INTRODUCTION TO QUANTUM INFORMATION SCIENCE

Exercise Set 3
HT 2020

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3.1. **Partial traces and reduced density operators.** Consider two qubits in the quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|1\rangle \otimes \left(\sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|1\rangle\right) + |0\rangle \otimes \left(\sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{1}{3}}|1\rangle\right)\right]. \quad (1)$$

(1) What is the density operator ρ of the two qubits corresponding to state $|\psi\rangle$? Write it in the Dirac notation and explicitly as a matrix in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.
(2) Find the reduced density operators ρ_1 and ρ_2 of the first and the second qubit, respectively. Again, write them in the Dirac notation and as matrices in the computational basis.

You obtain reduced density operators by taking partial traces, e.g. the partial trace over $\mathcal{H}_B$ is defined for the tensor product operators,
$\mathrm{Tr}_B\,(A \otimes B) = A\,(\mathrm{Tr}\,B)$
and extended to any other operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ by linearity. See the Prerequisite Material.

3.2. **Trace distance.** The trace norm of a matrix A is defined as

$$||A||_{tr} = \mathrm{Tr}\left(\sqrt{A^\dagger A}\right).$$

(1) Show that the trace norm of any self-adjoint matrix is the sum of the absolute values of its eigenvalues. What is the trace norm of a density matrix?
(2) The trace distance between density matrices ρ_1 and ρ_2 is defined as

$$d(\rho_1, \rho_2) = \frac{1}{2}||\rho_1 - \rho_2||_{tr}.$$

What is the trace distance between two pure states $|\phi\rangle$ and $|\psi\rangle$?

3.3. **How well can we distinguish two quantum states?** If a physical system is equally likely to be prepared either in state ρ_1 or state ρ_2 then a single measurement can distinguish between the two preparations with the probability at most

$$\frac{1}{2}\left[1 + d(\rho_1, \rho_2)\right].$$

(1) Suppose ρ_1 and ρ_2 commute. Use the spectral decomposition of ρ_1 and ρ_2 in their common eigenbasis and describe the optimal measurement that can distinguish between the two states. What is the probability of success?
(2) Suppose you are given one of the two, randomly selected, qubits of state $|\psi\rangle$ in Eq. (1). What is the maximal probability with which you can determine whether it is the first or the second qubit?

This special case is essentially a classical problem of differentiating between two probability distributions.

3.4. **Bloch vectors.** Any density matrix of a single qubit can be parametrised by the three real components of the Bloch vector $\vec{s} = (s_x, s_y, s_z)$ and written as

$$\varrho = \tfrac{1}{2}\left(\mathbb{1} + \vec{s}\cdot\vec{\sigma}\right),$$

where σ_x, σ_y and σ_z are the Pauli matrices, and $\vec{s}\cdot\vec{\sigma} = s_x\sigma_x + s_y\sigma_y + s_z\sigma_z$.

(1) Check that such parametrised ϱ has all the mathematical properties of a density matrix as long as the length of the Bloch vector does not exceed 1.
(2) Two qubits are in quantum states described by their respective Bloch vectors, $\vec{s}_1$ and $\vec{s}_2$. What is the trace distance between the two quantum states?

3.5. **Completely positive maps.** Any physically admissible operation on a qubit is described by a completely positive map which can always be written as

$$\varrho \mapsto \varrho' = \sum_k A_k \varrho A_k^\dagger,$$

where matrices A_k satisfy $\sum_k A_k^\dagger A_k = \mathbb{1}$.

(1) Show that this map preserves positivity and trace. Show that any composition of completely positive maps is also completely positive.
(2) A qubit in state ϱ is transmitted through a depolarising channel that effects a completely positive map

$$\varrho \mapsto (1-p)\varrho + \frac{p}{3}\left(\sigma_x \varrho \sigma_x + \sigma_y \varrho \sigma_y + \sigma_z \varrho \sigma_z\right),$$

for some $0 \le p \le 1$. Show that under this map the Bloch vector associated with ϱ shrinks by the factor $(3-4p)/3$.

3.6. **Positive but not completely positive maps.** Consider a map $\mathcal{N}$, called universal-NOT, which acts on single qubit density matrices and is defined by its action on the identity and the three Pauli matrices

Any 2×2 matrix can be written as a linear composition of the identity and the three Pauli matrices.

$$\mathcal{N}(\mathbb{1}) = \mathbb{1} \quad \mathcal{N}(\sigma_x) = -\sigma_x \quad \mathcal{N}(\sigma_y) = -\sigma_y \quad \mathcal{N}(\sigma_z) = -\sigma_z$$

(1) Describe the action of this map in terms of the Bloch vectors.
(2) Explain why $\mathcal{N}$, acting on a single qubit, maps density matrices to density matrices.
(3) The joint state of two qubits is described by the density matrix

$$\rho = \tfrac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z\right),$$

Apply $\mathcal{N}$ to the first qubit leaving the second qubit intact. Write the resulting matrix and explain why $\mathcal{N}$ is not a completely-positive map.

3.7. **Approximate cloning.** Consider a hypothetical universal quantum cloner that operates on two qubits and on some auxiliary system. Given one qubit in any quantum state $|\,\psi\rangle$ and the other one in a prescribed state $|\,0\rangle$ it maps

$$|\,\psi\rangle\,|\,0\rangle\,|\,R\rangle \mapsto |\,\psi\rangle\,|\,\psi\rangle\,|\,R'\rangle,$$

where $|\,R\rangle$ and $|\,R'\rangle$ are, respectively, the initial and the final state of any other auxiliary system that may participate in the cloning process ($|\,R'\rangle$ may depend on $|\,\psi\rangle$).

(1) Show that such a cloner is impossible.

But supposed we are willing to settle for an imperfect copy? It turns out that the best approximation to the universal quantum cloner is the following transformation

$$|\,\psi\rangle\,|\,0\rangle\,|\,0\rangle \mapsto \sqrt{\frac{2}{3}}\,|\,\psi\rangle\,|\,\psi\rangle\,|\,\psi\rangle + \sqrt{\frac{1}{6}}\left(|\,\psi\rangle\,\left|\,\psi^\perp\right\rangle + \left|\,\psi^\perp\right\rangle|\,\psi\rangle\right)\left|\,\psi^\perp\right\rangle$$

where $\left|\,\psi^\perp\right\rangle$ is a normalised state vector orthogonal to $|\,\psi\rangle$ and the auxiliary system is another qubit.

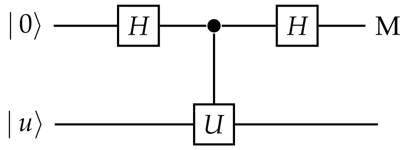
(2) Given the transformation above explain why the reduced density matrices of the first and the second qubit must be identical after the transformation.
(3) Show that the reduced density matrix of the first (and the second) qubit can be written as

$$\rho = \frac{5}{6}\,|\,\psi\rangle\langle\psi\,| + \frac{1}{6}\left|\,\psi^\perp\right\rangle\left\langle\psi^\perp\,\right|.$$

Perhaps our approximate cloning is also an approximate universal NOT operations described in the previous question? Think about it.

(4) What is the probability that the clone in state ρ will pass a test for being in the original state $|\,\psi\rangle$?
(5) What is the relation between the Bloch vectors of $|\,\psi\rangle\langle\psi\,|$ and ρ?

3.8. **Controlled unitaries revisited.** Consider the following quantum network composed of the two Hadamard gates, one controlled-U operation and the measurement M in the computational basis,



The top horizontal line represents a qubit and the bottom one an auxiliary physical system.

(1) Suppose $|u\rangle$ is an eigenvector of U, such that $U|u\rangle = e^{i\alpha}|u\rangle$. Step through the execution of this network, writing down quantum states of the qubit and the auxiliary system after each computational step. What is the probability for the qubit to be found in state $|0\rangle$?

Regardless the state of the auxiliary system, the probability P_0 for the qubit to be found in state $|0\rangle$, when the measurement M is performed, can be written as

$$P_0 = \frac{1}{2}(1 + v\cos\phi),$$

where v and ϕ depend on U and on the initial state of the auxiliary system.

(2) Show that for an arbitrary pure state $|u\rangle$ of the auxiliary system the quantities v and ϕ are given by the relation $ve^{i\phi} = \langle u|U|u\rangle$.

(3) Suppose the auxiliary system is prepared in a mixed state described by the density operator ρ,

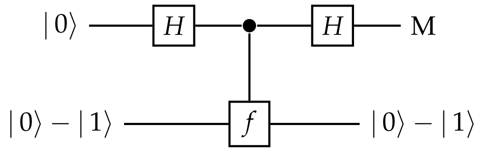
$$\rho = p_1|u_1\rangle\langle u_1| + p_2|u_2\rangle\langle u_2| + \ldots\ldots + p_n|u_n\rangle\langle u_n|,$$

where vectors $|u_k\rangle$ form an orthonormal basis, $p_k \geq 0$ and $\sum_{k=1}^{n} p_k = 1$. Show that

$$ve^{i\phi} = \mathrm{Tr}(\rho U).$$

(4) How would you modify the network in order to estimate $\mathrm{Tr}(\rho U)$? How would you estimate $\mathrm{Tr}\, U$?

3.9. **Deutsch's algorithm and decoherence.** Deutsch's algorithm with an oracle $f : \{0,1\} \mapsto \{0,1\}$, is implemented by the following network



Suppose that on between the Hadamard gates the top qubit undergoes decoherence by interacting with an environment in state $|e\rangle$,

$$|0\rangle|e\rangle \mapsto |0\rangle|e_0\rangle, \tag{2}$$

$$|1\rangle|e\rangle \mapsto |1\rangle|e_1\rangle, \tag{3}$$

where $|e_0\rangle$ and $|e_1\rangle$ are the new states of the environment which are normalised but not necessarily orthogonal, $\langle e_0|e_1\rangle = v$ for some real v. How reliably can you tell whether f is constant or balanced?