→ Without exercises in notes. If needed, see old solutions.

Plasma Kinetics Problem Set

1.)

$$\epsilon(p,k) = 1 + \sum_\alpha \frac{1+\zeta_\alpha Z(\zeta_\alpha)}{(k\lambda_{D\alpha})^2} \quad , \quad \zeta_\alpha = \frac{ip}{k v_{th\alpha}}$$

with

$$\omega_{p\alpha} = v_{th\alpha}/\sqrt{2}\lambda_{D\alpha}$$

$$Z(\zeta_\alpha) = \frac{1}{\sqrt{\pi}}\int_{C_L} du \frac{e^{-u^2}}{u-\zeta_\alpha} \quad , \quad u = \frac{v_z}{v_{th\alpha}}$$

which has the following properties:

$$Z'(\zeta) = -2[1+\zeta Z(\zeta)]$$

$$\lim_{|\zeta|\ll 1} Z(\zeta) = i\sqrt{\pi}e^{-\zeta^2} - 2\zeta\left(1 - \frac{2\zeta^2}{3} + \frac{4\zeta^4}{15} + \dots\right)$$

$$\lim_{|\zeta|\gg 1} Z(\zeta) = i\sqrt{\pi}e^{-\zeta^2} - \frac{1}{\zeta}\left(1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \dots\right)$$

Langmuir waves: $\omega \gg k v_{the}, k v_{thi} \Rightarrow |\zeta_\alpha| \gg 1$.
However, given $v_{the}/v_{thi} \gg 1$, neglect ion contributions, keeping only electron bulk contributions $(k\lambda_{Di}) \gg 1$

$$\epsilon \simeq 1 + \frac{(1+\zeta_e Z(\zeta_e))}{(k\lambda_{De})^2} \simeq 1 + \frac{1}{(k\lambda_{De})^2}\left[1 + \zeta_e\left(i\sqrt{\pi}e^{-\zeta_e^2} - \frac{1}{\zeta_e}\left(1+\frac{1}{2\zeta_e^2}+\frac{3}{4\zeta_e^4}+\dots\right)\right)\right]$$

$$= 1 + \frac{1}{(k\lambda_{De})^2}\left[i\zeta_e\sqrt{\pi}e^{-\zeta_e^2} - \frac{1}{2\zeta_e^2} - \frac{3}{4\zeta_e^4} + \dots\right]$$

For the frequency, $\mathrm{Re}\,\epsilon = 0 \Rightarrow 0 = 1 - \frac{1}{2\zeta_e^2}\left(\cancel{\frac{d}{\zeta_e}} + \frac{3}{2\zeta_e^2}\right)\frac{1}{(k\lambda_{De})^2}$

$$0 = 1 - \frac{1}{(\sqrt{2}\zeta_e k\lambda_{De})^2}\left(1+\frac{3}{2\zeta_e^2}\right) = 1 - \frac{\omega_{pe}^2}{\omega^2}\left(1 + \frac{3}{2}\frac{(k v_{the})^2}{\omega^2}\right)$$

which implies that

$$\boxed{\omega^2 = \omega_{pe}^2\left[1 + 3k^2\lambda_{De}^2\right]}$$

For the damping, $\gamma = -\operatorname{Im}\epsilon \left[\frac{\partial \operatorname{Re}\epsilon}{\partial \omega}\right]^{-1}$, $\frac{\partial}{\partial \omega} = \frac{1}{k v_{the}} \frac{\partial}{\partial \zeta_e}$

Now,

$$\operatorname{Re}\epsilon = 1 - \frac{1}{(k\lambda_{De})^2}\left[\frac{1}{2\zeta_e^2} + \frac{3}{4\zeta_e^4} + \dots\right]$$

$$\operatorname{Im}\epsilon = \frac{\zeta_e\sqrt{\pi}}{(k\lambda_{De})^2} e^{-\zeta_e^2}$$

$$\frac{\partial \operatorname{Re}\epsilon}{\partial \omega} \simeq \frac{1}{k v_{the}} \frac{1}{(k\lambda_{De})^2} \frac{1}{\zeta_e^3} \quad \text{to leading order.}$$

Then,

$$\gamma = -k v_{the}\, \zeta_e^4 \sqrt{\pi}\, e^{-\zeta_e^2} \underset{\omega \sim \omega_{pe}}{\simeq} -\sqrt{\pi}\frac{\omega_{pe}^4}{(k v_{the})^3} e^{-\frac{1}{2(k\lambda_{De})^2}}$$

which obtains the usual Landau damping result.

<u>For acoustic waves:</u> $v_{the} \gg \omega/k \gg v_{thi}$. $|\zeta_e| \ll 1$, $|\zeta_i| \gg 1$

$$\epsilon(p,k) = 1 + \frac{1}{(k\lambda_{De})^2}[1+\zeta_e Z(\zeta_e)] + \frac{1}{(k\lambda_{Di})^2}[1+\zeta_i Z(\zeta_i)]$$

$$= 1 + \frac{1}{(k\lambda_{De})^2}\left[i\sqrt{\pi}\zeta_e e^{-\zeta_e^2} + 1 - 2\zeta_e^2 + \frac{4\zeta_e^4}{3} - \dots\right]$$

$$+ \frac{1}{(k\lambda_{Di})^2}\left[i\sqrt{\pi}\zeta_i e^{-\zeta_i^2} - \frac{1}{2\zeta_i^2} - \frac{3}{4\zeta_i^4} + \dots\right]$$

$$= 1 + \frac{i\sqrt{\pi}\zeta_e e^{-\zeta_e^2}}{(k\lambda_{De})^2} + \frac{i\sqrt{\pi}\zeta_i e^{-\zeta_i^2}}{(k\lambda_{Di})^2} + \frac{2\omega_{pe}^2}{\omega^2}\left[\zeta_e^2 - 2\zeta_e^4 + \frac{4\zeta_e^6}{3} + \dots\right]$$

$$- \frac{\omega_{pi}^2}{\omega^2}\left[1 + 3(k^2\lambda_{Di}^2)\right]$$

assumed $\omega \sim \omega_{pi}$ in this term (small correction).

For the real frequency:

neglect as small.

$$0 = 1 + \frac{2\omega_{pe}^2}{\omega^2}\left[\zeta_e^2 - 2\zeta_e^4 + \dots\right] - \frac{\omega_{pi}^2}{\omega^2}\left[1 + \cancel{3(k\lambda_{Di})^2}\right] = 0$$

$$= \omega^2 + 2\omega_{pe}^2\zeta_e^2 - \omega_{pi}^2$$

so

$$\boxed{\omega^2 = \frac{\omega_{pi}^2}{1 + 1/(k\lambda_{De})^2} = \frac{k^2 c_s^2}{1+(k\lambda_{De})^2}} \rightarrow c_s = \omega_{pi}\lambda_{De} = \sqrt{\frac{Z_i T_e}{m_i}}$$

For the damping, $\mathrm{Re}\,\epsilon = 1 + \frac{1}{(k\lambda_{De})^2} - \frac{\omega_{pi}^2}{\omega^2}$ to lowest order

$$\frac{\partial \mathrm{Re}\,\epsilon}{\partial \omega} = \frac{2\omega_{pi}^2}{\omega^3}$$

Then,

$$\gamma = -\frac{\omega^3}{2\omega_{pi}^2}\frac{\sqrt{\pi}}{k^2}\left(\frac{\zeta_e e^{-\zeta_e^2}}{\lambda_{De}^2} + \frac{\zeta_i^2 e^{-\zeta_i^2}}{\lambda_{Di}^2}\right)$$

$$= -\sqrt{\pi}\frac{\omega^3}{2k^2}\frac{1}{\omega_{pe}^2}\left(\frac{m_i}{Z m_e}\right)\left(\frac{\zeta_e e^{-\zeta_e^2}}{\lambda_{De}^2} + \frac{\zeta_i^2 e^{-\zeta_i^2}}{\lambda_{Di}^2}\right) \quad \rangle\ \zeta_e = \frac{1}{v_{the}}\left(\frac{\omega}{k} - u_e\right)$$

$$= -\sqrt{\pi}\frac{\omega^4}{(k v_{the})^3}\frac{m_i}{Z m_e}\left[\left(1 - \frac{u_e}{\omega/k}\right)e^{-\zeta_e^2} + \left(\frac{v_{the}}{v_{thi}}\right)^3\left(\frac{\omega_{pi}}{\omega_{pe}}\right)^2 e^{-\zeta_i^2}\right]$$

However, given $\zeta_i \gg 1$, $\zeta_e \ll 1$, and letting $\omega/k \sim c_s$, we obtain the familiar result:

$$\boxed{\gamma = -\sqrt{\pi}\frac{\omega^4}{(k v_{the})^3}\frac{m_i}{Z m_e}\left(1 - \frac{u_e}{c_s}\right)}$$

Ion acoustic instability!

The ion contribution becomes important when:

$$e^{-\zeta_e^2} \sim \left(\frac{v_{the}}{v_{thi}}\right)^3\left(\frac{\omega_{pi}}{\omega_{pe}}\right)^2 e^{-\zeta_i^2} \sim \frac{Z m_e}{m_i}\left(\frac{T_e}{T_i}\frac{m_i}{m_e}\right)^{3/2} e^{-\zeta_e^2} \sim Z_i\left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2} e^{-\zeta_i^2}$$

so

$$\left| e^{\zeta_i^2} \sim Z_i\left(\frac{m_i}{m_e}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2} \right|$$

Ion langmuir waves: These occur for $k\lambda_{De} \gg 1$, then using

$$\mathrm{Re}\,\epsilon = 1 + \frac{1}{(k\lambda_{De})^2} - \frac{\omega_{pi}^2}{\omega^2}\left[1 + 3k^2\lambda_{Di}^2 + \dots\right]$$

we get the usual dispersion relation $\boxed{\omega^2 = \omega_{pi}^2\left[1 + 3(k\lambda_{Di})^2\right]}$.
Using the previous result for $k\lambda_{De} \gg 1$, and assuming that $\omega \approx \omega_{pi}$,
it follows that

$$\boxed{\gamma = -\sqrt{\pi}\,\frac{\omega_{pi}^4}{(k v_{thi})^3}\, e^{-\frac{1}{2(k\lambda_{Di})^2}}}$$

which is the familiar result.

2/3) <u>Plasma waves.</u>

$$\frac{\partial \delta f_\alpha}{\partial t} + i\underline{k}\cdot\underline{v}\,\delta f_\alpha + \frac{q_\alpha}{m_\alpha}\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right)\cdot\frac{\partial f_{0\alpha}}{\partial \underline{v}} = 0 \qquad (1)$$

$$i\underline{k}\cdot\underline{E} = 4\pi \sum_\alpha q_\alpha \int d^3\underline{v}\,\delta f_\alpha \qquad (2)$$

$$i\underline{k}\cdot\underline{B} = 0 \qquad (5)$$

$$i\underline{k}\times\underline{E} + \frac{1}{c}\frac{\partial \underline{B}}{\partial t} = 0 \qquad (3)$$

$$i\underline{k}\times\underline{B} - \frac{1}{c}\frac{\partial \underline{E}}{\partial t} = \frac{4\pi}{c}\underline{j} = \frac{4\pi}{c}\sum_\alpha q_\alpha \int d^3\underline{v}\,\underline{v}\,\delta f_\alpha \qquad (4)$$

Taking $i\underline{k}\times(3)$, and using (4):

$$0 = i\underline{k}\times(i\underline{k}\times\underline{E}) + \frac{1}{c}\frac{\partial}{\partial t}(i\underline{k}\times\underline{B}) = \overbrace{-\underline{k}\times(\underline{k}\times\underline{E})}^{k^2\underline{E}-\underline{k}(\underline{k}\cdot\underline{E})} + \frac{1}{c^2}\frac{\partial^2 \underline{E}}{\partial t^2} + \frac{4\pi}{c^2}\frac{\partial \underline{j}}{\partial t} = 0.$$

Taking a Laplace transformation:

$$\delta\hat{f}_\alpha = \frac{1}{p + i\underline{k}\cdot\underline{v}}\left[\delta f_\alpha(0) - \frac{q_\alpha}{m_\alpha}\left(\hat{\underline{E}} + \frac{\underline{v}\times\hat{\underline{B}}}{c}\right)\cdot\frac{\partial f_{0\alpha}}{\partial \underline{v}}\right]$$

$$i\underline{k}\cdot\hat{\underline{E}} = 4\pi\sum_\alpha q_\alpha \int d^3\underline{v}\,\delta\hat{f}_\alpha$$

$$i\underline{k}\cdot\hat{\underline{B}} = 0$$

$$i\underline{k}\times\hat{\underline{E}} + \frac{p}{c}\hat{\underline{B}} = \frac{\underline{B}(0)}{c}.$$

and

$$k^2\hat{\underline{E}} - \underline{k}(\underline{k}\cdot\hat{\underline{E}}) + \frac{1}{c^2}\left(p^2\hat{\underline{E}} - p\underline{E}(0) - \underline{E}'(0)\right) + \frac{4\pi}{c^2}\left(-\underline{j}(0) + p\hat{\underline{j}}\right) = 0.$$

Absorbing all of the initial conditions into $\underline{I}(0)$,

$$\underline{\hat{E}}\left(1+\frac{k^2c^2}{p^2}\right)-\frac{c^2}{p^2}\underline{k}(\underline{k}.\underline{\hat{E}})+\frac{4\pi}{p}\hat{\underline{j}}=\underline{\Gamma}(0). \qquad (*)$$

③

Now, taking $\underline{v}\times$ (laplace transform of ④),

$$\underline{v}\times(i\underline{k}\times\underline{\hat{E}})+p\frac{\underline{v}\times\underline{\hat{B}}}{c}=\frac{\underline{v}\times\underline{B}(0)}{c} \implies \frac{\underline{v}\times\underline{\hat{B}}}{c}=\frac{1}{p}\left[\frac{\underline{v}\times\underline{B}(0)}{c}-\underline{v}\times(i\underline{k}\times\underline{\hat{E}})\right]$$

Then,

$$\frac{1}{p+i\underline{k}.\underline{v}}\left(\underline{\hat{E}}+\frac{\underline{v}\times\underline{\hat{B}}}{c}\right)\cdot\frac{\partial f_{0\alpha}}{\partial\underline{v}}=\frac{1}{p+i\underline{k}.\underline{v}}\left[\underline{\hat{E}}-\frac{\underline{v}\times(i\underline{k}\times\underline{\hat{E}})}{p}+\frac{\underline{v}\times\underline{B}(0)}{pc}\right]\cdot\frac{\partial f_{0\alpha}}{\partial\underline{v}}$$

$$=\frac{1}{p+i\underline{k}.\underline{v}}\frac{1}{p}\left[\frac{\underline{v}\times\underline{B}(0)}{c}+p\underline{\hat{E}}-\left(i\underline{k}(\underline{v}.\underline{\hat{E}})-\underline{\hat{E}}(i\underline{k}.\underline{v})\right)\right]\cdot\frac{\partial f_{0\alpha}}{\partial\underline{v}}$$

$$=\frac{1}{p+i\underline{k}.\underline{v}}\frac{1}{p}\left[\frac{\underline{v}\times\underline{B}(0)}{c}+\underline{\hat{E}}(p+i\underline{k}.\underline{v})-i\underline{k}.(\underline{v}.\underline{\hat{E}})\right]\cdot\frac{\partial f_{0\alpha}}{\partial\underline{v}}$$

$$=\ldots+\frac{1}{p}\left[\underline{\hat{E}}-\frac{i\underline{k}}{p+i\underline{k}.\underline{v}}(\underline{v}.\underline{\hat{E}})\right]\cdot\frac{\partial f_{0\alpha}}{\partial\underline{v}}$$

Thus,

$$\delta\hat{f}_\alpha=\ldots+\frac{q_\alpha}{m_\alpha}\frac{1}{p}\left[\underline{\hat{E}}-\frac{i\underline{k}}{p+i\underline{k}.\underline{v}}(\underline{v}.\underline{\hat{E}})\right]\cdot\frac{\partial f_{0\alpha}}{\partial\underline{v}}$$

Substituting this into $(*)$ and absorbing the initial conditions into $\underline{\Gamma}(0)$,

$$\underline{\Gamma}(0)=\underline{\hat{E}}\left(1+\frac{k^2c^2}{p^2}\right)-\frac{c^2}{p^2}\underline{k}(\underline{k}.\underline{\hat{E}})+\frac{4\pi}{p}\sum_\alpha q_\alpha\int d^3\underline{v}\,\underline{v}\,\delta\hat{f}_\alpha$$

$$=\underline{\hat{E}}\left(1+\frac{k^2c^2}{p^2}\right)-\frac{c^2}{p^2}\underline{k}(\underline{k}.\underline{\hat{E}})-\frac{1}{p^2}\sum_\alpha\frac{4\pi q_\alpha^2}{m_\alpha}\int d^3\underline{v}\,\underline{v}\left[\underline{\hat{E}}-\frac{i\underline{k}(\underline{v}.\underline{\hat{E}})}{p+i\underline{k}.\underline{v}}\right]\cdot\frac{\partial f_{0\alpha}}{\partial\underline{v}}$$

We thus have that:

$$\underline{\Gamma}(0) = \hat{\underline{E}}\left(1+\frac{k^2c^2}{p^2}\right) - \frac{c^2}{p^2}\underline{k}(\underline{k}.\hat{\underline{E}}) - \frac{1}{p^2}\sum_\alpha \frac{\omega_{p\alpha}^2}{n_\alpha}\int d^3\underline{v}\ \underline{v}\left[\hat{\underline{E}} - \frac{i\underline{k}(\underline{v}.\hat{\underline{E}})}{p+i\underline{k}.\underline{v}}\right].\frac{\partial f_{0\alpha}}{\partial \underline{v}}$$

Now,

$$\int d^3\underline{v}\ \underline{v}\left[\hat{\underline{E}} - \frac{ik(\underline{v}.\hat{\underline{E}})}{p+i\underline{k}.\underline{v}}\right].\frac{\partial f_{0\alpha}}{\partial \underline{v}} \overset{\text{int. by parts}}{=} \int d^3\underline{v}\left[-\hat{\underline{E}} f_{0\alpha} - \frac{ik\,\underline{v}\,\underline{v}}{p+ikv_\parallel}\frac{\partial f_{0\alpha}}{\partial v_\parallel}.\hat{\underline{E}}\right]$$

$$= -\int d^3\underline{v}\left[\hat{\underline{E}} f_{0\alpha} + \frac{ik\,\underline{v}\,\underline{v}}{p+ikv_\parallel}\frac{\partial f_{0\alpha}}{\partial v_\parallel}.\hat{\underline{E}}\right] \quad \text{zero as } f_0 \text{ even in } \underline{v}_\perp$$

$$= -\int d^3\underline{v}\left[\hat{\underline{E}} f_{0\alpha} + \left(v_\parallel^2\underline{e}_\parallel\underline{e}_\parallel + \underline{v}_\perp\underline{v}_\perp + \cancel{v_\parallel\underline{e}_\parallel\underline{v}_\perp} + \cancel{v_\parallel\underline{v}_\perp\underline{e}_\parallel}\right)\frac{ik}{p+ikv_\parallel}.\hat{\underline{E}}\frac{\partial f_{0\alpha}}{\partial v_\parallel}\right]$$

$$= -\int d^3\underline{v}\left[\hat{\underline{E}} f_{0\alpha} + v_\parallel\frac{\partial f_{0\alpha}}{\partial v_\parallel}E_\parallel\underline{e}_\parallel - \frac{pv_\parallel}{p+ikv_\parallel}E_\parallel\underline{e}_\parallel\frac{\partial f_{0\alpha}}{\partial v_\parallel} + \frac{ik}{p+ikv_\parallel}\underline{v}_\perp\underline{v}_\perp.\hat{\underline{E}}_\perp\frac{\partial f_{0\alpha}}{\partial v_\parallel}\right]$$

$$= -\int d^3\underline{v}\left[\hat{\underline{E}} f_{0\alpha} - E_\parallel\underline{e}_\parallel f_{0\alpha} - \frac{pv_\parallel}{p+ikv_\parallel}\frac{\partial f_{0\alpha}}{\partial v_\parallel}E_\parallel\underline{e}_\parallel + \frac{ik\underline{v}_\perp}{p+ikv_\parallel}\underline{v}_\perp\frac{\partial f_{0\alpha}}{\partial v_\parallel}.\hat{\underline{E}}_\perp\right]$$

$$= -n_\alpha(\hat{\underline{E}} - E_\parallel\underline{e}_\parallel) + \int d^3\underline{v}\frac{pv_\parallel}{p+ikv_\parallel}\frac{\partial f_{0\alpha}}{\partial v_\parallel}E_\parallel\underline{e}_\parallel - \int d^3\underline{v}\frac{ik\underline{v}_\perp}{p+ikv_\parallel}\underline{v}_\perp\frac{\partial f_{0\alpha}}{\partial v_\parallel}.\underline{E}_\perp$$

Putting this together, we have that:

$$\underline{\Gamma}(0) = \hat{\underline{E}}\left(1+\frac{k^2c^2}{p^2}\right) - \frac{c^2}{p^2}\underline{k}(\underline{k}.\hat{\underline{E}}) + \frac{1}{p^2}\sum_\alpha \omega_{p\alpha}^2\Bigg[(\overbrace{\hat{\underline{E}} - E_\parallel\underline{e}_\parallel}^{\underline{E}_\perp})$$

$$- \frac{1}{n_\alpha}\int d^3\underline{v}\frac{pv_\parallel}{p+ikv_\parallel}\frac{\partial f_{0\alpha}}{\partial v_\parallel}\underline{E}_\parallel + \frac{1}{n_\alpha}\int d^3\underline{v}\frac{ik\underline{v}_\perp}{p+ikv_\parallel}\underline{v}_\perp\frac{\partial f_{0\alpha}}{\partial v_\parallel}.\underline{E}_\perp\Bigg]$$

$$= \hat{\underline{E}}\left(1+\frac{k^2c^2}{p^2}\right) - \frac{c^2}{p^2}\underline{k}(\underline{k}.\hat{\underline{E}}) + \frac{1}{p^2}\sum_\alpha \omega_{p\alpha}^2\Bigg[-\frac{1}{n_\alpha}\int d^3\underline{v}\frac{pv_\parallel}{p+ikv_\parallel}\frac{\partial f_{0\alpha}}{\partial v_\parallel}\underline{E}_\parallel$$

$$+ \left(\underline{E}_\perp + \frac{1}{n_\alpha}\int d^3\underline{v}\frac{ik\underline{v}_\perp\underline{v}_\perp}{p+ikv_\parallel}\frac{\partial f_{0\alpha}}{\partial v_\parallel}.\underline{E}_\perp\right)\Bigg]$$

Taking the ∥ component of this equation,

$$\Gamma_{\parallel}(\omega) = E_{\parallel} - \frac{1}{p^2}\sum_{\alpha}\frac{\omega_{p\alpha}^2}{n_\alpha}\int d^3\underline{v}\, \frac{p v_\parallel}{p+ikv_\parallel}\frac{\partial f_{0\alpha}}{\partial v_\parallel} E_\parallel$$

meaning that

$$\boxed{\epsilon_{LL}(p,\underline{k}) = 1 - \frac{1}{p^2}\sum_{\alpha}\frac{\omega_{p\alpha}^2}{n_\alpha}\int d^3\underline{v}\, \frac{p v_\parallel}{p+ikv_\parallel}\frac{\partial f_{0\alpha}}{\partial v_\parallel}}$$

Taking the ⊥ component,

$$\underline{\Gamma}_\perp(\omega) = \underline{E}_\perp\left(1+\frac{k^2c^2}{p^2}\right) + \frac{1}{p^2}\sum_\alpha \omega_{p\alpha}^2\left(\underline{E}_\perp + \frac{1}{n_\alpha}\int d^3\underline{v}\, \frac{ik\underline{v}_\perp\underline{v}_\perp}{p+ikv_\parallel}\frac{\partial f_{0\alpha}}{\partial v_\parallel}\cdot\underline{E}_\perp\right)$$

Now,

$$\begin{aligned}\int d^3\underline{v}\, \frac{ik\underline{v}_\perp\underline{v}_\perp}{p+ikv_\parallel}\frac{\partial f_{0\alpha}}{\partial v_\parallel} &= \int dv_\parallel \frac{ik}{p+ikv_\parallel}\frac{\partial}{\partial v_\parallel}\int d^2\underline{v}_\perp\, \underline{v}_\perp\underline{v}_\perp f_{0\alpha} \\ &= \int dv_\parallel \frac{ik}{p+ikv_\parallel}\frac{\partial}{\partial v_\parallel}\int d^2\underline{v}_\perp\, \frac{1}{2}(\hat{x}\hat{x}+\hat{y}\hat{y})v_\perp^2 f_{0\alpha} \\ &= \int dv_\parallel \frac{ik}{p+ikv_\parallel}\frac{\partial}{\partial v_\parallel}\int d^2\underline{v}_\perp\, \frac{v_\perp^2}{2} f_{0\alpha}\,(\underline{\underline{I}} - \underline{e}_\parallel\underline{e}_\parallel)\end{aligned}$$

meaning that

$$\boxed{\epsilon_{TT}(p,\underline{k}) = 1 + \frac{1}{p^2}\left[k^2c^2 + \sum_\alpha \omega_{p\alpha}^2\left(1+\frac{1}{n_\alpha}\int dv_\parallel \frac{ik}{p+ikv_\parallel}\frac{\partial}{\partial v_\parallel}\int d^2\underline{v}_\perp \frac{v_\perp^2}{2} f_{0\alpha}\right)\right]}$$

These are the general expressions for the longitudinal and transverse dielectric tensors. We are now well-equipped to deal with questions 2 and 3.

3.)

a.) $$f_{0\alpha} = \frac{n_\alpha}{\pi^{3/2} v_{th\perp\alpha}^2 v_{th\parallel\alpha}} e^{-v_\parallel^2/v_{th\parallel\alpha}^2 - v_\perp^2/v_{th\perp}^2}$$

so

$$\epsilon_L = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{n_\alpha} \frac{1}{ipk} \int d^3v \frac{v_\parallel}{v_\parallel - ip/k} \frac{\partial f_{0\alpha}}{\partial v_\parallel}$$

$$= 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{n_\alpha} \frac{1}{ipk} \int dv_\parallel \frac{v_\parallel}{v_\parallel - ip/k} \frac{\partial F_{0\alpha}}{\partial v_\parallel}$$

$u = v_\parallel / v_{th\parallel\alpha}$
$\zeta_\alpha = ip/kv_{th\parallel\alpha}$

$$= 1 + \sum_\alpha \frac{1}{ipk} \frac{1}{\lambda_{D\alpha}^2} \frac{v_{th\alpha}}{\sqrt{\pi}} \int du \frac{u^2 e^{-u^2}}{u - \zeta_\alpha}$$

$$= 1 + \sum_\alpha \frac{v_{th\alpha}}{ipk} \frac{1}{\lambda_{D\alpha}^2} \cancel{\zeta_\alpha} \zeta_\alpha [1 + \zeta_\alpha Z(\zeta_\alpha)]$$

such that

$$\boxed{\epsilon_L = 1 + \sum_\alpha \frac{1}{(k\lambda_{D\alpha})^2} [1 + \zeta_\alpha Z(\zeta_\alpha)]}$$

For the transverse dielectric function, note that

$$\int d^2v_\perp \frac{v_\perp^2}{2} f_{0\alpha} = \frac{1}{2} v_{th\alpha\perp}^2 F_{0\alpha}(v_\parallel) \Rightarrow \frac{\partial F_{0\alpha}}{\partial v_\parallel} = -\frac{2 v_\parallel}{v_{th\parallel}^2} F_{0\alpha}$$

meaning that:

$$\epsilon_T = 1 + \frac{1}{p^2} \left[k^2c^2 + \sum_\alpha \omega_{p\alpha}^2 \left(1 + \frac{1}{n_\alpha} \frac{v_{th\perp\alpha}^2}{v_{th\parallel\alpha}^2} (-1) \int dv_\parallel \frac{ikv_\parallel}{p + ikv_\parallel} F_{0\alpha} \right) \right]$$

Noting that $v_{th\perp\alpha}^2 / v_{th\parallel\alpha}^2 = T_{\perp\alpha}^{\cancel{2}} / T_{\parallel\alpha}^{2}$, such that

$$\boxed{\epsilon_T(p,k) = 1 + \frac{1}{p^2} \left[k^2c^2 + \sum_\alpha \omega_{p\alpha}^2 \left(1 - \frac{T_{\perp\alpha}}{T_{\parallel\alpha}} [1 + \zeta_\alpha Z(\zeta_\alpha)] \right) \right]}$$

as required.

b.) Considering $\xi_e \ll 1$, and neglecting ion terms (due to mass ratio):

keeping only leading ord

$$\epsilon_{TT} \simeq 1 + \frac{1}{p^2}\left[k^2c^2 + \omega_{pe}^2\left(1 - \frac{T_{\perp e}}{T_{\parallel e}} - \frac{T_{\perp e}}{T_{\parallel e}}\xi_e i\sqrt{\pi} + \dots\right)\right] = 0.$$

Given that $\omega_{pe}^2/p^2 \gg 1$, we have that

$$\frac{k^2c^2}{\omega_{pe}^2} + 1 - \frac{T_{\perp e}}{T_{\parallel e}}(1 + \xi_e i\sqrt{\pi}) = 0 \implies \xi_e = \frac{-i}{\sqrt{\pi}}\frac{T_{\parallel e}}{T_{\perp e}}\left(\frac{k^2c^2}{T_{\parallel e}} - \frac{T_{\perp e} - T_{\parallel e}}{T_{\parallel e}}\right)$$

meaning that

$$\boxed{p = \frac{k v_{th\parallel e}}{\sqrt{\pi}}\frac{T_{\parallel e}}{T_{\perp e}}\left(\Delta_e - (k d_e)^2\right), \quad d_e = \frac{c}{\omega_{pe}}, \quad \Delta_e = \frac{T_{\perp e} - T_{\parallel e}}{T_{\parallel e}}.}$$

Maximum growth rate: $\Delta_e - (kd_e)^2 - 3(kd_e)^2 = 0 \rightarrow (kd_e)^* = \frac{1}{\sqrt{3}}\Delta_e^{1/2}$

meaning that

$$\boxed{p_{max} = \frac{2 v_{th\parallel e}}{3\sqrt{3\pi}}\frac{T_{\parallel e}}{T_{\perp e}}\frac{\Delta_e^{3/2}}{d_e}}$$ assuming that $\Delta_e \ll 1$.

c.) If the anisotropy is strong, with $|\xi_e| \gg 1$, $\boxed{T_{\perp e}/T_{\parallel e} \gg 1}$, neglecting ion contributions,

$$0 = 1 + \frac{1}{p^2}\left[k^2c^2 + \omega_{pe}^2\left(\cancel{1} - \cancel{\frac{T_{\perp e}}{T_{\parallel e}}} - \frac{T_{\perp e}}{T_{\parallel e}}\left(\cancel{-1} - \frac{1}{2\xi_e^2} + \dots\right)\right)\right]$$

$$= 1 + \frac{1}{p^2}\left[k^2c^2 + \omega_{pe}^2\left(1 + \underbrace{\frac{T_{\perp e}}{T_{\parallel e}}\frac{1}{2\xi_e^2}}_{}\right)\right] \qquad 1 - \frac{T_{\perp e}}{T_{\parallel e}}\frac{k^2 v_{th\parallel e}^2}{2p^2}$$

so

$$p^4 + (k^2c^2 + \omega_{pe}^2) - \frac{T_{\perp e}}{T_{\parallel e}}\frac{k^2 v_{th\parallel e}^2}{2}\omega_{pe}^2 = 0.$$

with solutions

$$p^2 = \frac{1}{2}\left[-(k^2c^2+\omega_{pe}^2) \pm \sqrt{(k^2c^2+\omega_{pe}^2)^2 + 4\frac{T_{\perp e}}{T_{\parallel e}} \overbrace{\frac{k^2 v_{th\parallel e}^2}{2}\omega_{pe}^2}^{\text{smaller}}}\right]$$

$$\simeq -(k^2c^2+\omega_{pe}^2) \quad \text{or} \quad \frac{T_{\perp e}}{T_{\parallel e}}\frac{k^2 v_{th\parallel e}^2\omega_{pe}^2}{k^2c^2+\omega_{pe}^2}$$

so instability

$$p = \sqrt{\frac{T_{\perp e}}{T_{\parallel e}}}\frac{k v_{th\parallel e}\,\omega_{pe}}{\sqrt{k^2c^2+\omega_{pe}^2}} = \sqrt{\frac{T_{\perp e}}{T_{\parallel e}}}\frac{k v_{th\parallel e}}{\sqrt{1+(kd_e)^2}} \qquad \text{iff } \frac{T_{\perp e}}{T_{\parallel e}} \gg 1$$

d.) For isotropic ~~...~~ electrons but anisotropic ~~...~~ ions

assume $|\xi_e| \ll 1$

$$\epsilon_{TT} = 0 = 1 + \frac{1}{p^2}\left[k^2c^2 + \omega_{pe}^2 \underbrace{\xi_e Z(\xi_e)}_{\simeq i\sqrt{\pi}\xi_e} + \omega_{pi}^2\left(1 - \frac{T_{\perp i}}{T_{\parallel i}}\Big[1+\underbrace{\xi_i Z(\xi_i)}_{\simeq i\sqrt{\pi}\xi_i}\Big]\right)\right]$$

so

$$k^2c^2 - i\sqrt{\pi}\,\overbrace{\omega_{pe}^2\frac{v_{thi}^2}{v_{the}^2}}^{=\omega_{pi}^2\frac{1}{Z_i}\sqrt{\frac{T_{\parallel i}m_i}{T_e m_e}}}\xi_i + \omega_{pi}^2\left(1 - \frac{T_{\perp i}}{T_{\parallel i}} - i\sqrt{\pi}\frac{T_{\perp i}}{T_{\parallel i}}\xi_i\right) = 0$$

so

$$-i\sqrt{\pi}\,\xi_i\left[\frac{1}{Z_i}\sqrt{\frac{T_{\parallel i}m_i}{T_e m_e}} + \frac{T_{\perp i}}{T_{\parallel i}}\right] = \overbrace{\frac{T_{\perp i}}{T_{\parallel i}} - 1}^{\equiv \Delta_i} - k^2 d_i^2$$

meaning that

$$\boxed{p = \frac{k v_{th\parallel i}}{\sqrt{\pi}} Z_i \sqrt{\frac{T_e m_e}{T_{\parallel i} m_i}}\left(\Delta_i - (kd_i)^2\right)}$$

same as before, but $\sqrt{m_e/m_i}$ smaller, and constrained around kd_i^{-1}, so longer scales.

2.)

a.) Setting $T_{\parallel\alpha} = T_{\perp\alpha}$ in the previous expression for ϵ_{TT} gives the desired result, while ϵ_{LL} is the same as previously.

b.) $$\epsilon_{TT} = 0 = 1 - \frac{1}{\omega^2}\left[k^2c^2 - \sum_\alpha \omega_{p\alpha}^2 \xi_\alpha Z(\xi_\alpha)\right]$$

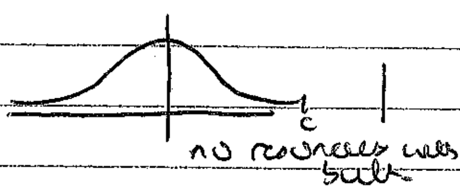
For $|\xi_e| \gg 1$, $\xi_\alpha Z(\xi_\alpha) \simeq i\sqrt{\pi}\,\xi_\alpha e^{-\xi_\alpha^2} - 1 - \frac{1}{2\xi_\alpha^2} + \dots$

Thus,

$$\omega^2 = k^2c^2 + \sum_\alpha \omega_{p\alpha}^2\left[1 + \frac{1}{2\xi_\alpha^2} + \dots\right] - i\sum_\alpha \omega_{p\alpha}^2\sqrt{\pi}\,\xi_\alpha e^{-\xi_\alpha^2}$$

To lowest order, and neglecting the ion contribution (small by mass ratio),

$$\boxed{\omega^2 = k^2c^2 + \omega_{pe}^2}$$



which are simply light waves, $\lim_{k\to 0}\omega \sim \omega_{pe}$, $\lim_{k\to\infty}\omega \sim kc$.
The light can "feel" the propagates through the plasma when these terms become comparable, so

$$k^2c^2 \sim \omega_{pe}^2 \Rightarrow \boxed{d_e \sim \frac{c}{\omega_{pe}}}$$ "skin depth"

These waves have vanishingly small damping since we are actually "outside the Maxwellian" and so no resonance can develop.

c.) $|\xi_e| \ll 1$, $\xi_\alpha Z(\xi_\alpha) \simeq -2\xi_\alpha^2 + i\sqrt{\pi}\,\xi_\alpha e^{-\xi_\alpha^2}$. Ignoring the ion contribution as usual,

$$0 = 1 - \frac{k^2c^2}{\omega^2} + \frac{\omega_{pe}^2}{\omega^2}\left[-2\xi_e^2 + i\sqrt{\pi}\,\xi_e e^{-\xi_e^2}\right]$$

$$= 1 - \frac{k^2c^2}{\omega^2} - \underbrace{\frac{2\omega_{pe}^2}{k^2 v_{the}^2}}_{=\frac{1}{(k\lambda_{De})^2}\text{ small}} + i\sqrt{\pi}\left(\frac{\omega_{pe}}{\omega}\right)^2 \xi_e e^{-\xi_e^2}$$

Ignoring the ξ_e^2 term,

$$0 = \underbrace{1}_{\text{small}} + \underbrace{\frac{c^2k^2}{p^2} + \frac{1}{k^2\lambda_{De}^2}\frac{i\sqrt{\pi}}{2\xi_e}}_{\text{large.}} + \dots$$

$$\frac{c^2k^2}{p^2} + \frac{1}{(k\lambda_{De})^2}\frac{\sqrt{\pi}\,k v_{the}}{2p} \Rightarrow p = -\frac{2k^3c^2\lambda_{De}^2}{\sqrt{\pi}\, v_{the}} = -\frac{k^3c^2 v_{the}}{\sqrt{\pi}\,\omega_{pe}^2}$$

$$\boxed{p = -\frac{k v_{the}\, k^2 d_e^2}{\sqrt{\pi}}}$$

This is an aperiodic / damped solution, valid if $\xi_e \ll 1$, $kd_e \ll 1$. i.e. perturbations with $\lambda \gg d_e$ are damped, and aperiodic. Since $\xi_e \ll 1$, this is a Landau resonance between perturbations and the bulk, which contains a large # of particles $\Rightarrow$ damping. This is essentially because the response of electrons on scales larger than d_e is too rapid for the wave to be supported. $\Rightarrow$ cancels out the affect of the wave too quickly.

4.)

a.) Consider the dielectric function:

$$\epsilon(p,k) = 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{n_\alpha} \frac{1}{k^2} \int d^3\underline{v} \, \frac{1}{p + i\underline{k}\cdot\underline{v}} \, i\underline{k}\cdot\frac{\partial f_{0\alpha}}{\partial \underline{v}} \qquad) \text{ 1D}$$

$$= 1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{n_\alpha} \frac{1}{k^2} \int_{C_L} dv_z \, \frac{1}{v_z - ip/k} \frac{\partial F_{0\alpha}}{\partial v_z}$$

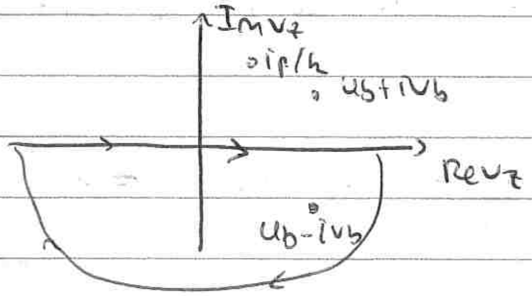
where $F_{0\alpha} = \int dv_x \int dv_y \, f_{0\alpha}(\underline{v})$

Given that the ions are maxwellian with $v_{thi} \ll u_b$, their contribution will be very small as weakly resonant $\Rightarrow$ neglect ion contributions

Define $v_p = \omega_{pe}/k$,

$$\epsilon(p,k) = 1 - \frac{v_p^2}{n_\alpha} \int_{C_L} dv_z \, \frac{1}{(v_z - ip/k)} \frac{\partial F_{0z}}{\partial v_z} = 1 - \frac{v_p^2}{n_\alpha} \int_{C_L} dv_z \frac{1}{(v_z - ip/k)^2} F_{0z}$$

$$= 1 - \frac{v_p^2}{\cancel{n_\alpha}} \int_{C_L} dv_z \, \frac{(n_e v_b/2\pi)}{(v_z - ip/k)} \left[\frac{1}{(v_z - u_b + iv_b)(v_z - u_b - iv_b)} + (u_b \to -u_b) \right]$$



We close the contour as follows: This is because whatever the sign of ip/k, it must stay above the contour, according to the Landau prescription. We thus pick up the $u_b - iv_b$ pole, with a minus sign due to the clockwise contour.

$$0 = 1 - \frac{v_p^2 v_b}{2\pi} (-2\pi i) \left[\frac{1}{(u_b - iv_b - ip/k)^2} \frac{1}{(-2iv_b)} + \frac{1}{(u_b + iv_b + ip/k)} \frac{1}{(-2iv_b)} \right]$$

$$= 1 - \frac{v_p^2}{2} \left[\frac{1}{(u_b - i\sigma)^2} + \frac{1}{(u_b + i\sigma)^2} \right]$$

$\sigma = v_b + \frac{p}{k}$

so $$0 = 1 - \frac{v_p^2}{2}\frac{(u_b - i\sigma)^2 + (u_b+i\sigma)^2}{(u_b^2+\sigma^2)^2} = 1 - v_p^2 \frac{(u_b^2-\sigma^2)}{(u_b^2+\sigma^2)^2}$$

Rearranging, we obtain the desired dispersion relation of:

$$\boxed{\sigma^4 + (2u_b^2+v_p^2)\sigma^2 + u_b^2(u_b^2-v_p^2) = 0}$$

ion cont.

$\frac{\omega_{pi}^2}{p^2} \ll \frac{v_b^2}{u_b^2}$

b.) The solution to this is

$$\sigma^2 = \frac{1}{2}\left[-(2u_b^2+v_p^2) \pm \sqrt{(2u_b^2+v_p)^2 - 4u_b^2(u_b^2-v_p^2)}\right]$$

$$= \frac{1}{2}\left[-(2u_b^2+v_p^2) \pm 2v_p\sqrt{2u_b^2+v_p^2/4}\right]$$

such that

$$\sigma^2 = -\left(u_b^2+\frac{v_p^2}{2}\right) \pm v_p\sqrt{2u_b^2+\frac{v_p^2}{4}}$$

Now, consider $k \ll \omega_{pe}/u_b \Rightarrow v_p \gg u_b$,

+ve sign for instability

$$\sigma^2 = -\left(u_b^2+\frac{v_p^2}{2}\right) \pm \frac{v_p^2}{2}\sqrt{1+\frac{8u_b^2}{v_p^2}} \simeq -\left(u_b^2+\cancel{\frac{v_p^2}{2}}\right) + \cancel{\frac{v_p^2}{2}}\left(1+\frac{1}{2}\frac{8u_b^2}{v_p^2}\right)+\dots$$
$$= u_b^2$$

so

$$\frac{p}{k} = \pm u_b - v_b \Rightarrow \boxed{p = k(u_b - v_b)}$$

There is thus an instability if $u_b > v_b$. This instability is hydrodynamic in nature as it does not rely on the presence of Landau resonances ~~[illegible]~~. Think generally.

c.) For $v_b = 0$, $\sigma = p/k$, so

$$p = k\sqrt{-\left(u_b^2 + \frac{v_p^2}{2}\right) + v_p\sqrt{2u_b^2 + \frac{v_p^2}{4}}}$$

again, since we want instability.

Instability if $\quad v_p\sqrt{2u_b^2+\frac{v_p^2}{4}} > u_b^2 + \frac{v_p^2}{2}$

$$2v_p^2u_b^2 + \cancel{\frac{v_p^4}{4}} > u_b^4 + \cancel{\frac{v_p^4}{4}} + u_b^2v_p^2$$

$$\boxed{v_p^2 > u_b^2} \Rightarrow k < \frac{\omega_{pe}}{u_b}.$$

$$\frac{\partial p}{\partial k} = 0 = \frac{\partial}{\partial k}\sqrt{-\left(u_b^2k^2 + \frac{\omega_{pe}^2}{2}\right) + \omega_{pe}\sqrt{2u_b^2k^2 + \frac{\omega_{pe}^2}{4}}}$$

$$= -2u_b^2\cancel{k} + \frac{\omega_{pe}}{\cancel{2}\sqrt{2u_b^2k^2+\frac{\omega_{pe}^2}{4}}}\,4u_b^2\cancel{k} = 0$$

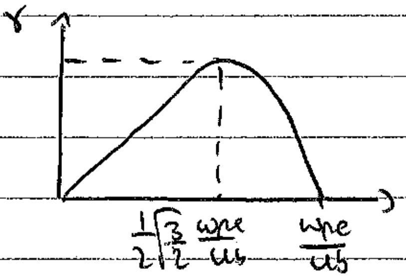
so

$$\omega_{pe} = \sqrt{2u_b^2k^2 + \omega_{pe}^2/4} \Rightarrow \boxed{k = \frac{1}{2}\sqrt{\frac{3}{2}}\,\frac{\omega_{pe}}{u_b}}$$

such that

$$p_{max} = \sqrt{-\frac{3}{8}\omega_{pe}^2 - \frac{\omega_{pe}^2}{2} + \omega_{pe}^2} = \frac{\omega_{pe}}{2\sqrt{2}}$$

Sketching the growth rate:



d.) For $v_b \neq 0$,

$$\frac{f}{k} = -v_b + \sqrt{-\left(u_b^2 + \frac{v_p^2}{2}\right) + v_p\sqrt{u_b^2 + \frac{v_p^2}{4}}} > 0$$

$$v_p\sqrt{2u_b^2+\frac{v_p^2}{4}} > u_b^2+\frac{v_p^2}{2}+v_b^2$$

$$2v_p^2u_b^2+\cancel{\frac{v_p^4}{4}} > (u_b^2+v_b^2)^2+\cancel{\frac{v_p^4}{4}}+(u_b^2+v_b^2)v_p^2$$

$$(u_b^2-v_b^2)\,v_p^2 > (u_b^2+v_b^2)^2$$

such that, with $v_p=\omega_{pe}/k$,

$$\boxed{k<\omega_{pe}\frac{\sqrt{u_b^2-v_b^2}}{u_b^2+v_b^2}} \quad \text{if } u_b>v_b.$$

As the beams get warmer ($v_b \to u_b$), the instability moves to longer wavelengths and gets weaker. For $v_b \geq u_b$, k is imaginary and so the mode is heavily damped.

5.)

b.) $$\frac{\partial \delta f_\alpha}{\partial t} + \underline{v}.\nabla \delta f_\alpha + \frac{q_\alpha}{m_\alpha}\left[\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right].\frac{\partial f_{0\alpha}}{\partial \underline{v}} = 0$$

$$\frac{\partial \underline{B}}{\partial t} = -c\nabla\times\underline{E}, \quad \frac{\partial \underline{E}}{\partial t} = c\nabla\times\underline{B} - 4\pi \underline{j}$$

Now,

$$\frac{d}{dt}\int d^3\underline{r}\int d^3\underline{v} \sum_\alpha \frac{T_\alpha \delta f_\alpha^2}{2 f_{0\alpha}} = \int d^3\underline{r}\int d^3\underline{v} \sum_\alpha \frac{T_\alpha}{f_{0\alpha}}\frac{\partial \delta f_\alpha}{\partial t}\delta f_\alpha$$

$$= \int d^3\underline{r}\int d^3\underline{v} \sum_\alpha \frac{T_\alpha}{f_{0\alpha}}\delta f_\alpha\left[-\underline{v}.\nabla\delta f_\alpha - \frac{q_\alpha}{m_\alpha}\left[\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right).\frac{\partial f_{0\alpha}}{\partial \underline{v}}\right]$$

$$= \int d^3\underline{r}\int d^3\underline{v} \sum_\alpha \frac{T_\alpha}{f_{0\alpha}}\left[\underbrace{-\underline{v}.\nabla\left(\frac{\delta f_\alpha^2}{2}\right)}_{\substack{=0 \\ \text{due to total} \\ \text{divergence}}} \underbrace{- \frac{q_\alpha}{m_\alpha}\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right).\frac{\partial f_{0\alpha}}{\partial \underline{v}}\delta f_\alpha}_{= -\frac{q_\alpha}{m_\alpha}\frac{2\underline{E}.\underline{v}}{v_{th\alpha}^2} f_{0\alpha}\delta f_\alpha}\right]$$

$$= \int d^3\underline{r}\int d^3\underline{v}\sum_\alpha \underline{v}.\underline{E}\, q_\alpha \delta f_\alpha = \int d^3\underline{r}\ \underline{j}.\underline{E}$$

Now,

$$\frac{d}{dt}\int d^3\underline{r}\ \frac{E^2+B^2}{8\pi} = \int d^3\underline{r}\ \frac{1}{4\pi}\left(\underline{E}.\frac{\partial \underline{E}}{\partial t} + \underline{B}.\frac{\partial \underline{B}}{\partial t}\right)$$

$$= \int d^3\underline{r}\ \frac{1}{4\pi}\left(-4\pi \underline{j}.\underline{E} + \underline{E}.c(\nabla\times\underline{B}) - c\,\underline{B}.(\nabla\times\underline{E})\right)$$

$$= -\int d^3\underline{r}\ \underline{j}.\underline{E} + \cancelto{0}{\int d^3\underline{r}\ \nabla.\left(\frac{c}{4\pi}\underline{E}\times\underline{B}\right)}$$

Thus,

$$\boxed{\frac{d}{dt}\int d^3\underline{r}\left[\sum_\alpha\int d^3\underline{v}\ \frac{T_\alpha \delta f_\alpha^2}{2 f_{0\alpha}} + \frac{E^2+B^2}{8\pi}\right] = 0}$$

QED

NB. For the result of a.), let $\underline{E} = -\nabla\phi$, $\underline{B} = 0$.

c.) Let G_α be some function such that $\partial_t G_\alpha = 0$, $\nabla G_\alpha = 0$. independent of space

$$\frac{d}{dt}\int d^3r \int d^3v \sum_\alpha G_\alpha \frac{\delta f_\alpha^2}{2} = \int d^3r \int d^3v \sum_\alpha G_\alpha \delta f_\alpha \frac{\partial \delta f_\alpha}{\partial t}$$

$$= -\int d^3r \int d^3v \sum_\alpha G_\alpha \delta f_\alpha \frac{q_\alpha}{m_\alpha}\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right)\frac{\partial f_{0\alpha}}{\partial \underline{v}}$$

letting

$$\frac{\partial f_{0\alpha}}{\partial \underline{v}} = \frac{\partial f_{0\alpha}}{\partial \varepsilon_\alpha}\frac{\partial \varepsilon_\alpha}{\partial \underline{v}} = m_\alpha \underline{v}\frac{\partial f_{0\alpha}}{\partial \varepsilon_\alpha}$$

$$= -\int d^3r \int d^3v \sum_\alpha G_\alpha \frac{\partial f_{0\alpha}}{\partial \varepsilon_\alpha} q_\alpha \underline{v}\cdot\underline{E}\, \delta f_\alpha \stackrel{!}{=} +\int d^3r\, \underline{j}\cdot\underline{E}$$

We thus require that $G_\alpha = -\left(\frac{\partial f_{0\alpha}}{\partial \varepsilon}\right)^{-1}$, so

$$\boxed{\frac{d}{dt}\int d^3r\left[\sum_\alpha \int d^3v \left(-\frac{\partial f_{0\alpha}}{\partial \varepsilon_\alpha}\right)^{-1}\frac{\delta f_\alpha^2}{2} + \frac{E^2+B^2}{8\pi}\right] = 0}$$

is the correct conserved invariant.

For stable perturbations, we require that $\frac{d}{dt}\int d^3r\, \frac{E^2+B^2}{8\pi} < 0$ (the free energy associated with the fields must be decreasing) meaning that

$$\frac{\partial f_{0\alpha}}{\partial \varepsilon_\alpha} < 0$$

is a sufficient (but not necessary) condition for stability.

6.)

~~xxxxx~~

a.) $$\frac{\partial f_\alpha}{\partial t} + v\cdot\nabla \delta f_\alpha - \frac{q_\alpha}{m_\alpha}\nabla\phi_{tot}\cdot\frac{\partial n_\alpha}{\partial v} = 0 \ , \quad -\nabla^2\phi = -\nabla^2\chi + 4\pi\sum_\alpha q_\alpha\int d^3v\, \delta f_\alpha$$

Fourier transform in space, and laplace transform in time, so

$$\hat{\delta f}_{\alpha k} = \frac{h_{\alpha k}(v)}{p+ik\cdot v} + \frac{q_\alpha}{m_\alpha}\frac{\hat{\phi}_k}{p+ik\cdot v}\, ik\cdot\frac{\partial n_\alpha}{\partial v} \ , \quad \hat{\phi}_k = \hat{\chi}_k + \frac{4\pi}{k^2}\sum_\alpha q_\alpha\int d^3v\, \delta f_\alpha$$

Now, ~~xxxx~~ substituting the perturbation into laplace's equation,

$$\hat{\phi}_k(p) = \hat{\chi}_k(p) + \frac{4\pi}{k^2}\sum_\alpha q_\alpha\int d^3v\left[\frac{h_{\alpha k}(v)}{p+ik\cdot v} + \frac{q_\alpha}{m_\alpha}\frac{\hat{\phi}_k(p)}{p+ik\cdot v}\, ik\cdot\frac{\partial n_\alpha}{\partial v}\right]$$

Defining the usual dielectric function, we have that

$$\hat{\phi}_k(p) = \frac{\hat{\chi}_k(p)}{\epsilon(p,k)} + \frac{4\pi}{k^2\epsilon(p,k)}\sum_\alpha q_\alpha\int d^3v\, \frac{h_{\alpha k}(v)}{p+ik\cdot v}.$$

Setting $h_{\alpha k}=0$, we arrive at the desired result: $\boxed{\hat{\phi}_k(p) = \frac{\hat{\chi}_k(p)}{\epsilon(p,k)}}$

b.)

$$\chi(t) = \chi_0 e^{-i\omega_0 t} \longrightarrow \hat{\chi}(p) = \int_0^\infty dt\, e^{-pt}\chi_0 e^{-i\omega_0 t} = \frac{\chi_0}{p+i\omega_0}$$

such that

$$\phi_k(t) = \int_{-i\infty+\sigma}^{i\infty+\sigma}\frac{dp}{2\pi i}\frac{e^{pt}}{\epsilon(p,k)}\frac{\chi_0}{p+i\omega_0} \simeq \frac{e^{-i\omega_0 t}\chi_0}{\epsilon(-i\omega_0,k)}.$$

after transients arising from $\epsilon(p,k)$ have decayed away, i.e. at $\gamma_j t \gg 1$, γ_j = ~~xxxx~~ smallest decay rate in the system.

c.) We now consider an external potential satisfying $\langle \chi(t,\underline{k}) \chi_{\underline{k}}(t')^* \rangle = 2D\delta(t-t')$. Then,

$$\langle |\phi_{\underline{k}}(t)|^2 \rangle$$

$\hat{\phi}_{-\underline{k}}(p) = \hat{\phi}_{\underline{k}}^*(p'^*)$ in order to satisfy the required conjugate property, see proof attached.

$$= \langle \phi_{\underline{k}}(t) \phi_{\underline{k}}^*(t) \rangle$$

$$= \int \frac{dp}{2\pi i} e^{pt} \int \frac{dp'}{2\pi i} e^{p't} \left\langle \frac{\hat{\chi}_{\underline{k}}(p)}{\epsilon(p,\underline{k})} \frac{\hat{\chi}_{\underline{k}}^*(p'^*)}{\epsilon(p'^*,\underline{k})^*} \right\rangle$$

$$= \int \frac{dp}{2\pi i} \int \frac{dp'}{2\pi i} \frac{e^{(p+p')t}}{\epsilon(p,\underline{k})\,\epsilon(p'^*,\underline{k})^*} \langle \hat{\chi}_{\underline{k}}(p) \underbrace{\hat{\chi}_{\underline{k}}^*(p'^*)}_{\hat{\chi}_{-\underline{k}}(p')} \rangle$$

Now,

$$\langle \hat{\chi}_{\underline{k}}(p) \hat{\chi}_{-\underline{k}}(p') \rangle = \int_0^\infty dt\, e^{-pt} \int_0^\infty dt'\, e^{-p't'} \langle \chi(t) \underbrace{\chi_{-\underline{k}}(t')}_{\chi_{\underline{k}}^*(t')} \rangle$$

$$= 2D \int_0^\infty dt\, e^{-(p+p')t} = \frac{2D}{p+p'}$$

so

$$\langle |\phi_{\underline{k}}(t)|^2 \rangle = \int \frac{dp}{2\pi i} \int \frac{dp'}{2\pi i} \frac{e^{(p+p')t}}{\epsilon(p,\underline{k})\,\epsilon(p'^*,\underline{k})^*} \frac{2D}{p+p'}$$

Poles arising from the dielectric function are assumed damped (and not marginal), so we evaluate the integral over p' at the obvious pole at $p' = -p$.

$$\langle |\phi_{\underline{k}}(t)|^2 \rangle = \int_{-i\infty+\sigma}^{i\infty+\sigma} \frac{dp}{2\pi i}\, 2D \frac{1}{\epsilon(p,\underline{k})} \frac{1}{\epsilon(-p^*,\underline{k})^*}$$

Now, by virtue of our assumptions, the only poles that will remain are purely imaginary, i.e. $\gamma \to 0$. Letting $\gamma \to 0, \sigma \to 0$, $p \to -i\omega$, $-p^* \to -i\omega$, and changing variables to $\omega = ip$, we arrive at the desired result

$$\boxed{\langle |\phi_k(t)|^2 \rangle = \frac{D}{\pi} \int_{-\infty}^{+\infty} d\omega \frac{1}{|\epsilon(-i\omega, k)|^2}}$$

d.) Let

$$\epsilon(p,k) = \epsilon_R(p,k) + i\,\epsilon_I(p,k) = \epsilon_R + i\epsilon_I$$

Then, for very small / infinitesimal damping,

$$\gamma_{ik} \equiv \frac{|\epsilon_I|}{\left|\frac{\partial \epsilon_R}{\partial \omega}\right|_{p=-i\omega}}$$

Then,

$$\frac{1}{|\epsilon(-i\omega,k)|^2} = \frac{1}{\epsilon_I^2 + \epsilon_R^2} \underset{\text{around } \omega=\omega_i}{\Big|} \simeq \sum_i \frac{1}{(\omega-\omega_i)^2 \left(\frac{\partial \epsilon_R}{\partial \omega}\right)^2 + \epsilon_I^2}$$

$$\simeq \sum_i \left(\frac{\partial \epsilon_R}{\partial \omega}\right)^{-2} \frac{1}{(\omega-\omega_i)^2 + \gamma_i^2}$$

$$= \sum_i \left(\frac{\partial \epsilon_R}{\partial \omega}\right)^{-2} \frac{\pi\, \delta(\omega-\omega_i)}{|\gamma_i|}$$

where we have used the delta function limit of the Lorentzian for vanishing γ_j. Using this result and evaluating the integral using the delta function,

$$\boxed{\langle |\phi_k(t)|^2 \rangle = \frac{D}{\pi} \sum_i \left(\frac{\partial \epsilon_R}{\partial \omega}\right)^{-2} \frac{\pi}{|\gamma_i|} = D \sum_i \frac{1}{|\gamma_i|} \left[\frac{\partial \,\mathrm{Re}\,\epsilon(-i\omega,k)}{\partial \omega}\right]^{-2}_{\omega=\omega_i}}$$

as required.

Proof of $\hat{\phi}_{-\underline{k}}(p) = \hat{\phi}^*_{\underline{k}}(p^*)$

$$\phi_{-\underline{k}}(t) = \int_{-i\infty+\sigma}^{i\infty+\sigma} \frac{dp}{2\pi i}\, e^{pt}\, \hat{\phi}_{-\underline{k}}(p) = \phi^*_{\underline{k}}(t) = \left(\int_{-i\infty+\sigma}^{i\infty+\sigma} \frac{dp}{2\pi i}\, e^{pt}\, \hat{\phi}_{\underline{k}}(p) \right)^*$$

$$= \int_{-i\infty+\sigma}^{i\infty+\sigma} \frac{dp^*}{2\pi i}\, e^{p^*t}\, \hat{\phi}^*_{\underline{k}}(p) = \int_{-i\infty+\sigma}^{i\infty+\sigma} \frac{dp}{2\pi i}\, e^{pt}\, \hat{\phi}^*_{\underline{k}}(p^*)$$

so

$$\hat{\phi}_{-\underline{k}}(p) = \hat{\phi}^*_{\underline{k}}(p^*).$$

Alternatively,

$$\hat{\phi}_{-\underline{k}}(p) = \int_0^\infty dt\, e^{-pt}\, \phi_{-\underline{k}}(t) = \int_0^\infty dt\, e^{-pt}\, \phi^*_{\underline{k}}(t)$$

$$= \left(\int_0^\infty dt\, e^{-p^*t}\, \phi_{\underline{k}}(t) \right)^* = \hat{\phi}^*_{\underline{k}}(p^*).$$

7.)

a.) Once again, ignore initial conditions, so

$$\delta\hat{f}_{\alpha\underline{k}} = \frac{q_\alpha}{m_\alpha}\frac{\hat{\phi}_{\underline{k}}}{p+i\underline{k}\cdot\underline{v}}\, i\underline{k}\cdot\frac{\partial f_{0\alpha}}{\partial \underline{v}} \underset{\underline{k}=k\hat{\underline{z}}}{=} \frac{q_\alpha}{m_\alpha}\frac{\hat{\phi}_k}{p+ikv_z}\, ik\frac{\partial f_{0\alpha}}{\partial v_z}.$$

Define $\delta\hat{f}_{\alpha\underline{k}}(v_z) = \int dv_x \int dv_y\ \delta\hat{f}_{\alpha\underline{k}}(\underline{v})$

$F_{0\alpha}(v_z) = \int dv_x \int dv_y\ f_{0\alpha}(\underline{v})$.

so

$$\delta\hat{f}_{\alpha\underline{k}}(v_z) = \frac{q_\alpha}{m_\alpha}\frac{\hat{\phi}_{\underline{k}}}{p+ikv_z}\, ik\frac{\partial F_{0\alpha}}{\partial v_z}$$

Then,

$F_{0\alpha} = \frac{n_\alpha}{\sqrt{\pi}v_{th\alpha}} e^{-u^2}$

$u = v_z/v_{th\alpha}$

$\gamma_\alpha = ip/kv_{th\alpha}$

$$\delta\hat{f}_{\alpha km} = \frac{1}{n_\alpha}\int dv_z \frac{H_m(u)\,\delta\hat{f}_{\alpha k}(v_z)}{\sqrt{2^m m!}}$$

$$= \frac{q_\alpha}{m_\alpha}\frac{1}{\cancel{n_\alpha}}\hat{\phi}_{\underline{k}}\int dv_z \frac{H_m(u)}{\sqrt{2^m m!}}\frac{ik}{p+ikv_z}\frac{\partial F_{0\alpha}}{\partial v_z}$$

$$= \frac{q_\alpha}{m_\alpha}\frac{1}{\sqrt{2^m m!}}\hat{\phi}_{\underline{k}}\frac{(-2)}{v_{th\alpha}^2}\int dv_z \frac{H_m(u)}{v_z - ip/k}\frac{v_z e^{-u^2}}{\sqrt{\pi}v_{th\alpha}}$$

Actually, let's do another trick.

$$= \frac{q_\alpha}{m_\alpha}\frac{1}{\cancel{n_\alpha}}\hat{\phi}_{\underline{k}}(p)\int dv_z \frac{H_m(u)}{\sqrt{2^m m!}}\frac{1}{p+ikv_z}\left[-\frac{2ikv_z}{v_{th\alpha}^2}\right]\frac{\cancel{n_\alpha}}{\sqrt{\pi}v_{th\alpha}} e^{-u^2}$$

$$= -\frac{q_\alpha}{T_\alpha}\frac{\hat{\phi}_{\underline{k}}(p)}{\sqrt{2^m m!}}\frac{1}{v_{th\alpha}^3}\frac{1}{\sqrt{\pi}}\int dv_z \frac{H_m(u)}{p+ikv_z}\left[p+ikv_z-p\right]e^{-u^2}$$

$$= \frac{-q_\alpha}{T_\alpha} \hat{\phi}_b(p) \frac{1}{\sqrt{2^m m!}} \int dv_z \frac{e^{-u^2}}{\sqrt{\pi} v_{th\alpha}} H_m(u) \left[H_0(u) - \frac{p}{p + ikv_z} \right]$$

$$= \frac{-q_\alpha}{T_\alpha} \hat{\phi}_b(p) \frac{1}{\sqrt{2^m m!}} \int dv_z \frac{e^{-u^2}}{\sqrt{\pi} v_{th\alpha}} H_m(u) \left[H_0(u) - \overset{\overset{\gamma_\alpha}{\shortparallel}}{\frac{p}{ik}} \frac{1}{v_z - ip/k} \right]$$

$$= \frac{-q_\alpha}{T_\alpha} \hat{\phi}_b(p) \frac{1}{\sqrt{2^m m!}} \frac{1}{\sqrt{\pi}} \int du\, e^{-u^2} H_m(u) \left[H_0(u) + \gamma_\alpha \frac{1}{u - \gamma_\alpha} \right]$$

$$= -\frac{q_\alpha}{T_\alpha} \hat{\phi}_b(p) \frac{1}{\sqrt{2^m m!}} \frac{1}{\sqrt{\pi}} \int du\, e^{-u^2} H_m(u) \left[H_0(u) + \frac{\gamma_\alpha}{u - \gamma_\alpha} \right]$$

Then,

$$\frac{1}{\sqrt{\pi}} \int du\, e^{-u^2} H_m(u) H_0(u) = 2^m m!\, \delta_{m0}$$

and

$$Z^{(m)}(\gamma_\alpha) = \frac{d^m}{d\gamma^m} \int \frac{du}{\sqrt{\pi}} \frac{e^{-u^2}}{u - \gamma_\alpha} = \int \frac{du}{\sqrt{\pi}} e^{-u^2} \frac{d^m}{d\gamma_\alpha^m} \frac{1}{u - \gamma_\alpha}$$

$$= (-1)^m \int \frac{du}{\sqrt{\pi}} e^{-u^2} \frac{d^m}{du^m} \frac{1}{u - \gamma_\alpha} = \int \frac{du}{\sqrt{\pi}} \frac{1}{u - \gamma_\alpha} (-1)^{2m} \frac{d^m}{du^m} e^{-u^2}$$

$$= (-1)^m \int \frac{du}{\sqrt{\pi}} \frac{H_m(u) e^{-u^2}}{u - \gamma_\alpha}$$

where we have used:

$$H_m(u) = (-1)^m e^{u^2} \frac{d^m}{du^m} e^{-u^2}.$$

Thus,

$$\boxed{\delta \hat{f}_{\alpha b m} = \frac{-q_\alpha}{T_\alpha} \frac{\hat{\chi}(p)}{\epsilon(p,b)} \frac{1}{\sqrt{2^m m!}} \left[2^m m!\, \delta_{m0} + \gamma_\alpha (-1)^m Z^{(m)}(\gamma_\alpha) \right]}$$

Considering $m \geq 1$, and using the same procedure as in the kinetic Langevin problem, the desired result follows quickly:

Q6.

$$\boxed{\langle |\delta \hat{f}_{2bm}|^2 \rangle = \frac{q_\alpha^2}{T_\alpha^2} \frac{1}{2^m m!} \frac{n}{\pi} \int_{-\infty}^{+\infty} d\omega \left| \frac{y_\alpha Z^{(m)}(y_\alpha)}{\epsilon(-i\omega, k)} \right|^2}$$

b.) For $m \gg 1$, $H_m(u) \sim \sqrt{2} \left(\frac{2m}{e}\right)^{m/2} \cos\left(\sqrt{2m}\,u - \frac{\pi m}{2}\right) e^{u^2/2}$ such that

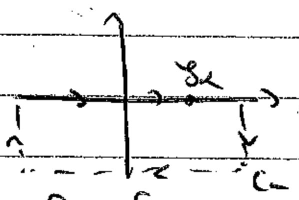
$$Z^{(m)}(y_\alpha) \simeq (-1)^m \sqrt{\frac{2}{\pi}} \left(\frac{2m}{e}\right)^{m/2} \int du \, \frac{\cos\left(\sqrt{2m}\,u - \frac{\pi m}{2}\right)}{u - y_\alpha} e^{-u^2/2}$$

We thus need to consider

$$\int_{C_L} du \, \frac{\cos\left(\sqrt{2m}\,u - \frac{\pi m}{2}\right)}{u - y_\alpha} e^{-u^2/2} = \int_{C_L} du \, \frac{1}{u - y_\alpha} \left[\underset{①}{e^{-u^2/2 + i\sqrt{2m}\,u}} + \underset{②}{e^{-u^2/2 - i\sqrt{2m}\,u}} \right]$$

For these integrals, we close differently due to the ~~[illegible]~~ $i\sqrt{2m}\,u$ factor.

$$\int_{C_L} du \, \frac{1}{u - y_\alpha} e^{-u^2/2 \mp i\sqrt{2m}\,u} = \int_{-\infty}^{+\infty} du' \, \frac{e^{-\frac{1}{2}u'^2}}{u' - y_\alpha + i\sqrt{2m}} e^{-m}$$



$\int_{C_L} - \int_{C_-} = 0.$

$$\simeq -e^{-m} \frac{\sqrt{2m}}{y_\alpha \mp i\sqrt{2m}} \quad \text{since } |u'| \ll \sqrt{m}$$

Then, $\int_{C_L} - \int_{C_+} = 2\pi i$ encloses pole.

$$\int_{C_L} du \, \frac{1}{u - y_\alpha} e^{-u^2/2 - i\sqrt{2m}\,u} = \int_{-\infty}^{+\infty} du' \, \frac{e^{-\frac{1}{2}u'^2}}{u' - y_\alpha - i\sqrt{2m}} e^{-m} + 2\pi i \, e^{-m} e^{-\frac{1}{2}(y_\alpha - i\sqrt{2m})^2}$$

$$\simeq -e^{-m} \frac{\sqrt{2m}}{y_\alpha + i\sqrt{2m}} + 2\pi i \, e^{-y_\alpha^2/2 + i y_\alpha \sqrt{2m}}.$$

Given that the other terms are exponentially small in comparison to the contribution from the pole, we have that

$$Z^{(m)}(y_\alpha) \simeq (-1)^m \sqrt{\frac{2}{\pi}} \left(\frac{2m}{e}\right)^{m/2} 2\pi i \, e^{-y_\alpha^2/2 + i y_\alpha \sqrt{2m}}$$

meaning that

$$|\gamma_a z^{(m)}(\gamma_a)|^2 = 2\pi\left(\frac{2m}{e}\right)^m e^{-\gamma_a^2}\gamma_a^2$$

Thus,

$$\langle|\hat{\delta x_{\underline{b}m}}|^2\rangle = \frac{q_\alpha^2}{k T_\alpha^2}\frac{1}{2^m m!}\frac{\Omega}{\Pi}\, 2\pi\left(\frac{2m}{e}\right)^m \int_{-\infty}^{+\infty} d\omega \left|\frac{\gamma_\alpha^2 e^{-\gamma_\alpha^2}}{\epsilon(-i\omega,h)}\right|^2$$

$$\sim \frac{1}{2^m m!}\frac{m^m}{e^m} \underset{\text{stirlings}}{\sim} \frac{1}{2^m\left(\sqrt{2\pi m}\,\cancel{\left(\frac{m}{e}\right)^m}\right)}\cancel{\left(\frac{m}{e}\right)^m} \sim \frac{1}{\sqrt{m}}.$$

Thus, we have shown that

$$\boxed{\lim_{m\to\infty}\langle|\delta x_{\underline{b}m}(t)|^2\rangle \sim m^{-1/2}}$$

as required.

8.)

a.) Consider $\frac{\partial f_{0\alpha}}{\partial t} = \frac{\partial}{\partial v} D_\alpha(v) \frac{\partial F_{0\alpha}}{\partial v}$ in 1D, with

$$D_\alpha(v) = \frac{q_\alpha^2}{m_\alpha^2} \sum_k |E_k|^2 \, \mathrm{Im} \left\langle \frac{1 - e^{-i(k \cdot v - \omega_j)t - \gamma_j t}}{kv - \omega_j - i\gamma_j} \right\rangle \quad \text{— } (\omega, \gamma_j) \text{ in notes}$$

NB.: $\omega_j \equiv \omega_j(k)$ with $\omega_j(-k) = -\omega_j(k)$
$\gamma_j \equiv \gamma_j(k)$ with $\gamma_j(-k) = \gamma_j(k)$.

The plateau forms around the resonant particles, for which $|\gamma_j| \ll |kv - \omega_j| \ll \omega_j \sim kv$, so

$$D_\alpha(v) \simeq \frac{q_\alpha^2}{m_\alpha^2} \sum_k |E_k|^2 \pi \delta(\omega_j - kv)$$

Define

$$W(k) = \frac{L}{2\pi} \frac{|E_k|^2}{4\pi}$$

such that

$\alpha = e$, electron plasma.

$$D_\alpha = \frac{q_\alpha^2}{m_\alpha^2} \int dk \, W(k) \, 4\pi^2 \delta(kv - \omega_j) = \frac{q_e^2}{m_e^2} \frac{4\pi^2}{v} W\left(\frac{\omega_j}{v}\right).$$

Assume that the system is evolving quasilinearly in the presence of Langmuir waves, so $\omega_j \sim \omega_{pe}$. Then,

$$\gamma_j = \frac{\pi}{2} \frac{\omega_{pe}^3}{k^2} \frac{1}{n_e} \frac{\partial F_{0e}(\omega_{pe}/k)}{\partial v_z} \longrightarrow \frac{\partial F_{0e}}{\partial v} = \left[\frac{2}{\pi} \frac{k^2}{\omega_{pe}^3} n_e \gamma_j \right]_{k = \omega_{pe}/v}$$

Then,

$$\frac{\partial}{\partial t} \left\{ \frac{\partial F_{0e}}{\partial t} - \frac{\partial}{\partial v} \left(\frac{q_e^2}{m_e^2} \frac{4\pi^2}{v} W\left(\frac{\omega_{pe}}{v}\right) \frac{2}{\pi} \frac{1}{\omega_{pe} v^2} n_e \gamma_j \right) \right\} = 0$$

$$\hookrightarrow \; \underbrace{2\gamma_j W\left(\frac{\omega_{pe}}{v}\right)}_{= \frac{\partial W}{\partial t}} \underbrace{\frac{4\pi n_e q_e^2}{m_e^2}}_{= \omega_{pe}^2} \frac{\omega_{pe}^{-1}}{m_e v^3} = \frac{\omega_{pe}}{m_e v^3} \frac{\partial W}{\partial t}.$$

Thus, we have that

$$\frac{\partial}{\partial t}\left[F_{0e} - \frac{\partial}{\partial v}\left(\frac{\omega_{pe}}{m_e v^3} W\left(\frac{\omega_{pe}}{v}\right)\right)\right] = 0$$

implying that

$$F_{0e}(v,t) = F_{0e}(v,0) + \frac{\partial}{\partial v}\left(\frac{\omega_{pe}}{m_e v^3}\left(W\left(t,\frac{\omega_{pe}}{v}\right) - W\left(\frac{\omega_{pe}}{v},0\right)\right)\right)$$

Thus,

$$F^{plat} = F(0,v_1) = F(0,v_2) \qquad \text{since } W_{pe}\left(\frac{\omega_{pe}}{v}\right) = 0 \text{ outside the interval.}$$

so

$$\boxed{F^{plat} = \frac{1}{v_2 - v_1}\int_{v_1}^{v_2} dv\, F(0,v)}$$

For the energy of the waves as $t \to \infty$,

$$\begin{aligned}
U_{res}(t) - U_{res}(0) &= \int_{v_1}^{v_2} dv\, \tfrac{1}{2} m_e v^2 \left[F(v,t) - F(v,0)\right] \\
&= +\int_{v_1}^{v_2} dv\, \frac{1}{2} \cancel{m_e} v^2 \frac{\partial}{\partial v} \frac{\omega_{pe}}{\cancel{m_e} v^3} \Delta W\left(\frac{\omega_{pe}}{v}\right) \\
&= -\omega_{pe}\int_{v_1}^{v_2} dv\, \frac{1}{v^2} \Delta W\left(\frac{\omega_{pe}}{v}\right) \\
&= -\int_{\omega_{pe}/v_2}^{\omega_{pe}/v_1} dk\, \Delta W(k)
\end{aligned}$$

But $$\int_{\omega_{pe}/v_2}^{\omega_{pe}/v_1} dk\, W(k,t) = 2\sum_k \frac{|E_k^{wave}(t)|^2}{8\pi} = 2\, U_{waves}(t).$$

Thus, $$U_{res}(t) - U_{res}(0) = -2\left[U_{waves}(t) - U_{waves}(0)\right]$$

so

$$\boxed{U_{waves}(\infty) = U_{waves}(0) - \frac{1}{2}\left(U_{res}(\infty) - U_{res}(0)\right)}$$

RHS ≥ 0 in order for the plateau to form.

b.) For the thermal particles, $\gamma_j \ll kv - \omega_j$, and so we neglect the exponential term, meaning that we obtain the familiar result that

$$D_e(v) = \frac{q_e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_j}{(kv-\omega_j)^2 + \gamma_j^2}$$

letting $\omega_j = \omega_{pe}$, and considering $kv \ll \omega_{pe}$, (bulk),

$$D_e(v) \simeq \frac{q_e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{\omega_{pe}^2} = \frac{1}{8\pi n_e m_e} \underbrace{2 \sum_k \gamma_j |E_k|^2}_{\frac{dU_{waves}}{dt}} = \frac{1}{8\pi n_e m_e} \frac{dU_{waves}}{dt}$$

Thus,

$$\frac{\partial F_{oe}}{\partial t} = \frac{1}{n_e m_e} \frac{dU_{waves}}{dt} \frac{\partial^2 F_{oe}}{\partial v^2}$$

Then, the thermal bulk satisfies:

$$\frac{dU_{th}}{dt} = \frac{d}{dt} \int dv \frac{1}{2} m_e v^2 F_{oe} = \frac{1}{n_e} \frac{dU_{waves}}{dt} \overbrace{\int dv \frac{1}{2} v^2 \frac{\partial F_{oe}}{\partial v^2}}^{n_e} = \frac{dU_{waves}}{dt}$$

$$\Rightarrow \boxed{U_{th}(t) - U_{th}(0) = U_{waves}(t) - U_{waves}(0)}$$

as required.

The energy lost by the thermal particles and the waves goes into the heating of the resonant particles. It is easy to see that

$$\Delta (U_{th} + U_{res}) = -\Delta U_{waves}.$$

meaning that energy is clearly conserved. Physically, thermal particles loose energy as the waves are supported by the bulk, and the waves damp onto resonant particles.

We will have slow damping of the perturbation if

width of resonant region in v

$$\frac{\Delta U_{waves}}{U_{waves}(0)} \ll 1 \Rightarrow U_{waves}(0) \gg \Delta U_{res} \sim \frac{1}{2} m_e v^2 \delta v\, F_{0e}(v)$$

Now,

$$\gamma_j = \frac{\pi}{2}\frac{\omega_{pe}^3}{k^2}\frac{1}{n_e}F_{0e}'\left(\frac{\omega_{pe}}{k}\right) \sim \omega_{pe}^3 v^2 \frac{1}{n_e} F_{0e}'(v) \sim \omega_{pe}\frac{v^3}{n_e v\, v_{the}^2} F_{0e}(v)$$

so

$$\boxed{U_{waves}(0) \gg \frac{\gamma}{\omega_{pe}} n_e T_e \frac{\delta v}{v}}$$

The timescale for quasilinear evolution is:

$$\frac{\partial}{\partial t} \sim D\frac{\partial^2}{\partial v^2} \sim \overbrace{\frac{1}{n_e T_e}\frac{dU_{waves}}{dt}}^{\text{in bulk}}\frac{1}{v_{the}^2} \sim \frac{\gamma}{n_e T_e} U_{waves}$$

$$\underbrace{\sim \frac{1}{\delta v^2}}_{\text{resonant}}\frac{q_e^2}{m_e^2}\frac{1}{v} W\left(\frac{\omega_{pe}}{v}\right)$$

$$\sim \frac{v^3}{\delta v^3}\frac{\omega_{pe}^2}{n_e m_e}\frac{1}{v^2}\frac{U_{waves}}{\omega_{pe}}$$

where we have used

$$\int dk\, W(k) \sim U_{waves} \Rightarrow W \sim \frac{U_{waves}}{\delta k} \sim \frac{U_{waves}}{\frac{\omega_{pe}}{v}\left(\frac{\delta v}{v}\right)} \sim \frac{v^2 U_{waves}}{\omega_{pe}\,\delta v}$$

Thus,

$$\frac{\partial}{\partial t} \sim \left(\frac{v}{\delta v}\right)^3 \omega_{pe} \left(\underbrace{\frac{v_{th}}{v}}_{\sim \omega_{pe}/k}\right)^2 \frac{U_{waves}}{n_e T_e} \ggg \gamma_j$$

so

$$\boxed{\frac{U_{waves}(0)}{n_e T_e} \gg \frac{\gamma_j}{\omega_{pe}}\left(\frac{\delta v}{v}\right)^3 \frac{1}{(k\lambda_{De})^2}}$$

as required.

9.) equation for the fluctuations

a.) $$\frac{\partial \delta f_\alpha}{\partial t} + \underline{v}\cdot\nabla \delta f_\alpha + \frac{q_\alpha}{m_\alpha}\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right)\cdot\frac{\partial f_{0\alpha}}{\partial \underline{v}} = 0.$$

Fourier transforming in space, and Laplace transforming in time,

$$\hat{\delta f}_{\alpha \underline{k}} = \frac{h_{\alpha\underline{k}}(v)}{p+i\underline{k}\cdot\underline{v}} - \frac{q_\alpha}{m_\alpha}\frac{1}{p+i\underline{k}\cdot\underline{v}}\left(\hat{\underline{E}}_{\underline{k}} + \frac{\underline{v}\times\hat{\underline{B}}_{\underline{k}}}{c}\right)\cdot\frac{\partial f_{0\alpha}}{\partial \underline{v}}$$

Assume growing perturbations (but not too quickly), and so we can ignore the ballistic contribution. For $p = p_j = -i\omega_j + \gamma_j$, it follows that:

$$\delta f_{\alpha\underline{k}}^{(t)} \simeq -\frac{q_\alpha}{m_\alpha}\frac{1}{i(\underline{k}\cdot\underline{v}-\omega_j)+\gamma_j}\left(\hat{\underline{E}} + \frac{\underline{v}\times\hat{\underline{B}}_{\underline{k}}}{c}\right)\cdot\frac{\partial f_{0\alpha}}{\partial \underline{v}}$$

Then, assuming that the eq. distribution is spatially homogeneous,

$$\frac{\partial f_{0\alpha}}{\partial t} = \frac{-q_\alpha}{m_\alpha}\sum_{\underline{k}}\left\langle\left(\underline{E}_{-\underline{k}} + \frac{\underline{v}\times\underline{B}_{-\underline{k}}}{c}\right)\cdot\frac{\partial f_{\alpha\underline{k}}}{\partial \underline{v}}\right\rangle = -\frac{q_\alpha}{m_\alpha}\sum_{\underline{k}}\left(\left(\underline{E}_{\underline{k}}^{*} + \frac{\underline{v}\times\underline{B}_{\underline{k}}^{*}}{c}\right)\cdot\frac{\partial \delta f_{\alpha\underline{k}}}{\partial \underline{v}}\right)$$

Using the result above, it follows that

$$\frac{\partial f_{0\alpha}}{\partial t} = \frac{\partial}{\partial \underline{v}}\cdot\underline{\underline{D}}_\alpha(\underline{v})\cdot\frac{\partial f_{0\alpha}}{\partial \underline{v}}$$

with

$$\boxed{\underline{\underline{D}}_\alpha(\underline{v}) = \frac{q_\alpha^2}{m_\alpha^2}\sum_{\underline{k}}\frac{1}{i(\underline{k}\cdot\underline{v}-\omega_j)+\gamma_j}\left(\underline{E}_{\underline{k}}^{*} + \frac{\underline{v}\times\underline{B}_{\underline{k}}^{*}}{c}\right)\left(\underline{E}_{\underline{k}} + \frac{\underline{v}\times\underline{B}_{\underline{k}}^{*}}{c}\right)}$$

as required.

b.) As we saw from Ex. 3, growing perturbations will have $\underline{E}_\perp$ only, so let us ignore contributions arising from $\underline{E}_\parallel$.

$$\frac{\partial \underline{B}_k}{\partial t} = -c\, i\underline{k}\times\underline{E}_k \quad , \quad \frac{\partial}{\partial t}(i\underline{k}\times\underline{B}_k) = -c\,\underline{k}\times(i\underline{k}\times\underline{E}_k)$$
$$= ick^2\, \underbrace{\underline{E}_k\cdot\left(\underline{\underline{I}} - \frac{\underline{k}\underline{k}}{k^2}\right)}_{\underline{E}_{k\perp}}$$

However, denote $\underline{E}_{k\perp} = \underline{E}_k$.

$$\underline{E}_k = \frac{-i}{ck^2}\frac{\partial}{\partial t}(\underline{k}\times\underline{B}_k) = \frac{-i\gamma_j}{ck^2}\,\underline{k}\times\underline{B}_k$$

eq. ~~[illegible]~~ is homogenous.

for a given perturbation growing @ γ_j. ~~the~~ without loss of generality, consider

$$\underline{B}_k = B_k\,\hat{y} \quad , \quad \underline{k} = k\hat{z} \quad , \quad \underline{E}_k = E_k\,\hat{e}_x$$

Then,

$$c\,\underline{E}_k + \underline{v}\times\underline{B}_k = \left(-\frac{i\gamma_j}{k^2}\underline{k} + \underline{v}\right)\times\underline{B}_k = \begin{pmatrix} -v_z + i\gamma_j/k \\ 0 \\ v_x \end{pmatrix} B_k$$

This means that

$$(\underline{\underline{D}}_\alpha)_{ij}(\underline{v}) = \frac{q_\alpha^2}{c^2 m_\alpha^2}\sum_{\underline{k}} \frac{|B_k|^2}{i(\underline{k}\cdot\underline{v}-\omega_j)+\gamma_j} \begin{pmatrix} -v_z - \frac{i\gamma_j}{k} \\ 0 \\ v_x \end{pmatrix} \otimes \begin{pmatrix} -v_z + \frac{i\gamma_j}{k} \\ 0 \\ v_x \end{pmatrix}$$

letting $\Omega_{k} = |q_\alpha B_k/m_\alpha c|$,

$$(\underline{\underline{D}}_\alpha)_{ab}(\underline{v}) = \sum_{\underline{k}} \frac{|\Omega_k|^2}{i(kv_z-\omega_j)+\gamma_j}\, M_{ab}$$

with

$$M_{ab} = \begin{pmatrix} v_z^2 + \gamma_j^2/k^2 & 0 & -v_x(v_z + i\gamma_j/k) \\ 0 & 0 & 0 \\ -v_x(v_z - i\gamma_j/k) & 0 & v_x^2 \end{pmatrix}$$

Given that we are considering a purely growing mode, $\det \omega_j' = 0$, and $\gamma_j = \gamma$. Then consider the components

$$D_{xx} = \sum_{\underline{k}} \frac{|\Omega_{\underline{k}}|^2}{\gamma_j + i k v_z} \; v_z^2 + \frac{\gamma_j^2}{k^2} = \sum_{\underline{k}} \frac{|\Omega_{\underline{k}}|^2}{\gamma_j + i k v_z} \left(v_z - \frac{i\gamma}{k}\right)\left(v_z + \frac{i\gamma}{k}\right)$$

$$= \sum_{\underline{k}} \left(\frac{-i}{k}\right) \frac{|\Omega_{\underline{k}}|^2}{\cancel{v_z - i\gamma/k}} \cancel{\left(v_z - \frac{i\gamma}{k}\right)}\left(v_z + \frac{i\gamma}{k}\right)$$

since terms odd in $\underline{k}$ vanish under the summation

$$\boxed{D_{xx} = \sum_{\underline{k}} |\Omega_{\underline{k}}|^2 \frac{\gamma}{k^2}}$$

$$\boxed{D_{zz} = \sum_{\underline{k}} \frac{|\Omega_{\underline{k}}|^2 v_x^2}{\gamma + i k v_z} = \sum_{\underline{k}} \frac{\gamma v_x^2}{\gamma^2 + k^2 v_z^2} |\Omega_{\underline{k}}|^2}$$

breaking the sum into two parts and changing $\underline{k} \leftrightarrow -\underline{k}$ in the second

$$D_{zx} = -\sum_{\underline{k}} \frac{|\Omega_{\underline{k}}|^2}{\gamma + i k v_z} v_x \left(v_z - \frac{i\gamma}{k}\right) = -\sum_{\underline{k}} \left(\frac{-i}{k}\right) \frac{|\Omega_{\underline{k}}|^2 v_x \cancel{(v_z - i\gamma/k)}}{\cancel{v_z - i\gamma/k}} = 0$$

odd in $\underline{k}$.

$$D_{xz} = -\sum_{\underline{k}} \frac{|\Omega_{\underline{k}}|^2}{\gamma + i k v_z} v_x \left(v_z + \frac{i\gamma}{k}\right) = -\sum_{\underline{k}} \left(\frac{-i}{k}\right) |\Omega_{\underline{k}}|^2 v_x \frac{v_z + i\gamma/k}{v_z - i\gamma/k}$$

$$= -\sum_{\underline{k}} |\Omega_{\underline{k}}|^2 v_x \left(\frac{-i}{k}\right) \frac{(v_z + i\gamma/k)^2}{v_z^2 + \gamma^2/k^2}$$

$$= -\sum_{\underline{k}} |\Omega_{\underline{k}}|^2 v_x \frac{1}{k^2} \frac{\gamma v_z}{v_z^2 + \gamma^2/k^2}$$

Thus,

$$\boxed{D_{xz} = -\sum_{\underline{k}} \frac{2\gamma v_x v_z}{\gamma^2 + k^2 v_z^2} |\Omega_{\underline{k}}|^2}$$

We have thus shown that

$$\frac{\partial f_{0\alpha}}{\partial t} = \frac{\partial}{\partial v_x}\left(D_{xx} \frac{\partial f_{0\alpha}}{\partial v_x} + D_{xz} \frac{\partial f_{0\alpha}}{\partial v_z}\right) + \frac{\partial}{\partial v_z}\left(D_{zz} \frac{\partial f_{0\alpha}}{\partial v_z}\right)$$

with the diffusion coefficients given by the above.

all other terms disappear as total divergences.

c.) Integrate by $\frac{1}{n_\alpha}\int d^3\underline{v}\, m_\alpha v_z^2$ throughout, so

$$\frac{\partial T_{\parallel\alpha}}{\partial t} = \frac{1}{n_\alpha}\int d^3\underline{v}\, m_\alpha v_z^2 \frac{\partial}{\partial v_z}\left(D_{zz}\frac{\partial f_{0\alpha}}{\partial v_z}\right)$$

$$= -\sum_{\underline{k}} |\Omega_{\alpha\underline{k}}|^2 \frac{\gamma}{k^2}\int d^3\underline{v}\, \frac{m_\alpha}{n_\alpha}\, 2v_z \frac{v_x^2}{v_z^2+\gamma^2/k^2}\frac{\partial f_{0\alpha}}{\partial v_z}$$

$$= -\sum_{\underline{k}} |\Omega_{\alpha\underline{k}}|^2 \frac{\gamma}{k^2}\frac{m_\alpha}{n_\alpha}\int d^2\underline{v}_\perp \int dv_z\, \frac{m_\alpha}{n_\alpha} v_x^2 \frac{2v_z}{v_z^2+\gamma^2/k^2}\frac{\partial f_{0\alpha}}{\partial v_z}$$

$$= -\sum_{\underline{k}} |\Omega_{\alpha\underline{k}}|^2 \frac{\gamma}{k^2}\frac{T_{\perp\alpha}}{\cancel{n_\alpha}}\int dv_z\, \frac{2v_z}{v_z^2+\gamma^2/k^2}\frac{\partial}{\partial v_z}\frac{\cancel{n_\alpha}}{\sqrt{\pi}\, v_{th\parallel\alpha}} e^{-v_z^2/v_{th\parallel\alpha}^2}$$

$$= 2\sum_{\underline{k}} \frac{2\gamma|\Omega_{\alpha\underline{k}}|^2}{(k v_{th\parallel\alpha})^2} T_{\perp\alpha}\int_{-\infty}^{\infty} dv_z\, \frac{v_z^2}{v_z^2+\gamma^2/k^2}\frac{e^{-v_z^2/v_{th\parallel\alpha}^2}}{\sqrt{\pi}\, v_{th\parallel\alpha}}$$

However, $\gamma \ll k v_{th\parallel\alpha}$ in order to preserve QL evolution, and so

$$\int_{-\infty}^{+\infty} dv_z\, \frac{v_z^2}{v_z^2+\gamma^2/k^2}\frac{e^{-v_z^2/v_{th\parallel\alpha}^2}}{\sqrt{\pi}\, v_{th\parallel\alpha}} \simeq \int_{-\infty}^{+\infty}\frac{du}{\sqrt{\pi}}\, e^{-u^2} = 1.$$

We thus obtain: $\boxed{\frac{\partial T_{\parallel\alpha}}{\partial t} = 2\lambda_\alpha T_{\perp\alpha}, \quad \lambda_\alpha = \sum_{\underline{k}} \frac{2\gamma|\Omega_{\alpha\underline{k}}|^2}{k^2 v_{th\parallel\alpha}^2}}$

Then, int. by parts and sub for $f_{0\alpha}$. v_y^2 contribution vanishes as one then has a total divergence

$$\frac{\partial T_{\perp\alpha}}{\partial t} = \frac{1}{2}\frac{m_\alpha}{n_\alpha}\sum_{\underline{k}} |\Omega_{\alpha\underline{k}}|^2 \frac{\gamma}{k^2}\int d^3\underline{v}\, v_x^2 \frac{\partial}{\partial v_x}\left(\frac{\partial}{\partial v_x} - \frac{2v_x v_z}{v_z^2+\gamma^2/k^2}\frac{\partial}{\partial v_z}\right) f_{0\alpha}$$

$$= \frac{1}{2}\frac{m_\alpha}{n_\alpha}\sum_{\underline{k}} |\Omega_{\alpha\underline{k}}|^2 \frac{\gamma}{k^2}\int d^3\underline{v}\left(-\frac{2v_x^2}{v_{th\perp\alpha}^2} + \frac{4v_z^2 v_x^2}{v_z^2+\gamma^2/k^2}\frac{1}{v_{th\parallel\alpha}^2}\right) f_{0\alpha}$$

$$= -2\sum_{\underline{k}} |\Omega_{\alpha\underline{k}}|^2 \frac{\gamma}{k^2}\frac{T_{\perp\alpha}}{v_{th\parallel\alpha}^2}\int dv_z \left(-\frac{T_{\parallel\alpha}}{T_{\perp\alpha}} + \frac{2v_z^2}{v_z^2+\gamma^2/k^2}\frac{e^{-v_z^2/v_{th\parallel\alpha}^2}}{\sqrt{\pi}\, v_{th\parallel\alpha}}\right)$$

$$= -\frac{1}{2}\lambda_\alpha T_{\perp\alpha}\left(2 - \frac{T_{\parallel\alpha}}{T_{\perp\alpha}}\right)$$

Assuming that $T_{\parallel\alpha} \sim T_{\perp\alpha}$, $\boxed{\frac{\partial T_{\perp\alpha}}{\partial t} = -\lambda_\alpha T_{\perp\alpha}}$

as required.

d.) We can only trust the solution that we have found as long as the Bi-Maxwellian remains valid => not very long!
Consider:

$$\Delta_\alpha = \frac{T_{\perp\alpha}}{T_{\parallel\alpha}} - 1$$

Then,

$$\frac{\partial \Delta_\alpha}{\partial t} = \frac{1}{T_{\parallel\alpha}}\frac{\partial T_{\perp\alpha}}{\partial t} - \frac{T_{\perp\alpha}}{T_{\parallel\alpha}^2}\frac{\partial T_{\parallel\alpha}}{\partial t} = -\frac{\lambda_\alpha T_{\perp\alpha}}{T_{\parallel\alpha}} - \frac{T_{\perp\alpha}}{T_{\parallel\alpha}^2}(2\lambda_\alpha T_{\perp\alpha})$$

$$= -\frac{\lambda_\alpha T_{\perp\alpha}}{T_{\parallel\alpha}}\left(1 + \frac{2T_{\perp\alpha}}{T_{\parallel\alpha}}\right) \simeq -3\lambda_\alpha$$

again for a weak anisotropy

Saturation occurs when $k_{max} \sim \rho_e^{-1} \sim \frac{1}{d_e}\sqrt{\Delta_e}$ (from Q8).
So

$$\Omega_\alpha \sim \frac{v_{the}}{d_e}\sqrt{\Delta} \implies \lambda \sim 2\gamma_B$$

For $T_\parallel/T_\perp \sim 1$, $\gamma \sim \frac{\sqrt{\Delta_e}}{d_e} v_{th\parallel e}\Delta_e \sim \frac{v_{th\parallel e}}{d_e}\Delta^{3/2}$ from Q3.
Thus,

$$\lambda \sim \Delta_e^{3/2}\frac{v_{the}}{d_e}$$

This means that

$$\frac{\partial \Delta_e}{\partial t} \simeq -\frac{v_{the}}{d_e}\Delta_e^{3/2} \implies \Delta_e \sim \frac{\Delta_e(0)}{\left(1 + \frac{v_{the}}{d_e}\sqrt{\Delta_e(0)}\,t\right)^2}$$

However, these estimates are already not satisfied by QLT, given that we need $\lambda \ll \gamma$, in order to allow there to be a timescale separation between the evolution of the mean and perturbations.

10.)

a.) Adopt the usual starting point,

$$\frac{\partial f_{0\alpha}}{\partial t} = -\frac{q_\alpha}{m_\alpha}\sum_{\underline{k}} i\underline{k}\cdot\frac{\partial}{\partial \underline{v}}\langle \delta f_{\alpha}^{(t)}\phi_{\underline{k}}^*(t)\rangle$$

and

we have neglected all nonlinearity, as per QLT.

$$\frac{\partial \delta f_{\alpha\underline{k}}}{\partial t} + i\underline{k}\cdot\underline{v}\,\delta f_{\alpha\underline{k}} = \frac{q_\alpha}{m_\alpha}\phi_{\underline{k}}\, i\underline{k}\cdot\frac{\partial f_{0\alpha}}{\partial \underline{v}}$$

Laplace transforming, and noting that $\delta f_{\alpha\underline{k}}(\underline{v}, t=0)=0$, we have that

$$\delta\hat{f}_{\alpha\underline{k}}(\underline{v},p) = \frac{q_\alpha}{m_\alpha}\frac{\hat{\phi}_{\underline{k}}(p)}{p+i\underline{k}\cdot\underline{v}}\, i\underline{k}\cdot\frac{\partial f_{0\alpha}}{\partial \underline{v}}$$

such that

$$\frac{\partial f_{0\alpha}}{\partial t} = -\frac{q_\alpha}{m_\alpha}\sum_{\underline{k}} i\underline{k}\cdot\frac{\partial}{\partial \underline{v}}\left\langle \int\frac{dp}{2\pi i}\, e^{pt}\,\delta\hat{f}_{\alpha\underline{k}}(\underline{v},p)\,\phi_{\underline{k}}^*(t)\right\rangle$$

$$= \frac{q_\alpha^2}{m_\alpha^2}\sum_{\underline{k}} \underline{k}\cdot\frac{\partial}{\partial \underline{v}}\int\frac{dp}{2\pi i}\,\frac{e^{pt}}{p+i\underline{k}\cdot\underline{v}}\,\langle\hat{\phi}_{\underline{k}}(p)\,\phi_{\underline{k}}^*(t)\rangle\,\underline{k}\cdot\frac{\partial f_{0\alpha}}{\partial \underline{v}}$$

This is clearly a diffusion equation with

$$\frac{\partial f_{0\alpha}}{\partial t} = \frac{\partial}{\partial \underline{v}}\cdot\underline{\underline{D}}_\alpha(\underline{v})\cdot\frac{\partial f_{0\alpha}}{\partial \underline{v}}$$

and

$$\underline{\underline{D}}_\alpha(\underline{v}) = \frac{q_\alpha^2}{m_\alpha^2}\sum_{\underline{k}}\underline{k}\underline{k}\int\frac{dp}{2\pi i}\,\frac{e^{pt}}{p+i\underline{k}\cdot\underline{v}}\int_0^\infty dt'\, e^{-pt'}\,\underbrace{\langle\phi_{\underline{k}}(t')\,\phi_{\underline{k}}^*(t)\rangle}_{=C_{\underline{k}}(t-t')}\quad \{\tau = t-t'$$

$$\Rightarrow\ \boxed{\underline{\underline{D}}_\alpha(\underline{v}) = \frac{q_\alpha^2}{m_\alpha^2}\sum_{\underline{k}}\underline{k}\underline{k}\int\frac{dp}{2\pi i}\,\frac{1}{p+i\underline{k}\cdot\underline{v}}\int_{-\infty}^{t} d\tau\, e^{p\tau}\, C_{\underline{k}}(\tau)}$$

which is the desired result

b.) Assume $C_{\underline{k}}(\tau) = A_{\underline{k}}\, e^{-\gamma_{\underline{k}}|\tau|}$.

$$\int \frac{dp}{2\pi i} \frac{e^{pt}}{p+i\underline{k}\cdot\underline{v}} \int_{-\infty}^{t} d\tau\, e^{-\gamma_{\underline{k}}|\tau|} \Rightarrow$$

$$= \int \frac{dp}{2\pi i} \frac{1}{p+i\underline{k}\cdot\underline{v}} \left[\int_0^t d\tau\, e^{(p-\gamma_{\underline{k}})\tau} + \int_{-\infty}^{0} d\tau\, e^{(p+\gamma_{\underline{k}})\tau} \right]$$

$$= \int \frac{dp}{2\pi i} \frac{1}{p+i\underline{k}\cdot\underline{v}} \left[\frac{1}{p+\gamma_{\underline{k}}} - \frac{1-e^{(p-\gamma_{\underline{k}})t}}{p-\gamma_{\underline{k}}} \right]$$

not a pole, as finite for $p \to \gamma_{\underline{k}}$.

$$= \cancel{\frac{1}{\gamma_{\underline{k}}-i\underline{k}\cdot\underline{v}}} + \cancel{\frac{1}{-\gamma_{\underline{k}}+i\underline{k}\cdot\underline{v}}} - \frac{1-e^{-(i\underline{k}\cdot\underline{v}+\gamma_{\underline{k}})t}}{-\gamma_{\underline{k}}-i\underline{k}\cdot\underline{v}}$$

$$= \frac{1-e^{-(i\underline{k}\cdot\underline{v}+\gamma_{\underline{k}})t}}{\gamma_{\underline{k}}+i\underline{k}\cdot\underline{v}} \underset{\gamma_{\underline{k}}t \gg 1}{\sim} \frac{1}{\gamma_{\underline{k}}+i\underline{k}\cdot\underline{v}}$$

Thus,

usual trick of splitting the sum, changing $\underline{k} \leftrightarrow -\underline{k}$ in the second. $A_{-\underline{k}} = A_{\underline{k}}$, $\gamma_{-\underline{k}} = \gamma_{\underline{k}}$

$$\boxed{\underline{\underline{D}}_\alpha(\underline{v}) = \frac{q_\alpha^2}{m_\alpha^2} \sum_{\underline{k}} \frac{\underline{k}\underline{k}\, A_{\underline{k}}}{\gamma_{\underline{k}}+i\underline{k}\cdot\underline{v}} = \frac{q_\alpha^2}{m_\alpha^2} \sum_{\underline{k}} \frac{\underline{k}\underline{k}\, A_{\underline{k}}\gamma_{\underline{k}}}{\gamma_{\underline{k}}^2+(\underline{k}\cdot\underline{v})^2}}.$$

as required.

c.) 1D, $\gamma_k \gg kv$, so $D_\alpha = \frac{q_\alpha^2}{m_\alpha^2} \sum_k \frac{k^2 A_k}{\gamma_k}$ = constant.

Thus,

$$\frac{\partial F_{0\alpha}}{\partial t} = D_\alpha \frac{\partial^2 F_{0\alpha}}{\partial v^2}.$$

Thus, for an initial Maxwellian of temperature $T_{0\alpha}$, will give a solution

$$F_{0\alpha}(t) = \frac{n_{0\alpha}}{\sqrt{\pi\left(\frac{2T_{0\alpha}}{m_\alpha}+4D_\alpha t\right)}}\, e^{-v^2/\left(\frac{2T_0}{m_\alpha}+4D_\alpha t\right)}$$

$$\boxed{F_{0\alpha}(t) = \frac{n_{0\alpha}}{\sqrt{2\pi T_\alpha(t)/m_\alpha}}\, e^{-m_\alpha v^2/2T_\alpha(t)}, \quad T_\alpha(t) = T_{0\alpha} + 2m_\alpha D_\alpha t}$$

The distribution is spreading as the species is being stochastically heated by the external potential.

~~Short correlation time~~ slow evolution of $F_{0\alpha}$ approx: $\Rightarrow \quad \gamma_k \gg t^{-1} \Rightarrow \gamma_k t \gg 1$

Short correlation time approx: $\quad \gamma_k \gg kv \sim k\sqrt{\frac{2T(t)}{m_\alpha}}$

$$T_\alpha(t) \ll \frac{m_\alpha \gamma_k^2}{k^2} \qquad D_\alpha \sim \frac{q_\alpha^2}{m_\alpha^2}\frac{k^2 A_k}{\gamma_k}$$

If this was initially satisfied at $t=0$, it will break down at

$$2m_\alpha D_\alpha t \sim \frac{m_\alpha \gamma_k^2}{k^2} \Rightarrow t \sim \frac{\gamma_k^2}{k^2 D_\alpha} \sim \frac{m_\alpha^2}{q_\alpha^2}\frac{\gamma_k^3}{k^4 A_k}.$$

This means that our solution is valid in the range:

$$\boxed{\gamma_k^{-1} \ll t \ll \frac{m_\alpha^2}{q_\alpha^2}\frac{\gamma_k^3}{k^4 A_k}} \qquad \text{Need } \frac{q|\delta\phi|}{m(\gamma_k/k)^2} \ll 1.$$

d.) For $\gamma_k \ll kv$, $\quad D_\alpha \triangleq \frac{q_\alpha^2}{m_\alpha^2}\sum_k \frac{A_k \gamma_k}{v^2} \equiv \frac{\beta}{v^2}.$
Thus:

$$\frac{\partial F_{0\alpha}}{\partial \tau} = \frac{\partial}{\partial v}\left(\frac{1}{v^2}\frac{\partial F_{0\alpha}}{\partial v}\right), \qquad \tau = \beta t.$$

Look for a solution of the form $F_{0\alpha} = \frac{1}{\tau^\lambda}\Phi(\kappa)$, $\kappa = \frac{v^4}{\tau}$ i.e. a self-similar solution. Then, one can show by subs. that:

$$16\kappa\Phi'' + (4+\kappa)\Phi' + \lambda\Phi = 0.$$

NB: This requires careful chain-rule application.

We can fix λ by demanding that

$$\int_{-\infty}^{+\infty} dv_\parallel\, F_{0\alpha} = n_\alpha \implies \frac{1}{\tau^{\lambda - 1/4}}\left(\frac{1}{2}\int_0^\infty d\kappa\, \kappa^{-3/4}\,\Phi(\kappa)\right) = n_\alpha = \text{constant}$$

Thus, $\lambda = 1/4$. Then, using the substitution

$$\psi' = \Phi' + \frac{1}{16}\Phi \implies \kappa\psi' + \frac{1}{4}\psi = 0 \implies \psi = \frac{c_1}{\kappa^{1/4}}$$

for some integration constant c_1. Integrating again, we find that:

$$\Phi(\kappa) = e^{-\kappa/16}\left(c_1\int_{-\infty}^{\kappa} d\kappa'\, \kappa'^{-1/4}\, e^{\kappa'/16} + c_2\right)$$

Letting $c_1 = 0$ (will be logarithmically divergent), we have

$$F_{0\alpha}(v,t) \propto \frac{e^{-v^4/16\beta\tau}}{(\beta\tau)^{1/4}} \implies \boxed{F_{0\alpha} = \frac{n_\alpha}{\Gamma(\frac{1}{4})}\frac{e^{-v^4/\hat{\alpha} t}}{(\hat{\alpha} t)^{1/4}}}, \quad \hat{\alpha} = \frac{g_\alpha^2}{m_\alpha^2}\sum_h \gamma_h A_h$$

Thus, an initially cold distribution will spread diffusively initially, until it enters the sub-diffusive regime, where it's spreading slows down (a lot of particles already heated around resonance).

$$\Phi(\kappa) \propto \int_{-\infty}^{\kappa} d\kappa'\, \kappa'^{-1/4}\, e^{-(\kappa-\kappa')/16}$$

$$\underset{\sigma = \kappa - \kappa'}{=} \kappa^{-1/4}\int_0^\infty d\sigma\, e^{-\sigma/16}$$

meaning that

$$\int_0^\infty d\kappa\, \kappa^{-3/4}\,\Phi(\kappa) \sim \int_0^\infty d\kappa\, \kappa^{-1} \sim \lim_{\kappa\to\infty}\log\kappa.$$