \vdash H}$$
   $$\frac{Re_1 \vdash F_1 \qquad \Gamma((Re_2, G_1); (\Delta(F \wedge G), \Gamma_1; F_1 -\!\!* G_1) \vdash H}{\Gamma(\Delta(F \wedge G), \Gamma_1; F_1 -\!\!* G_1) \vdash H}$$
   $$\frac{Re_1(F \wedge G) \vdash F_1 \qquad \Gamma((Re_2, G_1); (\Delta(F \wedge G), \Gamma_1; F_1 -\!\!* G_1)) \vdash H}{\Gamma(\Delta(F \wedge G), F_1 -\!\!* G_1) \vdash H}$$
   $$\frac{Re_1 \vdash F_1 \qquad \Gamma((Re_2(F \wedge G), G_1); (\Delta(F \wedge G), \Gamma_1; F_1 -\!\!* G_1)) \vdash H}{\Gamma(\Delta(F \wedge G), \Gamma_1; F_1 -\!\!* G_1) \vdash H}$$
   $$\frac{Re_1 \vdash F_1 \qquad \Gamma(F \wedge G)((Re_2, G); (\Delta, \Gamma_1; F_1 -\!\!* G_1)) \vdash H}{\Gamma(F \wedge G)(\Delta, \Gamma_1; F_1 -\!\!* G_1) \vdash H}$$
   For each, the required sequent results from induction hypothesis for occurrences of $F \wedge G$ on both of the premises, and then $-\!\!* L_1$ by appropriately carrying out its internal weakening to recover Δ (or $\Delta(F; G)$) from Re_1/Re_2 (*c.f.* Definition 5).

   h) $-\!\!* L_{2,3,4}$: Similar, but simpler.

   i) $\wedge R$: Similar to $\vee L$ in approach but simpler.

   j) $\vee R$: Similar.

   k) $\to R$: The derivation ends in:
   $$\frac{\Gamma(F \wedge G); F_1 \vdash G_1}{\Gamma(F \wedge G) \vdash F_1 \to G_1}\ \to R$$
   By induction hypothesis, $\Gamma(F; G); F_1 \vdash G_1$. Then, $\Gamma(F; G) \vdash F_1 \to G_1$ as required via $\to R$.

   l) $* R_1$: The derivation ends in one of the two below (associativity and commutativity of "," assumed):
   $$\frac{Re_1 \vdash F_1 \qquad Re_2 \vdash G_1}{\Delta(F \wedge G) \vdash F_1 * G_1}$$
   $$\frac{Re_1(F \wedge G) \vdash F_1 \qquad Re_2 \vdash G_1}{\Delta(F \wedge G) \vdash F_1 * G_1}$$
   Trivial for the first case. For the second, induction hypothesis on the left premise sequent produces $Re_1(F; G) \vdash F_1$. Then $* R_1$, appropriately carrying out its internal weakening to recover $\Delta(F; G)$ from $Re_1(F; G)$ and Re_2.

   m) $* R_2$: Trivial.

   n) $-\!\!* R$: Trivial.

   o) $EqAnt_2''$: The derivation ends in:
   $$\frac{\Gamma(F \wedge G)(\Delta) \vdash H}{\Gamma(F \wedge G)(\Delta, \emptyset) \vdash H}$$
   By induction hypothesis on the premise, $\Gamma(F; G)(\Delta) \vdash H$. Then, $EqAnt_2''$ produces $\Gamma(F; G)(\Delta, \emptyset) \vdash H$ as required.

   p) $\emptyset L$: The derivation ends in:
   $$\frac{\Gamma(F \wedge G)(\emptyset) \vdash H}{\Gamma(F \wedge G)(\emptyset; \emptyset) \vdash H}$$
   By induction hypothesis on the premise sequent, $\Gamma(F; G)(\emptyset) \vdash H$. Then $\Gamma(F; G)(\emptyset; \emptyset) \vdash H$ as required via $\emptyset L$.

2) A `LBI3` sequent $\Gamma(F_1 \vee F_2) \vdash H$: similar.

3) For a `LBI3` sequent $\Gamma(\Gamma_1; F \to G) \vdash H$, the base case is when it is an axiom, in which case the proof is trivial. For inductive cases, assume that it holds true for the derivation depths up to k, and show that it still holds true at the derivation depth of $k+1$. Consider what the last inference rule is.

   a) $\top L$: The derivation ends in one of the two below:
   $$\frac{\Gamma(\Gamma_1; F \to G)(\emptyset) \vdash H}{\Gamma(\Gamma_1; F \to G)(\top) \vdash H}$$
   $$\frac{\Gamma(\Gamma_1(\emptyset); F \to G) \vdash H}{\Gamma(\Gamma_1(\top); F \to G) \vdash H}$$

[6] A correction to an error in the submitted paper here.

For each, induction hypothesis on the premise and then $\top L$ to conclude.

b) $\to L$: If the additive structural layer for which the principal is a constituent coincides with $\Gamma_1; F \to G$ and if the principal coincides with the $F \to G$, then it is trivial via $\to L$. Otherwise, the derivation ends in either of the two below:

$$\dfrac{\dfrac{\Delta_1(\Gamma_1; F \to G); F_1 \to G_1 \vdash F_1 \qquad \Delta(\Delta_1(\Gamma_1; F \to G); G_1) \vdash H}{\Delta(\Delta_1(\Gamma_1; F \to G); F_1 \to G_1) \vdash H} \to L}{}$$

$$\dfrac{\Delta_1; F_1 \to G_1 \vdash F_1 \qquad \Delta(\Delta_1; F \to G)(\Gamma_1; G_1) \vdash H}{\Delta(\Delta_1; F \to G)(\Delta_1; F_1 \to G_1) \vdash H} \to L$$

Induction hypothesis on both of the premises in case the former, or on the right premise, and then $\to L$ to conclude.

c) $-\!\!*L_1$: Similar to the previous cases. Assume a maximal Re_1/Re_2 pair generation (Corollary 1) to simplify the proof.

d) The rest: Similar to the previous cases.

4) For a LBI3 sequent $\Gamma(F * G) \vdash H$, the base case is when it is an axiom for which a proof is trivial. For inductive cases, assume that it holds true for all the derivation depths up to k and show that the same still holds for the derivation depth of $k + 1$. Consider what the last inference rule is.

a) $\to L$: One of the following:

$$\dfrac{\Gamma_1(F * G); F_1 \to G_1 \vdash F_1 \qquad \Gamma(\Gamma_1(F * G); G_1) \vdash H}{\Gamma(\Gamma_1(F * G); F_1 \to G_1) \vdash H} \to L$$

$$\dfrac{\Gamma_1; F_1 \to G_1 \vdash H \qquad \Gamma(F * G)(\Gamma_1; G_1) \vdash H}{\Gamma(F * G)(\Gamma_1; F_1 \to G_1) \vdash H} \to L$$

For each, induction hypothesis on the premise(s) for each occurrence of $F * G$, and then $\to L$ to conclude.

b) $*L$: Trivial if the principal coincides with $F * G$. Otherwise, the derivation looks like:

$$\dfrac{\Gamma(F * G)(F_1, G_1) \vdash H}{\Gamma(F * G)(F_1 * G_1) \vdash H} *L$$

By induction hypothesis, $\Gamma(F, G)(F_1, G_1) \vdash H$. Then, $\Gamma(F, G)(F_1 * G_1) \vdash H$ as desired via $*L$.

c) The rest: Similar to the previous cases.

5) For a LBI3 sequent $\Gamma(\emptyset; \emptyset) \vdash H$ or $\Gamma(\Delta, \emptyset) \vdash H$, the base case is when they are an axiom. If the particular $\emptyset$ is not active in the axiom, then it is trivial to prove. Note that the $\emptyset$ in $\Gamma(\Delta, \emptyset) \vdash H$ cannot be active in an axiom because $\Delta, \emptyset$ is multiplicative which cannot be the antecedent part of the axiom conclusion sequent. If $\emptyset$ is active, then it is also straightforward to show that if $\top^* R$ is applicable on $\Gamma(\emptyset; \emptyset) \vdash H$, then it is also applicable on $\Gamma(\emptyset) \vdash H$. Inductive cases are straightforward in collation with the previous cases.

6) The rest: Similar.

■

APPENDIX B

Proof of Lemma 2

Proof:

$*R_1$: Under the assumption made, there exists a LBI3-derivable pair of $D_1 : Re_1 \vdash F$ and $D_2 : Re_2 \vdash G$ from the conclusion sequent $D : \Delta \vdash F * G$ such that $D \rightsquigarrow_{*R_1} D_1$ and $D \rightsquigarrow_{*R_1} D_2$. Internally (*c.f.* Definition 5) Re_1/Re_2 results from a finite number of $WkL_{\texttt{LBI}}$ applications on D as follows: $D \rightsquigarrow^*_{WkL_{\texttt{LBI}}} [D' : Re_1, Re_2 \vdash F * G]$. In D', notice that the outermost structural layer of the antecedent structure is multiplicative. If Δ in D was an additive structural layer, *i.e.* Δ denoting $\alpha_1; \ldots; \alpha_m; \mathcal{M}_1; \ldots; \mathcal{M}_n$ for $m + n \geq 2$, $m \geq 0$ and $n \geq 1$, then a finite number of $WkL_{\texttt{LBI}}$ applications must have taken place at this additive structural layer (which is the outermost structural layer in Δ) such that (in backward derivation) all but one multiplicative structural layer $\mathcal{M}_k$, $1 \leq k \leq n$ were weakened away. But this weakening process is also achieved via WkL_1. Once the outermost structural layer is multiplicative, it is either the case that some Re'_1/Re'_2 pair can be formed on the antecedent part for D'_1 and D'_2 such that $Re'_1 \vdash F$ and $Re'_2 \vdash G$ are both LBI3-derivable, or not. We are done if it can be formed. Otherwise, the current outermost multiplicative structural layer holds $\mathcal{A}$(s) as its constituent(s) whose $\mathcal{M}$ constituent (again only one of them) must be connected at the current outermost multiplicative structural layer, which is achieved through WkL_2. This incremental process eventually produces the Re'_1/Re'_2 pair on the antecedent part, *provided that a situation that satisfies all the below conditions does not arise.*

- there exists $D^* : Re^*_1, Re^*_2 \vdash F * G$ such that $D \rightsquigarrow^*_{\{WKL_1, WkL_2\}} D^*$ as the internal weakening process within $*R_1$.
- not both $D^*_1 : Re^*_1 \vdash F$ and $D^*_2 : Re^*_2 \vdash G$ are LBI3-derivable.
- there exists $D^{**} : Re^{**}_1, Re^{**}_2 \vdash F * G$ such that $D^* \rightsquigarrow^*_{WkL_{\texttt{LBI}}} D^{**}$ (as the internal weakening process within $*R_1$) and such that Re^{**}_1 (resp. Re^{**}_2) results from weakening Re^*_1 (resp. Re^*_2).
- both $D^{**}_1 : Re^{**}_1 \vdash F$ and $D^{**}_2 : Re^{**}_2 \vdash G$ are LBI3-derivable.

Suppose, by way of showing contradiction, that there exists a LBI3-derivation in which all the four conditions above satisfy. Then Proposition 2 dictates that LBI3-derivability of D^{**}_1 (resp. D^{**}_2) implies LBI3-derivability of D^*_1 (resp. D^*_2), a direct contradiction to the supposition.

$-\!\!*L_1$: Similar. The starting point for the implicit weakening in these rules is Δ in the conclusion sequent $D : \Gamma(\Delta, (\Gamma_1; F -\!\!* G)) \vdash H$. An application of WkL'_1 is mandatory (*c.f.* Definition 5) in case the principal is in an additive structural layer connected to Γ_1.

■

APPENDIX C

Proof for Theorem 1

Proof: By induction on $\texttt{der\_depth}(\Pi(D))$ where $\Pi(D)$ denotes a LBI3 derivation of D. The base cases are when it is 1, *i.e.* when D is the conclusion sequent of an axiom. Considering which axiom has been applied, if it is $\top R$, then it is trivial to show that if $\Gamma(\Delta;\Delta) \vdash \top$, then so is $\Gamma(\Delta) \vdash \top$. Also for $\bot L$, a single occurrence of $\bot$ on the antecedent part of D suffices for the $\bot L$ application, and the current theorem is trivially provable in this case, too. For both id and $\top^* R$, $\Pi(D)$ looks like:

$$\frac{}{\Gamma;\alpha \vdash F}$$

where (α, F) is (p,p) for id, and $(\emptyset, \top^*)$ for $\top^* R$. As the antecedent is either a single structure or its outermost structural layer, *i.e.* $\Gamma;\alpha$, is additive, irrespective of where Δ in D is, if D is LBI3-derivable, then so is D'.

For inductive cases, suppose that the current theorem has been proved for any derivation depth of up to k, it must be then demonstrated that it still holds for the derivation depth of $k+1$. Consider what the LBI3 inference rule applied last is, and, in case of a left inference rule, consider where the active structure Γ_1 of the inference rule is in $\Gamma(\Delta;\Delta)$.

1) $\top L$, and Γ_1 is $\top$: if it does not appear in Δ, induction hypothesis on the premise sequent concludes. Otherwise, $\Pi(D)$ looks like:

$$\frac{\vdots \atop D_1 : \Gamma(\Delta'(\emptyset);\Delta'(\top)) \vdash H}{D : \Gamma(\Delta'(\top);\Delta'(\top)) \vdash H}\ \top L$$

where $\Delta'(\top)$ represents Δ (assumed similarly for all the remaining cases). From LBI3 inversion lemma, if D_1 is derivable, so is $D_1' : \Gamma(\Delta'(\emptyset);\Delta'(\emptyset)) \vdash H$. By induction hypothesis on D_1', $D_1'' : \Gamma(\Delta'(\emptyset)) \vdash H$ is also derivable. Then a forward (as opposed to backward; assumed similarly for all the remaining cases) application of $\top L$ on D_1'', *i.e.* D_1'' as the premise sequent, deriving the conclusion sequent via $\top L$ at derivation depth $k+1$, concludes.

2) $\top^* L$, and Γ_1 is $\top^*$: similar to the case $\top$.

3) $\wedge L$, and Γ_1 is $F_1 \wedge F_2$: if it does not appear in Δ, induction hypothesis on the premise sequent. Otherwise, $\Pi(D)$ looks like:

$$\frac{\vdots \atop D_1 : \Gamma(\Delta'(F_1;F_2);\Delta'(F_1 \wedge F_2)) \vdash H}{D : \Gamma(\Delta'(F_1 \wedge F_2);\Delta'(F_1 \wedge F_2)) \vdash H}\ \wedge L$$

$D_1' : \Gamma(\Delta'(F_1;F_2);\Delta'(F_1;F_2)) \vdash H$ is LBI3-derivable (inversion lemma); $D_1'' : \Gamma(\Delta'(F_1;F_2)) \vdash H$ is also LBI3-derivable (induction hypothesis); then $\wedge L$ on D_1'' concludes.

4) $\emptyset L$, and Γ_1 is $\emptyset;\emptyset$: if it does not appear in Δ, the induction hypothesis on the premise sequents concludes. Otherwise, if it is entirely in Δ, then $\Pi(D)$ looks like:

$$\frac{\vdots \atop D_1 : \Gamma(\Delta'(\emptyset);\Delta'(\emptyset;\emptyset)) \vdash H}{D : \Gamma(\Delta'(\emptyset;\emptyset);\Delta'(\emptyset;\emptyset)) \vdash H}\ \emptyset L$$

$D_1' : \Gamma(\Delta'(\emptyset);\Delta'(\emptyset)) \vdash H$ is LBI3-derivable (inversion lemma); so is $D_1'' : \Gamma(\Delta'(\emptyset)) \vdash H$ (induction hypothesis); then $\emptyset L$ on D_1'' concludes. If only one of the $\emptyset$s appears in Δ, then it is perforce the case that $\Gamma(\Delta;\Delta) \rightleftharpoons \Gamma'(\emptyset;\Delta;\Delta) \rightleftharpoons \Gamma'(\emptyset;\emptyset;\Delta';\emptyset;\Delta')$, where "$\Gamma(\Delta;\Delta)$" denotes "$\Gamma'(\emptyset;\Delta;\Delta)$"; Δ does "$\Delta';\emptyset$"; and $\Gamma^* \rightleftharpoons \Gamma^{**}$ does that Γ^* is equivalent to Γ^{**} up to [coherent equivalence - additive/multiplicative units] (Definition 1). $\Pi(D)$ then looks like:

$$\frac{\vdots \atop D_1 : \Gamma'(\emptyset;\Delta';\emptyset;\Delta') \vdash H}{D : \Gamma'(\emptyset;\emptyset;\Delta';\emptyset;\Delta') \vdash H}\ \emptyset L$$

By induction hypothesis on D_1 twice, $D_1' : \Gamma'(\emptyset;\Delta') \vdash H$ upon which $\emptyset L$ applies for a conclusion.

5) $EqAnt_2''$, and Γ_1 is $\Gamma', \emptyset$: if it does not appear in Δ, then the induction hypothesis on the premise sequent concludes. Otherwise, $\Pi(D)$ looks like:

$$\frac{\vdots \atop D_1 : \Gamma(\Delta'(\Gamma');\Delta'(\Gamma',\emptyset)) \vdash H}{D : \Gamma(\Delta'(\Gamma',\emptyset);\Delta'(\Gamma',\emptyset)) \vdash H}\ EqAnt_2''$$

$D_1' : \Gamma(\Delta'(\Gamma');\Delta'(\Gamma')) \vdash H$ is LBI3-derivable (inversion lemma); so is $D_1'' : \Gamma(\Delta'(\Gamma')) \vdash H$ by induction hypothesis; then $EqAnt_2''$ on D_1'' concludes.

6) $\to L$, and Γ_1 is $\Gamma'; F \to G$: if it does not appear in Δ, then the induction hypothesis on both of the premises concludes. If it is entirely in Δ, then $\Pi(D)$ looks either like:

$$\frac{\vdots \atop D_1 : \Gamma'; F \to G \vdash F \qquad \vdots \atop D_2}{D : \Gamma(\Delta'(\Gamma'; F \to G);\Delta'(\Gamma'; F \to G)) \vdash H}\ \to L$$

where $D_2 : \Gamma(\Delta'(\Gamma'; G);\Delta'(\Gamma'; F \to G)) \vdash H$, or, in case Δ is $\Delta';\Gamma_1;F \to G$, like:

$$\frac{\vdots \atop D_1 : \Gamma';\Gamma';F \to G;F \to G \vdash F \qquad \vdots \atop D_2}{D : \Gamma(\Delta';\Gamma';F \to G;\Delta';\Gamma';F \to G) \vdash H}\ \to L$$

where $D_2 : \Gamma(\Delta';\Gamma';G;\Delta';\Gamma';F \to G) \vdash H$. In the former, $D_2' : \Gamma(\Delta'(\Gamma';G);\Delta'(\Gamma';G)) \vdash H$ (inversion lemma); $D_2'' : \Gamma(\Delta'(\Gamma';G)) \vdash H$ (induction hypothesis); then $\to L$ on D_1 and D_2'' concludes. In the latter, induction hypothesis on D_1, meanwhile the inversion lemma and the induction hypothesis on D_2; then via $\to L$ to conclude. Finally, if only a substructure of Γ_1 is in Δ with the rest spilling out of Δ, then similar to the latter case with appropriate applications of the inversion lemma.

7) $*R_1$: $\Pi(D)$ looks like:

$$\frac{\vdots \atop D_1 : Re_1 \vdash F_1 \qquad \vdots \atop D_2 : Re_2 \vdash F_2}{D : \Gamma(\Delta;\Delta) \vdash F_1 * F_2}\ *R_1$$

We show that the internalised weakening process to generate a maximal Re_1/Re_2 pair must either weaken away one Δ completely or preserve $\Delta;\Delta$ as a substructure of Re_1[7], and no other possibilities exist. But due to the formulation of the pair (*c.f.* Corollary 1), such must be the case. If $\Delta;\Delta$ is preserved in Re_1, then induction hypothesis on D_1 concludes; otherwise, it is trivial to see that only a single Δ needs to be present in D.

8) $-\!\!*L_1$, and Γ_1 is $\Delta_1, (\Gamma'; F -\!\!* G)$: if Γ_1 is not in Δ, then induction hypothesis on the right premise sequent concludes. If it is in Δ, $\Pi(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ D_1 : Re_1 \vdash F & & D_2\end{matrix}}{D : \Gamma(\Delta'(\Delta_1, (\Gamma'; F -\!\!* G)); \Delta'(\Delta_1, (\Gamma'; F -\!\!* G))) \vdash H} -\!\!*L_1$$

where D_2 is:

$$\Gamma(\Delta'((Re_2, G); (\Delta_1, (\Gamma'; F -\!\!* G))); \Delta'(\Delta_1, (\Gamma'; F -\!\!* G))) \vdash H$$

D_2' : $\Gamma(\Delta'((Re_2, G); (\Delta_1, (\Gamma'; F -\!\!* G))); \Delta'((Re_2, G); (\Delta_1, (\Gamma'; F -\!\!* G)))) \vdash H$ via Proposition 2 is also LBI3-derivable. D_2'' : $\Gamma(\Delta'((Re_2, G); (\Delta_1, (\Gamma'; F -\!\!* G)))) \vdash H$ via induction hypothesis. Then $-\!\!*L_1$ on D_1 and D_2'' concludes. If, on the other hand, Δ is in Γ_1, then it is either in Γ' or in Δ_1. But if it is in Γ', then it must be weakened away, and if it is in Δ_1, similar to the $*R_1$ case.

9) Other cases are similar to one of the cases already examined. Showing that if $\Gamma(\emptyset;\emptyset) \vdash F$, then $\Gamma(\emptyset) \vdash F$ is trivial due to Proposition 3.

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APPENDIX D

Proof of Proposition 4

Proof: Use a sequent $p_5; (\top -\!\!* p_1, \top -\!\!* (p_1 \to p_2), p_4) \vdash p_2$, and assume that every propositional variable is distinct. Then without contraction, there are several derivations of which two sensible ones are shown below.

1) $$\dfrac{\dfrac{}{p_3, \top -\!\!* (p_1 \to p_2) \vdash \top}\ \top R \qquad p_5; p_1 \vdash p_2}{D : p_5; (\top -\!\!* p_1, \top -\!\!* (p_1 \to p_2), p_3) \vdash p_2} -\!\!*L$$

2) $$\dfrac{\dfrac{}{p_3, \top -\!\!* p_1 \vdash \top}\ \top R \qquad \dfrac{p_5 \vdash p_1 \qquad \dfrac{\dfrac{}{p_2 \vdash p_2}\ id}{p_5; p_2 \vdash p_2}\ WkL}{p_5; p_1 \to p_2 \vdash p_2} \to L}{D : p_5; (\top -\!\!* p_1, \top -\!\!* (p_1 \to p_2), p_3) \vdash p_2} -\!\!*L$$

In both of the derivation trees above, one branch is open. Moreover, such holds true when only formula-level contraction is permitted in LBI. The sequent D cannot be derived under the given restriction. In the presence of structural contraction, however, another construction is possible:

$$\dfrac{\dfrac{\Pi(D_1) \qquad \Pi(D_2)}{p_5; (\top -\!\!* p_1, \top -\!\!* (p_1 \to p_2), p_3, p_4); (\top -\!\!* p_1, \top -\!\!* (p_1 \to p_2), p_3, p_4) \vdash p_2} -\!\!*L}{D : p_5; (\top -\!\!* p_1, \top -\!\!* (p_1 \to p_2), p_3, p_4) \vdash p_2}\ CtrL$$

where $\Pi(D_1)$ and $\Pi(D_2)$ are:

$\Pi(D_1)$:

$$\dfrac{}{\top -\!\!* (p_1 \to p_2), p_3 \vdash \top}\ \top R$$

$\Pi(D_2)$:

$$\dfrac{\dfrac{}{\top -\!\!* p_1, p_3 \vdash \top}\ \top R \qquad \dfrac{\dfrac{\dfrac{}{p_1 \vdash p_1}\ id}{p_5; p_1 \vdash p_1}\ WkL \qquad \dfrac{\dfrac{}{p_2 \vdash p_2}\ id}{p_5; p_1; p_2 \vdash p_2}\ WkL}{p_5; p_1; p_1 \to p_2 \vdash p_2} \to L}{p_5; p_1; (\top -\!\!* (p_1 \to p_2), p_3) \vdash p_2} -\!\!*L$$

where all the derivation tree branches are closed upward.

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APPENDIX E

Proof of Theorem 2

Proof: Into the *only if* direction, assume that D is LBI3-derivable, and then show that there is a LBI-derivation for each LBI3 derivation. But this is obvious because each LBI3 inference rule is derivable in LBI: $*R_{1,2\,\texttt{LBI3}}$, $-\!\!*L_{1,2,3,4\,\texttt{LBI3}}$, $\to L_{\texttt{LBI3}}$, $id_{\texttt{LBI3}}$ and $\top^* R_{\texttt{LBI3}}$ as have been stated in subsection A in Section III; $EqAnt''_{2\,\texttt{LBI3}}$ as one direction of $EqAnt_{2\,\texttt{LBI}}$; and $\emptyset L_{\texttt{LBI3}}$ as a special instance of $WkL_{\texttt{LBI}}$. All the other LBI3 rules are identical to LBI's.

Into the *if* direction, assume that D is LBI-derivable, and then show that there is a corresponding LBI3-derivation to each LBI derivation by induction on the derivation depth of $\Pi_{\texttt{LBI}}(D)$ ($\Pi_{\texttt{LBI}}(D)$ denotes a LBI-derivation of D).

If it is 1, *i.e.* if D is the conclusion sequent of an axiom, then $\bot L_{\texttt{LBI}}$ is identical to $\bot L_{\texttt{LBI3}}$; $id_{\texttt{LBI}}$ and $\top^* R_{\texttt{LBI}}$ via $id_{\texttt{LBI3}}$ and resp. $\top^* R_{\texttt{LBI3}}$ with Proposition 2; and $\top R_{\texttt{LBI}}$ as identical to $\top R_{\texttt{LBI3}}$. For inductive cases, assume that the *if* direction holds true up to the LBI-derivation depth of k, then it must be demonstrated that it still holds true for the LBI-derivation depth of $k+1$. Consider what the LBI rule applied last is:

1) $\to L_{\texttt{LBI}}$: $\Pi_{\texttt{LBI}}(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ D_1 : \Delta \vdash F & & D_2 : \Gamma(\Delta; G) \vdash H\end{matrix}}{D : \Gamma(\Delta; F \to G) \vdash H} \to L_{\texttt{LBI}}$$

By induction hypothesis, both D_1 and D_2 are also LBI3-derivable. Proposition 2 on D_1 in LBI3-space results in D_1' : $\Delta; F \to G \vdash F$. Then an application of $\to L_{\texttt{LBI3}}$ on D_1' and D_2 concludes in LBI3-space.

2) $-\!\!*L_{\texttt{LBI}}$: $\Pi_{\texttt{LBI}}(D)$ looks like:

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ D_1 : \Delta \vdash F & & D_2 : \Gamma(G) \vdash H\end{matrix}}{D : \Gamma(\Delta, F -\!\!* G) \vdash H} -\!\!*L_{\texttt{LBI}}$$

By induction hypothesis, D_1 and D_2 are also LBI3-derivable.

a) If $\Gamma(G)$ is G, *i.e.* if the antecedent part of D_2 is a single structure (G), then Proposition 2 on

[7] Re_1 or Re_2 if associativity and commutativity of "," is not assumed.

D_2 results in $D_2' : G; (\Delta, F \mathord{-\!\!*} G) \vdash H$ in LBI3-space. Then $\mathord{-\!\!*} L_{2\,\texttt{LBI3}}$ on D_1 and D_2' leads to $D' : \Delta, F \mathord{-\!\!*} G \vdash H$ as required. Instead of D_2', $D_2^* : G; F \mathord{-\!\!*} G \vdash H$ in case Δ is $\emptyset$, and $\mathord{-\!\!*} L_{4\,\texttt{LBI3}}$ instead of $\mathord{-\!\!*} L_{2\,\texttt{LBI3}}$.

b) If $\Gamma(G)$ is $\Gamma'(\Delta_1, G)$, then Proposition 2 on D_2 leads to $D_2' : \Gamma'((\Delta_1, G); (\Delta, \Delta_1, F \mathord{-\!\!*} G)) \vdash H$. Then $\mathord{-\!\!*} L_{1\,\texttt{LBI3}}$ on D_1 and D_2' leads to $D' : \Gamma'(\Delta, \Delta_1, F \mathord{-\!\!*} G) \vdash H$ as required. Instead of D_2', $D_2^* : \Gamma'((\Delta_1, G); (\Delta_1, F \mathord{-\!\!*} G)) \vdash H$ in case Δ is $\emptyset$, and $\mathord{-\!\!*} L_{3\,\texttt{LBI3}}$ instead of $\mathord{-\!\!*} L_{1\,\texttt{LBI3}}$.

c) Finally, if $\Gamma(G)$ is $\Gamma'(\Delta_1; G) \vdash H$, then Proposition 2 on D_2 leads to $D_2' : \Gamma'(\Delta_1; G; (\Delta, F \mathord{-\!\!*} G)) \vdash H$. Then $\mathord{-\!\!*} L_{2\,\texttt{LBI3}}$ on D_1 and D_2' leads to $D' : \Gamma'(\Delta_1; (\Delta, F \mathord{-\!\!*} G)) \vdash H$ as required. Instead of D_2', $D_2^* : \Gamma'(\Delta_1; G; F \mathord{-\!\!*} G) \vdash H$ in case Δ is $\emptyset$. Then $\mathord{-\!\!*} L_{4\,\texttt{LBI3}}$ instead of $\mathord{-\!\!*} L_{2\,\texttt{LBI3}}$.

3) $WkL_{\texttt{LBI}}$: Proposition 2.
4) $CtrL_{\texttt{LBI}}$: Theorem 1.
5) $EqAnt_{1\,\texttt{LBI}}$: Proposition 3.
6) $EqAnt_{2\,\texttt{LBI}}$: Proposition 3 and $EqAnt''_{2\,\texttt{LBI3}}$.
7) Others: straightforward.

■