Kinetic theory of self-gravitating systems

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Problem set solutions

1. Isochrone potential and Bertrand's theorem.

The differential of the energy can generically be written as

$$\begin{aligned} \mathrm{d}E &= \left(\frac{\partial E}{\partial J_r}\right)_L \mathrm{d}J_r + \left(\frac{\partial E}{\partial L}\right)_{J_r} \mathrm{d}L \\ &= \Omega_r\, \mathrm{d}J_r + \Omega_\phi\, \mathrm{d}L, \end{aligned} \tag{1}$$

where we used the definition of the orbital frequencies, $\mathbf{\Omega}{=}\partial E/\partial \mathbf{J}$. Moving terms around, we obtain the differential of J_r as

$$\begin{aligned} \mathrm{d}J_r &= \frac{1}{\Omega_r}\, \mathrm{d}E - \frac{\Omega_\phi}{\Omega_r}\, \mathrm{d}L \\ &= \left(\frac{\partial J_r}{\partial E}\right)_L \mathrm{d}E + \left(\frac{\partial J_r}{\partial L}\right)_E \mathrm{d}L, \end{aligned} \tag{2}$$

and by identification, we get the two needed relations.

Owing to Schwarz' theorem, we have

$$\frac{\partial}{\partial L}\frac{\partial J_r}{\partial E} = \frac{\partial}{\partial E}\frac{\partial J_r}{\partial L} \quad \Longrightarrow \quad \frac{\partial}{\partial L}\left(\frac{1}{\Omega_r}\right) = -\frac{\partial}{\partial E}\left(\frac{\Omega_\phi}{\Omega_r}\right). \tag{3}$$

An orbit is closed if the two orbital frequencies (Ω_r, Ω_ϕ) are in resonance, i.e. if there exists $\mathbf{k} \neq (0,0)$ such that $\mathbf{k}\cdot\mathbf{\Omega} = 0$. Said differently, an orbit is closed if the ratio Ω_ϕ/Ω_r is a rational number. For all the orbits to be closed, this rational number has to be a constant, so that one must have $(\Omega_\phi/\Omega_r)(E,L) = \mathrm{cst}$. A first necessary condition for this to be satisfied is to have

$$\frac{\partial}{\partial E}\left(\frac{\Omega_\phi}{\Omega_r}\right) = 0 \quad \Longleftrightarrow \quad \frac{\partial \Omega_r}{\partial L} = 0, \tag{4}$$

owing to Eq. (3). As proved by Michel Henon, the condition $\partial\Omega_r/\partial L = 0$ is exactly the one satisfied by the isochrone potential. As a conclusion, a necessary condition for a potential to support only closed orbits is for it to be the isochrone potential. Fortunately, for the isochrone potential, we know exactly the expression of the frequency ratio. It reads

$$\frac{\Omega_\phi}{\Omega_r} = \frac{1}{2}\left(1 + \frac{L}{\sqrt{L^2 + 4GM_{\mathrm{tot}}b}}\right). \tag{5}$$

For all the orbits to be closed, this ratio has to be constant. This is only possible if $b \to 0$ or $b \to +\infty$, which correspond respectively to

$$\Phi_{\mathrm{iso}}(r) = -\frac{GM_{\mathrm{tot}}}{b + \sqrt{b^2 + r^2}} \to \begin{cases} -\dfrac{GM_{\mathrm{tot}}}{r} & \text{for } b \to 0, \quad \text{(Keplerian potential)}, \\ \text{cst.} & \text{for } b \to +\infty, \quad \text{(harmonic potential)}. \end{cases} \tag{6}$$

To conclude the proof, it only remains to check that indeed orbits are closed for the Keplerian and harmonic potentials. In the limit $b \to 0$ (resp. $b \to +\infty$), corresponding to the Keplerian potential (resp. harmonic potential), we have $\Omega_\phi/\Omega_r = 1$ (resp. $\Omega_\phi/\Omega_r = 1/2$), which are both rational numbers, i.e. all orbits are closed.

2. Response matrix and dielectric function.

Since the mean potential vanishes, a test particle's mean motion is given by $(\mathbf{x}(t), \mathbf{v}(t)) = (\mathbf{x}_0 + \mathbf{v}t, \mathbf{v}_0)$. As a consequence, up to prefactors, we naturally want to interpret the position $\mathbf{x}$ as our angle $\boldsymbol{\theta}$, and the velocity $\mathbf{v}$ as our action $\mathbf{J}$. One requirement is that the angles are 2π-periodic, for one full oscillation for $\mathbf{x}$ going from 0 to L. This is the case if we pick $\boldsymbol{\theta} = \mathbf{x}\,(2\pi/L)$. The associated action is then given by the constraint of volume conservation, so that $\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v} = \mathrm{d}\boldsymbol{\theta}\mathrm{d}\mathbf{J}$, which gives $\mathbf{J} = \mathbf{v}\,(L/(2\pi))$. Finally, we know that the mean specific Hamiltonian reads $H = \frac{1}{2}|\mathbf{v}|^2 = \frac{1}{2}(2\pi/L)^2|\mathbf{J}|^2$. Then, the associated frequency is $\mathbf{\Omega} = \partial H/\partial\mathbf{J} = (2\pi/L)^2\mathbf{J} = (2\pi/L)\mathbf{v}$.

As we have assumed periodicity, we can write the Fourier decomposition of the potential, Φ, the density ρ, and the pairwise interaction potential as

$$\Phi(\boldsymbol{\theta}) = \sum_{\mathbf{k}} \Phi_{\mathbf{k}}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}} \quad ; \quad \rho(\boldsymbol{\theta}) = \sum_{\mathbf{k}} \rho_{\mathbf{k}}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}} \quad ; \quad U(\boldsymbol{\theta},\boldsymbol{\theta}') = \sum_{\mathbf{k}} U_{\mathbf{k}}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\boldsymbol{\theta}-\boldsymbol{\theta}')}. \tag{7}$$

In these units, the Laplacian reads $\Delta_{\mathbf{x}}\Phi = (2\pi/L)^2\Delta_{\boldsymbol{\theta}}\Phi$. As a result, Poisson's equation, $\Delta_{\mathbf{x}}\Phi = 4\pi G\rho$ reads in Fourier space

$$\Phi_{\mathbf{k}} = -\frac{GL^2}{\pi}\,\frac{1}{|\mathbf{k}|^2}\,\rho_{\mathbf{k}}. \tag{8}$$

Similarly, the self-consistent relation $\Phi(\mathbf{x}) = \int\!\mathrm{d}\mathbf{x}'\,\rho(\mathbf{x}')\,U(\mathbf{x},\mathbf{x}')$ can be written as

$$\Phi(\boldsymbol{\theta}) = L^3\!\int\!\frac{\mathrm{d}\boldsymbol{\theta}'}{(2\pi)^3}\,\rho(\boldsymbol{\theta}')\,U(\boldsymbol{\theta},\boldsymbol{\theta}'), \tag{9}$$

which in Fourier space gives

$$\Phi_{\mathbf{k}} = L^3\,\rho_{\mathbf{k}}\,U_{\mathbf{k}}. \tag{10}$$

Identifying the two equations, we therefore obtain the Fourier coefficients of the pairwise interaction as

$$U_{\mathbf{k}} = -\frac{G}{L\pi}\,\frac{1}{|\mathbf{k}|^2}. \tag{11}$$

Glancing back at Eq. (7), we can therefore rewrite the pairwise interaction from Eq. (7) as

$$U(\boldsymbol{\theta},\boldsymbol{\theta}') = -\sum_{\mathbf{p}\in\mathbb{Z}^3\backslash\{0\}} \psi^{(\mathbf{p})}(\boldsymbol{\theta})\,\psi^{(\mathbf{p})*}(\boldsymbol{\theta}), \tag{12}$$

where we introduced the natural basis elements, $\psi^{(\mathbf{p})}(\boldsymbol{\theta})$, as

$$\psi^{(\mathbf{p})}(\boldsymbol{\theta}) = \sqrt{\frac{G}{L\pi}}\,\frac{1}{|\mathbf{p}|}\,\mathrm{e}^{\mathrm{i}\mathbf{p}\cdot\boldsymbol{\theta}}. \tag{13}$$

The inhomogeneous response matrix asks us to compute the Fourier transformed basis elements, $\psi^{(\mathbf{p})}_{\mathbf{k}}(\mathbf{J})$, defined as

$$\psi^{(\mathbf{p})}_{\mathbf{k}}(\mathbf{J}) = \!\int\!\frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^3}\,\psi^{(\mathbf{p})}(\boldsymbol{\theta})\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}. \tag{14}$$

This integral is straigtforward and gives

$$\psi^{(p)}_{\mathbf{k}}(\mathbf{J}) = \sqrt{\frac{G}{L\pi}}\,\frac{1}{|\mathbf{p}|}, \tag{15}$$

which is independent of the action coordinate $\mathbf{J}$. When injected in the general expression of the response matrix, this gives us

$$\begin{aligned}
\widetilde{\mathbf{M}}_{\mathbf{pq}}(\omega) &= (2\pi)^3\sum_{\mathbf{k}}\!\int\!\mathrm{d}\mathbf{J}\,\frac{\mathbf{k}\cdot\partial F/\partial\mathbf{J}}{\omega-\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J})}\,\psi^{(\mathbf{p})*}_{\mathbf{k}}(\mathbf{J})\,\psi^{(\mathbf{q})}_{\mathbf{k}}(\mathbf{J}) \\
&= \delta_{\mathbf{pq}}\frac{GL^2}{\pi}\,\frac{1}{|\mathbf{p}|^2}\!\int\!\mathrm{d}\mathbf{v}\,\frac{\mathbf{p}\cdot\partial F/\partial\mathbf{J}}{\omega-\mathbf{p}\cdot\boldsymbol{\Omega}(\mathbf{J})} \\
&= \delta_{\mathbf{pq}}\frac{GL^2}{\pi}\,\frac{1}{|\mathbf{p}|^2}\!\int\!\mathrm{d}\mathbf{v}\,\frac{\mathbf{p}\cdot\partial F/\partial\mathbf{v}}{\overline{\omega}-\mathbf{p}\cdot\mathbf{v}},
\end{aligned} \tag{16}$$

with $\overline{\omega} = \omega L/(2\pi)$.

We assume that the background distribution function (DF) is a Maxwellian DF of the form

$$F(\mathbf{v}) = F(|\mathbf{v}|) = \frac{\rho_0}{(2\pi\sigma^2)^{3/2}}\,\mathrm{e}^{-|\mathbf{v}|^2/(2\sigma^2)}. \tag{17}$$

We can inject this DF into Eq. (16). In order to carry out the integral over $\mathrm{d}\mathbf{v}$, we make a change of coordinates, so that $v = v_z$ is aligned with the direction of $\mathbf{p}$. We obtain

$$\begin{aligned}
\widetilde{\mathbf{M}}_{\mathbf{pq}}(\omega) &= \delta_{\mathbf{pq}}\frac{GL^2}{\pi}\,\frac{1}{|\mathbf{p}|^2}\,\frac{\rho_0}{\sqrt{2\pi}\,\sigma}\!\int\!\mathrm{d}v\,\frac{|\mathbf{p}|\,\frac{\partial}{\partial v}\mathrm{e}^{-v^2/(2\sigma^2)}}{\overline{\omega}-|\mathbf{p}|v} \\
&= \delta_{\mathbf{pq}}\frac{GL^2}{\pi}\,\frac{1}{|\mathbf{p}|^2}\,\frac{\rho_0}{\sqrt{2\pi}\,\sigma^3}\!\int\!\mathrm{d}v\,\frac{v\,\mathrm{e}^{-v^2/(2\sigma^2)}}{v-\overline{\omega}/|\mathbf{p}|}.
\end{aligned} \tag{18}$$

Making the change of variables $u = v/(\sqrt{2}\sigma)$, we can rewrite this integral as

$$\widetilde{\mathbf{M}}_{\mathbf{pq}}(\omega) = \delta_{\mathbf{pq}} \frac{G L^2 \rho_0}{\pi \sigma^2} \frac{1}{|\mathbf{p}|^2} \frac{1}{\sqrt{\pi}} \int \mathrm{d}u \, \frac{u \, \mathrm{e}^{-u^2}}{u - \zeta} \tag{19}$$

where we introduced $\zeta = \overline{\omega}/(\sqrt{2}|\mathbf{p}|\sigma)$. The plasma dispersion function is defined as

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \mathrm{d}u \, \frac{\mathrm{e}^{-u^2}}{u - \zeta}. \tag{20}$$

As a consequence, the integral appearing in Eq. (19) can be written as

$$\frac{1}{\sqrt{\pi}} \int \mathrm{d}u \, \frac{u \, \mathrm{e}^{-u^2}}{u - \zeta} = \frac{1}{\sqrt{\pi}} \int \mathrm{d}u \, \frac{\big[(u - \zeta) + \zeta\big] \mathrm{e}^{-u^2}}{u - \zeta} = 1 + \zeta Z(\zeta). \tag{21}$$

As a consequence, the expression of the response matrix becomes

$$\widetilde{\mathbf{M}}_{\mathbf{pq}}(\omega) = \delta_{\mathbf{pq}} \frac{G L^2 \rho_0}{\pi \sigma^2} \frac{1}{|\mathbf{p}|^2} \big[1 + \zeta Z(\zeta)\big]. \tag{22}$$

Introducing the Jeans length as

$$L_{\mathrm{J}} = \sqrt{\frac{\pi \sigma^2}{G \rho_0}}, \tag{23}$$

we can finally rewrite this expression as

$$\widetilde{\mathbf{M}}_{\mathbf{pq}}(\omega) = \delta_{\mathbf{pq}} \left(\frac{L}{L_{\mathrm{J}}}\right)^2 \frac{1}{|\mathbf{p}|^2} \big[1 + \zeta Z(\zeta)\big]. \tag{24}$$

The main difference with the plasma case is the overall sign of the response matrix. It corresponds to the difference between an attractive and a repulsive force, where the present Jeans length acts as the dual of the Debye Length.

Let us now investigate the condition under which the present system is linearly stable. To do so, we must determine whether or not there exists $\zeta = \zeta_0 + \mathrm{i}s$, with $s \geq 0$ such that $\widetilde{\mathbf{M}}_{\mathbf{pp}}(\zeta) = 1$. First, we must necessarily have $\zeta_0 = 0$ for the coefficient to be real, and we note that the function $s \mapsto 1 + \mathrm{i}sZ(\mathrm{i}s)$ is a positive and decreasing function, with a maximum in $s = 0$ where it reaches the value 1. As a consequence, the present system will be linearly stable, iff $\forall \, \mathbf{p} \neq 0$ one has $\widetilde{\mathbf{M}}_{\mathbf{pp}}(0) < 1$. This gives the constraint $(L/L_{\mathrm{J}})^2/|\mathbf{p}|^2 < 1$. Given that $\mathbf{p}$ is an integer vector, we always have $|\mathbf{p}|^2 \geq 1$. As a conclusion, we conclude that the present system is linearly stable, iff one has $L < L_{\mathrm{J}}$. This corresponds to the scale of the Jeans instability, that states that on scales large enough, homogeneous self-gravitating system will unavoidably collapse on themselves. This is radically different from the repulsive plasma, where the Jeans scale is replaced by the Debye length, and there is no such large-scale instabilities. This Jeans instability also illustrates why the self-gravitating amplification is the strongest on large scales, so that in a self-gravitating system large-scale perturbations are the ones that get the most efficiently amplified by self-gravity.

3. BBGKY hierarchy and inhomogeneous Landau equation

a. The BBGKY hierarchy.

As a result of probability conservation, P_N evolves according to the continuity equation

$$\frac{\partial P_N}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial \mathbf{w}_i} \cdot \bigg[P_N \, \dot{\mathbf{w}}_i\bigg] = 0, \tag{25}$$

where $\dot{\mathbf{w}}_i = (\partial H_N/\partial \mathbf{p}_i, -\partial H_N/\partial \mathbf{q}_i)$ is given by Hamilton's individual equations of motion. The divergence from Eq. (25) can then be computed as

$$\begin{aligned}
\frac{\partial}{\partial \mathbf{w}_i} \cdot \bigg[P_N \, \dot{\mathbf{w}}_i\bigg] &= \frac{\partial}{\partial \mathbf{q}_i} \cdot \bigg[P_N \frac{\partial H_N}{\partial \mathbf{p}_i}\bigg] - \frac{\partial}{\partial \mathbf{p}_i} \cdot \bigg[P_N \frac{\partial H_N}{\partial \mathbf{q}_i}\bigg] \\
&= \frac{\partial P_N}{\partial \mathbf{q}_i} \cdot \frac{\partial H_N}{\partial \mathbf{p}_i} - \frac{\partial P_N}{\partial \mathbf{p}_i} \cdot \frac{\partial H_N}{\partial \mathbf{q}_i} + P_N \bigg(\frac{\partial}{\partial \mathbf{q}_i} \cdot \frac{\partial H_N}{\partial \mathbf{p}_i} - \frac{\partial}{\partial \mathbf{p}_i} \cdot \frac{\partial H_N}{\partial \mathbf{q}_i}\bigg) \\
&= \frac{\partial P_N}{\partial \mathbf{q}_i} \cdot \frac{\partial H_N}{\partial \mathbf{p}_i} - \frac{\partial P_N}{\partial \mathbf{p}_i} \cdot \frac{\partial H_N}{\partial \mathbf{q}_i},
\end{aligned} \tag{26}$$

where the last term vanished by Schwarz' theorem. All in all, this finally allows us to rewrite Eq. (25) as

$$\frac{\partial P_N}{\partial t} + \Big[P_N, H_N \Big]_N = 0. \tag{27}$$

In order to derive the BBGKY hierarchy, we will need the property that the integral of a Poisson bracket over phase space vanishes. One generically has

$$\begin{aligned}
\int \!\mathrm{d}\mathbf{w} \Big[f(\mathbf{w}), h(\mathbf{w}) \Big]_{\mathbf{w}} &= \int \!\mathrm{d}\mathbf{q}\mathrm{d}\mathbf{p} \left(\frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial h}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial h}{\partial \mathbf{q}} \right) \\
&= \int \!\mathrm{d}\mathbf{p} \!\int \!\mathrm{d}\mathbf{q}\, \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial h}{\partial \mathbf{p}} - \int \!\mathrm{d}\mathbf{q} \!\int \!\mathrm{d}\mathbf{p}\, \frac{\partial f}{\partial \mathbf{p}} \cdot \frac{\partial h}{\partial \mathbf{q}} \\
&= - \int \!\mathrm{d}\mathbf{p} \!\int \!\mathrm{d}\mathbf{q}\, f\, \frac{\partial}{\partial \mathbf{q}} \cdot \frac{\partial h}{\partial \mathbf{p}} + \int \!\mathrm{d}\mathbf{q} \!\int \!\mathrm{d}\mathbf{p}\, f\, \frac{\partial}{\partial \mathbf{p}} \cdot \frac{\partial h}{\partial \mathbf{q}} \\
&= 0.
\end{aligned} \tag{28}$$

where to get the third line, we performed an integration by parts. And to get the last line, we relied on Schwarz' theorem, so that $\partial/\partial\mathbf{q}\cdot\partial h/\partial\mathbf{p} = \partial/\partial\mathbf{p}\cdot\partial h/\partial\mathbf{q}$. Actually, this is not the nicer way of getting this property is to recall that Poisson brackets define a flow in action space. Indeed, we can write

$$\int \!\mathrm{d}\mathbf{w} \Big[f(\mathbf{w}), h(\mathbf{w}) \Big]_{\mathbf{w}} = \int \!\mathrm{d}\mathbf{w}\, \frac{\partial}{\partial \mathbf{w}} \cdot \left[f(\mathbf{w}) \left(\frac{\partial h}{\partial \mathbf{q}}, - \frac{\partial h}{\partial \mathbf{p}} \right) \right] = 0, \tag{29}$$

as it is the divergence of a flux.

As an intermediary step and to avoid getting confused in the normalisations, we introduce the reduced probability distribution functions (PDFs) as

$$P_n(\mathbf{w}_1, ..., \mathbf{w}_n) = \int \!\mathrm{d}\mathbf{w}_{n+1} \ldots \mathrm{d}\mathbf{w}_N\, P_N. \tag{30}$$

Integrating Eq. (27) w.r.t. all particles except the n first, we can write

$$\begin{aligned}
\frac{\partial P_n}{\partial t} + \int \!\mathrm{d}\mathbf{w}_{n+1} \ldots \mathrm{d}\mathbf{w}_N \bigg\{ & \Big[P_N, \sum_{1 \leq i \leq n} U_{\mathrm{ext}}(\mathbf{w}_i) \Big]_N \\
& + \Big[P_N, \sum_{n+1 \leq j \leq N} U_{\mathrm{ext}}(\mathbf{w}_j) \Big]_N \\
& + \Big[P_N, \sum_{1 \leq i < j \leq n} m\, U(\mathbf{w}_i, \mathbf{w}_j) \Big]_N \\
& + \Big[P_N, \sum_{n+1 \leq i < j \leq N} m\, U(\mathbf{w}_i, \mathbf{w}_j) \Big]_N \\
& + \Big[P_N, \sum_{1 \leq i \leq n < j \leq N} m\, U(\mathbf{w}_i, \mathbf{w}_j) \Big]_N \bigg\} = 0.
\end{aligned} \tag{31}$$

We can now compute each of the terms appearing in that equation. The first contribution becomes

$$\int \!\mathrm{d}\mathbf{w}_{n+1} \ldots \mathrm{d}\mathbf{w}_N \Big[P_N, \sum_{1 \leq i \leq n} U_{\mathrm{ext}}(\mathbf{w}_i) \Big]_N = \Big[P_n, \sum_{1 \leq i \leq n} U_{\mathrm{ext}}(\mathbf{w}_i) \Big]_n, \tag{32}$$

as the Hamiltonian term is independent of the particles $n+1 \leq j \leq N$. For the second contribution, a typical term is of the form

$$\int \!\mathrm{d}\mathbf{w}_j \Big[P_N, U_{\mathrm{ext}}(\mathbf{w}_j) \Big]_{\mathbf{w}_j} = 0, \tag{33}$$

As a consequence, we have

$$\int \!\mathrm{d}\mathbf{w}_{n+1} \ldots \mathrm{d}\mathbf{w}_N \Big[P_N, \sum_{n+1 \leq j \leq N} U_{\mathrm{ext}}(\mathbf{w}_j) \Big]_N = 0, \tag{34}$$

and this second term does not contribute. The third contribution immediately becomes

$$\int \!\mathrm{d}\mathbf{w}_{n+1} \ldots \mathrm{d}\mathbf{w}_N \Big[P_N, \sum_{1 \leq i < j \leq n} m\, U(\mathbf{w}_i, \mathbf{w}_j) \Big]_N = \Big[P_n, m \sum_{1 \leq i < j \leq n} U(\mathbf{w}_i, \mathbf{w}_j) \Big]_n, \tag{35}$$

as the Hamiltonian term is independent of the particles $n+1 \leq j \leq N$. For the fourth contribution, a typical term is of the form

$$\int \mathrm{d}\mathbf{w}_j \left[P_N, m\, U(\mathbf{w}_i, \mathbf{w}_j) \right]_{\mathbf{w}_j} = 0, \tag{36}$$

As a consequence, we have

$$\int \mathrm{d}\mathbf{w}_{n+1} \ldots \mathrm{d}\mathbf{w}_N \left[P_N, \sum_{n+1 \leq i < j \leq N} m\, U(\mathbf{w}_i, \mathbf{w}_j) \right]_N = 0, \tag{37}$$

and the fourth term does not contribute. Finally, for the fifth contribution, the interaction term $U(\mathbf{w}_i, \mathbf{w}_j)$ leads to two terms

$$\begin{aligned} \int \mathrm{d}\mathbf{w}_j \left\{ \left[P_N, U(\mathbf{w}_i, \mathbf{w}_j) \right]_{\mathbf{w}_i} + \left[P_N, U(\mathbf{w}_i, \mathbf{w}_j) \right]_{\mathbf{w}_j} \right\} &= \int \mathrm{d}\mathbf{w}_j \left[P_N, U(\mathbf{w}_i, \mathbf{w}_j) \right]_{\mathbf{w}_i} \\ &= \int \mathrm{d}\mathbf{w}_{n+1} \left[P_N, U(\mathbf{w}_i, \mathbf{w}_{n+1}) \right]_{\mathbf{w}_i}, \end{aligned} \tag{38}$$

where to get the second line, we used the fact that P_N is symmetric under any particle permutation. All in all, noting that there are $(N-n)$ such terms for each $1 \leq i \leq n$, the fifth contribution becomes

$$\int \mathrm{d}\mathbf{w}_{n+1} \ldots \mathrm{d}\mathbf{w}_N \left[P_N, \sum_{1 \leq i \leq n < j \leq N} m\, U(\mathbf{w}_i, \mathbf{w}_j) \right]_N = (N-n) \int \mathrm{d}\mathbf{w}_{n+1} \left[P_n, m \sum_{i=1}^{n} U(\mathbf{w}_i, \mathbf{w}_{n+1}) \right]_n. \tag{39}$$

Gathering all these elements, the evolution equation for P_n reads

$$\frac{\partial P_n}{\partial t} + \left[P_n, H_n \right]_n + (N-n)\, m \int \mathrm{d}\mathbf{w}_{n+1} \left[P_{n+1}, \delta H_{n+1} \right]_n = 0. \tag{40}$$

Relying on the definition

$$F_n = m^n \frac{N!}{(N-n)!} P_n, \tag{41}$$

we immediately obtain the evolution equation for F_n as

$$\frac{\partial F_n}{\partial t} + \left[F_n, H_n \right]_n + \int \mathrm{d}\mathbf{w}_{n+1} \left[F_{n+1}, \delta H_{n+1} \right]_n = 0. \tag{42}$$

b. The truncated BBGKY equations.

By definition, P_N satisfies the normalisation convention $\int \mathrm{d}\mathbf{w}_1 \ldots \mathrm{d}\mathbf{w}_N P_N = 1$. As a consequence, following the definition of F_n, the reduced DFs satisfy the normalisation

$$\begin{aligned} &\int \mathrm{d}\mathbf{w}_1\, F_1 = mN, \\ &\int \mathrm{d}\mathbf{w}_1 \mathrm{d}\mathbf{w}_2\, F_2 = m^2 N(N-1), \\ &\int \mathrm{d}\mathbf{w}_1 \mathrm{d}\mathbf{w}_2 \mathrm{d}\mathbf{w}_3\, F_3 = m^3 N(N-1)(N-2). \end{aligned} \tag{43}$$

Integrating the definition of G_2 w.r.t. $\mathrm{d}\mathbf{w}_1 \mathrm{d}\mathbf{w}_2$, we obtain

$$m^2\, N(N-1) = (mN)^2 + \int \mathrm{d}\mathbf{w}_1 \mathrm{d}\mathbf{w}_2\, G_2 \quad \Longrightarrow \quad \int \mathrm{d}\mathbf{w}_1 \mathrm{d}\mathbf{w}_2\, G_2 = -m^2\, N. \tag{44}$$

We place ourselves in the scaling limit where the system's total mass, M_{tot}, is fixed. Then, we have $m = M_{\mathrm{tot}}/N$, and therefore $\int \mathrm{d}\mathbf{w}_1 F_1 \simeq 1$, and $\int \mathrm{d}\mathbf{w}_1 \mathrm{d}\mathbf{w}_2 G_2 \simeq 1/N$, i.e. we have $|G_2| \ll |F_1|$ w.r.t. the small parameter $1/N$.

An appropriate definition for G_3 has to be symmetric w.r.t. permutations of particles, and satisfy the appropriate scaling w.r.t. N. Such an appropriate definition is

$$\begin{aligned} F_3(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = &\, F_1(\mathbf{w}_1)\, F_1(\mathbf{w}_2)\, F_1(\mathbf{w}_3) \\ &+ F_1(\mathbf{w}_1)\, G_2(\mathbf{w}_2, \mathbf{w}_3) + F_1(\mathbf{w}_2)\, G_2(\mathbf{w}_1, \mathbf{w}_3) + F_1(\mathbf{w}_3)\, G_2(\mathbf{w}_2, \mathbf{w}_3) \\ &+ G_3(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3). \end{aligned} \tag{45}$$

Integrating this equation w.r.t. $\mathrm{d}\mathbf{w}_1\mathrm{d}\mathbf{w}_2\mathrm{d}\mathbf{w}_3$, we obtain

$$m^3 N(N-1)(N-2) = (m\,N)^3 - 3\,m\,N\,m^2\,N + \!\int\!\mathrm{d}\mathbf{w}_1\mathrm{d}\mathbf{w}_2\mathrm{d}\mathbf{w}_3 G_3 \quad\Longrightarrow\quad \int\!\mathrm{d}\mathbf{w}_1\mathrm{d}\mathbf{w}_2\mathrm{d}\mathbf{w}_3\, G_3 = 2m^3 N. \tag{46}$$

As a consequence, we obtain $|G_3| \simeq 1/N^2 \ll |G_2| \ll |F_1|$, i.e. the correlation functions are correctly ordered.

Let us now rewrite the first equation of the BBGKY hierarchy. We naturally have

$$\Big[F(\mathbf{J}), \overline{H}(\mathbf{J})\Big]_{\mathbf{w}} = 0, \tag{47}$$

since both functions only depend on $\mathbf{J}$. Moreover, as $F(\mathbf{J},t)$ is independent of $\boldsymbol{\theta}$, we can average the first BBGKY equation w.r.t. $\boldsymbol{\theta}$ to obtain

$$\frac{\partial F(\mathbf{J},t)}{\partial t} + \!\int\!\frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^d}\!\int\!\mathrm{d}\mathbf{w}' \left\{ \frac{\partial G}{\partial \boldsymbol{\theta}}\cdot\frac{\partial U}{\partial \mathbf{J}} - \frac{\partial G}{\partial \mathbf{J}}\cdot\frac{\partial U}{\partial \boldsymbol{\theta}} \right\} = 0. \tag{48}$$

The integral term then takes the form

$$\begin{aligned}
\int\!\mathrm{d}\boldsymbol{\theta}\left\{ \frac{\partial G}{\partial \boldsymbol{\theta}}\cdot\frac{\partial U}{\partial \mathbf{J}} - \frac{\partial G}{\partial \mathbf{J}}\cdot\frac{\partial U}{\partial \boldsymbol{\theta}} \right\} &= \frac{\partial}{\partial \mathbf{J}}\cdot\left[\int\!\mathrm{d}\boldsymbol{\theta}\,\frac{\partial G}{\partial \boldsymbol{\theta}}\,U\right] - \!\int\!\mathrm{d}\boldsymbol{\theta}\,\frac{\partial G}{\partial \mathbf{J}}\cdot\frac{\partial U}{\partial \boldsymbol{\theta}} - \!\int\!\mathrm{d}\boldsymbol{\theta}\,\frac{\partial}{\partial \mathbf{J}}\cdot\frac{\partial G}{\partial \boldsymbol{\theta}}\,U \\
&= \frac{\partial}{\partial \mathbf{J}}\cdot\left[\int\!\mathrm{d}\boldsymbol{\theta}\,\frac{\partial G}{\partial \boldsymbol{\theta}}\,U\right] - \!\int\!\mathrm{d}\boldsymbol{\theta}\,\frac{\partial G}{\partial \mathbf{J}}\cdot\frac{\partial U}{\partial \boldsymbol{\theta}} - \!\int\!\mathrm{d}\boldsymbol{\theta}\,\frac{\partial}{\partial \boldsymbol{\theta}}\cdot\frac{\partial G}{\partial \mathbf{J}}\,U \\
&= \frac{\partial}{\partial \mathbf{J}}\cdot\left[\int\!\mathrm{d}\boldsymbol{\theta}\,\frac{\partial G}{\partial \boldsymbol{\theta}}\,U\right] - \!\int\!\mathrm{d}\boldsymbol{\theta}\,\frac{\partial G}{\partial \mathbf{J}}\cdot\frac{\partial U}{\partial \boldsymbol{\theta}} + \!\int\!\mathrm{d}\boldsymbol{\theta}\,\frac{\partial G}{\partial \mathbf{J}}\cdot\frac{\partial U}{\partial \boldsymbol{\theta}} \\
&= \frac{\partial}{\partial \mathbf{J}}\cdot\left[\int\!\mathrm{d}\boldsymbol{\theta}\,\frac{\partial G}{\partial \boldsymbol{\theta}}\,U\right],
\end{aligned} \tag{49}$$

where to get the second line, we relied on Schwarz' theorem, and to get the third line, we performed an integral by parts, and the boundary terms vanished because of the 2π-periodicity. All in all, we obtain the quoted result for that equation.

For the second equation of the hierarchy, the mean-field Poisson bracket takes the form

$$\begin{aligned}
\Big[G(\mathbf{w},\mathbf{w}'), \overline{H}(\mathbf{w}')\Big]_{\mathbf{w}} &= \frac{\partial G}{\partial \boldsymbol{\theta}}\cdot\frac{\partial \overline{H}}{\partial \mathbf{J}} - \frac{\partial G}{\partial \mathbf{J}}\cdot\frac{\partial \overline{H}}{\partial \boldsymbol{\theta}} \\
&= \frac{\partial G}{\partial \boldsymbol{\theta}}\cdot\boldsymbol{\Omega}(\mathbf{J}),
\end{aligned} \tag{50}$$

where we used the fact that the mean-field dynamics is integrable, i.e. $\overline{H}(\mathbf{w}) = \overline{H}(\mathbf{J})$, and introduced the mean-field orbital frequencies as $\boldsymbol{\Omega}(\mathbf{J}) = \partial\overline{H}/\partial\mathbf{J}$. From the assumption of neglecting collective effects, we also perform the simplification

$$\int\!\mathrm{d}\mathbf{w}''\,\Big[F(\mathbf{w})\,G(\mathbf{w}',\mathbf{w}''), U(\mathbf{w},\mathbf{w}'')\Big]_{\mathbf{w}} \quad\to\quad 0, \tag{51}$$

i.e. correlations are immune to the perturbations that they generate themselves. All in all, we obtain the quoted equation.

c. The dynamics of correlations.

We now perform a Fourier transform of the evolution equation for G by multiplying this equation by $\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\boldsymbol{\theta}-\mathbf{k}'\cdot\boldsymbol{\theta}')}$, and integrating w.r.t. $\int\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{\theta}'/((2\pi)^{2d})$. We obtain

$$\frac{\partial G_{-\mathbf{k}\mathbf{k}'}}{\partial t} - \mathrm{i}\,\Delta\Omega\,G_{-\mathbf{k}\mathbf{k}'} + m\,F(\mathbf{J}')\,\mathrm{i}\mathbf{k}\cdot\frac{\partial F(\mathbf{J})}{\partial \mathbf{J}}\,\psi_{-\mathbf{k}-\mathbf{k}'}(\mathbf{J},\mathbf{J}') - m\,F(\mathbf{J})\,\mathrm{i}\mathbf{k}'\cdot\frac{\partial F(\mathbf{J}')}{\partial \mathbf{J}'}\,\psi_{\mathbf{k}'\mathbf{k}}(\mathbf{J}',\mathbf{J}) = 0. \tag{52}$$

with $\Delta\Omega = \mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}) - \mathbf{k}'\cdot\boldsymbol{\Omega}(\mathbf{J}')$. Using the definition of the bare susceptibility coefficients, and the fact that the interaction potential U is real, we immediately have the symmetry relations $\psi_{-\mathbf{k}-\mathbf{k}'}(\mathbf{J},\mathbf{J}') = \psi^*_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')$, and $\psi_{\mathbf{k}'\mathbf{k}}(\mathbf{J}',\mathbf{J}) = \psi^*_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')$. As a consequence, the evolution equation for $G_{-\mathbf{k}\mathbf{k}'}$ reads

$$\frac{\partial G_{-\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')}{\partial t} - \mathrm{i}\,\Delta\Omega\,G_{-\mathbf{k}\mathbf{k}'} + \mathrm{i}m\,\psi^*_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} - \mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'}\right)F(\mathbf{J})\,F(\mathbf{J}') = 0. \tag{53}$$

With the initial condition $G(t=0)=0$, this evolution equation is straightforward to solve. It is of the form

$$\frac{\partial G}{\partial t} - \mathrm{i}\Delta\Omega\,G + S = 0. \tag{54}$$

Using the variation of the constant, we write $G(t) = G_0(t)\,\mathrm{e}^{\mathrm{i}\Delta\Omega t}$, and we obtain

$$\frac{\partial G_0}{\partial t}\,\mathrm{e}^{\mathrm{i}\Delta\Omega t} + S = 0. \tag{55}$$

Since the source term can be taken to be constant in time, and using the initial condition $G_0(t=0) = 0$, we obtain

$$G_0(t) = -S\int_0^t \mathrm{d}t'\,\mathrm{e}^{-\mathrm{i}\Delta\Omega t'} = S\,\frac{\mathrm{e}^{-\mathrm{i}\Delta\Omega t} - 1}{\mathrm{i}\Delta\Omega}. \tag{56}$$

All in all, we obtain

$$G_{-\mathbf{k}\mathbf{k}'} = m\,\psi^*_{\mathbf{k}\mathbf{k}'}\,\frac{\mathrm{e}^{\mathrm{i}\Delta\Omega t} - 1}{\Delta\Omega}\left(\mathbf{k}'\!\cdot\!\frac{\partial}{\partial\mathbf{J}'} - \mathbf{k}\!\cdot\!\frac{\partial}{\partial\mathbf{J}}\right) F(\mathbf{J})\,F(\mathbf{J}'). \tag{57}$$

d. The inhomogeneous Landau equation.

Rewriting the first BBGKY equation in Fourier space is also straightforward. It reads

$$\frac{\partial F(\mathbf{J})}{\partial t} = -(2\pi)^d\,\frac{\partial}{\partial\mathbf{J}}\cdot\left[\int\!\mathrm{d}\mathbf{J}'\!\int\!\frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^d}\,\frac{\mathrm{d}\boldsymbol{\theta}'}{(2\pi)^d}\sum_{\substack{\mathbf{k}_G,\mathbf{k}'_G\\ \mathbf{k}_U,\mathbf{k}'_U}} \mathrm{i}\mathbf{k}_G\,G_{\mathbf{k}_G\mathbf{k}'_G}\,\mathrm{e}^{\mathrm{i}(\mathbf{k}_G\cdot\boldsymbol{\theta}+\mathbf{k}'_G\cdot\boldsymbol{\theta}')}\,\psi_{\mathbf{k}_U\mathbf{k}'_U}\,\mathrm{e}^{\mathrm{i}(\mathbf{k}_U\cdot\boldsymbol{\theta}-\mathbf{k}'_U\cdot\boldsymbol{\theta}')}\right], \tag{58}$$

where one should pay attention to the prefactor $(2\pi)^d$. The angular integrals impose the constraints $\mathbf{k}_G = -\mathbf{k}_U$ and $\mathbf{k}'_G = \mathbf{k}'_U$, so paying attention to the overall sign, we finally obtain

$$\frac{\partial F(\mathbf{J})}{\partial t} = (2\pi)^d\,\frac{\partial}{\partial\mathbf{J}}\cdot\left[\sum_{\mathbf{k},\mathbf{k}'}\mathrm{i}\mathbf{k}\!\int\!\mathrm{d}\mathbf{J}'\,G_{-\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\,\psi_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}')\right]. \tag{59}$$

Obtaining the final expression for the inhomogeneous Landau equation is only a matter of reinjecting one equation into the other, using the asymptotic formula, and neglecting Cauchy principal value because the diffusion flux is real.

4. Conservations, H-Theorem, and Balescu-Lenard equation

As in the notes, we write the Balescu-Lenard equation as $\partial F/\partial = -\partial/\partial\mathbf{J}\cdot\mathbf{F}(\mathbf{J})$, The time-derivative of the total mass is straightforward to compute and reads

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\!\int\!\mathrm{d}\mathbf{J}\,\frac{\partial}{\partial\mathbf{J}}\cdot\mathbf{F}(\mathbf{J}) = 0, \tag{60}$$

which vanishes as it is the integral of a divergence.

The time derivative of the total energy reads

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= -\!\int\!\mathrm{d}\mathbf{J}\,H(\mathbf{J})\,\frac{\partial}{\partial\mathbf{J}}\cdot\mathbf{F}(\mathbf{J})\\ &= \!\int\!\mathrm{d}\mathbf{J}\,\boldsymbol{\Omega}(\mathbf{J})\cdot\mathbf{F}(\mathbf{J}),\end{aligned} \tag{61}$$

where we used an integration by parts, and the definition of the orbital frequencies as $\boldsymbol{\Omega}(\mathbf{J}) = \partial H/\partial\mathbf{J}$. Injecting the explicit expression of the Balescu-Lenard flux, we can rewrite this time derivative as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sum_{\mathbf{k},\mathbf{k}'}\!\int\!\mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,(\mathbf{k}\!\cdot\!\boldsymbol{\Omega})\,\big|\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\!\cdot\!\boldsymbol{\Omega})\big|^2\,\delta_{\mathrm{D}}(\mathbf{k}\!\cdot\!\boldsymbol{\Omega} - \mathbf{k}'\!\cdot\!\boldsymbol{\Omega}')\left(\mathbf{k}'\!\cdot\!\frac{\partial}{\partial\mathbf{J}'} - \mathbf{k}\!\cdot\!\frac{\partial}{\partial\mathbf{J}}\right)F(\mathbf{J})\,F(\mathbf{J}') \tag{62}$$

We can now symmetrise this expression by performing the change $(\mathbf{k},\mathbf{J}) \leftrightarrow (\mathbf{k}',\mathbf{J}')$. This gives

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}'}\!\int\!\mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,&\Big((\mathbf{k}\!\cdot\!\boldsymbol{\Omega})\,\big|\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\!\cdot\!\boldsymbol{\Omega})\big|^2 - (\mathbf{k}'\!\cdot\!\boldsymbol{\Omega}')\,\big|\psi^{\mathrm{d}}_{\mathbf{k}'\mathbf{k}}(\mathbf{J}',\mathbf{J},\mathbf{k}'\!\cdot\!\boldsymbol{\Omega}')\big|^2\Big)\\ &\times\delta_{\mathrm{D}}(\mathbf{k}\!\cdot\!\boldsymbol{\Omega} - \mathbf{k}'\!\cdot\!\boldsymbol{\Omega}')\left(\mathbf{k}'\!\cdot\!\frac{\partial}{\partial\mathbf{J}'} - \mathbf{k}\!\cdot\!\frac{\partial}{\partial\mathbf{J}}\right)F(\mathbf{J})\,F(\mathbf{J}').\end{aligned} \tag{63}$$

The dressed susceptibility coefficients satisfy the symmetry relation $\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\omega) = \psi^{\mathrm{d}*}_{\mathbf{k}'\mathbf{k}}(\mathbf{J}',\mathbf{J},\omega)$. Using this relation and the resonance condition to match the temporal frequency at which they are evaluated, we obtain

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}'}\!\int\!\mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\,(\mathbf{k}\!\cdot\!\boldsymbol{\Omega} - \mathbf{k}'\!\cdot\!\boldsymbol{\Omega}')\,\delta_{\mathrm{D}}(\mathbf{k}\!\cdot\!\boldsymbol{\Omega} - \mathbf{k}'\!\cdot\!\boldsymbol{\Omega}')\big|\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\!\cdot\!\boldsymbol{\Omega})\big|^2\left(\mathbf{k}'\!\cdot\!\frac{\partial}{\partial\mathbf{J}'} - \mathbf{k}\!\cdot\!\frac{\partial}{\partial\mathbf{J}}\right)F(\mathbf{J})\,F(\mathbf{J}'). \tag{64}$$

This finally gives us $\mathrm{d}E/\mathrm{d}t = 0$, owing to the vanishing term $(\mathbf{k}\cdot\mathbf{\Omega} - \mathbf{k}'\cdot\mathbf{\Omega}')\,\delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega} - \mathbf{k}'\cdot\mathbf{\Omega}')$.

We can follow a similar approach to compute the time derivative of the entropy. We write

$$
\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= \int \mathrm{d}\mathbf{J}\, s'(F(\mathbf{J}))\, \frac{\partial F(\mathbf{J})}{\partial t} \\
&= -\int \mathrm{d}\mathbf{J}\, s'(F(\mathbf{J}))\, \frac{\partial}{\partial \mathbf{J}} \cdot \mathbf{F}(\mathbf{J}) \\
&= \int \mathrm{d}\mathbf{J}\, s''(F(\mathbf{J}))\, \frac{\partial F(\mathbf{J})}{\partial \mathbf{J}} \cdot \mathbf{F}(\mathbf{J}) \\
&= -\int \mathrm{d}\mathbf{J}\, \frac{1}{F(\mathbf{J})}\, \frac{\partial F(\mathbf{J})}{\partial \mathbf{J}} \cdot \mathbf{F}(\mathbf{J}),
\end{aligned}
\tag{65}
$$

where we used the fact that $s''(F) = -1/F$. Using the explicit expression of the diffusion flux, we get

$$
\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} = \sum_{\mathbf{k},\mathbf{k}'} \int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\, & \frac{-\big(\mathbf{k}\cdot\partial F/\partial \mathbf{J}\big)\, F(\mathbf{J}')}{F(\mathbf{J})F(\mathbf{J}')} \left|\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\mathbf{\Omega})\right|^2 \\
& \times\, \delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega} - \mathbf{k}'\cdot\mathbf{\Omega}') \left(\mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'} - \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\right) F(\mathbf{J})\, F(\mathbf{J}').
\end{aligned}
\tag{66}
$$

Symmetrising w.r.t. $(\mathbf{k},\mathbf{J}) \leftrightarrow (\mathbf{k}',\mathbf{J}')$, and using the fact that the dressed susceptbility coefficients are left invariant under that change, we get

$$
\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{1}{2} \sum_{\mathbf{k},\mathbf{k}'} \int \mathrm{d}\mathbf{J}\mathrm{d}\mathbf{J}'\, & \frac{\left|\psi^{\mathrm{d}}_{\mathbf{k}\mathbf{k}'}(\mathbf{J},\mathbf{J}',\mathbf{k}\cdot\mathbf{\Omega})\right|^2}{F(\mathbf{J})\, F(\mathbf{J}')}\, \delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega} - \mathbf{k}'\cdot\mathbf{\Omega}') \\
& \times \left[\left(\mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'} - \mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}}\right) F(\mathbf{J})\, F(\mathbf{J}')\right]^2.
\end{aligned}
\tag{67}
$$

As a conclusion, we therefore recover that $\mathrm{d}S/\mathrm{d}t \geq 0$, i.e. the Balescu-Lenard satisfies a H-Theorem, highlighting the irreversible orbital relaxation undergone by the system.

Because $\partial H/\partial \mathbf{J} = \mathbf{\Omega}(\mathbf{J})$ for a Boltzmann distribution, we have $\partial F_{\mathrm{B}}/\partial t \propto \mathbf{\Omega}(\mathbf{J})\, F_{\mathrm{B}}(\mathbf{J})$. As a consequence, the associated Balescu-Lenard flux reads

$$
\begin{aligned}
\frac{\partial F_{\mathrm{B}}}{\partial t} &\propto \delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega} - \mathbf{k}'\cdot\mathbf{\Omega}') \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{J}} - \mathbf{k}'\cdot\frac{\partial}{\partial \mathbf{J}'}\right) F_{\mathrm{B}}(\mathbf{J})\, F_{\mathrm{B}}(\mathbf{J}') = 0 \\
&\propto \delta_{\mathrm{D}}(\mathbf{k}\cdot\mathbf{\Omega} - \mathbf{k}'\cdot\mathbf{\Omega}') \big(\mathbf{k}\cdot\mathbf{\Omega} - \mathbf{k}'\cdot\mathbf{\Omega}'\big)\, F_{\mathrm{B}}(\mathbf{J})\, F_{\mathrm{B}}(\mathbf{J}') \\
&= 0.
\end{aligned}
\tag{68}
$$

We recover therefore that Boltzmann's distributions are the expected endstates of relaxation. Yet, they can very rarely be reached, as they are rarely associated with physically admissible states.

5. Kinetic blockings

Let us first write the Balescu-Lenard equation for a $1D$ system. By direct analogy, it reads

$$
\begin{aligned}
\frac{\partial F(J,t)}{\partial t} = -2\pi^2\, m\, \frac{\partial}{\partial J}\Bigg[\sum_{k,k'} k \int \mathrm{d}J'\, & \left|\psi^{\mathrm{d}}_{kk'}(J,J',k\,\Omega(J))\right|^2 \delta_{\mathrm{D}}(k'\Omega' - k\Omega) \\
& \times \left(k\frac{\partial}{\partial J} - k'\frac{\partial}{\partial J'}\right) F(J,t)\, F(J',t)\Bigg],
\end{aligned}
\tag{69}
$$

where we emphasise that the action, J, the orbital frequency, $\Omega(J)$, and the resonance vectors, k and k', are now scalar quantities.

Similarly, we can also write the system's dressed susceptibility coefficients that read

$$
\psi^{\mathrm{d}}_{kk'}(J,J',\omega) = \sum_{p,q} \psi^{(p)}_{k}(J)\, \big[\mathbf{I} - \widetilde{\mathbf{M}}(\omega)\big]^{-1}_{pq}\, \psi^{(q)*}_{k'}(J'),
\tag{70}
$$

where the response matrix, $\widetilde{\mathbf{M}}_{pq}(\omega)$, reads

$$
\widetilde{\mathbf{M}}_{pq}(\omega) = 2\pi \sum_{k} \int \mathrm{d}J\, \frac{k\partial F/\partial J}{\omega - k\Omega(J)}\, \psi^{(p)*}_{k}(J)\, \psi^{(q)}_{k}(J).
\tag{71}
$$

Because the basis elements satisfy the symmetry constraint $\psi_k^{(p)} \propto \delta_p^k$, we immediately get that the response matrix is diagonal, i.e. $\widetilde{\mathbf{M}}_{pq}(\omega) \propto \delta_p^q$. Using this constraint in the expression of the dressed susceptibility coefficients, we similarly get $\psi^{\mathrm{d}}_{kk'}(J, J') \propto \delta_k^{k'}$. This allows us to get rid of the sum over k' in Eq. (69). Finally, using the relation $\delta_{\mathrm{D}}(k(\Omega - \Omega') = \delta_{\mathrm{D}}(\Omega - \Omega')/|k|$, we can rewrite the Balescu-Lenard equation as

$$\frac{\partial F(J)}{\partial t} = -2\pi^2\, m\, \frac{\partial}{\partial J} \bigg[\int \! \mathrm{d}J' \left| \psi^{\mathrm{tot}}(J, J') \right|^2 \delta_{\mathrm{D}}(\Omega(J) - \Omega(J')) \left(\frac{\partial}{\partial J'} - \frac{\partial}{\partial J} \right) F(J)\, F(J') \bigg], \tag{72}$$

where we introduced the total coupling coefficients, $\left| \psi^{\mathrm{tot}}(J, J') \right|^2$ as

$$\psi^{\mathrm{tot}}(J, J') = \sum_k |k| \left| \psi^{\mathrm{d}}_{kk}(J, J', k\Omega(J)) \right|^2. \tag{73}$$

Generically, a $1D$ resonance condition over the orbital frequencies can be rewritten as

$$\delta_{\mathrm{D}}(\Omega(J) - \Omega(J')) = \sum_{J'_* | \Omega(J'_*) = \Omega(J)} \frac{\delta_{\mathrm{D}}(J' - J'_*)}{|\partial\Omega/\partial J|_{J'_*}}, \tag{74}$$

where J'_* are the locations in orbital space where the resonance condition $\Omega(J'_*) = \Omega(J)$ is satisfied. If the frequency profile is monotonic, the constraint $\Omega(J'_*) = \Omega(J)$ naturally translates into $J'_* = J$. As a consequence, the resonance condition becomes

$$\delta_{\mathrm{D}}(\Omega(J) - \Omega(J')) = \frac{\delta_{\mathrm{D}}(J - J')}{|\partial\Omega/\partial J|_J}. \tag{75}$$

When injected into Eq. (72), we immediately note that the crossed gradient term exactly vanishes. As a consequence, we have $\partial F(J, t)/\partial t = 0$. This implies that such a system cannot relax under the Balescu-Lenard dynamics as its diffusion flux exactly vanishes. This does not mean that the system's DF cannot keep evolving, it only means that $1/N$ effects are inefficient at sourcing its evolution. It is only under the effects of higher-order contributions, e.g., $1/N^2$ effects, that the system can (slowly) relax towards its thermodynamical equilibrium, should it exist.