CALABI–YAU EMBEDDING OF SOME REDUCIBLE SURFACES VIA ORBIFOLDS

1. Introduction

The purpose of this note is to introduce the local Gromov–Witten invariants of a local del Pezzo orbifold and to prove embeddability of some higher rank surfaces considered in [6]. Here, a del Pezzo orbifold is a 2 dimensional orbifold constructed in the following way. Given a smooth projective surface S, a normal crossing divisor $D = D_1 + \dots D_n$, and a vector $\vec{r} \in \mathbb{Z}^n$, we can attach a stabilizer group of $\mathbb{Z}_{r_i} := \mathbb{Z}/r_i\mathbb{Z}$ to D, and construct a smooth DM stack $\mathcal{S}_{D,\vec{r}}$ it is called a root stack [2]. For our application, we will only concern when $n = 1$, $r = 2$ so we write simply $\mathcal{S}$ for $\mathcal{S}_{D,\vec{r}}$. We work over $\mathbb{C}$.

2. Geometry of orbifolds

Here we recall a bit of well known facts on orbifolds. A good reference is [4], chapter 4. See also [2] on root stacks. The root stack construction comes with a map c to its coarse moduli space, which is just the underlying projective surface S

$$c : \mathcal{S} \to S.$$

It is an isomorphism on $S \setminus D$ and thus the generic stabilizer is trivial. Hence, it is a Gorenstein orbifold. Its inertia stack is given by the disjoint union

$$I\mathcal{S} = \mathcal{S} \sqcup D.$$

It carries a canonical line bundle $\omega_{\mathcal{S}}$ or canonical divisor $K_{\mathcal{S}}$. The orbi-line bundle $\omega_{\mathcal{S}}$ is given by

$$\omega_{\mathcal{S}} = c^*\omega_S \otimes \mathcal{L}$$

where $\mathcal{L}^{\otimes 2} \simeq \mathcal{O}_S(D)$ is an orbi-line bundle which is not a pullback of a usual line bundle on S. Let (z_1, z_2) be local coordinates with D given by $z_1 = 0$ and let (x_1, x_2) be a corresponding local coodrinate system of the upstair orbifold. Then we have

$$c^*(dz_1 \wedge dz_2) = 2x_1 dx_1 \wedge dx_2$$

and the ramification divisor is given by $x_1 = 0$. This shows that orbifold line bundle $\omega_{\mathcal{S}}$ is *not locally trivial* around points on D and it has the induced $\mathbb{Z}_2$ action from the downstair. On $S \setminus D$, it is just the usual line bundle.

By the same way, the canonical divisor is given by

$$K_{\mathcal{S}} = c^*K_S + \frac{1}{2}D$$

which is a $\mathbb{Q}$-divisor. The anticanonical orbibundle $K_{\mathcal{S}}^{-1}$ can also be defined.

The notion of Chern class also generalizes, having value in $H^i(X, \mathbb{Q})$.

We say a $\mathbb{Q}$-Cartier divisor D is ample if it is locally ample and some positive multiple of it is an integral ample Cartier divisor.

Definition 2.1. Let $\mathcal{S}$ be an orbifold constructed from the root stack construction on (S, D). If $K_{\mathcal{S}}^{-1}$ is an ample divisor, then we say $\mathcal{S}$ is a del Pezzo orbifold.

3. Calabi–Yau embedding of some reducible surfaces

Let $\mathcal{X}$ be the total space of $K_{\mathcal{S}}$ on $\mathcal{S}$ which is again an orbifold. It is a Calabi-Yau orbifold of dimension 3 which we call an local (orbi)surface. Its coarse moduli space X is known to be $\mathbb{Q}$-factorial and it has a singularity along D in the zero section and there exists a crepant resolution by a single blow-up. Resolving this singularity, we get a genuine (non-stacky, non-compact) Calabi–Yau 3-fold X which has $S \cup_D R$ where R is a ruled surface over D.

Theorem 3.1. *Let S be a smooth projective surface and D be a smooth divisor with self intersection $D^2 = d$. Then the above construction gives a (non-compact) Calabi–Yau 3-fold containing $S \cup_D R$ where R is a ruled surface over D such that the self-intersection of D in R is $2 - d$.*

In the case $D^2 \geq -1$, the Calabi–Yau condition completely determines the ruled surface and the glued curve.

Theorem 3.2. *Let S be a smooth projective surface and D be a smooth divisor with self-intersection $D^2 = d \geq -1$. Then there is a non-compact Calabi-Yau 3-fold X containing $S \cup \mathbb{F}_{-2-d}$ where $S \cap \mathbb{F}_{-2-d} = D$ is the unique negative curve in $\mathbb{F}_{-2-d}$.*

Example 3.3. Let $S = \mathbb{P}^2$ and D be a smooth conic. In this case, the root stack $\mathcal{S}$ is actually a global quotient stack $\mathcal{S} = [\mathbb{F}_0/\langle i \rangle]$ where $p : \mathbb{F}_0 \to \mathbb{P}^2$ is the double cover of $\mathbb{P}^2$ branched along D and $i : \mathbb{F}_0 \to \mathbb{F}_0$ is the involution that comes with the double cover. Although scheme-theoretic quotient is just $\mathbb{P}^2$, it has nontrivial stabilizer attached to D. Consider the toal space of line bundle $\omega_{\mathcal{S}}$ on $\mathcal{S}$. Outside D, it is just the usual line bundle. Thus, the orbifold structure is supported over D. In local coordinate, the stabilizer just acts as multiplication by (-1) on the fiber and the defining equation. Thus, the coarse moduli space of $\mathcal{X}$ has a transverse A_1 singularity. Being Gorenstein Calabi–Yau, it admits a crepant resolution which is just the blowing up the singular locus in this case. The resulting blow-up is $\mathbb{F}_6$ by the Calabi–Yau condition and we get a normal crossing divisor $\mathbb{P}^2 \cup \mathbb{F}_6$.

This example can be constructed from $\mathrm{Tot}K_{\mathbb{F}_0}/\langle \tau \rangle$ where τ is acting on the fiber by multiplication by (-1) as well.

Hence the invariant for $\mathbb{P}^2 \cup \mathbb{F}_6$ in [6] could have been defined using this Calabi–Yau embedding.

Example 3.4. Let $S = \mathbb{P}^2$ and $D = \ell$ be a line. In this case, there does not exist a double cover of $\mathbb{P}^2$ branched along a line, so it cannot be constructed as a global quotient of a variety. However, there is such a double cover locally, and it glues in the category of orbifolds. Let $\mathcal{S}$ be such an orbifold, and $\mathcal{X} = \mathrm{Tot}(\omega_{\mathcal{S}})$. The coarse moduli space X has transverse A_1 singularity along a line. Blowing up this line, we get a divisor $\mathbb{P}^2 \cup \mathbb{F}_3$ in the resolution $\widetilde{X}$.

Since a coordinate axis in $\mathbb{P}^2$ is a torus invariant divisor, the orbifold $\mathcal{X}$ has toric description, as a partial resolution of isolated quotient singularity $\mathbb{C}^3/\mathbb{Z}_5$.

Example 3.5. Let $S = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and D be a smooth curve of class $f_1 + f_2$, where f_i is a ruling of $\mathbb{F}_0$. It is not a torus invariant divisor, and there is no double cover in branched along D in the categories of varieties. Let $\mathcal{S}$ be the (square) root stack along D and let $\mathcal{X}$ be the total space of canonical bundle on $\mathcal{S}$. Blowing up D in the zero section, we get $\mathbb{F}_0 \cup \mathbb{F}_4$.

Example 3.6. It can be extended to root stack of higher order of stabilizer group. In this case, the canonical bundle becomes $K_{\mathcal{S}} = c^* K_S + \left(1 - \frac{1}{n}\right) D$ and thus the zero section has A_{n-1} singularity. we need more than one blowups to resolve the singularity, and we get more than one ruled surface attached to S. For instance, the partial resolution of $\mathbb{C}^3/\mathbb{Z}_{2k+1}$ can be obtained from this orbifold construction, by considering $\mathbb{P}^2_{\ell,k}$

In particular, we have the following.

Proposition 3.7. *The snc del Pezzo surfaces in* [6] *can all be embedded into a non-compact Calabi–Yau 3-fold.*

Proof. The only remaining case is $\mathbb{F}_0 \cup \mathbb{F}_4$. Since the gluing curve is of class $f_1 + f_2$ which has self-intersection 2 in $\mathbb{F}_0$, Theorem 3.2. produces an embedding. $\square$

We expect to produce all Calabi–Yau 3-fold containing shrinkable surfaces. However, we do not yet know how to deal with surfaces with some points on a component blown-up.

References

[1] T. Bridgeland, A. King, M. Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. 14 (3) (2001) 535–554.

[2] C. Cadman, *Gromov-Witten invariants of* $\mathbb{P}^2$*-stacks*, Compos. Math. 143 (2007), no. 2, 495–514.

[3] F. Nironi, *Moduli spaces of semistable sheaves on projective deligne-mumford stacks*, math.AG/0811.1949 (2008).

[4] C. Boyer, K. Galicki, Sasakian geometry, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008, MR2382957.

[5] A. Kresch *On the geometry of DM stacks*

[6] S. Katz and S. Nam, *Local enumerative invariants of some simple normal crossing del Pezzo surfaces*, arXiv:2209.13031.

[7] J. Ross, R. Thomas, *Weighted Projective Embeddings, Stability of Orbifolds, and Constant Scarla Curvature Kär Metrics*