Socioeconomic And Biological Scaling Laws

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I just wanted to TEX up the math behind the sigmoidal scaling behavior of biological systems. I go through, in excruciating detail, how to set up and solve the differential growth equation. Along the way, I consider the cases of linear, sub-linear, and super-linear behavior, and discuss what I gleaned from Geoffrey West's paper [1]. I conclude by finding a closed-form expression for the dependence of the inflection point on the set of initial parameters that define the biological problem—these parameters would either need to be theoretically predicted by some other theory or empirically found by fitting to the derived functional forms.

I. SETTING UP THE GROWTH EQUATION

Growth is constrained by the availability of resources and their rates of consumption. Resources, Y, are used for both maintenance and growth. If, on average, it requires a quantity R per unit time to maintain an individual and a quantity E to add a new one to the population, then this allocation of resources is expressed as $Y = RN + E(\frac{dN}{dt})$, where $\frac{dN}{dt}$ is the population growth rate. Assuming that the resources denoted by Y scale as $Y(t) = Y_0 N(t)^\beta$, we then find a differential equation for $N(t)$:

$$\frac{dN}{dt} = \left(\frac{Y_0}{E}\right) N(t)^\beta - \left(\frac{R}{E}\right) N(t) \tag{1}$$

This is the growth equation, and it is this differential equation that we will be principally concerned with in these notes. Observe that inherent in this derivation is an assumption that the maintenance cost scales linearly with $N(t)$; this may or may not be true, but this is how Geoffrey West proceeds, and so I will continue with this assumption as well. He addresses the possibility for different power law scalings in his supplementary text, but he claims that it doesn't qualitatively change any of the results.

II. SOLVING THE GROWTH EQUATION

In order to solve the growth equation analytically, we have to consider two base cases: (1) $\beta = 1$ and (2) $\beta \neq 1$. I start with (1) since it's the simplest, but the really interesting behavior is in case (2).

A. First Case: Growth Equation With $\beta = 1$

In this case, the growth equation simplifies into the form

$$\frac{dN}{dt} = \left(\frac{Y_0 - R}{E}\right) N(t), \tag{2}$$

which has as its solution either decaying ($Y_0 < R$) or growing ($Y_0 > R$) exponentials. There really isn't too much more to say about this case, but I just wanted to include it for completeness.

B. Second Case: Growth Equation With $\beta \neq 1$

To proceed for the case with $\beta \neq 1$, we begin by doing a variable substitution $x(t) \equiv N(t)^{1-\beta}$, which will greatly simplify the original differential equation. This substitution implies that $\dot{N}(t) = \left(\frac{1}{1-\beta}\right)\left(x(t)^{\frac{\beta}{1-\beta}}\right)\dot{x}(t)$, and so, plugging into the growth equation, we have

$$\frac{dx}{dt} = \left(\frac{1-\beta}{E}\right)(Y_0 - Rx(t)). \tag{3}$$

This is just a linear, first-order differential equation, and so can be solved by standard procedures. My favorite method is to guess a solution $x_p(t) = \text{Constant}$ that is a particular solution to the differential equation and then add to it a solution to the homogeneous equation $\dot{x}_h(t) = -(\frac{1-\beta}{E})Rx_h(t)$. Since $x_p(t) = \frac{Y_0}{R}$ and $x_h(t) = Ce^{-\frac{R(1-\beta)t}{E}}$ are particular and homogeneous solutions, respectively, with C a constant of integration to be determined, we see that the general solution to Equation 3 is

$$x(t) = \frac{Y_0}{R} + Ce^{-\frac{R(1-\beta)t}{E}}. \tag{4}$$

Re-expressing Equation 4 in terms of the population $N(t)$ (including the constant of integration C), we arrive at Geoffrey West's solution, which has either sigmoidal or super-exponential growth depending on whether $\beta < 1$ or $\beta > 1$, respectively:

$$N(t) = \left(\frac{Y_0}{R} + \left(N(0)^{1-\beta} - \frac{Y_0}{R}\right)e^{-\frac{R(1-\beta)t}{E}}\right)^{\frac{1}{1-\beta}}. \tag{5}$$

In Equation 5, $N(0)$ denotes the population at $t = 0$.

III. SOCIOECONOMIC SOLUTION: SUPER-EXPONENTIALLY GROWING, $\beta > 1$

Take the case of $\beta > 1$ first, which means dealing with the equation

$$N(t) = \frac{1}{\left(\frac{Y_0}{R} + \left(\frac{1}{N(0)^{\beta-1}} - \frac{Y_0}{R}\right) e^{\frac{R(\beta-1)t}{E}}\right)^{\frac{1}{\beta-1}}}, \quad (6)$$

where all I have done is rearrange the $1-\beta$ terms in Equation 5 to $-(\beta-1)$ so that we are dealing everywhere with positive quantities. In this form, it's clearer that the finite time singularity happens when the denominator of Equation 6 is zero. One subtlety that I hadn't appreciated before working through the math is that the $\beta > 1$ condition isn't enough to guarantee super-exponential growth—we also need $\frac{1}{N(0)^{\beta-1}} - \frac{Y_0}{R} < 0$, or else the denominator of Equation 6 never equals zero and we never have a finite time singularity. It's not obvious to me why this condition (which relates the population at $t = 0$ ($N(0)$), the fixed total resources at $t = 0$ (Y_0), and the quantity required per unit time to maintain an individual (R)) is important, but it definitely comes out of the math.

Because of this, suppose that both the $\beta > 1$ and $\frac{1}{N(0)^{\beta-1}} - \frac{Y_0}{R} < 0$ conditions are true. In this case, we can solve for the time at which the denominator equals zero, with the answer that

$$\begin{aligned} t_{\text{singularity}} &= -\left(\frac{E}{R(\beta-1)}\right) \ln\left[1 - \frac{R}{Y_0 N(0)^{\beta-1}}\right] \\ &\approx \frac{E}{Y_0(\beta-1)N(0)^{\beta-1}}. \end{aligned} \quad (7)$$

The approximation comes from doing a Taylor expansion for $\ln[1-x]$ in the small parameter $x = \frac{R}{Y_0 N(0)^{\beta-1}}$, but Geoffrey West's answer has a typo.

The super-exponential behavior persists until the finite resources are used up, at which point the condition $\frac{1}{N(0)^{\beta-1}} - \frac{Y_0}{R} < 0$ is no longer true and the total population experiences a collapse.

IV. BIOLOGICAL SOLUTION: SIGMOIDAL EVOLUTION, $\beta < 1$

Finally, for the case of $\beta < 1$ which characterizes biological systems (and companies), we obtain sigmoidal behavior with a carrying capacity of $\lim_{t\to+\infty} N(t) = \left(\frac{Y_0}{R}\right)^{\frac{1}{1-\beta}}$. For the purposes of modeling company growth, one of the interesting characteristics of the curve is the point of inflection (t_i), which delimits the exponential growth region at early times from the limiting asymptotic behavior at large times. This point of inflection is found by taking the second derivative of $N(t)$ and setting it equal to zero. I was lazy, and so I did the second differentiation in Mathematica; I obtained:

$$\begin{aligned} \ddot{N}(t) = \; & \left(\frac{Y_0}{R} + \left(N(0)^{1-\beta} - \frac{Y_0}{R}\right) e^{\frac{-R(1-\beta)t}{E}}\right)^{\frac{1}{1-\beta}} \\ \times \; & \left(N(0)R - N(0)^{\beta} Y_0 \left(1 - (1-\beta)\, e^{\frac{R(1-\beta)t}{E}}\right)\right) \\ \times \; & \frac{R^2 \left(N(0)R - N(0)^{\beta} Y_0\right)}{E^2 \left(N(0)R + N(0)^{\beta} Y_0 \left(e^{\frac{R(1-\beta)t}{E}} - 1\right)\right)^2}. \end{aligned} \quad (8)$$

From this, we find the inflection point t_i by solving $\ddot{N}(t_i) = 0$, which I also did in Mathematica:

$$t_i = -\frac{E}{(1-\beta)R} \ln\left[\frac{N(0)^{\beta} Y_0 (1-\beta)}{N(0)^{\beta} Y_0 - N(0)R}\right]. \quad (9)$$

This result is essentially the one I was looking for. From a theoretical model or from a multi-variable fit, if we can obtain five parameters (E, R, β, $N(0)$, and Y_0), then we can effectively find the time scale for population (or company) growth. Since t_i would be a natural benchmark time to use when deciding to divest from the company, this could be a useful closed-form expression.

V. CONCLUSION

I think I said pretty much all I wanted to. I hope you found this useful.

[1] "Growth, innovation, scaling, and the pace of life in cities", Luís M. A. Bettencourt, José Lobo, Dirk Helbing, Christian Kühnert, and Geoffrey B. West, http://www.pnas.org/content/104/17/7301.full.pdf?with-ds=yes