

### III. NONEQUILIBRIUM KINETICS

We have so far studied the statistical properties of Brownian motion by both solving stochastic equations of motion with random forces modelled by Gaussian white noise and studying the time evolution of the probability of finding the particle in a given configuration. In this section, we will develop a more general probabilistic approach that could be used for studying any stochastic system.

#### A. Master equation

Let us consider a physical system that can be in one of many possible states, denoted by the variable  $a$ . The set of all possible states  $\{a\}$  is called the *configuration space* of the system. Examples of configuration spaces are: i) the set of all positions of a Brownian particle, ii) the set of all chemical states of a chemically reacting system, or iii) the set of all electronic states of an organic semiconductor. As the system evolves, it must be in one of the possible states at any instant of time. However, due to its random motion, we cannot write down a deterministic equation of motion for the system. The dynamics of the system is characterized by ‘random’ jumps from one possible state,  $a$  to another,  $b$ . The rate of transition from state  $b$  to state  $a$  is denoted by  $k_{b \rightarrow a}$ .

The probability of the system being in state  $a$  at time  $t$  is given by  $\mathcal{P}_a(t)$ , and it is possible to write down a deterministic equation of motion for  $\mathcal{P}_a(t)$  as follows:

$$\frac{d}{dt}\mathcal{P}_a(t) = \sum_b [k_{b \rightarrow a} \mathcal{P}_b(t) - k_{a \rightarrow b} \mathcal{P}_a(t)] \quad , \quad (\text{III.1})$$

which is called the *Master equation*. The above equation states that the rate of change of the probability of finding the system in a given state is equal to the total current of transitions to that state minus the current of transitions away from that state; it is a statement of the conservation of probability.

In the stationary state we have  $\frac{d}{dt}\mathcal{P}_a = 0$ , which *can* be satisfied if the condition of *detailed balance* holds:

$$\frac{k_{b \rightarrow a}}{k_{a \rightarrow b}} = \frac{\mathcal{P}_a}{\mathcal{P}_b} \quad . \quad (\text{III.2})$$

If the stationary state corresponds to thermodynamic equilibrium at a temperature  $T$ , then  $\mathcal{P}_a \propto e^{-E_a/k_B T}$ , where  $E_a$  is the internal energy of state  $a$ , and the detailed balance condition gives

$$\frac{k_{b \rightarrow a}}{k_{a \rightarrow b}} = e^{-(E_a - E_b)/k_B T} \quad . \quad (\text{III.3})$$

This means that any kinetic process that satisfies Eq. (III.3) will lead to thermodynamic equilibrium of the system at temperature  $T$ . This is used extensively in Monte Carlo simulation of model systems (e.g. Ising model) to obtain equilibrium states. Two common algorithms for simulating Ising models are Glauber dynamics and Kawasaki dynamics, which both satisfy the above detailed balance condition.

##### 1. Two-state process: the telegraph noise

A particularly common class of random processes in which the configuration space has two states deserves special attention. The process comprises discrete jumps between the two states, which can be chosen to be 0 and 1 without loss of generality, and can be represented as follows

$$0 \xrightleftharpoons[\mu]{\lambda} 1 \quad . \quad (\text{III.4})$$

This will describe a stochastic binary variable  $n(t)$ , which can, for example, represent the occupation number of a molecular binding site.

The master equation for a system that has been prepared to have  $n(t=0) = n_0$  reads

$$\begin{aligned}\frac{d}{dt}\mathcal{P}(0, t|n_0, 0) &= -\lambda\mathcal{P}(0, t|n_0, 0) + \mu\mathcal{P}(1, t|n_0, 0) \quad , \\ \frac{d}{dt}\mathcal{P}(1, t|n_0, 0) &= \lambda\mathcal{P}(0, t|n_0, 0) - \mu\mathcal{P}(1, t|n_0, 0) \quad ,\end{aligned}\tag{III.5}$$

where the conditional probabilities, which enforce the initial condition, are subject to the normalization condition

$$\mathcal{P}(0, t|n_0, 0) + \mathcal{P}(1, t|n_0, 0) = 1 \quad .\tag{III.6}$$

In stationary state, detailed balance and normalization can be used to solve for the (stationary) probabilities and the average transition flux

$$\mathcal{P}_0^s = \frac{\mu}{\lambda + \mu} \quad , \quad \mathcal{P}_1^s = \frac{\lambda}{\lambda + \mu} \quad , \quad J = \lambda\mathcal{P}_0^s = \mu\mathcal{P}_1^s = \frac{\lambda\mu}{\lambda + \mu} \quad ,\tag{III.7}$$

which can be used to calculate the stationary averages

$$\langle n \rangle^s = \frac{\lambda}{\lambda + \mu} \quad , \quad \langle n^2 \rangle_c^s = \frac{\lambda\mu}{(\lambda + \mu)^2} = \langle n \rangle^s [1 - \langle n \rangle^s] \quad .\tag{III.8}$$

The probabilities can be readily solved at any time; they read

$$\begin{aligned}\mathcal{P}(0, t|n_0, 0) &= \frac{\mu}{\lambda + \mu} - \left[ n_0 - \frac{\lambda}{\lambda + \mu} \right] e^{-(\lambda + \mu)t} \quad , \\ \mathcal{P}(1, t|n_0, 0) &= \frac{\lambda}{\lambda + \mu} + \left[ n_0 - \frac{\lambda}{\lambda + \mu} \right] e^{-(\lambda + \mu)t} \quad .\end{aligned}\tag{III.9}$$

They can be used to calculate the time dependent average of the binary signal

$$\langle n(t) \rangle = \sum_{n=0,1} n \mathcal{P}(n, t|n_0, 0) = \frac{\lambda}{\lambda + \mu} + \left[ n_0 - \frac{\lambda}{\lambda + \mu} \right] e^{-(\lambda + \mu)t} \quad ,\tag{III.10}$$

and the auto-correlation function

$$\langle n(t)n(t_0) \rangle^s = \sum_{n=0,1} \sum_{m=0,1} n m \mathcal{P}(n, t|m, t_0) \mathcal{P}_m^s = \left( \frac{\lambda}{\lambda + \mu} \right)^2 + \frac{\lambda\mu}{(\lambda + \mu)^2} e^{-(\lambda + \mu)|t - t_0|} \quad ,\tag{III.11}$$

calculated in the stationary regime. We thus find

$$\langle n(t)n(t_0) \rangle_c^s = \langle n^2 \rangle_c^s e^{-(\lambda + \mu)|t - t_0|} \quad .\tag{III.12}$$

## B. Chemical reactions

The Master equation formalism has many applications in studies of the kinetics of chemical reactions.

Let us start by reviewing chemical reactions at equilibrium. As an example, consider the reaction



which describes the breakup of molecule  $S$  into two molecules  $P$  and  $Q$ . At thermodynamic equilibrium all intensive variables are equal, which implies that the sum of the chemical potentials of the reactants should equal the sum of the chemical potentials of the product, i.e.

$$\mu_S = \mu_P + \mu_Q \quad . \quad (\text{III.14})$$

Chemical reactions typically occur in dilute solutions, in which low concentrations of reactive molecules are dissolved in a solvent. Therefore, each chemical species can be modelled as an ‘ideal gas’ of well separated molecules immersed in the solvent.

Recalling that the chemical potential for an ideal gas is given by  $\mu = k_B T \ln(C/C_0)$  where  $C$  is the concentration, we thus have

$$k_B T \ln \left( \frac{C_S}{C_{0S}} \right) = k_B T \ln \left( \frac{C_P}{C_{0P}} \right) + k_B T \ln \left( \frac{C_Q}{C_{0Q}} \right) \quad . \quad (\text{III.15})$$

This implies

$$\frac{C_P C_Q}{C_S} = \frac{C_{0P} C_{0Q}}{C_{0S}} \equiv K_{\text{eq}} \quad , \quad (\text{III.16})$$

where  $K_{\text{eq}}$  is the *equilibrium constant* for the reaction,  $S \rightleftharpoons P + Q$ .

Equation (III.16) is called the *law of mass action*, which determines the balance of the chemical reaction. If a solution of  $S$ ,  $P$ , and  $Q$  is prepared in an initial state in which the concentrations satisfy Eq. (III.16), then the solution will be at equilibrium. What happens if the initial state does not satisfy the law of mass action? The forward and backward reactions will proceed such that the molecules with concentrations below the equilibrium values will be produced and those with concentrations above the equilibrium values will be consumed, until the system arrives at equilibrium as defined by Eq. (III.16).

To understand the process of equilibration and its timescale we need to consider the kinetics of chemical reactions. This can be done within a similar framework to the Master equation described above. We will approximate the probabilities by concentrations, which is valid in the low density ‘ideal’ regime considered here. Reactions involving more than one molecule can occur when the molecules ‘collide’, i.e. when they are within ‘touching’ distance of each other. This means that the rate at which a multi-particle reaction proceeds is proportional to the product of the concentrations of the reactants, as we discussed in Sec. I.E.1. We can describe the reactions proceeding inside a volume  $dV = d^3\mathbf{r}$  as the change in the number of molecules of type  $i$  inside that volume. Hence, the rate of change in the number of molecules of  $S$ ,  $N_S$ , in the volume  $dV$ , can be written as;  $\dot{N}_S = d^3\mathbf{r} \dot{C}_S(t) = d^3\mathbf{r} (-k_+ C_S + k_- C_P C_Q)$ , with similar equations for  $P$  and  $Q$ .

We can now write kinetic rate equations for the evolution of concentrations of the three species

$$\begin{aligned} \frac{d}{dt} C_S(t) &= k_- C_P C_Q - k_+ C_S \quad , \\ \frac{d}{dt} C_P(t) &= k_+ C_S - k_- C_P C_Q \quad , \\ \frac{d}{dt} C_Q(t) &= k_+ C_S - k_- C_P C_Q \quad . \end{aligned} \quad (\text{III.17})$$

Note that the rates  $k_+$  and  $k_-$  necessarily have different dimensions. Equations (III.17) imply the conservation laws

$$\frac{d}{dt} [C_P + C_S] = \frac{d}{dt} [C_Q + C_S] = 0 \quad , \quad (\text{III.18})$$

$$\Rightarrow C_P + C_S = \text{constant} \quad , \quad C_Q + C_S = \text{constant} \quad (\text{III.19})$$

as the chemical reactions require  $\delta N_S = -\delta N_P = -\delta N_Q$ .

Furthermore, at equilibrium we have  $\frac{d}{dt}C_i(t) = 0$  and Eqs. (III.17) give the detailed balance condition

$$\frac{C_P C_Q}{C_S} = \frac{k_+}{k_-} , \quad (\text{III.20})$$

which is identical to the law of mass action with  $K_{\text{eq}} = \frac{k_+}{k_-}$ . For a more physical picture of the meaning of the reaction rates and the equilibrium constant it is instructive to consider the free energy landscape of the problem. For reactions in solution, the Gibbs free energy is more useful as the process does not conserve particle numbers in each species.

We can define a (simplified) one dimensional reaction coordinate going from the initial state  $i$  (corresponding to reactant  $S$ ) to the final state  $f$  (corresponding to the products  $P$  and  $Q$ ) and a free energy landscape over that reaction coordinate. The free energies of the initial and final states are  $G_i$  and  $G_f$ , respectively. There is a ‘free energy barrier’ of height  $G_b > G_f \& G_i$  between them. The rate of a transition occurring is proportional to the probability of achieving a thermally activated jump across the barrier, after which they will be trapped by the local minima on the other side  $i$  or  $f$ :

$$k_+ \propto e^{-(G_b - G_i)/k_B T} , \quad (\text{III.21})$$

$$k_- \propto e^{-(G_b - G_f)/k_B T} , \quad (\text{III.22})$$

which leads to

$$K_{\text{eq}} = \frac{k_+}{k_-} \propto e^{-(G_f - G_i)/k_B T} , \quad (\text{III.23})$$

to which you can compare the detailed balance condition from the Master equation in Eq. (III.3). The detailed balance condition here involves free energies rather than internal energies, which reflects the fact that we have ‘integrated’ out some microscopic degrees of freedom to simplify the problem to a simple one-dimensional reaction coordinate.

### 1. Relaxation to equilibrium

Now let us consider a system which is prepared away from equilibrium and consider its dynamics as it relaxes towards equilibrium. For simplicity, we consider the system slightly away from equilibrium and write the concentrations as equilibrium values plus small perturbations  $\delta C_i(t) \ll C_i^{\text{eq}}$ , as

$$\begin{aligned} C_S(t) &= C_S^{\text{eq}} + \delta C_S(t) , \\ C_P(t) &= C_P^{\text{eq}} + \delta C_P(t) , \\ C_Q(t) &= C_Q^{\text{eq}} + \delta C_Q(t) . \end{aligned} \quad (\text{III.24})$$

Detailed balance requires  $\frac{C_P^{\text{eq}} C_Q^{\text{eq}}}{C_S^{\text{eq}}} = \frac{k_+}{k_-}$ , and the conservation laws of Eqs. (III.18) and (III.19) yield

$$\delta C_S(t) = -\delta C_P(t) = -\delta C_Q(t) . \quad (\text{III.25})$$

Due to these constraints there is only one independent equation of motion for the linearized dynamics

$$\frac{d}{dt} \delta C_S(t) = - \left[ k_+ + k_- C_P^{\text{eq}} + k_- C_Q^{\text{eq}} \right] \delta C_S(t) . \quad (\text{III.26})$$

This means that the concentration fluctuations relax to equilibrium over a timescale  $\mathcal{T}$ —the relaxation time—given by

$$\frac{1}{\mathcal{T}} = k_+ + k_- C_P^{\text{eq}} + k_- C_Q^{\text{eq}} \quad , \quad (\text{III.27})$$

which depends on the equilibrium concentrations of the chemical species.

## 2. Enzyme kinetics

The interior of a living cell is an environment where a multitude of chemical reactions takes place and a tight control mechanism is needed to regulate them such that the cell maintains its nonequilibrium stationary state. To achieve this, nature has developed enzymes, which are machines that manipulate the rate of chemical reactions, such that reaction paths that are practically dormant will be activated when the right enzyme comes along. To perform this catalytic function, the enzyme binds to the reactants in such a way that will effectively lower the height of the free energy barrier that the reaction needs to overcome, hence speeding the reaction. Lowering the free energy barrier affects the reaction rates in both directions, with the net effect of favouring one route compared to the other and driving the system away from detailed balance (equilibrium).

Let us consider the breakup of a molecule  $S$  (often called *substrate* in the Biochemistry literature) into two product molecules  $P$  and  $Q$ , facilitated by the enzyme  $E$



The reaction consists of two stages; the complexation of  $S$  and  $E$  and the subsequent breakup into product molecules plus the free enzyme. Each enzyme is described by an internal two-state process; it is free (and ready to form a complex) with probability  $\mathcal{P}_E$  and occupied with probability  $\mathcal{P}_{SE}$ . Naturally,  $\mathcal{P}_E + \mathcal{P}_{SE} = 1$ . The kinetic rate equations read

$$\begin{aligned} \frac{d}{dt} C_S(t) &= -k_1 \mathcal{P}_E C_E C_S \quad , \\ \frac{d}{dt} C_P(t) &= k_2 \mathcal{P}_{SE} C_E \quad , \\ \frac{d}{dt} C_Q(t) &= k_2 \mathcal{P}_{SE} C_E \quad , \\ \frac{d}{dt} \mathcal{P}_E(t) &= -\frac{d}{dt} \mathcal{P}_{SE}(t) = -k_1 C_S \mathcal{P}_E + k_2 \mathcal{P}_{SE} \quad . \end{aligned} \quad (\text{III.29})$$

In stationary state (see Sec. III.A.1), we find

$$\mathcal{P}_E^s = \frac{k_2}{k_1 C_S + k_2} \quad , \quad \mathcal{P}_{SE}^s = \frac{k_1 C_S}{k_1 C_S + k_2} \quad , \quad J = k_1 C_S \mathcal{P}_E^s = k_2 \mathcal{P}_{SE}^s = \frac{k_2 k_1 C_S}{k_1 C_S + k_2} \quad . \quad (\text{III.30})$$

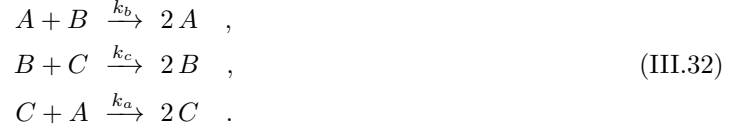
Therefore, the overall rate (per unit volume) of catalysis in the solution can be written as

$$k_{\text{eff}} = \frac{k_2 C_S C_E}{C_S + K_M} \quad , \quad (\text{III.31})$$

which is the celebrated **Michaelis-Menten rule** of enzymatic reactions, with  $K_M = k_2/k_1$  being called the **Michaelis constant**.

### 3. Oscillating chemical reactions

Until the middle of the twentieth century it was generally thought that the exponential relaxation to equilibrium that we described in Sec. III.B.1 is the only possible dynamics for chemical reactions, until the oscillating reaction system that has come to be known as the Belousov-Zhabotinsky reaction was discovered. A simplest representation of an oscillating chemical reaction can be constructed using the following cyclic kinetic scheme



Note that there are no back-reactions; the reactions are assumed to be irreversible.

The chemical kinetic equations corresponding to Eq. (III.32) are

$$\begin{aligned} \frac{d}{dt}C_A(t) &= k_b C_A C_B - k_a C_A C_C \quad , \\ \frac{d}{dt}C_B(t) &= k_c C_C C_B - k_b C_A C_B \quad , \\ \frac{d}{dt}C_C(t) &= k_a C_A C_C - k_c C_B C_C \quad , \end{aligned} \tag{III.33}$$

which imply the conservation law

$$\frac{d}{dt}[C_A + C_B + C_C] = 0 \quad , \tag{III.34}$$

$$C_A(t) + C_B(t) + C_C(t) = \text{constant} \quad . \tag{III.35}$$

Detailed balance for Eq. (III.33) requires

$$\begin{aligned} k_b C_B^{\text{eq}} &= k_a C_C^{\text{eq}} \quad , \\ k_c C_C^{\text{eq}} &= k_b C_A^{\text{eq}} \quad , \\ k_a C_C^{\text{eq}} &= k_c C_B^{\text{eq}} \quad , \end{aligned} \tag{III.36}$$

where  $C_i^{\text{eq}}$  are the equilibrium concentrations for  $i = A, B, C$ . Now, we examine what happens when the system is perturbed away from equilibrium (e.g. an initial condition not quite at equilibrium). We expand the concentrations around their equilibrium values,  $C_i = C_i^{\text{eq}} + \delta C_i$  with  $\delta C_i(t) \ll C_i^{\text{eq}}$ . To linear order, we obtain

$$\begin{aligned} \frac{d}{dt}\delta C_A(t) &= C_A^{\text{eq}}(k_b \delta C_B - k_a \delta C_C) \quad , \\ \frac{d}{dt}\delta C_B(t) &= C_B^{\text{eq}}(k_c \delta C_C - k_b \delta C_A) \quad , \\ \frac{d}{dt}\delta C_C(t) &= C_C^{\text{eq}}(k_a \delta C_A - k_c \delta C_B) \quad . \end{aligned} \tag{III.37}$$

The conservation law constraint from Eq. (III.34) leads to

$$\delta C_A(t) + \delta C_B(t) + \delta C_C(t) = 0 \quad , \tag{III.38}$$

which can be used to eliminate one of the concentrations. We thus obtain the coupled equations for

$\delta C_A(t)$  and  $\delta C_B(t)$ :

$$\frac{d}{dt} \begin{bmatrix} \delta C_A(t) \\ \delta C_B(t) \end{bmatrix} = \begin{bmatrix} k_a C_A^{\text{eq}} & k_a C_A^{\text{eq}} + k_c C_C^{\text{eq}} \\ -k_a C_A^{\text{eq}} - k_b C_B^{\text{eq}} & -k_a C_A^{\text{eq}} \end{bmatrix} \begin{bmatrix} \delta C_A(t) \\ \delta C_B(t) \end{bmatrix} , \quad (\text{III.39})$$

where we have also used the equilibrium conditions, Eq. (III.36) to make the equations more symmetric.

To study the dynamics of the system of equations, we try a solution of the form

$$\begin{bmatrix} \delta C_A(t) \\ \delta C_B(t) \end{bmatrix} = e^{\lambda t} \begin{bmatrix} \delta C_A(0) \\ \delta C_B(0) \end{bmatrix} , \quad (\text{III.40})$$

which gives the following equation

$$\begin{bmatrix} -\lambda + k_a C_A^{\text{eq}} & k_a C_A^{\text{eq}} + k_c C_C^{\text{eq}} \\ -k_a C_A^{\text{eq}} - k_b C_B^{\text{eq}} & -\lambda - k_a C_A^{\text{eq}} \end{bmatrix} \begin{bmatrix} \delta C_A(0) \\ \delta C_B(0) \end{bmatrix} \equiv \mathbf{M} \cdot \begin{bmatrix} \delta C_A(0) \\ \delta C_B(0) \end{bmatrix} = 0 . \quad (\text{III.41})$$

To have non-vanishing solutions, the above equation requires  $\det \mathbf{M} = 0$ . We thus find,

$$\lambda = \pm i\omega_0 , \quad \omega_0 = \sqrt{k_a C_A^{\text{eq}} k_c C_C^{\text{eq}} + k_a C_A^{\text{eq}} k_b C_B^{\text{eq}} + k_c C_C^{\text{eq}} k_b C_B^{\text{eq}}} . \quad (\text{III.42})$$

Hence, a small deviation from equilibrium does not lead to an exponential relaxation to equilibrium; it leads to a self-sustaining oscillatory cycle, which lasts as long as the free energy required for the catalytic reactions is supplied.

### C. One-step processes

We are often interested in modelling stochastic processes in which transitions only occur between states in configuration space which are somehow *near* each other. This makes the configuration space of these systems much simpler and amenable to mathematical analysis. This was the case when we looked at Brownian motion of a microscopic particle as the random kicks do not cause the particle to move far from its initial position in a small time period. Particularly relevant is the stochastic dynamics of particles hopping on graphs in which the particles can only hop between neighbouring nodes. A periodic lattice is a particularly simple graph. Examples of such systems include molecular motors ‘walking’ on biopolymer tracks, electrons hopping in (possibly disordered) semiconductors, or the evolution of a set of chemical reactants where the states correspond to the number of molecules of a particular type.

A particularly useful set of problems consists of *one dimensional* models in which the particles can only hop between neighbouring sites. These models have provided great insights into the behaviour of driven far from equilibrium systems as well as having many applications in physics, biology and chemistry. The one dimensional ‘track’ can be described by the set of integers  $n \in \mathbb{Z}$ . In these systems, a particle at state  $n$  can only move to the state  $n + 1$  with rate  $q_n$  or to state  $n - 1$  with rate  $r_n$ .

The Master equation for the dynamics of the probability of finding the system in state  $n$ ,  $\mathcal{P}_n(t)$ , is given as in Eq. (III.1), with the following transition rates

$$k_{n \rightarrow m} = q_n \delta_{m, n+1} + r_n \delta_{m, n-1} , \quad (\text{III.43})$$

where  $\delta_{n,m}$  is the Kronecker delta-function.

This yields

$$\frac{d}{dt} \mathcal{P}_n(t) = r_{n+1} \mathcal{P}_{n+1}(t) + q_{n-1} \mathcal{P}_{n-1}(t) - (q_n + r_n) \mathcal{P}_n(t) , \quad (\text{III.44})$$

An unbiased one dimensional random walk with homogeneous hopping rates corresponding to  $q_n = r_n = k$  gives

$$\frac{d}{dt}\mathcal{P}_n(t) = k \mathcal{P}_{n+1}(t) + k \mathcal{P}_{n-1}(t) - 2k \mathcal{P}_n(t) \quad , \quad (\text{III.45})$$

which is a discrete version of the 1D diffusion equation

$$\partial_t \mathcal{P}(x, t) = D \partial_x^2 \mathcal{P}(x, t) \quad , \quad (\text{III.46})$$

with the effective diffusion coefficient  $D = ka^2$ , and  $a$  being the lattice base ( $na \rightarrow x$ ). A biased variant, with homogeneous hopping rates  $q_n = k + \frac{1}{2}\Delta k$  and  $r_n = k - \frac{1}{2}\Delta k$  reads

$$\frac{d}{dt}\mathcal{P}_n(t) = \left(k - \frac{1}{2}\Delta k\right) \mathcal{P}_{n+1}(t) + \left(k + \frac{1}{2}\Delta k\right) \mathcal{P}_{n-1}(t) - 2k \mathcal{P}_n(t) \quad , \quad (\text{III.47})$$

which goes to the 1D Fokker-Planck equation

$$\partial_t \mathcal{P}(x, t) = -v \partial_x \mathcal{P}(x, t) + D \partial_x^2 \mathcal{P}(x, t) \quad , \quad (\text{III.48})$$

in the continuum limit, with  $D = ka^2$  as before and  $v = a\Delta k$  as the effective drift velocity. As discussed in Sec. II.B, Eq. (III.48) has the form of a continuity equation  $\partial_t \mathcal{P}(x, t) + \partial_x \mathcal{J} = 0$ , where the flux  $\mathcal{J} = v\mathcal{P} - D\partial_x \mathcal{P}$  has deterministic (drift) and stochastic (diffusive) components.

### 1. Poisson process

A very important and paradigmatic example is the *Poisson process* in which there is a constant transition (hopping) rate in one direction (e.g. the forward rate) such that  $q_n = g$  and  $r_n = 0$ . This is a good model for a stochastic ‘counter’. For example, it is used as a model to count the number of particles in a chemically reactive system producing them at a constant rate.

The Master equation is then

$$\frac{d}{dt}\mathcal{P}_n(t) = -g [\mathcal{P}_n(t) - \mathcal{P}_{n-1}(t)] \quad . \quad (\text{III.49})$$

With an initial condition of  $n = 0$  at  $t = 0$ , i.e  $\mathcal{P}_n(0) = \delta_{n,0}$ , it is easy to show by substitution that

$$\mathcal{P}_n(t) = \frac{(gt)^n}{n!} e^{-gt} \quad , \quad (\text{III.50})$$

is the appropriate solution of Eq. (III.49). This is called the *Poisson distribution*.

Note that  $\sum_{n=0}^{\infty} \mathcal{P}_n(t) = 1$  so that  $\mathcal{P}_n(t)$  is indeed a probability.

We can calculate various averages using the Poisson distribution. The average number of counts at time  $t$  is given

$$\begin{aligned} \bar{n}(t) \equiv \langle n(t) \rangle &= \sum_{n=0}^{\infty} n \mathcal{P}_n(t) = \sum_{n=0}^{\infty} n \frac{(gt)^n}{n!} e^{-gt} \\ \Rightarrow \bar{n}(t) &= gt \end{aligned} \quad (\text{III.51})$$



Similarly, we can calculate

$$\begin{aligned} \langle n(t)^2 \rangle &= \sum_{n=0}^{\infty} n^2 \mathcal{P}_n(t) = \sum_{n=0}^{\infty} [n(n-1) + n] \frac{(gt)^n}{n!} e^{-gt} = (gt)^2 + (gt) \\ \Rightarrow \Delta n(t)^2 &\equiv \langle n(t)^2 \rangle_c = \langle n(t)^2 \rangle - \langle n(t) \rangle^2 = gt = \bar{n}(t) \quad , \end{aligned} \quad (\text{III.52})$$

Therefore, the variance of the Poisson distribution is equal to its mean. The width of the distribution is controlled by

$$\Delta n(t) = \sqrt{\bar{n}(t)} = \sqrt{gt} \quad . \quad (\text{III.53})$$

So the Poisson process starts with a distribution that is narrow and localized around  $n = 0$  initially, and moves away from  $n = 0$  at the same time as it spreads. The peak of the distribution moves with a constant ‘speed’,  $g$  so that the peak will be at position  $gt$  at time  $t$ . We also have

$$\frac{\Delta n}{\bar{n}} = \frac{1}{\sqrt{\bar{n}(t)}} = \frac{1}{\sqrt{gt}} \quad . \quad (\text{III.54})$$

#### D. Number fluctuations

The simple example of the Poisson process in Sec. III.C.1 suggests that number fluctuations could be a significant source of stochasticity, and that deterministic descriptions of kinetic processes that involve particle number changes, such as the one we used to describe the kinetics of chemical reactions in Sec. III.B, do not capture these fluctuations. Here, we investigate the effect of number fluctuations in the context of kinetic processes using a number of simple models.

##### 1. Birth-death process: an exact solution

Consider a population of some species  $A$  that can undergo the following two processes:

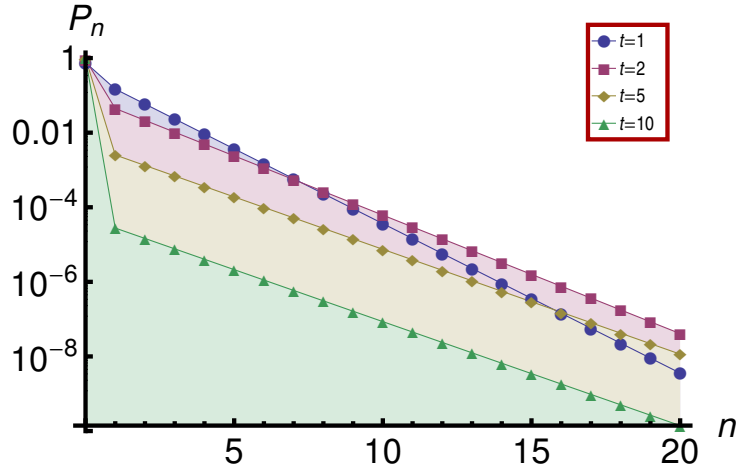
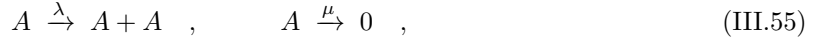


FIG. III.1 **The death-dominated distribution.** The probability distribution of a population that starts from a single member at  $t = 0$ , corresponding to  $\lambda = 1$  and  $\mu = 1.9$ . Time is in unit of the division time (since we set  $\lambda = 1$ ). We observe that the population is always exponentially distributed, and has a long-time *extinction* probability of unity.

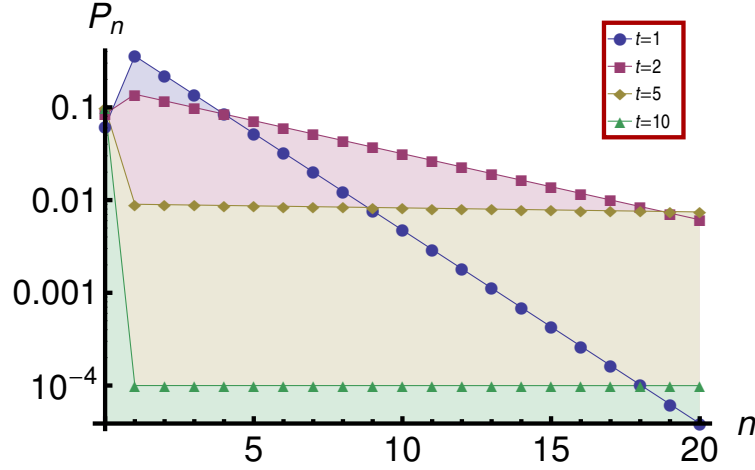


FIG. III.2 **The birth-dominated distribution.** The probability distribution of a population that starts from a single member at  $t = 0$ , corresponding to  $\lambda = 1$  and  $\mu = 0.1$ . Time is in unit of the birth time (since we set  $\lambda = 1$ ). We observe that the probability distribution of the population is flat, i.e. number fluctuations are as large as the average numbers, and that the population has a long-time *extinction* probability of  $\mu/\lambda = 0.1$ .

i.e., each  $A$  can give birth to another  $A$  with rate  $\lambda$ , or, it can die with rate  $\mu$ , at any time. Ignoring the stochasticity, the average behaviour of the system is described by exponential birth and death. The population  $\bar{n}(t)$  can be determined as follows:

$$\frac{d}{dt}\bar{n}(t) = (\lambda - \mu)\bar{n}(t) \Rightarrow \bar{n}(t) = e^{(\lambda - \mu)t} \quad , \quad (\text{III.56})$$

where we have assumed that the initial size of the population is one. As this is a one-step process, the probability of finding  $n$  copies of  $A$  in the sample at time  $t$  satisfies Eq. (III.44) with  $r_n = \mu n$  and  $q_n = \lambda n$ , namely

$$\frac{d}{dt}\mathcal{P}_n(t) = \mu(n+1)\mathcal{P}_{n+1}(t) + \lambda(n-1)\mathcal{P}_{n-1}(t) - (\mu + \lambda)n\mathcal{P}_n(t) \quad , \quad (\text{III.57})$$

The factor of  $n$  is needed because the birth or death could happen to anyone. Equation (III.57) can be solved by an *ansatz* of the form  $\mathcal{P}_n \sim f^n$  for  $n \geq 1$ , which together with the initial condition  $\mathcal{P}_n(0) = \delta_{n,1}$  gives us the solution as

$$\mathcal{P}_n(t) = \frac{\bar{n}(1 - \mu/\lambda)^2}{(\bar{n} - 1)(\bar{n} - \mu/\lambda)} \left( \frac{\bar{n} - 1}{\bar{n} - \mu/\lambda} \right)^n \quad . \quad (\text{III.58})$$

The distribution can be used to calculate the first two moments

$$\langle n(t) \rangle = \sum_n n \mathcal{P}_n(t) = \bar{n}(t) = e^{(\lambda - \mu)t} \quad , \quad (\text{III.59})$$

$$\Delta n^2 = \langle n(t)^2 \rangle - \langle n(t) \rangle^2 = \left( \frac{\lambda + \mu}{\lambda - \mu} \right) e^{(\lambda - \mu)t} \left[ e^{(\lambda - \mu)t} - 1 \right] \quad , \quad (\text{III.60})$$

which reveal more interesting features about the system. First, it is reassuring that the average population size behaves according to the mean-field description above that predicted exponential growth or decay. A quantity of interest is

$$\frac{\Delta n^2}{\bar{n}} = \left( \frac{\lambda + \mu}{\lambda - \mu} \right) \left[ e^{(\lambda - \mu)t} - 1 \right] \quad , \quad (\text{III.61})$$

which probes whether number fluctuations follow a characteristic Poisson behaviour. In the long time limit, we have

$$\frac{\Delta n^2}{\bar{n}} = \begin{cases} \infty & ; \quad \lambda > \mu \\ \frac{\mu + \lambda}{\mu - \lambda} & ; \quad \lambda < \mu \end{cases} \quad (\text{III.62})$$

which shows that while a decaying population that corresponds to  $\lambda < \mu$  has a Poisson behaviour, a growing population corresponding to  $\lambda > \mu$  has *giant number fluctuations*, which can be characterized via

$$\frac{\Delta n}{\bar{n}} = \sqrt{\frac{\lambda + \mu}{\lambda - \mu}} \quad (\text{III.63})$$

in the long time limit. In other words, the fluctuations scale with the average population size when  $\lambda > \mu$ , and with the square root of the average population size when  $\lambda < \mu$ .

The above solution allows us to calculate *the extinction probability* of the population  $\mathcal{P}_0(t)$ , which is an *absorbing state*. We find

$$\mathcal{P}_0(t) = 1 - \sum_{n=1}^{\infty} \mathcal{P}_n(t) = \frac{\mu}{\lambda} \frac{\bar{n} - 1}{\bar{n} - \mu/\lambda} \quad (\text{III.64})$$

which is a very interesting result. When  $\mu > \lambda$ ,  $\bar{n} \rightarrow 0$  at long times, and we obtain  $\mathcal{P}_0 = 1$ . It is no surprise that extinction at long times is a certainty when the death rate is larger than the birth rate. However, when  $\mu < \lambda$ ,  $\bar{n} \rightarrow \infty$  at long times, and we obtain  $\mathcal{P}_0 = \mu/\lambda$ ; a result that is in contradiction with the prediction of the average behaviour of the system, which is exponential growth. So, number fluctuations could completely annihilate an exponentially growing population!

The distribution is plotted for a number of time slices in Fig. III.1 for the case where  $\lambda < \mu$ . In this case, we expect the population to decay as time goes on, as the average behaviour predicts. Figure III.1 shows that after a short transient period, the population exhibits an exponential distribution which is consistent with an exponentially decaying average population size. Figure III.2 shows the distribution for a number of time slices for  $\lambda > \mu$ . In this case, the distribution of the population makes a transition from an initial exponential form to a flat dependence on size, implying that in a growing population any size is as likely as any other size.

Another interesting feature of the exact solution is that in the special case where  $\mu = \lambda$ , and we expect a static population, the distribution reads

$$\mathcal{P}_n(t) = \frac{(\lambda t)^{n-1}}{(1 + \lambda t)^{n+1}} \quad (\text{III.65})$$

which gives a constant  $\langle n(t) \rangle = 1$  and  $\Delta n^2 = 2\lambda t$ . In this limit, the extinction probability is

$$\mathcal{P}_0(t) = \frac{\lambda t}{1 + \lambda t} \quad (\text{III.66})$$

## 2. Logistic growth and the van Kampen expansion

A more widely used model to study population growth includes a process of annihilation that is controlled by interactions between the particles. This process can stabilize the exponential growth and lead to a saturation of the population at a value called the *carrying capacity*.

Let us first study the interplay between stochastic birth and coagulation that leads to death. Consider a volume  $V$  in which we study the growth-coagulation process described as



where  $\mu/V$  is taken to be the coagulation rate of distinct pairs of particles within the volume. We can write the mean-field kinetic equations for the population  $\bar{n}(t)$  and the concentration  $\bar{C} = \bar{n}/V$  as follows:

$$\frac{d}{dt}\bar{n}(t) = \lambda\bar{n} - \frac{\mu}{V}\bar{n}^2 \Rightarrow \frac{d}{dt}\bar{C}(t) = \lambda\bar{C} - \mu\bar{C}^2 \quad . \quad (\text{III.68})$$

The above equations describe the logistic growth process, giving us a saturation at  $\bar{C}_0 = \lambda/\mu$ , which is the stable fixed-point of the dynamics ( $\bar{C} = 0$  is unstable).

The stochastic equation for the probability  $\mathcal{P}_n(t)$  for having  $n$  particles at time  $t$  in that volume satisfies Eq. (III.44) with  $r_n = \left(\frac{\mu}{V}\right)n(n-1)$  and  $q_n = \lambda n$ , namely

$$\partial_t \mathcal{P}_n(t) = \left(\frac{\mu}{V}\right)(n+1)n \mathcal{P}_{n+1}(t) + \lambda(n-1) \mathcal{P}_{n-1}(t) - \left[\left(\frac{\mu}{V}\right)n(n-1) + \lambda n\right] \mathcal{P}_n(t) \quad . \quad (\text{III.69})$$

Rather than attempting to solve the equation exactly, which is far from trivial in this case, we can use alternative approximation techniques to understand the behaviour of the system. Since we are interested in probing deviations from the mean-field solution  $\bar{C}$ , we can perform an expansion that gives us the continuum limit of  $n$ . Expanding Eq. (III.69) up to second order in  $n$ , gives us the Fokker-Planck equation

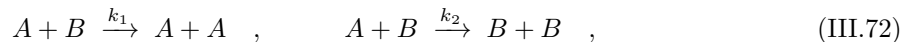
$$\partial_t \mathcal{P}(n, t) = -\partial_n \left\{ \left[ \lambda n - \left(\frac{\mu}{V}\right)n(n-1) \right] \mathcal{P}(n, t) \right\} + \frac{1}{2} \partial_n^2 \left\{ \left[ \lambda n + \left(\frac{\mu}{V}\right)n(n-1) \right] \mathcal{P}(n, t) \right\} \quad , \quad (\text{III.70})$$

which we can interpret as the corresponding Fokker-Planck equation of the following Langevin equation for a stochastic density  $C(t) = n(t)/V$ :

$$\frac{d}{dt}C(t) = (\lambda + \mu/V)C(t) - \mu C(t)^2 + \frac{1}{\sqrt{V}} \sqrt{(\lambda - \mu/V)C(t) + \mu C(t)^2} \zeta(t) \quad , \quad (\text{III.71})$$

where  $\zeta(t)$  is a Gaussian white noise, i.e.  $\langle \zeta(t) \rangle = 0$  and  $\langle \zeta(t)\zeta(t') \rangle = \delta(t-t')$ . Equation (III.71) has a very interesting structure. It gives us the mean-field logistic growth kinetics as the average description of the dynamics, and provides us with a way to study fluctuations around that. The noise term in Eq. (III.71) is proportional to  $1/\sqrt{V}$ , which manifests that its effect disappears in the thermodynamic limit and that fluctuations are important in finite-size systems. Moreover, the strength of the noise term depends on the stochastic concentration itself, which leads to inhomogeneous fluctuations. This is an example of stochastic dynamics with multiplicative noise.

There are several microscopic processes that are described by logistic growth at the mean-field level. The above method helps us differentiate between them, as the exact form of the density dependence of the noise that describes density fluctuations turn out to depend on the specific process. To see this, let us consider the reaction between two species  $A$  and  $B$  described by the process



which is similar to both Eq. (III.55) and Eq. (III.67), but also subtly different. Let us define  $C_A = n_A/V$ ,  $C_B = n_B/V$ , and  $C_0 = N/V$ , where  $n_A$  and  $n_B$  are the number of particles in each species at any instant. The reaction strictly conserves  $n_A + n_B = N$ , hence we can describe the system using the number of  $A$

particles, which we will denote as  $n$ . At the mean-field level, we have

$$\frac{d}{dt}\bar{C}_A(t) = -\frac{d}{dt}\bar{C}_B(t) = (k_1 - k_2) \bar{C}_A \bar{C}_B = (k_1 - k_2) \bar{C}_A (C_0 - \bar{C}_A) \quad , \quad (\text{III.73})$$

which is the logistic growth equation. The stochastic equation for the probability  $\mathcal{P}_n(t)$  reads

$$\partial_t \mathcal{P}_n(t) = \left(\frac{k_2}{V}\right) (n+1)(N-n-1) \mathcal{P}_{n+1}(t) + \left(\frac{k_1}{V}\right) (n-1)(N-n+1) \mathcal{P}_{n-1}(t) - \left(\frac{k_1 + k_2}{V}\right) n(N-n) \mathcal{P}_n(t) \quad . \quad (\text{III.74})$$

$$\partial_t \mathcal{P}(n, t) = -\partial_n \left[ \frac{1}{V} (k_1 - k_2) n(N-n) \mathcal{P}(n, t) \right] + \frac{1}{2} \partial_n^2 \left[ \frac{1}{V} (k_1 + k_2) n(N-n) \mathcal{P}(n, t) \right] \quad , \quad (\text{III.75})$$

which corresponds to the following Langevin equation for a stochastic density

$$\frac{d}{dt} C(t) = (k_1 - k_2) C(t) [C_0 - C(t)] + \frac{1}{\sqrt{V}} \sqrt{(k_1 + k_2) C(t) [C_0 - C(t)]} \zeta(t) \quad , \quad (\text{III.76})$$

with a different density dependence of the noise strength. Notably, in Eq. (III.76) the strength of the noise vanishes at the same time as the average term when  $C(t) = 0$  or  $C(t) = C_0$ . This feature was missing in Eq. (III.71).

### 3. Aggregation

In a process of irreversible aggregation, particles and their clusters can join to make larger clusters. For a single species  $A$ , we can represent this process as



for any  $i$  and  $j$ . Consider a system in which  $N$  particles are initially distributed uniformly in a volume  $V$ , giving the single-particle average concentration  $C = N/V$ . The master equation for the cluster size distribution  $\mathcal{C}_n(t)$ , defined as the average concentration of clusters of size  $n$  in volume  $V$ , is nonlinear:

$$\frac{d}{dt} \mathcal{C}_n(t) = \frac{1}{2} \sum_{i+j=n} k_{ij} \mathcal{C}_i \mathcal{C}_j - \sum_i k_{in} \mathcal{C}_i \mathcal{C}_n \quad . \quad (\text{III.78})$$

The total concentration of the particles in the system  $C = \sum_n n \mathcal{C}_n(t)$  is a conserved quantity; this can be checked by inspection.

For diffusive aggregation, and assuming that each cluster of size  $n$  is a sphere of radius  $a_n$ , the rates can be calculated as (see Sec. I.E.1)

$$k_{ij} = 4\pi(D_i + D_j)(a_i + a_j) = \frac{2k_B T}{3\eta} \left( \frac{1}{a_i} + \frac{1}{a_j} \right) (a_i + a_j) \quad . \quad (\text{III.79})$$

In 1917, Smoluchowski argued that the process of aggregation is dominated by collision of clusters that are of nearly equal size, and used this justification to use the following approximation  $\left(\frac{1}{a_i} + \frac{1}{a_j}\right)(a_i + a_j) \simeq 4$ , which gives a constant rate with which he could solve Eq. (III.78) exactly. The solution can be constructed using the same *ansatz* as that used in Sec. III.D.1. The Smoluchowski cluster distribution reads

$$\mathcal{C}_n(t) = \frac{C(\lambda t)^{n-1}}{(1 + \lambda t)^{n+1}} \quad , \quad (\text{III.80})$$

where

$$\lambda = \frac{4k_B T}{3\eta} C \quad , \quad (\text{III.81})$$

is an inverse characteristic time scale, that roughly corresponds to the time it takes for a particle of size  $C^{-1/3}$  to diffuse this distance. Interestingly, this is identical to Eq. (III.65). The total concentration of clusters (i.e. the total number of clusters per unit volume) can be calculated as

$$C_{\text{cluster}} = \sum_{n=1}^{\infty} c_n(t) = \frac{C}{1 + \lambda t} \quad , \quad (\text{III.82})$$

which decays very slowly.

#### 4. Exclusion processes

A simple way to model non-equilibrium behaviour of a collection of particles that interact with each other via short range excluded volume is to discretize space into a lattice and assume that every lattice point could be at most singly occupied. For equilibrium systems, such a simple lattice-gas model (that is related to the Ising-model) allows us to study a variety of phenomena, such as the liquid-gas condensation transition. The model has also served as a very powerful playground for studying non-equilibrium transport phenomena, in particular in 1D where exact solutions could be found. These results helped us understand many novel aspects of nonequilibrium steady states.

The Asymmetric Simple Exclusion Process (ASEP) is constructed using an occupation number  $n_j(t)$ , which can be 0 or 1, on a 1D lattice. A particle can hop to the neighbouring site if it is unoccupied. Let us denote the rates as  $k_+ = k + \frac{1}{2}\Delta k$  for hopping to the right and  $k_- = k - \frac{1}{2}\Delta k$  for hopping to the left. We can write the governing equation for the evolution of the occupation number at any site as follows:

$$\frac{d}{dt}n_j(t) = k_+n_{j-1}(1 - n_j) + k_-n_{j+1}(1 - n_j) - k_+n_j(1 - n_{j+1}) - k_-n_j(1 - n_{j-1}) \quad . \quad (\text{III.83})$$

The exclusion has made this process nonlinear, but in a very interesting way, as can be seen from this rendition of the equation

$$\frac{d}{dt}n_j(t) = k(n_{j+1} - 2n_j + n_{j-1}) + \frac{1}{2}\Delta k(2n_jn_{j+1} - 2n_jn_{j-1} + n_{j-1} - n_{j+1}) \quad , \quad (\text{III.84})$$

which shows that the nonlinearity only corresponds to the net transport properties controlled by  $\Delta k$ . For the Symmetric Exclusion Process (SEP) that corresponds to  $\Delta k = 0$ , the process becomes free diffusion despite the presence of exclusion.

To understand the structure of the nonlinear term, let us introduce an average density profile  $\rho(x = ja, t) = \langle n_j(t) \rangle$  and take the continuum limit of Eq. (III.84). Note that by definition, we have  $0 \leq \rho(x, t) \leq 1$ . To achieve that, we will need to use the approximation  $\langle n_j(t)(n_{j+1} - n_{j-1}) \rangle \simeq \langle n_j(t) \rangle \langle (n_{j+1} - n_{j-1}) \rangle$  that is equivalent to the closure scheme in Eq. (II.49). Taking the continuum limit, we find

$$\partial_t \rho(x, t) + \partial_x [v\rho(1 - \rho)] - D\partial_x^2 \rho = 0 \quad , \quad (\text{III.85})$$

with  $D = ka^2$  and  $v = a\Delta k$ , as in Sec. III.C. The deterministic component of the flux

$$J = v\rho(1 - \rho) \quad , \quad (\text{III.86})$$

has the expected component  $\rho v$  times the probability that the next site is empty, which is  $1 - \rho$ . This interesting structure turns out to be valid (at the global scale) for the exact solution of ASEP, which is a surprising finding given that our derivation of the continuum limit involved the truncation approximation.