Problem sheet 4 Solutions

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1. a) The metric is

$$g = -\left(1 - \frac{2M}{r}\right) \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)$$

and its inverse is

$$g^{-1} = +2\partial_v\partial_r + \left(1 - \frac{2M}{r}\right) \partial_r^2 + r^{-2} \left(\partial_\theta^2 + (\sin\theta)^{-2}\partial_\phi^2\right)$$

Both g and g^{-1} have smooth components in ingoing Eddington-Finkelstein coordinates at $r = 2M$, so they allow us to deal with the region $r < 2M$ as well as $r > 2M$.

b) The easiest way to find the geodesic equations is to use the Lagrangian

$$\mathcal{L} = g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}$$

which (writing a 'dot' for an s derivative) is

$$\mathcal{L} = -\left(1 - \frac{2M}{r}\right) \dot{v}^2 + 2\dot{v}\dot{r} + r^2 \left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)$$

i) Looking at the Euler-Lagrange equations associated with varying θ, we find that

$$2r^2\ddot{\theta} + 4r\dot{\theta}\dot{r} = 2r^2\sin\theta\cos\theta\dot{\phi}^2$$

and $\theta \equiv \frac{\pi}{2}$ solves this equation with initial data $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$. By uniqueness this is the only solution with this initial data.

Using the form of the Lagrangian, we can immediately identify three conserved quantities:

ii) Since $\mathcal{L}$ is independent of v, we have a conserved quantity E,

$$E = \left(1 - \frac{2M}{r}\right) \dot{v} - \dot{r}$$

Any quantity proportional to E is also conserved and would be a correct answer to this question.

iii) Since $\mathcal{L}$ is independent of ϕ, we have a conserved quantity m,

$$m = r^2\sin^2\theta\dot{\phi}$$

As above, any quantity proportional to m is also conserved. If we set $\theta = \frac{\pi}{2}$, then for such orbits, $m = r^2\dot{\phi}$.

iv) Since the curve $x^a(s)$ is an affinely parametrised null geodesic, $\mathcal{L} = 0$. Hence

$$0 = -\left(1 - \frac{2M}{r}\right)^{-1} (E + \dot{r})^2 + 2\left(1 - \frac{2M}{r}\right)^{-1} (E + \dot{r})\dot{r} + r^{-2}m^2$$

and therefore

$$\left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2-\left(1-\frac{2M}{r}\right)^{-1}E^2+r^{-2}m^2=0$$

$$\Rightarrow \dot{r}^2=E^2-r^{-2}\left(1-\frac{2M}{r}\right)m^2$$

or

$$\frac{\mathrm{d}r}{\mathrm{d}s}=\pm\sqrt{E^2-r^{-2}\left(1-\frac{2M}{r}\right)m^2}$$

c) One way to do this question is to return to the full geodesic equations. Alternatively, we can use the work done above: by applying an isometry, we can take the constant θ coordinate to be $\theta_0=\frac{\pi}{2}$. If $m=0$ then the equation for ϕ is simply $\dot{\phi}=0$, so ϕ maintains a constant value along the geodesic. Finally, examining the equation for $\dot{r}$ and setting $m=0$ we see that $\dot{r}=0$ if we set $E=0$.

Substituting this into the equation defining E, we find that

$$\left(1-\frac{2M}{r_0}\right)\dot{v}=0$$

now, if r_0 takes any value other than $2M$, we would be forced to have $\dot{v}=0$, but this would mean that the tangent vector to the curve was simply zero, and not a null vector. However, since $r_0=2M$, we do not necessarily have to have $\dot{v}=0$.

Now, by examining the Euler-Lagrange equation associated with varying r (and setting $\theta=\frac{\pi}{2}$, $\phi=\phi_0$), we see that v satisfies

$$2\ddot{v}=-\frac{2M}{r^2}\dot{v}^2$$

setting $r=2M$, this is simply

$$\ddot{v}+\frac{1}{4M}\dot{v}^2=0$$

Setting $w=\dot{v}$, we have

$$\frac{1}{4M}(s-s_0)=\int_0^w-\frac{1}{(w')^2}\mathrm{d}w'=\frac{1}{w}$$

and so

$$\frac{\mathrm{d}v}{\mathrm{d}s}=4M\frac{1}{(s+4M(v_0)^{-1})}$$

is the solution with $v=v_0$ when $s=0$. Hence

$$v=4M\log\left(\frac{s+4M(v_0)^{-1}}{4M(v_0)^{-1}}\right)$$

is the solution with the initial data $v=0$ when $s=0$. We can also invert this relationship to obtain

$$s=4M(v_0)^{-1}\left(e^{\frac{v}{4M}}-1\right)$$

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2. a) The metric in Schwarzschild coordinates is

$$g=-\left(\frac{2M}{r}-1\right)^{-1}\mathrm{d}r^2+\left(\frac{2M}{r}-1\right)\mathrm{d}t^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right)$$

b) Let $\gamma(\tau)$ be a timelike curve, represented in local coordinates by $x^a(\tau)$, where τ is proper time along the curve. The tangent to this curve has components X^a, where $X^a=\frac{\mathrm{d}x^a}{\mathrm{d}\tau}$. Since this curve is timelike and τ is the proper time, we have

$$\begin{aligned}
-1 &= g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau} \\
&= -\left(\frac{2M}{r}-1\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \left(\frac{2M}{r}-1\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + r^2\left(\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right)
\end{aligned}$$

Since $r < 2M$, the first term on the right hand side is the only one which can be negative, and so we see that $\left|\frac{\mathrm{d}r}{\mathrm{d}\tau}\right| > 0$. Rearranging the equation above, we see that

$$\begin{aligned}
\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 &= \left(\frac{2M}{r}-1\right) + \left(\frac{2M}{r}-1\right)^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(\frac{2M}{r}-1\right)r^2\left(\left(\frac{\mathrm{d}\theta}{\mathrm{d}\tau}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right) \\
&\geq \left(\frac{2M}{r}-1\right)
\end{aligned}$$

Since r decreases with τ (as the geodesic is future-directed), we have

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} \leq -\sqrt{\frac{2M}{r}-1}$$

c) Using the previous part, we see that the proper time taken to move from $r = 2M$ to $r = 0$ is bounded above by

$$\begin{aligned}
\tau &\leq \int_0^{2M} \frac{1}{\sqrt{\frac{2M}{r}-1}}\mathrm{d}r \\
&\leq \frac{1}{\sqrt{2M}}\int_0^{2M}\frac{\sqrt{r}}{\sqrt{1-\frac{r}{2M}}}\mathrm{d}r
\end{aligned}$$

Now set $u = \sqrt{1-\frac{r}{2M}}$, i.e. $r = 2M(1-u^2)$. So $\mathrm{d}r = -4Mu\mathrm{d}u$, and the integral above becomes

$$\begin{aligned}
\tau &\leq \int_0^1 \frac{\sqrt{1-u^2}}{u}\cdot 4Mu\mathrm{d}u \\
&\leq 4M\int_0^1\sqrt{1-u^2}\mathrm{d}u
\end{aligned}$$

Now we can set $u = \sin\theta$ to obtain

$$\begin{aligned}
\tau &\leq 4M\int_0^{\frac{\pi}{2}}\cos^2\theta\mathrm{d}\theta \\
&\leq M\pi
\end{aligned}$$

If you fall into a larger black hole (with larger mass M), then you can live a longer proper time before you hit the singularity. So it is better to fall into a larger black hole!

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3. a) We can compute

$$r = \rho + M + \frac{M^2}{4\rho}$$

and so

$$\mathrm{d}r = \left(1-\frac{M^2}{4\rho^2}\right)\mathrm{d}\rho = \frac{(2\rho+M)(2\rho-M)}{4\rho^2}\mathrm{d}\rho$$

We can also compute

$$1 - \frac{2M}{r} = \left(\frac{2\rho - M}{2\rho + M}\right)^2$$

Putting these together, we can write the Schwarzschild metric as

$$\begin{aligned} g &= -\left(\frac{(2\rho - M)}{(2\rho + M)}\right)^2 \mathrm{d}t^2 + \left(\frac{2\rho + M}{2\rho}\right)^4 \left(\mathrm{d}\rho^2 + \rho^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\right) \\ &= -\left(\frac{(2\rho - M)}{(2\rho + M)}\right)^2 \mathrm{d}t^2 + \left(\frac{2\rho + M}{2\rho}\right)^4 \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \end{aligned}$$

where in the second line we could also substitute for ρ, using

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

Since the coordinate r lies in the range $r \in (2M, \infty)$, the coordinate ρ lies in the range $\rho \in \left(\frac{1}{2}M, \infty\right)$. At the inner boundary of this range, i.e. as $\rho \to \frac{1}{2}M$, the metric in these coordinates degenerates: the component $g_{tt} \to 0$, and the component of the inverse metric $(g^{-1})^{tt} \to \infty$.

b) The light cone at the point p is, by definition, the set of vector fields such that $g(X, X) = 0$ at p. In isotropic coordinates, this means that, for $X \in C_p$,

$$\begin{aligned} -\left(\frac{(2\rho - M)}{(2\rho + M)}\right)^2 (X^t)^2 + \left(\frac{2\rho + M}{2\rho}\right)^4 \left((X^x)^2 + (X^y)^2 + (X^z)^2\right) = 0 \\ \Rightarrow (X^t)^2 = \frac{(2\rho + M)^6}{16\rho^4(2\rho - M)^2} \left((X^x)^2 + (X^y)^2 + (X^z)^2\right) \end{aligned}$$

so the light cones are isotropic.

c) Let $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ be the usual rectangular coordinates on $\mathbb{R}^3$. Consider the map

$$\begin{aligned} p \mapsto (\tilde{x}(p), \tilde{y}(p), \tilde{z}(p)) \\ (\tilde{x}(p), \tilde{y}(p), \tilde{z}(p)) = (x(p), y(p), z(p)) \end{aligned}$$

In other words, we simply use the coordinates (x, y, z) to define the map to $\mathbb{R}^3$.

Now, for any vectors X, Y, we can compute

$$h(X, Y) = \frac{(2\rho + M)^6}{16\rho^4(2\rho - M)^2} \left(X^x Y^x + X^y Y^y + X^z Y^z\right)$$

and so, if ϑ is the angle between the vectors X and Y as defined in the example sheet, then

$$\begin{aligned} \cos\vartheta &= \frac{h(X, Y)}{\sqrt{h(X, X)h(Y, Y)}} \\ &= \frac{X^x Y^x + X^y Y^y + X^z Y^z}{\sqrt{\left((X^x)^2 + (X^y)^2 + (X^z)^2\right)\left((Y^x)^2 + (Y^y)^2 + (Y^z)^2\right)}} \\ &= \frac{\boldsymbol{X} \cdot \boldsymbol{Y}}{|\boldsymbol{X}||\boldsymbol{Y}|} \end{aligned}$$

which is the usual expression for $\cos\theta$, where θ is the angle between the vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\mathbb{R}^3$. Hence $\vartheta = \theta$, i.e. this map preserves angles.

4.

The Friedmann equations with $k = 1$, $p = 0$ are

$$3\frac{\dot{a}^2 + 1}{a^2} - \Lambda = 8\pi\rho$$
$$2a\ddot{a} + \dot{a}^2 + 1 - a^2\Lambda = 0$$

a) Setting $\dot{a} = \ddot{a} = 0$, we obtain

$$\frac{3}{a^2} - \Lambda = 8\pi\rho$$
$$1 - a^2\Lambda = 0$$

From the second equation we obtain

$$a = \Lambda^{-\frac{1}{2}} = a_0$$

and from the first equation we obtain

$$\rho = \frac{1}{4\pi}\Lambda = \rho_0$$

b) Expanding the Friedmann equations around this solution, we obtain

$$\rho_1 = \frac{3\Lambda^{\frac{3}{2}}}{4\pi}a_1$$
$$\ddot{a}_1 - \Lambda a_1 = 0$$

The general solution to which is

$$a_1 = Ae^{\sqrt{\Lambda}\tau} + Be^{-\sqrt{\Lambda}\tau}$$
$$\rho_1 = \frac{3\Lambda^{\frac{3}{2}}}{4\pi}\left(Ae^{\sqrt{\Lambda}\tau} + Be^{-\sqrt{\Lambda}\tau}\right)$$

for constants A and B. So for a generic solution ($A \neq 0$) both the scale factor a and the matter density ρ diverge exponentially from the background values a_0 and ρ_0.

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5. a) The Robertson-Walker metric is

$$g = -\mathrm{d}\tau^2 + a^2\left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right)$$

Setting $\mathrm{d}\tau = a\mathrm{d}\eta$, we obtain the expression

$$g = a^2\left(-\mathrm{d}\eta^2 + \frac{\mathrm{d}r^2}{1 - kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right)$$

b) If $\sin\chi = r$, then $\mathrm{d}r = \cos\chi\mathrm{d}\chi$. Hence

$$g = a^2\left(-\mathrm{d}\eta^2 + \mathrm{d}\chi^2 + \sin^2\chi\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right)$$

i) The spatial part of the metric is proportional to the standard metric on the 3-sphere $\mathbb{S}^3$, expressed in "hyperspherical coordinates".

ii) We can find geodesics by extremising the action associated with the Lagrangian

$$\mathcal{L} = a^2\left(-\left(\frac{\mathrm{d}\eta}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}\chi}{\mathrm{d}\lambda}\right)^2 + \sin^2\chi\left(\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\right)\right)$$

where λ is an affine parameter. The Euler-Lagrange equation for χ is

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}\lambda^2} + 2a^{-1}\frac{\mathrm{d}a}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}\lambda}\frac{\mathrm{d}\chi}{\mathrm{d}\lambda} - \sin\chi\cos\chi\left(\left(\frac{\mathrm{d}\theta}{\mathrm{d}\lambda}\right)^2 + \sin^2\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2\right) = 0$$

$\chi \equiv \frac{\pi}{2}$ solves this equation with initial data $\chi = \frac{\pi}{2}$, $\frac{\mathrm{d}\chi}{\mathrm{d}\lambda} = 0$. By uniqueness of solutions to ODEs, this is the unique solution with this initial data.

The Euler-Lagrange equation for θ is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\lambda^2} + 2a^{-1}\frac{\mathrm{d}a}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}\lambda}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} + 2\cot\chi\frac{\mathrm{d}\chi}{\mathrm{d}\lambda}\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} - \sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 = 0$$

Note that, if $\chi \equiv \frac{\pi}{2}$, then the third term vanishes identically. So, for a solution where $\chi \equiv \frac{\pi}{2}$, $\theta \equiv \frac{\pi}{2}$ solves the Euler-Lagrange equation with initial data $\theta = \frac{\pi}{2}$, $\frac{\mathrm{d}\theta}{\mathrm{d}\lambda} = 0$.

Along a null, equatorial geodesic, we have $\mathcal{L} = 0$, so

$$\begin{aligned} 0 &= -\left(\frac{\mathrm{d}\eta}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 \\ \Rightarrow 0 &= -1 + \left(\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right)^2 \end{aligned}$$

where in the second line we have written $\lambda = \lambda(\eta)$ and used the chain rule. Thus $\frac{\mathrm{d}\phi}{\mathrm{d}\eta} = \pm 1$.

c) In the radiation dominated universe, the equation of state for matter is

$$p = \frac{1}{3}\rho$$

Hence, from the equation (which itself follows from the Friedmann equations)

$$\dot{\rho} = -3\frac{\dot{a}}{a}(p + \rho)$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\log\rho = -4\frac{\mathrm{d}}{\mathrm{d}\tau}\log a$$

and so

$$\rho = \rho_0 a^{-4}$$

for some constant ρ_0.

d) The Friedmann equations give

$$3\frac{\dot{a}^2 + 1}{a^2} = 8\pi\rho$$

where $\dot{a} = \frac{\mathrm{d}a}{\mathrm{d}\tau}$. But we can calculate

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = \frac{\mathrm{d}a}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = a^{-1}\frac{\mathrm{d}a}{\mathrm{d}\eta}$$

so the Friedmann equations give

$$\begin{aligned} 3\frac{\left(\frac{\mathrm{d}a}{\mathrm{d}\eta}\right)^2 + a^2}{a^4} &= 8\pi\rho_0 a^{-4} \\ \Rightarrow 3\left(\frac{\mathrm{d}a}{\mathrm{d}\eta}\right)^2 + 3a^2 &= 8\pi\rho_0 \end{aligned} \tag{1}$$

We can write this equation as

$$\frac{1}{\sqrt{\frac{8\pi\rho_0}{3} - a^2}}\frac{\mathrm{d}a}{\mathrm{d}\eta} = \pm 1$$

Choosing the positive root for now (since $\dot{a} > 0$ initially), we find that the solution with $a = 0$ at $\eta = 0$ is

$$a = \sqrt{\frac{8\pi\rho_0}{3}} \sin\eta$$

In fact, this is a solution to the equation 1 for $\dot{a} < 0$ also.

The "big crunch" occurs when $a = 0$ and $\eta > 0$, which occurs when $\eta = \pi$.

e) A photon moving on an equatorial orbit has $\phi - \phi_0 = \pm\eta$. Since the big crunch occurs when $\eta = \pi$, this photon can travel exactly half way around the universe before the end of time.

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6. a) i) We can calculate

$$\begin{aligned}\mathrm{d}x^0 = &\left(\sqrt{\frac{3}{\Lambda}}\sinh\left(\sqrt{\frac{\Lambda}{3}}\tau\right) + \frac{1}{2}\sqrt{\frac{\Lambda}{3}}r^2e^{\sqrt{\frac{\Lambda}{3}}\tau}\right)\mathrm{d}\rho + \rho\left(\cosh\left(\sqrt{\frac{\Lambda}{3}}\tau\right) + \frac{\Lambda}{6}r^2e^{\sqrt{\frac{\Lambda}{3}}\tau}\right)\mathrm{d}\tau\\ &+ \rho\left(\sqrt{\frac{\Lambda}{3}}re^{\sqrt{\frac{\Lambda}{3}}\tau}\right)\mathrm{d}r\end{aligned}$$

$$\begin{aligned}\mathrm{d}x^1 = &\left(\sqrt{\frac{3}{\Lambda}}\cosh\left(\sqrt{\frac{\Lambda}{3}}\tau\right) - \frac{1}{2}\sqrt{\frac{\Lambda}{3}}r^2e^{\sqrt{\frac{\Lambda}{3}}\tau}\right)\mathrm{d}\rho + \rho\left(\sinh\left(\sqrt{\frac{\Lambda}{3}}\tau\right) - \frac{\Lambda}{6}r^2e^{\sqrt{\frac{\Lambda}{3}}\tau}\right)\mathrm{d}\tau\\ &- \rho\left(\sqrt{\frac{\Lambda}{3}}re^{\sqrt{\frac{\Lambda}{3}}\tau}\right)\mathrm{d}r\end{aligned}$$

$$\begin{aligned}\mathrm{d}x^2 = &\left(e^{\sqrt{\frac{\Lambda}{3}}\tau}r\sin\theta\cos\phi\right)\mathrm{d}\rho + \left(\sqrt{\frac{\Lambda}{3}}\rho e^{\sqrt{\frac{\Lambda}{3}}\tau}r\sin\theta\cos\phi\right)\mathrm{d}\tau + \left(\rho e^{\sqrt{\frac{\Lambda}{3}}\tau}\sin\theta\cos\phi\right)\mathrm{d}r\\ &+ \left(\rho e^{\sqrt{\frac{\Lambda}{3}}\tau}r\cos\theta\cos\phi\right)\mathrm{d}\theta - \left(\rho e^{\sqrt{\frac{\Lambda}{3}}\tau}r\sin\theta\sin\phi\right)\mathrm{d}\phi\end{aligned}$$

$$\begin{aligned}\mathrm{d}x^3 = &\left(e^{\sqrt{\frac{\Lambda}{3}}\tau}r\sin\theta\sin\phi\right)\mathrm{d}\rho + \left(\sqrt{\frac{\Lambda}{3}}\rho e^{\sqrt{\frac{\Lambda}{3}}\tau}r\sin\theta\sin\phi\right)\mathrm{d}\tau + \left(\rho e^{\sqrt{\frac{\Lambda}{3}}\tau}\sin\theta\sin\phi\right)\mathrm{d}r\\ &+ \left(\rho e^{\sqrt{\frac{\Lambda}{3}}\tau}r\cos\theta\sin\phi\right)\mathrm{d}\theta + \left(\rho e^{\sqrt{\frac{\Lambda}{3}}\tau}r\sin\theta\cos\phi\right)\mathrm{d}\phi\end{aligned}$$

$$\begin{aligned}\mathrm{d}x^4 = &\left(e^{\sqrt{\frac{\Lambda}{3}}\tau}r\cos\theta\right)\mathrm{d}\rho + \left(\sqrt{\frac{\Lambda}{3}}\rho e^{\sqrt{\frac{\Lambda}{3}}\tau}r\cos\theta\right)\mathrm{d}\tau + \left(\rho e^{\sqrt{\frac{\Lambda}{3}}\tau}\cos\theta\right)\mathrm{d}r\\ &- \left(\rho e^{\sqrt{\frac{\Lambda}{3}}\tau}r\sin\theta\right)\mathrm{d}\theta\end{aligned}$$

i) This allows us to calculate (after a bit of work)

$$-(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2 + (\mathrm{d}x^4)^2 = -\rho^2\mathrm{d}\tau^2 + \frac{3}{\lambda}\mathrm{d}\rho^2 + \rho^2e^{2\sqrt{\frac{\Lambda}{3}}\tau}\left(\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right)$$

ii) We can calculate

$$-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = \rho^2\frac{3}{\Lambda}$$

So the surface $-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = \frac{3}{\Lambda}$ is given by $\rho = \pm 1$. In fact, we have

$$\rho = \sqrt{\frac{\Lambda}{3}\Big(-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2\Big)}$$

and so $\rho \geq 0$, and $\rho = 1$ is the surface $-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = \frac{3}{\Lambda}$

Hence the restriction of the metric to the hypersurface $\rho = 1$ is

$$-\mathrm{d}\tau^2 + e^{2\sqrt{\frac{\Lambda}{3}}\tau}\left(\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right)$$

iii) This metric is of the Robertson-Walker form, with $k = 0$ (i.e. it is *flat*) and with the scale factor

$$a(\tau) = e^{\sqrt{\frac{\Lambda}{3}}\tau}$$

b) i) Using the conformal time, the De Sitter metric becomes

$$\left(1 - \sqrt{\frac{\Lambda}{3}}\eta\right)^{-2}\left(-\mathrm{d}\eta^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right)$$

ii) Past-directed radial null geodesics from the point $r = 0$, $\eta = \eta_0$ satisfy

$$0 = \left(1 - \sqrt{\frac{\Lambda}{3}}\eta\right)^{-2}\left(-\left(\frac{\mathrm{d}\eta}{\mathrm{d}\lambda}\right)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2\right)$$

from which it follows that $\frac{\mathrm{d}r}{\mathrm{d}\eta} = -1$ along past directed radial null geodesics. Hence the particle horizon at the conformal time η_0 is given by $r = \eta_0 - \eta$.

iii) We can compute

$$\eta = \sqrt{\frac{3}{\Lambda}}\left(1 - e^{-\sqrt{\frac{\Lambda}{3}}\tau}\right)$$

and so the particle horizon at the (proper) time τ is given by the surface

$$r = \sqrt{\frac{3}{\Lambda}}\left(e^{-\sqrt{\frac{\Lambda}{3}}\tau} - e^{-\sqrt{\frac{\Lambda}{3}}\tau_0}\right)$$

with $\tau \leq \tau_0$. As $\tau_0 \to \infty$, the particle horizon approaches the surface

$$r = \sqrt{\frac{3}{\Lambda}}e^{-\sqrt{\frac{\Lambda}{3}}\tau}$$

Hence there are events which always lie outside all the particle horizons of this observer. For example, the event with coordinates $(\tau, r, \theta, \phi) = (1, 2\sqrt{\frac{3}{\Lambda}}e^{-\sqrt{\frac{\Lambda}{3}}}, \theta_0, \phi_0)$ lies outside of all the particle horizons (for any choice of the angular coordinates θ_0 and ϕ_0), and hence can never be observed by this observer.

***7. a)** The dual basis is given by

$$\begin{aligned}
e_0 &:= \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}\partial_t \\
e_1 &:= \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}\partial_r \\
e_2 &:= r^{-1}\partial_\theta \\
e_3 &:= r^{-1}(\sin\theta)^{-1}\partial_\theta
\end{aligned}$$

From this we can calculate

$$
\begin{aligned}
(e_0)^\sharp &:= -\left(1-\frac{2M}{r}\right)^{\frac{1}{2}} \mathrm{d}t = m_{0A} f^A \\
(e_1)^\sharp &:= -\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} \mathrm{d}r = m_{1A} f^A \\
(e_2)^\sharp &= r\mathrm{d}\theta = m_{2A} f^A \\
(e_3)^\sharp &= r\sin\theta \mathrm{d}\phi = m_{3A} f^A
\end{aligned}
$$

b) We have

$$
\begin{aligned}
m_{AB} f^A f^B &= -(f^0)^2 + (f^1)^2 + (f^2)^2 + (f^3)^2 \\
&= -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2 \\
&= g
\end{aligned}
$$

and likewise

$$
\begin{aligned}
(m^{-1})^{AB} e_A e_B &= -(e_0)^2 + (e_1)^2 + (e_2)^2 + (e_3)^2 \\
&= -\left(1-\frac{2M}{r}\right)^{-1}(\partial_t)^2 + \left(1-\frac{2M}{r}\right)(\partial_r)^2 + r^{-2}(\partial_\theta)^2 + r^{-2}(\sin\theta)^{-2}(\partial\phi)^2 \\
&= g^{-1}
\end{aligned}
$$

c) i) We have

$$
\eta_{ABC} + \eta_{BAC} = \omega_{AB}(e_C) - \bar{\omega}_{AB}(e_C) + \omega_{BA}(e_C) - \bar{\omega}_{BA}(e_C)
$$

but $\omega_{BA} = -\omega_{AB}$ and $\bar{\omega}_{BA} = -\bar{\omega}_{AB}$ so the expression above vanishes, and $\eta_{ABC} = -\eta_{BAC}$.

We also have

$$
\begin{aligned}
0 &= (\omega_{AB} - \bar{\omega}_{AB}) \wedge f^B \\
&= \eta_{ABC} f^C \wedge f^B
\end{aligned}
$$

Acting with this two form on $e_C \otimes e_B$, we find

$$
0 = \eta_{ABC} - \eta_{ACB}
$$

iii) Using these symmetries, we get

$$
\begin{aligned}
\eta_{ABC} &= -\eta_{BAC} \\
&= -\eta_{BCA} \\
&= \eta_{CBA} \\
&= \eta_{CAB} \\
&= -\eta_{ACB} \\
&= -\eta_{ABC}
\end{aligned}
$$

and so $\eta_{ABC} = 0$.

d) We can compute

$$
\begin{aligned}
g(\nabla_C e_A, e_B) &= e_C\Big(g(e_A, e_B)\Big) - g(e_A, \nabla_C e_B) \\
&= e_C(m_{AB}) - g(\nabla_C e_B, e_A) \\
&= -g(\nabla_C e_B, e_A)
\end{aligned}
$$

using the fact that the m_{AB} are constants.

e) (approach 2) We can compute

$$m_{AB}\mathrm{d}f^A = \begin{pmatrix} \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} f^0\wedge f^1 \\ 0 \\ \frac{1}{r}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}} f^1\wedge f^2 \\ \frac{1}{r}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}} f^1\wedge f^3 + \frac{1}{r}\cot\theta f^2\wedge f^3 \end{pmatrix}$$

and so, from $m_{AB}\mathrm{d}f^A = -\omega_{AB}\wedge f^B$ we have

$$\begin{pmatrix} \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} f^0\wedge f^1 \\ 0 \\ \frac{1}{r}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}} f^1\wedge f^2 \\ \frac{1}{r}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}} f^1\wedge f^3 + \frac{1}{r}\cot\theta f^2\wedge f^3 \end{pmatrix} = \begin{pmatrix} -\omega_{01}\wedge f^1 - \omega_{02}\wedge f^2 - \omega_{03}\wedge f^3 \\ \omega_{01}\wedge f^0 - \omega_{12}\wedge f^2 - \omega_{13}\wedge f^3 \\ \omega_{02}\wedge f^0 + \omega_{12}\wedge f^1 - \omega_{23}\wedge f^3 \\ \omega_{03}\wedge f^0 + \omega_{13}\wedge f^1 + \omega_{23}\wedge f^2 \end{pmatrix}$$

From this it is fairly easy to guess that the connection coefficients are given by

$$\begin{aligned}
\omega_{01} &= -\frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-\frac{1}{2}} f^0 \\
\omega_{12} &= -\frac{1}{r}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}} f^2 \\
\omega_{13} &= -\frac{1}{r}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}} f^3 \\
\omega_{23} &= -\frac{1}{r}\cot\theta f^3
\end{aligned}$$

with the other connection coefficients either given by (anti)symmetry or vanishing. We can also write these in terms of the coordinate induced covector fields:

$$\begin{aligned}
\omega_{01} &= -\frac{M}{r^2}\mathrm{d}t \\
\omega_{12} &= -\left(1-\frac{2M}{r}\right)^{\frac{1}{2}} \mathrm{d}\theta \\
\omega_{13} &= -\left(1-\frac{2M}{r}\right)^{\frac{1}{2}} \sin\theta\mathrm{d}\phi \\
\omega_{23} &= -\cos\theta\mathrm{d}\phi
\end{aligned}$$

f) Now we can compute

$$\mathrm{d}\omega_{AB} = \left(\begin{matrix} 0 & -\frac{2M}{r^3}f^0\wedge f^1 \\ \frac{2M}{r^3}f^0\wedge f^1 & 0 \\ 0 & \frac{M}{r^3}f^1\wedge f^2 \\ 0 & \frac{M}{r^3}f^1\wedge f^3 + \frac{1}{r^2}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\cot\theta f^2\wedge f^3 \end{matrix}\right.$$

$$\left.\begin{matrix} 0 & 0 \\ -\frac{M}{r^3}f^1\wedge f^2 & -\frac{M}{r^3}f^1\wedge f^3 - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}\cot\theta f^2\wedge f^3 \\ 0 & \frac{1}{r^2}f^2\wedge f^3 \\ -\frac{1}{r^2}f^2\wedge f^3 & 0 \end{matrix}\right)$$

We can also calculate

$$
\begin{aligned}
\omega_{AC} \wedge \omega^{C}{}_{B} &= \begin{pmatrix} 0 & \omega_{01} & 0 & 0 \\ -\omega_{01} & 0 & \omega_{12} & \omega_{13} \\ 0 & -\omega_{12} & 0 & \omega_{23} \\ 0 & -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & -\omega_{01} & 0 & 0 \\ -\omega_{01} & 0 & \omega_{12} & \omega_{13} \\ 0 & -\omega_{12} & 0 & \omega_{23} \\ 0 & -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & \omega_{01} \wedge \omega_{12} & \omega_{01} \wedge \omega_{13} \\ 0 & 0 & -\omega_{13} \wedge \omega_{23} & \omega_{12} \wedge \omega_{23} \\ -\omega_{01} \wedge \omega_{12} & -\omega_{13} \wedge \omega_{23} & 0 & -\omega_{12} \wedge \omega_{13} \\ -\omega_{01} \wedge \omega_{13} & -\omega_{12} \wedge \omega_{23} & \omega_{12} \wedge \omega_{13} & 0 \end{pmatrix} \\
&= \left(\begin{matrix} 0 & 0 \\ 0 & 0 \\ -\frac{M}{r^3} f^0 \wedge f^2 & 0 \\ -\frac{M}{r^3} f^0 \wedge f^3 & -\frac{1}{r^2}\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \cot\theta f^2 \wedge f^3 \end{matrix} \right| \\
&\qquad \left| \begin{matrix} \frac{M}{r^3} f^0 \wedge f^2 & \frac{M}{r^3} f^0 \wedge f^3 \\ 0 & \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \cot\theta f^2 \wedge f^3 \\ 0 & -\frac{1}{r^2}\left(1 - \frac{2M}{r}\right) f^2 \wedge f^3 \\ \frac{1}{r^2}\left(1 - \frac{2M}{r}\right) f^2 \wedge f^3 & 0 \end{matrix} \right)
\end{aligned}
$$

Putting these together, we can calculate

$$
\Omega_{AB} = \begin{pmatrix} 0 & -\frac{2M}{r^3} f^0 \wedge f^1 & \frac{M}{r^3} f^0 \wedge f^2 & \frac{M}{r^3} f^0 \wedge f^3 \\ \frac{2M}{r^3} f^0 \wedge f^1 & 0 & -\frac{M}{r^3} f^1 \wedge f^2 & -\frac{M}{r^3} f^1 \wedge f^3 \\ -\frac{M}{r^3} f^0 \wedge f^2 & \frac{M}{r^3} f^1 \wedge f^2 & 0 & \frac{2M}{r^3} f^2 \wedge f^3 \\ -\frac{M}{r^3} f^0 \wedge f^3 & \frac{M}{r^3} f^1 \wedge f^3 & -\frac{2M}{r^3} f^2 \wedge f^3 & 0 \end{pmatrix}
$$

To calculate the components of the Ricci tensor R_{AB} with respect to this orthonormal frame, we note that

$$
\begin{aligned}
R_{AB} &= -R_{A0B0} + R_{A1B1} + R_{A2B2} + R_{A3B3} \\
&= -(\Omega_{A0})_{B0} + (\Omega_{A1})_{B1} + (\Omega_{A2})_{B2} + (\Omega_{A3})_{B3} \\
&= -\Omega_{A0}(e_B, e_0) + \Omega_{A1}(e_B, e_1) + \Omega_{A2}(e_B, e_2) + \Omega_{A3}(e_B, e_3)
\end{aligned}
$$

and we can calculate

$$-\Omega_{A0}(e_B, e_0) = \begin{pmatrix} 0 \\ -\frac{2M}{r^3} f^0 \wedge f^1 \\ \frac{M}{r^3} f^0 \wedge f^2 \\ \frac{M}{r^3} f^0 \wedge f^3 \end{pmatrix} \begin{pmatrix} (e_0, e_0) & (e_1, e_0) & (e_2, e_0) & (e_3, e_0) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2M}{r^3} & 0 & 0 \\ 0 & 0 & -\frac{M}{r^3} & 0 \\ 0 & 0 & 0 & -\frac{M}{r^3} \end{pmatrix}$$

$$\Omega_{A1}(e_B, e_1) = \begin{pmatrix} -\frac{2M}{r^3} f^0 \wedge f^1 \\ 0 \\ \frac{M}{r^3} f^1 \wedge f^2 \\ \frac{M}{r^3} f^1 \wedge f^3 \end{pmatrix} \begin{pmatrix} (e_0, e_1) & (e_1, e_1) & (e_2, e_1) & (e_3, e_1) \end{pmatrix}$$
$$= \begin{pmatrix} -\frac{2M}{r^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{M}{r^3} & 0 \\ 0 & 0 & 0 & -\frac{M}{r^3} \end{pmatrix}$$

$$\Omega_{A2}(e_B, e_2) = \begin{pmatrix} \frac{M}{r^3} f^0 \wedge f^2 \\ -\frac{M}{r^3} f^1 \wedge f^2 \\ 0 \\ -\frac{2M}{r^3} f^2 \wedge f^3 \end{pmatrix} \begin{pmatrix} (e_0, e_2) & (e_1, e_2) & (e_2, e_2) & (e_3, e_2) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{M}{r^3} & 0 & 0 & 0 \\ 0 & -\frac{M}{r^3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2M}{r^3} \end{pmatrix}$$

$$\Omega_{A3}(e_B, e_3) = \begin{pmatrix} \frac{M}{r^3} f^0 \wedge f^3 \\ -\frac{M}{r^3} f^1 \wedge f^3 \\ \frac{2M}{r^3} f^2 \wedge f^3 \\ 0 \end{pmatrix} \begin{pmatrix} (e_0, e_3) & (e_1, e_3) & (e_2, e_3) & (e_3, e_3) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{M}{r^3} & 0 & 0 & 0 \\ 0 & -\frac{M}{r^3} & 0 & 0 \\ 0 & 0 & \frac{2M}{r^3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so, adding these four matrices together, we see that $R_{AB} = 0$, i.e. the components of the Ricci tensor with respect to this orthonormal basis vanish, and hence $R_{\mu\nu} = 0$. Therefore the Schwarzschild metric solves the vacuum Einstein equations!