Problem sheet 3 Solutions

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1. a) and **b)** Here's the solution to part **b)** – part **a)** follows from setting $d = 4$.

First, consider the first two equations. From these, we see that we can write the components of the Riemann tensor as

$$R_{abcd} = R_{[ab][cd]} = R_{AB}$$

where the indices A and B label an *antisymmetric pair* of the original indices a, b and c, d .

Since the original $a, b\dots$ indices take d values, an antisymmetric pair of these indices can take $\frac{1}{2}d(d-1)$ values. Thus we could take the indices A and B to take values in the set $\{1, 2, \dots, \frac{1}{2}d(d-1)\}$.

Now, R_{AB} is itself a symmetric matrix: $R_{AB} = R_{BA}$. Hence, setting $n = \frac{1}{2}d(d-1)$, we see that the number of independent components (ignoring the third symmetry of the Riemann tensor for now) is

$$\frac{1}{2}n(n+1) = \frac{1}{8}d\left(d^3 - 2d^2 + 3d - 2\right)$$

Now, as stated in the question, the third symmetry of the Riemann tensor gives a new condition if and only if all four components of the Riemann tensor take unique values. It doesn't matter which order these values come in, since the other symmetries of the Riemann tensor can be used to reorder the indices. Hence there are

$$^{d}C_4 = \frac{d(d-1)(d-2)(d-3)}{4!} = \frac{1}{24}d\left(d^3 - 6d^2 + 11d - 6\right)$$

extra conditions given by the final symmetry of the Riemann tensor. Each one of these conditions is a linear equation for the components of the Riemann tensor, so each condition eliminates one independent component. Hence we are left with

$$\begin{aligned}&\frac{1}{8}d\left(d^3 - 2d^2 + 3d - 2\right) - \frac{1}{24}d\left(d^3 - 6d^2 + 11d - 6\right)\\&= \frac{1}{12}d^2\left(d^2 - 1\right)\end{aligned}$$

independent components.

Setting $d = 4$ we find that there are 20 independent components in 4 spacetime dimensions.

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2. a) i) From question 1 we see that, in two spacetime dimensions, there is only one independent component of the Riemann tensor. Hence all $(0, 4)$ tensors which obey the same symmetries as the Riemann tensor must be proportional to the Riemann tensor.

It is easy to check that the tensor field

$$A_{\mu\nu\rho\sigma} = g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}$$

satisfies all of the required symmetries. Hence we must have

$$R_{\mu\nu\rho\sigma} = KA_{\mu\nu\rho\sigma} = K\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right)$$

for some scalar function K. We can find this function by contracting the indices μ and ρ, and the indices ν and σ, obtaining

$$R = 2K$$

Hence

$$R_{\mu\nu\rho\sigma} = \frac{1}{2} R \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)$$

as required.

ii) Using the expression for the Riemann tensor in part **i)**, we find that the Ricci tensor is given by

$$R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu}$$

meaning that the (vacuum) Einstein equations are always satisfied, on any geometry.

(In other words, there is no way to include matter in two spacetime dimensions!)

b) i) in 3 spacetime dimensions, using the answer to part **a)** we find that the Riemann tensor has 6 independent components. On the other hand, the components of the Ricci tensor form a symmetric 3×3 matrix, which therefore has $\frac{1}{2} 3 \times 4 = 6$ independent components – the same number!

Let us set

$$B_{\mu\nu\rho\sigma} = R_{\nu\sigma} g_{\mu\rho} + R_{\mu\rho} g_{\nu\sigma} - R_{\nu\rho} g_{\mu\sigma} - R_{\mu\sigma} g_{\nu\rho} - \frac{1}{2} R \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)$$

As before, it is easy to check that $B_{\mu\nu\rho\sigma}$ obeys all of the required symmetries. What's more, if we contract two indices (say μ and ρ – we could also have contracted ν and σ) then we obtain

$$(g^{-1})^{\mu\rho} B_{\mu\nu\rho\sigma} = R_{\nu\sigma}$$

Now, contracting two indices cannot *increase* the number of independent components: the components of the tensor $(g^{-1})^{\mu\rho} B_{\mu\nu\rho\sigma}$ are simply sums of the components of the tensor $B_{\mu\nu\rho\sigma}$. From the equation above we see that $(g^{-1})^{\mu\rho} B_{\mu\nu\rho\sigma}$ has 6 independent components, and $B_{\mu\nu\rho\sigma}$ has *at least* 6 independent components. On the other hand, since the tensor B satisfies the same symmetries as the Riemann tensor, it has *at most* 6 independent components. Hence it has exactly 6 independent components.

Now, $B_{\mu\nu\rho\sigma}$ has 6 independent components and satisfies the 6 linear equations

$$(g^{-1})^{\mu\rho} B_{\mu\nu\rho\sigma} = R_{\nu\sigma}$$

but these same equations are satisfied by the 6 independent components of the Riemann tensor. Hence we must have

$$B_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}$$

ii) The Einstein equations are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}$$

Contracting indices, and writing $T = (g^{-1})^{\mu\nu} T_{\mu\nu}$ we have

$$-\frac{1}{2} R = T$$

and so we can rewrite the Einstein equations as

$$R_{\mu\nu} = T_{\mu\nu} - T g_{\mu\nu}$$

Substituting this back into the expression for the Riemann tensor, we find

$$R_{\mu\nu\rho\sigma} = T_{\nu\sigma} g_{\mu\rho} + T_{\mu\rho} g_{\nu\sigma} - T_{\nu\rho} g_{\mu\sigma} - T_{\mu\sigma} g_{\nu\rho} - T \left(g_{mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)$$

From this expression it follows that, if $T_{\mu\nu} = 0$ in some region, then the Riemann tensor $R_{\mu\nu\rho\sigma} = 0$ in that same region, and hence this region is locally isometric to Minkowski space.

(It also follows from this expression that, in three spacetime dimensions, the spacetime curvature (i.e. the Riemann tensor) is completely determined by the matter content. This is not the case in four spacetime dimensions: in the 4D case, even if we know the matter distribution completely, this is not enough information to evolve the system forward in time: there are extra degrees of freedom in the gravitational field itself, which must be specified as initial data.)

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3. a)

Expanding in terms of some local coordinates

$$\begin{aligned}\Box_g\phi &= (g^{-1})^{\mu\nu}\nabla_\mu\nabla_\nu\phi \\ &= (g^{-1})^{ab}(\partial_a)^\mu(\partial_b)^\nu\nabla_\mu\nabla_\nu\phi \\ &= (g^{-1})^{ab}(\partial_a)^\mu\nabla_\mu\left((\partial_b)^\nu\nabla_\nu\phi\right) - (g^{-1})^{ab}(\partial_a)^\mu\left(\nabla_\mu(\partial_b)^\nu\right)\nabla_\nu\phi \\ &= (g^{-1})^{ab}\left(\partial_a\partial_b\phi\right) - (g^{-1})^{ab}\Gamma^c_{ab}\partial_c\phi\end{aligned}$$

Now, if these coordinates are normal coordinates at a point p, then $g = \mathrm{diag}(-1,1,1,1)$ at p, and the Christoffel symbols vanish at p. Hence, at p, if the coordinates are $(x^0,x^1,x^2,x^3) = (t,x,y,z)$, then

$$\Box_g\phi\Big|_p = \Big(-\partial_t^2\phi + \partial_x^2\phi + \partial_y^2\phi + \partial_z^2\phi\Big)\Big|_p$$

b) i) We can calculate

$$\begin{aligned}\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} &= \nabla_\mu\nabla_\nu A_\rho - \nabla_\mu\nabla_\rho A_\nu + \nabla_\nu\nabla_\rho A_\mu - \nabla_\nu\nabla_\mu A_\rho + \nabla_\rho\nabla_\mu A_\nu - \nabla_\rho\nabla_\nu A_\mu \\ &= R_\mu{}^\alpha{}_{\nu\rho}A_\alpha + R_\nu{}^\alpha{}_{\rho\mu}A_\alpha + R_\rho{}^\alpha{}_{\mu\nu}A_\alpha \\ &= -\left(R^\alpha{}_{\mu\nu\rho} + R^\alpha{}_{\nu\rho\mu} + R^\alpha{}_{\rho\mu\nu}\right)A_\alpha = 0\end{aligned}$$

using the algebraic Bianchi identity.

ii) Under a gauge transformation we have

$$\begin{aligned}F_{\mu\nu} \mapsto F_{\mu\nu} + \nabla_\mu(\mathrm{d}f)_\nu - \nabla_\nu(\mathrm{d}f)_\mu &= F_{\mu\nu} + \nabla_\mu\nabla_\nu f - \nabla_\nu\nabla_\mu f \\ &= F_{\mu\nu} + T^\rho{}_{\mu\nu}\nabla_\rho f\end{aligned}$$

where $T^\rho{}_{\mu\nu}$ is the torsion tensor. But since we are working with the Levi-Civita connection, this tensor vanishes, and F is invariant under a gauge transformation. Since the Maxwell equations are equations for F (and not A), these equations are invariant under gauge transformations.

iii) We have

$$\begin{aligned}\nabla^\mu F_{\mu\nu} &= \nabla^\mu\nabla_\mu A_\nu - \nabla^\mu\nabla_\nu A_\mu \\ &= \nabla^\mu\nabla_\mu A_\nu - \left(\nabla_\mu\nabla_\nu A^\mu - \nabla_\nu\nabla_\mu A^\mu\right) \\ &= \Box_g A_\nu - R^\mu{}_{\rho\mu\nu}A^\rho \\ &= \Box_g A_\nu - R_\nu{}^\rho A_\rho\end{aligned}$$

iv) We can rearrange the Einstein equations to obtain (in 4 spacetime dimensions)

$$R_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}$$

where $T = (g^{-1})^{\mu\nu}T_{\mu\nu}$ is the trace of the energy momentum tensor. Actually, with the Maxwell energy momentum tensor, we find that $T = 0$. Hence

$$\begin{aligned}J_\mu &= \Box_g A_\mu - R^\nu{}_\mu A_\nu \\ &= \Box_g A_\mu - \left(F^{\nu\rho}F_{\mu\rho} - \frac{1}{4}\delta^\nu_\mu F^{\rho\sigma}F_{\rho\sigma}\right)A_\nu\end{aligned}$$

Recalling that $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, we see that this is a nonlinear wave equation for A (with quadratic terms in A).

c) i) Recall the Bianchi identity

$$\nabla_\mu R_{\nu\rho\alpha\beta} + \nabla_\nu R_{\rho\mu\alpha\beta} + \nabla_\rho R_{\mu\nu\alpha\beta} = 0 \tag{1}$$

Contracting the indices μ and α, and using the symmetries of the Riemann tensor:

$$\nabla^\mu R_{\nu\rho\mu\beta} - \nabla_\nu R_{\rho\beta} + \nabla_\rho R_{\nu\beta} = 0$$

Now we recall that, in the vacuum, the Ricci tensor vanishes $R_{\mu\nu} = 0$. Hence

$$\nabla^\mu R_{\nu\rho\mu\beta} = \nabla^\mu R_{\mu\beta\nu\rho} = 0$$

ii) Returning to the Bianchi identity (1) and taking the covariant derivative ∇^μ, shifting some indices up and down and using the divergence-free property from part **i)** we find

$$\begin{aligned}
0 &= \Box_g R_{\nu\rho\alpha\beta} + \nabla_\mu \nabla_\nu R_\rho{}^\mu{}_{\alpha\beta} + \nabla_\mu \nabla_\rho R^\mu{}_{\nu\alpha\beta} \\
&= \Box_g R_{\nu\rho\alpha\beta} + (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) R_\rho{}^\mu{}_{\alpha\beta} + (\nabla_\mu \nabla_\rho - \nabla_\rho \nabla_\mu) R^\mu{}_{\nu\alpha\beta} \\
&= \Box_g R_{\nu\rho\alpha\beta} + R_\rho{}^\gamma{}_{\mu\nu} R_\gamma{}^\mu{}_{\alpha\beta} + R^\mu{}_{\gamma\mu\nu} R_\rho{}^\gamma{}_{\alpha\beta} + R_\alpha{}^\gamma{}_{\mu\nu} R_\rho{}^\mu{}_{\gamma\beta} + R_\beta{}^\gamma{}_{\mu\nu} R_\rho{}^\mu{}_{\alpha\gamma} + R^\mu{}_{\gamma\mu\rho} R^\gamma{}_{\nu\alpha\beta} \\
&\quad + R_\nu{}^\gamma{}_{\mu\rho} R^\mu{}_{\gamma\alpha\beta} + R_\alpha{}^\gamma{}_{\mu\rho} R^\mu{}_{\nu\gamma\beta} + R_\beta{}^\gamma{}_{\mu\rho} R^\mu{}_{\nu\alpha\gamma} \\
&= \Box_g R_{\nu\rho\alpha\beta} - 2 R_\nu{}^\gamma{}_\rho{}^\sigma R_{\gamma\sigma\alpha\beta} - 2 R_\nu{}^\gamma{}_\beta{}^\sigma R_{\rho\gamma\sigma\alpha} - 2 R_\nu{}^\gamma{}_\alpha{}^\sigma R_{\rho\gamma\beta\sigma}
\end{aligned}$$

where we've used the fact that the Ricci tensor vanishes.

(This equation indicates that the Riemann tensor itself obeys a kind of nonlinear wave equation, and is a strong hint that the Einstein equations have a wave-like character.)

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4. a) Contracting the indices μ and ρ, we find

$$R_{\nu\rho} = K\,(4 g_{\nu\rho} - g_{\nu\rho}) = 3K g_{\nu\rho}$$

Contracting ν and ρ, we find that $R = 12K$, i.e. $K = \frac{1}{12}R$.

b) The geodesic deviation equation is

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} J^a = R^a{}_{bcd} X^b X^c J^d$$

where X is the tangent to a timelike geodesic parametrised by proper time. Substituting the expression for the Riemann curvature:

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} J^a &= \frac{1}{12} R\,(\delta^a_c g_{bd} - \delta^a_d g_{bc})\, X^b X^c J^d \\
&= \frac{1}{12} R\,(g(X,J) X^a - g(X,X) J^a) \\
&= \frac{1}{12} R J^a
\end{aligned}$$

Now, since R is constant and negative, the general solution to this equation is

$$J^a = A^a \sin\left(\sqrt{\frac{R}{12}}\tau\right) + B^a \cos\left(\sqrt{\frac{R}{12}}\tau\right)$$

for constants A^a and B^a.

Physically, this means that nearby geodesics oscillate around one another – in particular, since $J^a = 0$ at some point in time, nearby geodesics will come together.

ii) In the case of positive Ricci scalar, the general solution to the geodesic deviation equation is

$$J^a = A^a \exp\left(\sqrt{\frac{R}{12}}\tau\right) + B^a \exp\left(-\sqrt{\frac{R}{12}}\tau\right)$$

for constants A^a and B^a. This means that generic nearby geodesics (that is, ones which are not fine-tuned so that $A^a = 0$ for all a) will diverge from one another exponentially.

Finally, in the case of vanishing Ricci scalar, the general solution is

$$J^a = A^a + B^a\tau$$

So nearby geodesics will, in general, diverge from one another linearly, unless they are initially parallel (so $B^a = 0$) in which case they neither diverge nor converge, but remain parallel forever. These results are exact in Minkowski space.

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5. a) Differentiating the expression $r \sin\phi = b$ with respect to an arbitrary parameter s along the curve, we obtain

$$\frac{\mathrm{d}r}{\partial \mathrm{d}} \sin\theta + r\cos\theta \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = 0$$

substituting back $\sin\phi = \frac{b}{r}$:

$$\frac{b}{r}\frac{\mathrm{d}r}{\mathrm{d}\lambda} + \sqrt{r^2 - b^2}\frac{\mathrm{d}\phi}{\mathrm{d}\lambda} = 0$$

rearranging gives the required expression.

b) Since the light ray moves along a null curve, we must have $g(X, X) = 0$ where X is the tangent to the light ray. Moreover, since it moves in the plane $\theta = \frac{\pi}{2}$, the θ component of X vanishes. Hence

$$-\left(1 - \frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}\right)^2 = 0$$

here, as above, λ can be *any* parameter along the curve – it does not necessarily have to be an affine parameter, since in any case the vector X will be null.

In particular, we can choose $\lambda = r$. Then, substituting in for $\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}$ using part **a)**, we find that

$$\left(\frac{\mathrm{d}t}{\mathrm{d}r}\right)^2 = \left(1 - \frac{2M}{r}\right)^{-2} + \left(1 - \frac{2M}{r}\right)^{-1}\frac{b^2}{r^2 - b^2} = 0$$

Now, expanding in powers of $\frac{M}{r}$

$$\begin{aligned}
\left(\frac{\partial t}{\partial r}\right)^2 &= 1 + \frac{4M}{r} + \frac{b^2}{r^2 - b^2}\left(1 + \frac{2M}{r}\right) + \mathcal{O}\left(\left(\frac{M}{r}\right)^2\right) \\
&= +\frac{2M}{r} + \frac{r^2}{r^2 - b^2}\left(1 + \frac{2M}{r}\right) + \mathcal{O}\left(\left(\frac{M}{r}\right)^2\right) \\
&= \frac{r^2}{r^2 - b^2}\left(1 + \frac{4M}{r} - \frac{2Mb^2}{r^3}\right) + \mathcal{O}\left(\left(\frac{M}{r}\right)^2\right)
\end{aligned}$$

Taking the square root and expanding again:

$$\frac{\partial t}{\partial r} = \pm\frac{r}{\sqrt{r^2 - b^2}}\left(1 + \frac{2M}{r} - \frac{Mb^2}{r^3}\right) + \mathcal{O}\left(\left(\frac{M}{r}\right)^2\right)$$

c) Neglecting terms of order $\left(\frac{M}{r}\right)^2$ and doing the integral above (taking the positive root, since r is increasing with time):

$$\Delta t = \int_b^{r_1} \frac{r}{\sqrt{r^2-b^2}}\left(1+\frac{2M}{r}-\frac{Mb^2}{r^3}\right)\mathrm{d}r$$

We can integrate each of the three terms in turn.

First, we calculate

$$\int_b^{r_1} \frac{r}{\sqrt{r^2-b^2}}\mathrm{d}r = \left[\sqrt{r^2-b^2}\right]_b^{r_1} = \sqrt{(r_1)^2-b^2}$$

Next we can calculate

$$\int_b^{r_1} \frac{2M}{\sqrt{r^2-b^2}}\mathrm{d}r = \int_0^{\operatorname{arcosh}\left(\frac{r_1}{b}\right)} \frac{2M}{b}\frac{b\sinh y}{\sqrt{\cosh^2 y-1}}\mathrm{d}y = 2M\operatorname{arcosh}\left(\frac{r_1}{b}\right)$$

where we used the substitution $r = b\cosh y$.

Finally we can deal with the third term

$$\int_b^{r_1} -\frac{Mb^2}{r^2\sqrt{r^2-b^2}}\mathrm{d}r = \int_b^{r_1} -\frac{\mathrm{d}}{\mathrm{d}r}\left(M\left(1-\frac{b^2}{r^2}\right)^{\frac{1}{2}}\right)\mathrm{d}r = -\frac{M}{r_1}\sqrt{r_1^2-b^2}$$

Putting together these three integrals gives the required expression.

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6. a) We can calculate

$$g(\ddot{\gamma},\dot{\gamma}) = g(\nabla_{\dot{\gamma}}\dot{\gamma},\dot{\gamma}) = \frac{1}{2}\dot{\gamma}\left(g(\dot{\gamma},\dot{\gamma})\right) = \frac{1}{2}\dot{\gamma}(-1) = 0$$

or, if you prefer to work with indices,

$$\ddot{\gamma}_\mu\dot{\gamma}^\mu = \gamma^\nu(\nabla_\nu\dot{\gamma}^\mu)\dot{\gamma}_\mu = \frac{1}{2}\gamma^\nu\nabla_\nu\left(\dot{\gamma}^\mu\dot{\gamma}_\mu\right) = \frac{1}{2}\gamma^\nu\partial_\nu(-1) = 0$$

b) i) Working in Schwarzschild coordinates, the worldline of the observer is given by $(t,r,\theta,\phi) = (t(\tau), r_0, \theta_0, \phi_0)$ where r_0, θ_0 and ϕ_0 are constants. Hence

$$\dot{\gamma} = \begin{pmatrix}\dot{t}\\0\\0\\0\end{pmatrix}$$

Since $g(\dot{\gamma},\dot{\gamma}) = -1$ we have

$$-1 = -\dot{t}^2\left(1-\frac{2M}{r_0}\right)$$
$$\Rightarrow \dot{t} = \left(1-\frac{2M}{r_0}\right)^{-\frac{1}{2}}$$

since t increases as τ increases.

Now we can calculate

$$\dot{\gamma}^b\nabla_b\dot{\gamma}^a = \dot{\gamma}^b\partial_b\dot{\gamma}^a + \Gamma^a_{bc}\dot{\gamma}^b\dot{\gamma}^c = \left(1-\frac{2M}{r_0}\right)^{-\frac{1}{2}}\partial_t\dot{\gamma}^a + \left(1-\frac{2M}{r_0}\right)^{-1}\Gamma^a_{00}$$

The first term on the right hand side vanishes (all of the components of $\dot{\gamma}$ are constant in Schwarzschild coordinates). To compute the second term we recall

$$
\begin{aligned}
\Gamma^a_{00} &= \frac{1}{2}(g^{-1})^{ab}\left(2\partial_0 g_{0b} - \partial_b g_{00}\right) \\
&= -\frac{1}{2}(g^{-1})^{a1}\partial_r g_{00} \\
&= \left(1-\frac{2M}{r}\right)\begin{pmatrix} 0 \\ \frac{M}{r^2} \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

So the components of the proper acceleration vector relative to Schwarzschild coordinates are

$$
\ddot{\gamma}^a = \begin{pmatrix} 0 \\ \frac{M}{r_0^2} \\ 0 \\ 0 \end{pmatrix}
$$

ii) The magnitude of $\ddot{\gamma}$ is

$$
\sqrt{g(\ddot{\gamma}, \ddot{\gamma}} = \sqrt{\frac{M^2}{r_0^4}\left(1-\frac{2M}{r_0}\right)^{-1}} = \frac{M}{r_0^{\frac{3}{2}}\sqrt{r_0 - 2M}}
$$

iii) It is in fact true that an observer who is "stationary" at the surface of the Earth is accelerating upwards (the magnitude of this acceleration is calculated above). In GR, there is no force of gravity, so the only forces acting on an observer who is stationary at the surface of the Earth are the 'normal reaction forces', which act in an upwards direction. (The Earth is, of course, not flat. . .)

iv) As $r_0 \to 2M$, the acceleration required in order to remain 'stationary' (i.e. at $r = r_0$) tends to infinity. Hence, no matter how powerful your rocket is, you will not be able to remain at the surface $r = r_0$!

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7. a)

The geodesic equation is

$$
\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} = 0
$$

Examining the t (or 0) components of this equation, it is

$$
\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + \Gamma^0_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} = 0
$$

Now, some computation reveals that the only nonzero Christoffel symbols with 0 as an 'up' index are

$$
\Gamma^0_{01} = \Gamma^0_{10} = \frac{M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}
$$

so the 0 component of the geodesic equation is

$$
\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + \frac{2M}{r^2}\left(1-\frac{2M}{r}\right)^{-1}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau} = 0
$$

Now, we can also compute the τ derivative of the quantity $\frac{E}{m}$:

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{E}{m}\right) = \left(1-\frac{2M}{r}\right)\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} + \frac{2M}{r^2}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}r}{\mathrm{d}\tau}
$$

which vanishes, as seen in the geodesic equation.

b) Since the particle is following a radial geodesic, it satisfies

$$\begin{aligned}
-1 &= g_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + g_{11}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 \\
&= -\left(1-\frac{2M}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 \\
&= -\left(1-\frac{2M}{r}\right)^{-1}\frac{E^2}{m^2} + \left(1-\frac{2M}{r}\right)^{-1}\frac{p^2}{m^2}
\end{aligned}$$

Rearranging things

$$E^2 = p^2 + m^2\left(1-\frac{2M}{r}\right)$$

c) When r reaches its maximum value, $p=0$, so

$$E^2 = m^2\left(1-\frac{2M}{r_{\max}}\right)$$

On the other hand, initially $r=R$ and $p=mv$. Hence

$$\frac{E^2}{m^2} = v^2 + \left(1-\frac{2M}{R}\right)$$

Since E is constant, we find that

$$\left(1-\frac{2M}{r_{\max}}\right) = v^2 + \left(1-\frac{2M}{R}\right)$$

and so

$$r_{\max} = \frac{2M}{\frac{2M}{R} - v^2}$$

This is exactly the same formula for the maximum radius reached by a particle moving in Newtonian gravity! (See example sheet 0).

If $v^2 > \frac{2M}{R}$, the particle excapes to infinity, i.e. the escape velocity is $\frac{\mathrm{d}r}{\mathrm{d}\tau} = \sqrt{\frac{2M}{R}}$.

d) i) If normal coordinates are set up at the point p, then the coordinate-induced one-forms are orthonormal at the point p. So the only things to check are that, at the point p,

$$\begin{aligned}
g^{-1}(\mathrm{d}\tilde{t},\mathrm{d}\tilde{t}) &= -1 \\
g^{-1}(\mathrm{d}\tilde{r},\mathrm{d}\tilde{r}) &= -1 \\
g^{-1}(\mathrm{d}\tilde{t},\mathrm{d}\tilde{r}) &= 0
\end{aligned}$$

but we can check

$$\begin{aligned}
g^{-1}(\mathrm{d}\tilde{t},\mathrm{d}\tilde{t}) &= \left(1-\frac{2M}{R}\right)g^{-1}(\mathrm{d}t,\mathrm{d}t) = -1 \\
g^{-1}(\mathrm{d}\tilde{x},\mathrm{d}\tilde{x}) &= \left(1-\frac{2M}{R}\right)^{-1}g^{-1}(\mathrm{d}r,\mathrm{d}r) = 0 \\
g^{-1}(\mathrm{d}\tilde{t},\mathrm{d}\tilde{r}) &= g^{-1}(\mathrm{d}t,\mathrm{d}r) = 0
\end{aligned}$$

Similarly, we can choose the remaining coordinates $\tilde{y}$ and $\tilde{z}$ so that their differentials are given by

$$\begin{aligned}
\mathrm{d}\tilde{y} &= R\mathrm{d}\theta \\
\mathrm{d}\tilde{z} &= R\sin\theta\mathrm{d}\theta
\end{aligned}$$

then it is clear that all of these coordinate differentials form an orthonormal basis for the cotangent space, as is satisfied by orthonormal coordinates at the point p.

(To extend these coordinates away from the point p in a nice way, see the appendix of the lecture notes. It shouldn't be hard to convince yourself that these coordinates can be extended in *some* (arbitrary) way away from the point p.)

ii) The coordinate velocity is

$$\begin{aligned}
\tilde{v} &= \frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tilde{t}} \\
&= \left(\frac{\mathrm{d}\tilde{x}}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}\tilde{t}}{\mathrm{d}\tau}\right)^{-1} \\
&= \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^{-1} \\
&= \frac{vm}{E} \\
&= \frac{v}{\sqrt{v^2 + \left(1 - \frac{2M}{R}\right)}}
\end{aligned}$$

iii) We can invert the relationship derived in part **ii)** to obtain

$$v^2 = \tilde{v}^2\left(\frac{1 - \frac{2m}{R}}{1 - \tilde{v}^2}\right)$$

Using $\tilde{v}$ instead of v, we find that $r_{\max}$ is given by

$$r_{\max} = \frac{2M(1 - \tilde{v}^2)}{\frac{2M}{R} - \tilde{v}^2}$$

The velocity $\tilde{v}$ is more meaningful, physically, than the coordinate velocity $\tilde{v}$ since it corresponds the the velocity measured by an observer using an orthonormal frame. The additional factor $(1 - \tilde{v}^2)$ in the expression above (relative to the Newtonian one) indicates that objects will not escape as far as they do in Newtonian theory (at least if we think of the coordinate r as measuring something physically meaningful), so in this sense gravity is 'stronger'.

***8. a)** Using the coordinates $(x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)$, the nonzero Christoffel symbols are

$$\begin{aligned}
\Gamma^0_{01} = \Gamma^0_{10} &= \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1} \\
\Gamma^1_{00} &= -\frac{2M}{r^2}\left(1 - \frac{2M}{r}\right) \\
\Gamma^1_{11} &= -\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1} \\
\Gamma^1_{22} &= -(r - 2M) \\
\Gamma^1_{33} &= -(r - 2M)\sin^2\theta \\
\Gamma^2_{12} = \Gamma^2_{21} &= \frac{1}{r} \\
\Gamma^2_{33} &= -\sin\theta\cos\theta \\
\Gamma^3_{13} = \Gamma^3_{31} &= \frac{1}{r} \\
\Gamma^3_{23} = \Gamma^3_{32} &= \cot\theta
\end{aligned}$$

Those not listed above vanish identically.

b) The components of the Riemann tensor are

$$
\begin{aligned}
R_{0101} &= -\frac{2M}{r^3} \\
R_{0202} &= \frac{M}{r}\left(1 - \frac{2M}{r}\right) \\
R_{0303} &= \frac{M}{r}\left(1 - \frac{2M}{r}\right)\sin^2\theta \\
R_{1212} &= -\frac{M}{r}\left(1 - \frac{2M}{r}\right)^{-1} \\
R_{1313} &= -\frac{M}{r}\left(1 - \frac{2M}{r}\right)^{-1}\sin^2\theta \\
R_{2323} &= 2Mr\sin^2\theta
\end{aligned}
$$

All the others either vanish or are found easily by symmetry from those given above.

c) The components of the Ricci tensor vanish identically.