Problem sheet 1 Solutions

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1. a) A general linear map is given by

$$t' = at + b_i x^i$$
$$(x^i)' = A_j{}^i x^j + c^i t$$

where a is a constant, b_i is a constant 3 dimensional (co)vector, c^i is a constant 3 dimensional vector and $A_j{}^i$ is a constant 3×3 matrix.

In these new coordinates, the vector T, which previously had coordinates $(1, \mathbf{0})$, now has coordinates $(a, \boldsymbol{c})$. So, if the form of T is preserved, then $a = 1$ and $\boldsymbol{c} = 0$.

Similarly, if the form of the metric g is preserved, then we must have

$$\begin{pmatrix} X^0 + \boldsymbol{b} \cdot \boldsymbol{X}, & \boldsymbol{X} A^T \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X^0 + \boldsymbol{b} \cdot \boldsymbol{X} \\ A\boldsymbol{X} \end{pmatrix} = \boldsymbol{X} \cdot \boldsymbol{X}$$

which implies that $A^T A = I$, i.e. A is an orthogonal matrix.

b) If $(q-p)$ has components $\big((q-p)^0, (q-p)^i\big)$ in the original (t, x^i) coordinates, then in the new $(t', (x')^i)$ coordinates it will have components $\Big((q-p) + b_i (q-p)^i\,,\ R_j{}^i (q-p)^j\Big)$. Hence the time between events q and p will be measured to be

$$(q-p) + b_i (q-p)^i$$

while the distance will be

$$\sqrt{R_k{}^i (q-p)^k R_l{}^j (q-p)^l \delta_{ij}} = \sqrt{(q-p)^i (q-p)^j \delta_{ij}}$$

as for the original observer.

c) Calculating

$$\begin{aligned} \partial_t &= \frac{\partial t'}{\partial t}\Big|_{x'} \partial'_{t'} + \frac{\partial (x')^{i'}}{\partial t}\Big|_{x'} \partial'_{i'} \\ &= \partial_t \end{aligned}$$

$$\begin{aligned} \partial_i &= \left(\frac{\partial t'}{\partial x^i}\Big|_{t, (x')^j, j \neq i} \right) \partial'_{t'} + \left(\frac{\partial (x')^{i'}}{\partial x^i}\Big|_{t, (x')^j, j \neq i} \right) \partial'_{i'} \\ &= b_i \partial'_{t'} + R_i{}^{i'} \partial'_{i'} \end{aligned}$$

so the only first order operators which are invariant are those proportional to ∂_t.

Now consider the second order differential operator

$$A\partial_t^2 + B^i \partial_t \partial_i + C^{ij} \partial_i \partial_j$$

Under a Carrollian transformation this transforms to

$$\left(A + B^i b_i + C^{ij} b_i b_j\right) \partial_{t'}'^2 + R_i^{\ i'} \left(B^i + 2C^{ij} b_j\right) \partial_{t'}' \partial_{i'}' + C^{ij} R_i^{\ i'} R_j^{\ j'} \partial_{i'}' \partial_{j'}'$$

For this to be invariant under *any* Carroll transformation, we must have

$$A = A + B^i b_i + C^{ij} b_i b_j$$

for *all* b_i, and so $B^i = 0$ and $C^{ij} = 0$.

iii) A Galilean transformation is given by

$$\begin{aligned} t' &= t + t_0 \\ (x')^{i'} &= R_i^{\ i'} x^i + v^{i'} t + x_0^{i'} \end{aligned}$$

in this case we can compute

$$\begin{aligned} \partial_t &= \partial_{t'}' + v^{i'} \partial_{i'}' \\ \partial_i &= R_i^{\ i'} \partial_{i'}' \end{aligned}$$

so there are no (nonzero) first order differential operators which are invariant. On the other hand, the second order differential operator

$$A\partial_t^2 + B^i \partial_t \partial_i + C^{ij} \partial_i \partial_j$$

transforms under a Galilean transformation to

$$A\partial_t^2 + \left(R_i^{\ i'} B^i + 2Av^{i'}\right) \partial_{t'}' \partial_{i'}' + \left(Av^{i'} v^{j'} + 2R_i^{\ i'} B^i v^{j'} + R_i^{\ i'} R_j^{\ j'} C^{ij}\right) \partial_{i'}' \partial_{j'}'$$

For this to be invariant, we clearly must have $A = B = 0$. On the other hand, C must satisfy

$$R_i^{\ i'} R_j^{\ j'} C^{ij} = C^{i'j'}$$

that is, it must be fixed by rotations. All such tensors are proportional to $\delta^{i'j'}$. So, all second order differential operators which are invariant under Galilean transformations are proportional to the Laplacian $\Delta = \delta^{ij} \partial_i \partial_j$.

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2. a) i) By conservation of energy,

$$\frac{1}{2} m \left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 - \frac{Mm}{r} = -\frac{Mm}{r_0}$$

and so (using the fact that $\dot{r}$ is negative)

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -\sqrt{\frac{2M}{r} - \frac{2M}{r_0}}$$

Since $r = r_0$ when $t = t_0$ we can integrate to obtain

$$t = \int_{r'}^{r_0} \left(\frac{2M}{r'} - \frac{2M}{r_0}\right)^{-\frac{1}{2}} \mathrm{d}r'$$

ii) Set

$$x = \frac{r_0 - r}{r_0}$$

Then x is supposed to be small. We have

$$\begin{aligned} t &= r_0 \int_0^{\frac{r_0 - r}{r_0}} x^{-\frac{1}{2}} (1 - x)^{\frac{1}{2}} \mathrm{d}x \\ &= 2r_0 \sqrt{\frac{r_0}{2M}} \left(\sqrt{\frac{r_0 - r}{r_0}} + \mathcal{O}\left(\left(\frac{r_0 - r}{r_0}\right)^{\frac{3}{2}}\right)\right) \end{aligned}$$

so to leading order

$$
\begin{aligned}
t &= 2r_0\sqrt{\frac{r_0 - r}{2M}} \\
\Rightarrow r &= r_0 - M\frac{t^2}{2r_0^2}
\end{aligned}
$$

b) Work in polar coordinates with the origin at the centre of the Earth. Suppose that the angle between the two particles is θ. Note that, since each particle falls radially, this angle remains fixed. We have

$$x = 2r\sin\frac{\theta}{2}$$

and so

$$
\begin{aligned}
\ddot{x} &= 2\ddot{r}\sin\frac{\theta}{2} \\
&= \frac{\ddot{r}}{r}x \\
&= -\frac{M}{r_0^2 r}x + \mathcal{O}(t) \\
&= -\frac{M}{r_0^3}x + \mathcal{O}(t)
\end{aligned}
$$

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3. Using Bob's inertial coordinates, Alice's worldline is given by

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ v\frac{T}{2\pi}\sin\left(\frac{2\pi t}{T}\right) \\ v\frac{T}{2\pi}\left(\cos\left(\frac{2\pi t}{T}\right) - 1\right) \\ 0 \end{pmatrix}$$

If we parametrise this path by Alice's proper time τ, then we can set $t = t(\tau)$. Then the tangent vector to Alice's worldline is

$$\begin{pmatrix} \dot{t} \\ \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \dot{t}\begin{pmatrix} 1 \\ v\cos\left(\frac{2\pi t}{T}\right) \\ -v\sin\left(\frac{2\pi t}{T}\right) \\ 0 \end{pmatrix}$$

where dots represent derivatives with respect to τ. Since τ is proper time, this vector should have (Lorentzian) norm -1, i.e.

$$(\dot{t})^2\left(-1 + v^2\right) = -1$$

and so

$$\tau = t\sqrt{1 - v^2}$$

In particular, when $t = T$, $\tau = T\sqrt{1 - v^2}$. Since τ is proper time along Alice's worldline, it measures the time that passes according to Alice.

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4. a) The energy of a particle with 3-velocity v and rest mass m is

$$E^2 = \frac{m^2}{1 - v^2}$$

Now, since the particle falls a height H with constant acceleration $-g$ starting from rest, when it reaches the bottom of the tower its velocity is

$$v = -\sqrt{2gH}$$

and so

$$E = \frac{m}{\sqrt{1 - 2gH}}$$

Since H is small (*perhaps this should have been made clearer in the question*) we have

$$E = m\,(1 + gH) + \mathcal{O}(H^2)$$

b) The gravitational field doesn't enter Maxwell's equations, so it should not affect photons or their frequencies.

c) The extra energy is simply $\frac{m}{\sqrt{1-2gH}} - m$. Since H is small, this is $mgH + \mathcal{O}(H^2)$.

d) Suppose N photons are emitted. At the end of the process, if Alice is correct, then $Nh\nu = m$. On the other hand, when the photons are emitted, we have

$$Nh\nu_0 = \frac{m}{\sqrt{1 - 2gH}}$$

and so

$$\begin{aligned} \nu &= \nu_0(1 - 2gH)^{\frac{1}{2}} \\ &= \nu_0(1 - gH) + \mathcal{O}(H^2) \end{aligned}$$

e) In a freely-falling reference frame, the top of the tower is at

$$z = H + \frac{1}{2}gt^2$$

Since the top of the tower accelerates upwards, we should expect a redshift of photons travelling upwards.

We can actually calculate the expected redshift using the equivalence principle. Doing calculations in a freely falling frame and treating it as an 'inertial frame', if two pulses of light are emitted from the bottom of the tower at times $t = 0$ and $t = \epsilon$, they will reach the top of the tower at $t = t_0$ and $t = t_1$ respectively, where

$$\begin{aligned} H + \frac{1}{2}gt_0^2 &= t_0 \\ \Rightarrow t_0 &= H + \mathcal{O}(H^2) \end{aligned}$$

$$\begin{aligned} H + \frac{1}{2}gt_1^2 &= t_1 - \epsilon \\ \Rightarrow t_1 &= H + \epsilon + Hg\epsilon + \mathcal{O}(H^2) \end{aligned}$$

so the time delay between the pulses at the top of the tower is $\epsilon(1 + Hg)$, i.e. the frequency has shifted by a factor of $(1 + Hg)^{-1} \sim 1 - Hg$, agreeing with the other calculation to leading order.

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5. a) Calculating

$$\begin{aligned} \partial^a T_{ab} &= (\Box\phi)(\partial_b\phi) + (\partial^a\phi)(\partial_a\partial_b\phi) - (\partial_b\partial_a\phi)(\partial^a\phi) - \mu^2\phi(\partial_b\phi) \\ &= \left(\Box\phi - \mu^2\phi\right)(\partial_b\phi) + (\partial^a\phi)\left(\partial_a\partial_b\phi - \partial_b\partial_a\phi\right) = 0 \end{aligned}$$

where the first line vanishes due to the equations of motion, and the second line vanishes because partial derivatives commute.

b)

$$\begin{aligned} \partial^a T_{ab} &= (\partial^a F_{ac})F_b{}^c + F^{ac}(\partial_a F_{bc}) - \frac{1}{2}F^{cd}(\partial_b F_{cd}) \\ &= (\partial^a F_{ac})F_b{}^c + \frac{1}{2}F^{ac}\left(\partial_a F_{bc} - \partial_c F_{ba} - \partial_b F_{ac}\right) \end{aligned}$$

where we have relabelled dummy indices and use the antisymmetry of F. Making further use of the antisymmetry of F we have

$$\begin{aligned}\partial^a T_{ab} &= (\partial^a F_{ac}) F_b{}^c + \frac{1}{2} F^{ac} \left(\partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca} \right) \\ &= 0\end{aligned}$$

using the equations of motion.

c) i)

$$\partial^a T_{ab} = u^a \partial_a \left(\rho + p \right) u_b + (\rho + p) u^a \partial_a u_b + (\rho + p)(\partial_a u^a) u_b + \partial_b p = 0$$

To find the component of this equation in the direction parallel to u, we can contract with $-u^b$ and use the fact that $u_a u^a = -1$:

$$\begin{aligned}-u^b \partial^a T_{ab} &= u^a \partial_a \left(\rho + p \right) - (\rho + p) u^a u^b \partial_a u_b + (\rho + p)(\partial_a u^a) - u^a \partial_a p \\ &= u^a \partial_a \rho - \frac{1}{2} (\rho + p) u^a \partial_a (u^b u_b) + (\rho + p) \partial_a u^a \\ &= \partial_a (\rho u^a) + p \partial_a u^a = 0\end{aligned}$$

where we have used the fact that $u_a u^a = -1$ (so derivatives of $u_a u^a$ vanish). This gives the first equation required.

To find the components orthogonal to u, we can contract with the projection operator $\delta_a^b + u_a u^b$. This gives us

$$\begin{aligned}\left(\delta_a^b + u_a u^b \right) \partial^c T_{cb} &= (\rho + p) u^b \partial_b u_a + (\rho + p) u^b u^c (\partial_b u_c) u_a + \partial_a p + u^b (\partial_b p) u_a \\ &= (\rho + p) u^b \partial_b u_a + \frac{1}{2} (\rho + p) u^b \left(\partial_b (u^c u_c) \right) u_a + \partial_a p + u^b (\partial_b p) u_a \\ &= (\rho + p) u^b \partial_b u_a + \left(\delta_a^b + u_a u^b \right) \partial_b p\end{aligned}$$

Raising the index a (i.e. contracting with $(m^{-1})^{ab}$ and relabelling indices) gives us the second required equation.

ii) Setting $u = \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix}$ with $|\mathbf{u}| = \epsilon \ll 1$ means that, in the inertial frame we are using, the fluid moves at a speed much slower than the speed of light (a stationary fluid would have $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$).

Indeed, if we wish to restore the speed of light, then the dimensions of the equations can be made consistent by setting

$$\begin{pmatrix} \boldsymbol{u} \\ \rho \\ p \end{pmatrix} = \begin{pmatrix} c^{-1} \tilde{\boldsymbol{u}} \\ \tilde{\rho} \\ c^{-2} \tilde{p} \end{pmatrix}$$

where $\tilde{\boldsymbol{u}}$ and $\tilde{p}$ are the values of the fluid velocity and the pressure when we work in units such that the speed of light is c rather than 1. If, in such units, the speed of light is very large compared to the values of the various quantities, then we can set the values of the quantities with tildes to be $\mathcal{O}(1)$, while $c = \mathcal{O}(\epsilon^{-1})$, and this gives the required smallness.

Additionally, if we restore the speed of light, then for dimensional consistency time derivatives must appear with an additional factor of $\frac{1}{c}$ (i.e. $\partial_t \mapsto c^{-1} \partial_t$) relative to spatial derivatives. Hence these are also smaller by a factor of ϵ.

iii) Expanding the first equation

$$\partial_t \left(\rho u^0 \right) + \partial_i \left(\rho u^i \right) + p (\partial_t u^0 + \partial_i u^i) = 0$$

Now, recalling that $u^0 = 1$, $u^i = \mathcal{O}(\epsilon)$ and $p = \mathcal{O}(\epsilon^2)$, the equation to leading order (which, in this case, is $\mathcal{O}(\epsilon)$) is

$$\partial_t \rho + \partial_i (\rho u^i) = 0$$

Similarly, we can expand the second equation. Note that the time component of this equation vanishes, while the spatial components are given by

$$(\rho + p)(\partial_t u^i + u^j \partial_j u^i) + (\delta^{ij} + u^i u^j)\partial_j p = 0$$

and this equation, to leading order in ϵ (which, this time, is $\mathcal{O}(\epsilon^2)$) is

$$\rho(\partial_t u^i + u^j \partial_j u^i) + \partial^i p = 0$$

***6. a)** Using the definition of a derivative

$$\dot{\tilde{\gamma}}_U = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\gamma}_U\big|_{t=0} = \lim_{\epsilon\to 0}\frac{\tilde{\gamma}_U(t+\epsilon) - \tilde{\gamma}_U(t)}{\epsilon}$$

but $\tilde{\gamma}_U(t+\epsilon)$ and $\tilde{\gamma}_U(t)$ are both points in $\mathbb{E}\times\mathbb{E}^3$, so their difference defines a vector in $\mathbb{R}\times\mathbb{R}^3$. So, if this limit exists, then it takes a value in $\mathbb{R}\times\mathbb{R}^3$.

b) We can write

$$\begin{aligned}
\mathrm{d}f\big|_p(\dot{\gamma}) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(f\circ\gamma\right)(\tau)\big|_{\tau=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(f\circ\phi_U^{-1}\circ\phi_U\circ\gamma\right)(\tau)\big|_{\tau=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(f\circ\phi_U^{-1}\circ\phi_U\circ\gamma\right)(\tau)\big|_{\tau=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\tilde{f}_U\circ\tilde{\gamma}_U\right)(\tau)\big|_{\tau=0} \\
&= \mathrm{d}\tilde{f}_U\big|_q(\dot{\tilde{\gamma}})
\end{aligned}$$

Note that $\tilde{f}_U : \mathbb{E}\times\mathbb{E}^3 \to \mathbb{R}$. Here, $\mathrm{d}\tilde{f}_U\big|_q$ is *defined* as the linear map such that

$$\tilde{f}_U(q+\epsilon X) = \tilde{f}_U(q) + \epsilon\mathrm{d}\tilde{f}_U\big|_q(X) + \mathcal{O}(\epsilon^2)$$

Note that here $\epsilon\in\mathbb{R}$, $q\in\mathbb{E}\times\mathbb{E}^3$ and $q+\epsilon X\in\mathbb{E}\times\mathbb{E}^3$, so $X\in\mathbb{R}\times\mathbb{R}^3$. Additionally, $\tilde{f}_U(q)\in\mathbb{R}$ and $\epsilon\mathrm{d}\tilde{f}_U\big|_q(X)\in\mathbb{R}$.

In summary, $\mathrm{d}\tilde{f}_U\big|_q$ is a linear map from $\mathbb{R}\times\mathbb{R}^3$ to $\mathbb{R}$.

c) i) We have

$$\begin{aligned}
\dot{\tilde{\gamma}}_V &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\phi_V\circ\gamma\right)(\tau)\big|_{\tau=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\phi_V\circ\phi_U^{-1}\circ\phi_U\circ\gamma\right)(\tau)\big|_{\tau=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\psi_{U,V}\circ\tilde{\gamma}_U\right)(\tau)\big|_{\tau=0} \\
&= \mathrm{d}\psi_{U,V}\big|_{\phi_U(p)}(\dot{\tilde{\gamma}}_U)
\end{aligned}$$

where in the last line we used the chain rule. The differential $\mathrm{d}\psi_{U,V}$ is defined like $\mathrm{d}f$ above: since $\psi_{U,V}$ maps $\mathbb{E}\times\mathbb{E}^3$ to itself, we write

$$\psi_{U,V}(\phi_U(p)+\epsilon X) = \psi_{U,V}(\phi_U(p)) + \epsilon\mathrm{d}\psi_{U,V}\big|_{\phi_U(p)}(X) + \mathcal{O}(\epsilon^2)$$

where, as before, $\epsilon\in\mathbb{R}$ and $X\in\mathbb{R}\times\mathbb{R}^3$.

Note that $\psi_{U,V}(\phi_U(p)) \in \mathbb{E}\times\mathbb{E}^3$, and also $\psi_{U,V}(\phi_U(p)) + \epsilon \mathrm{d}\psi_{U,V}\big|_{\phi_U(p)}(X) \in \mathbb{E}\times\mathbb{E}^3$, and so $\mathrm{d}\psi_{U,V}\big|_{\phi_U(p)}(X) \in \mathbb{R}\times\mathbb{R}^3$. Hence $\mathrm{d}\psi_{U,V}\big|_{\phi_U(p)}$ is a linear map from $\mathbb{R}\times\mathbb{R}^3$ to itself.

ii) Writing

$$\tilde{\gamma}_U(\tau) = \begin{pmatrix} t_U(\tau) \\ x_U(\tau) \end{pmatrix}$$

with $t_U(\tau) \in \mathbb{E}$, $x_U(\tau) \in \mathbb{E}^3$. Hence

$$\begin{aligned} t_U(\tau) &= \tilde{\pi}_U(\tilde{\gamma}_U(\tau)) \\ \Rightarrow \dot{t}_U = \frac{\mathrm{d}}{\mathrm{d}\tau} t_U(\tau) &= \mathrm{d}\tilde{\pi}_U\big|_{\phi_U(p)}\left(\dot{\tilde{\gamma}}_U\right) \end{aligned}$$

where in the last line we used the chain rule.

Now, we know that the projection map commutes with the trivializations, so

$$\begin{aligned} \tilde{\pi}_U(\tilde{\gamma}_U(\tau)) &= \pi(\gamma(\tau)) \\ &= \tilde{\pi}_V(\tilde{\gamma}_V(\tau)) \end{aligned}$$

so, taking derivatives with respect to τ on both sides, we have $\dot{t}_U = \dot{t}_V$.

Hence, we have

$$\mathrm{d}\psi_{U,V}\big|_{\phi_U(p)} \begin{pmatrix} \dot{t}_U \\ \dot{x}_U \end{pmatrix} = \begin{pmatrix} \dot{t}_U \\ \dot{x}_V \end{pmatrix}$$

From this it follows that $\mathrm{d}\psi_{U,V}\big|_{\phi_U(p)}$ is of the form

$$\mathrm{d}\psi_{U,V}\big|_{\phi_U(p)} = \left(\begin{array}{c|c} 1 & 0 \\ \hline A_{U,V}(p) & R_{U,V}(p) \end{array}\right)$$

but, by considering $\mathrm{d}\psi_{V,U}\big|_{\phi_V(p)}$, which is the inverse matrix but which must be of the same form, we conclude that $A_{U,V} = 0$ as required.

d) i) If the inner product is preserved then

$$\langle \dot{\gamma}, \dot{\xi} \rangle = \langle R(p)\dot{\gamma}, R(p)\dot{\xi} \rangle$$

but this defines the group $O(3)$.

Now we recall that $p = \gamma(0)$. In general, we can write $R(p) = R(\gamma(\tau))$, or, if we wish, $R(\tau)$. Then we have

$$\ddot{x}_V = \dot{R}\dot{x}_U + R\ddot{x}_U$$

so if, for all curves $x_U(\tau)$ such that $ddotx_U = 0$ it is also the case that $\ddot{x}_V = 0$, then we must have $\dot{R} = 0$ everywhere (and in every direction), and so R is constant.

ii) If (V, ϕ_V) preserves both the inner product and 'straight lines', we have shown above that

$$\mathrm{d}\psi_{U,V} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right)$$

where $R \in O(3)$ is constant. Choosing some arbitrary origin so that points in $\mathbb{E}\times\mathbb{E}^3$ can be identified with points in $\mathbb{R}\times\mathbb{R}^3$ and integrating, it follows that

$$\psi_{U,V}\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}\begin{pmatrix} t \\ x \end{pmatrix} + \begin{pmatrix} t_0 \\ x_0 \end{pmatrix}$$

where $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$.

iii) Now $\dot{x}_W = R(\tau)\dot{x}_U$, where $R(\tau) \in O(3)$ no longer needs to be constant. Hence we have

$$\ddot{x}_W = \dot{R}\dot{x}_U + R\ddot{x}_U$$

so, if $\ddot{x}_U = 0$, then

$$\begin{aligned}\ddot{x}_W &= \dot{R}\dot{x}_U \\ &= \dot{R}R^T\dot{x}_W\end{aligned}$$

since $R^T = R^{-1}$.

Next we write

$$\begin{aligned}R(\tau) = R((\gamma(\tau)) &= (R \circ \gamma)(\tau) \\ &= (R \circ \phi_W^{-1} \circ \phi_W \circ \gamma)(\tau) \\ &= (\tilde{R}_W \circ \tilde{\gamma}_W)(\tau)\end{aligned}$$

where $\tilde{R}_W : \mathbb{E} \times \mathbb{E}^3 \to O(3)$. So, using the chain rule,

$$\dot{R}(\tau) = \mathrm{d}\tilde{R}_W\big|_{\phi_W(\gamma(\tau))}(\dot{\tilde{\gamma}}_W)$$

where

$$\tilde{R}_W(q + \epsilon X) = \tilde{R}_W(q) + \epsilon \mathrm{d}\tilde{R}_W\big|_q(X) + \mathcal{O}(\epsilon^2)$$

as before, we find that $X \in \mathbb{R} \times \mathbb{R}^3$. On the other hand, we should have

$$\left(\tilde{R}_W(q) + \epsilon \mathrm{d}\tilde{R}_W\big|_q(X) + \mathcal{O}(\epsilon^2)\right)\left((\tilde{R}_W(q))^T + \epsilon\left(\mathrm{d}\tilde{R}_W\big|_q(X)\right)^T + \mathcal{O}(\epsilon^2)\right) = I$$

from which it follows that

$$\left(\tilde{R}_W(q)\right)\left(\mathrm{d}\tilde{R}_W\big|_q(X)\right)^T + \left(\mathrm{d}\tilde{R}_W\big|_q(X)\right)\left(\tilde{R}_W(q)\right)^T = 0$$

In other words, $\mathrm{d}\tilde{R}_W\big|_q$ is a linear map

$$\begin{aligned}\mathrm{d}\tilde{R}_W\big|_q &: \mathbb{R} \times \mathbb{R}^3 \to M \\ M &= \left\{3 \times 3 \text{ matrices } A \text{ such that } \quad A\left(\tilde{R}_W(q)\right)^T \quad \text{is skew-symmetric}\right\}\end{aligned}$$

Since $\mathrm{d}\tilde{R}_W$ is a linear map, we can further write

$$\mathrm{d}\tilde{R}_W\begin{pmatrix}\dot{t} \\ \dot{x}\end{pmatrix} = \Gamma_t(\dot{t})R + \Gamma_x(\dot{x})R$$

Putting this all together, we find that

$$\dot{R}(\tau)R^T = \Gamma_t(\dot{t}_W) + \Gamma_x(\dot{x}_W)$$

and so the equation for 'straight lines' is

$$\ddot{x}_W = \left(\Gamma_t(\dot{t}_W) + \Gamma_x(\dot{x}_W)\right)\dot{x}_W$$

where $\Gamma_t(\dot{t}_W)$ and $\Gamma_x(\dot{x}_W)$ are skew-symmetric.