INTRODUCTION TO QFT 2019: PROBLEM SET 4

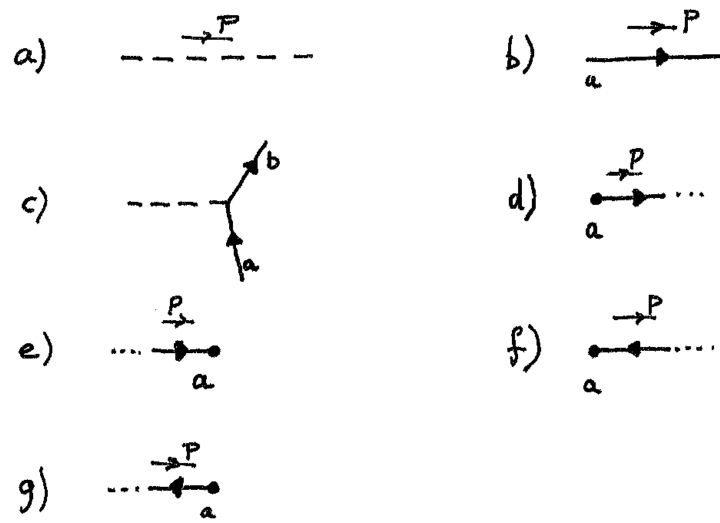
JOHN WHEATER

Please hand in your answers at the Reception Desk of the Physics Department by 5pm on *Wednesday* Week 7 (27 November) preferably using the cover sheet on the website. Your class tutor may have given you different instructions for hand-in times – in that case please do as they ask.

This Problem Set is all to do with a field theory consisting of a scalar ϕ of mass m and a Dirac fermion ψ of mass m described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\phi\,\partial_\mu\phi - m^2\phi^2) + \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi - g\phi\overline{\psi}\psi + \mathcal{L}_{Int}(\phi) \tag{1}$$

Remember that γ^μ are four-by-four matrices and the indices a, b run over $\{1, 2, 3, 4\}$ covering all the components). Fermion lines have an arrow on them which denotes the flow of particle number; the momentum in the propagator is the value of the momentum in the direction of the arrow. The Feynman rules are



a) Scalar propagator (internal line)
$$i\frac{1}{p^2 - m^2 + i\epsilon} \tag{2}$$
b) Fermion propagator (internal line)
$$i\frac{(\not{p} + m)_{a,b}}{p^2 - m^2 + i\epsilon} \tag{3}$$
c) Scalar fermion vertex $-ig\delta_{a,b}$
d) Initial state fermion (external line) $u^s(p)_a$
e) Final state fermion (external line) $\overline{u}^s(p)_a$
f) Initial state anti-fermion (external line) $\overline{v}^s(p)_a$

g) Final state anti-fermion (external line) $v^s(p)_a$

h) Closed fermion loops have an extra factor -1, and have a Trace over the spinor indices.

1. *Gamma matrix manipulations* Hard-core theorists will derive these results for themselves once in their life! However it is permissible to jump to question 2 and simply use these formulae! The γ^μ matrices are traceless and satisfy

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{I} \tag{4}$$

where $\mathbb{I}$ is the four-by-four identity matrix.

a) By contracting $p^\mu p^\nu$ with (4) show that $\not{p}\not{p} = p^2\mathbb{I}$.

b) By contracting $p^\mu q^\nu$ with (4) and taking the Trace show that

$$\mathrm{Tr}\,\not{p}\not{q} = 4p\cdot q \tag{5}$$

c) Show that the matrix $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ anticommutes with γ^μ and that $\gamma^5\gamma^5 = -\mathbb{I}$.

d) By considering $\mathrm{Tr}\gamma^5\gamma^5\gamma^\lambda\gamma^\mu\gamma^\nu$, and moving one of the γ^5s to the right through the other matrices, and finally using the cyclic property of the Trace show that

$$\mathrm{Tr}\gamma^\lambda\gamma^\mu\gamma^\nu = 0 \tag{6}$$

Generalise the argument to show that the Trace of any odd number of gamma matrices is zero.

e) Using (4) show that $\mathrm{Tr}\,\not{p}\not{q}\not{r}\not{s} = \mathrm{Tr}\,\not{p}\not{q}(2s\cdot r - \not{s}\not{r})$. Repeat the manipulation to "walk" $\not{s}$ through to the left hand end, then put it back at the right hand end by the cyclic property of the trace. Hence show that

$$\mathrm{Tr}\,\not{p}\not{q}\not{r}\not{s} = 4(p\cdot q\, s\cdot r - p\cdot r\, q\cdot s + p\cdot s\, q\cdot r) \tag{7}$$

2. *Scalar-fermion scattering.* Draw the Feynman Diagrams(s) for scattering of a scalar with four-momentum k off a fermion with four-momentum p, spin s, to final states described by k', p', s' respectively.

a) Show that the scattering matrix element can be written

$$\widetilde{M} = \widetilde{M}_1 + \widetilde{M}_2 \tag{8}$$

where

$$\widetilde{M}_1 = -ig^2\frac{\overline{u}^{s'}(p')(\not{k}' + 2m)u^s(p)}{S - m^2}, \qquad \widetilde{M}_2 = -ig^2\frac{\overline{u}^{s'}(p')(-\not{k}' + 2m)u^s(p)}{U - m^2} \tag{9}$$

where we define $S = (p+k)^2$, $T = (p'-p)^2$ and $U = (p'-k)^2$.

b) Use the result $\sum_s u^s_a(p)\overline{u}^s_b(p) = (\not{p} + m)_{ab}$ for Dirac spinors $u^s(p)$ and the trace formulae you found in Q.1 to show that

$$\begin{aligned} A &= \sum_{s,s'}|\widetilde{M}_1|^2 = \frac{g^4}{(S-m^2)^2} \\ &\quad\times\big((S+2m^2)^2 + (U-6m^2)^2 - T(T+6m^2)\big) \end{aligned}$$

where $T = (p'-p)^2$ and $U = (p'-k)^2$.

c) In an experiment the initial fermion spin direction is unknown, and the final fermion spin direction is not measured. Explain why $\frac{1}{2}\sum_{s,s'}|\widetilde{M}|^2$ is the correct quantity to insert in the formula for the cross-section.

3. *Fermion-Fermion scattering* Draw the Feynman Diagrams for scattering of two fermions with spin and four-momenta s_1, p_1 s_2, p_2 respectively to two final state fermions with spin and four-momenta s_3, p_3 and s_4, p_4 .

   a) Use the Feynman Rules to write down the matrix element – but take care that these are identical fermions so the matrix element must be antisymmetric under $1 \leftrightarrow 2$, or $3 \leftrightarrow 4$.

   b) Calculate $\sum_{s_1,s_2,s_3,s_4} |\widetilde{M}|^2$

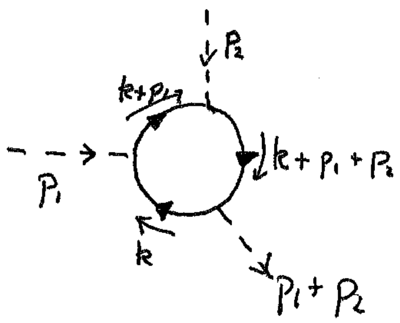
4. *One Loop*

   a) Draw the Feynman diagram for the one fermion loop correction to the scalar two point function and show that the 1PI part can be written

$$i\Sigma(p) = 4g^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 - M^2} + \frac{2M^2}{(k^2 - M^2)^2} \right) \tag{10}$$

   where $M^2 = m^2 - p^2 x(1 - x)$. Hence show that as well as a mass shift of order $\lambda\Lambda^2$ the kinetic (ie p^2) part of D(p) is multiplied by a cut-off dependent factor $(1 + \mathit{const}\, \lambda \log(\Lambda^2/m^2))$

   b) Consider the 1PI contribution to the scalar three point function



   Write down the expression for this graph using the Feynman rules. Now focus on the leading powers of k; and using the Trace rules from Q.1 show that the k^3 term in the numerator vanishes. Hence show that this graph has a divergent part $\mathit{const}\, m\lambda^3 \log(\Lambda^2/m^2)$ (it is not necessary to find the full result). It follows that we are forced to add a ϕ^3 counterterm to $\mathcal{L}$.

   c) Repeat the previous part for the scalar four point function and show that it has a divergent part $\mathit{const}\, \lambda^4 \log(\Lambda^2/m^2)$, and will therefore also need a counterterm.