1) $\langle O \rangle = \int dV_1 \ldots dV_N \; \rho(\underline{p}, \underline{q}, t) \, O(\underline{p}, \underline{q})$

$$0 = \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \{\rho, H\}$$

$$= \frac{\partial \rho}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial \rho}{\partial \underline{q}_i} \cdot \frac{\partial H}{\partial \underline{p}_i} - \frac{\partial \rho}{\partial \underline{p}_i} \cdot \frac{\partial H}{\partial \underline{q}_i} \right)$$

$$\frac{d}{dt} \langle O \rangle = \int dV_1 \ldots dV_N \; \frac{\partial \rho}{\partial t} \, O(\underline{p}, \underline{q})$$

$$= \int dV_1 \ldots dV_N \left(\sum_{i=1}^{N} \frac{\partial \rho}{\partial \underline{p}_i} \cdot \frac{\partial H}{\partial \underline{q}_i} - \frac{\partial \rho}{\partial \underline{q}_i} \cdot \frac{\partial H}{\partial \underline{p}_i} \right) O$$

$$= \sum_{i=1}^{N} \int dV_1 \ldots dV_N \; \rho \left(\frac{\partial}{\partial \underline{q}_i} \cdot \left(O \frac{\partial H}{\partial \underline{p}_i} \right) - \frac{\partial}{\partial \underline{p}_i} \cdot \left(O \frac{\partial H}{\partial \underline{q}_i} \right) \right)$$

$$= \sum_{i=1}^{N} \int dV_1 \ldots dV_N \; \rho \quad \frac{\partial O}{\partial \underline{q}_i} \cdot \frac{\partial H}{\partial \underline{p}_i} - \frac{\partial O}{\partial \underline{p}_i} \cdot \frac{\partial H}{\partial \underline{q}_i}$$

$$= \int dV_1 \ldots dV_N \; \rho \, \{O, H\}$$

$$= \langle \{O, H\} \rangle$$

2) $$\langle H \rangle = \int dV_1 \ldots dV_N \, \rho \left(\sum_{i=1}^{N} \frac{|p_i|^2}{2m} + \frac{1}{2} \sum_{i \neq j} \phi(|q_i - q_j|) \right)$$

$$f_1(\underline{p_1}, \underline{q_1}, t) = N \int dV_2 \ldots dV_N \, \rho(\underline{p}, \underline{q}, t)$$

$$f_2(\underline{p_1}, \underline{q_1}, \underline{p_2}, \underline{q_2}, t) = N(N-1) \int dV_3 \ldots dV_N \, \rho$$

where factors of N and $N(N-1)$ are the numbers of different ways of choosing 1 or 2 particles (without replacement).

$$\frac{d}{dt} \langle H \rangle = \frac{d}{dt} \int dV_1 \ldots dV_N \rho \left(N \frac{|\underline{p_1}|^2}{2m} + \frac{N(N-1)}{2} \phi(\underline{q_1} - \underline{q_2}) \right)$$

$$= \frac{d}{dt} \int dV_1 \frac{|\underline{p_1}|^2}{2m} N \int dV_2 \ldots dV_N \, \rho$$

$$+ \frac{d}{dt} \frac{1}{2} \int dV_1 \, dV_2 \, \phi(|\underline{q_1} - \underline{q_2}|) \, N(N-1) \int dV_3 \ldots dV_N \, \rho$$

$$= \frac{d}{dt} \int dV_1 \frac{|\underline{p_1}|^2}{2m} f_1$$

$$+ \frac{d}{dt} \frac{1}{2} \int dV_1 \, dV_2 \, \phi(|\underline{q_1} - \underline{q_2}|) f_2$$

$$= \int dV_1 \frac{|\underline{p_1}|^2}{2m} \frac{\partial f_1}{\partial t} + \frac{1}{2} \int dV_1 dV_2 \, \phi(|\underline{q} - \underline{q_2}|) \frac{\partial f_2}{\partial t}$$

$$\frac{d}{dt}\langle H\rangle = \int dV_1 \, \frac{|\underline{p}_1|^2}{2m} \frac{\partial f_1}{\partial t} + \frac{1}{2}\int dV_1 \, dV_2 \, \phi(|\underline{q}_1-\underline{q}_2|) \, \frac{\partial f_2}{\partial t}$$

$$= \int dV_1 \, \frac{|\underline{p}_1|^2}{2m} \Bigg(\underbrace{- \frac{\underline{p}_1}{m} \cdot \frac{\partial f_1}{\partial \underline{q}_1}}_{q_1} + \int dV_2 \, \frac{\partial \phi(|\underline{q}_1-\underline{q}_2|)}{\partial \underline{q}_1} \cdot \frac{\partial f_2}{\partial \underline{p}_1} \Bigg)$$

$$+ \frac{1}{2}\int dV_1 \, dV_2 \, \phi(|\underline{q}_1-\underline{q}_2|) \Bigg(- \frac{\underline{p}_1}{m} \cdot \frac{\partial f_2}{\partial \underline{q}_1} - \frac{\underline{p}_2}{m} \cdot \frac{\partial f_2}{\partial \underline{q}_2}$$

$$+ \underbrace{\frac{\partial \phi(|\underline{q}_1-\underline{q}_2|)}{\partial \underline{q}_1} \cdot \left(\frac{\partial}{\partial \underline{p}_1} - \frac{\partial}{\partial \underline{p}_2} \right) f_2}_{p_1 \text{ and } p_2}$$

$$+ \int dV_3 \, \underbrace{\frac{\partial \phi(|\underline{q}_1-\underline{q}_3|)}{\partial \underline{q}_1} \cdot \frac{\partial f_3}{\partial \underline{p}_1}}_{p_1} + \underbrace{\frac{\partial \phi(|\underline{q}_2-\underline{q}_3|)}{\partial \underline{q}_2} \cdot \frac{\partial f_3}{\partial \underline{p}_2}}_{p_2} \Bigg)$$

Many terms are integrals of divergences (—)

Integrating by parts on the remaining terms:

$$\frac{d}{dt}\langle H\rangle = \int dV_1 dV_2 \, - \frac{\underline{p}_1}{m} \cdot \frac{\partial \phi(|\underline{q}_1-\underline{q}_2|)}{\partial \underline{q}_1} \, f_2$$

$$+ \frac{1}{2} \frac{\underline{p}_1}{m} \cdot \frac{\partial \phi(|\underline{q}_1-\underline{q}_2|)}{\partial \underline{q}_1} \, f_2$$

$$+ \frac{1}{2} \frac{\underline{p}_2}{m} \cdot \frac{\partial \phi(|\underline{q}_1-\underline{q}_2|)}{\partial \underline{q}_2} \, f_2$$

$$= \frac{1}{2}\int dV_1 \, dV_2 \, f_2 \underbrace{\left(\frac{\underline{p}_2}{m} \cdot \frac{\partial \phi(|\underline{q}_1-\underline{q}_2|)}{\partial \underline{q}_2} - \frac{\underline{p}_1}{m} \cdot \frac{\partial \phi(|\underline{q}_2-\underline{q}_1|)}{\partial \underline{q}_1} \right)}_{\text{antisymmetric under } 1 \leftrightarrow 2}$$

$= 0$ symmetric under $1 \leftrightarrow 2$

3) Put $f_2(\underline{p}_1, \underline{q}_1, \underline{p}_2, \underline{q}_2, t) = f_1(\underline{p}_1, \underline{q}_1, t) f_1(\underline{p}_2, \underline{q}_2, t)$

into

$$\left(\partial_t + \frac{\underline{p}_1}{m} \cdot \frac{\partial}{\partial \underline{q}_1}\right) f_1 = \int dV_2 \, \frac{\partial \phi(|\underline{q}_1 - \underline{q}_2|)}{\partial \underline{q}_1} \cdot \frac{\partial f_2}{\partial \underline{p}_1}$$

$$= \int dV_2 \, \frac{\partial \phi(|\underline{q}_1 - \underline{q}_2|)}{\partial \underline{q}_1} \cdot \frac{\partial}{\partial \underline{p}_1} \left(f_1(\underline{p}_1, \underline{q}_2, t) f_1(\underline{p}_2, \underline{q}_2, t) \right)$$

$$= \left(\int dV_2 \, f_1(\underline{p}_2, \underline{q}_2, t) \frac{\partial \phi(|\underline{q}_1 - \underline{q}_2|)}{\partial \underline{q}_1} \right) \cdot \frac{\partial f_1}{\partial \underline{p}_1}$$

$$= \frac{\partial}{\partial \underline{q}_1} \left(\int d\underline{q}_2 \, n(\underline{q}_2, t) \, \phi(|\underline{q}_1 - \underline{q}_2|) \right) \cdot \frac{\partial f_1}{\partial \underline{p}_1}$$

where $n(\underline{q}_2, t) = \int d\underline{p}_2 \, f_1(\underline{p}_2, \underline{q}_2, t)$

For Coulomb interactions this is equivalent to the electrostatic Vlasov equation

$$\left(\partial_t + \frac{\underline{p}_1}{m} \cdot \frac{\partial}{\partial \underline{q}_1} + \underline{E} \cdot \frac{\partial}{\partial \underline{p}_1}\right) f_1 = 0$$

for $\underline{E} = -\nabla \Phi$, particles of unit charge, and $\Phi(\underline{q}_1, t) = \int d\underline{q}_2 \, n(\underline{q}_2, t) \phi(|\underline{q}_1 - \underline{q}_2|)$

$$= \int d\underline{q}_2 \, \frac{n(\underline{q}_2, t)}{|\underline{q}_1 - \underline{q}_2|}$$

up to $4\pi\varepsilon_0$-type constants.

The number density $n \to \infty$ as $N \to \infty$

We need to scale the particle charge with $1/N$ so the charge density tends to a finite value as $N \to \infty$.

This corresponds to rescaling

$$H = \sum_{i=1}^{N} \frac{|p_i|^2}{2m} + \frac{1}{N} \sum_{i<j} \phi(|q_i - q_j|)$$

The potential energy term then scales in proportion to N, like the kinetic energy.

By contrast, the Boltzmann scaling corresponds to

$$H = \sum_{i=1}^{N} \frac{|p_i|^2}{2m} + \sum_{i<j} \Phi\left(\frac{|q_i - q_j|}{d} \right)$$

with the interaction distance d scaling so that $nd^2 \sim \frac{1}{\lambda_{mfp}}$ is constant as $N \to \infty$.

The potential energy then scales like $N(N-1)d^3 \sim N(Nd^3) \ll N$, so is much smaller than the kinetic energy.

4) "Physicist's" Hermite polynomials

$$H_n(v) = (-1)^n e^{v^2} \left(\frac{d}{dv}\right)^n e^{-v^2}$$

$$H_0(v) = 1, \quad H_1(v) = 2v, \quad H_2(v) = 4v^2 - 2$$

etc.

Consider $f_n(v) = H_n(v) e^{-v^2}$

$$L f_n = \frac{d}{dv}\left(v + \frac{1}{2}\frac{d}{dv}\right) f_n$$

$$= \frac{d}{dv}\left(v + \frac{1}{2}\frac{d}{dv}\right)(-1)^n \left(\frac{d}{dv}\right)^n e^{-v^2}$$

$$= (-1)^n \left(\frac{1}{2}\left(\frac{d}{dv}\right)^{n+2} + v\left(\frac{d}{dv}\right)^{n+1} + \left(\frac{d}{dv}\right)^n\right) e^{-v^2}$$

Use $H_n''(v) - 2v H_n'(v) = -2n H_n(v)$

$$L f_n = \frac{d}{dv}\left(v + \frac{1}{2}\frac{d}{dv}\right)\left(H_n e^{-v^2}\right)$$

$$= \frac{d}{dv}\left(v H_n e^{-v^2} + \frac{1}{2} H_n' e^{-v^2} - v H_n e^{-v^2}\right)$$

$$= \frac{d}{dv}\left(\frac{1}{2} H_n' e^{-v^2}\right)$$

$$= \frac{1}{2} H_n'' e^{-v^2} - v H_n' e^{-v^2}$$

$$= \frac{1}{2}\left(H_n'' - 2v H_n'\right) e^{-v^2}$$

$$= -n H_n e^{-v^2}$$

$$= -n f_n$$ so the eigenvalues are $-n$

Going back to the first approach:

$$\left(\frac{1}{2}\left(\frac{d}{dv}\right)^{n+2} + v\left(\frac{d}{dv}\right)^{n+1} + \left(\frac{d}{dv}\right)^{n}\right) e^{-v^2}$$

$$= \left(\frac{1}{2}\left(\frac{d}{dv}\right)^{n+2} + \left(\frac{d}{dv}\right)^{n}\right) e^{-v^2}$$

$$+ \left(\frac{d}{dv}\right)^{n+1}\left(ve^{-v^2}\right) - (n+1)\left(\frac{d}{dv}\right)^{n} e^{-v^2}$$

$$= \frac{1}{2}\left(\frac{d}{dv}\right)^{n+2} e^{-v^2} + \left(\frac{d}{dv}\right)^{n+1}\left(ve^{-v^2}\right)$$

$$- n\left(\frac{d}{dv}\right)^{n} e^{-v^2}$$

$$= \frac{1}{2}\left(\frac{d}{dv}\right)^{n+2} e^{-v^2} - \left(\frac{d}{dv}\right)^{n+1}\left(\frac{1}{2}\frac{d}{dv} e^{-v^2}\right)$$

$$- n\left(\frac{d}{dv}\right)^{n} e^{-v^2}$$

$$= -n\left(\frac{d}{dv}\right)^{n} e^{-v^2}$$

so we didn't need to assume the Hermite polynomials satisfied $H_n'' - 2v H_n' = -2n H_n$

Alternatively, consider the Kirkwood operator

$$Kf = \frac{d}{dv}\left((v-u) + T\frac{d}{dv}\right) f$$

where $n = \int f \, dv$, $nu = \int v f \, dv$

$$nT = \int |v-u|^2 f \, dv$$

This now conserves momentum and energy

5) Start from

$$\partial_t P_{ij} + \partial_k (u_k P_{ij} + Q_{ijk}) + P_{ik} \frac{\partial u_j}{\partial x_k} + P_{kj} \frac{\partial u_i}{\partial x_k}$$

$$= -\frac{1}{\tau} (P_{ij} - \theta \rho \delta_{ij})$$

There is a solution with ρ, $\underline{\underline{P}}$ and $\underline{\underline{Q}}$ all spatially uniform, $\underline{u} = \gamma y \hat{\underline{x}}$.

Hence $\partial_k (u_k P_{ij} + Q_{ijk}) = \nabla \cdot \underline{u} \, P_{ij} + \partial_k Q_{ijk} + u_k \partial_k P_{ij}$

$= 0$

$$P_{ij} + \tau \left(\partial_t P_{ij} + P_{ik} \frac{\partial u_j}{\partial x_k} + P_{kj} \frac{\partial u_i}{\partial x_k} \right)$$

$$= \rho \theta \delta_{ij}$$

$$\underline{\underline{P}} + \tau \left(\partial_t \underline{\underline{P}} + \underline{\underline{P}} \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} + \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} \underline{\underline{P}} \right)$$

$$= \rho \theta \underline{\underline{I}}$$

$$\begin{pmatrix} P_{xx} & P_{xy} \\ P_{xy} & P_{yy} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix} + \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P_{xx} & P_{xy} \\ P_{xy} & P_{yy} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma P_{xy} & 0 \\ \gamma P_{yy} & 0 \end{pmatrix} + \begin{pmatrix} \gamma P_{xy} & \gamma P_{yy} \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2\gamma P_{xy} & \gamma P_{yy} \\ \gamma P_{yy} & 0 \end{pmatrix}$$

$$\underline{\underline{P}} + \tau \left(\partial_t \underline{\underline{P}} + \gamma \begin{pmatrix} 2P_{xy} & P_{yy} \\ P_{yy} & 0 \end{pmatrix} \right) = \rho \theta \underline{\underline{I}}$$

$$P_{xx} + P_{yy} + P_{zz} = 3 \rho \theta$$

$$P_{zz} + \tau \partial_t P_{zz} = \frac{1}{3}(P_{xx} + P_{yy} + P_{zz})$$

$$P_{xx} + \tau(\partial_t P_{xx} + 2\gamma P_{xy}) = \frac{1}{3}(P_{xx} + P_{yy} + P_{zz})$$

$$P_{yy} + \tau \partial_t P_{yy} = \frac{1}{3}(P_{xx} + P_{yy} + P_{zz})$$

$$P_{xy} + \tau(\partial_t P_{xy} + \gamma P_{yy}) = 0$$

Seek solutions proportional to $e^{\chi t/\tau}$
so $\tau \partial_t \to \chi$

$$\begin{pmatrix} \frac{2}{3}+\chi & -\frac{1}{3} & -\frac{1}{3} & 2\gamma\tau \\ -\frac{1}{3} & \frac{2}{3}+\chi & -\frac{1}{3} & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}+\chi & 0 \\ 0 & \gamma\tau & 0 & 1+\chi \end{pmatrix} \begin{pmatrix} P_{xx} \\ P_{yy} \\ P_{zz} \\ P_{xy} \end{pmatrix} = 0$$

$$\cancel{\left(\tfrac{2}{3}+\chi\right)\left(\left(\tfrac{2}{3}+\chi\right)^2(1+\chi)\right.}$$

$$0 = (1+\chi)\left(\chi^3 + 2\chi^2 + \chi - \frac{2}{3}(\gamma\tau)^2\right)$$

Roots $\chi = -1$ or $\chi(1+\chi)^2 = \frac{2}{3}(\gamma\tau)^2$

When $\gamma\tau \ll 1$ there are 3 roots near -1, and a positive root
$\chi \approx \frac{2}{3}(\gamma\tau)^2$.

Putting this root into the P_{xy} equation

gives $P_{xy} \approx -\gamma\tau P_{yy} \approx -\gamma\tau\rho\theta$

$$= -\gamma\mu$$

6) Straightforward (Hilbert) expansion of

$$u_t + im = 0, \quad m_t = -\frac{1}{\varepsilon\tau}(m-u)$$

Expand $u = u^{(0)} + \varepsilon u^{(1)} + \dots$
$m = m^{(0)} + \varepsilon m^{(1)} + \dots$

① $u_t^{(0)} + \varepsilon u_t^{(1)} + \dots + im^{(0)} + i\varepsilon m^{(1)} + \dots = 0$

② $m_t^{(0)} + \varepsilon m_t^{(1)} + \dots = -\frac{1}{\varepsilon\tau}(m^{(0)} - u^{(0)}) - \frac{1}{\tau}(m^{(1)} - u^{(1)}) + \dots$

① $\Rightarrow$ $u_t^{(0)} + im^{(0)} = 0$ at $O(1)$
$u_t^{(1)} + im^{(1)} = 0$ at $O(\varepsilon)$

② $\Rightarrow$ $m^{(0)} = u^{(0)}$ at $O(1/\varepsilon)$

$\therefore$ $u_t^{(0)} + iu^{(0)} = 0$

Solution $u_t^{(0)} = Ue^{-it}$ with $u^{(0)}(0) = U$

② $\Rightarrow$ $m_t^{(0)} = -\frac{1}{\tau}(m^{(1)} - u^{(1)})$ at $O(1)$

$$\Rightarrow m^{(1)} = u^{(1)} - \tau m_t^{(0)}$$
$$= u^{(1)} - \tau u_t^{(0)}$$
$$= u^{(1)} + i\tau U e^{-it}$$

$\therefore$ $u_t^{(1)} + iu^{(1)} = \tau U e^{-it}$ from ① at $O(\varepsilon)$

$$u_t^{(1)} + i u^{(1)} = \tau U e^{-it}$$

Integrating factor e^{it}

$$\frac{d}{dt}\left(u^{(1)} e^{it}\right) = \tau U$$

$$u^{(1)} e^{it} = \tau U t + A$$

$$u^{(1)} = (A + \tau U t) e^{-it}$$

Expansion becomes disordered when $t \sim 1/\varepsilon$

cont. 6) Multiple scales expansion of

$$u_t + i m = 0, \quad m_t = -\frac{1}{\varepsilon\tau}(m-u)$$

The solvability condition says we don't expand u, as it's conserved under collisions.

$$\partial_t = \partial_{t_0} + \varepsilon \partial_{t_1} + \dots \quad \text{and} \quad m = m^{(0)} + \varepsilon m^{(1)} + \dots$$

① $(\partial_{t_0} u + \varepsilon \partial_{t_1} u + \dots) + i(m^{(0)} + \varepsilon m^{(1)} + \dots) = 0$

② $\partial_{t_0} m^{(0)} + \varepsilon(\partial_{t_0} m^{(1)} + \partial_{t_1} m^{(0)}) + \dots$

$$= -\frac{1}{\varepsilon\tau}(m^{(0)} - u) - \frac{1}{\tau} m^{(1)} + \dots$$

Using ① at $O(1)$ and ② at $O(1/\varepsilon)$

$$\partial_{t_0} u + i m^{(0)} = 0, \quad \text{and} \quad m^{(0)} = u$$

$$\therefore \quad \partial_{t_0} u + i u = 0, \quad u = U e^{-it_0}$$

with U possibly dependent on t_1.

Now ② at $O(1) \Rightarrow \partial_{t_0} m^{(0)} = -\frac{1}{\tau} m^{(1)}$

$$\therefore \quad m^{(1)} = -\tau \partial_{t_0} m^{(0)} = -\tau \partial_{t_0} u$$

$$= i\tau U e^{-it_0}$$

Finally, ① at $O(\varepsilon)$ $\Rightarrow$

$$\partial_{t_1} u = -i m^{(1)}$$

$$\partial_{t_1} U(t_1) e^{-it_0} = \tau U(t_1) e^{-it_0}$$

$$\frac{dU}{dt_1} = \tau U$$

$$U(t_1) = U(0) e^{t_1 \tau}$$

Complete solution : $u(t) = U(0) e^{t_1 \tau} e^{-it_0}$

$$= U(0) e^{(\varepsilon\tau - i)t}$$

Compare with the exact solution :

$$\partial_t \begin{pmatrix} u \\ m \end{pmatrix} = \begin{pmatrix} 0 & -i \\ \frac{1}{\varepsilon\tau} & -\frac{1}{\varepsilon\tau} \end{pmatrix} \begin{pmatrix} u \\ m \end{pmatrix}$$

Eigenvalues λ given by

$$0 = \begin{vmatrix} -\lambda & -i \\ \frac{1}{\varepsilon\tau} & -\frac{1}{\varepsilon\tau} - \lambda \end{vmatrix} = \lambda^2 + \frac{\lambda}{\varepsilon\tau} + \frac{i}{\varepsilon\tau}$$

$$\lambda = \frac{1}{2}\left(-\frac{1}{\varepsilon\tau} \pm \sqrt{\frac{1}{\varepsilon^2\tau^2} - \frac{4i}{\varepsilon\tau}}\right)$$

$$= \frac{1}{2\varepsilon\tau}\left(-1 \pm \sqrt{1 - 4i\varepsilon\tau}\right)$$

$$= \frac{1}{2\varepsilon\tau}\left(-1 \pm \left(1 - 2i\varepsilon\tau + \dots\right)\right)$$

$$= -\frac{1}{\varepsilon\tau} + i \quad (\text{fast}) \quad \text{or} \quad -i + \varepsilon\tau \quad (\text{slow})$$

7a) $$f^{(1)} = -\tau \left[\partial_{t_0} + \underline{v} \cdot \nabla \right] f^{(0)}$$

$$f^{(0)} = \frac{\rho/m}{(2\pi\theta)^{3/2}} \exp\left(-|\underline{v}-\underline{u}|^2/(2\theta)\right)$$

Euler equations:

$$\partial_{t_0} \rho = -\nabla\cdot(\rho\underline{u}), \quad \partial_{t_0}(\rho\underline{u}) + \nabla\cdot\left(\rho\underline{u}\underline{u} + \theta\rho\underline{\underline{I}}\right) = 0$$

$$\partial_{t_0}\theta + \underline{u}\cdot\nabla\theta + \frac{2}{3}\theta\nabla\cdot\underline{u} = 0$$

$$-\frac{1}{\tau} f^{(1)} = \left[\partial_{t_0} + \underline{v}\cdot\nabla\right] f^{(0)}$$

$$= f^{(0)} \left[\partial_{t_0} + \underline{v}\cdot\nabla\right] \operatorname{Log} f^{(0)}$$

$$= f^{(0)} \left[\partial_{t_0} + \underline{v}\cdot\nabla\right] \left(\operatorname{Log}\rho - \frac{3}{2}\operatorname{Log}\theta - \frac{|\underline{v}-\underline{u}|^2}{2\theta} - \operatorname{Log} m - \frac{3}{2}\operatorname{Log} 2\pi \right)$$

$$= f^{(0)} \left\{ \frac{1}{\rho}(\partial_{t_0} + \underline{v}\cdot\nabla)\rho - \frac{3}{2\theta}(\partial_{t_0} + \underline{v}\cdot\nabla)\theta - \frac{1}{\theta}(\underline{v}-\underline{u})\cdot(\partial_{t_0} + \underline{v}\cdot\nabla)(\underline{v}-\underline{u}) + \frac{1}{2\theta^2}|\underline{v}-\underline{u}|^2(\partial_{t_0} + \underline{v}\cdot\nabla)\theta \right\}$$

$$= f^{(0)} \left\{ \frac{1}{\rho}\left(-\nabla\cdot(\rho\underline{u}) + \underline{v}\cdot\nabla\rho\right) + \frac{1}{\theta}(\underline{v}-\underline{u})\cdot(\partial_{t_0} + \underline{v}\cdot\nabla)\underline{u} + \left(\frac{1}{2\theta^2}|\underline{v}-\underline{u}|^2 - \frac{3}{2\theta}\right)\left((\underline{v}-\underline{u})\cdot\nabla\theta - \frac{2}{3}\theta\nabla\cdot\underline{u}\right) \right\}$$

7b) $$-\frac{1}{\tau} f^{(1)} = f^{(0)} \Big\{ \frac{1}{\rho}(\underline{v}-\underline{u})\cdot\nabla\rho + \nabla\cdot\underline{u}$$

$$+ \frac{1}{\theta}(\underline{v}-\underline{u})\cdot\Big((\underline{v}-\underline{u})\cdot\nabla\underline{u} - \frac{1}{\rho}\nabla(\rho\theta)\Big)$$

$$+ \frac{1}{2\theta^2}\left(|\underline{v}-\underline{u}|^2 - 3\theta\right)\Big((\underline{v}-\underline{u})\cdot\nabla\theta - \frac{2}{3}\theta\nabla\cdot\underline{u}\Big)\Big\}$$

$$= f^{(0)} \Big\{ \underline{\frac{1}{\rho}(\underline{v}-\underline{u})\cdot\nabla\rho} \boxed{-\nabla\cdot\underline{u}}$$

$$+\frac{1}{\theta}(\underline{v}-\underline{u})\cdot\Big((\underline{v}-\underline{u})\cdot\nabla\underline{u}\Big) \underbrace{- \frac{1}{\rho}(\underline{v}-\underline{u})\cdot\nabla\rho}_{\text{cancels}}$$

$$\underline{-\frac{1}{\theta}(\underline{v}-\underline{u})\cdot\nabla\theta}$$

$$\underbrace{- \frac{1}{2\theta^2}\left(|\underline{v}-\underline{u}|^2 - 3\theta\right)}_{\text{combine}} (\underline{v}-\underline{u})\cdot\nabla\theta$$

$$\cancel{+\frac{1}{2\theta^2}|\underline{v}-\underline{u}|^2(\underline{v}-\underline{u})\cdot\nabla\theta}$$

$$- \frac{1}{3\theta}|\underline{v}-\underline{u}|^2\nabla\cdot\underline{u} \underbrace{\boxed{+\nabla\cdot\underline{u}}}_{\text{cancels}} \Big\}$$

$$= f^{(0)} \Big\{ - \frac{1}{2\theta^2}\left(|\underline{v}-\underline{u}|^2 - \theta\right)(\underline{v}-\underline{u})\cdot\nabla\theta$$

$$+ \frac{1}{\theta}(v_i - u_i)(v_j - u_j)\frac{\partial u_i}{\partial x_j}$$

$$- \frac{1}{3\theta}|\underline{v}-\underline{u}|^2 \nabla\cdot\underline{u} \Big\}$$

$$= f^{(0)} \Big\{ - \frac{1}{2\theta^2}\left(|\underline{v}-\underline{u}|^2 - \theta\right)(\underline{v}-\underline{u})\cdot\nabla\theta$$

$$+\frac{1}{2\theta}\Big((v_i - u_i)(v_j - u_j) \underbrace{- \theta\delta_{ij}}_{\text{optional}} \Big)\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\underline{u}\Big)\Big\}$$

7c) $$-\frac{1}{\tau} f^{(1)} = f^{(0)} \left\{ -\frac{1}{2\theta^2}\left(|\underline{w}|^2 - \theta\right) \underline{w}\cdot\nabla\theta + \frac{1}{\theta}\left(w_i w_j - \theta\delta_{ij}\right) E_{ij} \right\}$$

where $\underline{w} = \underline{v} - \underline{u}$, $2E_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\nabla\cdot u\,\delta_{ij}$

$w_i w_j - \theta\delta_{ij}$ and $(|\underline{w}|^2 - \theta)\underline{w}$

are Grad's tensor Hermite polynomials, orthogonal w.r.t. $\frac{1}{(2\pi\theta)^{3/2}} e^{-|\underline{w}|^2/2\theta}$

$\therefore \underline{\underline{P}}^{(1)} = \int d\underline{w}\, f^{(1)}\, \underline{w}\,\underline{w} = \tau\rho\theta\,\underline{\underline{E}}$

since equal to $\int d\underline{w}\,(\underline{w}\,\underline{w} - \theta\underline{\underline{I}}) f^{(1)}$

using $\mathrm{Tr}\,\underline{\underline{E}} = 0$.

$$\underline{q}^{(1)} = \int d\underline{w}\, f^{(1)} (|\underline{w}|^2 - \theta)\underline{w}$$
$$= -\frac{1}{2\theta^2}\int d\underline{w}\,(|\underline{w}|^2 - \theta)^2\, \underline{w}\,\underline{w}\cdot\overset{\nabla}{\theta}\, \frac{1}{(2\pi\theta)^{3/2}} e^{-\frac{|\underline{w}|^2}{2\theta}}$$
$$= -\frac{5}{2}\tau\rho\theta\nabla\theta$$

should be $\underline{w}\,\underline{w}\cdot\nabla\theta$