Lectures on Kinetic Theory and Magnetohydrodynamics of Plasmas

(Oxford MMathPhys/MSc in Mathematical and Theoretical Physics)

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These are the notes for my lectures on Kinetic Theory of Plasmas and on Magnetohydrodynamics, taught since 2014 as part of the MMathPhys programme at Oxford. Part I contains the lectures on plasma kinetics that formed part of the course on Kinetic Theory, taught jointly with Paul Dellar and James Binney. Part II is an introduction to magnetohydrodynamics, which was part of the course on Advanced Fluid Dynamics, taught jointly with Paul Dellar. These notes have evolved from two earlier courses: "Advanced Plasma Theory," taught as a graduate course at Imperial College in 2008, and "Magnetohydrodynamics and Turbulence," taught as a Mathematics Part III course at Cambridge in 2005-06. Extracts from these notes have also been used in (and some cases written for) my lectures at successive plasma-physics sessions of École de Physique des Houches in 2017 and 2019. I will be grateful for any feedback from students, tutors or sympathisers.

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PART I
Kinetic Theory of Plasmas

1. Kinetic Description of a Plasma

We shall study a gas consisting of charged particles—ions and electrons. In general, there may be many different species of ions, with different masses and charges, and, of course, only one type of electrons.

I will index particle species by α ($\alpha = e$ for electrons, $\alpha = i$ for ions). Each is characterised by its mass m_α and charge $q_\alpha = Z_\alpha e$, where e is the magnitude of the electron charge and Z_α is a positive or negative integer (e.g., $Z_e = -1$).

1.1. *Quasineutrality*

We shall always assume that plasma is neutral overall:

$$\sum_\alpha q_\alpha N_\alpha = eV \sum_\alpha Z_\alpha \bar{n}_\alpha = 0, \tag{1.1}$$

where N_α is the number of the particles of species α, $\bar{n}_\alpha = N_\alpha/V$ is their mean number density and V the volume of the plasma. This condition is known as *quasineutrality*.

1.2. *Weak Interactions*

Interaction between charged particles is governed by the Coulomb potential:

$$U\big(|\boldsymbol{r}_i^{(\alpha)} - \boldsymbol{r}_j^{(\alpha')}|\big) = -\frac{q_\alpha q_{\alpha'}}{|\boldsymbol{r}_i^{(\alpha)} - \boldsymbol{r}_j^{(\alpha')}|}, \tag{1.2}$$

where by $\boldsymbol{r}_i^{(\alpha)}$ I mean the position of the i-th particle of species α. It is a safe bet that we will only be able to have a nice closed kinetic description if the gas is approximately ideal, i.e., if particles interact weakly, viz.,

$$k_{\mathrm{B}}T \gg U \sim \frac{e^2}{\Delta r} \sim e^2 n^{1/3}, \tag{1.3}$$

where k_{B} is the Boltzmann constant, which will henceforth be absobed into the temperature T, and $\Delta r \sim n^{-1/3}$ is the typical interparticle distance. Let us see what this condition means and implies physically.

1.3. *Debye Shielding*

Let us consider a plasma in thermodynamic equilibrium (as one does in statistical mechanics, I will refuse to discuss, for the time being, how exactly it got there). Take one particular particle, of species α. It creates an electric field around itself, $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi$; all other particles sit in this field (Fig. 1)—and, indeed, also affect it, as we will see below. In equilibrium, the densities of these particles ought to satisfy Boltzmann's formula:

$$n_{\alpha'}(\boldsymbol{r}) = \bar{n}_{\alpha'} \, e^{-q_{\alpha'}\varphi(\boldsymbol{r})/T} \approx \bar{n}_{\alpha'} - \frac{\bar{n}_{\alpha'}q_{\alpha'}\varphi}{T}, \tag{1.4}$$

where $\bar{n}_{\alpha'}$ is the mean density of the particles of species α' and $\varphi(\boldsymbol{r})$ is the electrostatic potential, which depends on the distance $\boldsymbol{r}$ from our "central" particle. As $r \to \infty$, $\varphi \to 0$ and $n_{\alpha'} \to \bar{n}_{\alpha'}$. The exponential can be Taylor-expanded provided the weak-interaction condition (1.3) is satisfied ($e\varphi \ll T$).
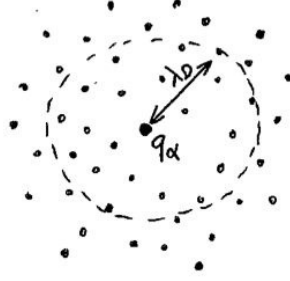
FIGURE 1. A particle amongst particles and its Debye sphere.

By the Gauss–Poisson law, we have

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = -\nabla^2 \varphi = 4\pi q_\alpha \delta(\boldsymbol{r}) + 4\pi \sum_{\alpha'} q_{\alpha'} n_{\alpha'}$$

$$\approx 4\pi q_\alpha \delta(\boldsymbol{r}) + \underbrace{4\pi \sum_{\alpha'} q_{\alpha'} \bar{n}_{\alpha'}}_{\substack{= 0 \text{ by} \\ \text{quasineutral-} \\ \text{ity}}} - \underbrace{\left(\sum_{\alpha'} \frac{4\pi \bar{n}_{\alpha'} q_{\alpha'}^2}{T} \right)}_{\equiv 1/\lambda_{\mathrm{D}}^2} \varphi. \qquad (1.5)$$

In the first line of this equation, the first term on the right-hand side is the charge density associated with the "central" particle and the second term is the charge density of the rest of the particles. In the second line, I used the Taylor-expanded Boltzmann expression (1.4) for the particle densities and then the quasineutrality (1.1) to establish the vanishing of the second term. The combination that has arisen in the last term as a prefactor of φ has dimensions of inverse square length, so we define the *Debye length* to be

$$\boxed{\lambda_{\mathrm{D}} \equiv \left(\sum_\alpha \frac{4\pi \bar{n}_\alpha q_\alpha^2}{T} \right)^{-1/2}.} \qquad (1.6)$$

Using also the obvious fact that the solution of (1.5) must be spherically symmetric, we recast this equation as follows

$$\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial \varphi}{\partial r} - \frac{1}{\lambda_{\mathrm{D}}^2} \varphi = -4\pi q_\alpha \delta(\boldsymbol{r}). \qquad (1.7)$$

The solution to this that asymptotes to the Coulomb potential $\varphi \to q_\alpha/r$ as $r \to 0$ and to zero as $r \to \infty$ is

$$\varphi = \frac{q_\alpha}{r} e^{-r/\lambda_{\mathrm{D}}}. \qquad (1.8)$$

Thus, in a quasineutral plasma, charges are shielded on typical distances $\sim \lambda_{\mathrm{D}}$.

Obviously, this statistical calculation only makes sense if the "Debye sphere" has very many particles in it, viz., if

$$\boxed{n\lambda_{\mathrm{D}}^3 \gg 1.} \qquad (1.9)$$

Let us check that this is the case:

$$n\lambda_{\mathrm{D}}^3 \sim n \left(\frac{T}{ne^2} \right)^{3/2} = \left(\frac{T}{n^{1/3} e^2} \right)^{3/2} \gg 1, \qquad (1.10)$$

provided the weak-interaction condition (1.3) is satisfied. The quantity $n\lambda_{\mathrm{D}}^3$ is called the *plasma parameter*.

1.4. *Micro- and Macroscopic Fields*

This calculation tells us something very important about electromagnetic fields in a plasma. Let $\boldsymbol{E}^{(\mathrm{micro})}(\boldsymbol{r}, t)$ and $\boldsymbol{B}^{(\mathrm{micro})}(\boldsymbol{r}, t)$ be the exact *microscopic fields* at a given location $\boldsymbol{r}$ and time t. These fields are responsible for interactions between particles. On distances $l \ll \lambda_{\mathrm{D}}$, these will be just the two-particle interactions—binary collisions between particles in vacuo, just like in a neutral gas (except the interparticle potential is the Coulomb potential). In contrast, on distances $l \gtrsim \lambda_{\mathrm{D}}$, individual particles' fields are shielded and what remains are fields due to collective influence of large numbers of particles—*macroscopic fields:*

$$\boldsymbol{E}^{(\mathrm{micro})} = \underbrace{\langle \boldsymbol{E}^{(\mathrm{micro})} \rangle}_{\equiv \boldsymbol{E}} + \delta\boldsymbol{E}, \quad \boldsymbol{B}^{(\mathrm{micro})} = \underbrace{\langle \boldsymbol{B}^{(\mathrm{micro})} \rangle}_{\equiv \boldsymbol{B}} + \delta\boldsymbol{B}, \tag{1.11}$$

where the macroscopic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are averages over some intermediate scale l such that

$$\Delta r \sim n^{-1/3} \ll l \ll \lambda_{\mathrm{D}}. \tag{1.12}$$

Such averaging (or "coarse-graining") is made possible by the condition (1.9).

Thus, plasma has a new feature compared to neutral gas: because the Coulomb potential is long-range ($\propto 1/r$), the fields decay on a length scale that is long compared to the interparticle distances [$\lambda_{\mathrm{D}} \gg \Delta r \sim n^{-1/3}$ according to (1.9)] and so, besides interactions between individual particles, there are also collective effects: interaction of particles with mean macroscopic fields due to all other particles.

Before I use this approach to construct a description of the plasma as a continuum (on scales $\gg l$), let us check that particles travel sufficiently long distances between collisions in order to feel the macroscopic fields, viz., that their mean free path is $\lambda_{\mathrm{mfp}} \gg \lambda_{\mathrm{D}}$. The mean free path can be estimated in terms of the collision cross-section σ:

$$\lambda_{\mathrm{mfp}} \sim \frac{1}{n\sigma} \sim \frac{T^2}{ne^4} \tag{1.13}$$

because $\sigma \sim d^2$ and the effective distance d by which particles have to approach each other in order to have significant Coulomb interaction is inferred by balancing the Coulomb potential energy (1.2) with the particle temperature, $e^2/d \sim T$. Using (1.13) and (1.6), we find

$$\frac{\lambda_{\mathrm{mfp}}}{\lambda_{\mathrm{D}}} \sim \frac{T^2}{ne^4} \left(\frac{ne^2}{T} \right)^{1/2} \sim n\lambda_{\mathrm{D}}^3 \gg 1, \quad \text{q.e.d.} \tag{1.14}$$

Thus, it makes sense to talk about a particle travelling long distances experiencing the macroscopic fields exerted by the rest of the plasma collectively before being deflected by a much larger, but also much shorter-range, microscopic field of another individual particle.

1.5. *Maxwell's Equations*

The exact microscopic fields satisfy *Maxwell's equations* and, since Maxwell's equations are linear, so do the macroscopic fields: by direct averaging,

$$\boldsymbol{\nabla} \cdot \langle \boldsymbol{E}^{(\mathrm{micro})} \rangle = 4\pi \langle \sigma^{(\mathrm{micro})} \rangle, \tag{1.15}$$

$$\boldsymbol{\nabla} \cdot \langle \boldsymbol{B}^{(\mathrm{micro})} \rangle = 0, \tag{1.16}$$

$$\boldsymbol{\nabla} \times \langle \boldsymbol{E}^{(\mathrm{micro})} \rangle + \frac{1}{c} \frac{\partial \langle \boldsymbol{B}^{(\mathrm{micro})} \rangle}{\partial t} = 0, \tag{1.17}$$

$$\boldsymbol{\nabla} \times \langle \boldsymbol{B}^{(\mathrm{micro})} \rangle - \frac{1}{c} \frac{\partial \langle \boldsymbol{E}^{(\mathrm{micro})} \rangle}{\partial t} = \frac{4\pi}{c} \langle \boldsymbol{j}^{(\mathrm{micro})} \rangle. \tag{1.18}$$

The new quantities here are the averages of the microscopic charge density $\sigma^{(\mathrm{micro})}$ and the microscopic current density $\boldsymbol{j}^{(\mathrm{micro})}$. How do we calculate them?

Clearly, they depend on where all the particles are at any given time and how fast these particles move. We can assemble all this information in one function:

$$F_\alpha(\boldsymbol{r}, \boldsymbol{v}, t) = \sum_{i=1}^{N_\alpha} \delta^3 \big(\boldsymbol{r} - \boldsymbol{r}_i^{(\alpha)}(t) \big) \delta^3 \big(\boldsymbol{v} - \boldsymbol{v}_i^{(\alpha)}(t) \big), \tag{1.19}$$

where $\boldsymbol{r}_i^{(\alpha)}(t)$ and $\boldsymbol{v}_i^{(\alpha)}(t)$ are the exact phase-space coordinates of particle i of species α at time t, i.e., these are the solutions of the exact equations of motion for all these particles moving in the microscopic fields $\boldsymbol{E}^{(\mathrm{micro})}(t, \boldsymbol{r})$ and $\boldsymbol{B}^{(\mathrm{micro})}(t, \boldsymbol{r})$. The function F_α is called the *Klimontovich distribution function*. It is a random object (i.e., it fluctuates on scales $\ll \lambda_{\mathrm{D}}$) because it depends on the exact particle trajectories, which depend on the exact microscopic fields. In terms of this distribution function,

$$\sigma^{(\mathrm{micro})}(\boldsymbol{r}, t) = \sum_\alpha q_\alpha \int \mathrm{d}^3 \boldsymbol{v} \, F_\alpha(\boldsymbol{r}, \boldsymbol{v}, t), \tag{1.20}$$

$$\boldsymbol{j}^{(\mathrm{micro})}(\boldsymbol{r}, t) = \sum_\alpha q_\alpha \int \mathrm{d}^3 \boldsymbol{v} \, \boldsymbol{v} F_\alpha(\boldsymbol{r}, \boldsymbol{v}, t). \tag{1.21}$$

We now need to average these quantities for use in (1.15) and (1.18). We shall assume that the average over microscales (1.12) and the ensemble average (i.e., the average over many different initial conditions) are the same. The ensemble average of F_α is an object familiar from the kinetic theory of gases (Dellar 2015), the so-called *one-particle distribution function*:

$$\langle F_\alpha \rangle = f_{1\alpha}(\boldsymbol{r}, \boldsymbol{v}, t) \tag{1.22}$$

(I shall henceforth omit the subscript 1). If we learn how to compute f_α, then we can average (1.20) and (1.21), substitute them into (1.15) and (1.18), and have the following set of macroscopic Maxwell's equations:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi \sum_\alpha q_\alpha \int \mathrm{d}^3 \boldsymbol{v} \, f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t), \tag{1.23}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{1.24}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} + \frac{1}{c} \frac{\partial \boldsymbol{B}}{\partial t} = 0, \tag{1.25}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} - \frac{1}{c} \frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c} \sum_\alpha q_\alpha \int \mathrm{d}^3 \boldsymbol{v} \, \boldsymbol{v} f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t). \tag{1.26}$$

1.6. *Vlasov–Landau Equation*

We now need an evolution equation for $f_\alpha(\boldsymbol{r}, \boldsymbol{v}, t)$, hopefully in terms of the macroscopic fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$, so we can couple it to (1.23–1.26) and thus have a closed system of equations describing our plasma.

The process of deriving it starts with Liouville's theorem and is a direct generalisation of the BBGKY procedure familiar from gas kinetics (e.g., Dellar 2015)[1] to the somewhat more cumbersome case of a plasma:

—many species α;

—presence of forces due to the macroscopic fields $\boldsymbol{E}$ and $\boldsymbol{B}$;

—Coulomb potential for interparticle collisions, with some attendant complications to do with its long-range nature: in brief, use Rutherford's cross section and cut off long-range interactions at $\lambda_{\rm D}$ (this is described in many textbooks and plasma-physics courses: see, e.g., Parra 2019*a* or a shortcut in §8.6.3 of these Lectures).

The result of this derivation is

$$\frac{\partial f_\alpha}{\partial t} + \{f_\alpha, H_{1\alpha}\} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\rm c}. \tag{1.27}$$

The Poisson bracket contains $H_{1\alpha}$, the Hamiltonian for a single particle of species α moving in the *macroscopic* fields $\boldsymbol{E}$ and $\boldsymbol{B}$—all the microscopic fields $\delta\boldsymbol{E}$[2] are gone into the collision operator on the right-hand side, of which more will be said shortly (§1.8).

Technically speaking, one ought to be working with canonical variables, but dealing with canonical momenta of charged particles in a magnetic field is an unnecessary complication, so I shall stick to the $(\boldsymbol{r}, \boldsymbol{v})$ representation of the phase space. Then (1.27) takes the form of Liouville's equation, but with microscopic fields hidden inside the collision operator [see (1.41)]:

$$\frac{\partial f_\alpha}{\partial t} + \frac{\partial}{\partial \boldsymbol{r}} \cdot \left(\dot{\boldsymbol{r}} f_\alpha\right) + \frac{\partial}{\partial \boldsymbol{v}} \cdot \left(\dot{\boldsymbol{v}} f_\alpha\right) = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\rm c}, \tag{1.28}$$

where

$$\dot{\boldsymbol{r}} = \boldsymbol{v}, \quad \dot{\boldsymbol{v}} = \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{v} \times \boldsymbol{B}}{c}\right). \tag{1.29}$$

This gives us the *Vlasov–Landau equation:*

$$\boxed{\frac{\partial f_\alpha}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f_\alpha + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{v} \times \boldsymbol{B}}{c}\right) \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{v}} = \left(\frac{\partial f_\alpha}{\partial t}\right)_{\rm c}.} \tag{1.30}$$

Any other macroscopic force that the plasma might be subject to (e.g., gravity) can be added to the Lorentz force in the third term on the left-hand side, as long as its divergence in velocity space is $(\partial/\partial\boldsymbol{v}) \cdot \text{force} = 0$. Equation (1.30) is closed by Maxwell's equations (1.23–1.26).

1.7. *Klimontovich's Version of BBGKY*

By way of a technical digression, let me outline the (beginning of the) derivation of (1.30) due to Klimontovich (1967). Consider the Klimontovich distribution function (1.19) and calculate

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[1]In §1.7, I will sketch Klimontovich's version of this procedure (Klimontovich 1967).

[2]$\delta\boldsymbol{B}$ turns out to be irrelevant as long as the particle motion is non-relativistic, $v/c \ll 1$.

its time derivative: by the chain rule,

$$
\frac{\partial F_\alpha}{\partial t} = -\sum_i \frac{\mathrm{d}\boldsymbol{r}_i^{(\alpha)}(t)}{\mathrm{d}t} \cdot \left[\frac{\partial}{\partial \boldsymbol{r}}\, \delta^3\big(\boldsymbol{r} - \boldsymbol{r}_i^{(\alpha)}(t)\big) \delta^3\big(\boldsymbol{v} - \boldsymbol{v}_i^{(\alpha)}(t)\big) \right]
$$
$$
- \sum_i \frac{\mathrm{d}\boldsymbol{v}_i^{(\alpha)}(t)}{\mathrm{d}t} \cdot \left[\frac{\partial}{\partial \boldsymbol{v}}\, \delta^3\big(\boldsymbol{r} - \boldsymbol{r}_i^{(\alpha)}(t)\big) \delta^3\big(\boldsymbol{v} - \boldsymbol{v}_i^{(\alpha)}(t)\big) \right]. \tag{1.31}
$$

First, because $\boldsymbol{r}_i^{(\alpha)}(t)$ and $\boldsymbol{v}_i^{(\alpha)}(t)$ obviously do not depend on the phase-space variables $\boldsymbol{r}$ and $\boldsymbol{v}$, the derivatives $\partial/\partial \boldsymbol{r}$ and $\partial/\partial \boldsymbol{v}$ can be pulled outside, so the right-hand side of (1.31) can be written as a divergence in phase space. Secondly, the particle equations of motion give us

$$
\frac{\mathrm{d}\boldsymbol{r}_i^{(\alpha)}(t)}{\mathrm{d}t} = \boldsymbol{v}_i^{(\alpha)}(t), \tag{1.32}
$$
$$
\frac{\mathrm{d}\boldsymbol{v}_i^{(\alpha)}(t)}{\mathrm{d}t} = \frac{q_\alpha}{m_\alpha} \left[\boldsymbol{E}^{(\mathrm{micro})}\big(\boldsymbol{r}_i^{(\alpha)}(t), t\big) + \frac{\boldsymbol{v}_i^{(\alpha)}(t) \times \boldsymbol{B}^{(\mathrm{micro})}\big(\boldsymbol{r}_i^{(\alpha)}(t), t\big)}{c} \right], \tag{1.33}
$$

which are to be substituted into the right-hand side of (1.31)—*after* it is written in the divergence form. Since the time derivatives of $\boldsymbol{r}_i^{(\alpha)}(t)$ and $\boldsymbol{v}_i^{(\alpha)}(t)$ inside the divergence multiply delta functions identifying $\boldsymbol{r}_i^{(\alpha)}(t)$ with $\boldsymbol{r}$ and $\boldsymbol{v}_i^{(\alpha)}(t)$ with $\boldsymbol{v}$, $\boldsymbol{r}_i^{(\alpha)}(t)$ may be replaced by $\boldsymbol{r}$ and $\boldsymbol{v}_i^{(\alpha)}(t)$ by $\boldsymbol{v}$ in the right-hand sides of (1.32) and (1.33) when they go into (1.31). This gives (wrapping all the sums of delta functions back into F_α)

$$
\frac{\partial F_\alpha}{\partial t} = -\boldsymbol{\nabla} \cdot (\boldsymbol{v} F_\alpha) - \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[\frac{q_\alpha}{m_\alpha} \left(\boldsymbol{E}^{(\mathrm{micro})}(\boldsymbol{r}, t) + \frac{\boldsymbol{v} \times \boldsymbol{B}^{(\mathrm{micro})}(\boldsymbol{r}, t)}{c} \right) F_\alpha \right]. \tag{1.34}
$$

Finally, because $\boldsymbol{r}$ and $\boldsymbol{v}$ are independent variables and the Lorentz force has zero divergence in $\boldsymbol{v}$ space, F_α satisfies *exactly*

$$
\boxed{\frac{\partial F_\alpha}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} F_\alpha + \frac{q_\alpha}{m_\alpha} \left(\boldsymbol{E}^{(\mathrm{micro})} + \frac{\boldsymbol{v} \times \boldsymbol{B}^{(\mathrm{micro})}}{c} \right) \cdot \frac{\partial F_\alpha}{\partial \boldsymbol{v}} = 0}. \tag{1.35}
$$

This is the *Klimontovich equation*. There is no collision integral here because microscopic fields are explicitly present. The equation is closed by the microscopic Maxwell's equations:

$$
\boldsymbol{\nabla} \cdot \boldsymbol{E}^{(\mathrm{micro})} = 4\pi \sum_\alpha q_\alpha \int \mathrm{d}^3\boldsymbol{v}\, F_\alpha(\boldsymbol{r}, \boldsymbol{v}, t), \tag{1.36}
$$
$$
\boldsymbol{\nabla} \cdot \boldsymbol{B}^{(\mathrm{micro})} = 0, \tag{1.37}
$$
$$
\boldsymbol{\nabla} \times \boldsymbol{E}^{(\mathrm{micro})} + \frac{1}{c} \frac{\partial \boldsymbol{B}^{(\mathrm{micro})}}{\partial t} = 0, \tag{1.38}
$$
$$
\boldsymbol{\nabla} \times \boldsymbol{B}^{(\mathrm{micro})} - \frac{1}{c} \frac{\partial \boldsymbol{E}^{(\mathrm{micro})}}{\partial t} = \frac{4\pi}{c} \sum_\alpha q_\alpha \int \mathrm{d}^3\boldsymbol{v}\, \boldsymbol{v} F_\alpha(\boldsymbol{r}, \boldsymbol{v}, t). \tag{1.39}
$$

Now let us separate the microscopic fields into mean (macroscopic) and fluctuating parts according to (1.11); also

$$
F_\alpha = \underbrace{\langle F_\alpha \rangle}_{\equiv\, f_\alpha} + \delta F_\alpha. \tag{1.40}
$$

Maxwell's equations are linear, so averaging them gives the same equations for $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of f_α [see (1.23–1.26)] and for $\delta\boldsymbol{E}$ and $\delta\boldsymbol{B}$ in terms of δF_α. Averaging the Klimontovich equation (1.35) gives the Vlasov–Landau equation:

$$
\frac{\partial f_\alpha}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f_\alpha + \frac{q_\alpha}{m_\alpha} \left(\boldsymbol{E} + \frac{\boldsymbol{v} \times \boldsymbol{B}}{c} \right) \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{v}}
$$
$$
= -\frac{q_\alpha}{m_\alpha} \left\langle \left(\delta\boldsymbol{E} + \frac{\boldsymbol{v} \times \delta\boldsymbol{B}}{c} \right) \cdot \frac{\partial \delta F_\alpha}{\partial \boldsymbol{v}} \right\rangle \equiv \left(\frac{\partial f_\alpha}{\partial t} \right)_{\mathrm{c}}. \tag{1.41}
$$

The macroscopic fields in the left-hand side satisfy the macroscopic Maxwell's equations (1.23–1.26). The microscopic fluctuating fields $\delta\boldsymbol{E}$ and $\delta\boldsymbol{B}$ inside the average in the right-hand side satisfy microscopic Maxwell's equations with fluctuating charge and current densities expressed in terms of δF_α. Thus, the right-hand side is quadratic in δF_α. In order to close this equation, we need an expression for the correlation function $\langle\delta F_\alpha\delta F_{\alpha'}\rangle$ in terms of f_α and $f_{\alpha'}$. This is basically what the BBGKY procedure plus truncation of velocity integrals based on an expansion in $1/n\lambda_{\mathrm{D}}^3$ achieve. The result is the Landau collision operator (or the more precise Lenard–Balescu one; see §§8.6.2 and 8.6.3).

Further details are a bit complicated (see Klimontovich 1967), but my aim here was just to show how the fields are split into macroscopic and microscopic ones, with the former appearing explicitly in the kinetic equation and the latter wrapped up inside the collision operator. The presence of the macroscopic fields and the consequent necessity for coupling the kinetic equation with Maxwell's equations for these fields is the main mathematical difference between the kinetics of neutral gases and the kinetics of plasmas.

1.8. *Collision Operator*

Finally, a few words about the plasma collision operator (or "collision integral"), first derived explicitly by Landau (1936) (the same considerations apply to the the more general operator due to Lenard 1960 and Balescu 1960). It describes two-particle collisions both within the species α and with other species α' and so depends both on f_α and on all other $f_{\alpha'}$. Its derivation is left to you as an exercise in BBGKY'ing, calculating cross sections and velocity integrals (or in googling; shortcut: see Parra 2019*a* or Swanson 2008).[3] In these Lectures, I shall focus on collisionless aspects of plasma kinetics. Whenever a need arises for invoking the collision operator, the important things about it for us will be its properties:

- *conservation of particles* (within each species α),

$$\int \mathrm{d}^3\boldsymbol{v}\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} = 0; \tag{1.42}$$

- *conservation of momentum,*

$$\sum_\alpha \int \mathrm{d}^3\boldsymbol{v}\, m_\alpha\boldsymbol{v}\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} = 0 \tag{1.43}$$

(same-species collisions conserve momentum, whereas different-species collisions conserve it only after summation over species—there is friction of one species against another; e.g., the friction of electrons against the ions is the Ohmic resistivity of the plasma, known as "Spitzer resistivity": see Parra 2019*a* or Helander & Sigmar 2005);

- *conservation of energy,*

$$\sum_\alpha \int \mathrm{d}^3\boldsymbol{v}\, \frac{m_\alpha v^2}{2}\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} = 0; \tag{1.44}$$

- Boltzmann's *H-theorem:* the kinetic entropy

$$S = -\sum_\alpha \int \mathrm{d}^3\boldsymbol{r} \int \mathrm{d}^3\boldsymbol{v}\, f_\alpha \ln f_\alpha \tag{1.45}$$

cannot decrease, and, as S is conserved by all the collisionless terms in (1.30), the collision

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[3]A somewhat unorthodox derivation of both the Lenard–Balescu and Landau operators will be given in §§8.6.2 and 8.6.3, respectively, as a by-product of a discussion of collisionless (*sic*) relaxation.

operator must have the property that

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -\sum_\alpha \int \mathrm{d}^3\boldsymbol{r} \int \mathrm{d}^3\boldsymbol{v} \left(\frac{\partial f_\alpha}{\partial t}\right)_\mathrm{c} \ln f_\alpha \geqslant 0, \tag{1.46}$$

with equality obtained if and only if f_α is a local Maxwellian;

• unlike the Boltzmann operator for neutral gases, the Landau operator expresses the cumulative effect of many glancing (rather than "head-on") collisions (due to the long-range nature of the Coulomb interaction) and so it is a Fokker–Planck operator:[4]

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_\mathrm{c} = \frac{\partial}{\partial\boldsymbol{v}} \cdot \sum_{\alpha'} \left(\boldsymbol{A}_{\alpha\alpha'}[f_{\alpha'}] + \mathsf{D}_{\alpha\alpha'}[f_{\alpha'}] \cdot \frac{\partial}{\partial\boldsymbol{v}}\right) f_\alpha, \tag{1.47}$$

where the drag $\boldsymbol{A}_{\alpha\alpha'}$ (vector) and diffusion $\mathsf{D}_{\alpha\alpha'}$ (matrix) coefficients are integral (in $\boldsymbol{v}$ space) functionals of $f_{\alpha'}$. The Fokker–Planck form (1.47) of the Landau operator means that it describes diffusion in velocity space and so will erase sharp gradients in f_α with respect to $\boldsymbol{v}$—a property that we will find very important in §5.

1.9. *So What's New and What Now?*

Let me summarise the new features that have appeared in the kinetic description of a plasma compared to that of a neutral gas.

• First, particles are charged, so they interact via Coulomb potential. The collision operator is, therefore, different: the cross-section is the Rutherford cross-section, most collisions are glancing (with interaction on distances up to the Debye length), leading to diffusion of the particle distribution function in velocity space. Mathematically, this is manifested in the collision operator in (1.30) having the Fokker–Planck structure (1.47).

One can spin out of the Vlasov–Landau equation (1.30) a theory that is analogous to what is done with Boltzmann's equation in gas kinetics (Dellar 2015): derive *fluid equations*, calculate viscosity, thermal conductivity, Ohmic resistivity, etc., of a collisionally dominated plasma, i.e., of a plasma in which the collision frequency of the particles is much greater than all other relevant time scales. This is done in the same way as in gas kinetics, but now applying the Chapman–Enskog procedure to the Landau collision operator. This is quite a lot of work—and constitutes core textbook material (see Parra 2019*a*). In magnetised plasmas especially, the resulting fluid dynamics of the plasma are quite interesting and quite different from neutral fluids—we shall see some of this in Part II of these Lectures, while the classic treatment of the transport theory can be found in Braginskii (1965); a great textbook on collisional transport is Helander & Sigmar (2005) (see Krommes 2018 for a modernist approach).

• Secondly, Coulomb potential is long-range, so the electric and magnetic fields have a macroscopic (mean) part on scales longer than the Debye length—a particle experiencing these fields is not undergoing a collision in the sense of bouncing off another particle, but is, rather, interacting, via the fields, with the collective of all the other particles. Mathematically, this manifests itself as a Lorentz-force term appearing in the right-hand side of the Vlasov–Landau kinetic equation (1.30). The macroscopic $\boldsymbol{E}$ and $\boldsymbol{B}$ fields that

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[4]The simplest example that I can think of in which the collision operator is a velocity-space diffusion operator of this kind is the gas of Brownian particles [each with velocity described by Langevin's equation (10.61)]. This is treated in detail in §6.9 of Schekochihin (2019). In these Lectures, the general Fokker–Planck form (1.47) emerges in (8.61); the Landau operator is (8.76).

figure in it are determined by the particles via their mean charge and current densities that enter the macroscopic Maxwell's equations (1.23–1.26).

In the case of neutral gas, all the interesting kinetic physics is in the collision operator, hence the focus on transport theory in gas-kinetic literature (see, e.g., the classic monograph by Chapman & Cowling 1991 if you want an overdose of this). In the collisionless limit, the kinetic equation for a neutral gas,

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f = 0, \tag{1.48}$$

simply describes particles with some initial distribution individualistically flying in straight lines along their initial directions of travel. In contrast, for a plasma, even the *collisionless kinetics* (and, indeed, especially the collisionless—or *weakly collisional*—kinetics) is interesting and nontrivial because the particles, via the average properties of their distribution—charge densities and currents,—collectively modify $\boldsymbol{E}$ and $\boldsymbol{B}$, which then act on individual particles and thus modify f_α, etc. This "plasma democracy" is a whole new conceptual world and it is on the effects involving interactions between particles and fields that I shall focus here, in pursuit of maximum novelty.[5]

I shall also be in pursuit of maximum simplicity (well, to use Einstein's dictum, "as simple as possible, but not simpler"!) and so will mostly restrict my considerations to the *"electrostatic approximation"*:

$$\boldsymbol{B} = 0, \quad \boldsymbol{E} = -\boldsymbol{\nabla}\varphi. \tag{1.49}$$

This, of course, eliminates a huge number of interesting and important phenomena without which plasma physics would not be the voluminous subject that it is, but I cannot do them justice in just a few lectures (so see Parra 2019*b* for a course largely devoted to collisionless magnetised plasmas). Some opportunities for generalising electrostatic theory to electromagnetic one will be provided in Q2 and Q3.

Thus, we shall henceforth consider a simplified kinetic system, called the *Vlasov–Poisson system*:

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f_\alpha - \frac{q_\alpha}{m_\alpha}(\boldsymbol{\nabla}\varphi) \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{v}} = 0, \tag{1.50}$$

$$-\nabla^2 \varphi = 4\pi \sum_\alpha q_\alpha \int \mathrm{d}^3\boldsymbol{v}\, f_\alpha. \tag{1.51}$$

Formally, considering a collisionless plasma[6] would appear to be legitimate as long as the collision frequency is small compared to the characteristic frequencies of any other evolution that might be going on. What are the characteristic time scales (and length scales) in a plasma and what phenomena occur on these scales? These questions bring us to our next theme.

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[5]Similarly interesting things happen when the field tying the particles together is gravity—an even more complicated situation because, while the potential is long-range, rather like the Coulomb potential, gravity is not shielded and so all particles feel each other at all distances. This gives rise to remarkably interesting theory (Binney 2016; Fouvry 2019).

[6]Or, I stress again, a weakly collisional plasma. The collision operator is dropped in (1.50), but let us not forget about it entirely even if the collision frequency is small; it will make a come back in §5.
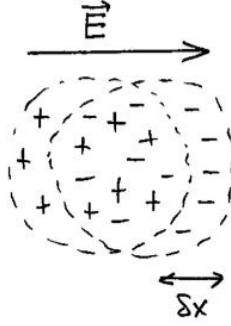
FIGURE 2. A displaced population of electrons will set up a quasineutrality-restoring electric field, leading to plasma oscillations.

2. Equilibrium and Fluctuations

2.1. *Plasma Frequency*

Consider a plasma in equilibrium, in a happy quasineutral state. Suppose a population of electrons strays from this equilibrium and upsets quasineutrality a bit (Fig. 2). If they have shifted by distance δx, the restoring force on each electron will be

$$m_e \delta \ddot{x} = -eE = -4\pi e^2 n_e \delta x \quad \Rightarrow \quad \delta \ddot{x} = -\underbrace{\frac{4\pi e^2 n_e}{m_e}}_{\equiv\, \omega_{\text{pe}}^2} \delta x, \tag{2.1}$$

so there will be oscillations at what is known as the (electron) *plasma frequency*:

$$\boxed{\omega_{\text{pe}} = \sqrt{\frac{4\pi e^2 n_e}{m_e}}}. \tag{2.2}$$

Thus, we expect *fluctuations* of electric field in a plasma with characteristic frequencies $\omega \sim \omega_{\text{pe}}$ (these are Langmuir waves; I will derive their dispersion relation formally in §3.4). These fluctuations are due to *collective* motions of the particles—so they are still *macroscopic* fields in the nomenclature of §1.4.

The time scale associated with ω_{pe} is the scale of restoration of quasineutrality. The distance an electron can travel over this time scale before the restoring force kicks in, i.e., the distance over which quasineutrality can be violated, is (using the *thermal speed* $v_{\text{the}} \sim \sqrt{T/m_e}$ to estimate the electron's velocity)

$$\frac{v_{\text{the}}}{\omega_{\text{pe}}} \sim \sqrt{\frac{T}{m_e}} \sqrt{\frac{m_e}{e^2 n_e}} = \sqrt{\frac{T}{e^2 n_e}} \sim \lambda_{\text{D}}, \tag{2.3}$$

the Debye length (1.6)—not surprising, as this is, indeed, the scale on which microscopic fields are shielded and plasma is quasineutral (§1.3).

Finally, let us check that the plasma oscillations happen on collisionless time scales. The collision frequency of the electrons is, using (2.3) and (1.14),

$$\nu_e \sim \frac{v_{\text{the}}}{\lambda_{\text{mfp}}} = \frac{v_{\text{the}}}{\omega_{\text{pe}}} \frac{\omega_{\text{pe}}}{\lambda_{\text{mfp}}} \sim \frac{\lambda_{\text{D}}}{\lambda_{\text{mfp}}} \omega_{\text{pe}} \ll \omega_{\text{pe}}, \quad \text{q.e.d.} \tag{2.4}$$

2.2. *Slow vs. Fast*

The plasma frequency ω_{pe} is only one of the characteristic frequencies (the largest) of the fluctuations that can occur in plasmas. We will think of the scales of all these

fluctuations as short and of the associated variation in time and space as *fast*. They occur against the background of some *equilibrium state*,[7] which is either constant or varies *slowly* in time and space. The slow evolution and spatial variation of the equilibrium state can be due to slowly changing, large-scale external conditions that gave rise to this state or, as we will discover soon, it can be due to the average effect of a sea of small fluctuations.

Formally, what we are embarking on is an attempt to set up a mean-field theory, separating slow (large-scale) and fast (small-scale) parts of the distribution function:

$$f(\boldsymbol{r}, \boldsymbol{v}, t) = f_0(\epsilon^a \boldsymbol{r}, \boldsymbol{v}, \epsilon t) + \delta f(\boldsymbol{r}, \boldsymbol{v}, t), \qquad (2.5)$$

where ϵ is some small parameter characterising the scale separation between fast and slow variation (note that this separation need not be the same for spatial and time scales, hence ϵ^a). To avoid clutter, I shall drop the species index where this does not lead to ambiguity.

For simplicity, I will abolish the spatial dependence of the equilibrium distribution altogether and consider *homogeneous* systems:

$$f_0 = f_0(\boldsymbol{v}, \epsilon t), \qquad (2.6)$$

which also means $\boldsymbol{E}_0 = 0$ (there is no equilibrium electric field). Equivalently, all our considerations are restricted to scales much smaller than the characteristic system size. Formally, this equilibrium distribution can be defined as the average of the exact distribution over the volume of space that we are considering and over time scales intermediate between the fast and the slow ones:[8]

$$f_0(\boldsymbol{v}, t) = \langle f(\boldsymbol{r}, \boldsymbol{v}, t) \rangle \equiv \frac{1}{\Delta t} \int_{t - \Delta t/2}^{t + \Delta t/2} \mathrm{d}t' \int \frac{\mathrm{d}^3 \boldsymbol{r}}{V} f(\boldsymbol{r}, \boldsymbol{v}, t'), \qquad (2.7)$$

where $\omega^{-1} \ll \Delta t \ll t_{\mathrm{eq}}$, where t_{eq} is the equilibrium time scale.

2.3. *Multiscale Dynamics*

It is convenient to work in Fourier space:

$$\varphi(\boldsymbol{r}, t) = \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \varphi_{\boldsymbol{k}}(t), \quad f(\boldsymbol{r}, \boldsymbol{v}, t) = f_0(\boldsymbol{v}, t) + \sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \delta f_{\boldsymbol{k}}(\boldsymbol{v}, t). \qquad (2.8)$$

Then the Poisson equation (1.51) becomes

$$\varphi_{\boldsymbol{k}} = \frac{4\pi}{k^2} \sum_{\alpha} q_\alpha \int \mathrm{d}^3 \boldsymbol{v} \, \delta f_{\boldsymbol{k}\alpha} \qquad (2.9)$$

and the Vlasov equation (1.50) written for $\boldsymbol{k} = 0$ (i.e., the spatial average of the equation) is

$$\frac{\partial f_0}{\partial t} + \frac{\partial \delta f_{\boldsymbol{k}=0}}{\partial t} = -\frac{q}{m} \sum_{\boldsymbol{k}} \varphi_{-\boldsymbol{k}} i\boldsymbol{k} \cdot \frac{\partial \delta f_{\boldsymbol{k}}}{\partial \boldsymbol{v}}, \qquad (2.10)$$

---

[7]Or even just an initial state that is slow to change.

[8]I use anglular brackets to denote this average, but it should be clear that this is not the same thing as the average (1.11) that separated the macroscopic fields from the microscopic ones. The latter average was over sub-Debye scales, whereas the new average (2.7) is over scales that are larger than fluctuation scales but smaller than the system size; both fluctuations and equilibrium are "macroscopic" in the language of §1.4.

where we can replace $\varphi_{-\mathbf{k}} = \varphi_{\mathbf{k}}^*$ because $\varphi(\mathbf{r}, t)$ is a real field. Averaging over time according to (2.7) eliminates fast variation and gives

$$\boxed{\frac{\partial f_0}{\partial t} = -\frac{q}{m} \sum_{\mathbf{k}} \left\langle \varphi_{\mathbf{k}}^* i\mathbf{k} \cdot \frac{\partial \delta f_{\mathbf{k}}}{\partial \mathbf{v}} \right\rangle}. \tag{2.11}$$

The right-hand side of (2.11) describes the slow evolution of the equilibrium (mean) distribution due to the effect of fluctuations (see §§7 and 8.6). In practice, the main question is often how the equilibrium evolves and so we need a closed equation for the evolution of f_0. This should be obtainable at least in principle because the fluctuating fields appearing in the right-hand side of (2.11) themselves depend on f_0: indeed, writing the Vlasov equation (1.50) for the $\mathbf{k} \neq 0$ modes, we find the following evolution equation for the fluctuations:

$$\boxed{\frac{\partial \delta f_{\mathbf{k}}}{\partial t} + \underbrace{i\mathbf{k} \cdot \mathbf{v}\, \delta f_{\mathbf{k}}}_{\substack{\text{particle} \\ \text{streaming} \\ \text{(phase} \\ \text{mixing)}}} = \underbrace{\frac{q}{m}\, \varphi_{\mathbf{k}} i\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}}_{\substack{\text{wave-particle} \\ \text{interaction} \\ \text{(linear)}}} + \underbrace{\frac{q}{m} \sum_{\mathbf{k}'} \varphi_{\mathbf{k}'} i\mathbf{k}' \cdot \frac{\partial \delta f_{\mathbf{k}-\mathbf{k}'}}{\partial \mathbf{v}}}_{\substack{\text{nonlinear} \\ \text{interactions}}}}. \tag{2.12}$$

The three terms that control the evolution of the perturbed distribution function in (2.12) represent the three physical effects that I shall focus on in these Lectures. The second term on the left-hand side describes the free ballistic motion of particles ("streaming"). It gives rise to the phenomenon of phase mixing (§5) and, in its interplay with plasma waves, to Landau damping and kinetic instabilities (§3). The first term on the right-hand side contains the interaction of the electric-field perturbations (waves) with the equilibrium particle distribution (§3). The second term on the right-hand side captures the nonlinear interactions between the fluctuating fields and the perturbed distribution—it is negligible only when fluctuation amplitudes are small enough (which, sadly, they rarely are) and is responsible for *plasma turbulence* (§§9.2 and 10) and other nonlinear phenomena (§6).

The programme for determining the slow evolution of the equilibrium is "simple": solve (2.12) together with (2.9), calculate the *correlation function* of the fluctuations, $\langle \varphi_{\mathbf{k}}^* \delta f_{\mathbf{k}} \rangle$, as a functional of f_0, and use it to close (2.11); then proceed to solve the latter. Obviously, this is impossible to do in most cases. But it is possible to construct a hierarchy of approximations to the answer and learn much interesting physics in the process.

2.4. *Hierarchy of Approximations*

2.4.1. *Linear Theory*

Consider first infinitesimal perturbations of the equilibrium. All nonlinear terms can then be ignored, (2.11) turns into $f_0 = $ const and (2.12) becomes

$$\frac{\partial \delta f_{\mathbf{k}}}{\partial t} + i\mathbf{k} \cdot \mathbf{v}\, \delta f_{\mathbf{k}} = \frac{q}{m}\, \varphi_{\mathbf{k}} i\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}}, \tag{2.13}$$

the *linearised kinetic equation.* Solving this together with (2.9) allows one to find oscillating and/or growing/decaying[9] perturbations of a particular equilibrium f_0. The theory for doing this is very well developed and contains some of the core ideas that give plasma physics its intellectual shape (§3).

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[9] We shall see (§5) that growing/decaying linear solutions imply the equilibrium distribution giving/receiving energy to/from the fluctuations.

Physically, the linear solutions will describe what happens over short term, viz., on times t such that

$$\omega^{-1} \ll t \ll t_{\text{eq}} \text{ or } t_{\text{nl}}, \tag{2.14}$$

where ω is the characteristic frequency of the perturbations, t_{eq} is the time after which the equilibrium starts getting modified by the perturbations via (2.11) [which depends on the amplitude to which they can grow; if perturbations do grow, i.e., the equilibrium is unstable, they can modify the equilibrium by this mechanism so as to render it stable], and t_{nl} is the time at which perturbation amplitudes become large enough for nonlinear interactions between individual modes to matter [second term on the right-hand side of (2.12); if perturbations grow, they can saturate by this mechanism].

2.4.2. *Quasilinear Theory (QLT)*

Suppose

$$t_{\text{eq}} \ll t_{\text{nl}}, \tag{2.15}$$

i.e., *growing perturbations start modifying the equilibrium before they saturate nonlinearly.* Then the strategy is to solve (2.13) [together with (2.9)] for the perturbations, use the result to calculate their correlation function needed in the right-hand side of (2.11), then work out how the equilibrium therefore evolves and hence how large the perturbations must grow in order for this evolution to turn the unstable equilibrium into a stable one. This is a classic piece of theory, important conceptually—I will describe it in detail and do one example in §7. In reality, however, it happens relatively rarely that unstable perturbations saturate at amplitudes small enough for the nonlinear interactions not to matter [i.e., for (2.15) to hold true].

2.4.3. *Weak-Turbulence (WT) Theory*

Sometimes, one is not lucky enough to get away with QLT (i.e., alas, $t_{\text{nl}} \ll t_{\text{eq}}$ or $t_{\text{nl}} \sim t_{\text{eq}}$), but is lucky enough to have perturbations saturating nonlinearly at a *small amplitude* such that[10]

$$t_{\text{nl}} \gg \omega^{-1}, \tag{2.16}$$

i.e., *perturbations oscillate linearly faster than they interact nonlinearly* (this can happen, e.g., because propagating wave packets do not stay together long enough to break up completely in one encounter). Because waves are fast compared to nonlinear evolution in this approximation, it is possible to "quantise" them, i.e., to treat a nonlinear turbulent state of the plasma as a cocktail consisting of both "true" particles (ions and electrons) and "quasiparticles" representing electromagnetic excitations (§9).

In this approximation, one can do perturbation theory treating the nonlinear term in (2.12) as small and expanding in the small parameter $(\omega t_{\text{nl}})^{-1}$. The resulting *weak (or "wave") turbulence theory* is quite an analytical *tour de force*—but it is a lot of work to do it properly! I will provide an introduction to WT in §9.2. Classic texts on this are Kadomtsev (1965) (early but lucid) and Zakharov *et al.* (1992) (mathematically definitive); a recent textbook in Zakharov's tradition is Nazarenko (2011), while the quasiparticle approach (with Feynman diagrams and all that) can be learned from Tsytovich (1995) or Kingsep (2004). Specifically on weak turbulence of Langmuir waves, there is a long, mushy review by Musher *et al.* (1995); my attempt of tackling this subject is §10.5.

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[10]Note that the nonlinear time scale is typically inversely proportional to the amplitude; see (2.12).

(a) (b)

FIGURE 3. Lev Landau (1908-1968), great Soviet physicist, quintessential theoretician, author of the Book, cult figure. It is a minor feature of his scientific biography that he wrote the two most important plasma-physics papers of all time (Landau 1936, 1946). He also got a Nobel Prize (1962), but not for plasma physics. (a) Cartoon by A. A. Yuzefovich (from Landau & Lifshitz 1976); the caption says "[And] Dau spake..." (...unto the students, also depicted). (b) Landau's mugshot from NKVD prison (1938), where he ended up for seditious talk and from whence he was released in 1939 after Peter Kapitsa's personal appeal to Stalin.

Note that because the nonlinear term couples perturbations at different $\boldsymbol{k}$'s (scales), this theory will lead to multi-scale (usually, power-law) fluctuation spectra.

2.4.4. *Strong-Turbulence Theory*

If perturbations manage to grow to a level at which

$$t_{\mathrm{nl}} \sim \omega^{-1}, \tag{2.17}$$

we are a facing *strong turbulence*. This is actually what mostly happens (including in WT systems, where turbulence often transitions into the strong regime at small enough, or large enough, scales). Theory of such regimes tends to be of phenomenological/scaling kind, often in the spirit of the classic Kolmogorov (1941) theory of hydrodynamic turbulence.[11] Here are two examples, not necessarily the best or most relevant, just mine: Schekochihin *et al.* (2009, 2016). No one really knows how to move very far beyond this sort of approach—and not for lack of trying (a recent but historically aware review is Krommes 2015).

3. Linear Theory: Waves, Landau Damping and Kinetic Instablities

Enough idle chatter, let us calculate! In this section, we are concerned with the linearised Vlasov–Poisson system, (2.13) and (2.9):

$$\frac{\partial \delta f_{\boldsymbol{k}\alpha}}{\partial t} + i\boldsymbol{k} \cdot \boldsymbol{v}\, \delta f_{\boldsymbol{k}\alpha} = \frac{q_\alpha}{m_\alpha}\, \varphi_{\boldsymbol{k}} i\boldsymbol{k} \cdot \frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}, \tag{3.1}$$

$$\varphi_{\boldsymbol{k}} = \frac{4\pi}{k^2} \sum_\alpha q_\alpha \int \mathrm{d}^3\boldsymbol{v}\, \delta f_{\boldsymbol{k}\alpha}. \tag{3.2}$$

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[11] A kind of exception is a very special case of strong Langmuir turbulence, which was extremely popular in the 1970s and 80s. The founding documents on this are Zakharov (1972) and Kingsep *et al.* (1973), but there is a huge and sophisticated literature that followed. There is, alas, no particularly good review, but see Thornhill & ter Haar (1978), Rudakov & Tsytovich (1978), Goldman (1984), Zakharov *et al.* (1985) and Robinson (1997) (I find the first of these the most readable of the lot). I will give an introduction to this topic in §10.
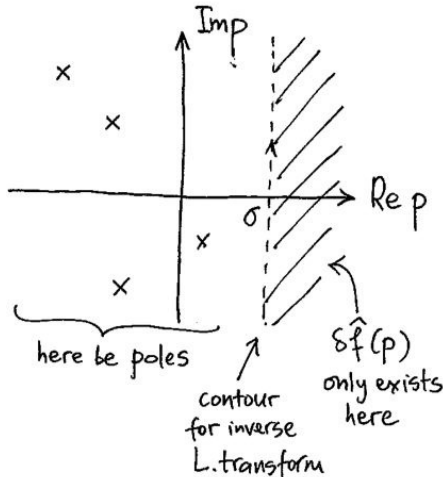
FIGURE 4. Layout of the complex-p plane: $\delta\hat{f}(p)$ is analytic for $\operatorname{Re} p \geqslant \sigma$. At $\operatorname{Re} p < \sigma$, $\delta\hat{f}(p)$ may have singularities (poles).

For compactness of notation, I will drop both the species index α and the wave number $\boldsymbol{k}$ in the subscripts, unless they are necessary for understanding.

We will discover that electrostatic perturbations in a plasma described by (3.1) and (3.2) oscillate, can pass their energy to particles (damp) or even grow, sucking energy from the particles. We will also discover that it is useful to know some complex analysis.

3.1. *Initial-Value Problem*

We shall follow Landau's original paper (Landau 1946) in considering an initial-value problem—because, as we will see, perturbations can be damped or grow, so it is not appropriate to think of them over $t \in [-\infty, +\infty]$ (and—NB!!!—the damped perturbations are *not* pure eigenmodes; see §5.3). So we look for $\delta f(\boldsymbol{v}, t)$ satisfying (3.1) with the initial condition

$$\delta f(\boldsymbol{v}, t = 0) = g(\boldsymbol{v}). \tag{3.3}$$

It is, therefore, appropriate to use the *Laplace transform* to solve (3.1):

$$\boxed{\delta\hat{f}(p) = \int_0^\infty \mathrm{d}t\, e^{-pt} \delta f(t)}. \tag{3.4}$$

It is a mathematical certainty that if there exists a real number $\sigma > 0$ such that

$$|\delta f(t)| < e^{\sigma t} \text{ as } t \to \infty, \tag{3.5}$$

then the integral (3.4) exists (i.e., is finite) for all values of p such that $\operatorname{Re} p \geqslant \sigma$. The inverse Laplace transform, giving us back our distribution function as a function of time, is then

$$\delta f(t) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} \mathrm{d}p\, e^{pt} \delta\hat{f}(p), \tag{3.6}$$

where the integral in the complex plane is along a straight line parallel to the imaginary axis and intersecting the real axis at $\operatorname{Re} p = \sigma$ (Fig. 4).

Since we expect to be able to recover our desired time-dependent function $\delta f(\boldsymbol{v}, t)$ from its Laplace transform, it is worth knowing the latter. To find it, we Laplace-

transform (3.1):

$$\text{l.h.s.} = \int_0^\infty \mathrm{d}t\, e^{-pt} \frac{\partial \delta f}{\partial t} = \left[e^{-pt}\delta f\right]_0^\infty + p\int_0^\infty \mathrm{d}t\, e^{-pt}\delta f = -g + p\,\delta\hat{f},$$

$$\text{r.h.s.} = -i\boldsymbol{k}\cdot\boldsymbol{v}\,\delta\hat{f} + \frac{q}{m}\,\hat{\varphi}\,i\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}}. \tag{3.7}$$

Equating these two expressions, we find the solution:

$$\boxed{\delta\hat{f}(p) = \frac{1}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}\left[i\,\frac{q}{m}\,\hat{\varphi}(p)\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}} + g\right].} \tag{3.8}$$

The Laplace transform of the potential, $\hat{\varphi}(p)$, itself depends on $\delta\hat{f}$ via (3.2):

$$\hat{\varphi}(p) = \int_0^\infty \mathrm{d}t\, e^{-pt}\,\varphi(t) = \frac{4\pi}{k^2}\sum_\alpha q_\alpha \int \mathrm{d}^3\boldsymbol{v}\,\delta\hat{f}_\alpha(p)$$

$$= \frac{4\pi}{k^2}\sum_\alpha q_\alpha \int \mathrm{d}^3\boldsymbol{v}\,\frac{1}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}\left[i\,\frac{q_\alpha}{m_\alpha}\,\hat{\varphi}(p)\,\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}} + g_\alpha\right]. \tag{3.9}$$

This is an algebraic equation for $\hat{\varphi}(p)$. Collecting terms, we get

$$\underbrace{\left[1 - \sum_\alpha \frac{4\pi q_\alpha^2}{k^2 m_\alpha}\,i\int \mathrm{d}^3\boldsymbol{v}\,\frac{1}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}}\right]}_{\equiv\,\epsilon(p,\boldsymbol{k})}\hat{\varphi}(p) = \frac{4\pi}{k^2}\sum_\alpha q_\alpha \int \mathrm{d}^3\boldsymbol{v}\,\frac{g_\alpha}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}. \tag{3.10}$$

The prefactor in the left-hand side, which I denote $\epsilon(p,\boldsymbol{k})$, is called the *dielectric function*, because it encodes all the self-consistent charge-density perturbations that plasma sets up in response to an electric field. This is going to be an important function, so let us write it out beautifully:

$$\boxed{\epsilon(p,\boldsymbol{k}) = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\,\frac{i}{n_\alpha}\int \mathrm{d}^3\boldsymbol{v}\,\frac{1}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial\boldsymbol{v}},} \tag{3.11}$$

where the plasma frequency of species α is defined by [cf. (2.2)]

$$\omega_{\mathrm{p}\alpha}^2 = \frac{4\pi q_\alpha^2 n_\alpha}{m_\alpha}. \tag{3.12}$$

The solution of (3.10) is

$$\hat{\varphi}(p) = \frac{4\pi}{k^2\epsilon(p,\boldsymbol{k})}\sum_\alpha q_\alpha \int \mathrm{d}^3\boldsymbol{v}\,\frac{g_\alpha}{p+i\boldsymbol{k}\cdot\boldsymbol{v}}. \tag{3.13}$$

To calculate $\varphi(t)$, we need to inverse-Laplace-transform $\hat{\varphi}$: similarly to (3.6),

$$\varphi(t) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma} \mathrm{d}p\, e^{pt}\hat{\varphi}(p). \tag{3.14}$$

How do we do this integral? Recall that $\delta\hat{f}$ and, therefore, $\hat{\varphi}$ only exists (i.e., is finite) for $\mathrm{Re}\,p \geqslant \sigma$, whereas at $\mathrm{Re}\,p < \sigma$, it can have singularities, i.e., poles—let us call them p_i, indexed by i. If we analytically continue $\hat{\varphi}(p)$ everywhere to $\mathrm{Re}\,p < \sigma$ except those poles, the result must have the form

$$\hat{\varphi}(p) = \sum_i \frac{c_i}{p-p_i} + A(p), \tag{3.15}$$
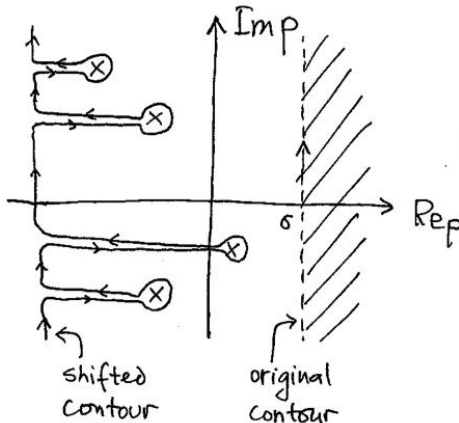
FIGURE 5. New contour for the inverse Laplace transform.

where c_i are some coefficients (residues) and $A(p)$ is the analytic part of the solution. The integration contour in (3.14) can be shifted to $\operatorname{Re} p \to -\infty$ but with the proviso that it cannot cross the poles, as shown in Fig. 5 (this is proven by making a closed loop out of the old and the new contours, joining them at $\pm i\infty$, and noting that this loop encloses no poles). Then the contributions to the integral from the vertical segments of the contour are exponentially small,[12] the contributions from the segments leading towards and away from the poles cancel, and the contributions from the circles around the poles can, by Cauchy's formula, be expressed in terms of the poles and residues:

$$\boxed{\varphi(t) = \sum_i c_i e^{p_i t}} . \tag{3.16}$$

Thus, in the long-time limit, *perturbations of the potential will evolve* $\propto e^{p_i t}$, *where p_i are poles of $\hat\varphi(p)$.* In general, $p_i = -i\omega_i + \gamma_i$, where ω_i is a real frequency (giving wave-like behaviour of perturbations), $\gamma_i < 0$ represents damping and $\gamma_i > 0$ growth of the perturbations (instability).

Note that we need not be particularly interested in what c_i's are because, if we set up an initial perturbation with a given $\boldsymbol{k}$ and then wait long enough, only the fastest-growing or, failing growth, the slowest-damped mode will survive, with all others having exponentially small amplitudes. Thus, a typical outcome of the linear theory is $\varphi(t)$ oscillating at some frequency and growing or decaying at some unique rate. Since this is a solution of a linear equation, the prefactor in front of the exponential can be scaled arbitrarily and so does not matter.

Going back to (3.13), we realise that the poles of $\hat\varphi(p)$ are zeros of the dielectric function:

$$\boxed{\epsilon(p_i, \boldsymbol{k}) = 0} \quad \Rightarrow \quad p_i = p_i(\boldsymbol{k}) = -i\omega_i(\boldsymbol{k}) + \gamma_i(\boldsymbol{k}). \tag{3.17}$$

---

[12]They are exponentially small *in time* as $t \to \infty$ because the integrand of the inverse Laplace transform (3.14) contains a factor of $e^{\operatorname{Re} p t}$, which decays faster than any of the "modes" in (3.16). If $\hat\varphi(p)$ does not grow too fast at large p, the integral along the vertical part of the contour may also vanish at any finite t, but that is not guaranteed in general: indeed, looking ahead to the explicit expression (3.27) for $\hat\varphi(p)$, with the Landau prescription for analytic continuation to $\operatorname{Re} p < 0$ analogous to (3.20), we see that $\hat\varphi(p)$ will contain a term $\propto G_\alpha(ip/k)$, which can be large at large $\operatorname{Re} p$, e.g., if $G_\alpha(v_z)$ is a Maxwellian.

(a) $\operatorname{Re} p > 0$ (b) $\operatorname{Re} p = 0$ (c) $\operatorname{Re} p < 0$
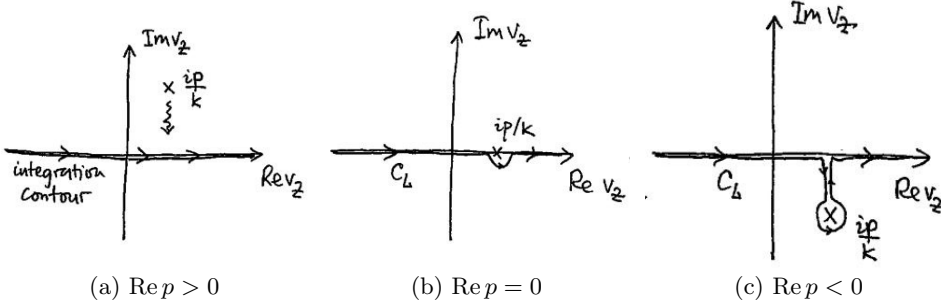
FIGURE 6. The Landau prescription for the contour of integration in (3.18).

To find the wave frequencies ω_i and the damping/growth rates γ_i, we must solve this equation, which is called the *plasma dispersion relation*.

3.2. *Calculating the Dielectric Function: the "Landau Prescription"*

In order to be able to solve $\epsilon(p, \boldsymbol{k}) = 0$, we must learn how to calculate $\epsilon(p, \boldsymbol{k})$ for any given p and $\boldsymbol{k}$. Before I wrote (3.15), I said that $\hat{\varphi}$, given by (3.13), had to be analytically continued to the entire complex plane from the area where its analyticity was guaranteed ($\operatorname{Re} p \geqslant \sigma$), but I did not explain how this was to be done. In order to do it, we must learn how to calculate the velocity integral in (3.11)—if we want $\epsilon(p, \boldsymbol{k})$ and, therefore, its zeros p_i—and also how to calculate the similar integral in (3.13) containing g_α.

First of all, let us turn these integrals into a 1D form. Given $\boldsymbol{k}$, we can always choose the z axis to be along $\boldsymbol{k}$.[13] Then

$$\int \mathrm{d}^3 \boldsymbol{v} \, \frac{1}{p + i\boldsymbol{k} \cdot \boldsymbol{v}} \, \boldsymbol{k} \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} = \int \mathrm{d} v_z \, \frac{1}{p + ikv_z} \, k \frac{\partial}{\partial v_z} \underbrace{\int \mathrm{d} v_x \int \mathrm{d} v_y \, f_0(v_x, v_y, v_z)}_{\equiv F(v_z)}$$

$$= -i \int_{-\infty}^{+\infty} \mathrm{d} v_z \, \frac{F'(v_z)}{v_z - ip/k}. \tag{3.18}$$

Assuming, reasonably, that $F'(v_z)$ is a nice (analytic) function everywhere, we conclude that the integrand in (3.18) has one pole, $v_z = ip/k$. When $\operatorname{Re} p \geqslant \sigma > 0$, this pole is harmless because, in the complex plane associated with the v_z variable, it lies above the integration contour, which is the real axis, $v_z \in (-\infty, +\infty)$. We can think of analytically continuing the above integral to $\operatorname{Re} p < \sigma$ as moving the pole $v_z = ip/k$ down, towards and below the real axis. As long as $\operatorname{Re} p > 0$, this can be done with impunity, in the sense that the pole stays above the integration contour, and so the analytic continuation is simply the same integral (3.18), still along the real axis. However, if the pole moves so far down that $\operatorname{Re} p = 0$ or $\operatorname{Re} p < 0$, we must deform the contour of integration in such a way as to keep the pole always above it, as shown in Fig. 6. This is called the *Landau prescription* and the contour thus deformed is called the *Landau contour*, C_{L}.

Let me prove that this is indeed an analytic continuation, i.e., that the integral (3.18), adjusted to be along C_{L}, is an analytic function for all values of p. Let us cut the Landau contour at $v_z = \pm R$ and close it in the upper half-plane with a semicircle C_R of radius $R > \sigma/k$ (Fig. 7). Then, with integration running along the truncated C_{L} and

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[13]NB: This means that in what follows, $k \geqslant 0$ by definition.
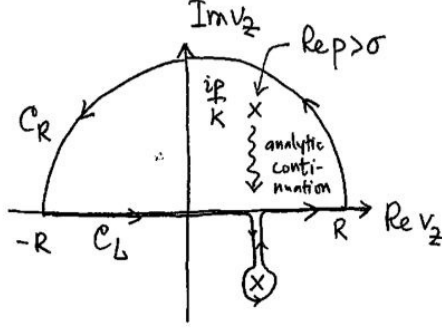
FIGURE 7. Proof of Landau's prescription [see (3.19)].

counterclockwise along C_R, we get, by Cauchy's formula,

$$\int_{C_L} dv_z \frac{F'(v_z)}{v_z - ip/k} + \int_{C_R} dv_z \frac{F'(v_z)}{v_z - ip/k} = 2\pi i\, F'\left(\frac{ip}{k}\right). \qquad (3.19)$$

Since analyticity is guaranteed for $\operatorname{Re} p \geqslant \sigma$, the integral along C_R is analytic. The right-hand side is also analytic, by assumption. Therefore, the integral along C_L is analytic—this is the integral along the Landau contour if we take $R \to \infty$. Q.e.d.

With the Landau prescription, our integral is calculated as follows:

$$\int_{C_L} dv_z \frac{F'(v_z)}{v_z - ip/k} = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - ip/k} & \text{if} \quad \operatorname{Re} p > 0, \\[2ex] \displaystyle\mathcal{P}\int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - ip/k} + i\pi F'\left(\frac{ip}{k}\right) & \text{if} \quad \operatorname{Re} p = 0, \\[2ex] \displaystyle\int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - ip/k} + i2\pi F'\left(\frac{ip}{k}\right) & \text{if} \quad \operatorname{Re} p < 0, \end{cases} \qquad (3.20)$$

where the integrals are again over the real axis and the imaginary bits come from the contour making a half (when $\operatorname{Re} p = 0$) or a full (when $\operatorname{Re} p < 0$) circle around the pole. In the case of $\operatorname{Re} p = 0$, or $ip = \omega$, the integral along the real axis is formally divergent and so we take its *principal value*, defined as

$$\mathcal{P}\int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - \omega/k} = \lim_{\varepsilon \to 0} \left[\int_{-\infty}^{\omega/k-\varepsilon} + \int_{\omega/k+\varepsilon}^{+\infty}\right] dv_z \frac{F'(v_z)}{v_z - \omega/k}. \qquad (3.21)$$

The difference between (3.21) and the usual Lebesgue definition of an integral is that the latter would be

$$\int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - \omega/k} = \left[\lim_{\varepsilon_1 \to 0} \int_{-\infty}^{\omega/k-\varepsilon_1} + \lim_{\varepsilon_2 \to 0} \int_{\omega/k+\varepsilon_2}^{+\infty}\right] dv_z \frac{F'(v_z)}{v_z - \omega/k}, \qquad (3.22)$$

and this, with, in general, $\varepsilon_1 \neq \varepsilon_2$, diverges logarithmically, whereas in (3.21), the divergences neatly cancel.

The $\operatorname{Re} p = 0$ case in (3.20),

$$\int_{C_L} dv_z \frac{F'(v_z)}{v_z - \omega/k} = \mathcal{P}\int_{-\infty}^{+\infty} dv_z \frac{F'(v_z)}{v_z - \omega/k} + i\pi F'\left(\frac{\omega}{k}\right), \qquad (3.23)$$

which tends to be of most use in analytical theory, is a particular instance of *Plemelj's formula:* for a real ζ and a well-behaved function f (no poles on or near the real axis),

$$\lim_{\varepsilon \to +0} \int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{f(x)}{x - \zeta \mp i\varepsilon} = \mathcal{P} \int_{-\infty}^{+\infty} \mathrm{d}x \, \frac{f(x)}{x - \zeta} \pm i\pi f(\zeta), \tag{3.24}$$

also sometimes written as

$$\lim_{\varepsilon \to +0} \frac{1}{x - \zeta \mp i\varepsilon} = \mathcal{P} \frac{1}{x - \zeta} \pm i\pi \delta(x - \zeta). \tag{3.25}$$

Finally, armed with Landau's prescription, we are ready to calculate. The dielectric function (3.11) becomes

$$\boxed{\epsilon(p, \boldsymbol{k}) = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2} \frac{1}{n_\alpha} \int_{C_{\mathrm{L}}} \mathrm{d}v_z \, \frac{F_\alpha'(v_z)}{v_z - ip/k}}, \tag{3.26}$$

and, analogously, our Laplace-transformed solution (3.13) becomes

$$\hat{\varphi}(p) = -\frac{4\pi i}{k^3 \epsilon(p, \boldsymbol{k})} \sum_\alpha q_\alpha \int_{C_{\mathrm{L}}} \mathrm{d}v_z \, \frac{G_\alpha(v_z)}{v_z - ip/k}, \tag{3.27}$$

where $G_\alpha(v_z) = \int \mathrm{d}v_x \int \mathrm{d}v_y \, g_\alpha(v_x, v_y, v_z)$.

3.3. *Solving the Dispersion Relation: the Limit of Slow Damping/Growth*

A particularly analytically solvable and physically interesting case is one in which, for $p = -i\omega + \gamma$, $\gamma \ll \omega$ and $\gamma \ll k v_{\mathrm{th}\alpha}$, i.e., the case of the damping or growth time of the waves being longer than either their period or the time particles take to cross them. In this limit, the dispersion relation (3.17) is

$$\epsilon(p, \boldsymbol{k}) \approx \epsilon(-i\omega, \boldsymbol{k}) + i\gamma \frac{\partial}{\partial \omega} \epsilon(-i\omega, \boldsymbol{k}) = 0. \tag{3.28}$$

Setting the real part of (3.28) to zero gives the equation for the real frequency:

$$\boxed{\operatorname{Re} \epsilon(-i\omega, \boldsymbol{k}) = 0}. \tag{3.29}$$

Setting the imaginary part of (3.28) to zero gives us the damping/growth rate in terms of the real frequency:

$$\boxed{\gamma = -\operatorname{Im} \epsilon(-i\omega, \boldsymbol{k}) \left[\frac{\partial}{\partial \omega} \operatorname{Re} \epsilon(-i\omega, \boldsymbol{k}) \right]^{-1}}. \tag{3.30}$$

Thus, we now only need $\epsilon(p, \boldsymbol{k})$ with $p = -i\omega$. Using (3.23), we get

$$\operatorname{Re} \epsilon = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2} \frac{1}{n_\alpha} \mathcal{P} \int \mathrm{d}v_z \, \frac{F_\alpha'(v_z)}{v_z - \omega/k}, \tag{3.31}$$

$$\operatorname{Im} \epsilon = -\sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2} \frac{\pi}{n_\alpha} F_\alpha'\left(\frac{\omega}{k}\right). \tag{3.32}$$

Let us consider a two-species plasma, consisting of electrons and a single species of ions. There will be two interesting limits:

- *"High-frequency"* electron waves: $\omega \gg k v_{\mathrm{the}}$, where $v_{\mathrm{the}} = \sqrt{2T_e/m_e}$ is the "thermal

speed" of the electrons;[14] this limit will give us Langmuir waves (§3.4), slowly damped or growing (§3.5).

• *"Low-frequency"* ion waves: a particularly tractable limit will be that of "hot" electrons and "cold" ions, viz., $kv_{\text{the}} \gg \omega \gg kv_{\text{thi}}$, where $v_{\text{thi}} = \sqrt{2T_i/m_i}$ is the "thermal speed" of the ions; this limit will give us the sound ("ion-acoustic waves"; §3.8), which also can be damped or growing (§3.9).

3.4. *Langmuir Waves*

Consider the limit

$$\frac{\omega}{k} \gg v_{\text{the}}, \tag{3.33}$$

i.e., the phase velocity of the waves is much greater than the typical velocity of a particle from the "thermal bulk" of the distribution. This means that in (3.31), we can expand in $v_z \sim v_{\text{the}}$ being small compared to ω/k (higher values of v_z are cut off by the "thermal" fall-off of the equilibrium distribution function). Note that $\omega \gg kv_{\text{the}}$ also implies $\omega \gg kv_{\text{thi}}$ because

$$\frac{v_{\text{thi}}}{v_{\text{the}}} = \sqrt{\frac{T_i}{T_e}\frac{m_e}{m_i}} \ll 1 \tag{3.34}$$

as long as T_i/T_e is not huge.[15] Thus, (3.31) becomes

$$
\begin{aligned}
\operatorname{Re}\epsilon &= 1 + \sum_\alpha \frac{\omega_{\text{p}\alpha}^2}{k^2}\frac{1}{n_\alpha}\frac{k}{\omega}\, \mathcal{P}\!\int \mathrm{d}v_z\, F_\alpha'(v_z)\left[1 + \frac{kv_z}{\omega} + \left(\frac{kv_z}{\omega}\right)^2 + \left(\frac{kv_z}{\omega}\right)^3 + \dots\right] \\
&= 1 + \sum_\alpha \frac{\omega_{\text{p}\alpha}^2}{k\omega}\Bigg[\underbrace{\frac{1}{n_\alpha}\int \mathrm{d}v_z\, F_\alpha'(v_z)}_{=\,0} - \frac{k}{\omega}\underbrace{\frac{1}{n_\alpha}\int \mathrm{d}v_z\, F_\alpha(v_z)}_{=\,1} \\
&\qquad\quad - 2\frac{k^2}{\omega^2}\underbrace{\frac{1}{n_\alpha}\int \mathrm{d}v_z\, v_z F_\alpha(v_z)}_{=\,0} - 3\frac{k^3}{\omega^3}\underbrace{\frac{1}{n_\alpha}\int \mathrm{d}v_z\, v_z^2 F_\alpha(v_z)}_{=\,v_{\text{th}\alpha}^2/2} + \dots\Bigg] \\
&= 1 - \sum_\alpha \frac{\omega_{\text{p}\alpha}^2}{\omega^2}\left[1 + \frac{3}{2}\frac{k^2 v_{\text{th}\alpha}^2}{\omega^2} + \dots\right],
\end{aligned}
\tag{3.35}
$$

where we have integrated by parts everywhere, assumed that there are no mean flows, $\langle v_z\rangle = 0$, and, in the last term, used

$$\langle v_z^2\rangle = \frac{v_{\text{th}\alpha}^2}{2}, \tag{3.36}$$

which is indeed the case for a Maxwellian F_α or, if F_α is not a Maxwellian, can be viewed as the definition of $v_{\text{th}\alpha}$.

The ion contribution to (3.35) is small because

$$\frac{\omega_{\text{p}i}^2}{\omega_{\text{p}e}^2} = \frac{Zm_e}{m_i} \ll 1, \tag{3.37}$$

---

[14]This is a standard well-defined quantity for a Maxwellian equilibrium distribution $F_e(v_z) = (n_e/\sqrt{\pi}\, v_{\text{the}}) \exp(-v_z^2/v_{\text{the}}^2)$, but if we wish to consider a non-Maxwellian F_e, let v_{the} be some typical speed characterising the width of the equilibrium distribution, defined by, e.g., (3.36).

[15]For hydrogen plasma, $\sqrt{m_i/m_e} \approx 42$, the answer to the Ultimate Question of Life, Universe and Everything (Adams 1979).

so ions do not participate in this dynamics at all. Therefore, to lowest order, the dispersion relation (3.29) becomes

$$\mathrm{Re}\,\epsilon \approx 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2} = 0 \quad \Rightarrow \quad \boxed{\omega^2 = \omega_{\mathrm{pe}}^2 = \frac{4\pi e^2 n_e}{m_e}}. \tag{3.38}$$

This is the Tonks & Langmuir (1929) dispersion relation for what is known as *Langmuir, or plasma, oscillations*. This is the formal derivation of the result that we already had, on less mathematically rigorous, physical grounds, in §2.1.

We can do a little better if we retain the (small) k-dependent term in (3.35):

$$\mathrm{Re}\,\epsilon \approx 1 - \frac{\omega_{\mathrm{pe}}^2}{\omega^2}\left(1 + \underbrace{\frac{3}{2}\frac{k^2 v_{\mathrm{the}}^2}{\omega^2}}_{\substack{\text{use}\\ \omega^2 \approx \omega_{\mathrm{pe}}^2}}\right) = 0 \quad \Rightarrow \quad \boxed{\omega^2 \approx \omega_{\mathrm{pe}}^2(1 + 3k^2\lambda_{\mathrm{D}e}^2)}, \tag{3.39}$$

where $\lambda_{\mathrm{D}e} = v_{\mathrm{the}}/\sqrt{2}\,\omega_{\mathrm{pe}} = \sqrt{T_e/4\pi e^2 n_e}$ is the "electron Debye length" [cf. (1.6)]. Equation (3.39) is the Bohm & Gross (1949a) dispersion relation, describing an upgrade of the Langmuir oscillations to dispersive *Langmuir waves*, which have a non-zero group velocity (this effect is due to electron pressure: see Exercise 3.1).

Note that all this is only valid for $\omega \gg k v_{\mathrm{the}}$, which we now see is equivalent to

$$k\lambda_{\mathrm{D}e} \ll 1 \tag{3.40}$$

(the wave length of the perturbation is long compared to the Debye length).

Exercise 3.1. Langmuir hydrodynamics.[16] Starting from the linearised kinetic equation for electrons and ignoring perturbations of the ion distribution function completely, work out the fluid equations for electrons (i.e., the evolution equations for the electron density n_e and velocity u_e) and show that you can recover the Langmuir waves (3.39) if you assume that electrons behave as a 1D adiabatic fluid (i.e., have the equation of state $p_e n_e^{-\gamma} = \mathrm{const}$ with $\gamma = 3$). You can prove that they indeed do this by calculating their density and pressure directly from the Landau solution for the perturbed distribution function (see §§5.3 and 5.6), ignoring resonant particles. The "hydrodynamic" description of Langmuir waves will reappear in §10.

3.5. *Landau Damping and Kinetic Instabilities*

Now let us calculate the damping rate of Langmuir waves using (3.30), (3.38) and (3.32):

$$\frac{\partial\mathrm{Re}\,\epsilon}{\partial\omega} \approx \frac{2\omega_{\mathrm{pe}}^2}{\omega^3}, \quad \mathrm{Im}\,\epsilon \approx -\frac{\omega_{\mathrm{pe}}^2}{k^2}\frac{\pi}{n_e}F_e'\!\left(\frac{\omega}{k}\right) \quad \Rightarrow \quad \boxed{\gamma \approx \frac{\pi}{2}\frac{\omega^3}{k^2}\frac{1}{n_e}F_e'\!\left(\frac{\omega}{k}\right)}, \tag{3.41}$$

where ω is given by (3.39). Provided $\omega F'(\omega/k) < 0$ (as would be the case, e.g., for any distribution monotonically decreasing with $|v_z|$; see Fig. 8a), $\gamma < 0$ and so this is indeed a damping rate, the celebrated *Landau damping* (Landau 1946; it was confirmed experimentally two decades later, by Malmberg & Wharton 1964).

The same theory also describes a class of *kinetic instabilities*: if $\omega F'(\omega/k) > 0$, then $\gamma > 0$, so perturbations grow exponentially with time. An iconic example is the *bump-on-tail instability* (Fig. 8b), which arises when a high-energy ($v_z \gg v_{\mathrm{the}}$) electron beam

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[16]This is based on the 2017 exam question.

(a) $\omega F'(\omega/k) < 0$: Landau damping   (b) $\omega F'(\omega/k) > 0$: instability
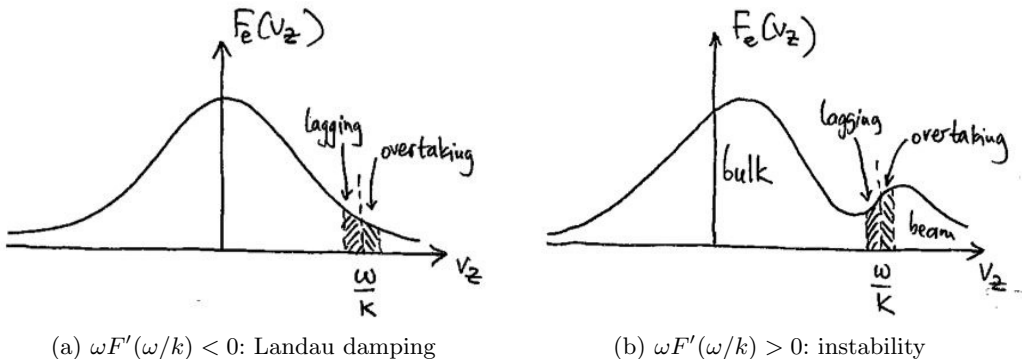
FIGURE 8. The Landau resonance (particle velocities equalling phase speed of the wave $v_z = \omega/k$) leads to damping of the wave if more particles lag just behind than overtake the wave and to instability in the opposite case.

is injected into a plasma[17] and whose quasilinear saturation we will study in great detail in §7.

We see that the damping or growth of plasma waves occur via their interaction with the particles whose velocities coincide with the phase velocity of the wave (*"Landau resonance"*). Because such particles are moving in phase with the wave, its electric field is stationary in their reference frame and so can do work on these particles, giving its energy to them (damping) or receiving energy from them (instability). In contrast, other, out-of-phase, particles experience no mean energy change over time because the field that they "see" is oscillating. It turns out (§3.6) that the process works in the spirit of socialist redistrbution: the particles slightly lagging behind the wave will, on average, receive energy from it, damping the wave, whereas those overtaking the wave will have some of their energy taken away, amplifying the wave. The condition $\omega F'(\omega/k) < 0$ corresponds to the stragglers being more numerous than the strivers, leading to net damping; $\omega F'(\omega/k) > 0$ implies the opposite, leading to an instability (which then leads to flattening of the distribution; see §7).

Let us note again that these results are quantitatively valid only in the limit (3.33), or, equivalently, (3.40). It makes sense that damping should be slow ($\gamma \ll \omega$) in the limit where the waves propagate much faster than the majority of the electrons ($\omega/k \gg v_{\mathrm{th}e}$) and so can interact only with a small number of particularly fast particles (for a Maxwellian equilibrium distribution, it is an exponentially small number $\sim e^{-\omega^2/k^2 v_{\mathrm{th}e}^2}$). If, on the other hand, $\omega/k \sim v_{\mathrm{th}e}$, the waves interact with the majority population and the damping should be strong: *a priori*, we might expect $\gamma \sim k v_{\mathrm{th}e}$.[18]

Exercise 3.2. Stability of isotropic distributions. Prove that if $f_{0e}(v_x, v_y, v_z) = f_{0e}(v)$, i.e., if it is a 3D-isotropic distribution, monotonic or otherwise, the Langmuir waves at $k\lambda_{\mathrm{D}e} \ll 1$ are always damped (this is solved in Lifshitz & Pitaevskii 1981; the statement of stability of isotropic distributions is in fact valid much more generally than just for long-wavelength Langmuir waves: see Exercise 4.2).

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[17]Here we are dealing with the case of a "warm beam" (meaning that it has a finite width). It turns out that there exists also another instability, leading to growth of perturbations with ω/k to the right of the bump's peak, due to a different, "fluid" kind of resonance and possible even for "cold beams" (i.e., beams of particles that all have the same velocity): see §3.7.

[18]This is indeed correct. You can confirm it numerically using (3.83) and (3.89).
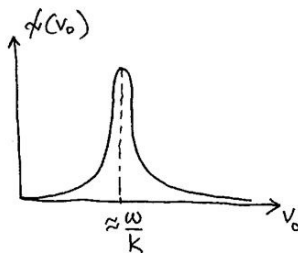
FIGURE 9. The function $\chi(v_0)$ defined in (3.49).

Landau's method of working out waves and damping in collisionless plasmas has always elicited a degree of dissatisfaction in the minds of some mathematically inclined physicists and motivated them to search for alternatives. Perhaps the earliest and best known such alternative is the formalism due to van Kampen. His objective was more mathematical rigour—but even if this is of limited appeal to you, the book by van Kampen & Felderhof (1967) is still a good read and a good chance to question and re-examine your understanding of how it all works.[19]

Landau damping became a *cause célèbre* in the hard-core mathematics community, as well as in the wider science world, with the award of the Fields Medal in 2010 to Cédric Villani, who proved (with C. Mouhot) that, basically, Landau's solution of the linearised Vlasov equation survived as a solution of the full nonlinear Vlasov equation for small enough and regular enough initial perturbations: see a "popular" account of this by Villani (2014). The regularity restriction is apparently important and the result can break down in interesting ways (Bedrossian 2016). The culprit is plasma echo, of which more will be said in §6.2 (without any claim to mathematical rigour; see also Schekochihin *et al.* 2016 and Adkins & Schekochihin 2018).

Even in the linear approximation, Landau damping depends on mathematical assumptions that are relaxed at one's peril: you might find further enlightenment, or at least enjoyment, in the recent paper by Ramos & White (2018), where the Landau problem is recast as a proper eigenvalue problem—and it is shown, amongst other mathematical delights, that if one fiddles cleverly with initial conditions, one can obtain solutions that do not decay at the Landau rate and, in fact, can have any time evolution that one cares to specify!

3.6. *Physical Picture of Landau Damping*

The following simple argument (Lifshitz & Pitaevskii 1981) illustrates the physical mechanism of Landau damping.

Consider an electron moving along the z axis, subject to a wave-like electric field:

$$\frac{dz}{dt} = v_z, \tag{3.42}$$

$$\frac{dv_z}{dt} = -\frac{e}{m_e} E(z, t) = -\frac{e}{m_e} E_0 \cos(\omega t - kz) e^{\varepsilon t}. \tag{3.43}$$

I have given the electric field a slow time dependence, $E \propto e^{\varepsilon t}$, but will later take $\varepsilon \to +0$—this describes the field switching on infinitely slowly from $t = -\infty$. Let us assume that the amplitude E_0 of the electric field is so small that it changes the electron's trajectory only a little over several wave periods. Then the equations of motion can be solved perturbatively.

The lowest-order ($E_0 = 0$) solution is

$$v_z(t) = v_0 = \text{const}, \quad z(t) = v_0 t. \tag{3.44}$$

In the next order, let

$$v_z(t) = v_0 + \delta v_z(t), \quad z(t) = v_0 t + \delta z(t). \tag{3.45}$$

---

[19]Blithely skipping half a century of literature, let me mention a recent paper by Heninger & Morrison (2018), which (following up on Morrison 1994, 2000) recast van Kampen's scheme as a new transform (called "G-transform") being used instead of the Laplace transform to solve Landau's initial-value problem.

Equation (3.43) becomes

$$\frac{\mathrm{d}\delta v_z}{\mathrm{d}t} = -\frac{e}{m_e}\, E(z(t), t) \approx -\frac{e}{m_e}\, E(v_0 t, t) = -\frac{eE_0}{m_e}\, \mathrm{Re}\, e^{[i(\omega - kv_0) + \varepsilon]t}. \tag{3.46}$$

Integrating this gives

$$\begin{aligned}
\delta v_z(t) &= -\frac{eE_0}{m_e} \int_0^t \mathrm{d}t'\, \mathrm{Re}\, e^{[i(\omega - kv_0) + \varepsilon]t'} \\
&= -\frac{eE_0}{m_e}\, \mathrm{Re}\, \frac{e^{[i(\omega - kv_0) + \varepsilon]t} - 1}{i(\omega - kv_0) + \varepsilon} \\
&= -\frac{eE_0}{m_e}\, \frac{\varepsilon e^{\varepsilon t} \cos[(\omega - kv_0)t] - \varepsilon + (\omega - kv_0)e^{\varepsilon t} \sin[(\omega - kv_0)t]}{(\omega - kv_0)^2 + \varepsilon^2}. \tag{3.47}
\end{aligned}$$

Integrating again, one gets

$$\begin{aligned}
\delta z(t) &= \int_0^t \mathrm{d}t'\, \delta v_z(t') \\
&= -\frac{eE_0}{m_e} \int_0^t \mathrm{d}t'\, \mathrm{Re}\, \frac{e^{[i(\omega - kv_0) + \varepsilon]t'} - 1}{i(\omega - kv_0) + \varepsilon} \\
&= -\frac{eE_0}{m_e} \left\{ \mathrm{Re}\, \frac{e^{[i(\omega - kv_0) + \varepsilon]t} - 1}{[i(\omega - kv_0) + \varepsilon]^2} - \frac{\varepsilon t}{(\omega - kv_0)^2 + \varepsilon^2} \right\} \\
&= -\frac{eE_0}{m_e} \left\{ \frac{[\varepsilon^2 - (\omega - kv_0)^2]\left\{ e^{\varepsilon t}\cos[(\omega - kv_0)t] - 1 \right\} + 2\varepsilon(\omega - kv_0)e^{\varepsilon t}\sin[(\omega - kv_0)t]}{[(\omega - kv_0)^2 + \varepsilon^2]^2} \right. \\
&\qquad\qquad \left. -\frac{\varepsilon t}{(\omega - kv_0)^2 + \varepsilon^2} \right\}. \tag{3.48}
\end{aligned}$$

The work done by the field on the electron per unit time, averaged over time, is the power gained by the electron:

$$\begin{aligned}
\delta P(v_0) &= -e\, \langle E(z(t), t) v_z(t) \rangle \\
&\approx -e \left\langle \left[E(v_0 t, t) + \delta z(t)\, \frac{\partial E}{\partial z}(v_0 t, t) \right] [v_0 + \delta v_z(t)] \right\rangle \\
&= -eE_0 e^{\varepsilon t} \left\langle \underbrace{v_0 \cos[(\omega - kv_0)t]}_{\substack{\text{vanishes} \\ \text{under} \\ \text{averaging}}} + \underbrace{\delta v_z(t) \cos[(\omega - kv_0)t]}_{\substack{\text{only cos term} \\ \text{from (3.47)} \\ \text{survives} \\ \text{averaging}}} + \underbrace{v_0 \delta z(t) k \sin[(\omega - kv_0)t]}_{\substack{\text{only sin term} \\ \text{from (3.48)} \\ \text{survives} \\ \text{averaging}}} \right\rangle \\
&= \frac{e^2 E_0^2}{2m_e}\, e^{2\varepsilon t} \left\{ \frac{\varepsilon}{(\omega - kv_0)^2 + \varepsilon^2} + \frac{2kv_0 \varepsilon(\omega - kv_0)}{[(\omega - kv_0)^2 + \varepsilon^2]^2} \right\} \\
&= \frac{e^2 E_0^2}{2m_e}\, e^{2\varepsilon t}\, \frac{\mathrm{d}}{\mathrm{d}v_0} \underbrace{\frac{\varepsilon v_0}{(\omega - kv_0)^2 + \varepsilon^2}}_{\equiv\, \chi(v_0)}. \tag{3.49}
\end{aligned}$$

We see (Fig. 9) that
—if the electron is lagging behind the wave, $v_0 \lesssim \omega/k$, then $\chi'(v_0) > 0 \;\Rightarrow\; \delta P(v_0) > 0$, so energy goes from the field to the electron (the wave is damped);
—if the electron is overtaking the wave, $v_0 \gtrsim \omega/k$, then $\chi'(v_0) < 0 \;\Rightarrow\; \delta P(v_0) < 0$, so energy goes from the electron to the field (the wave is amplified).
   Now remember that we have a whole distribution of these electrons, $F(v_z)$. So the total power
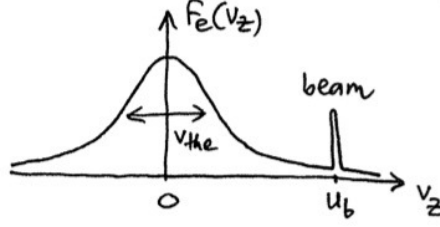
FIGURE 10. Electron distribution with a cold beam; see (3.54).

per unit volume going into (or out of) them is

$$P = \int \mathrm{d}v_z \, F(v_z) \delta P(v_z) = \frac{e^2 E_0^2 e^{2\varepsilon t}}{2m_e} \int \mathrm{d}v_z \, F(v_z) \chi'(v_z)$$
$$= -\frac{e^2 E_0^2 e^{2\varepsilon t}}{2m_e} \int \mathrm{d}v_z \, F'(v_z) \chi(v_z). \tag{3.50}$$

Noticing that, by Plemelj's formula (3.25), in the limit $\varepsilon \to +0$,

$$\chi(v_z) = \frac{\varepsilon v_z}{(\omega - k v_z)^2 + \varepsilon^2} = -\frac{i v_z}{2} \left(\frac{1}{k v_z - \omega - i\varepsilon} - \frac{1}{k v_z - \omega + i\varepsilon} \right) \to \pi \frac{\omega}{k^2} \, \delta \left(v_z - \frac{\omega}{k} \right), \tag{3.51}$$

we conclude

$$P = -\frac{e^2 E_0^2}{2m_e k^2} \, \pi \omega F' \left(\frac{\omega}{k} \right). \tag{3.52}$$

As in §3.5, we find damping if $\omega F'(\omega/k) < 0$ and instability if $\omega F'(\omega/k) > 0$.

Thus, around the wave-particle resonance $v_z = \omega/k$, the particles just lagging behind the wave receive energy from the wave and those just overtaking it give up energy to it. Therefore, qualitatively, damping occurs if the former particles are more numerous than the latter. We see that Landau's mathematics in §§3.1–3.5 led us to a result that makes physical sense.

3.7. *Hot and Cold Beams*

Let us return to the unstable situation, when a high-energy beam produces a bump on the tail of the distribution function and thus electrostatic perturbations can suck energy out of the beam and grow in the region of wave numbers where $v_0 < \omega/k < u_{\mathrm{b}}$. Here v_0 is the point of the minimum of the distribution in Fig. 8(b) and u_{b} is the point of the maximum of the bump, which is the velocity of the beam; we are assuming that $u_{\mathrm{b}} \gg v_{\mathrm{the}}$. In view of (3.41), the instability will have a greater growth rate if the bump's slope is steeper, i.e., if the beam is colder (narrower in v_z space).

Imagine modelling the beam by a little Maxwellian distribution with mean velocity u_{b}, tucked onto the bulk distribution:[20]

$$F_e(v_z) = \frac{n_e - n_{\mathrm{b}}}{\sqrt{\pi} \, v_{\mathrm{the}}} \exp \left(-\frac{v_z^2}{v_{\mathrm{the}}^2} \right) + \frac{n_{\mathrm{b}}}{\sqrt{\pi} \, v_{\mathrm{b}}} \exp \left[-\frac{(v_z - u_{\mathrm{b}})^2}{v_{\mathrm{b}}^2} \right], \tag{3.53}$$

where n_{b} is the density of the beam, v_{b} is its width, and so $T_{\mathrm{b}} = m_e v_{\mathrm{b}}^2/2$ is its "temperature", just like $T_e = m_e v_{\mathrm{the}}^2/2$ is the temperature of the thermal bulk. A colder beam will have less of a thermal spread around u_{b}. It turns out that if the width of

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[20]The fact that we are working in 1D means that we are restricting our consideration to perturbations whose wave numbers $\boldsymbol{k}$ are parallel to the beam's velocity. In general, allowing transverse wave numbers brings into play the transverse (electromagnetic) part of the dielectric tensor (see Q2). However, for non-relativistic beams, the fastest-growing modes will still be the longitudinal, electrostatic ones (see, e.g., Alexandrov *et al.* 1984, §32).
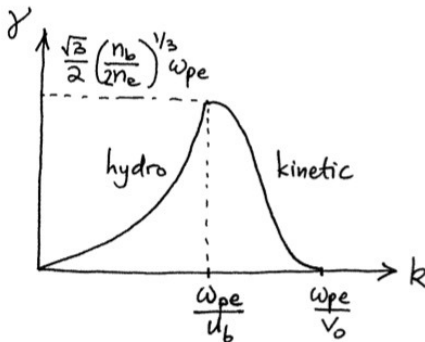
FIGURE 11. Sketch of the growth rate of the hydrodynamic and kinetic beam instabilities: see (3.57) for $k < \omega_{\mathrm{pe}}/u_{\mathrm{b}}$, (3.59) for $k = \omega_{\mathrm{pe}}/u_{\mathrm{b}}$, and (3.41) for $\omega_{\mathrm{pe}}/u_{\mathrm{b}} < k < \omega_{\mathrm{pe}}/v_0$, where v_0 is the point of the minimum of the distribution in Fig. 8(b) and u_{b} is the point of the maximum of the bump.

the beam is sufficiently small, another instability appears, whose origin is hydrodynamic rather than kinetic. Let us work it out.

Consider a very simple limiting case of the distribution (3.53): let $v_{\mathrm{b}} \to 0$ and $n_{\mathrm{b}} \ll n_e$. Then (Fig. 10)

$$F_e(v_z) = F_{\mathrm{M}}(v_z) + n_{\mathrm{b}}\delta(v_z - u_{\mathrm{b}}), \tag{3.54}$$

where F_{M} is the bulk Maxwellian from (3.53) (with density $\approx n_e$, neglecting n_{b} in comparison). Let us substitute the distribution (3.54) into the dielectric function (3.26), seek solutions with $p/k \gg v_{\mathrm{th}e}$, expand the part containing F_{M} in the same way as we did in §3.4,[21] neglect the ion contribution for the same reason as we did there, and deal with $\delta'(v_z - u_{\mathrm{b}})$ in the integrand via integration by parts. The resulting dispersion relation is

$$\epsilon \approx 1 + \frac{\omega_{\mathrm{pe}}^2}{p^2} - \frac{n_{\mathrm{b}}}{n_e}\frac{\omega_{\mathrm{pe}}^2}{(ku_{\mathrm{b}} - ip)^2} = 0. \tag{3.55}$$

Since $n_{\mathrm{b}} \ll n_e$, the last term can only matter for those perturbations that are close to resonance with the beam (this is called the *Cherenkov resonance*):

$$p = -iku_{\mathrm{b}} + \gamma, \quad \gamma \ll ku_{\mathrm{b}}. \tag{3.56}$$

This turns (3.55) into

$$1 - \frac{\omega_{\mathrm{pe}}^2}{k^2u_{\mathrm{b}}^2} + \frac{n_{\mathrm{b}}}{n_e}\frac{\omega_{\mathrm{pe}}^2}{\gamma^2} = 0 \quad \Rightarrow \quad \gamma = \pm\sqrt{\frac{n_{\mathrm{b}}}{n_e}}\left(\frac{1}{k^2u_{\mathrm{b}}^2} - \frac{1}{\omega_{\mathrm{pe}}^2}\right)^{-1/2}. \tag{3.57}$$

The expression under the square root is positive and so there is a growing mode only if $k < \omega_{\mathrm{pe}}/u_{\mathrm{b}}$. This is in contrast to the case of a hot (or warm) beam in §3.5: there, having a kinetic instability required $\omega F_e'(\omega/k) > 0$, which was only possible at $k > \omega_{\mathrm{pe}}/u_{\mathrm{b}}$ (the phase speed of the perturbations had to be to the left of the bump's maximum). The new instability that we have found—the *hydrodynamic beam instability*—has the largest growth rate at $ku_{\mathrm{b}} = \omega_{\mathrm{pe}}$, i.e., when the beam and the plasma oscillations are in resonance, in which case, to resolve the singularity, we need to retain γ in the second

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[21]We can treat the Landau contour as simply running along the real axis because we are expecting to find a solution with $\mathrm{Re}\,p > 0$ [see (3.20)], for reasons independent of the Landau resonance.

term in (3.55). Doing so and expanding in γ, we get

$$\epsilon \approx 1 - \frac{\omega_{\text{pe}}^2}{(\omega_{\text{pe}} + i\gamma)^2} + \frac{n_{\text{b}}}{n_e} \frac{\omega_{\text{pe}}^2}{\gamma^2} \approx \frac{2i\gamma}{\omega_{\text{pe}}} + \frac{n_{\text{b}}}{n_e} \frac{\omega_{\text{pe}}^2}{\gamma^2} = 0. \tag{3.58}$$

Solution:

$$\boxed{\gamma = \left(\frac{\pm\sqrt{3} + i}{2}, -i \right) \left(\frac{n_{\text{b}}}{2n_e} \right)^{1/3} \omega_{\text{pe}}}. \tag{3.59}$$

The unstable root ($\text{Re}\,\gamma > 0$) is the interesting one. The growth rate of the combined beam instability, hydrodynamic and kinetic, is sketched in Fig. 11.

Exercise 3.3. This instability is called "hydrodynamic" because it can be derived from fluid equations (cf. Exercise 3.1) describing cold electrons ($v_{\text{the}} = 0$) and a cold beam ($v_{\text{b}} = 0$). Convince yourself that this is the case.

Exercise 3.4. Using the model distribution (3.53), work out the conditions on v_{b} and n_{b} that must be satisfied in order for our derivation of the hydrodynamic beam instability to be valid, i.e., for (3.55) to be a good approximation to the true dispersion relation. What is the effect of finite v_{b} on the hydrodynamic instability? Sketch the growth rate of unstable perturbations as a function of k, taking into account both the hydrodynamic instability and the kinetic one, as well as the Landau damping.

Exercise 3.5. Two-stream instability. This is a popular instability[22] that arises, e.g., in a situation where the plasma consists of two cold counter-streams of electrons propagating against a quasineutrality-enforcing background of effectively immobile ions (Fig. 12a). Model the corresponding electron distribution by

$$F_e(v_z) = \frac{n_e}{2} \left[\delta(v_z - u_{\text{b}}) + \delta(v_z + u_{\text{b}}) \right] \tag{3.60}$$

and solve the resulting dispersion relation (where the ion terms can be neglected for the same reason as in §3.4). Find the wave number at which perturbations grow fastest and the corresponding growth rate. Find also the maximum wave number at which perturbations can grow. If you want to know what happens when the two streams are warm (have a finite thermal spread v_{b}; Fig. 12b), a nice fully tractable quantitative model of such a situation is the double-Lorentzian distribution (4.16). The dispersion relation for it can be solved exactly: this is done in Q4. You will again find a hydrodynamic instability, but is there also a kinetic one (due to Landau resonance)? It is an interesting and non-trivial question why not.

3.8. *Ion-Acoustic Waves*

Let us now see what happens at lower frequencies,

$$v_{\text{the}} \gg \frac{\omega}{k} \gg v_{\text{thi}}, \tag{3.61}$$

i.e., when the waves propagate slower than the bulk of the electron distribution but faster than the bulk of the ion one (Fig. 13). This is another regime in which we might expect to find weakly damped waves: they are out of phase with the majority of the ions, so $F_i'(\omega/k)$ might be small because $F_i(\omega/k)$ is small, while as far as the electrons are concerned, the phase speed of the waves is deep in the core of the distribution, perhaps

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[22]It was discovered by engineers (Haeff 1949; Pierce & Hebenstreit 1949) and quickly adopted by physicists (Bohm & Gross 1949*b*). Buneman (1958) realised that a case with an electron and an ion stream (i.e., with plasma carrying a current) is unstable in an analogous way. The kinetic version of the latter situation is the ion-acoustic instability derived in §3.9. In §4.4, I will discuss in a more general way the stability of distributions featuring streams.

(a) Cold streams: see (3.60)   (b) Hot streams: see (4.16) and Q4
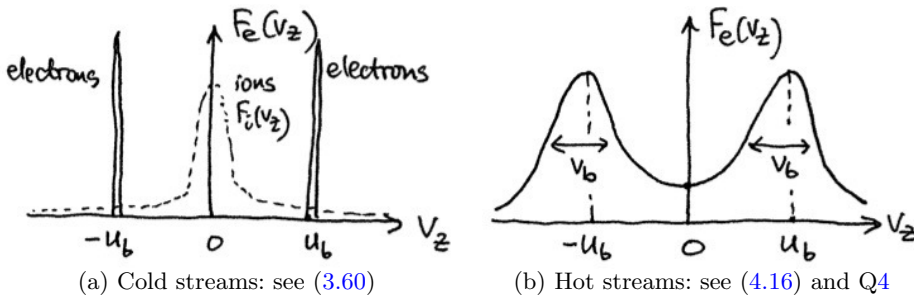
FIGURE 12. Two streams.

close to its maximum at $v_z = 0$ (if that is where its maximum is) and so $F'_e(\omega/k)$ might turn out to be small because $F_e(v_z)$ changes slowly in that region.

To make this more specific, let us consider Maxwellian electrons:

$$F_e(v_z) = \frac{n_e}{\sqrt{\pi}\, v_{\mathrm{the}}} \exp\left[-\frac{(v_z - u_e)^2}{v_{\mathrm{the}}^2}\right], \tag{3.62}$$

where we are, in general, allowing the electrons to have a mean flow (current). We will assume that $u_e \ll v_{\mathrm{the}}$ but allow $u_e \sim \omega/k$. We can anticipate that this will give us an interesting new effect. Indeed,

$$F'_e(v_z) = -\frac{2(v_z - u_e)}{v_{\mathrm{the}}^2}\, F_e(v_z). \tag{3.63}$$

For resonant particles, $v_z = \omega/k$, the prefactor will be small, so we can hope for $\gamma \ll \omega$, as anticipated above, but note that its sign will depend on the relative size of u_e and ω/k and so we might (we will!) get an instability (§3.9).

But let us not get ahead of ourselves: we must first calculate the real frequency $\omega(k)$ of these waves, from (3.29) and (3.31):

$$\mathrm{Re}\,\epsilon = 1 - \frac{\omega_{\mathrm{pe}}^2}{k^2}\frac{1}{n_e}\mathcal{P}\!\int \mathrm{d}v_z\,\frac{F'_e(v_z)}{v_z - \omega/k} - \underbrace{\frac{\omega_{\mathrm{pi}}^2}{k^2}\frac{1}{n_i}\mathcal{P}\!\int \mathrm{d}v_z\,\frac{F'_i(v_z)}{v_z - \omega/k}}_{\approx \dfrac{\omega_{\mathrm{pi}}^2}{\omega^2}\left(1 + 3k^2\lambda_{\mathrm{D}i}^2\right)} = 0. \tag{3.64}$$

The last (ion) term in this equation can be expanded in $kv_z/\omega \ll 1$ in exactly the same way as it was done in (3.35). The expansion is valid provided

$$k\lambda_{\mathrm{D}i} \ll 1, \tag{3.65}$$

and I will retain only the lowest-order term, dropping the $k^2\lambda_{\mathrm{D}i}^2$ correction. The second (electron) term in (3.64) is subject to the opposite limit, $v_z \gg \omega/k$, so, using (3.63),

$$\frac{\omega_{\mathrm{pe}}^2}{k^2}\frac{1}{n_e}\mathcal{P}\!\int \mathrm{d}v_z\,\frac{F'_e(v_z)}{v_z - \omega/k} \approx -\frac{\omega_{\mathrm{pe}}^2}{k^2}\frac{1}{n_e}\mathcal{P}\!\int \mathrm{d}v_z\,\frac{2(v_z - u_e)}{v_{\mathrm{the}}^2 v_z}\, F_e(v_z) \approx -\frac{2\omega_{\mathrm{pe}}^2}{k^2 v_{\mathrm{the}}^2} = -\frac{1}{k^2\lambda_{\mathrm{De}}^2}, \tag{3.66}$$

where we have neglected $u_e \ll v_z$ because this integral is over the thermal bulk of the electron distribution.

With all these approximations, (3.64) becomes

$$\mathrm{Re}\,\epsilon = 1 + \frac{1}{k^2\lambda_{\mathrm{De}}^2} - \frac{\omega_{\mathrm{pi}}^2}{\omega^2} = 0. \tag{3.67}$$

The dispersion relation is then

$$\omega^2 = \frac{\omega_{\mathrm{p}i}^2}{1 + 1/k^2\lambda_{\mathrm{D}e}^2} = \frac{k^2 c_{\mathrm{s}}^2}{1 + k^2\lambda_{\mathrm{D}e}^2}, \tag{3.68}$$

where

$$c_{\mathrm{s}} = \omega_{\mathrm{p}i}\lambda_{\mathrm{D}e} = \sqrt{\frac{ZT_e}{m_i}} \tag{3.69}$$

is the *sound speed*, called that because, if $k\lambda_{\mathrm{D}e} \ll 1$, (3.68) describes a wave that is very obviously a sound, or *ion-acoustic, wave:*

$$\omega = \pm k c_{\mathrm{s}}. \tag{3.70}$$

The phase speed of this wave is the sound speed, $\omega/k = c_{\mathrm{s}}$. That the expression (3.69) for c_{s} combines electron temperature and ion mass is a hint as to the underlying physics of sound propagation in plasma: ions provide the inertia, electrons the pressure (see Exercise 3.6).

We can now check under what circumstances the condition (3.61) is indeed satisfied:

$$\frac{c_{\mathrm{s}}}{v_{\mathrm{th}e}} = \sqrt{\frac{Zm_e}{2m_i}} \ll 1, \quad \frac{c_{\mathrm{s}}}{v_{\mathrm{th}i}} = \sqrt{\frac{ZT_e}{2T_i}} \gg 1, \tag{3.71}$$

with the latter condition requiring that the ions should be colder than the electrons.

Exercise 3.6. Hydrodynamics of sound waves. Starting from the linearised kinetic equations for ions and electrons, work out the fluid equations for the plasma, i.e., the evolution equations for its mass density and mass flow velocity. Assuming $m_e \ll m_i$ (negligible electron inertia) and $T_i \ll T_e$ (cold ions), show that these equations are

$$\frac{\partial \delta n_i}{\partial t} + n_i \boldsymbol{\nabla} \cdot \boldsymbol{u}_i = 0, \tag{3.72}$$

$$m_i n_i \frac{\partial \boldsymbol{u}_i}{\partial t} + \boldsymbol{\nabla}\delta p_e = 0, \tag{3.73}$$

and that the sound waves (3.70) with c_{s} given by (3.69) are recovered if electrons have the equation of state of an isothermal fluid. Why, and under what assumptions, should they be isothermal physically? Prove mathematically that they indeed are. Why is the equation of state for electrons different in a sound wave than in a Langmuir wave (see Exercise 3.1)? We will revisit ion hydrodynamics in §10.

3.9. *Damping of Ion-Acoustic Waves and Ion-Acoustic Instability*

Are ion acoustic waves damped? Can they grow? We have a standard protocol for answering this question: calculate $\mathrm{Re}\,\epsilon$ and $\mathrm{Im}\,\epsilon$ and substitute into (3.30). Using (3.67) and (3.32), we find

$$\frac{\partial \mathrm{Re}\,\epsilon}{\partial \omega} = \frac{2\omega_{\mathrm{p}i}^2}{\omega^3}, \quad \mathrm{Im}\,\epsilon = -\frac{\omega_{\mathrm{p}e}^2}{k^2}\frac{\pi}{n_e}F_e'\left(\frac{\omega}{k}\right) - \frac{\omega_{\mathrm{p}i}^2}{k^2}\frac{\pi}{n_i}F_i'\left(\frac{\omega}{k}\right). \tag{3.74}$$

The two terms in $\mathrm{Im}\,\epsilon$ represent the interaction between the waves and, respectively, electrons and ions. The ion term is small both on account of $\omega_{\mathrm{p}i} \ll \omega_{\mathrm{p}e}$ and, assuming Maxwellian ions, of the exponential smallness of $F_i(\omega/k) \propto \exp[-(\omega/kv_{\mathrm{th}i})^2]$. We are then left with

$$\gamma = -\frac{\mathrm{Im}\,\epsilon}{\partial(\mathrm{Re}\,\epsilon)/\partial\omega} = -\sqrt{\pi}\,\frac{\omega^3}{k^2 v_{\mathrm{th}e}^3}\frac{m_i}{Zm_e}\left(\frac{\omega}{k} - u_e\right), \tag{3.75}$$
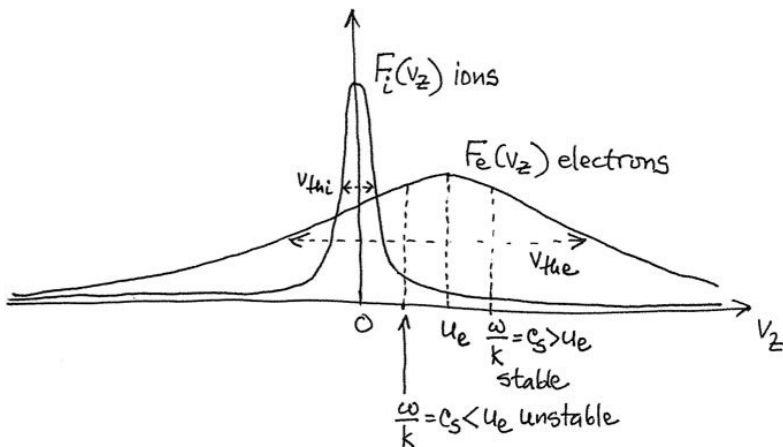
FIGURE 13. Ion-acoustic resonance: damping ($c_\mathrm{s} > u_e$) or instability ($c_\mathrm{s} < u_e$). Ion Landau damping is weak because $c_\mathrm{s} \gg v_{\mathrm{th}i}$, so in the tail of $F_i(v_z)$; electron damping/instability is also weak because $u_e, c_\mathrm{s} \ll v_{\mathrm{th}e}$, so close to the peak $F_e(v_z)$.

where we have used (3.63). In the long-wavelength limit, $k\lambda_{\mathrm{D}e} \ll 1$, we have $\omega = \pm kc_\mathrm{s}$, and so, for the "+" mode,

$$\boxed{\gamma = -\sqrt{\frac{\pi Z m_e}{8 m_i}}\, k\,(c_\mathrm{s} - u_e)}\,. \tag{3.76}$$

If the electron flow is subsonic, $u_e < c_\mathrm{s}$, this describes the *Landau damping of ion acoustic waves on hot electrons.* If, on the other hand, the electron flow is supersonic, the sign of γ reverses[23] and we discover the *ion-acoustic instability:* excitation of ion acoustic waves by a fast electron current. The instability belongs to the same general class as, e.g., the bump-on-tail instability (§3.5) in that it involves waves sucking energy from particles, but the new conceptual feature here is that such energy conversion can result from a collaboration between different particle species (electrons supplying the energy, ions carrying the wave).

There is a host of related instabilities involving various combinations of electron and ion beams, currents, streams and counter-streams—excellent treatments of them can be found in the textbooks by Krall & Trivelpiece (1973) and by Alexandrov *et al.* (1984) or in the review by Davidson (1983). I shall return to this topic in §4.4.

Exercise 3.7. Damping of sound waves on ions.[24] Find the ion contribution to the damping of ion-acoustic waves. Under what conditions does it become comparable to, or larger than, the electron contribution?

3.10. *Ion Langmuir Waves*

Note that since

$$\frac{\lambda_{\mathrm{D}e}}{\lambda_{\mathrm{D}i}} = \frac{v_{\mathrm{th}e}\omega_{\mathrm{p}i}}{v_{\mathrm{th}i}\omega_{\mathrm{p}e}} = \sqrt{\frac{Z T_e}{T_i}}, \tag{3.77}$$

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[23]Recall that $k > 0$ by the choice of the z axis.

[24]The 2016 exam question was loosely based on this.
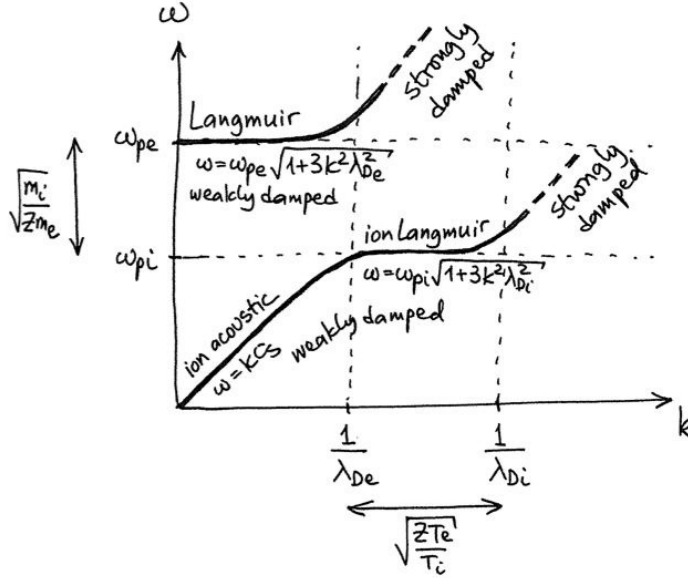
FIGURE 14. Electrostatic (longitudinal) plasma waves.

the condition (3.65) need not entail $k\lambda_{\mathrm{D}e} \ll 1$ in the limit of cold ions [see (3.71)]—in this case, the size of the Debye sphere (1.6) is set by the ions, rather than by the electrons, and so we can have perfectly macroscopic (in the language of §1.4) perturbations on scales both larger and smaller than $\lambda_{\mathrm{D}e}$. At larger scales, we have found sound waves (3.70). At smaller scales, $k\lambda_{\mathrm{D}e} \gg 1$, the dispersion relation (3.68) gives us *ion Langmuir oscillations*:

$$\omega^2 = \omega_{\mathrm{p}i}^2 = \frac{4\pi Z^2 e^2 n_i}{m_i}, \tag{3.78}$$

which are analogous to the electron Langmuir oscillations (3.38) and, like them, turn into dispersive *ion Langmuir waves* if the small $k^2\lambda_{\mathrm{D}i}^2$ correction in (3.64) is retained, leading to the Bohm–Gross dispersion relation (3.39), but with ion quantities this time.

Exercise 3.8. Derive the dispersion relation for ion Langmuir waves. Investigate their damping/instability.

3.11. *Summary of Electrostatic (Longitudinal) Plasma Waves*

We have achieved what turns out to be a complete characterisation of electrostatic (also known as "longitudinal", in the sense that $\boldsymbol{k} \parallel \boldsymbol{E}$) waves in an unmagnetised plasma. These are summarised in Fig. 14. In the limit of short wavelengths, $k\lambda_{\mathrm{D}e} \gg 1$ and $k\lambda_{\mathrm{D}i} \gg 1$, the electron and ion branches, respectively, becomes dispersive, their damping rates increase and eventually stop being small. This corresponds to waves having phase speeds that are comparable to the speeds of the particles in the thermal bulk of their distributions, so a great number of particles are available to have Landau resonance with the waves and absorb their energy—the damping becomes strong.

Note that if the cold-ion condition $T_i \ll T_e$ is not satisfied, the sound speed is comparable to the ion thermal speed $c_{\mathrm{s}} \sim v_{\mathrm{th}i}$, and so the ion-acoustic waves are strongly

damped at all wave numbers—it is well-nigh impossible to propagate sound through a collisionless hot plasma (in such an environment, no one will hear you scream)!

3.12. *Plasma Dispersion Function: Putting Linear Theory on Industrial Basis*

Clearly, we have entered the realm of practical calculation—it is now easy to imagine an industry of solving the plasma dispersion relation

$$\epsilon(p, \boldsymbol{k}) = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2} \frac{1}{n_\alpha} \int_{C_{\mathrm{L}}} \mathrm{d}v_z \frac{F_\alpha'(v_z)}{v_z - ip/k} = 0 \tag{3.79}$$

and similar dispersion relations arising from, e.g., considering electromagnetic perturbations (see Q2), magnetised plasmas (see Parra 2019b), different equilibria F_α (see Q3 and Q4), etc.

A Maxwellian equilibrium is obviously an extremely important special case because that is, after all, the distribution towards which plasma is pushed by collisions on long time scales:

$$f_{0\alpha}(\boldsymbol{v}) = \frac{n_\alpha}{(\pi v_{\mathrm{th}\alpha}^2)^{3/2}} e^{-v^2/v_{\mathrm{th}\alpha}^2} \quad \Rightarrow \quad F_\alpha(v_z) = \frac{n_\alpha}{\sqrt{\pi} v_{\mathrm{th}\alpha}} e^{-v_z^2/v_{\mathrm{th}\alpha}^2}. \tag{3.80}$$

For this case, we would like to introduce a new "special function" that would incorporate the Landau prescription for calculating the velocity integral in (3.79) and that we could in some sense "tabulate" once and for all.[25]

Taking F_α to be (3.80) and letting $u = v_z/v_{\mathrm{th}\alpha}$ and $\zeta_\alpha = ip/kv_{\mathrm{th}\alpha}$, we can rewrite the velocity integral in (3.79) as follows

$$\frac{1}{n_\alpha} \int_{C_{\mathrm{L}}} \mathrm{d}v_z \frac{F_\alpha'(v_z)}{v_z - ip/k} = -\frac{2}{\sqrt{\pi} v_{\mathrm{th}\alpha}^2} \int_{C_{\mathrm{L}}} \mathrm{d}u \frac{u e^{-u^2}}{u - \zeta_\alpha} = -\frac{2}{v_{\mathrm{th}\alpha}^2} \left[1 + \zeta_\alpha Z(\zeta_\alpha) \right], \tag{3.81}$$

where the *plasma dispersion function* is defined to be

$$\boxed{Z(\zeta) = \frac{1}{\sqrt{\pi}} \int \mathrm{d}u \frac{e^{-u^2}}{u - \zeta}}. \tag{3.82}$$

In these terms, the plasma dispersion relation (3.79) becomes

$$\boxed{\epsilon = 1 + \sum_\alpha \frac{1 + \zeta_\alpha Z(\zeta_\alpha)}{k^2 \lambda_{\mathrm{D}\alpha}^2} = 0}. \tag{3.83}$$

Note that if the Maxwellian distribution (3.80) has a mean flow, as it did, e.g., in (3.62), this amounts to a shift by some mean velocity u_α and all one needs to do to adjust the above results is to shift the argument of Z accordingly:

$$\zeta_\alpha \to \zeta_\alpha - \frac{u_\alpha}{v_{\mathrm{th}\alpha}}. \tag{3.84}$$

3.12.1. *Some Properties of $Z(\zeta)$*

It is not hard to see that

$$Z'(\zeta) = -\frac{1}{\sqrt{\pi}} \int \mathrm{d}u\, e^{-u^2} \frac{\partial}{\partial u} \frac{1}{u - \zeta} = -\frac{2}{\sqrt{\pi}} \int \mathrm{d}u \frac{u e^{-u^2}}{u - \zeta} = -2 \left[1 + \zeta Z(\zeta) \right]. \tag{3.85}$$

Let us treat this identity as a differential equation: the integrating factor is e^{ζ^2}, so

$$e^{\zeta^2} Z(\zeta) = -2 \int_0^\zeta \mathrm{d}t\, e^{t^2} + Z(0). \tag{3.86}$$

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[25]In the olden days, one would literally tabulate it (Fried & Conte 1961). In the 21st century, we could just teach a computer to compute it [see (3.89)] and make an app.

We know the boundary condition at $\zeta = 0$ from (3.23): for real ζ,

$$\frac{1}{\sqrt{\pi}} \int du \frac{e^{-u^2}}{u - \zeta} = \frac{1}{\sqrt{\pi}} \underbrace{\mathcal{P} \int_{-\infty}^{+\infty} du \frac{e^{-u^2}}{u - \zeta}}_{\substack{= 0 \text{ for } \zeta = 0 \\ \text{because} \\ \text{integrand is} \\ \text{odd}}} + i\sqrt{\pi}\, e^{-\zeta^2} \quad \Rightarrow \quad Z(0) = i\sqrt{\pi}. \tag{3.87}$$

Using this in (3.86) and changing the integration variable $t = -ix$, we find

$$Z(\zeta) = e^{-\zeta^2} \left(i\sqrt{\pi} + 2i \int_0^{i\zeta} dx\, e^{-x^2} \right) = 2i\, e^{-\zeta^2} \int_{-\infty}^{i\zeta} dx\, e^{-x^2}. \tag{3.88}$$

This turns out to be a uniformly valid expression for $Z(\zeta)$: our function is simply a complex erf! Here is a MATHEMATICA script for calculating it:

$$\texttt{Z[zeta\_] := I Sqrt[Pi] Exp[}{-}\texttt{zeta}^2\texttt{](1 + I Erfi[zeta]).} \tag{3.89}$$

You can use this to code up (3.83) and explore, e.g., the strongly damped solutions ($\zeta \sim 1$, $\gamma \sim \omega$).

3.12.2. *Asymptotics of $Z(\zeta)$*

If you are a devotee of the ancient art of asymptotic theory, you will find most useful (as, effectively, we did in §§3.4–3.9) various limiting forms of $Z(\zeta)$. At small argument $|\zeta| \ll 1$, the Taylor series is

$$\boxed{Z(\zeta) = i\sqrt{\pi}\, e^{-\zeta^2} - 2\zeta \left(1 - \frac{2\zeta^2}{3} + \frac{4\zeta^4}{15} - \frac{8\zeta^6}{105} + \dots \right)}. \tag{3.90}$$

At large argument, $|\zeta| \gg 1$, $|\text{Re}\,\zeta| \gg |\text{Im}\,\zeta|$, the asymptotic series is

$$\boxed{Z(\zeta) = i\sqrt{\pi}\, e^{-\zeta^2} - \frac{1}{\zeta} \left(1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \frac{15}{8\zeta^6} + \dots \right)}. \tag{3.91}$$

All the results (for a Maxwellian equilibrium) that I derived in §§3.4–3.10 can be readily obtained from (3.83) by using the above limiting cases (see Q1). It is, indeed, a general practical strategy for studying this and similar plasma dispersion relations to look for solutions in the limits $\zeta_\alpha \to 0$ or $\zeta_\alpha \to \infty$, then check under what physical conditions the solutions thus obtained are valid (i.e., that they indeed satisfy $|\zeta_\alpha| \ll 1$ or $|\zeta_\alpha| \gg 1$, $|\text{Re}\,\zeta| \gg |\text{Im}\,\zeta|$), and then fill in the non-asymptotic blanks in the same way that an experienced hunter espying antlers sticking out above the shrubbery can reconstruct, in contour outline, the rest of the hiding deer.

Exercise 3.9. Work out the Taylor series (3.90). A useful step might be to prove this interesting formula (which also turns out to be handy in other calculations; see, e.g., Q7):

$$\frac{d^m Z}{d\zeta^m} = \frac{(-1)^m}{\sqrt{\pi}} \int_{C_{\text{L}}} du \frac{H_m(u)\, e^{-u^2}}{u - \zeta}, \tag{3.92}$$

where $H_m(u)$ are Hermite polynomials [defined in (10.70)].

Exercise 3.10. Work out the asymptotic series (3.91) using the Landau prescription (3.20) and expanding the principal-value integral similarly to the way it was done in (3.35). Work out also (or look up; e.g., Fried & Conte 1961) other asymptotic forms of $Z(\zeta)$, relaxing the condition $|\text{Re}\,\zeta| \gg |\text{Im}\,\zeta|$.

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4. Linear Theory: Stability

In §3, we learned how to perturb some given equilibrium distribution $f_{0\alpha}$ infinitesimally and work out whether this perturbation will decay, grow, oscillate, and how quickly. Let

me now pose the question in a more general way. In a collisionless plasma, there can be infinitely many possible equilibria, including quite complicated ones. If we set one up, will it persist, i.e., is it stable? If it is not stable, what modification do we expect it to undergo in order to become stable? Other than solving the dispersion relation (3.17) to answer the first question and developing various types of nonlinear theories to answer the second (along the lines advertised in §2.4 and developed in §7 and subsequent sections), both of which can be quite complicated and often intractable technical challenges, do we have at our disposal any general principles that allow us to pronounce on stability? Is there a general insight that we can cultivate as to what sort of distributions are likely to be stable or unstable and to what sorts of perturbations?

We have had glimpses of such general principles already. For example, in §3.5, it was ascertained, by an explicit calculation, that one could encounter a situation with a (small) growth rate if the equilibrium distribution had a positive derivative somewhere along the direction (z) of the wave number of the perturbation, viz., $v_z F_e'(v_z) > 0$. I developed this further in §3.7, finding that not only hot but also cold beams and streams triggered instabilities. In Exercise 3.2, I dropped a hint that general statements could perhaps be made about certain general classes of distributions: 3D-isotropic equilibria could be proven stable (we shall prove this again, by a different method, in Exercise 4.2). How general are such statements? Are they sufficient or also necessary criteria? Is there a universal stability litmus test? Let us attack the problem of kinetic stability with an aspiration to generality—although still, for now, for electrostatic perturbations only. We shall also, for now, limit our ambition to determining *linear* stability of generic equilibria, i.e., their stability against infinitesimal perturbations. *Nonlinear* stability will have to wait till §8.1.

4.1. *Nyquist's Method*

The problem of linear stability comes down to the question of whether the dispersion relation (3.17) has any unstable solutions: roots with growth rates $\gamma_i(\boldsymbol{k}) > 0$.

It is going to be useful to write the dielectric function (3.26) as follows

$$\epsilon(p, \boldsymbol{k}) = 1 - \frac{\omega_{\mathrm{pe}}^2}{k^2} \int_{C_{\mathrm{L}}} \mathrm{d}v_z \frac{\bar{F}'(v_z)}{v_z - ip/k}, \tag{4.1}$$

$$\bar{F} = \frac{1}{n_e} \sum_\alpha Z_\alpha^2 \frac{m_e}{m_\alpha} F_\alpha = \frac{F_e}{n_e} + \frac{Zm_e}{m_i} \frac{F_i}{n_i}, \tag{4.2}$$

where the last expression in (4.2) is for the case of a two-species plasma. Thus, the distribution functions of different species come into the linear problem additively, weighted by their species' charges and (inverse) masses.

Let us develop a method (due to Nyquist 1932) for counting zeros of $\epsilon(p)$ (I will henceforth suppress $\boldsymbol{k}$ in the argument) in the half-plane $\mathrm{Re}\, p > 0$ (the unstable roots of the dispersion relation). Observe that $\epsilon(p)$ is analytic (by virtue of our efforts in §3.2 to make it so) and that if $p = p_i$ is its zero of order N_i, then in its vicinity,

$$\epsilon(p) = \mathrm{const}\,(p - p_i)^{N_i} + \dots \quad \Rightarrow \quad \frac{\partial \ln \epsilon(p)}{\partial p} = \frac{N_i}{p - p_i} + \dots, \tag{4.3}$$

so zeros of $\epsilon(p)$ are poles of $\partial \ln \epsilon(p)/\partial p$; the latter function has no other poles because $\epsilon(p)$ is analytic. If we now integrate this function over a closed contour C_R running along the imaginary axis (and just to the right of it: $p = -i\omega + 0$) in the complex p plane from iR to $-iR$ and then along a semicircle of radius R back to iR (Fig. 15), we will, in the
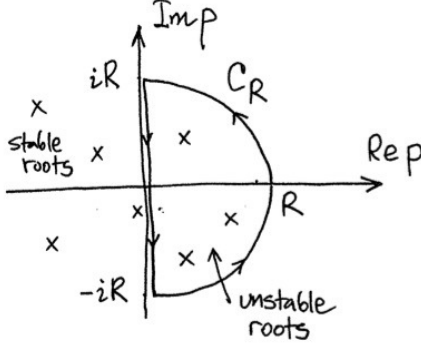
FIGURE 15. Integration contour for (4.4).

limit $R \to \infty$, capture all these poles:

$$\lim_{R \to \infty} \int_{C_R} \mathrm{d}p \, \frac{\partial \ln \epsilon(p)}{\partial p} = 2\pi i \sum_i N_i = 2\pi i N, \tag{4.4}$$

where N is the total number of zeros of $\epsilon(p)$ in the half-plane $\mathrm{Re}\, p > 0$. It turns out (as I shall prove in a moment) that the contribution to the integral over C_R from the semicircle vanishes at $R \to \infty$ and so we need only integrate along the imaginary axis:

$$\boxed{2\pi i N = \int_{+i\infty+0}^{-i\infty+0} \mathrm{d}p \, \frac{\partial \ln \epsilon(p)}{\partial p} = \ln \frac{\epsilon(-i\infty)}{\epsilon(+i\infty)}}. \tag{4.5}$$

Proof. All we need to show is that

$$|p| \frac{\partial \ln \epsilon(p)}{\partial p} \to 0 \quad \text{as} \quad |p| \to \infty, \; \mathrm{Re}\, p > 0. \tag{4.6}$$

Indeed, using (4.1) and the Landau integration rule (3.20), we have in this limit:

$$\epsilon(p) = 1 - \frac{\omega_{\mathrm{pe}}^2}{k^2} \int_{-\infty}^{+\infty} \mathrm{d}v_z \, \bar{F}'(v_z) \frac{ik}{p} \left(1 - \frac{ikv_z}{p} + \dots\right) \approx 1 + \frac{1}{p^2} \sum_\alpha \omega_{\mathrm{p}\alpha}^2, \tag{4.7}$$

where I have integrated by parts and used $\int \mathrm{d}v_z \, F_\alpha = n_\alpha$. Manifestly, the condition (4.6) is satisfied.

Note that, along the imaginary axis $p = -i\omega$, by the same expansion and using also the Plemelj formula (3.23), we have

$$\epsilon(-i\omega) \approx 1 - \frac{1}{\omega^2} \sum_\alpha \omega_{\mathrm{p}\alpha}^2 - i\pi \frac{\omega_{\mathrm{pe}}^2}{k^2} \bar{F}'\left(\frac{\omega}{k}\right) \to 1 \mp i0 \quad \text{as} \quad \omega \to \mp\infty. \tag{4.8}$$

This is going to be useful shortly.

In view of (4.8) and of our newly proven formula (4.5), as the function $\epsilon(-i\omega)$ runs along the real line in ω, it changes from

$$\epsilon(i\infty) = 1 - i0 \quad \text{at} \quad \omega = -\infty, \tag{4.9}$$

where I have arbitrarily fixed its phase, to

$$\epsilon(-i\infty) = e^{2\pi i N} + i0 \quad \text{at} \quad \omega = +\infty, \tag{4.10}$$
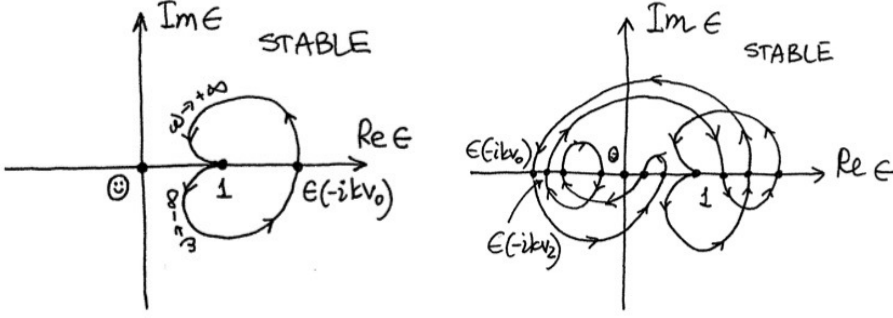
(a) Single-maximum, stable equilibrium   (b) Strange but stable equilibrium

FIGURE 16. Two examples of Nyquist diagrams showing stability (because failing to circle zero): (a) the case of a monotonically decreasing distribution (§4.2, Fig. 17a); (b) another stable case, even though very complicated (it also illustrates the argument in §4.3).

where N is the number of times the function

$$\epsilon(-i\omega) = 1 - \frac{\omega_{pe}^2}{k^2}\left[\mathcal{P}\int_{-\infty}^{+\infty} dv_z \frac{\bar{F}'(v_z)}{v_z - \omega/k} + i\pi\bar{F}'\left(\frac{\omega}{k}\right)\right] \tag{4.11}$$

circles around the origin in the complex ϵ plane. Since N is also the number of unstable roots of the dispersion relation, this gives us a way to count these roots by sketching $\epsilon(-i\omega)$—this sketch is called the *Nyquist diagram*. Two examples of Nyquist diagrams implying stability are given in Fig. 16: the curve $\epsilon(-i\omega)$ departs from $1 - i0$ and comes back to $1 + i0$ via a path that, however complicated, never makes a full circle around zero. Two examples of unstable situations appear in Fig. 18(b,d): in these cases, zero is circumnavigated, implying that the equilibrium distribution $\bar{F}$ is unstable (at a given value of k).

In order to work out whether the Nyquist curve circles zero (and how many times), all one needs to do is find $\mathrm{Re}\,\epsilon(-i\omega)$ at all points ω where $\mathrm{Im}\,\epsilon(-i\omega) = 0$, i.e., where the curve intersects the real line, and hence sketch the Nyquist diagram. We shall see in a moment, with the aid of some important examples, how this is done, but let us do a little bit of preparatory work first.

It follows immediately from (4.11) that these crossings happen whenever $\omega/k = v_*$ is a velocity at which $\bar{F}(v_z)$ has an extremum, $\bar{F}'(v_*) = 0$. At these points, the dielectric function (4.11) is real and can be expressed so:

$$\boxed{\epsilon(-ikv_*) = 1 + \frac{\omega_{pe}^2}{k^2}P(v_*)}. \tag{4.12}$$

Here $P(v_*)$ is (minus) the principal-value integral in (4.11), which can be manipulated as follows:

$$\begin{aligned}
P(v_*) &= -\mathcal{P}\int_{-\infty}^{+\infty} dv_z \frac{\bar{F}'(v_z)}{v_z - v_*} = -\mathcal{P}\int_{-\infty}^{+\infty} dv_z \frac{1}{v_z - v_*}\frac{\partial}{\partial v_z}\left[\bar{F}(v_z) - \bar{F}(v_*)\right] \\
&= \int_{-\infty}^{+\infty} dv_z \frac{\bar{F}(v_*) - \bar{F}(v_z)}{(v_z - v_*)^2},
\end{aligned} \tag{4.13}$$

where I have integrated by parts; the additional term $\bar{F}(v_*)$ was inserted under the

(a) Single-maximum distribution (§4.2)   (b) Single-minimum distribution (§4.3)
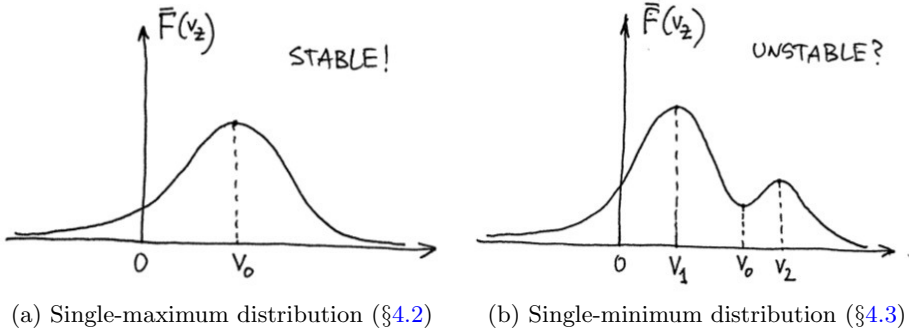
FIGURE 17. Two examples of equilibrium distributions.

derivative in order to eliminate the boundary terms arising in this integration by parts around the pole $v_z = v_*$.[26]

Now we are ready to analyse particular (and, as we shall see, also generic) equilibrium distributions $\bar{F}(v_z)$.

4.2. *Stability of Monotonically Decreasing Distributions*

Consider first a distribution function that has a single maximum at $v_z = v_0$ and monotonically decays in both directions away from it (Fig. 17a): $\bar{F}'(v_0) = 0$, $\bar{F}''(v_0) < 0$. This means that, besides at $\omega = \mp\infty$, $\mathrm{Im}\,\epsilon(-i\omega) \propto \bar{F}'(\omega/k)$ also vanishes at $\omega = kv_0$. It is then clear that

$$\epsilon(-ikv_0) = 1 + \frac{\omega_{\mathrm{pe}}^2}{k^2}P(v_0) > 1 \tag{4.14}$$

because $\bar{F}(v_0) > \bar{F}(v)$ for all v_z and so $P(v_0) > 0$. Thus, the Nyquist curve departs from $1 - i0$ at $\omega = -\infty$, intersects the real line once at $\omega = kv_0$ and then comes back to $1 + i0$ without circling zero; the corresponding Nyquist digram is sketched in Fig. 16(a). Conclusion:

> *Monotonically decreasing distributions are stable against electrostatic perturbations.*

We do not in fact need all this mathematical machinery just to prove the stability of monotonically decreasing distributions (in §8.2, we shall see that this is a very robust result)— but it will come handy when dealing with less simple cases. Parenthetically, let us work out some direct proofs of stability.

Exercise 4.1. Direct proof of linear stability of monotonically decreasing distributions. (a) Consider the dielectric function (4.1) with $p = -i\omega + \gamma$ and assume $\gamma > 0$ (so the Landau contour is just the real axis). Work out the real and imaginary parts of the dispersion relation $\epsilon(p) = 0$ and show that it can never be satisfied if $v_z\bar{F}'(v_z) \leqslant 0$, i.e., that any equilibrium distribution that has a maximum at zero and decreases monotonically on both sides of it is stable against electrostatic perturbations.[27]

(b) What if the maximum is at $v_z = v_0 \neq 0$?

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[26]Note that in the final expression in (4.13), there is no longer a need for principal-value integration because, v_* being a point of extremum of $\bar{F}$, the numerator of the integrand is quadratic in $v_z - v_*$ in the vicinity of v_*.

[27]This kind of argument can also be useful in stability considerations applying to more complicated situations, e.g., magnetised plasmas (Bernstein 1958).
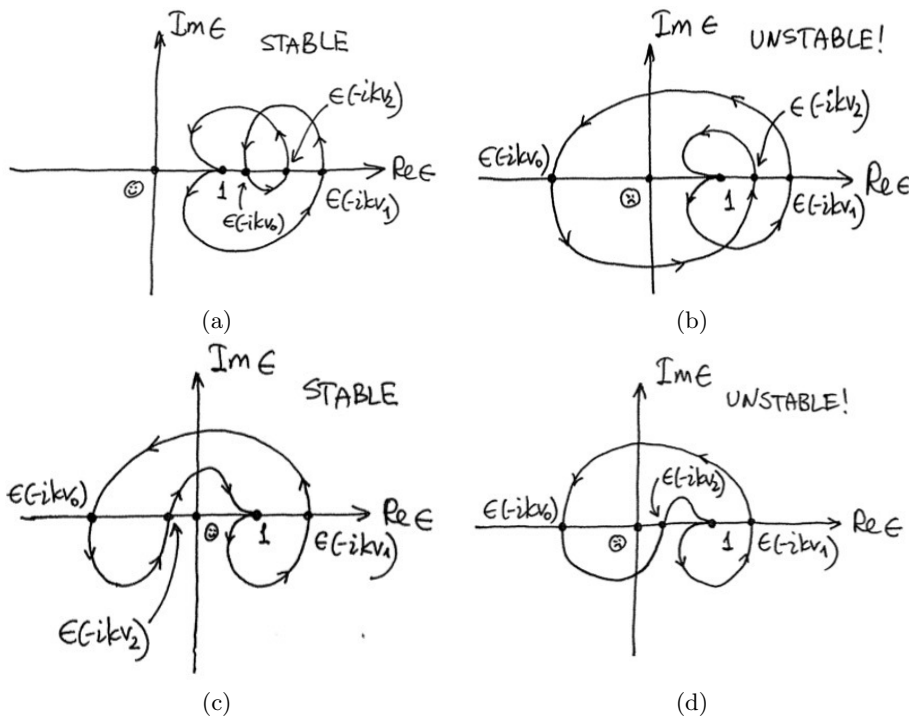
FIGURE 18. Various possible forms of the Nyquist diagram for a single-minimum distribution sketched in Fig. 17b: (a) $\epsilon(-ikv_0) > 1$, stable; (b) $\epsilon(-ikv_0) < 0$, $\epsilon(-ikv_2) > 1$, unstable; (c) $\epsilon(-ikv_0) < \epsilon(-ikv_2) < 0$, stable; (d) $\epsilon(-ikv_0) < 0 < \epsilon(-ikv_2) < 1$, unstable.

Exercise 4.2. Direct proof of linear stability of isotropic distributions. (a) Recall Exercise 3.2 and show that all homogeneous, 3D-isotropic (in velocity) equilibria are stable against electrostatic perturbations (with no need to assume long wave lengths).

(b) Prove, in the same way, that isotropic equilibria are also stable against *electromagnetic* perturbations. You will need to derive the transverse dielectric function in the same way as in Q2 or Q3, but for a general equilibrium distribution $f_{0\alpha}(v_x, v_y, v_z)$; failing that, you can look it up in a book, e.g., Krall & Trivelpiece (1973) or Davidson (1983).

4.3. *Penrose's Instability Criterion*

It would be good to learn how to test for stability generic distributions that have multiple minima and maxima: the simplest of them is shown in Fig. 17b, evoking the bump-on-tail situation discussed in §3.5 and thus posing a risk (but, as we are about to see, not a certainty!) of being unstable.

The Nyquist curve $\epsilon(-i\omega)$ departs from $1 - i0$ at $\omega = -\infty$, then crosses the real line for the first time at $\omega = kv_1$, corresponding to the leftmost maximum of $\bar{F}$.[28] This crossing is upwards, from the lower to the upper half-plane, and it is not hard to see that a maximum will always correspond to such an upward crossing and a minimum to a downward one, from the upper to the lower half-plane: this follows directly from the change of sign of $\text{Im}\,\epsilon$ in Eq. (4.11) because $\bar{F}'(\omega/k)$ goes from positive to negative at any point of maximum and vice versa at any minimum. After a few crossings back and forth,

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[28]For the distribution sketched in Fig. 17(b), this maximum is global, so $P(v_1) > 0$ and, therefore, $\epsilon(-ikv_1) > 1$. This is the rightmost such crossing when v_1 is the global maximum.

corresponding to local minima and maxima (if any), the Nyquist curve will come to the the downward crossing corresponding to the *global* minimum (other than at $v_z = \pm\infty$) of the distribution function at, say, $\omega = kv_0$. If at this point $P(v_0) > 0$, then $\epsilon(-ikv_0) > 1$ and the same is true at all other crossing points v_* because v_0 is the global minimum of $\bar{F}$ and so $P(v_*) > P(v_0) > 0$ for all other extrema. In this situation, illustrated in Fig. 18(a), the Nyquist curve never circumnavigates zero and, therefore, $P(v_0) > 0$ is a sufficient condition of stability. It is also the necessary one, which is proved in the following way.

Suppose $P(v_0) < 0$. Then, in (4.12), we can always find a range of k that are small enough that $\epsilon(-ikv_0) < 0$, so the downward crossing at v_0 happens on the negative side of zero in the ϵ plane. After this downward crossing, the Nyquist curve will make more crossings, until it finally comes to rest at $1 + i0$ as $\omega = +\infty$. Let us denote by $v_2 > v_0$ the point of extremum for which the corresponding crossing occurs at a point on the $\mathrm{Re}\,\epsilon$ axis that is closest to (but always will be to the right of) $\epsilon(-ikv_0) < 0$. If $\epsilon(-ikv_2) > 0$, then there is no way back, zero has been fully circumnavigated and so there must be at least one unstable root (see Fig. 18b,d). If $\epsilon(-ikv_2) < 0$, there is in principle some wiggle room for the Nyquist curve to avoid circling zero (see Fig. 18c for a single-minimum distribution of Fig. 17b—or Fig. 16b for some serious wiggles). However, since $P(v_2) > P(v_0)$ for any v_2 (because v_0 is the global minimum of $\bar{F}$), we can always increase k in (4.12) just enough so $\epsilon(-ikv_2) > 0$ even though $\epsilon(-ikv_0) < 0$ still (this corresponds to turning Fig. 18c into Fig. 18d). Thus, if $P(v_0) < 0$, there will always be some range of k inside which there is an instability.

We have obtained a sufficient and necessary condition of instability of an equilibrium $\bar{F}(v_z)$ against electrostatic perturbations: if v_0 is the point of global minimum of $\bar{F}$,[29]

$$\boxed{P(v_0) = \int_{-\infty}^{+\infty} \mathrm{d}v_z \, \frac{\bar{F}(v_0) - \bar{F}(v_z)}{(v_z - v_0)^2} < 0 \quad \Leftrightarrow \quad \bar{F} \text{ is unstable}} . \qquad (4.15)$$

This is the famous *Penrose's instability criterion* (the famous criterion, not the famous Penrose; it was proved by *Oliver* Penrose 1960, in a stylistically somewhat different way than I did it here). Note that considerations of the kind presented above can be used to work out the wave-number intervals, corresponding to various troughs in $\bar{F}$, in which instabilities exist.

Intuitively, the criterion (4.15) says that, in order for a distribution to be unstable, it needs to have a trough and this trough must be deep enough. Thus, if $F(v_0) = 0$, i.e., if the distribution has a "hole", it is always unstable (an extreme example of this is the two-stream instability; see Exercise 3.5). Another corollary is that you cannot stablise a distribution by just adding some particles in a narrow interval around v_0, as this would create two minima nearby, which, the filled interval being narrow, are still going to render the system unstable. To change that, you must fill the trough substantially with particles—hence the tendency to flatten bumps into plateaus, which we will discover in §7 (this answers, albeit in very broad strokes, the question posed at the beginning of §4 about the types of stable distributions towards which the unstable ones will be pushed as the instabilities saturate).

Exercise 4.3. Consider a single-minimum distribution like the one in Fig. 17(b), but with the global maximum on the right and the lesser maximum on the left of the minimum.

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[29]Another way of putting this is: *a distribution $\bar{F}$ is unstable iff it has a minimum at some v_0 for which $P(v_0) < 0$.* Obviously, if $P(v_0) < 0$ at some minimum, it is also negative at the global minimum.

Draw various possible Nyquist diagrams and convince yourself that Penrose's criterion works. If you enjoy this, think of a distribution that would give rise to the Nyquist diagram in Fig. 16(b).

Exercise 4.4. What happens if the distribution function $\bar{F}$ has an inflection point, i.e., $\bar{F}(v_0) = 0$, $\bar{F}'(v_0) = 0$, $\bar{F}''(v_0) = 0$?

Exercise 4.5. What happens if the distribution function has a trough with a flat bottom (i.e., a flat minimum over some interval of velocities)?

4.4. *Bumps, Beams, Streams and Flows*

An elementary example of the use of Penrose's criterion is the two-stream instability, first introduced in Exercise 3.5. The case of two cold streams, represented by (3.60) and Fig. 12(a), is obviously unstable because there is a gaping hole in this distribution. What if we now give these streams some thermal width? This can be modeled by the double-Lorentzian distribution (Fig. 12b)

$$F_e(v_z) = \frac{n_e v_{\rm b}}{2\pi} \left[\frac{1}{(v_z - u_{\rm b})^2 + v_{\rm b}^2} + \frac{1}{(v_z + u_{\rm b})^2 + v_{\rm b}^2} \right], \tag{4.16}$$

which is particularly easy to handle analytically. For the moment, we will consider the ions to be infinitely heavy, so $\bar{F} = F_e$.

Since the distribution (4.16) is symmetric, it can only have its minimum at $v_0 = 0$. Asking that it should indeed be a minimum, rather than a maximum, i.e., $\bar{F}'(0) > 0$, one finds that the condition for this is

$$u_{\rm b} > \frac{v_{\rm b}}{\sqrt{3}}. \tag{4.17}$$

Otherwise, the two streams are too wide (in velocity space) and the distribution is monotonically decreasing, so, according to §4.2, it is stable.

If the condition (4.17) is satisfied, the distribution has two bumps, but is this enough to make it unstable? Substituting this distribution into Penrose's criterion (4.15) and doing the integral exactly,[30] we get the necessary and sufficient instability condition:

$$P(0) = -\frac{u_{\rm b}^2 - v_{\rm b}^2}{(u_{\rm b}^2 + v_{\rm b}^2)^2} < 0 \quad \Leftrightarrow \quad \boxed{u_{\rm b} > v_{\rm b}}. \tag{4.18}$$

Thus, if the streams are sufficiently fast and/or their thermal spread is sufficiently narrow, an instability will occur, but it is not quite enough just to have a little trough. Note, by the way, that Penrose's criterion does not differentiate between hydrodynamic (cold) and kinetic (hot) instability mechanisms (§3.7).

Exercise 4.6. Use Nyquist's method to work out the range of wave numbers at which perturbations will grow for the two-stream instability (you will find the answer in Jackson 1960—yes, *that* Jackson). Convince yourself that this is all in accord with the explicit solution of the dispersion relation obtained in Q4.

It is obvious how these considerations can be generalised to more complicated situations, e.g., to cases where the streams have different velocities, where one of them is, in fact, the thermal bulk of the distribution and the other is a little bump on its tail (§3.7), where there are more than two streams, etc. The streams also need not be composed

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[30]The easiest way to do it is to turn the integration path along the real axis into a loop by completing it with a semicircle at positive or negative complex infinity, where the integrand vanishes, and use Cauchy's formula.
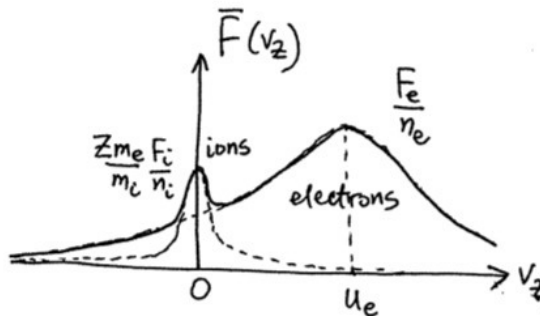
FIGURE 19. Combined distribution (4.2) for cold ions and hot electrons (cf. Fig. 13).

of the particles of the same species: indeed, as we saw in (4.1), in the linear theory, the distributions of all species are additively combined into $\bar{F}$ with weights that are inversely proportional to their masses [see (4.2)]. Thus, the ion-acoustic instability (§3.9) is also just a kind of of two-stream—or, if you like, bump-on-tail—instability, with the entire hot and mighty electron distribution making up a magnificent bump on the tail of the cold, m_e/m_i-weighted ion one (Fig. 19).[31] When the streams/beams have thermal spreads, they are more commonly thought of as mean flows—or currents, if the electron flows are not compensated by the ion ones.

Exercise 4.7. Construct an equilibrium distribution to model your favorite plasma system with flows and/or beams and investigate its stability: find the growth rate as a function of wave number, instability conditions, etc.

4.5. *Anisotropies*

So we have found that various holes, bumps, streams, beams, flows, currents and other such nonmonotonic features in the (combined, multispecies) equilibrium distribution present an instability risk, unless they are sufficiently small, shallow, wide and/or close enough to the thermal bulk. All of these are, of course, *anisotropic* features—indeed, as we saw in Exercise 4.2, 3D-isotropic distributions are harmless, instability-wise. It turns out that anisotropies of the distribution function in velocity space are dangerous even when the distribution decays monotonically in all directions.[32] However, the instabilities that occur in such situations are *electromagnetic*, rather than electrostatic, and so require an investigation into the properties of the transverse dielectric function of the kind derived in Q2 or Q3, but for a general equilibrium. A nice treatment of anisotropy-driven instabilities can be found in Krall & Trivelpiece (1973) and an even more thorough one in Davidson (1983). In §§8.2.2 and 8.4.1, I will show in quite a simple way that, at least in principle, there is always energy available to be extracted from anisotropic distributions.

Exercise 4.8. Criterion of instability of anisotropic distributions. This is an independent-study topic. Consider linear stability of general distribution functions to *electromagnetic* perturbations and work out the stability criterion in the spirit of §4. You should discover that anisotropic distributions such as, e.g., the bi-Maxwellian (10.43), tend to be unstable. Krall & Trivelpiece (1973, §9.10) would be a good place to read about it, but do range beyond.

---

[31] In fact, when the two species' temperatures are the same, there is still an instability, whose criterion can again be obtained by the Nyquist-Penrose method: see Jackson (1960).

[32] In Q3, you have an opportunity to derive the most famous of all instabilities triggered by anisotropy.

5. Energy, Entropy, Heating, Irreversibility and Phase Mixing

While we are done with the "calculational" part of linear theory (calculating whether the field perturbations oscillate, decay or grow, and at what rates), we are not yet done with the "conceptual" part: what exactly is going on, mathematically and physically? The plan of addressing this question in this section is as follows.

• I will show that Landau damping of perturbations of a plasma in thermal equilibrium leads to the heating of this equilibrium—basically, that energy is conserved. This is not a surprise, but it is useful to see explicitly how this works (§5.1).

• I will then ask how it is possible to have heating (an irreversible process) in a plasma that was assumed collisionless and must conserve entropy. In other words, why, physically, is Landau damping a damping? This will lead us to consider entropy evolution in our system and to introduce an important concept of free energy (§5.2).

• In the above context, we will examine (§§5.3 and 5.6) the Laplace-transform solution (3.8) for the perturbed distribution function and establish the phenomenon of *phase mixing*—emergence of fine structure in velocity (phase) space. This will allow collisions and, therefore, irreversiblity back in (§5.5). We will also see that the Landau-damped solutions are *not* eigenmodes (while growing solutions can be), and so conclude that it made sense to insist on using an initial-value-problem formalism.

5.1. *Energy Conservation and Heating*

Let us go back to the full, nonlinear Vlasov–Poisson system, with the collision term restored:

$$\frac{\partial f_\alpha}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f_\alpha - \frac{q_\alpha}{m_\alpha} (\boldsymbol{\nabla}\varphi) \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{v}} = \left(\frac{\partial f_\alpha}{\partial t} \right)_{\mathrm{c}} , \tag{5.1}$$

$$-\nabla^2 \varphi = 4\pi \sum_\alpha q_\alpha \int \mathrm{d}^3 \boldsymbol{v} \, f_\alpha. \tag{5.2}$$

Let us calculate the rate of change of the electric energy:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3\boldsymbol{r} \, \frac{E^2}{8\pi} = \underbrace{\int \mathrm{d}^3\boldsymbol{r} \, \frac{\boldsymbol{\nabla}\varphi}{4\pi} \cdot \frac{\partial(\boldsymbol{\nabla}\varphi)}{\partial t}}_{\text{by parts}} = -\underbrace{\int \mathrm{d}^3\boldsymbol{r} \, \frac{\varphi}{4\pi} \frac{\partial}{\partial t} \nabla^2\varphi}_{\text{use (5.2)}} = \sum_\alpha q_\alpha \underbrace{\iint \mathrm{d}^3\boldsymbol{r} \, \mathrm{d}^3\boldsymbol{v} \, \varphi \, \frac{\partial f_\alpha}{\partial t}}_{\text{use (5.1)}}$$

$$= \sum_\alpha q_\alpha \iint \mathrm{d}^3\boldsymbol{r} \, \mathrm{d}^3\boldsymbol{v} \, \varphi \left[\underbrace{-\boldsymbol{v} \cdot \boldsymbol{\nabla} f_\alpha}_{\substack{\text{by parts}}} + \underbrace{\frac{q_\alpha}{m_\alpha}(\boldsymbol{\nabla}\varphi) \cdot \frac{\partial f_\alpha}{\partial \boldsymbol{v}}}_{\substack{\text{vanishes} \\ \text{because} \\ f(\pm\infty) = 0}} + \underbrace{\left(\frac{\partial f_\alpha}{\partial t} \right)_{\mathrm{c}}}_{\substack{\text{vanishes} \\ \text{because} \\ \text{number of} \\ \text{particles is} \\ \text{conserved}}} \right]$$

$$= \sum_\alpha q_\alpha \iint \mathrm{d}^3\boldsymbol{r} \, \mathrm{d}^3\boldsymbol{v} \, f_\alpha \boldsymbol{v} \cdot \boldsymbol{\nabla}\varphi = -\int \mathrm{d}^3\boldsymbol{r} \, \boldsymbol{E} \cdot \boldsymbol{j}, \tag{5.3}$$

where $\boldsymbol{j}$ is the current density. So the rate of change of the electric field is minus the rate at which electric field does work on the charges, a.k.a. Joule heating—not a surprising result. The energy goes into accelerating particles, of course: the rate of change of their

kinetic energy is

$$\frac{dK}{dt} = \sum_\alpha \iint d^3r\, d^3v\, \frac{m_\alpha v^2}{2} \underbrace{\frac{\partial f_\alpha}{\partial t}}_{\text{use (5.1)}}$$

$$= \sum_\alpha \iint d^3r\, d^3v\, \frac{m_\alpha v^2}{2} \Bigg[\underbrace{-\boldsymbol{v}\cdot\boldsymbol{\nabla}f_\alpha}_{\substack{\text{vanishes}\\\text{because}\\\text{full}\\\text{divergence}}} + \underbrace{\frac{q_\alpha}{m_\alpha}(\boldsymbol{\nabla}\varphi)\cdot\frac{\partial f_\alpha}{\partial\boldsymbol{v}}}_{\text{by parts in }\boldsymbol{v}} + \underbrace{\left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}}}_{\substack{\text{vanishes}\\\text{because}\\\text{energy is}\\\text{conserved}}} \Bigg]$$

$$= -\sum_\alpha q_\alpha \iint d^3r\, d^3v\, f_\alpha \boldsymbol{v}\cdot\boldsymbol{\nabla}\varphi = \int d^3r\, \boldsymbol{E}\cdot\boldsymbol{j}. \tag{5.4}$$

Combining (5.3) and (5.4) gives us the law of energy conservation:

$$\boxed{\frac{d}{dt}\left(K + \int d^3r\, \frac{E^2}{8\pi}\right) = 0}. \tag{5.5}$$

Exercise 5.1. Demonstrate energy conservation for the more general case in which magnetic-field perturbations are also allowed.

Thus, if the perturbations are damped, the energy of the particles must increase—this is usually called *heating*. Strictly speaking, however, heating is a slow, irreversible increase in the mean temperature of the thermal equilibrium. Let us make this statement quantitative. Consider a Maxwellian plasma, homogeneous in space but possibly with some slow dependence on time (cf. §2):

$$f_{0\alpha} = \frac{n_\alpha}{(\pi v_{\mathrm{th}\alpha}^2)^{3/2}}\, e^{-v^2/v_{\mathrm{th}\alpha}^2} = n_\alpha \left(\frac{m_\alpha}{2\pi T_\alpha}\right)^{3/2} e^{-m_\alpha v^2/2T_\alpha}. \tag{5.6}$$

In a homogeneous system with a fixed volume, the density n_α is constant in time because the number of particles is constant: $d(V n_\alpha)/dt = 0$. The temperature, however, is allowed to change: $T_\alpha = T_\alpha(t)$. The total kinetic energy of the particles is

$$K = V \sum_\alpha \underbrace{\int d^3v\, \frac{m_\alpha v^2}{2}\, f_{0\alpha}}_{=\frac{3}{2}n_\alpha T_\alpha} + \sum_\alpha \iint d^3r\, d^3v\, \frac{m_\alpha v^2}{2}\, \delta f_\alpha. \tag{5.7}$$

Let us average this over time, as per (2.7): the perturbed part vanishes and we have

$$\langle K\rangle = V \sum_\alpha \frac{3}{2} n_\alpha T_\alpha. \tag{5.8}$$

Averaging also (5.5) and using (5.8), we get

$$\sum_\alpha \frac{3}{2} n_\alpha \frac{dT_\alpha}{dt} = -\frac{d}{dt}\frac{1}{V}\int d^3r\, \frac{\langle E^2\rangle}{8\pi}, \tag{5.9}$$

so the heating rate of the equilibrium equals the rate of decrease of the mean energy of the perturbations.

We saw that the perturbations evolve according to (3.16). If we wait for a while, only the slowest-damped mode will matter, with all others exponentially small in comparison. Let us call its frequency and its damping rate $\omega_{\boldsymbol{k}}$ and $\gamma_{\boldsymbol{k}} < 0$, respectively, so $\boldsymbol{E}_{\boldsymbol{k}} \propto e^{-i\omega_{\boldsymbol{k}}t + \gamma_{\boldsymbol{k}}t}$. If $|\gamma_{\boldsymbol{k}}| \ll \omega_{\boldsymbol{k}}$, the time average (2.7) can be defined in such a way that $\omega_{\boldsymbol{k}}^{-1} \ll \Delta t \ll |\gamma_{\boldsymbol{k}}|^{-1}$. Then (5.9) becomes

$$\sum_{\alpha} \frac{3}{2}\, n_\alpha \frac{\mathrm{d}T_\alpha}{\mathrm{d}t} = -\sum_{\boldsymbol{k}} 2\gamma_{\boldsymbol{k}} \frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi} > 0. \tag{5.10}$$

The Landau damping rate of the electric-field perturbations is the heating rate of the equilibrium.[33]

This result, while at first glance utterly obvious, might, on reflection, appear to be paradoxical: surely, the heating of the equilibrium implies increasing entropy—but the damping that is leading to the heating is collisionless and, in a collisionless system, in view of the H theorem, how can the entropy change?

5.2. *Entropy and Free Energy*

The kinetic entropy for each species of particles is defined to be

$$S_\alpha = -\iint \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{v}\, f_\alpha \ln f_\alpha. \tag{5.11}$$

This quantity [or, indeed, the full-phase-space integral of any quantity that is a function only of f_α; see (8.8)] can only be changed by collisions and, furthermore, the plasma-physics version of Boltzmann's H theorem says that

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{\alpha} S_\alpha = -\sum_{\alpha} \iint \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{v} \left(\frac{\partial f_\alpha}{\partial t}\right)_{\mathrm{c}} \ln f_\alpha \geqslant 0, \tag{5.12}$$

where equality is achieved iff all f_α are Maxwellian with the same temperature $T_\alpha = T$.

Thus, *if collisions are ignored, the total entropy stays constant and everything that happens is, in principle, reversible. So how can there be net damping of waves and, worse still, net heating of the equilibrium particle distribution?!* Presumably, any damping solution can be turned into a growing solution by reversing all particle trajectories—so should the overall perturbation level not stay constant?

As I already noted in §5.1, strictly speaking, heating is the increase of the equilibrium temperature—and, therefore, of the equilibrium entropy. Indeed, for each species, the equilibrium entropy is

$$S_0 = -\iint \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{v}\, f_0 \ln f_0 = -\iint \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{v}\, f_0 \left\{ \ln\left[n\left(\frac{m}{2\pi}\right)^{3/2} \right] - \frac{3}{2}\ln T - \frac{mv^2}{2T} \right\}$$

$$= V\left\{ -n\ln\left[n\left(\frac{m}{2\pi}\right)^{3/2} \right] + \frac{3}{2}\, n\ln T + \frac{3}{2}\, n \right\}, \tag{5.13}$$

where I have used $\int \mathrm{d}^3\boldsymbol{v}\, (mv^2/2)f_0 = (3/2)nT$. Since $n = \mathrm{const}$,

$$T\frac{\mathrm{d}S_0}{\mathrm{d}t} = V\frac{3}{2}\, n\frac{\mathrm{d}T}{\mathrm{d}t}, \tag{5.14}$$

so heating is indeed associated with an increase of S_0.

Since, according to (5.10), this can be successfully accomplished by collisionless damping and since entropy overall can only increase due to collisions, we must search for the

---

[33]Obviously, the damping of waves on particles of species α increases only the temperature of that species.

"missing entropy" (or, rather, for the missing *decrease* of entropy) in the perturbed part of the distribution. The mean entropy associated with the perturbed distribution is[34]

$$
\langle \delta S \rangle = \langle S - S_0 \rangle = - \iint \mathrm{d}^3 \boldsymbol{r} \, \mathrm{d}^3 \boldsymbol{v} \, \langle (f_0 + \delta f) \ln(f_0 + \delta f) - f_0 \ln f_0 \rangle
$$
$$
= - \iint \mathrm{d}^3 \boldsymbol{r} \, \mathrm{d}^3 \boldsymbol{v} \left\langle (f_0 + \delta f) \left[\ln f_0 + \frac{\delta f}{f_0} - \frac{\delta f^2}{2 f_0^2} + \dots \right] - f_0 \ln f_0 \right\rangle
$$
$$
\approx - \iint \mathrm{d}^3 \boldsymbol{r} \, \mathrm{d}^3 \boldsymbol{v} \, \frac{\langle \delta f^2 \rangle}{2 f_0}, \tag{5.15}
$$

after expanding to second order in small $\delta f / f_0$ and using $\langle \delta f \rangle = 0$. The total mean entropy of each species, $\langle S \rangle = S_0 + \langle \delta S \rangle$, can only by changed by collisions, so, if collisions are ignored, any heating of a given species, i.e., any increase in its S_0 [see (5.14)] must be compensated by a *decrease* in its $\langle \delta S \rangle$. The latter can only be achieved by *increasing* $\langle \delta f^2 \rangle$: indeed, using (5.14) and (5.15), we find[35]

$$
T \left(\frac{\mathrm{d} S_0}{\mathrm{d} t} + \frac{\mathrm{d} \langle \delta S \rangle}{\mathrm{d} t} \right) = V \frac{3}{2} n \frac{\mathrm{d} T}{\mathrm{d} t} - \frac{\mathrm{d}}{\mathrm{d} t} \iint \mathrm{d}^3 \boldsymbol{r} \, \mathrm{d}^3 \boldsymbol{v} \, \frac{T \langle \delta f^2 \rangle}{2 f_0}
$$
$$
= - \iint \mathrm{d}^3 \boldsymbol{r} \, \mathrm{d}^3 \boldsymbol{v} \, T \left\langle \left(\frac{\partial f}{\partial t} \right)_{\mathrm{c}} \ln f \right\rangle. \tag{5.16}
$$

If the right-hand side is ignored, T can only increase if $\langle \delta f^2 \rangle$ increases too.

It is useful to work out the collision term in (5.16) in terms of f_0 and δf: using the fact that $\langle \delta f \rangle = 0$ by definition and that the number of particles is conserved by the collision operator, we get

$$
\iint \mathrm{d}^3 \boldsymbol{r} \, \mathrm{d}^3 \boldsymbol{v} \, T \left\langle \left(\frac{\partial f}{\partial t} \right)_{\mathrm{c}} \ln f \right\rangle \approx \iint \mathrm{d}^3 \boldsymbol{r} \, \mathrm{d}^3 \boldsymbol{v} \left[T \left(\frac{\partial f_0}{\partial t} \right)_{\mathrm{c}} \ln f_0 + \left\langle \frac{T \delta f}{f_0} \left(\frac{\partial \delta f}{\partial t} \right)_{\mathrm{c}} \right\rangle \right]
$$
$$
= V \int \mathrm{d}^3 \boldsymbol{v} \, \frac{m v^2}{2} \left(\frac{\partial f_0}{\partial t} \right)_{\mathrm{c}} + \iint \mathrm{d}^3 \boldsymbol{r} \, \mathrm{d}^3 \boldsymbol{v} \left\langle \frac{T \delta f}{f_0} \left(\frac{\partial \delta f}{\partial t} \right)_{\mathrm{c}} \right\rangle. \tag{5.17}
$$

The second term is the collisional damping of δf, of which more will be said soon. The first term is the collisional energy exchange between the equilibrium distributions of different species (intra-species collisions conserve energy, but inter-species ones do not, because there is friction between species). If the species under consideration is α, this energy exchange can be represented as $\sum_{\alpha'} \nu_{\alpha \alpha'} (T_\alpha - T_{\alpha'})$ (see, e.g., Helander & Sigmar 2005) and will act to equilibrate temperatures between species as the system strives toward thermal equilibrium. If the collision frequencies $\nu_{\alpha \alpha'}$ are small, this is a slow effect. Due to overall energy conservation, the energy-exchange terms vanish exactly if (5.17) is summed over species.

Finally, let us sum (5.16) over species and use (5.9) to relate the total heating to the rate

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[34] As an aside, note that this piece of the calculation is entirely idependent of what f_0 is. It simply demonstrates that the entropy of an averaged distribution f_0 is always larger than that of the exact distribution f, as long as $\delta f \ll f_0$. If the average is reinterpreted as a coarse graining over phase space, this argument is sometimes viewed as a kind of "proof" (or illustration) of the second law of thermodynamics. Indeed, take f_0 and f to be the same at some initial time t. Then $S_0(t) = \langle S(t) \rangle$. Now advance to time $t + \delta t$. Some small δf arises, but coarse graining "deletes" the information contained in it and (5.15) shows that $S_0(t + \delta t) > \langle S(t + \delta t) \rangle$ (cf. the general statistical-mechanical argument to the same effect: Schekochihin 2019, §§12.4 and 13.4).

[35] In the second term, T can be brought inside the time derivative because $\langle \delta f^2 \rangle / f_0 \ll f_0$.

of change of the electric-perturbation energy:

$$
\boxed{
\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left[\sum_{\alpha}\iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{v}\,\frac{T_\alpha\langle\delta f_\alpha^2\rangle}{2f_{0\alpha}}+\int \mathrm{d}^3\boldsymbol{r}\,\frac{\langle E^2\rangle}{8\pi}\right]}_{\equiv\,W}=\sum_\alpha\iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{v}\,\left\langle\frac{T_\alpha\delta f_\alpha}{f_{0\alpha}}\left(\frac{\partial\delta f_\alpha}{\partial t}\right)_{\mathrm{c}}\right\rangle\leqslant 0
}\,,
$$

$$(5.18)$$

where we used (5.17) in the right-hand side (with the total equilibrium collisional energy-exchange terms vanishing upon summation over species). The right-hand side must be non-positive-definite because collisions cannot decrease entropy [see (5.12)].

Equation (5.18) is a way to express the idea that, except for the effect of collisions, the change in the electric-perturbation energy ($=-$heating) must be compensated by the change in $\langle\delta f^2\rangle$, in terms of a *conservation law* of a quadratic positive-definite quantity, W, that measures the total amount of perturbation in the system (a quadratic norm of the perturbed solution).[36] It is not hard to realise that this quantity is the *free energy* of the perturbed state, comprising the entropy of the perturbed distribution and the energy of the electric field:

$$
W=\mathcal{E}-\sum_\alpha T_\alpha\langle\delta S_\alpha\rangle,\quad \mathcal{E}=\int \mathrm{d}^3\boldsymbol{r}\,\frac{\langle E^2\rangle}{8\pi}.
\tag{5.19}
$$

It is quite a typical situation in non-equilibrium systems that there is an energy-like (quadratic in the relevant fields and positive definite) quantity, which is conserved except for dissipation. For example, in hydrodynamics, the motions of a fluid are governed by the Navier–Stokes equation:

$$
\rho\left(\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right)=-\boldsymbol{\nabla}p+\mu\Delta\boldsymbol{u},
\tag{5.20}
$$

where $\boldsymbol{u}$ is velocity, ρ mass density ($\rho=$ const for an incompressible fluid), p pressure and μ the dynamical viscosity of the fluid. The conservation law is

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{r}\,\frac{\rho u^2}{2}=-\mu\int \mathrm{d}^3\boldsymbol{r}\,|\boldsymbol{\nabla}\boldsymbol{u}|^2\leqslant 0.
\tag{5.21}
$$

The conserved quadratic quantity is kinetic energy and the negative-definite dissipation (leading to net entropy production) is viscous heating.[37]

Exercise 5.2. Free Energy and Kinetic Energy of Mean Plasma Flow. Suppose the perturbation δf contains a mean flow of particles, with velocity $\boldsymbol{u}$. Show that it is then always formally possible to decompose

$$
\delta f=\frac{2\boldsymbol{u}\cdot\boldsymbol{v}}{v_{\mathrm{th}}^2}\,f_0+h,
\tag{5.22}
$$

where $\int \mathrm{d}^3\boldsymbol{v}\,\boldsymbol{v}h=0$. Hence show that

$$
\iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{v}\,\frac{T\langle\delta f^2\rangle}{2f_0}=\int \mathrm{d}^3\boldsymbol{r}\,\frac{mn\langle u^2\rangle}{2}+\iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{v}\,\frac{T\langle h^2\rangle}{2f_0},
\tag{5.23}
$$

---

[36] Note that the existence of such a quantity implies that the Maxwellian equilibrium is stable: if a quadratic norm of the perturbed solution cannot grow, clearly there cannot be any exponentially growing solutions. This is known as Newcomb's theorem, first communicated to the world in the paper by Bernstein (1958, Appendix I). A generalisation of this principle to isotropic distributions is the subject of Q5(c) and of §8.3, where the conserved quantity W will reemerge in a different way, confirming its status as a Platonic entity that cannot be avoided.

[37] You will find a similar conservation law for incompressible MHD if, in §11.11, you work out the time evolution of $\int \mathrm{d}^3\boldsymbol{r}\,(\rho u^2/2+B^2/8\pi)$ assuming $\rho=$ const and $\boldsymbol{\nabla}\cdot\boldsymbol{u}=0$.
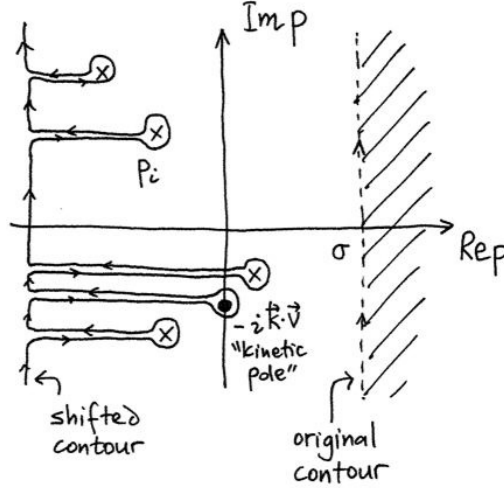
FIGURE 20. Shifting the integration contour in (5.25). This is analogous to Fig. 5 but note the additional "kinetic" pole.

i.e., the entropic part of the free energy is equal to the kinetic energy of the mean plasma flow plus the free entropy associated with the part of the perturbed distribution that has no mean velocity. This means that "fluid" energy budgets such as (5.21) are not just analogs, but particular cases, of the free-energy budget (5.18). In §5.7, I will show how these ideas play out for Langmuir waves.

Thus, as the electric perturbations decay via Landau damping, the perturbed distribution function must grow. This calls for going back to our solution for it (§3.1) and analysing carefully the behaviour of δf.

5.3. *Structure of Perturbed Distribution Function*

Start with our solution (3.8) for $\delta f(p)$ and substitute into it the solution (3.15) for $\varphi(p)$:

$$\delta \hat{f}(p) = \underbrace{\frac{1}{p + i\boldsymbol{k} \cdot \boldsymbol{v}}}_{\substack{\text{"kinetic"} \\ \text{(ballistic)} \\ \text{pole}}} \left\{ i \frac{q}{m} \left[\underbrace{\sum_i \frac{c_i}{p - p_i}}_{\substack{\text{poles} \\ \text{representing} \\ \text{linear} \\ \text{modes, see} \\ \text{(3.17)}}} + A(p) \right] \boldsymbol{k} \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} + g \right\}. \qquad (5.24)$$

To compute the inverse Laplace transform (3.6), we adopt the same method as in §3.1 (Fig. 5), viz., shift the integration contour to large negative Re p as shown in Fig. 20 and

use Cauchy's formula:

$$\delta f(t) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{i\infty+\sigma} \mathrm{d}p\, e^{pt} \delta\hat{f}(p) = \underbrace{i\frac{q}{m} \sum_i \frac{c_i e^{p_i t}}{p_i + i\boldsymbol{k}\cdot\boldsymbol{v}} \boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}}}_{\substack{\text{eigenmode-like} \\ \text{solution, comes from} \\ \text{the poles of } \varphi(p)}}$$

$$+ \underbrace{e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}\left\{ g - i\frac{q}{m}\left[\sum_i \frac{c_i}{p_i + i\boldsymbol{k}\cdot\boldsymbol{v}} + A(-i\boldsymbol{k}\cdot\boldsymbol{v})\right]\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}}\right\}}_{\substack{\text{ballistic response,} \\ \text{comes from} \\ p = -i\boldsymbol{k}\cdot\boldsymbol{v}}}. \qquad (5.25)$$

A perceptive reader has spotted that this formula does not seem to satisfy $\delta f(t = 0) = g$ unless $A(-i\boldsymbol{k}\cdot\boldsymbol{v}) = 0$. This is because, as explained in fotnote 12, the method for calculating the inverse Laplace transform that involves discarding the integral along the vertical part of the shifted contour in Fig. 20 only works in the limit of long times. It is an amusing exercise in complex analysis to show that, in the (overly restrictive) case of $\hat{\varphi}(p)$ decaying quickly at $\mathrm{Re}\, p \to -\infty$, the solution (5.25) is also valid at finite t and, accordingly, $A(-i\boldsymbol{k}\cdot\boldsymbol{v}) = 0$, i.e., $A(p)$ vanishes for any purely imaginary p.

The solution (5.25) teaches us two important things.

1) First, *the Landau-damped solution is not an eigenmode.* Even though the evolution of the potential, given by (3.16), does look like a sum of damped eigenmodes of the form $\varphi \propto e^{p_i t}$, $\mathrm{Re}\, p_i < 0$, the full solution of the Vlasov–Poisson system does *not* decay: there is a part of $\delta f(t)$, the "ballistic response" $\propto e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}$, that oscillates without decaying—in fact, we shall see in §5.6 that δf even has a growing part! It is this part that is responsible for keeping free energy conserved, as per (5.18) without collisions (§5.7). Thus, you may think of Landau damping as a process of transferring (free) energy from the electric-field perturbations to the perturbations of the distribution function.

In contrast to the case of damping, a growing solution ($\mathrm{Re}\, p_i > 0$) can be viewed as an eigenmode because, after a few growth times, the first term in (5.25) will be exponentially larger than the ballistic term. This will allow us to ignore the latter in our treatment of QLT (§7.1)—a handy, although not necessary (see Q8), simplification. Note that reversibility is not an issue for the growing solutions: so, there may be (and often are) damped solutions as well, so what? We only care about the growing modes because they will be all that is left if we wait long enough.

2) Secondly, the δf perturbations have *fine structure in velocity (phase) space.* This structure gets finer with time: roughly speaking, if $\delta f \propto e^{-ikvt}$, then

$$\frac{1}{\delta f}\frac{\partial \delta f}{\partial v} \sim ikt \to \infty \quad \text{as} \quad t \to \infty. \qquad (5.26)$$

This phenomenon is called *phase mixing.* You can think of the basic mechanism responsible for it as a shearing in phase space: the homogeneous part of the linearised kinetic equation,

$$\frac{\partial \delta f}{\partial t} + v\frac{\partial \delta f}{\partial z} = \dots, \qquad (5.27)$$

describes advection of δf by a linear shear flow in the the (z, v) plane. This turns any δf structure in this plane into long thin filaments, with large gradients in v (Fig. 21).
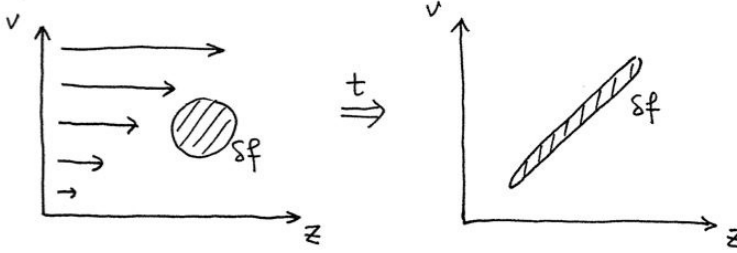
FIGURE 21. Shearing of δf in phase space.

5.4. *Landau Damping Is Phase Mixing*

Phase mixing helps us make sense of the notion that, even though φ is the velocity integral of δf, the former can be decaying while the latter is not:

$$\varphi = \frac{4\pi}{k^2} \sum_{\alpha} q_{\alpha} \underbrace{\int \mathrm{d}^3\boldsymbol{v}\, \delta f_{\alpha}}_{\substack{\text{fine} \\ \text{structure} \\ \text{cancels}}} \propto e^{-\gamma t} \to 0. \tag{5.28}$$

The velocity integral over the fine structure increasingly cancels as time goes on—a perturbation initially "visible" as φ phase-mixes away, disappearing into the negative entropy associated with the fine velocity dependence of δf [see (5.15)].

More generally speaking, one can similarly argue that the refinement of velocity dependence of δf causes lower velocity moments of δf (density, flow velocity, pressure, heat flux, and so on) to decrease with time, transferring free energy to higher moments (ever higher as time goes on). One way to formalise this statement neatly is in terms of Hermite moments: since Hermite polynomials are orthogonal, the free energy of the perturbed distribution can be written as a sum of "energies" of the Hermite moments [see (10.73)]. It is then possible to represent the Landau-damped perturbations as having a broad spectrum in Hermite space, with the majority of the free energy residing in high-order moments—infinitely high in the formal limit of zero collisionality and infinite time (see Q7 and Kanekar *et al.* 2015).

Since the mth-order Hermite moment can, for $m \gg 1$, be asymptotically represented as a cosine function in v space oscillating with the "frequency" $\sqrt{2m}/v_{\text{th}}$ [see (10.74)], (5.26) implies that the typical order of the moment in which the free energy resides grows with time as $m \sim (kv_{\text{th}}t)^2$.

Taking Hermite (or other kind of) moments of the kinetic equation is essentially the procedure for deriving "fluid" equations for the plasma—or, rather, plasma becomes a fluid if this procedure can be stopped after a few moments (e.g., in the limit of strong collisionality, this happens at the third moment; see Dellar 2015 and Parra 2019*a*). Since Landau damping is a long-time effect of this phase-mixing process, it cannot be captured by any fluid approximation to the kinetic system involving a truncation of the hierarchy of moment equations at some finite-order moment—it is an essentially kinetic effect "beyond all orders".

One useful way to see this is by examining the structure of Langmuir hydrodynamics, which was the subject of Exercise 3.1. The moment hierarchy can be truncated by assuming $kv_{\text{th}e}/\omega \gg 1$, but one can never capture Landau damping however many moments one keeps: indeed, the
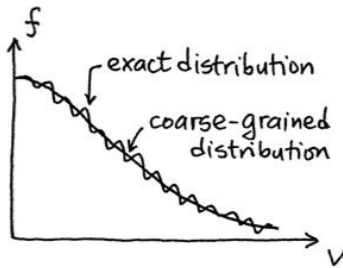
FIGURE 22. Coarse graining of the distribution function.

Landau damping rate (3.41) for, say, a Maxwellian plasma will be $\gamma \propto \exp(-\omega^2/k^2 v_{\mathrm{th}e}^2)$, all coefficients in the Taylor expansion of which in powers of $k v_{\mathrm{th}e}/\omega$ are zero.

5.5. *Role of Collisions*

As ever larger velocity-space gradients emerge, it becomes inevitable that at some point they will become so large that *collisions can no longer be ignored*. Indeed, the Landau collision operator is a Fokker–Planck (diffusion) operator in velocity space [see (1.47)] and so it will eventually wipe out the fine structure in $\boldsymbol{v}$, however small is the collision frequency ν. Let us estimate how long this takes.

The size of the velocity-space gradients of δf due to ballistic response is given by (5.26). Then the collision term is

$$\left(\frac{\partial \delta f}{\partial t}\right)_{\mathrm{c}} \sim \nu v_{\mathrm{th}}^2 \frac{\partial^2 \delta f}{\partial v^2} \sim -\nu v_{\mathrm{th}}^2 k^2 t^2 \delta f. \tag{5.29}$$

Solving for the time evolution of the perturbed distribution function due to collisions, we get

$$\frac{\partial \delta f}{\partial t} \sim -\nu (k v_{\mathrm{th}} t)^2 \delta f \quad \Rightarrow \quad \delta f \sim \exp\left(-\frac{1}{3}\nu k^2 v_{\mathrm{th}}^2 t^3\right) \equiv e^{-(t/t_{\mathrm{c}})^3}. \tag{5.30}$$

Therefore, the characteristic collisional decay time is

$$\boxed{t_{\mathrm{c}} \sim \frac{1}{\nu^{1/3}(k v_{\mathrm{th}})^{2/3}}}. \tag{5.31}$$

Note that $t_{\mathrm{c}} \ll \nu^{-1}$ provided $\nu \ll k v_{\mathrm{th}}$, i.e., t_{c} is within the range of times over which our "collisionless" theory is valid. After time t_{c}, "collisionless" damping becomes irreversible because the part of δf that is fast-varying in velocity space is lost (entropy has grown) and so it is no longer possible, even in principle, to invert all particle trajectories, have the system retrace back its steps, "phase-unmix" and thus "undamp" the damped perturbation.

In a sufficiently collisionless system, phase unmixing is, in fact, possible if nonlinearity is allowed—giving rise to the beautiful phenomenon of *plasma echo*, in which perturbations can first appear to be damped away but then come back from phase space (§6.2). This effect is a source of much preoccupation to pure mathematicians (Villani 2014; Bedrossian 2016): indeed the validity of the linearised Vlasov equation (3.1) as a sensible approximation to the full nonlinear one (2.12) is in question if the velocity derivative $\partial \delta f/\partial \boldsymbol{v}$ in the last term of the latter starts growing uncontrollably. Phase unmixing has also recently turned out to have interesting consequences for the role of Landau damping in plasma turbulence (Schekochihin *et al.* 2016; Adkins & Schekochihin 2018).

Some rather purist theoreticians sometimes choose to replace collisional estimates of the type discussed above by a stipulation that $\delta f(\boldsymbol{v})$ must be *"coarse-grained"* beyond some suitably chosen scale in $\boldsymbol{v}$ (Fig. 22)—this is equivalent to saying that the formation of the fine-structured phase-space part of δf constitutes a loss of information and so leads to growth of entropy (i.e., loss of negative entropy associated with $\langle \delta f^2 \rangle$). Somewhat non-rigorously, this means that we can just consider the ballistic term in (5.25) to have been wiped out and use the coarse-grained (i.e., velocity-space-averaged) version of δf:[38]

$$\overline{\delta f} = i\,\frac{q}{m}\sum_i \frac{c_i e^{p_i t}}{p_i + i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}. \tag{5.32}$$

We can check that the correct solution (3.16) for the potential can be recovered from this:

$$\begin{aligned}
\varphi &= \frac{4\pi}{k^2}\sum_\alpha q_\alpha \int \mathrm{d}^3\boldsymbol{v}\,\overline{\delta f}_\alpha \\
&= \sum_i c_i e^{p_i t}\Bigg[\underbrace{\sum_\alpha i\,\frac{4\pi q_\alpha^2}{m_\alpha k^2}\int \mathrm{d}^3\boldsymbol{v}\,\frac{1}{p_i + i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}} - 1}_{=\,-\epsilon(p_i,\boldsymbol{k})\,=\,0\ \text{by definition of}\ p_i} + 1\Bigg] = \sum_i c_i e^{p_i t}.
\end{aligned} \tag{5.33}$$

If you are wondering how this works without the coarse-graining kludge, read on.

5.6. *Further Analysis of δf: Case–van Kampen Mode*

Having given a rather qualitative analysis of the structure and consequences of the solution (5.25), I anticipate a degree of dissatisfaction from a perceptive reader. Yes, there is a non-decaying piece of δf. But conservation of free energy in a collisionless system in the face of Landau damping in fact requires $\langle \delta f^2 \rangle$ to grow, not just to fail to decay [see (5.18)]. How do we see that this does indeed happen? The analysis that follows addresses this question. These considerations are not really necessary for most practical plasma-physics calculations (see, however, Q8), but it may be necessary for your peace of mind and greater comfort with this whole conceptual framework.

Let us rearrange the solution (5.25) as follows:

$$\boxed{\delta f(t) = i\,\frac{q}{m}\sum_i c_i\,\frac{e^{p_i t} - e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}}{p_i + i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} + (g+\dots)e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}.} \tag{5.34}$$

The second term is the ballistic evolution of perturbations (particles flying apart in straight lines at different velocities)—a homogeneous solution of the kinetic equation (3.1). This develops a lot of fine-scale velocity-space structure, but obviously does not grow. The first term, a particular solution arising from the (linear) wave-particle interaction, is more interesting, especially around the resonances $\operatorname{Re} p_i + \boldsymbol{k}\cdot\boldsymbol{v} = 0$.

Consider one of the modes, $p_i = -i\omega + \gamma$, and assume $\gamma \ll \boldsymbol{k}\cdot\boldsymbol{v} \sim \omega$. This allows us to introduce "intermediate" times:

$$\frac{1}{\boldsymbol{k}\cdot\boldsymbol{v}} \ll t \ll \frac{1}{\gamma}. \tag{5.35}$$

This means that the wave has had time to oscillate, phase mixing has got underway, but the perturbation has not yet been damped away significantly. We have then, for the

---

[38]With an understanding that any integral involving the resonant denominator must be taken along the Landau contour (see Q8). If you adopt this shorthand, you can, nonrigorously but often expeditiously, use Fourier transforms into frequency space, rather than Laplace transforms.
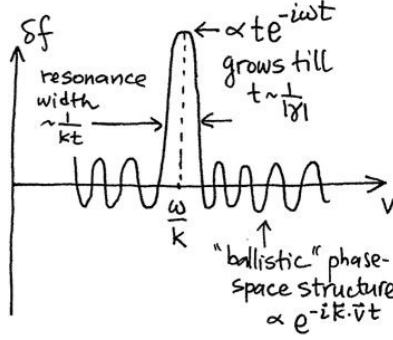
FIGURE 23. Emergence of the Case–van Kampen mode.

relevant piece of the perturbed distribution (5.34),

$$\delta f \propto \frac{e^{p_i t} - e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}}{p_i + i\boldsymbol{k}\cdot\boldsymbol{v}} = -ie^{-i\omega t}\frac{e^{\gamma t} - e^{-i(\boldsymbol{k}\cdot\boldsymbol{v}-\omega)t}}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega - i\gamma} \approx -ie^{-i\omega t}\frac{1 - e^{-i(\boldsymbol{k}\cdot\boldsymbol{v}-\omega)t}}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega}, \qquad (5.36)$$

with the last, approximate, expression valid at the intermediate times (5.35), assuming also that, even though we might be close to the resonance, we shall not come closer than γ, viz., $|\boldsymbol{k}\cdot\boldsymbol{v} - \omega| \gg \gamma$. Respecting this ordering, but taking $|\boldsymbol{k}\cdot\boldsymbol{v} - \omega| \ll 1/t$, we find

$$\delta f \propto t\,e^{-i\omega t}. \qquad (5.37)$$

Thus, δf has a peak that *grows with time*, emerging from the sea of fine-scale but constant-amplitude structures (Fig. 23). The width of this peak is obviously $|\boldsymbol{k}\cdot\boldsymbol{v} - \omega| \sim 1/t$ and so δf around the resonance develops a sharp structure, which, in the formal limit $t \to \infty$ (but respecting $\gamma t \ll 1$, i.e., with infinitesimal damping), tends to a delta function:

$$\boxed{\delta f \propto -ie^{-i\omega t}\frac{1 - e^{-i(\boldsymbol{k}\cdot\boldsymbol{v}-\omega)t}}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega} \to e^{-i\omega t}\pi\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \omega)} \quad \text{as} \quad t \to \infty. \qquad (5.38)$$

Here is a "formal" proof:

$$\frac{1 - e^{-ixt}}{x} = \underbrace{\frac{1 - \cos xt}{x}}_{\substack{\text{finite as} \\ t \to \infty, \\ \text{even at} \\ x = 0}} + i\underbrace{\frac{\sin xt}{x}}_{\substack{= it \text{ at} \\ x = 0, \\ \text{so dom-} \\ \text{inant}}} \approx \frac{e^{ixt} - e^{-ixt}}{2x} = \frac{i}{2}\int_{-t}^{t} \mathrm{d}t'e^{ixt'} \to i\pi\delta(x) \quad \text{as} \quad t \to \infty.$$

$$(5.39)$$

The delta-function solution (5.38) is an instance of a *Case–van Kampen mode* (van Kampen 1955; Case 1959)—an object that belongs to the mathematical realms briefly alluded to at the end of §3.5. Note that writing the solution in the vicinity of the resonance in this form is tantamount to stipulating that any integral taken with respect to $\boldsymbol{v}$ (or $\boldsymbol{k}$) and involving δf must always be done along the Landau contour, circumventing the pole from below [cf. (3.23)]. We will find the representation (5.38) of δf useful in working out the quasilinear theory of Landau damping in Q8.

If we restore finite damping, all this goes on until $t \sim 1/\gamma$, with the delta function reaching the height $\propto 1/\gamma$ and width $\propto \gamma$. In the limit $t \gg 1/\gamma$, the damped part of the solution decays, $e^{\gamma t} \to 0$, and we are left with just the ballistic part, the second term in (5.25).

5.7. *Free-Energy Conservation for Landau-Damped Langmuir Waves*

Finally, let us convince ourselves that, if we ignore collisions, we can recover (5.18) with a zero right-hand side from the full collisionless Landau-damped solution given by (3.16) and (5.34). For simplicity, let us consider the case of electron Langmuir waves and prove that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3\boldsymbol{v}\, \frac{T|\delta f_{\boldsymbol{k}}|^2}{2 f_0} = -2\gamma_{\boldsymbol{k}} \frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi} = -\frac{\mathrm{d}}{\mathrm{d}t} \frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi}}. \tag{5.40}$$

In (5.34), let the relevant root of the dispersion relation be $p_i = -i\omega_{\mathrm{pe}} + \gamma_{\boldsymbol{k}}$, where $\gamma_{\boldsymbol{k}}$ is given by (3.41), and assume a Maxwellian f_0. Based on the discussion §5.6, we should expect $\delta f_{\boldsymbol{k}}$ to develop a growing delta-like peak around the resonance $\boldsymbol{k} \cdot \boldsymbol{v} \approx \omega_{\mathrm{pe}}$. In this region of velocity space, the distribution function (5.34) for electrons ($q = -e$) is

$$\delta f_{\boldsymbol{k}}^{(\mathrm{res})} \approx \frac{e}{m_e} \underbrace{c_i e^{p_i t}}_{=\,\varphi_{\boldsymbol{k}}} \frac{1 - e^{-i(\boldsymbol{k} \cdot \boldsymbol{v} - i p_i)t}}{\boldsymbol{k} \cdot \boldsymbol{v} - i p_i} \frac{2\boldsymbol{k} \cdot \boldsymbol{v}}{v_{\mathrm{the}}^2} f_0 \approx \frac{e\varphi_{\boldsymbol{k}}}{T} \boldsymbol{k} \cdot \boldsymbol{v} \underbrace{\frac{1 - e^{-i(\boldsymbol{k} \cdot \boldsymbol{v} - \omega_{\mathrm{pe}})t}}{\boldsymbol{k} \cdot \boldsymbol{v} - \omega_{\mathrm{pe}}}}_{\approx\, i\pi\delta(\boldsymbol{k} \cdot \boldsymbol{v} - \omega_{\mathrm{pe}})} f_0. \tag{5.41}$$

We are going to have to compute $|\delta f_{\boldsymbol{k}}|^2$ and squaring delta functions is a dangerous game belonging to the class of games that one must play *veeery carefully.*[39] Here is how:

$$\frac{\partial}{\partial t} \left| \frac{1 - e^{-ixt}}{x} \right|^2 = \frac{\partial}{\partial t} \frac{4}{x^2} \sin^2 \frac{xt}{2} = \frac{2\sin xt}{x} \xrightarrow{t \to \infty} 2\pi\delta(x) \quad \Rightarrow \quad \left| \frac{1 - e^{-ixt}}{x} \right|^2 \xrightarrow{t \to \infty} 2\pi t\delta(x). \tag{5.42}$$

Using this prescription,

$$
\begin{aligned}
\int \mathrm{d}^3\boldsymbol{v}\, \frac{T|\delta f_{\boldsymbol{k}}^{(\mathrm{res})}|^2}{2 f_0} &= \int \mathrm{d}^3\boldsymbol{v}\, \frac{e^2|\varphi_{\boldsymbol{k}}|^2}{2T}(\boldsymbol{k} \cdot \boldsymbol{v})^2\, 2\pi t\delta(\boldsymbol{k} \cdot \boldsymbol{v} - \omega_{\mathrm{pe}}) f_0 \\
&= t|\varphi_{\boldsymbol{k}}|^2 \frac{2\pi e^2 \omega_{\mathrm{pe}}^2}{m_e v_{\mathrm{the}}^2 k} F\left(\frac{\omega_{\mathrm{pe}}}{k}\right) \\
&= 4t \frac{k^2|\varphi_{\boldsymbol{k}}|^2}{8\pi} \underbrace{\frac{\omega_{\mathrm{pe}}^4}{k^3} \frac{\pi}{n_e v_{\mathrm{the}}^2} F\left(\frac{\omega_{\mathrm{pe}}}{k}\right)}_{=\,-\gamma_{\boldsymbol{k}} > 0;\ \text{see (3.41)}} \\
&= -4\gamma_{\boldsymbol{k}} t \frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi}.
\end{aligned}
\tag{5.43}
$$

Thus, the entropic part of the free energy grows secularly with time (assuming still $\gamma_{\boldsymbol{k}} t \ll 1$). Its time derivative is

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3\boldsymbol{v}\, \frac{T|\delta f_{\boldsymbol{k}}^{(\mathrm{res})}|^2}{2 f_0} \approx -4\gamma_{\boldsymbol{k}} \frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi} = -2\frac{\mathrm{d}}{\mathrm{d}t} \frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{8\pi}. \tag{5.44}$$

Despite what it looks like, the extra factor of 2 in (5.44) compared to (5.40) is a feature, not a bug. If you have done Exercise 3.1 (or even just paid attention in §2.1), you know that a Langmuir oscillations involve some mean (oscillating) flows of the plasma, and so a sloshing of energy between potential, $|\boldsymbol{E}_{\boldsymbol{k}}|^2/8\pi$, and kinetic, $n_e m_e |\boldsymbol{u}_{\boldsymbol{k}}|^2/2$, where $\boldsymbol{u}_{\boldsymbol{k}} = (1/n_e) \int \mathrm{d}^3\boldsymbol{v}\, \boldsymbol{v}\, \delta f_{\boldsymbol{k}}$. These flows are contained in the non-resonant ("thermal") part of $\delta f_{\boldsymbol{k}}$, i.e., in $\delta f_{\boldsymbol{k}}$ at velocities such that $\boldsymbol{k} \cdot \boldsymbol{v} \ll \omega_{\mathrm{pe}}$. In this region of velocity space, let us write the distribution function (5.34) as follows:

$$\delta f_{\boldsymbol{k}}^{(\mathrm{th})} = \frac{e}{m_e} \underbrace{c_i e^{p_i t}}_{=\,\varphi_{\boldsymbol{k}}} \frac{1}{\boldsymbol{k} \cdot \boldsymbol{v} - i p_i} \frac{2\boldsymbol{k} \cdot \boldsymbol{v}}{v_{\mathrm{the}}^2} f_0 + e^{-i\boldsymbol{k} \cdot \boldsymbol{v} t} h_{\boldsymbol{k}} \approx -\frac{e\varphi_{\boldsymbol{k}}}{T} \frac{\boldsymbol{k} \cdot \boldsymbol{v}}{\omega_{\mathrm{pe}}} f_0 + e^{-i\boldsymbol{k} \cdot \boldsymbol{v} t} h_{\boldsymbol{k}}, \tag{5.45}$$

where $h_{\boldsymbol{k}}$ denotes everything in (5.34) that multiplies $e^{-i\boldsymbol{k} \cdot \boldsymbol{v} t}$. This should remind you of

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[39] I am grateful to Glenn Wagner for making me practice what I preach and work out this derivation correctly, with all the meaningful factors of 2.

Exercise 5.2. The first term in (5.45) is precisely the plasma flow:

$$\boldsymbol{u_k} = \frac{1}{n_e}\int \mathrm{d}^3\boldsymbol{v}\,\boldsymbol{v}\,\delta f_{\boldsymbol{k}}^{(\mathrm{th})} = -\frac{e\varphi_{\boldsymbol{k}}}{T\omega_{\mathrm{pe}}n_e}\boldsymbol{k}\cdot\underbrace{\int \mathrm{d}^3\boldsymbol{v}\,\boldsymbol{v}\boldsymbol{v}f_0}_{=\,\mathsf{I}n_e v_{\mathrm{the}}^2/2} = \frac{e\boldsymbol{E_k}}{i\omega_{\mathrm{pe}}m_e} \qquad (5.46)$$

The contribution from the second term in (5.45) to the velocity integral has vanished in the limit $\boldsymbol{k}\cdot\boldsymbol{v}t \gg 1$. Note that (5.46) just says that $m_e\dot{\boldsymbol{u}}_{\boldsymbol{k}} = -e\boldsymbol{E_k}$, as indeed is the case in a Langmuir oscillation. It is not hard to check that $n_e m_e|\boldsymbol{u_k}|^2/2 = |\boldsymbol{E_k}|^2/8\pi$.

Let us now work out the contribution of (5.45) to the free energy [cf. (5.23)]:

$$\int\mathrm{d}^3\boldsymbol{v}\,\frac{T|\delta f_{\boldsymbol{k}}^{(\mathrm{th})}|^2}{2f_0} = \int\mathrm{d}^3\boldsymbol{v}\left[\frac{e^2|\varphi_{\boldsymbol{k}}|^2}{2T\omega_{\mathrm{pe}}^2}(\boldsymbol{k}\cdot\boldsymbol{v})^2 f_0 + \frac{T|h_{\boldsymbol{k}}|^2}{2f_0} + e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}(\dots) + e^{i\boldsymbol{k}\cdot\boldsymbol{v}t}(\dots)^*\right]$$
$$= \frac{|\boldsymbol{E_k}|^2}{8\pi} + \int\mathrm{d}^3\boldsymbol{v}\,\frac{T|h_{\boldsymbol{k}}|^2}{2f_0}, \qquad (5.47)$$

where the velocity integral has been done in the same way as in (5.46) and the contribution from the terms that oscillate in $\boldsymbol{v}$ has been integrated away. The salient property of $h_{\boldsymbol{k}}$ is that it does not depend on time. Therefore, its contribution to the time derivative of the free energy vanishes and we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\mathrm{d}^3\boldsymbol{v}\,\frac{T|\delta f_{\boldsymbol{k}}^{(\mathrm{th})}|^2}{2f_0} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{|\boldsymbol{E_k}|^2}{8\pi}, \qquad (5.48)$$

i.e., the kinetic energy of the Langmuir oscillations decays at the same rate as their potential (electric) energy.

Finally, adding (5.44) and (5.48), we get (5.40), q.e.d.

Exercise 5.3. Free-Energy Conservation for Sound Waves. Consider an ion-acoustic wave (§3.8) damped on electrons according to (3.76) (with $u_e = 0$). Work out the contributions to free energy from ions and from electrons. Check that free energy is conserved.

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6. Nonlinear Theory: Two Pretty Nuggets

Nonlinear theory of anything is, of course, hard—indeed, in most cases, intractable. These days, an impatient researcher's answer to being faced with a hard question is to outsource it to a computer. This sometimes leads to spectacular successes, but also, somewhat more frequently, to spectacular confusion about how to interpret the output. In dealing with a steady stream of data produced by ever more powerful machines, one is sometimes helped by the residual memory of analytical results obtained in the prehistoric era when computation was harder than theory and plasma physicists had to find ingenious ways to solve nonlinear problems "by hand"—which usually required finding ingenious ways of posing problems that were solvable. These could be separated into two broad categories: interesting particular cases of nonlinear behaviour involving usually just a few interacting waves and systems of very many waves amenable to some approximate statistical treatment.[40] Here I will give two very pretty examples of the former, before moving on to an extended presentation of the latter in §§7, 8.6 and onwards.

6.1. *Nonlinear Landau Damping*

Coming soon. See O'Neil (1965); Mazitov (1965).

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[40]The third kind is asking for general criteria of certain kinds of behaviour, such as stability or otherwise—we shall dabble in this type of nonlinear theory in §§8.1–8.4.
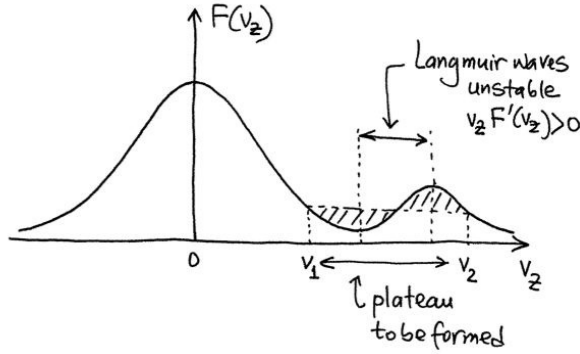
FIGURE 24. An unstable distribution with a bump on its tail.

6.2. *Plasma Echo*

Coming soon. See Gould *et al.* (1967); Malmberg *et al.* (1968).

7. Quasilinear Theory

7.1. *General Scheme of QLT*

In §§3 and 5, I discussed at length the structure of the linear solution corresponding to a Landau-damped initial perturbation. This could be adequately done for a Maxwellian plasma and the result was that, after some interesting transient time-dependent phase-space dynamics, perturbations damped away and their energy turned into heat, increasing somewhat the temperature of the equilibrium (see, however, Q8).

Let us now turn to a different problem: an *unstable* (and so decidedly non-Maxwellian) equilibrium distribution giving rise to exponentially growing perturbations. The specific example on which we shall focus is the *bump-on-tail instability,* which involves generation of unstable Langmuir waves with phase velocities corresponding to instances of positive derivative of the equilibrium distribution function (Fig. 24). The energy of the waves grows exponentially:

$$\frac{\partial |\boldsymbol{E_k}|^2}{\partial t} = 2\gamma_{\boldsymbol{k}} |\boldsymbol{E_k}|^2, \quad \gamma_{\boldsymbol{k}} = \frac{\pi}{2} \frac{\omega_{\mathrm{pe}}^3}{k^2} \frac{1}{n_e} F'\!\left(\frac{\omega_{\mathrm{pe}}}{k}\right), \tag{7.1}$$

where $F(v_z) = \int \mathrm{d}v_x \int \mathrm{d}v_y \, f_0(\boldsymbol{v})$ [see (3.41)]. In the absence of collisions, the only way for the system to achieve a nontrivial steady state (i.e., such that $|\boldsymbol{E_k}|^2$ is not just zero everywhere) is by adjusting the equilibrium distribution so that

$$\gamma_{\boldsymbol{k}} = 0 \quad \Leftrightarrow \quad F'\!\left(\frac{\omega_{\mathrm{pe}}}{k}\right) = 0 \tag{7.2}$$

at all k where $|\boldsymbol{E_k}|^2 \neq 0$, say, $k \in [k_2, k_1]$. If we translate this range into velocities, $v = \omega_{\mathrm{pe}}/k$, we see that the equilibrium must develop a flat spot:

$$F'(v) = 0 \quad \text{for} \quad v \in [v_1, v_2] = \left[\frac{\omega_{\mathrm{pe}}}{k_1}, \frac{\omega_{\mathrm{pe}}}{k_2} \right]. \tag{7.3}$$

This is called a *quasilinear plateau* (§7.4). Obviously, the rest of the equilibrium distribution may (and will) also be modified in some, to be determined, way (§§7.6, 7.7).

These modifications of the original (initial) equilibrium distribution can be accomplished by the growing fluctuations via the feedback mechanism already discussed in

§2.3, namely, the equilibrium distribution will evolve slowly according to (2.11):

$$\frac{\partial f_0}{\partial t} = -\frac{q}{m} \sum_{\boldsymbol{k}} \left\langle \varphi_{\boldsymbol{k}}^* i\boldsymbol{k} \cdot \frac{\partial \delta f_{\boldsymbol{k}}}{\partial \boldsymbol{v}} \right\rangle. \tag{7.4}$$

The time averaging here [see (2.7)] is over $\omega_{\mathrm{pe}}^{-1} \ll \Delta t \ll \gamma_{\boldsymbol{k}}^{-1}$.

The general scheme of QLT is:

- start with an unstable equilibrium f_0,
- use the linearised equations (3.1) and (3.2) to work out the linear solution for the growing perturbations $\varphi_{\boldsymbol{k}}$ and $\delta f_{\boldsymbol{k}}$ in terms of f_0,
- use this solution in (7.4) to evolve f_0, leading, if everything works as it is supposed to, to an ever less unstable equilibrium.

We shall keep only the fastest growing mode (all others are exponentially small after a while), and so the solution (3.16) for the electric perturbations is

$$\varphi_{\boldsymbol{k}} = c_{\boldsymbol{k}} e^{(-i\omega_{\boldsymbol{k}} + \gamma_{\boldsymbol{k}})t}. \tag{7.5}$$

In the solution (5.25) for the perturbed distribution function, we may ignore the ballistic term because the exponentially growing piece (the first term) will eventually leave all this velocity-space structure behind,[41] so

$$\delta f_{\boldsymbol{k}} = i\frac{q}{m} \frac{c_{\boldsymbol{k}} e^{(-i\omega_{\boldsymbol{k}} + \gamma_{\boldsymbol{k}})t}}{-i\omega_{\boldsymbol{k}} + \gamma_{\boldsymbol{k}} + i\boldsymbol{k}\cdot\boldsymbol{v}} \boldsymbol{k} \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} = \frac{q}{m} \frac{\varphi_{\boldsymbol{k}}}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}} - i\gamma_{\boldsymbol{k}}} \boldsymbol{k} \cdot \frac{\partial f_0}{\partial \boldsymbol{v}}. \tag{7.6}$$

Substituting (7.6) into (7.4), we get

$$\frac{\partial f_0}{\partial t} = -\frac{q^2}{m^2} \sum_{\boldsymbol{k}} |\varphi_{\boldsymbol{k}}|^2 i\boldsymbol{k} \cdot \frac{\partial}{\partial \boldsymbol{v}} \frac{1}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}} - i\gamma_{\boldsymbol{k}}} \boldsymbol{k} \cdot \frac{\partial f_0}{\partial \boldsymbol{v}} = \frac{\partial}{\partial \boldsymbol{v}} \cdot \mathsf{D}(\boldsymbol{v}) \cdot \frac{\partial f_0}{\partial \boldsymbol{v}}. \tag{7.7}$$

This is a diffusion equation in velocity space, with a velocity-dependent diffusion matrix

$$\begin{aligned}
\mathsf{D}(\boldsymbol{v}) &= -\frac{q^2}{m^2} \sum_{\boldsymbol{k}} i\boldsymbol{k}\boldsymbol{k} |\varphi_{\boldsymbol{k}}|^2 \frac{1}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}} - i\gamma_{\boldsymbol{k}}} \\
&= -\frac{q^2}{m^2} \sum_{\boldsymbol{k}} i\boldsymbol{k}\boldsymbol{k} |\varphi_{\boldsymbol{k}}|^2 \frac{1}{2} \left(\frac{1}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}} - i\gamma_{\boldsymbol{k}}} + \underbrace{\frac{1}{-\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{-\boldsymbol{k}} - i\gamma_{-\boldsymbol{k}}}}_{\substack{\text{here I changed} \\ \text{variables } \boldsymbol{k} \to -\boldsymbol{k}}} \right) \\
&= -\frac{q^2}{m^2} \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\boldsymbol{k}}{k^2} |\boldsymbol{E}_{\boldsymbol{k}}|^2 \frac{i}{2} \left(\frac{1}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}} - i\gamma_{\boldsymbol{k}}} - \frac{1}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}} + i\gamma_{\boldsymbol{k}}} \right) \\
&= \frac{q^2}{m^2} \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\boldsymbol{k}}{k^2} |\boldsymbol{E}_{\boldsymbol{k}}|^2 \operatorname{Im} \frac{1}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}} - i\gamma_{\boldsymbol{k}}} \\
&= \frac{q^2}{m^2} \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\boldsymbol{k}}{k^2} |\boldsymbol{E}_{\boldsymbol{k}}|^2 \frac{\gamma_{\boldsymbol{k}}}{(\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}})^2 + \gamma_{\boldsymbol{k}}^2}. \tag{7.8}
\end{aligned}$$

To obtain these expressions, I used the fact that the wave-number sum could just as well be over $-\boldsymbol{k}$ instead of $\boldsymbol{k}$ and that $\omega_{-\boldsymbol{k}} = -\omega_{\boldsymbol{k}}$, $\gamma_{-\boldsymbol{k}} = \gamma_{\boldsymbol{k}}$ [because $\varphi_{-\boldsymbol{k}} = \varphi_{\boldsymbol{k}}^*$, where $\varphi_{\boldsymbol{k}}$ is given by (7.5)]. The matrix D is manifestly positive definite—this adds credence to

---

[41] See, however, Q8 on how to avoid having to wait for this to happen: in fact, the results below are valid for $\gamma_k t \lesssim 1$ as well.

our *a priori* expectation that a plateau will form: diffusion will smooth the bump in the equilibrium distribution function.

The question of validity of the QL approximation is quite nontrivial and rife with subtle issues, all of which I have swept under the carpet. They mostly have to do with whether coupling between waves [the last term in (2.12)] truly remains unimportant throughout the quasilinear evolution, especially as the plateau regime is approached and the growth rate of the waves becomes infinitesimally small. If you wish to investigate further—and in the process gain a finer appreciation of nonlinear plasma theory,—the article by Besse *et al.* (2011) (as far as I know, the most recent substantial contribution to the topic) is a good starting point, from which you can follow the paper trail backwards in time and decide for yourself whether you trust the QLT.

7.2. *Conservation Laws*

When we get to the stage of solving a specific problem (§7.3), we shall see that paying attention to energy and momentum budgets leads one to important discoveries about the QL evolution of the particle distribution. With this prospect in mind, as well as by way of a consistency check, let us check that the quasilinear kinetic equation (7.7) conserves energy and momentum.

7.2.1. *Energy Conservation*

The rate of change of the particle energy associated with the equilibrium distribution is

$$
\begin{aligned}
\frac{dK}{dt} &\equiv \frac{d}{dt} \sum_\alpha \iint d^3\boldsymbol{r}\, d^3\boldsymbol{v}\, \frac{m_\alpha v^2}{2} f_{0\alpha} = \sum_\alpha \iint d^3\boldsymbol{r}\, d^3\boldsymbol{v}\, \frac{m_\alpha v^2}{2} \frac{\partial}{\partial \boldsymbol{v}} \cdot \mathsf{D}_\alpha(\boldsymbol{v}) \cdot \frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}} \\
&= -\sum_\alpha \iint d^3\boldsymbol{r}\, d^3\boldsymbol{v}\, m_\alpha \boldsymbol{v} \cdot \mathsf{D}_\alpha(\boldsymbol{v}) \cdot \frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}} \\
&= -V \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \sum_{\boldsymbol{k}} \frac{|\boldsymbol{E_k}|^2}{k^2} \int d^3\boldsymbol{v}\, \underbrace{\mathrm{Im} \frac{\boldsymbol{k} \cdot \boldsymbol{v}}{\boldsymbol{k} \cdot \boldsymbol{v} - \omega_{\boldsymbol{k}} - i\gamma_{\boldsymbol{k}}}}_{\substack{\text{add and substract} \\ \omega_{\boldsymbol{k}} + i\gamma_{\boldsymbol{k}} \text{ in the} \\ \text{numerator}}} \boldsymbol{k} \cdot \frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}} \\
&= -V \sum_{\boldsymbol{k}} \frac{|\boldsymbol{E_k}|^2}{4\pi} \mathrm{Im}\Bigg[(\omega_{\boldsymbol{k}} + i\gamma_{\boldsymbol{k}}) \underbrace{\sum_\alpha \frac{\omega_{p\alpha}^2}{k^2} \frac{1}{n_\alpha} \int d^3\boldsymbol{v}\, \frac{1}{\boldsymbol{k} \cdot \boldsymbol{v} - \omega_{\boldsymbol{k}} - i\gamma_{\boldsymbol{k}}} \boldsymbol{k} \cdot \frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}}_{\substack{= 1 - \epsilon(-i\omega_{\boldsymbol{k}} + \gamma_{\boldsymbol{k}}, \boldsymbol{k}) = 1 \\ \text{because } -i\omega_{\boldsymbol{k}} + i\gamma_{\boldsymbol{k}} \text{ is a solution of} \\ \text{dispersion relation } \epsilon = 0}}\Bigg] \\
&= -V \sum_{\boldsymbol{k}} 2\gamma_{\boldsymbol{k}} \frac{|\boldsymbol{E_k}|^2}{8\pi} = -\frac{d}{dt} \int d^3\boldsymbol{r}\, \frac{E^2}{8\pi}, \quad \text{q.e.d.,}
\end{aligned}
\tag{7.9}
$$

viz., the total energy $K + \int d^3\boldsymbol{r}\, E^2/8\pi = \text{const.}$ This will motivate §7.6.

7.2.2. *Momentum Conservation*

Since unstable distributions like the one with a bump on its tail can carry net momentum, it is useful to calculate its rate of change:

$$
\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}t} \equiv \frac{\mathrm{d}}{\mathrm{d}t} \sum_\alpha \iint \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{v}\, m_\alpha \boldsymbol{v} f_{0\alpha} = \sum_\alpha \iint \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{v}\, m_\alpha \boldsymbol{v}\, \frac{\partial}{\partial \boldsymbol{v}} \cdot \mathsf{D}_\alpha(\boldsymbol{v}) \cdot \frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}
$$

$$
= -\sum_\alpha \iint \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{v}\, m_\alpha \mathsf{D}_\alpha(\boldsymbol{v}) \cdot \frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}
$$

$$
= -V \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}|\boldsymbol{E_k}|^2}{k^2} \int \mathrm{d}^3\boldsymbol{v}\, \mathrm{Im}\frac{1}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}} - i\gamma_{\boldsymbol{k}}} \boldsymbol{k} \cdot \frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}
$$

$$
= -V \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}|\boldsymbol{E_k}|^2}{4\pi} \mathrm{Im}\underbrace{\sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2} \frac{1}{n_\alpha} \int \mathrm{d}^3\boldsymbol{v}\, \frac{1}{\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}} - i\gamma_{\boldsymbol{k}}} \boldsymbol{k} \cdot \frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}}}_{= 1 - \epsilon(-i\omega_{\boldsymbol{k}} + \gamma_{\boldsymbol{k}}, \boldsymbol{k}) = 1} = 0, \quad \text{q.e.d.,} \qquad (7.10)
$$

so momentum can only be redistributed between particles. This will motivate §7.7.

7.3. *Quasilinear Equations for the Bump-on-Tail Instability in 1D*

What follows is the iconic QL calculation due to Vedenov *et al.* (1962) and Drummond & Pines (1962).

These two papers, published in the same year, are a spectacular example of the "great minds think alike" principle. They both appeared in the Proceedings of the 1961 IAEA conference in Salzburg, one of those early international gatherings in which the Soviets (grudgingly allowed out) and the Westerners (eager to meet them) were telling each other about their achievements in the recently declassified controlled-nuclear-fusion research. The entire Proceedings are now online (http://www-naweb.iaea.org/napc/physics/FEC/1961.pdf)—a remarkable historical document and a great read, containing, besides the papers (in three languages), a record of the discussions that were held. The Vedenov *et al.* (1962) paper is in Russian, but you will find a very similar exposition in English in the review by Vedenov (1963) published shortly thereafter. Two other lucid accounts of quasilinear theory belonging to the same historical (and historic!) period are in the books by Kadomtsev (1965) and by Sagdeev & Galeev (1969).

As promised in §7.1, I shall consider electron Langmuir oscillations in 1D, triggered by the bump-on-tail instability, so $\boldsymbol{k} = k\hat{\boldsymbol{z}}$, $\omega_k = \omega_{\mathrm{p}e}$, γ_k is given by (7.1), and the QL diffusion equation (7.7) becomes

$$
\boxed{\frac{\partial F}{\partial t} = \frac{\partial}{\partial v} D(v) \frac{\partial F}{\partial v}}, \qquad (7.11)
$$

where $F(v)$ is the 1D version of the distribution function, $v = v_z$ and the diffusion coefficient, now a scalar, is given by

$$
\boxed{D(v) = \frac{e^2}{m_e^2} \sum_k |E_k|^2 \,\mathrm{Im}\frac{1}{kv - \omega_{\mathrm{p}e} - i\gamma_k}}. \qquad (7.12)
$$

As I explained when discussing (7.1), if the fluctuation field has reached a steady state, it must be the case that

$$
\frac{\partial |E_k|^2}{\partial t} = 2\gamma_k |E_k|^2 = 0 \quad \Leftrightarrow \quad |E_k|^2 = 0 \quad \text{or} \quad \gamma_k = 0, \qquad (7.13)
$$

i.e., either there are no fluctuations or there is no growth (or damping) rate. The result is a non-zero spectrum of fluctuations in the interval $k \in [k_2, k_1]$ and a plateau in the
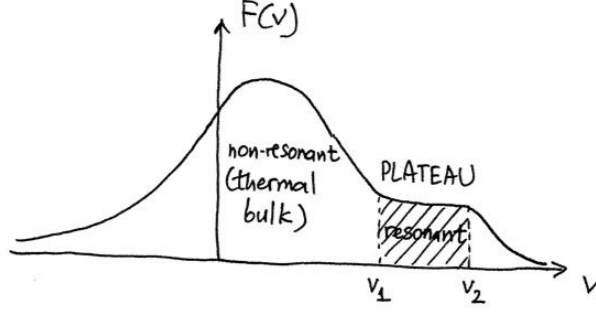
FIGURE 25. Quasilinear plateau.

distribution function in the corresponding velocity interval $v \in [v_1, v_2] = [\omega_{\mathrm{pe}}/k_1, \omega_{\mathrm{pe}}/k_2]$ [see (7.3) and Fig. 25]. The particles in this interval are *resonant* with Langmuir waves; those in the ("thermal") bulk of the distribution outside this interval are *non-resonant*. We will have solved the problem completely if we find

- F^{plateau}, the value of the distribution function in the interval $[v_1, v_2]$,
- the extent of the plateau $[v_1, v_2]$,
- the functional form of the spectrum $|E_k|^2$ in the interval $[k_2, k_1]$,
- any modifications of the distribution function $F(v)$ of the nonresonant particles.

7.4. *Resonant Region: QL Plateau and Spectrum*

Consider first the velocities $v \in [v_1, v_2]$ for which $|E_{k=\omega_{\mathrm{pe}}/v}|^2 \neq 0$. If L is the linear size of the system, the wave-number sum in (7.12) can be replaced by an integral according to

$$\sum_k = \sum_k \frac{\Delta k}{2\pi/L} = \frac{L}{2\pi} \int \mathrm{d}k. \tag{7.14}$$

Defining the continuous energy spectrum of the Langmuir waves[42]

$$W(k) = \frac{L}{2\pi} \frac{|E_k|^2}{4\pi}, \tag{7.15}$$

we rewrite the QL diffusion coefficient (7.12) in the following form:

$$D(v) = \frac{e^2}{m_e^2} \frac{1}{v} \operatorname{Im} \int \mathrm{d}k \, \frac{4\pi W(k)}{k - \omega_{\mathrm{pe}}/v - i\gamma_k/v} = \frac{e^2}{m_e^2} \frac{4\pi^2}{v} W\left(\frac{\omega_{\mathrm{pe}}}{v}\right). \tag{7.16}$$

The last expression is obtained by applying Plemelj's formula (3.25) to the wave-number integral taken in the limit $\gamma_k/v \to +0$.[43] Substituting now this expression into (7.11) and using also (7.1) to express

$$\gamma_k = \frac{\pi}{2} \frac{\omega_{\mathrm{pe}}^3}{k^2} \frac{1}{n_e} F'\left(\frac{\omega_{\mathrm{pe}}}{k}\right) \quad \Rightarrow \quad \frac{\partial F}{\partial v} = \left[\frac{2}{\pi} \frac{k^2}{\omega_{\mathrm{pe}}^3} n_e \gamma_k \right]_{k=\omega_{\mathrm{pe}}/v}, \tag{7.17}$$

---

[42]Why the prefactor is $1/4\pi$, rather than $1/8\pi$, will become clear at the end of §7.5.

[43]In fact, the wave-number integral must be taken along the Landau contour (i.e., keeping the contour below the pole) regardless of the sign of γ_k: see Q8, where you get to work out the QL theory for Landau-damped, rather than growing, perturbations.
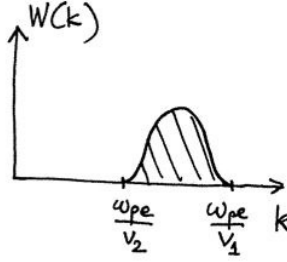
FIGURE 26. Quasilinear spectrum.

we get

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial v}\frac{e^2}{m_e^2}\frac{4\pi^2}{v}W\left(\frac{\omega_{\mathrm{pe}}}{v}\right)\left[\frac{2}{\pi}\frac{k^2}{\omega_{\mathrm{pe}}^3}n_e\gamma_k\right]_{k=\omega_{pe}/v} = \frac{\partial}{\partial v}\frac{\omega_{\mathrm{pe}}}{m_e v^3}\underbrace{2\gamma_{\omega_{\mathrm{pe}}/v}W\left(\frac{\omega_{\mathrm{pe}}}{v}\right)}_{=\,\partial W/\partial t}. \tag{7.18}$$

Rearranging, we arrive at

$$\frac{\partial}{\partial t}\left[F - \frac{\partial}{\partial v}\frac{\omega_{\mathrm{pe}}}{m_e v^3}W\left(\frac{\omega_{\mathrm{pe}}}{v}\right)\right] = 0. \tag{7.19}$$

Thus, during QL evolution, the expression in the square brackets stays constant in time. Since at $t = 0$, there are no waves, $W = 0$, we find

$$F(0, v) + \frac{\partial}{\partial v}\frac{\omega_{\mathrm{pe}}}{m_e v^3}W\left(t, \frac{\omega_{\mathrm{pe}}}{v}\right) = F(t, v) \to F^{\mathrm{plateau}} \quad \text{as} \quad t \to \infty. \tag{7.20}$$

In the saturated state ($t \to \infty$), $W(\omega_{\mathrm{pe}}/v) = 0$ outside the interval $v \in [v_1, v_2]$. Therefore, (7.20) gives us two implicit equations for v_1 and v_2:

$$\boxed{F(0, v_1) = F(0, v_2) = F^{\mathrm{plateau}}} \tag{7.21}$$

and, after integration over velocities, also an equation for F^{plateau}:[44]

$$\int_{v_1}^{v_2}\mathrm{d}v\left[F^{\mathrm{plateau}} - F(0, v)\right] = 0 \quad \Rightarrow \quad \boxed{F^{\mathrm{plateau}} = \frac{1}{v_2 - v_1}\int_{v_1}^{v_2}\mathrm{d}v\, F(0, v)}. \tag{7.22}$$

Finally, integrating (7.20) with respect to v and using the boundary condition $W(\omega_{\mathrm{pe}}/v_1) = 0$, we get, at $t \to \infty$,

$$W\left(\frac{\omega_{\mathrm{pe}}}{v}\right) = \frac{m_e v^3}{\omega_{\mathrm{pe}}}\int_{v_1}^{v}\mathrm{d}v'\left[F^{\mathrm{plateau}} - F(0, v')\right]. \tag{7.23}$$

Hence the spectrum is

$$\boxed{W(k) = \frac{m_e\omega_{\mathrm{pe}}^2}{k^3}\int_{v_1}^{\omega_{\mathrm{pe}}/k}\mathrm{d}v\left[F^{\mathrm{plateau}} - F(0, v)\right]} \quad \text{for} \quad k \in \left[\frac{\omega_{\mathrm{pe}}}{v_2}, \frac{\omega_{\mathrm{pe}}}{v_1}\right] \tag{7.24}$$

and $W(k) = 0$ everywhere else (Fig. 26).

Thus, we have completed the first three items of the programme formulated at the

---

[44]This is somewhat reminiscent of the "Maxwell construction" in thermodynamics of real gases: the plateau sits at such a level that the integral under it, i.e., the number of particles involved, stays the same as it was for the same velocities in the initial state; see Fig. 24.

end of §7.3. What about the particle distribution outside the resonant region? How is it modified by the quasilinear evolution? Is it modified at all? The following calculation shows that it must be.

7.5. *Energy of Resonant Particles*

Since feeding the instability requires extracting energy from the resonant particles, their energy must change. We calculate this change by taking the $m_e v^2/2$ moment of (7.20):

$$
\begin{aligned}
K_{\mathrm{res}}(\infty) - K_{\mathrm{res}}(0) &= \int_{v_1}^{v_2} \mathrm{d}v\, \frac{m_e v^2}{2} \left[F^{\mathrm{plateau}} - F(0,v) \right] \\
&= \int_{v_1}^{v_2} \mathrm{d}v\, \frac{m_e v^2}{2} \frac{\partial}{\partial v} \frac{\omega_{\mathrm{pe}}}{m_e v^3} W\left(\frac{\omega_{\mathrm{pe}}}{v}\right) \\
&= -\omega_{\mathrm{pe}} \int_{v_1}^{v_2} \mathrm{d}v\, \frac{1}{v^2} W\left(\frac{\omega_{\mathrm{pe}}}{v}\right) \\
&= -\int_{\omega_{\mathrm{pe}}/v_2}^{\omega_{\mathrm{pe}}/v_1} \mathrm{d}k\, W(k) = -2 \sum_k \frac{|E_k|^2}{8\pi} \equiv -2\,\mathcal{E}(\infty).
\end{aligned} \tag{7.25}
$$

Thus, only half of the energy lost by the resonant particles goes into the electric-field energy of the waves,

$$
\boxed{\mathcal{E}(\infty) = \frac{K_{\mathrm{res}}(0) - K_{\mathrm{res}}(\infty)}{2}}. \tag{7.26}
$$

Since the energy must be conserved overall [see (7.9)], we must account for the missing half: this is easy to do physically, as, obviously, the electric energy of the waves is their potential energy, which is half of their total energy—the other half being the kinetic energy of the oscillatory plasma motions associated with the wave (in §5.7, this was worked out explicitly). These oscillations are enabled by the non-resonant, "thermal-bulk" particles, and so we must be able to show that, as a result of QL evolution, these particles pick up the total of $\mathcal{E}(\infty)$ of energy—one might say that the plasma is *heated*.

7.6. *Heating of Non-Resonant Particles*

Consider the thermal bulk of the distribution, $v \ll v_1$ (assuming that the bump is indeed far out in the tail of the distribution). The QL diffusion coefficient (7.12) becomes, assuming now $\gamma_k, kv \ll \omega_{\mathrm{pe}}$ and using the last expression in Eq. (7.8),

$$
\begin{aligned}
D(v) &= \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{(kv - \omega_{\mathrm{pe}})^2 + \gamma_k^2} \approx \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{\omega_{\mathrm{pe}}^2} \\
&= \frac{e^2}{m_e^2 \omega_{\mathrm{pe}}^2} \sum_k \frac{1}{2} \frac{\partial |E_k|^2}{\partial t} = \frac{4\pi e^2}{m_e^2 \omega_{\mathrm{pe}}^2} \frac{\mathrm{d}}{\mathrm{d}t} \sum_k \frac{|E_k|^2}{8\pi} = \frac{1}{m_e n_e} \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t},
\end{aligned} \tag{7.27}
$$

independent of v. The QL evolution equation (7.11) for the bulk distribution is then[45]

$$
\frac{\partial F}{\partial t} = \frac{1}{m_e n_e} \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} \frac{\partial^2 F}{\partial v^2}. \tag{7.28}
$$

Equation (7.28) describes slow diffusion of the bulk distribution, i.e., as the wave field

---

[45]Note that this implies $\mathrm{d}\int \mathrm{d}v F(v)/\mathrm{d}t = 0$, so the number of these particles is conserved, there is no exchange between the non-resonant and resonant populations.

grows, the bulk distribution gets a little broader (which is what heating is). Namely, the "thermal" energy satisfies

$$\frac{\mathrm{d}K_{\mathrm{th}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}v\,\frac{m_e v^2}{2}\,F = \frac{1}{m_e n_e}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}\underbrace{\int \mathrm{d}v\,\frac{m_e v^2}{2}\frac{\partial^2 F}{\partial v^2}}_{=\,m_e n_e} = \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}. \tag{7.29}$$

$$\text{(by parts twice)}$$

Integrating this with respect to time, we find that the missing half of the energy lost by the resonant particles indeed goes into the thermal bulk:

$$\boxed{K_{\mathrm{th}}(\infty) - K_{\mathrm{th}}(0) = \mathcal{E}(\infty) = \frac{K_{\mathrm{res}}(0) - K_{\mathrm{res}}(\infty)}{2}}. \tag{7.30}$$

Overall, the energy is, of course, conserved:

$$K_{\mathrm{th}}(\infty) + K_{\mathrm{res}}(\infty) + \mathcal{E}(\infty) = K_{\mathrm{th}}(0) + K_{\mathrm{res}}(0), \tag{7.31}$$

as it shoud be, in accordance with (7.9).

Equation (7.28) can be explicitly solved: changing the time variable to $\tau = \mathcal{E}(t)/m_e n_e$ turns it into a simple diffusion equation

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial v^2}. \tag{7.32}$$

If we let the initial distribution be a Maxwellian and ignore the bump on its tail, the solution is

$$F(\tau, v) = \int \mathrm{d}v'\, F(0, v')\,\frac{e^{-(v-v')^2/4\tau}}{\sqrt{4\pi\tau}} = \int \mathrm{d}v'\,\frac{n_e}{\sqrt{\pi v_{\mathrm{the}}^2 4\pi\tau}}\exp\left[-\frac{v'^2}{v_{\mathrm{the}}^2} - \frac{(v-v')^2}{4\tau}\right]$$

$$= \frac{n_e}{\sqrt{\pi(v_{\mathrm{the}}^2 + 4\tau)}}\exp\left[-\frac{v^2}{v_{\mathrm{the}}^2 + 4\tau}\right]. \tag{7.33}$$

Since

$$v_{\mathrm{the}}^2 + 4\tau = \frac{2T_e}{m_e} + \frac{4\mathcal{E}(t)}{m_e n_e} = \frac{2}{m_e}\left[T_e + \frac{2\mathcal{E}(t)}{n_e}\right], \tag{7.34}$$

one concludes that an initially Maxwellian bulk stays Maxwellian but its temperature grows as the wave energy grows, reaching in saturation

$$\boxed{T_e(\infty) = T_e(0) + \frac{2\mathcal{E}(\infty)}{n_e}}. \tag{7.35}$$

7.7. Momentum Conservation

The bump-on-tail configuration is in general asymmetric in v and so the particles in the bump carry a net mean momentum. Let us find out whether this momentum changes. Taking the $m_e v$ moment of (7.20), we calculate the total momentum lost by the resonant
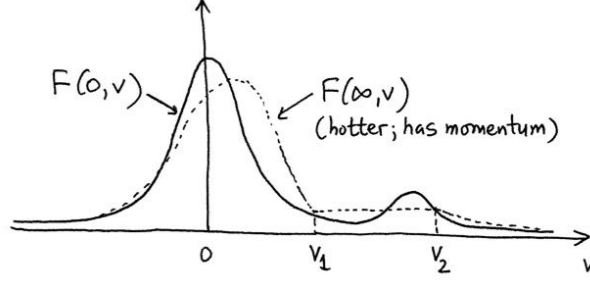
FIGURE 27. The initial distribution and the final outcome of the QL evolution: its bulk hotter and shifted towards the plateau in the tail.

particles:

$$
\begin{aligned}
\mathcal{P}_{\mathrm{res}}(\infty) - \mathcal{P}_{\mathrm{res}}(0) &= \int_{v_1}^{v_2} \mathrm{d}v\, m_e v \left[F^{\mathrm{plateau}} - F(0,v) \right] \\
&= \int_{v_1}^{v_2} \mathrm{d}v\, m_e v \frac{\partial}{\partial v} \frac{\omega_{\mathrm{pe}}}{m_e v^3} W\!\left(\frac{\omega_{\mathrm{pe}}}{v}\right) \\
&= -\omega_{\mathrm{pe}} \int_{v_1}^{v_2} \mathrm{d}v\, \frac{1}{v^3}\, W\!\left(\frac{\omega_{\mathrm{pe}}}{v}\right) \\
&= -\int_{\omega_{\mathrm{pe}}/v_2}^{\omega_{\mathrm{pe}}/v_1} \mathrm{d}k\, \frac{kW(k)}{\omega_{\mathrm{pe}}} < 0. \tag{7.36}
\end{aligned}
$$

This is negative, so momentum is indeed lost. Since it cannot go into electric field [see (7.10)], it must all get transferred to the thermal particles. Let us confirm this.

Going back to the QL diffusion equation (7.28) for the non-resonant particles, at first glance, we have a problem: the diffusion coefficient is independent of v and so momentum is conserved. However, one should never take zero for an answer when dealing with asymptotic expansions—indeed, it turns out here that we ought to work to higher order in our calculation of $D(v)$. Keeping next-order terms in (7.27), we get

$$
\begin{aligned}
D(v) &= \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{(kv - \omega_{\mathrm{pe}})^2 + \gamma_k^2} = \frac{e^2}{m_e^2} \sum_k |E_k|^2 \frac{\gamma_k}{\omega_{\mathrm{pe}}^2} \left(1 + \frac{2kv}{\omega_{\mathrm{pe}}} + \dots \right) \\
&\approx \frac{4\pi e^2}{m_e^2 \omega_{\mathrm{pe}}^2} \frac{\mathrm{d}}{\mathrm{d}t} \left[\sum_k \frac{|E_k|^2}{8\pi} + v \sum_k \frac{k|E_k|^2}{4\pi \omega_{\mathrm{pe}}} \right] = \frac{1}{m_e n_e} \frac{\mathrm{d}}{\mathrm{d}t} \left[\mathcal{E} + v \int \mathrm{d}k\, \frac{kW(k)}{\omega_{\mathrm{pe}}} \right]. \tag{7.37}
\end{aligned}
$$

Thus, there is a wave-induced drag term in the QL diffusion equation (7.11), which indeed turns out to impart to the thermal particles the small additional momentum that, according to (7.36), the resonant particles lose when rearranging themselves to produce the QL plateau:

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{P}_{\mathrm{th}}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}v\, m_e v F = \int \mathrm{d}v\, m_e v \frac{\partial}{\partial v} D(v) \frac{\partial F}{\partial v} = -m_e \int \mathrm{d}v\, D(v) \frac{\partial F}{\partial v} \\
&= -\left[\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}k\, \frac{kW(k)}{\omega_{\mathrm{pe}}} \right] \frac{1}{n_e} \int \mathrm{d}v\, v \frac{\partial F}{\partial v} = \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}k\, \frac{kW(k)}{\omega_{\mathrm{pe}}}, \tag{7.38}
\end{aligned}
$$

whence, integrating and comparing with (7.36),

$$\boxed{\mathcal{P}_{\text{th}}(\infty) - \mathcal{P}_{\text{th}}(0) = \int \mathrm{d}k \, \frac{kW(k)}{\omega_{\text{pe}}} = \mathcal{P}_{\text{res}}(0) - \mathcal{P}_{\text{res}}(\infty)}. \tag{7.39}$$

This means that *the thermal bulk of the final distribution is not only slightly broader (hotter) than that of the initial one (§7.6), but it is also slightly shifted towards the plateau* (Fig. 27).

In a collisionless plasma, this is the steady state. However, as this steady state is approached, $\gamma_k \to 0$, so the QL evolution becomes ever slower and even a very small collision frequency can become important. Eventually, collisions will erode the plateau and return the plasma to a global Maxwellian equilibrium—which is the fate of all things.

8. Nonlinear Stability and Collisionless Relaxation

Let me now go back to the generalist agenda first articulated at the beginning of §4: What kind of equilibria are stable?[46] Are there universal distributions to which a collisionless plasma will relax? This time I shall ask the stability question while forbidding myself any recourse to linear theory (§§8.1–8.4). This will push us towards certain distributions that will turn out to make some sense statistical-mechanically (§8.5) and that I will then show can be obtained by recourse to QLT (§8.6).

8.1. *Nonlinear Stability Theory: Thermodynamic Method*

The general idea of the method is to find, for a given initial equilibrium distribution f_0, an upper bound on the amount of energy that might be transferred into electromagnetic perturbations (not necessarily small). If that bound is zero, the system is stable; if it is not zero but is sharp enough to be nontrivial, it gives us a constraint on the amplitude of the perturbations in the saturated state.

Here is how it is done.[47] Let us introduce a functional

$$H = \underbrace{\int \mathrm{d}^3\boldsymbol{r} \, \frac{E^2 + B^2}{8\pi}}_{\equiv \, \mathcal{E}} + \underbrace{\iint \mathrm{d}^3\boldsymbol{r} \, \mathrm{d}^3\boldsymbol{v} \, [\mathcal{A}(\boldsymbol{r}, \boldsymbol{v}, f) - \mathcal{A}(\boldsymbol{r}, \boldsymbol{v}, f_*)]}_{\equiv \, A[f, f_*]} = \mathcal{E} + A[f, f_*], \tag{8.1}$$

where f_* is some trial distribution, which will represent our best guess about the properties of the stable distribution towards which the system wants to evolve and/or in the general vicinity of which we are interested in investigating stability. The function $\mathcal{A}(\boldsymbol{r}, \boldsymbol{v}, f)$ is chosen in such way that for any f,

$$A[f, f_*] \geqslant 0. \tag{8.2}$$

If it is also chosen so that H is conserved by the (collisionless) Vlasov–Maxwell equations, then $H(t) = H(0)$ and the inequality (8.2) gives us an upper bound on the field energy

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[46]In Q5, isotropic, monotonically decreasing equilibria were found to be stable not just against infinitesimal (linear), electrostatic perturbations, but also against small but finite electromagnetic ones, giving us a taste of a powerful nonlinear constraint.

[47]These ideas appear to have crystallised in the papers by T. K. Fowler in the early 1960s (see his review, Fowler 1968; his reminiscences and speculations on the subject 50 years later can be found in Fowler 2016), although a number of founding fathers of plasma physics were thinking along these lines around the same time (references are given in opportune places below).

at time t:

$$\boxed{\mathcal{E}(t) - \mathcal{E}(0) = A[f_0, f_*] - A[f(t), f_*] \leqslant A[f_0, f_*]}, \tag{8.3}$$

where f_0 is the initial ($t = 0$) equilibrium whose stability is under investigation.

The bound (8.3) implies stability if $A[f_0, f_*] = 0$,[48] i.e., certainly for $f_0 = f_*$. This guarantees stability of any f_* for which a functional $A[f, f_*]$ satisfying (8.2) and giving a conserved H can be produced.

Physically, the above construction is nontrivial if the bound (8.3) is smaller than the total initial kinetic energy of the particles:

$$A[f_0, f_*] < \sum_\alpha \iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{v}\,\frac{m_\alpha v^2}{2} f_{0\alpha} \equiv K(0). \tag{8.4}$$

It is obvious that one cannot extract from a distribution more energy than $K(0)$, but the above tells us that, in fact, one might only be able to extract less. $A[f_0, f_*]$ is an upper bound on the *available energy* of the distribution f_0. The sharper it can be made, the closer we are to learning something useful. Thus, the idea is to identify some suitable functional $A[f, f_*]$ for which H is conserved, and some class of trial distributions f_* for which (8.2) holds, then minimise $A[f_0, f_*]$ within that class, subject to whatever physical constraints one can reasonably expect to hold: e.g., conservation of particles, momentum, and/or any other (possibly approximate) invariants of the system (e.g., its adiabatic invariants; see Helander 2017).[49]

To make some steps towards a practical implementation of this programme, let us investigate how to choose $\mathcal{A}$ in such way as to ensure conservation of H:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} + \sum_\alpha \iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{v}\,\frac{\partial\mathcal{A}}{\partial f_\alpha}\frac{\partial f_\alpha}{\partial t} = \sum_\alpha \iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{v}\,\left(\frac{\partial\mathcal{A}}{\partial f_\alpha} - \frac{m_\alpha v^2}{2}\right)\frac{\partial f_\alpha}{\partial t} = 0. \tag{8.5}$$

The second equality was obtained by using the conservation of total energy,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{E} + K) = 0, \quad K = \sum_\alpha \iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{v}\,\frac{m_\alpha v^2}{2} f_\alpha, \tag{8.6}$$

where K is the kinetic energy of the particles. Now (8.5) tells us how to choose $\mathcal{A}$:

$$\boxed{\mathcal{A}(\boldsymbol{r}, \boldsymbol{v}, f) = \sum_\alpha \left[\frac{m_\alpha v^2}{2} f_\alpha + G_\alpha(f_\alpha)\right]}, \tag{8.7}$$

where $G_\alpha(f_\alpha)$ are arbitrary functions of f_α. These can be added to $\mathcal{A}$ because Vlasov's

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[48]This statement is based on the assumption that if the total electromagnetic enegy decreases, that corresponds to initial perturbations decaying. You might wonder what happens if $\mathcal{E}(0)$ contains some equilibrium magnetic field and if that equilibrium is unstable: can the equilibrium field's energy be tapped and transferred partially into unstable perturbations of kinetic energy in such a way that $\mathcal{E}(t) < \mathcal{E}(0)$ even though the system is unstable? I do not know how to isolate formally the set of conditions under which this is impossible (you may wish to think about this question; §14 might help). To avoid this problem, we could just restrict applicability of all considerations in this section to unmagnetised initial equilibria.

[49]Krall & Trivelpiece (1973) comment with a slight air of resignation that, with the rules of the game much vaguer than in linear theory, the thermodynamical approach to stability is "more art than science". In the Russian translation of their textbook, this statement provokes a disapproving footnote from the scientific editor (A. M. Dykhne), who observes that the *right* way to put it would be "more art than craft".
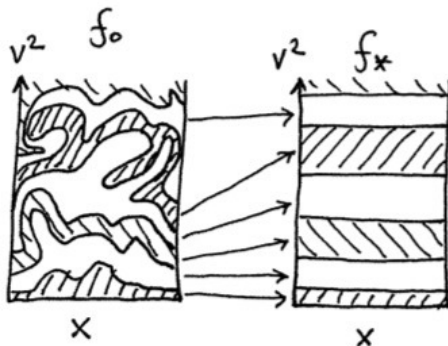
FIGURE 28. Gardner's rearrangement of the distribution, conserving phase-space volume.

equation has an infinite number of invariants: for any (sufficiently smooth) $G_\alpha(f_\alpha)$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint \mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{v}\,G_\alpha(f_\alpha) = 0. \tag{8.8}$$

This follows from the fact that, in the absence of collisions, the kinetic equation (1.30) expresses the conservation of phase volume in $(\boldsymbol{r}, \boldsymbol{v})$ space (the flow in this phase space is divergence-free).

Exercise 8.1. Prove the conservation law (8.8), assuming that the system is isolated.

The existence of an infinite number of conservation laws suggests that the evolution of a collisionless system in phase space is much more constrained than that of a collisional one. In the latter case, the evolution is constrained only by conservation of particles, momentum and energy and the requirement (5.12) that entropy must not decrease. I shall return shortly to the question of how available energy might be related to entropy.

A quick sanity check is to try $G_\alpha(f_\alpha) = 0$. The inequality (8.2) is then certainly satisfied for $f_* \propto \delta(\boldsymbol{v})$ and the bound (8.3) becomes

$$\mathcal{E}(t) - \mathcal{E}(0) \leqslant K(0), \tag{8.9}$$

i.e., one cannot extract any more energy than the total energy contained in the distribution—indeed, one cannot. Let us now move on to more nontrivial results.

8.2. *Gardner's Theorem*

Gardner (1963), in a classic two-page paper, proved that if the equilibrium distributions of all species are isotropic and decrease monotonically as functions of the particle energy $\varepsilon_\alpha = m_\alpha v^2/2$, the system is stable:[50]

$$\boxed{\frac{\partial f_{0\alpha}}{\partial \varepsilon_\alpha} < 0 \quad \Rightarrow \quad \text{stability}}. \tag{8.10}$$

---

[50]The stability of *Maxwellian* equlibria against small perturbations was first proved by W. Newcomb, whose argument was published as Appendix I of Bernstein (1958) (and followed by Fowler 1963, who proved stability against large perturbations). Gardner (1963) attributes the first appearance of the stability condition (8.10) to an obscure 1960 report by M. N. Rosenbluth, although the same condition was derived also by Kruskal & Oberman (1958), more or less in the manner described in §8.3. Many great minds were clearly thinking alike in those glory days of plasma physics.

Proof. For every species (suppressing species indices), let me again take $G(f) = 0$ in (8.7), but construct a nontrivial f_* that satisfies (8.2) for $f(t)$ at every time t since the beginning of its evolution from the initial distribution f_0.

For any given f_0, define f_* to be a monotonically decreasing function of v^2 (i.e., energy), such that for any $\Lambda > 0$, the volume of the region in the phase space $(\boldsymbol{r}, \boldsymbol{v})$ where $f_* > \Lambda$ is the same as the volume of the phase-space region where $f_0 > \Lambda$. Then f_* is the distribution with the smallest kinetic energy, denoted here by K_*, that can be reached from f_0 while preserving phase-space volume:

$$K(t) \geqslant K_*. \tag{8.11}$$

Indeed, while the phase-space volume occupied by any given value of the probability density is the same for f_0 and for f_*, the corresponding energy is always lower for f_* than for f_0 or for any other f that can evolve from it, because in f_*, the values of the probability density are rearranged in such a way as to put the largest of them at the lowest values of v^2, thus minimising the velocity integral in (8.6). A vivid analogy is to think of the evolution of f under the collisionless kinetic equation (1.28) as the evolution of a mixture of "fluids" of different densities (values of f) advected in a 6D phase-space $(\boldsymbol{r}, \boldsymbol{v})$ by a divergence-free flow $(\dot{\boldsymbol{r}}, \dot{\boldsymbol{v}})$. The lowest-energy state is the one in which these fluids are arranged in layers of density decreasing with increasing v^2, the heaviest at the bottom, the lightest at the top (Fig. 28).

In view of (8.11), and since $\mathcal{A}$ is given by (8.7) with $G(f) = 0$,

$$A[f, f_*] = K(t) - K_* \geqslant 0, \tag{8.12}$$

so (8.2) holds and (8.3) follows. When $f_0 = f_*$, i.e., the equilibrium distribution satisfies (8.10), the system is stable, q.e.d. The available energy is $A[f_0, f_*] = K(0) - K_*$.

Note that the condition (8.10) is sufficient, but not necessary, as we already know from, e.g., Exercise 4.2.

8.2.1. *Helander's Take on Gardner*

In a recent paper, Helander (2017) developed an elegant scheme for calculating "ground states" (the states of minimum energy) of Vlasov's equation, i.e., for determining Gardner's f_* and then calculating K_* to work out specific values of the available energy.

The idea is to look for a distribution f_* such that the kinetic energy of any distribution evolving from it cannot increase. So, let us set $f(t = 0) = f_*$ and evolve $f(t)$ forward a short time δt. The collisionless kinetic equation can be written simply as [cf. (1.28)]

$$\frac{\partial f}{\partial t} + \dot{\boldsymbol{q}} \cdot \frac{\partial f}{\partial \boldsymbol{q}} = 0 \quad \Rightarrow \quad f(\delta t) \approx f_* - \delta t \, \dot{\boldsymbol{q}} \cdot \frac{\partial f_*}{\partial \boldsymbol{q}}, \tag{8.13}$$

where $\boldsymbol{q} = (\boldsymbol{r}, \boldsymbol{v})$ is the phase-space variable. The first-order (in δt) kinetic-energy change from f_* to $f(\delta t)$ is then

$$\delta K[\delta \boldsymbol{q}] = -\int \mathrm{d}^6 \boldsymbol{q} \, \varepsilon(\boldsymbol{q}) \delta \boldsymbol{q} \cdot \frac{\partial f_*}{\partial \boldsymbol{q}}, \tag{8.14}$$

where $\varepsilon(\boldsymbol{q}) = mv^2/2$ and $\delta \boldsymbol{q} = \delta t \, \dot{\boldsymbol{q}}$. We want to minimise K, so we need $\delta K = 0$. This will be achieved for f_* such that $\delta K[\delta \boldsymbol{q}] = 0$ for any phase-space vector $\delta \boldsymbol{q}$ that behaves appropriately (vanishes) at the boundaries and satisfies $(\partial / \partial \boldsymbol{q}) \cdot \delta \boldsymbol{q} = 0$—the latter condition is imposed because the phase-space velocity field in (8.13) must be divergence-free: $(\partial / \partial \boldsymbol{q}) \cdot \dot{\boldsymbol{q}}$ (the system is Hamiltonian). This last condition can be enforced by means of a Langrange multiplier $\lambda(\boldsymbol{q})$:

$$\delta K[\delta \boldsymbol{q}] - \int \mathrm{d}^6 \boldsymbol{q} \, \lambda(\boldsymbol{q}) \frac{\partial}{\partial \boldsymbol{q}} \cdot \delta \boldsymbol{q} = 0 \quad \Leftrightarrow \quad \varepsilon(\boldsymbol{q}) \frac{\partial f_*}{\partial \boldsymbol{q}} = \frac{\partial \lambda}{\partial \boldsymbol{q}} \quad \Rightarrow \quad f_* = f_*(\varepsilon(\boldsymbol{q})), \tag{8.15}$$

i.e., the desired minimum-energy distribution must be a function of the particle energy only, as anticipated by Gardner.

Thus, any $f_*(\varepsilon)$ is a minimum-energy state, but we now must find one that is accessible from a given initial distribution f_0 via collisionless evolution, i.e., conserving phase-space volumes. This condition can be written in the form of the conservation law (8.8) with $G(f) = H(f(\boldsymbol{q}) - \Lambda)$, where H is the Heaviside function, picking out the volume of phase space where $f > \Lambda$:

$$\Gamma[f, \Lambda] \equiv \int d^6\boldsymbol{q}\, H(f(\boldsymbol{q}) - \Lambda) = \text{const}. \tag{8.16}$$

Since this is conserved, for any f_* accessible from a given f_0, $\Gamma[f_*, \Lambda] = \Gamma[f_0, \Lambda]$. Notice now that if $\partial f_*/\partial\varepsilon < 0$, as it should be if f_* is a Gardner function, $H(f_*(\varepsilon) - \Lambda) = H(\varepsilon_\Lambda - \varepsilon)$, where ε_Λ is such an energy that $f_*(\varepsilon_\Lambda) = \Lambda$. Therefore,

$$\Gamma[f_*, \Lambda] = \int d^6\boldsymbol{q}\, H(\varepsilon_\Lambda - \varepsilon(\boldsymbol{q})) \equiv \Omega(\varepsilon_\Lambda), \tag{8.17}$$

where the function $\Omega(\varepsilon_\Lambda)$ is entirely independent of f_*, being just the integrated density of states corresponding to the energy ε_Λ: since $\varepsilon(\boldsymbol{q}) = mv^2/2$,

$$\Omega(\varepsilon) = \frac{4\pi V}{3}\left(\frac{2\varepsilon}{m}\right)^{3/2}. \tag{8.18}$$

Collecting all these relations, we conclude that $\Omega(\varepsilon_\Lambda) = \Gamma[f_*, \Lambda] = \Gamma[f_0, \Lambda] = \Gamma[f_0, f_*(\varepsilon_\Lambda)]$, or, for any ε,

$$\boxed{\Gamma[f_0, f_*(\varepsilon)] = \Omega(\varepsilon)}. \tag{8.19}$$

This is an integral equation for the Gardner function $f_*(\varepsilon)$ accessible from the initial distribution f_0.[51]

8.2.2. *Anisotropic Equilibria*

Let me give an example of the use of Helander's scheme for an anisotropic initial distribution—the case that, at the end of §4, I had to relegate to Exercise 4.8 as it needed substantial extra work if it were to be handled by the method developed there.

Consider a *bi-Maxwellian distribution*, a useful and certainly the simplest model for anisotropic equilibria:

$$f_0 = \overline{C}\exp\left(-\frac{mv_\perp^2}{2T_\perp} - \frac{mv_\parallel^2}{2T_\parallel}\right), \quad \overline{C} = n\left(\frac{m}{2\pi\overline{T}}\right)^{3/2}, \tag{8.20}$$

where $\overline{T} = T_\perp^{2/3}T_\parallel^{1/3}$ and $T_\perp$ and $T_\parallel$ are the "temperatures" of particle motion perpendicular and parallel to some special direction. Is this distribution unstable? (Yes: see Q3.) To work out the Gardner distribution corresponding to it, observe that the volume $\Gamma[f_0, \Lambda]$ of the part of phase space where $f_0 > \Lambda$ is V times the volume of the velocity-space ellipsoid

$$\frac{mv_\perp^2}{2T_\perp} + \frac{mv_\parallel^2}{2T_\parallel} = \ln\frac{\overline{C}}{\Lambda} \quad \Rightarrow \quad \Gamma[f_0, \Lambda] = \frac{4\pi V}{3}\left(\frac{2\overline{T}}{m}\ln\frac{\overline{C}}{\Lambda}\right)^{3/2}. \tag{8.21}$$

Letting $\Lambda = f_*(\varepsilon)$ and, according to (8.19), equating $\Gamma[f_0, f_*(\varepsilon)]$ to (8.18), we find

$$f_*(\varepsilon) = \overline{C}\exp\left(-\frac{\varepsilon}{\overline{T}}\right). \tag{8.22}$$

This is an interesting, if perhaps somewhat rigged, example of a system "wanting" to go to a Maxwellian equilibrium even in the absence of collisions.

The upper bound on the available energy is

$$A[f_0, f_*] = K(0) - K_* = \frac{3}{2}Vn\left(\frac{2}{3}T_\perp + \frac{1}{3}T_\parallel - T_\perp^{2/3}T_\parallel^{1/3}\right). \tag{8.23}$$

The bound is zero when $T_\perp = T_\parallel$ and is always positive otherwise (because it is the difference between an arithmetic and a geometric mean of the two temperatures). We do not, of course,

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[51] In the form $\Omega'(\varepsilon) = f_*'(\varepsilon)\partial\Gamma[f_0, f_*(\varepsilon)]/\partial f_*$, it was first derived by Dodin & Fisch (2005).

have any way of knowing how good an approximation this is to the true saturated level of whatever instability (if any) might exist here in any particular physical regime,[52] but this does suggest that temperature anisotropy is a viable source of free energy.

In Helander (2017), you will find other examples, in particular, a nice demonstration that Maxwellian equilibria with spatially dependent density and temperature have available energy.

8.3. *Thermodynamics of Small Perturbations*

There is a neat development (due, it seems, to Kruskal & Oberman 1958 and Fowler 1963) of the formalism presented at the beginning of this section that leads again to Gardner's result (8.10), but also puts us in contact with some familiar themes from §5.

Let us investigate the stability of *isotropic* distributions with respect to small (but not necessarily infinitesimal) perturbations, i.e., take $f(t) = f_0 + \delta f$, $\delta f \ll f_0$, and also $f_* = f_0$, so the bound (8.3) will imply stability if we can find $G(f)$ such that (8.2) holds.

In (8.7), we expand

$$G(f) = G(f_0) + G'(f_0)\delta f + G''(f_0)\frac{\delta f^2}{2} + \dots \tag{8.24}$$

and use this to obtain, keeping terms up to second order,

$$A[f(t), f_0] = \sum_\alpha \iint \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{v} \left\{ \left[\frac{m_\alpha v^2}{2} + G'_\alpha(f_{0\alpha}) \right] \delta f_\alpha + G''_\alpha(f_{0\alpha})\frac{\delta f_\alpha^2}{2} \right\}. \tag{8.25}$$

Suppose we contrive to pick $G_\alpha(f_{0\alpha})$ in such a way that

$$G'_\alpha(f_{0\alpha}) = -\frac{m_\alpha v^2}{2} \equiv -\varepsilon_\alpha, \tag{8.26}$$

obliterating the first-order term in (8.25). Then, since $f_{0\alpha} = f_{0\alpha}(\varepsilon_\alpha)$ by assumption (it is isotropic), differentiating the above condition with respect to $f_{0\alpha}$ gives

$$G''_\alpha(f_{0\alpha}) = -\frac{1}{\partial f_{0\alpha}/\partial \varepsilon_\alpha} \quad \Rightarrow \quad A[f(t), f_0] = \sum_\alpha \iint \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{v}\, \frac{\delta f_\alpha^2}{2(-\partial f_{0\alpha}/\partial \varepsilon_\alpha)}. \tag{8.27}$$

We see that $A[f(t), f_0] \geqslant 0$ and, therefore, (8.3) with $f_* = f_0$ implies stability if, again, $f_{0\alpha}(\varepsilon_\alpha)$ is monotonically decreasing for all species.

Besides stability, this construction has given us an interesting quadratic conserved quantity for our system:

$$\boxed{H = \mathcal{E} + A[f, f_0] = \int \mathrm{d}^3\boldsymbol{r}\, \frac{E^2 + B^2}{8\pi} + \sum_\alpha \iint \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{v}\, \frac{\delta f_\alpha^2}{2(-\partial f_{0\alpha}/\partial \varepsilon_\alpha)}}. \tag{8.28}$$

The condition (8.10) makes H positive definite and so no wonder the system is stable: perturbations around f_0 have a conserved norm! For a Maxwellian equilibrium, $-\partial f_{0\alpha}/\partial \varepsilon_\alpha = f_{0\alpha}/T_\alpha$, so this H is none other than W, (the electromagnetic version of) the free energy (5.19), and so it is tempting to think of (8.28) as providing a natural generalisation of free energy to non-Maxwellian plasmas.

In Q5, the results of this section are obtained in a more straightforward way, directly from the

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[52]Nominally, this calculation applies with equal validity to many different instabilities that can be triggered by temperature anisotropy in both unmagnetised and magnetised plasmas—and indeed also to some anisotropic distributions that can, in fact, be proved stable (which is common in magnetised plasmas where the externally imposed magnetic field is sufficiently large).

Vlasov–Maxwell equations.

This style of thinking has been having a revival lately: see, e.g., the discussion of firehose and mirror stability of a magnetised plasma in Kunz *et al.* (2015). Generalised energy invariants like H are important not just for stability calculations, but also for theories of kinetic turbulence in weakly collisional environments, e.g., the solar wind (see, e.g., Schekochihin *et al.* 2009).

8.4. *Thermodynamics of Finite Perturbations*

One might wonder at this point whether the condition (8.26) is fulfillable and also whether anything can be done without assuming small perturbations. An answer to both questions is provided by the following argument.

The realisation in §8.3 that our conserved quantity H is a generalisation of free energy nudges us in the direction of a particular choice of functions $G_\alpha(f_\alpha)$ and trial equilibria $f_{*\alpha}$, fully inspired by conventional thermodynamics. Namely, in (8.7), let

$$G_\alpha(f_\alpha) = T_\alpha f_\alpha \left(\ln \frac{f_\alpha}{C_\alpha} - 1 \right), \quad f_{*\alpha} = C_\alpha \exp \left(-\frac{m_\alpha v^2}{2 T_\alpha} \right), \tag{8.29}$$

where C_α and T_α are constants independent of space. It is then certainly true that $G'_\alpha(f_{*\alpha}) = -\varepsilon_\alpha$. It is also straightforward to show that the inequality (8.2) is always satisfied: essentially, this follows from the fact that the Maxwellian distribution $f_{*\alpha}$ maximises the entropy $-\iint \mathrm{d}^3 r \, \mathrm{d}^3 v \, f_\alpha \ln f_\alpha$, subject to fixed energy, $1/T_\alpha$ being the corresponding Lagrange multiplier.

Exercise 8.2. Prove formally that if G_α and $f_{*\alpha}$ are given by (8.29), then

$$A[f, f_*] = \sum_\alpha \iint \mathrm{d}^3 r \, \mathrm{d}^3 v \left[\frac{m_\alpha v^2}{2} (f_\alpha - f_{*\alpha}) + G_\alpha(f_\alpha) - G_\alpha(f_{*\alpha}) \right] \geqslant 0 \tag{8.30}$$

for any values of C_α and T_α.

Thus, Eq. (8.3) provides an upper bound on the energy of the electromagnetic fields that can be extracted from any given initial distribution $f_{0\alpha}$. In order to make this bound as sharp as possible, one picks the constants C_α and T_α (and, therefore, determines $f_{*\alpha}$) so as to minimise $A[f_0, f_*]$ subject to constraints that cannot change: e.g., freezing the number of particles of each species gives

$$C_\alpha = \left(\frac{m_\alpha}{2\pi T_\alpha} \right)^{3/2} \frac{1}{V} \iint \mathrm{d}^3 r \, \mathrm{d}^3 v \, f_{0\alpha}. \tag{8.31}$$

8.4.1. *Anisotropic Equilibria*

To test-drive this method, let us go back to the bi-Maxwellian distribution (8.20) and assume it is the initial distribution for every species α. To obtain an upper bound on the energy available for extraction from it, substitute this distribution into (8.29), use also (8.31), and find

$$A[f_0, f_*] = V \sum_\alpha n_\alpha \left[\frac{3}{2} T_\alpha \left(\ln \frac{T_\alpha}{\overline{T}_\alpha} - 1 \right) + T_{\perp\alpha} + \frac{T_{\parallel\alpha}}{2} \right]. \tag{8.32}$$

This is minimised by $T_\alpha = \overline{T}_\alpha$, resulting in the following estimate of the available energy:

$$\mathcal{E}(t) - \mathcal{E}(0) \leqslant \min_{T_\alpha} A[f_0, f_*] = \frac{3}{2} V \sum_\alpha n_\alpha \left(\frac{2}{3} T_{\perp\alpha} + \frac{1}{3} T_{\parallel\alpha} - \overline{T}_\alpha \right). \tag{8.33}$$

This is the same result as (8.23), because, in this case, the target distribution $f_{*\alpha}$ was also the Gardner distribution (not, generally speaking, an absolute requirement).

Further examples of such calculations can be found in Krall & Trivelpiece (1973, §9.14) and Fowler (1968). A certain further development of the methodology discussed above allows one to derive upper bounds not just on the energy of perturbations but also on their growth rates (Fowler 1964, 1968).

8.5. *Statistical Mechanics of Collisionless Relaxation*

In §8.2.2, and again in §8.4.1, I considered a simple example in which an initially anisotropic distribution appeared keen to evolve towards a Maxwellian, even though its relaxation was assumed completely collisionless. Is there any fundamental physical reason for collisionless plasmas to privilege Maxwellian distributions? Perhaps, in the same way that the Maxwellian emerges in statistical mechanics as a universal equilibrium state. Let me work though a statistical argument to that effect, proposed originally by Lynden-Bell (1967) in the context of collisionless relaxation of kinetic systems of mutually gravitating objects (e.g., stars in a galaxy).

Let us start by discretising the phase space into a very large number of micro-cells, each with phase volume $\delta\Gamma$. Let us assume also (in what is a rather drastic simplifying step) that the exact distribution function in each of these micro-cells is equal to either zero or some constant (the same constant in all micro-cells):

$$f(\boldsymbol{q}) = \eta \quad \text{or} \quad 0 \tag{8.34}$$

(this is known as a *waterbag distribution*—a constant probability density in a finite subvolume of phase space). Then

$$N = \int \mathrm{d}^6 \boldsymbol{q}\, f = \eta\, \delta\Gamma \mathcal{N}, \tag{8.35}$$

where N is the number of particles and $\mathcal{N}$ is the number of micro-cells with non-zero density. We are going to think of our plasma as a statistical-mechanical system of $\mathcal{N}$ phase-density elements, which are allowed, under collisionless evolution, to move around phase space subject to the usual constraints: conservation of energy and conservation of phase volume. The latter constraint in this language means that phase-density elements can never occupy the same micro-cell, i.e., that they are subject to the Pauli-like exclusion principle. Thus, they are fermions, except that they are distinguishable (by their initial position in phase space).

Let us now coarse-grain our phase space into macro-cells, each containing $\mathcal{M}$ micro-cells. We can represent the coarse-grained distribution $\bar{f}$ as a set of occupation numbers $\mathcal{N}_i \leqslant \mathcal{M}$ of the i-th macro-cell, namely, the density in the i-th macro-cell is

$$\bar{f}_i = \frac{\eta \mathcal{N}_i}{\mathcal{M}} \leqslant \eta. \tag{8.36}$$

The total number of ways of setting up a particular distribution $\{\mathcal{N}_i\}$ is

$$W = \frac{\mathcal{N}!}{\prod_i \mathcal{N}_i!} \prod_i W_i, \quad W_i = \frac{\mathcal{M}!}{(\mathcal{M} - \mathcal{N}_i)!}. \tag{8.37}$$

Here the first factor is the number of ways of distributing $\mathcal{N}$ phase-density elements amongst the macro-cells and W_i is the number of ways to distribute $\mathcal{N}_i$ distinguishable elements between the micro-cells in the i-th macro-cell. Assuming that $\mathcal{N}, \mathcal{N}_i, \mathcal{M} > \mathcal{N}_i$

are all large, we can use Stirling's formula ($\ln \mathcal{N}! \approx \mathcal{N} \ln \mathcal{N} - \mathcal{N}$) to find the Boltzmann entropy for our system:

$$S = \ln W \approx \mathcal{N}(\ln \mathcal{N} - 1) - \mathcal{M} \sum_i \left[\frac{\mathcal{N}_i}{\mathcal{M}} \ln \frac{\mathcal{N}_i}{\mathcal{M}} + \left(1 - \frac{\mathcal{N}_i}{\mathcal{M}} \right) \ln \left(1 - \frac{\mathcal{N}_i}{\mathcal{M}} \right) \right]$$

$$= \mathcal{N}(\ln \mathcal{N} - 1) - \frac{1}{\delta \Gamma} \int \mathrm{d}^6 \boldsymbol{q} \left[\frac{\bar{f}}{\eta} \ln \frac{\bar{f}}{\eta} + \left(1 - \frac{\bar{f}}{\eta} \right) \ln \left(1 - \frac{\bar{f}}{\eta} \right) \right], \qquad (8.38)$$

where $\int \mathrm{d}^6 \boldsymbol{q} = \mathcal{M} \delta \Gamma \sum_i$. This is to be maximised under the constraints of a fixed number of particles N and energy K in the distribution.[53] The problem is exactly the same as for a Fermi gas[54] and its solution is the Fermi–Dirac distribution:

$$\boxed{\bar{f} = \frac{\eta}{e^{(\varepsilon - \mu)/T} + 1}}, \qquad (8.39)$$

where ε is the particle energy corresponding to the given macro-cell, and T ("temperature") and μ ("chemical potential") are Lagrange multipliers that are determined by fixing N and K:

$$N = \int \mathrm{d}^6 \boldsymbol{q} \, \bar{f}, \quad K = \int \mathrm{d}^6 \boldsymbol{q} \, \varepsilon \bar{f}. \qquad (8.40)$$

8.5.1. *Non-degenerate Limit*

Just like in the case of Fermi–Dirac statistics, the Maxwellian (non-degenerate) limit is recovered when the initial waterbag distribution is sparse in phase space—i.e., when it is sufficiently spread out around the part of phase space that is available subject to given energy K. Mathematically, this limit is achieved by letting

$$e^{-\mu/T} \gg 1 \quad \Rightarrow \quad \bar{f} \approx \eta e^{\mu/T} e^{-\varepsilon/T}. \qquad (8.41)$$

Then, from (8.40), after doing the integrals (with $\mathrm{d}^6 \boldsymbol{q} = \mathrm{d}^3 \boldsymbol{r} \mathrm{d}^3 \boldsymbol{v}$ and $\varepsilon = mv^2/2$),

$$T = \frac{2}{3} \frac{K}{N}, \quad e^{\mu/T} = \frac{n}{(2\pi T/m)^{3/2} \eta}. \qquad (8.42)$$

Thus, the non-degenerate, Maxwellian limit (8.41) is

$$\frac{n}{(2\pi T/m)^{3/2}} \ll \eta \quad \Rightarrow \quad \boxed{\bar{f} = \frac{n}{(2\pi T/m)^{3/2}} e^{-\varepsilon/T}}. \qquad (8.43)$$

Of course it is not a particular surprise that a Maxwellian distribution emerges from a statistical mechanics of completely randomised objects with fixed overall energy. This might appear to be a argument in favour of universality in collisionless plasmas—but that hinges on the validity of the assumption that the phase-density elements of which

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[53]For simplicity, we consider a (statistically) homogeneous system. Then let each phase-space macro-cell contain the entire position space and, consequently, $\bar{f}$ be independent of position. In an electrostatic system, $\bar{f}$ then gives rise to no fields and so K is kinetic energy. A more general consideration is possible in which the energy is the total energy, kinetic plus electric—the latter is included in manner similar to how Lynden-Bell (1967) handles gravitational energy, except now with mutiple particle species that have different charges.

[54]The distinguishability of the phase-density elements turns out not to matter: in (8.37), the factor of $1/\mathcal{N}_i!$ that would appear in W_i for indistinguishable fermions is recovered in the prefactor that expresses the number of ways of populating the macro-cells. For a tutorial on Fermi gases, see Schekochihin (2019, §§16–17).

the distribution consists are independent and free to sample the entirety of phase space without fear or prejudice.[55]

Exercise 8.3. Multi-waterbag statistics (Lynden-Bell 1967). The above construction contained a very restrictive assumption of an initial waterbag distribution. This restriction is, however, not hard to remove. Let us discretise the values that the distribution function can take and index them by J, so a general distribution function is represented as a superposition of waterbags:

$$f(\boldsymbol{q}) = \sum_J f_J(\boldsymbol{q}), \quad f_J(\boldsymbol{q}) = \eta_J \quad \text{or} \quad 0. \tag{8.44}$$

If there are $\mathcal{N}_J$ phase elements with density η_J, then $\delta\Gamma\mathcal{N}_J$ is the phase-space volume occupied by the J-th waterbag, i.e., the phase-space volume where $f = \eta_J$. This is conserved by collisionless evolution of f. The corresponding number of particles is $N_J = \eta_J\delta\Gamma\mathcal{N}_J$. As before, we may now coarse-grain f over groups (macro-cells) of $\mathcal{M}$ microcells and represent the resulting $\bar{f}$ in terms occupation numbers $\mathcal{N}_{iJ}$ of the i-th macro-cell by elements of phase density η_J. Show that

$$\bar{f} = \sum_J \bar{f}_J, \quad \bar{f}_J = \frac{\eta_J e^{-\beta\eta_J(\varepsilon-\mu_J)}}{1 + \sum_{J'} e^{-\beta\eta_{J'}(\varepsilon-\mu_{J'})}}, \tag{8.45}$$

where μ_J and β are determined by

$$N_J = \int \mathrm{d}^6\boldsymbol{q}\, \bar{f}_J, \quad K = \int \mathrm{d}^6\boldsymbol{q}\, \varepsilon\bar{f}. \tag{8.46}$$

Thus, the more general equilibrium distribution is a kind of superposition of many Fermi–Dirac distributions or, in the non-degenerate limit, of Maxwellians, with effective temperatures $1/\beta\eta_J$. Considering the Maxwellian limit, or otherwise, propose a sensible way to define the overall temperature (Lynden-Bell 1967 has an answer to this; Adkins & Schekochihin 2019 have a different one).

Exercise 8.4. Gardner distribution is a Lynden-Bell distribution. Consider a Gardner distribution, i.e., a distribution function that is a monotonically decreasing function of ε alone (cf. §8.2). Discretise it to be a multi-waterbag distribution (8.44). Show that, under an appropriate coarse-graining, the corresponding Lynden-Bell equilibrium (8.45) is, in fact, the same distribution, i.e., that Lynden-Bell's entropy-maximisation procedure does not change Gardner's distribution.[56] Thinking of Vlasov–Maxwell equations, explain why that is.

8.5.2. *Warm and Cold Beams*

Let me offer a very simple example of a physically plausible waterbag distribution: a pair of beams (streams) homogeneous in position space with density n_{b}, with velocities $\pm u_{\mathrm{b}}$ in some direction, and constant velocity-space density over a region of width v_{b} around $\pm u_{\mathrm{b}}$—similar to what is illustrated in Fig. 12b but with square-top (Π-shaped) humps, in 3D. This is a waterbag with

$$\eta \sim \frac{n_{\mathrm{b}}}{v_{\mathrm{b}}^3}, \quad K \sim n_{\mathrm{b}} V m u_{\mathrm{b}}^2. \tag{8.47}$$

It is non-degenerate if $v_{\mathrm{b}} \ll u_{\mathrm{b}}$. i.e., if the beams are "cold"—as is obvious from (8.43). One therefore expects to see a distribution close to a Maxwellian with $T \sim m u_{\mathrm{b}}^2$ at the end of the relaxation process. In contrast, "warm" beams, $v_{\mathrm{b}} \sim u_{\mathrm{b}}$, should tend to a

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[55]One way in which this might not be true would be if our system had further invariants that constrained its dynamics (at least on the time scales of interest). An example is a magntised plasma in which particles conserve adiabatic invariants (cf. Helander 2017). Another is galactic dynamics—Lynden-Bell's original preoccupation—where stars are approximately tied to the Keplerian ellipses they follow around the central black hole (Binney 2016).

[56]Intuitively, this should be the case, but is, in fact, neither obvious nor trivial to prove. I owe this result to a conversation with Per Helander (2019).

(a) Cold beams: $v_b \ll u_b$ (b) Warm beams: $v_b \sim u_b$
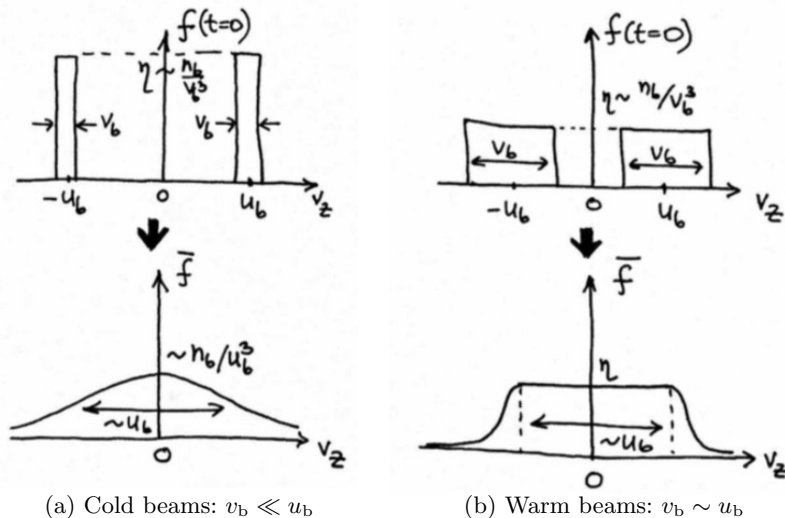
FIGURE 29. Waterbag beams and their Lynden-Bell equilibria.

Fermi–Dirac flat-topped equilibrium with Fermi energy $\varepsilon_F \sim mu_b^2$. Remarkably, there appears to be some recent numerical evidence that something like this might indeed be happening (Skoutnev *et al.* 2019).

Exercise 8.5. For some convenient specific model of the v_b-sized beam shape in velocity space, work out and plot (or at least sketch) T/mu_b^2 and μ/mu_b^2 as functions of v_b/u_b, and hence work out the Lynden-Bell equilibria. In the limits $v_b \ll u_b$ or $T \ll mu_b^2$, everything should be doable analytically (in the latter case, via Sommerfeld expansion; see, e.g., §17.3.3 of Schekochihin 2019).

8.6. *QL Relaxation*

I am now going to go back to the QL scheme and use it to derive a kind of "collisionless collision integral" that evolves the mean distribution function of a collisional, homogeneous, electrostatic plasma towards some equilibria that could perhaps be argued to be universal—for an initial waterbag distribution, we will recover Fermi–Dirac statistics (8.39) as a solution of our collision integral.

So the equilibrium (slow, space- and time-averaged) distribution f_0 again evolves according to (2.11), which I now would like to rewrite as follows

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[-\frac{q}{m} \sum_{\boldsymbol{k}} i\boldsymbol{k} \langle \varphi_{\boldsymbol{k}}^* \delta f_{\boldsymbol{k}} \rangle \right] = \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[-\frac{e}{m} \sum_{\boldsymbol{k}} \boldsymbol{k} \, \mathrm{Im} \langle \varphi_{\boldsymbol{k}}^* \delta f_{\boldsymbol{k}} \rangle \right], \tag{8.48}$$

where the imaginary part was distilled out of the $\boldsymbol{k}$ integral by splitting the latter in two equal parts and replacing $\boldsymbol{k} \to -\boldsymbol{k}$ in one of them [cf. (7.8)]; I have also specialised to electrons ($q = -e$) and will assume, for simplicity, that ions play no role apart from providing a homogeneous neutralising background. The perturbation $\delta f = f - f_0$ satisfies (2.12) exactly and its linear part (2.13) approximately, within the QL approximation.

The Laplace-transformed solution of this equation is given by (3.8) and (3.13), which,

assuming for simplicity that only electrons have δf, are

$$\delta \hat{f}_{\boldsymbol{k}}(p) = -i\frac{e}{m}\frac{\hat{\varphi}_{\boldsymbol{k}}(p)}{p + i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} + \hat{h}_{\boldsymbol{k}}(p), \tag{8.49}$$

$$\hat{h}_{\boldsymbol{k}}(p) = \frac{g_{\boldsymbol{k}}(\boldsymbol{v})}{p + i\boldsymbol{k}\cdot\boldsymbol{v}}, \tag{8.50}$$

$$\hat{\varphi}_{\boldsymbol{k}}(p) = -\frac{4\pi e}{k^2\epsilon(p,\boldsymbol{k})}\int \mathrm{d}^3\boldsymbol{v}\,\hat{h}_{\boldsymbol{k}}(p), \tag{8.51}$$

$$\epsilon(p,\boldsymbol{k}) = 1 - i\frac{4\pi e^2}{mk^2}\int \mathrm{d}^3\boldsymbol{v}\,\frac{1}{p + i\boldsymbol{k}\cdot\boldsymbol{v}}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}. \tag{8.52}$$

The time-dependent solution is recovered via the inverse Laplace transform (3.14), where I now wish to change the integration variable to $p = -i\omega + \sigma$:

$$\delta f_{\boldsymbol{k}}(t) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\,e^{(-i\omega+\sigma)t}\delta f_{\boldsymbol{k}\omega}, \quad \varphi_{\boldsymbol{k}}(t) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\,e^{(-i\omega+\sigma)t}\varphi_{\boldsymbol{k}\omega}, \tag{8.53}$$

where the Laplace transformed functions have been redefined to look almost like Fourier transformed ones:

$$\delta f_{\boldsymbol{k}\omega} = \delta\hat{f}_{\boldsymbol{k}}(-i\omega + \sigma) = \frac{e}{m}\frac{\varphi_{\boldsymbol{k}\omega}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\sigma}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} + h_{\boldsymbol{k}\omega}, \tag{8.54}$$

$$h_{\boldsymbol{k}\omega} = \hat{h}_{\boldsymbol{k}}(-i\omega + \sigma) = \frac{ig_{\boldsymbol{k}}(\boldsymbol{v})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\sigma}, \tag{8.55}$$

$$\hat{\varphi}_{\boldsymbol{k}\omega} = \hat{\varphi}_{\boldsymbol{k}}(-i\omega + \sigma) = -\frac{4\pi e}{k^2\epsilon_{\boldsymbol{k}\omega}}\int \mathrm{d}^3\boldsymbol{v}\,h_{\boldsymbol{k}\omega}, \tag{8.56}$$

$$\epsilon_{\boldsymbol{k}\omega} = \epsilon(-i\omega + \sigma, \boldsymbol{k}) = 1 + \frac{4\pi e^2}{mk^2}\int \mathrm{d}^3\boldsymbol{v}\,\frac{1}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\sigma}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}. \tag{8.57}$$

So far I have not introduced any extra assumptions, but the reason for this rearrangement is that I now want to assume that $\hat{\varphi}_{\boldsymbol{k}}(p)$ and, therefore, $\delta\hat{f}_{\boldsymbol{k}}(p)$ have no poles at $\mathrm{Re}\,p > 0$, i.e., f_0 supports no instabilities, so I may let $\sigma \to +0$ and integrate along the real line in ω, only needing the imfinitesimal σ in the denominators to tell me how to circumvent the ballistic pole $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$.

In the context of collisional relaxation of f_0, this assumption is justified as follows. The initial distribution may well be unstable, but we can always wait for its instabilities to get excited, saturate and change f_0 in such a way to shut themselves down (as they did in §7). After that, we have a distribution function that is stable or, at worst, marginally stable, i.e., $\epsilon_{\boldsymbol{k}\omega}$ might have zeros on the real-ω line and certainly at $\mathrm{Im}\,\omega < 0$ ($\mathrm{Re}\,p \leqslant 0$), but not at $\mathrm{Im}\,\omega > 0$ ($\mathrm{Re}\,p > 0$).

We are ready to roll. Substituting (8.53) and (8.54) into (8.48), we get[57]

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}}\cdot\sum_{\boldsymbol{k}}\boldsymbol{k}\,\mathrm{Im}\iint\frac{\mathrm{d}\omega\mathrm{d}\omega'}{(2\pi)^2}\,e^{i(\omega'-\omega)t}\left(-\frac{e^2}{m^2}\frac{\langle\varphi_{\boldsymbol{k}\omega'}^*\varphi_{\boldsymbol{k}\omega}\rangle}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\sigma}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}} - \frac{e}{m}\langle\varphi_{\boldsymbol{k}\omega'}^* h_{\boldsymbol{k}\omega}\rangle\right). \tag{8.58}$$

The first term is the familiar QL diffusion (here written for a general-form φ) whereas the second term turns out to be a kind of drag [cf. (1.47)], whose form we shall now

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[57] A conscientious reader might at this point become worried about what exactly is meant by averages in (8.58). She will find some succour at the beginning of §8.6.1.

elaborate a little further. Using (8.56), we get

$$
-\frac{e}{m}\langle\varphi_{\boldsymbol{k}\omega'}^* h_{\boldsymbol{k}\omega}(\boldsymbol{v})\rangle = \frac{4\pi e^2}{mk^2\epsilon_{\boldsymbol{k}\omega'}^*}\int \mathrm{d}^3\boldsymbol{v}' \left\langle h_{\boldsymbol{k}\omega'}^*(\boldsymbol{v}')h_{\boldsymbol{k}\omega}(\boldsymbol{v})\right\rangle
$$

$$
= \frac{4\pi e^2}{mk^2\epsilon_{\boldsymbol{k}\omega'}^*\epsilon_{\boldsymbol{k}\omega}}\int \mathrm{d}^3\boldsymbol{v}' \left[C_{\boldsymbol{k}\omega'\omega}(\boldsymbol{v}',\boldsymbol{v}) + \frac{4\pi e^2}{mk^2}\int \mathrm{d}^3\boldsymbol{v}'' \frac{C_{\boldsymbol{k}\omega'\omega}(\boldsymbol{v}',\boldsymbol{v})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}'' + i\sigma}\,\boldsymbol{k}\cdot\frac{\partial f_0(\boldsymbol{v}'')}{\partial\boldsymbol{v}''}\right],
\tag{8.59}
$$

where $C_{\boldsymbol{k}\omega'\omega}(\boldsymbol{v}',\boldsymbol{v}) = \left\langle h_{\boldsymbol{k}\omega'}^*(\boldsymbol{v}')h_{\boldsymbol{k}\omega}(\boldsymbol{v})\right\rangle$ and the last expression was obtained by multiplying and dividing by $\epsilon_{\boldsymbol{k}\omega}$. The reason for this seemingly gratuitous manipulation is two-fold. First, the first term in (8.59) in fact vanishes after it is plugged into (8.58), integrated over frequencies and taken the imaginary part of.

Exercise 8.6. Show that this is true. An assumption needed for that is that the correlation function $\langle h(\boldsymbol{v})h(\boldsymbol{v}')\rangle$ is symmetric with respect to swapping velocities, $\boldsymbol{v}\leftrightarrow\boldsymbol{v}'$. Hint: it is useful to introduce $\tilde{h}_{\boldsymbol{k}}(t) = (1/2\pi)\int \mathrm{d}\omega\, e^{-i\omega t}h_{\boldsymbol{k}\omega}/\epsilon_{\boldsymbol{k}\omega}$.

Secondly, the first term in (8.58), upon insertion of (8.56), turns into an expression of a similar form to the second term in (8.59):

$$
-\frac{e^2}{m^2}\frac{\langle\varphi_{\boldsymbol{k}\omega'}^*\varphi_{\boldsymbol{k}\omega}\rangle}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\sigma}\,\boldsymbol{k}\cdot\frac{\partial f_0}{\partial\boldsymbol{v}} = -\frac{16\pi^2 e^4}{m^2 k^4\epsilon_{\boldsymbol{k}\omega'}^*\epsilon_{\boldsymbol{k}\omega}}\iint \mathrm{d}^3\boldsymbol{v}'\mathrm{d}^3\boldsymbol{v}'' \frac{C_{\boldsymbol{k}\omega'\omega}(\boldsymbol{v}',\boldsymbol{v}'')}{\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\sigma}\,\boldsymbol{k}\cdot\frac{\partial f_0(\boldsymbol{v})}{\partial\boldsymbol{v}}.
\tag{8.60}
$$

Putting all this together, we end up with (8.58) in the following form:

$$
\boxed{\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial\boldsymbol{v}}\cdot\int \mathrm{d}^3\boldsymbol{v}'' \left[\mathsf{D}(\boldsymbol{v},\boldsymbol{v}'')\cdot\frac{\partial f_0(\boldsymbol{v})}{\partial\boldsymbol{v}} - \mathsf{D}(\boldsymbol{v}'',\boldsymbol{v})\cdot\frac{\partial f_0(\boldsymbol{v}'')}{\partial\boldsymbol{v}''}\right]},
\tag{8.61}
$$

where the "diffusion kernel" is

$$
\mathsf{D}(\boldsymbol{v}'',\boldsymbol{v}) = -\frac{16\pi^2 e^4}{m^2}\,\mathrm{Im}\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}\boldsymbol{k}}{k^4}\iint\frac{\mathrm{d}\omega\mathrm{d}\omega'}{(2\pi)^2}\frac{e^{i(\omega'-\omega)t}}{\epsilon_{\boldsymbol{k}\omega'}^*\epsilon_{\boldsymbol{k}\omega}}\int \mathrm{d}^3\boldsymbol{v}' \frac{C_{\boldsymbol{k}\omega'\omega}(\boldsymbol{v}',\boldsymbol{v})}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}'' + i\sigma}.
\tag{8.62}
$$

Note that the QL diffusion coefficient similar to the one appearing in (7.7) is then $\int \mathrm{d}^3\boldsymbol{v}''\,\mathsf{D}(\boldsymbol{v},\boldsymbol{v}'')$.

There is a further simplification available. Using (8.55), we can rewrite

$$
C_{\boldsymbol{k}\omega'\omega}(\boldsymbol{v}',\boldsymbol{v}) = \frac{C_{\boldsymbol{k}}(\boldsymbol{v}',\boldsymbol{v})}{(\omega' - \boldsymbol{k}\cdot\boldsymbol{v}' - i\sigma)(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\sigma)}, \quad C_{\boldsymbol{k}}(\boldsymbol{v}',\boldsymbol{v}) = \left\langle g_{\boldsymbol{k}}^*(\boldsymbol{v}')g_{\boldsymbol{k}}(\boldsymbol{v})\right\rangle, \tag{8.63}
$$

and, therefore,

$$
\mathsf{D}(\boldsymbol{v}'',\boldsymbol{v}) = -\frac{16\pi^2 e^4}{m^2}\,\mathrm{Im}\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}\boldsymbol{k}}{k^4}\int \mathrm{d}^3\boldsymbol{v}'\,C_{\boldsymbol{k}}(\boldsymbol{v}',\boldsymbol{v})I_{\boldsymbol{k}}(\boldsymbol{v}'',\boldsymbol{v}',\boldsymbol{v}),
\tag{8.64}
$$

$$
I_{\boldsymbol{k}}(\boldsymbol{v}'',\boldsymbol{v}',\boldsymbol{v}) = \iint\frac{\mathrm{d}\omega\mathrm{d}\omega'}{(2\pi)^2}\frac{e^{i(\omega'-\omega)t}}{\epsilon_{\boldsymbol{k}\omega'}^*\epsilon_{\boldsymbol{k}\omega}(\omega - \boldsymbol{k}\cdot\boldsymbol{v}'' + i\sigma)(\omega' - \boldsymbol{k}\cdot\boldsymbol{v}' - i\sigma)(\omega - \boldsymbol{k}\cdot\boldsymbol{v} + i\sigma)}.
\tag{8.65}
$$

The remaining work is now in calculating the double integral (8.65). If $\epsilon_{\boldsymbol{k}\omega}$ has no poles at real ω (i.e., f_0 is a stable distribution), the integral is done by shifting the ω' integration contour upwards to $\mathrm{Im}\,\omega\to+\infty$, but snagging on the pole at $\omega' = \boldsymbol{k}\cdot\boldsymbol{v}'$ and the ω contour downwards to $\mathrm{Im}\,\omega\to-\infty$, snagging on the poles at $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$ and $\omega = \boldsymbol{k}\cdot\boldsymbol{v}''$
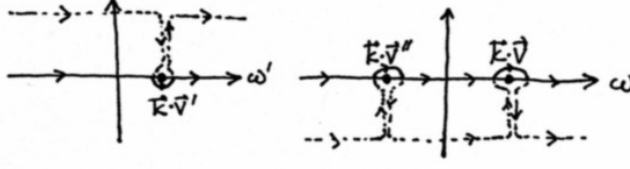
FIGURE 30. The contours for calculating the double integral (8.65).

(see Fig. 30). The result is

$$I_{\boldsymbol{k}}(\boldsymbol{v}'', \boldsymbol{v}', \boldsymbol{v}) = \frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{v}'-\boldsymbol{v})t}}{\epsilon^*_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}'}\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}} \underbrace{\frac{1}{\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')}\left(1 - e^{i\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')t}\frac{\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}}{\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}''}}\right)}_{\to\, -i\pi\delta\left(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')\right)\text{ as } t\to\infty}, \qquad (8.66)$$

where the δ-function approximation is obtained in the same way as in (5.39). Thus, (8.64) becomes

$$\mathsf{D}(\boldsymbol{v}'', \boldsymbol{v}) = \frac{16\pi^3 e^4}{m^2}\,\mathrm{Re}\sum_{\boldsymbol{k}}\frac{\boldsymbol{k}\boldsymbol{k}}{k^4}\,\delta\left(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')\right)\int\mathrm{d}^3\boldsymbol{v}'\,\frac{C_{\boldsymbol{k}}(\boldsymbol{v}',\boldsymbol{v})e^{i\boldsymbol{k}\cdot(\boldsymbol{v}'-\boldsymbol{v})t}}{\epsilon^*_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}'}\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}}. \qquad (8.67)$$

8.6.1. *Kadomtsev–Pogutse "Collision" Integral*

In order to make further progress, we must discuss what is meant by the "initial" distribution $g_{\boldsymbol{k}}(\boldsymbol{v})$ and its correlation function $C_{\boldsymbol{k}}(\boldsymbol{v}', \boldsymbol{v})$. Let us imagine that we start the evolution of our collisionless system with some initial distribution—generally speaking, unstable—and let it proceed for a while. Instabilities might flare up and saturate, particles will stream and phase-mix the distribution, etc., so fairly quickly the exact f will become stable but extremely chopped up and fine-structured in phase space.[58] It is at such a point that we pick it up and treat it as an "initial" state, from which we then examine its further evolution. This involves the evolution of f_0, which is an average over space and "fast" times (meaning frequencies associated with any plasma processes, e.g., $\sim \omega_{\mathrm{pe}}$, and with particle streaming, $\sim \boldsymbol{k}\cdot\boldsymbol{v}$), and the evolution of $\delta f = f - f_0$, whose "initial" state is g and which contains all the fine structure in phase space. It is then reasonable to think of g as essentially random.

The right-hand side of (8.61) encodes further evolution of δf and its effect on f_0 over time t that is long compared to streaming times ($\boldsymbol{k}\cdot\boldsymbol{v}\,t \gg 1$) but short compared to the times over which f_0 changes significantly.[59] In order to calculate (8.67) and, therefore, (8.61), we need to know the correlation function $C_{\boldsymbol{k}}(\boldsymbol{v}', \boldsymbol{v})$ of g. In a somewhat bold move, let us approximate

$$\langle g(\boldsymbol{r},\boldsymbol{v})g(\boldsymbol{r}',\boldsymbol{v}')\rangle \approx \langle g(\boldsymbol{r},\boldsymbol{v})^2\rangle\Delta\Gamma\,\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(\boldsymbol{v}-\boldsymbol{v}'), \qquad (8.68)$$

where $\Delta\Gamma$ is the phase-space volume representing the "width" of the two delta functions. A way to make (8.68) *almost* true by construction is to redefine our average as coarse-graining over phase-space macro-cells of at least the volume $\Delta\Gamma = \Delta r^3 \Delta v^3$, where Δr and Δv are, respectively, the position- and velocity-space correlation scales of g. I said

---

[58]This is the part of the evolution that Lynden-Bell (1967) calls "violent relaxation".

[59]Note that the integrand of the $\boldsymbol{v}'$ integral in (8.67) is the correlation function of $g_{\boldsymbol{k}}(\boldsymbol{v})e^{-i\boldsymbol{k}\cdot\boldsymbol{v}t}/\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}$, which is simply the "initial" distribution evolved ballistically (see §5.3) and "attenuated" by the dielectric response.

"almost" because the whole scheme depends on it being possible also to make both Δr and Δv sufficiently small:

$$\Delta r \ll V^{1/3}, \quad \Delta v \ll \left(\frac{K}{mN}\right)^{1/2}, \tag{8.69}$$

where V is the system's volume and K its total energy. In other words, we are still assuming that, as a result of phase mixing, g would lose any system-scale correlations in either position or velocity space.

With this assumption and this definition of averaging, f_0 is the same as $\bar{f}$ in §8.5 and the macro-cells over which it is coarse-grained are the same randomly filled macro-cells as in the Lynden-Bell statistics (so, in the language of §8.5, $\Delta\Gamma = \mathcal{M}\delta\Gamma$). Unsurprisingly, the same result is about to pop out. Indeed, let us, for simplicity (and for lack of a better idea), assume that the exact distribution is a waterbag (8.34). Then[60]

$$\langle g^2 \rangle = \langle (f - f_0)^2 \rangle = \langle f^2 \rangle - f_0^2 = (\eta - f_0)f_0, \tag{8.70}$$

because, for a waterbag distribution, $\langle f^2 \rangle = \langle \eta f \rangle = \eta f_0$.

Finally, using (8.68) and then (8.70), we get

$$C_{\boldsymbol{k}}(\boldsymbol{v}', \boldsymbol{v}) = \left\langle g_{\boldsymbol{k}}^*(\boldsymbol{v}') g_{\boldsymbol{k}}(\boldsymbol{v}) \right\rangle = \iint \frac{\mathrm{d}^3\boldsymbol{r}\mathrm{d}^3\boldsymbol{r}'}{V^2} \, e^{-\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')} \langle g(\boldsymbol{r}, \boldsymbol{v})g(\boldsymbol{r}', \boldsymbol{v}') \rangle$$

$$= \langle g^2 \rangle \frac{\Delta\Gamma}{V} \, \delta(\boldsymbol{v} - \boldsymbol{v}') = \frac{\Delta\Gamma}{V} \big(\eta - f_0(\boldsymbol{v})\big) f_0(\boldsymbol{v}) \, \delta(\boldsymbol{v} - \boldsymbol{v}'). \tag{8.71}$$

With this model of phase-space correlations, (8.61) combined with (8.67) turns into

$$\frac{\partial f_0}{\partial t} = \frac{16\pi^3 e^4 \Delta\Gamma}{m^2 V} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \mathrm{d}^3\boldsymbol{v}'' \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\boldsymbol{k}}{k^4} \frac{\delta\big(\boldsymbol{k}\cdot(\boldsymbol{v} - \boldsymbol{v}'')\big)}{|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}|^2}$$

$$\cdot \left[\big(\eta - f_0(\boldsymbol{v}'')\big) f_0(\boldsymbol{v}'') \frac{\partial f_0(\boldsymbol{v})}{\partial \boldsymbol{v}} - \big(\eta - f_0(\boldsymbol{v})\big) f_0(\boldsymbol{v}) \frac{\partial f_0(\boldsymbol{v}'')}{\partial \boldsymbol{v}''} \right]. \tag{8.72}$$

This is the "collisionless collision integral" of Kadomtsev & Pogutse (1970) (whose derivation I have more or less followed). It is not hard to show that it has an H-theorem with the entropy (8.38) and that this entropy is maximised—and the right-hand-side of (8.72) is annihilated—by the Fermi–Dirac distribution (8.39).

Exercise 8.7. Show this. If in difficulty, look up how to do it in Adkins & Schekochihin (2019).

Exercise 8.8. Generalise the above calculation to multiple species (electrons and ions).

Exercise 8.9. Generalise the above calculation to multi-waterbag statistics of Exercise 8.3.

Exercise 8.10. Is the Hermite spectrum (10.77) obtained in Q7 consistent with the approximation (8.68)?

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[60]It has not escaped my perceptive reader that a potential mathematical illegality is being perpetrated: is coarse-grain average of a coarse-grain average equal to itself, and is it true that $\langle f f_0 \rangle = \langle f_0^2 \rangle = f_0^2$? In general, no, not when the coarse-graining is done by convolution with some continuous shape function. However, it is fine if we formally do this as we did in §8.5, by grouping fixed discrete micro-cells into fixed discrete macro-cells. Alternatively, we may argue that the differences between f_0 and $\langle f_0 \rangle$ and similar quantities are small enough to be ignored or, cleaner still, resort to an ensemble average over many possible realisations of f, assuming it will do the coarse-graining job automatically. Note that our time average (2.7) in fact suffered from the same problem. Again, we may do an ensemble average and assume that that will automatically have slow time evolution and no spatial dependence (in a homogeneous system).

8.6.2. *Lenard–Balescu Collision Integral*

Just as in §8.5, the non-degenerate limit is $f_0 \ll \eta$. This turns (8.72) into

$$\frac{\partial f_0}{\partial t} = \frac{16\pi^3 e^4 \eta \Delta \Gamma}{m^2 V} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \mathrm{d}^3 \boldsymbol{v}'' \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\boldsymbol{k}}{k^4} \frac{\delta(\boldsymbol{k} \cdot (\boldsymbol{v} - \boldsymbol{v}''))}{|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}|^2} \cdot \left[f_0(\boldsymbol{v}'') \frac{\partial f_0}{\partial \boldsymbol{v}} - f_0(\boldsymbol{v}) \frac{\partial f_0}{\partial \boldsymbol{v}''} \right]. \tag{8.73}$$

This has a functional form that is identical to the standard Lenard–Balescu collision integral (Lenard 1960; Balescu 1960) and its solution is, obviously, a Maxwellian.

How is it possible that collisionless and collisional behaviour turns out to be the same? Mathematically, this is not hard to understand. Let us recall what is meant by collisions in plasma physics. When a plasma (or, rather, a collection of individual particles) is described by its Klimontovich distribution function (1.19), the latter satisfies a "collisionless" Vlasov equation involving microscopic electromagnetic field—the Klimontovich equation (1.35). Collisions acquire a specific mathematical meaning when this equation is averaged (coarse-grained) over the Debye scale, leading to (1.41), where the collision integral is defined as the correlation function of the differences between the averaged ("macroscopic") and the exact ("microscopic") fields and distribution. This is basically the same procedure as the one that led to (8.48) and hence to the "collisionless collision integal" (8.61) and its descendants (8.72) and (8.73). If we interpret the exact distribution f as the Klimontovich distribution, then the difference between it and f_0 will satisfy (8.68) automatically, because the Klimontovich distribution is only non-zero at the exact positions of the particles in phase space and so the correlation function of δf must needs be a delta function. In this interpretation, $\Delta \Gamma$ is the effective width of the delta functions associated with individual particles, while η is their height (so the Klimontovich distribution is a kind of waterbag). Clearly, they are related by

$$\eta \Delta \Gamma = 1, \tag{8.74}$$

which is all we need to know in (8.73), finally turning it into *the* Lenard–Balescu collision integral. It is not surprising that the solution is a Maxwellian because the Klimontovich distribution is extremely non-degenerate: all these delta functions always occupy a negligible fraction of the available phase space.

8.6.3. *Landau's Collision Integral*

While we are at it, let me show for completeness how the Landau (1936) collision integral is recovered from the Lenard–Balescu one. If we want to use (8.73) (with $\eta \, \delta \Gamma = 1$) as an expression for *bona fide* collisions, i.e., if δf and φ are interpreted as fluctuating fields below the Debye scale, we must restrict the $\boldsymbol{k}$ summation to $k \lambda_{\mathrm{D}} \gg 1$. In this approximation, $|\epsilon_{\boldsymbol{k},\boldsymbol{k}\cdot\boldsymbol{v}}|^2 \approx 1$. The $\boldsymbol{k}$ sum then becomes tractable: denoting $\boldsymbol{v} - \boldsymbol{v}'' = \boldsymbol{w}$, we get

$$\frac{1}{V} \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\boldsymbol{k}}{k^4} \delta(\boldsymbol{k}\cdot(\boldsymbol{v}-\boldsymbol{v}'')) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \frac{\boldsymbol{k}\boldsymbol{k}}{k^4} \delta(\boldsymbol{k}\cdot\boldsymbol{w}) = \frac{1}{w} \int \frac{\mathrm{d}^2\boldsymbol{k}_\perp}{(2\pi)^3} \frac{\boldsymbol{k}_\perp \boldsymbol{k}_\perp}{k_\perp^4} = \frac{1}{8\pi^2 w} \left(\boldsymbol{\mathsf{I}} - \frac{\boldsymbol{w}\boldsymbol{w}}{w^2} \right) \int \frac{\mathrm{d}k_\perp}{k_\perp}, \tag{8.75}$$

where $\boldsymbol{k}_\perp = \boldsymbol{k} \cdot (\boldsymbol{\mathsf{I}} - \boldsymbol{w}\boldsymbol{w}/w^2)$. The divergent integral is the Coulomb logarithm $\Lambda = \ln(\lambda_{\mathrm{D}}/d)$ if the integration cut off at the distance of closest approach d [see text after (1.13)] and the Debye scale λ_{D}. Consequently, (8.73) becomes

$$\frac{\partial f_0}{\partial t} = \frac{2\pi e^4 \Lambda}{m^2} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \frac{\mathrm{d}^3\boldsymbol{v}''}{w} \left(\boldsymbol{\mathsf{I}} - \frac{\boldsymbol{w}\boldsymbol{w}}{w^2} \right) \cdot \left[f_0(\boldsymbol{v}'') \frac{\partial f_0}{\partial \boldsymbol{v}} - f_0(\boldsymbol{v}) \frac{\partial f_0}{\partial \boldsymbol{v}''} \right]. \tag{8.76}$$

This is the Landau (1936) collision integral [cf. (1.47)].

8.6.4. *So What Does It All Mean and Where Do We Go from Here?*

Mathematics and some (perhaps) reasonable assumptions got us as far as equation (8.72) [or (8.73)] for the evolution of f_0. This equation tells us that some form of "effective collisionality" associated with the fine-scale structure accumulated all over the phase space pushes the mean distribution function of a collisionless plasma towards a universal distribution—Maxwellian if the phase density is not too tightly packed into the available phase space, Fermi–Dirac if it is and the waterbag model is adopted, or, more generally, a kind of multi-temperature Fermi–Dirac distribution (8.45) (provided Exercise 8.9 is a success).

Suspending for now further doubts about the assumptions that have gone into this, let us note that the progress achieved in comparison with the statistical-mechanical calculations of §8.5 is that we can now follow the relaxation of f_0 in time, at least after some initial period during which all the instabilities of an initial state sort themselves out and f_0 becomes stable. The rate at which f_0 tends to the Fermi–Dirac distribution can be read off from (8.72): ignoring any factors of order unity and using (8.75) to estimate the size of the $\boldsymbol{k}$ sum,[61] we get

$$\boxed{\nu_{\text{eff}} \sim \nu \eta \Delta\Gamma} \;, \tag{8.77}$$

where ν is the "true" collision frequency. In the "true" collisional limit, $\eta\Delta\Gamma = 1$ [see (8.74)], but in the collisionless regime, $\eta\Delta\Gamma$ (the number of particles there would be in a fully occupied macro-cell whose size is the correlation volume of g) must be large, so the "collisionless collision frequency" ν_{eff} is generally speaking much larger than ν.

Can we estimate ν_{eff} in terms of some physical (measurable) parameters?

Since we are dealing with collisionless dynamics, initial distribution will matter and determine η, assuming of course that a waterbag is at all a good model for it (it usually will not be, but let us ignore that). One good physical example is some collection of spatially homogeneous beams, all with the same phase density—then, in terms of the beam number density n_{b} and width v_{b}, $\eta \sim n_{\text{b}}/v_{\text{b}}^3$ [see (8.47)].

The correlation volume $\Delta\Gamma = \Delta r^3 \Delta v^3$ is a trickier quantity to make sense of. As the evolution of a collisionless system proceeds, one might argue that g would be getting ever more fine-scaled ("phase-mixed"; cf. §5.3), i.e., that Δv (and possibly also Δr) would be decreasing with time (eventually, Δr would be limited by $\sim \lambda_{\text{D}}$ and Δv by collisions, however small they are—see §5.5—but for now let us assume that collisionless dynamics can continue as long as we like). Under this scenario, $\Delta\Gamma(t)$ will decrease with time and so the convergence of f_0 towards equilibrium will slow down as time goes on.

It remains an open research topic exactly how to calculate $\Delta v(t)$ and $\Delta r(t)$, or, indeed, more generally, $\langle g(\boldsymbol{r}, \boldsymbol{v})g(\boldsymbol{r}', \boldsymbol{v}')\rangle$, i.e., whether (8.68) is at all adequate. A key challenge in this context is to go beyond the QL approximation. One could reinterpret $h_{k\omega}$ in (8.54) as containing not just the ballistic evolution (8.55) but also the "nonlinear part" of $\delta f_{k\omega}$. The question is what is its correlation function $C_{\boldsymbol{k}\omega'\omega}(\boldsymbol{v}', \boldsymbol{v})$, which then goes into the general collision integral (8.61). You will find some thoughts on this in the classic papers by Kadomtsev & Pogutse (1971) and Dupree (1972), although I recommend them with some hesitation: neither is particularly transparent, so you might end up with less clarity rather than more.[62] The basic idea put forward in these papers is that the correlation scales Δr and Δv will decrease in time much less quickly than you

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[61]This includes the $1/|\epsilon_{\boldsymbol{k}, \boldsymbol{k}\cdot\boldsymbol{v}}|^2$ factor. For $k\lambda_{\text{D}} \gg 1$, $|\epsilon_{\boldsymbol{k}, \boldsymbol{k}\cdot\boldsymbol{v}}| \approx 1$ and the contribution to the $\boldsymbol{k}$ sum from these wave numbers is $\sim \Lambda$; for $k\lambda_{\text{D}} \ll 1$, $|\epsilon_{\boldsymbol{k}, \boldsymbol{k}\cdot\boldsymbol{v}}| \sim 1/(k\lambda_{\text{D}})^2$ and the contribution from these wave numbers is ~ 1.

[62]The textbook by Diamond *et al.* (2010, Chapter 8) is on an ideological continuum from Dupree

might imagine based on naïve phase-mixing estimates—long-time correlated phase-space "clumps" (or "granulations") will form. The "collisionless collision integral" (8.61) with an appropriate clump correlation function $C_{\boldsymbol{k}\omega'\omega}(\boldsymbol{v}', \boldsymbol{v})$ then represents some effective scattering of particles by these clumps. The status of all this is rather uncertain: there was a lot of analytical work done in the 1970s and 1980s along and around these lines, but most of it has remained hypothetical, due to the difficulty of nonlinear theory and impossibility of kinetic numerical simulations capable of resolving anything at the time. The latter impediment to progress appears set to be lifted in the near future, so now probably is a good time to revisit the old theories and attempt new ones.

8.6.5. *Relation to Standard QL Calculations*

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9. Quasiparticle Kinetics

9.1. *QLT in the Language of Quasiparticles*

First I would like to outline a neat way of reformulating the QL theory, which both sheds some light on the meaning of what was done in §7 and opens up promising avenues for theorising further about nonlinear plasma states.

Let us reimagine our system of particles and waves as a mixture of two interacting gases: "true" particles (electrons) and *quasiparticles*, or *plasmons*, which will be the "quantised" version of Langmuir waves. If each of these plasmons has momentum $\hbar\boldsymbol{k}$ and energy $\hbar\omega_{\boldsymbol{k}}$, we can declare

$$N_{\boldsymbol{k}} = \frac{V|\boldsymbol{E}_{\boldsymbol{k}}|^2/4\pi}{\hbar\omega_{\boldsymbol{k}}} \tag{9.1}$$

to be the mean occupation number of plasmons with wave number $\boldsymbol{k}$ (in a box of volume V). The total energy of the plasmons is then

$$\sum_{\boldsymbol{k}} \hbar\omega_{\boldsymbol{k}} N_{\boldsymbol{k}} = V \sum_{\boldsymbol{k}} \frac{|\boldsymbol{E}_{\boldsymbol{k}}|^2}{4\pi}, \tag{9.2}$$

twice the total electric energy in the system (twice because it includes the energy of the mean oscillatory motion of electrons within a wave; see discussion at the end of §7.5). Similarly, the total momentum of the plasmons is

$$\sum_{\boldsymbol{k}} \hbar\boldsymbol{k} N_{\boldsymbol{k}} = V \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}|\boldsymbol{E}_{\boldsymbol{k}}|^2}{4\pi\omega_{\boldsymbol{k}}}. \tag{9.3}$$

This is indeed in line with our previous calculations [see (7.39)]. Note that the role of $\hbar$ here is simply to define a splitting of wave energy into individual plasmons—this can be done in an arbitrary way, provided $\hbar$ is small enough to ensure $N_{\boldsymbol{k}} \gg 1$. Since there is nothing quantum-mechanical about our system, all our results will in the end have to be independent of $\hbar$, so we will use $\hbar$ as an arbitrarily small parmeter, in which it will be convenient to expand, expecting it eventually to cancel out in all physically meaningful relationships.

We may now think of the QL evolution (or indeed generally of the nonlinear evolution) of our plasma in terms of interactions between plasmons and electrons. These are

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(1972), and, while also not a transparent read (imho), is probably the most up-to-date exposition of the status of that line of thinking.

(a) absorption of a plasmon by an electron   (b) emission of a plasmon by an electron

FIGURE 31. Diagrams for (9.6).

resonant electrons; the thermal bulk only participates via its supporting role of enabling oscillatory plasma motions associated with plasmons. The electrons are described by their distribution function $f_0(\boldsymbol{v})$, which we can, to make our formalism nicely uniform, recast in terms of occupation numbers: if the wave number corresponding to velocity $\boldsymbol{v}$ is $\boldsymbol{p} = m_e \boldsymbol{v}/\hbar$, then its occupation number is

$$n_{\boldsymbol{p}} = \left(\frac{2\pi\hbar}{m_e}\right)^3 f_0(\boldsymbol{v}) \quad \Rightarrow \quad \sum_{\boldsymbol{p}} n_{\boldsymbol{p}} = \frac{V}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{p}\, n_{\boldsymbol{p}} = V\int \mathrm{d}^3\boldsymbol{v}\, f_0(\boldsymbol{v}) = V n_e. \quad (9.4)$$

It is understood that $n_{\boldsymbol{p}} \ll 1$ (our electron gas is non-degenerate).

The QL evolution of the plasmon and electron distributions is controlled by two processes: absorption or emission of a plasmon by an electron (known as Cherenkov absorption/emission). Diagrammatically, these can be depicted as shown in Fig. 31. As we know from §7.2, they are subject to momentum conservation, $\boldsymbol{p} = \boldsymbol{k} + (\boldsymbol{p} - \boldsymbol{k})$, and energy conservation:

$$0 = \varepsilon_{\boldsymbol{p}}^e - \varepsilon_{\boldsymbol{k}}^l - \varepsilon_{\boldsymbol{p}-\boldsymbol{k}}^e = \frac{\hbar^2 p^2}{2m_e} - \hbar\omega_{\boldsymbol{k}} - \frac{\hbar^2|\boldsymbol{p}-\boldsymbol{k}|^2}{2m_e} = \hbar\left(-\omega_{\boldsymbol{k}} + \frac{\hbar\boldsymbol{p}\cdot\boldsymbol{k}}{m_e} - \frac{\hbar k^2}{2m_e}\right)$$
$$= \hbar(\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}}) + O(\hbar^2). \quad (9.5)$$

This is the familiar resonance condition $\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}} = 0$. The superscripts e and l stand for electrons and (Langmuir) plasmons.

9.1.1. *Plasmon Distribution*

We may now write an equation for the evolution of the plasmon occupation number:

$$\frac{\partial N_{\boldsymbol{k}}}{\partial t} = -\underbrace{\sum_{\boldsymbol{p}} \mathfrak{w}(\boldsymbol{p}-\boldsymbol{k}, \boldsymbol{k} \to \boldsymbol{p})\delta(\varepsilon_{\boldsymbol{p}-\boldsymbol{k}}^e + \varepsilon_{\boldsymbol{k}}^l - \varepsilon_{\boldsymbol{p}}^e)n_{\boldsymbol{p}-\boldsymbol{k}}N_{\boldsymbol{k}}}_{\text{Fig. 31(a)}}$$

$$+\underbrace{\sum_{\boldsymbol{p}} \mathfrak{w}(\boldsymbol{p} \to \boldsymbol{k}, \boldsymbol{p}-\boldsymbol{k})\delta(\varepsilon_{\boldsymbol{p}}^e - \varepsilon_{\boldsymbol{k}}^l - \varepsilon_{\boldsymbol{p}-\boldsymbol{k}}^e)n_{\boldsymbol{p}}(N_{\boldsymbol{k}} + 1)}_{\text{Fig. 31(b)}}, \quad (9.6)$$

where $\mathfrak{w}$ are the probabilities of absorption and emission and must be equal:

$$\mathfrak{w}(\boldsymbol{p}-\boldsymbol{k}, \boldsymbol{k} \to \boldsymbol{p}) = \mathfrak{w}(\boldsymbol{p} \to \boldsymbol{k}, \boldsymbol{p}-\boldsymbol{k}) \equiv w(\boldsymbol{p}, \boldsymbol{k}). \quad (9.7)$$

The first term in the right-hand side of (9.6) describes the absorption of one of (indistinguishable) $N_{\boldsymbol{k}}$ plasmons by one of $n_{\boldsymbol{p}-\boldsymbol{k}}$ electrons, the second term desribes the emission by one of $n_{\boldsymbol{p}}$ electrons of one of $N_{\boldsymbol{k}} + 1$ plasmons. The $+1$ is, of course, a small correction to $N_{\boldsymbol{k}} \gg 1$ and can be neglected, although sometimes, in analogous but more complicated
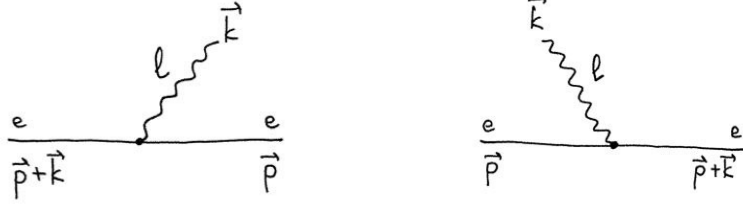
(a) emission of a plasmon by an electron (b) absorption of a plasmon by an electron

FIGURE 32. Additional diagrams for (9.10).

calculations, it has to be kept because lowest-order terms cancel. Using (9.7), (9.5) and (9.4), we find

$$
\frac{\partial N_{\boldsymbol{k}}}{\partial t} \approx \sum_{\boldsymbol{p}} w(\boldsymbol{p}, \boldsymbol{k}) \delta(\varepsilon_{\boldsymbol{p}}^e - \varepsilon_{\boldsymbol{k}}^l - \varepsilon_{\boldsymbol{p}-\boldsymbol{k}}^e)(n_{\boldsymbol{p}} - n_{\boldsymbol{p}-\boldsymbol{k}})N_{\boldsymbol{k}}
$$

$$
= V \int \mathrm{d}^3\boldsymbol{v}\, w\left(\frac{m_e \boldsymbol{v}}{\hbar}, \boldsymbol{k}\right) \delta\big(\hbar(\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}})\big) \left[f_0(\boldsymbol{v}) - f_0\left(\boldsymbol{v} - \frac{\hbar\boldsymbol{k}}{m_e}\right)\right] N_{\boldsymbol{k}}
$$

$$
\approx V \int \mathrm{d}^3\boldsymbol{v}\, w\left(\frac{m_e \boldsymbol{v}}{\hbar}, \boldsymbol{k}\right) \frac{1}{\hbar}\delta(\boldsymbol{k}\cdot\boldsymbol{v} - \omega_{\boldsymbol{k}}) \frac{\hbar}{m_e}\boldsymbol{k}\cdot\frac{\partial f_0}{\partial \boldsymbol{v}}\, N_{\boldsymbol{k}}
$$

$$
= \underbrace{\frac{V}{m_e} w\left(\frac{m_e \omega_{\mathrm{pe}}}{\hbar k}, k\right) F'\left(\frac{\omega_{\mathrm{pe}}}{k}\right)}_{=\,2\gamma_{\boldsymbol{k}}} N_{\boldsymbol{k}} \qquad \Rightarrow \qquad \boxed{\frac{\partial N_{\boldsymbol{k}}}{\partial t} = 2\gamma_{\boldsymbol{k}} N_{\boldsymbol{k}}}. \qquad (9.8)
$$

Note that $\hbar$ has disappeared from our equations, after being used as an expansion parameter.

Since $N_{\boldsymbol{k}} \propto |\boldsymbol{E}_{\boldsymbol{k}}|^2$ [see (9.1)], the prefactor in (9.8) is clearly just the (twice) growth or damping rate of the waves. Comparing with (7.1), we read off the expression for the absorption/emission probability:

$$
w\left(\frac{m_e \omega_{\mathrm{pe}}}{\hbar k}, k\right) = \frac{\pi m_e \omega_{\mathrm{pe}}^3}{V n_e k^2}. \qquad (9.9)
$$

Thus, our calculation of Landau damping could be thought of as a calculation of this probability. Whether there is damping or an instability is decided by whether it is absorption or emission of plasmons that occurs more frequently—and that depends on whether, for any given $\boldsymbol{k}$, there are more electrons that are slightly slower or slightly faster than the plasmons with wave number $\boldsymbol{k}$. Note that getting the correct sign of the damping rate is automatic in this approach, since the probability w must obviously be positive.

9.1.2. *Electron Distribution*

The evolution equation for the occupation number of electrons can be derived in a similar fashion, if we itemise the processes that lead to an electron ending up in a state with a given wave number $\boldsymbol{p} = m_e \boldsymbol{v}/\hbar$ or moving from this state to one with a different wave number. The four relevant diagrams are the two in Fig. 31 and the additional two shown in Fig. 32. The absorption and emission probabilities are the same as before and

so are the energy conservation conditions. We have

$$
\frac{\partial n_{\boldsymbol{p}}}{\partial t} = \underbrace{\sum_{\boldsymbol{k}} \mathfrak{w}(\boldsymbol{p}-\boldsymbol{k}, \boldsymbol{k} \to \boldsymbol{p})\delta(\varepsilon_{\boldsymbol{p}-\boldsymbol{k}}^e + \varepsilon_{\boldsymbol{k}}^l - \varepsilon_{\boldsymbol{p}}^e)n_{\boldsymbol{p}-\boldsymbol{k}}N_{\boldsymbol{k}}}_{\text{Fig. 31(a)}}
$$

$$
+ \underbrace{\sum_{\boldsymbol{k}} \mathfrak{w}(\boldsymbol{p}+\boldsymbol{k} \to \boldsymbol{k}, \boldsymbol{p})\delta(\varepsilon_{\boldsymbol{p}+\boldsymbol{k}}^e - \varepsilon_{\boldsymbol{k}}^l - \varepsilon_{\boldsymbol{p}}^e)n_{\boldsymbol{p}+\boldsymbol{k}}(N_{\boldsymbol{k}}+1)}_{\text{Fig. 32(a)}}
$$

$$
- \underbrace{\sum_{\boldsymbol{k}} \mathfrak{w}(\boldsymbol{p}, \boldsymbol{k} \to \boldsymbol{p}+\boldsymbol{k})\delta(\varepsilon_{\boldsymbol{p}}^e + \varepsilon_{\boldsymbol{k}}^l - \varepsilon_{\boldsymbol{p}+\boldsymbol{k}}^e)n_{\boldsymbol{p}}N_{\boldsymbol{k}}}_{\text{Fig. 32(b)}}
$$

$$
- \underbrace{\sum_{\boldsymbol{k}} \mathfrak{w}(\boldsymbol{p} \to \boldsymbol{k}, \boldsymbol{p}-\boldsymbol{k})\delta(\varepsilon_{\boldsymbol{p}}^e - \varepsilon_{\boldsymbol{k}}^l - \varepsilon_{\boldsymbol{p}-\boldsymbol{k}}^e)n_{\boldsymbol{p}}(N_{\boldsymbol{k}}+1)}_{\text{Fig. 31(b)}}
$$

$$
\approx \sum_{\boldsymbol{k}} w(\boldsymbol{p}+\boldsymbol{k}, \boldsymbol{k})\delta(\varepsilon_{\boldsymbol{p}+\boldsymbol{k}}^e - \varepsilon_{\boldsymbol{k}}^l - \varepsilon_{\boldsymbol{p}}^e)(n_{\boldsymbol{p}+\boldsymbol{k}} - n_{\boldsymbol{p}})N_{\boldsymbol{k}}
$$

$$
- \sum_{\boldsymbol{k}} w(\boldsymbol{p}, \boldsymbol{k})\delta(\varepsilon_{\boldsymbol{p}}^e - \varepsilon_{\boldsymbol{k}}^l - \varepsilon_{\boldsymbol{p}-\boldsymbol{k}}^e)(n_{\boldsymbol{p}} - n_{\boldsymbol{p}-\boldsymbol{k}})N_{\boldsymbol{k}}
$$

$$
\approx \sum_{\boldsymbol{k}} \boldsymbol{k} \cdot \frac{\partial}{\partial \boldsymbol{p}}\, w(\boldsymbol{p}, \boldsymbol{k})\delta(\varepsilon_{\boldsymbol{p}}^e - \varepsilon_{\boldsymbol{k}}^l - \varepsilon_{\boldsymbol{p}-\boldsymbol{k}}^e)\boldsymbol{k} \cdot \frac{\partial n_{\boldsymbol{p}}}{\partial \boldsymbol{p}}\, N_{\boldsymbol{k}}, \tag{9.10}
$$

where we have expanded twice in small $\boldsymbol{k}$ (i.e., in $\hbar$). This is a diffusion equation in $\boldsymbol{p}$ (or, equivalently, $\boldsymbol{v} = \hbar\boldsymbol{p}/m_e$) space. In view of (9.4), (9.10) has the same form as (7.7), viz.,

$$
\boxed{\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}} \cdot \mathsf{D}(\boldsymbol{v}) \cdot \frac{\partial f_0}{\partial \boldsymbol{v}}}, \tag{9.11}
$$

where the diffusion matrix is

$$
\mathsf{D}(\boldsymbol{v}) = \sum_{\boldsymbol{k}} \boldsymbol{k}\boldsymbol{k}\, \frac{\hbar N_{\boldsymbol{k}}}{m_e^2}\, w\!\left(\frac{m_e \boldsymbol{v}}{\hbar}, \boldsymbol{k}\right) \delta(\boldsymbol{k} \cdot \boldsymbol{v} - \omega_{\boldsymbol{k}}) = \frac{e^2}{m_e^2} \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\boldsymbol{k}}{k^2}|\boldsymbol{E}_{\boldsymbol{k}}|^2 \pi \delta(\boldsymbol{k} \cdot \boldsymbol{v} - \omega_{\boldsymbol{k}}). \tag{9.12}
$$

The last expression is identical to the resonant form of the QL diffusion matrix (7.8) [cf. (7.16) and (10.86)]. To derive it, we used the definition (9.1) of $N_{\boldsymbol{k}}$ and the absorption/emission probability (9.7), already known from linear theory.

Thus, we are able to recover the (resonant part of the) QL theory from our new electron-plasmon interaction approach. There is more to this approach than a pretty "field-theoretic" reformulation of already-derived earlier results. The diagram technique and the interpretation of the nonlinear state of the plasma as arising from interactions between particles and quasiparticles can be readily generalised to situations in which the nonlinear interactions in (2.12) cannot be neglected and/or more than one type of waves is present. In this new language, the nonlinear interactions would be manifested as interactions between plasmons (rather than only between plasmons and electrons) contributing to the rate of change of $N_{\boldsymbol{k}}$. There are many possibilities: four-plasmon interactions, interactions between plasmons and phonons (sound waves), as well as between the latter and electrons and/or ions, etc. Some of these will be further explored in §9.2 and onwards. A comprehensive monograph on this subject is Tsytovich (1995)

(see also Kingsep 2004, which is a much more human-scale exposition, although it is only available in the original Russian).

I have introduced the language of kinetics of quasiparticles and their interactions with "true" particles as a reformulation of QLT for plasmas. The method is much more general and originates, as far as I know, from condensed-matter physics, the classic problem being the kinetics of electrons and phonons in metals—the founding texts on this subject are Peierls (1955) and Ziman (1960).

9.2. *Weak Turbulence*

Work in progress. See books by Zakharov *et al.* (1992); Tsytovich (1995); Kingsep (2004); Nazarenko (2011).

9.3. *General Scheme for Calculating Probabilities in WT*

10. Langmuir Turbulence

10.1. *Electrons and Ions Must Talk to Each Other*

10.2. *Zakharov's Equations*

10.3. *Derivation of Zakharov's Equations*

Here I provide a systematic perturbative derivation of the Zakharov (1972) equations, which is surprisingly difficult to find in the literature.

10.3.1. *Scale Separations*

The problem has four characteristic timescales: the plasma oscillation frequency, the electron streaming rate, the ion sound frequency and the ion streaming rate:

$$\omega_{\mathrm{p}e} \gg k v_{\mathrm{th}e} \gg k c_{\mathrm{s}} \sim k v_{\mathrm{th}i}, \tag{10.1}$$

where $v_{\mathrm{th}\alpha} = (2T_\alpha/m_\alpha)^{1/2}$ and $c_{\mathrm{s}} = (T_e/m_i)^{1/2}$. The relative size of these frequencies is controlled by the following three independent parameters:

$$\frac{k v_{\mathrm{th}e}}{\omega_{\mathrm{p}e}} \sim k\lambda_{\mathrm{D}e} \ll 1, \quad \frac{k c_{\mathrm{s}}}{k v_{\mathrm{th}e}} \sim \sqrt{\frac{m_e}{m_i}} \ll 1, \quad \frac{k v_{\mathrm{th}i}}{k c_{\mathrm{s}}} \sim \sqrt{\frac{T_i}{T_e}} \sim 1. \tag{10.2}$$

The scale separation between ions and electrons is non-negotiable as the mass ratio is always small. As long as $k\lambda_{\mathrm{D}e} \ll 1$, which we will assume here, the electron Landau damping is exponentially small and the electrons will be fluid (as we will see shortly; it is no surprise, given what we know from §3.5). Ions too behave as a fluid if they are cold ($T_i \ll T_e$; cf. §3.8), which is the limit most often considered in the context of Zakharov's equations, if not necessarily one that is most relevant physically.

10.3.2. *Electron kinetics and ordering*

We split the electron distribution function and the electrostatic potential into two parts: the time-averaged ("slow", denoted by overbars) and fluctuating ("fast", denoted by overtildes):

$$f_e = \bar{f}_e + \tilde{f}_e, \quad \varphi = \bar{\varphi} + \tilde{\varphi}. \tag{10.3}$$

The time average is taken over time scales longer than both $\omega_{\mathrm{p}e}^{-1}$ and $(k v_{\mathrm{th}e})^{-1}$ but shorter than $(k c_{\mathrm{s}})^{-1}$ or $(k v_{\mathrm{th}i})^{-1}$, i.e., $\bar{f}_e$ and $\bar{\varphi}$ are the electron distribution and potential that the ions will "see". The slow part of the electron distribution is assumed to consist of a homogeneous Maxwellian equlibrium (5.6) and a perturbation:

$$\bar{f}_e = f_{0e} + \delta\bar{f}_e. \tag{10.4}$$

The slow and fast distribution functions satisfy the following equations, which are obtained

by time averaging the Vlasov equation (1.50) for electrons ($\alpha = e$, $q_\alpha = -e$) and subtracting the average from the exact equation:

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} \delta \bar{f}_e + \frac{e}{m_e}(\boldsymbol{\nabla}\bar{\varphi}) \cdot \frac{\partial \bar{f}_e}{\partial \boldsymbol{v}} + \frac{e}{m_e}\overline{(\boldsymbol{\nabla}\tilde{\varphi}) \cdot \frac{\partial \tilde{f}_e}{\partial \boldsymbol{v}}} = 0, \tag{10.5}$$

$$\frac{\partial \tilde{f}_e}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \tilde{f}_e + \frac{e}{m_e}(\boldsymbol{\nabla}\tilde{\varphi}) \cdot \frac{\partial \bar{f}_e}{\partial \boldsymbol{v}} + \frac{e}{m_e}(\boldsymbol{\nabla}\tilde{\varphi}) \cdot \frac{\partial \tilde{f}_e}{\partial \boldsymbol{v}} + \frac{e}{m_e}\widetilde{(\boldsymbol{\nabla}\tilde{\varphi}) \cdot \frac{\partial \tilde{f}_e}{\partial \boldsymbol{v}}} = 0, \tag{10.6}$$

where all time evolution on ion scales is neglected. The slow and fast parts of the Poisson equation (1.51) are

$$-\nabla^2 \bar{\varphi} = 4\pi e(Z\delta n_i - \delta \bar{n}_e) = 4\pi e \left(Z \int \mathrm{d}^3 \boldsymbol{v} \, \delta f_i - \int \mathrm{d}^3 \boldsymbol{v} \, \delta \bar{f}_e \right), \tag{10.7}$$

$$-\nabla^2 \tilde{\varphi} = -4\pi e \tilde{n}_e = -4\pi e \int \mathrm{d}^3 \boldsymbol{v} \, \tilde{f}_e, \tag{10.8}$$

where δf_i is the perturbed ion distribution function and δn_i its density. We shall solve (10.6) and (10.8) for $\tilde{\varphi}$ and $\tilde{f}_e$, use that to calculate the last term in (10.5), which will give rise to an average effect of the fast oscillations known as the poderomotive force, then solve (10.5) for $\bar{f}_e$ in terms of $\bar{\varphi}$, and finally use that solution in (10.7) to get an expression for $\bar{\varphi}$ in terms of f_i. The latter can then be coupled with the ion Vlasov–Landau equation (5.1) ($\alpha = i$, $q_\alpha = Ze$), giving rise to a closed "hybrid" system for kinetic ions and "fluid" electrons.

In order to implement this plan, we carry out a perturbation expansion of the above equations in the small parameter

$$\varepsilon = k\lambda_{\mathrm{D}e}. \tag{10.9}$$

The algebra becomes more compact if we first make the following ansatz, designed to remove the third (the largest, as we will see) term in (10.6):[63]

$$\tilde{f}_e = -\boldsymbol{u} \cdot \frac{\partial \bar{f}_e}{\partial \boldsymbol{v}} + h, \tag{10.10}$$

where $\boldsymbol{u}$ is, by definition, the velocity associated with the plasma oscillation:

$$\frac{\partial \boldsymbol{u}}{\partial t} = \frac{e}{m_e}\boldsymbol{\nabla}\tilde{\varphi} \tag{10.11}$$

[cf. (5.22) and (5.46)]. Then (10.6) becomes

$$\frac{\partial h}{\partial t} = \underbrace{\boldsymbol{v} \cdot \boldsymbol{\nabla}\left(\boldsymbol{u} \cdot \frac{\partial \bar{f}_e}{\partial \boldsymbol{v}}\right)}_{\varepsilon^2} - \underbrace{\boldsymbol{v} \cdot \boldsymbol{\nabla} h}_{\varepsilon^3} + \underbrace{\frac{e}{m_e}(\boldsymbol{\nabla}\bar{\varphi}) \cdot \frac{\partial}{\partial \boldsymbol{v}}\boldsymbol{u} \cdot \frac{\partial \bar{f}_e}{\partial \boldsymbol{v}}}_{\varepsilon^4} - \underbrace{\frac{e}{m_e}(\boldsymbol{\nabla}\bar{\varphi}) \cdot \frac{\partial h}{\partial \boldsymbol{v}}}_{\varepsilon^5}$$

$$+ \underbrace{\frac{e}{m_e}\widetilde{(\boldsymbol{\nabla}\tilde{\varphi}) \cdot \frac{\partial}{\partial \boldsymbol{v}}}\boldsymbol{u} \cdot \frac{\partial \bar{f}_e}{\partial \boldsymbol{v}}}_{\varepsilon^2} - \underbrace{\frac{e}{m_e}\widetilde{(\boldsymbol{\nabla}\tilde{\varphi}) \cdot \frac{\partial h}{\partial \boldsymbol{v}}}}_{\varepsilon^3}, \tag{10.12}$$

where we have indicated the ordering of each term in the small parameter (10.9), based on the following assumptions. The plasma-oscillation velocity (10.11) is

$$\frac{u}{v_{\mathrm{the}}} \sim \frac{ke\tilde{\varphi}}{m_e v_{\mathrm{the}}\omega_{\mathrm{pe}}} \sim k\lambda_{\mathrm{D}e}\frac{e\tilde{\varphi}}{T_e} \sim \varepsilon, \tag{10.13}$$

if, in general,

$$\frac{e\tilde{\varphi}}{T_e} \sim 1. \tag{10.14}$$

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[63]This is equivalent to splitting the electron distribution function into fast and slow parts using as the velocity variable of $\bar{f}_e$ the peculiar velocity of the particle around a centre oscillating with velocity $\boldsymbol{u}$ (cf. DuBois *et al.* 1995): namely, set $f_e = \bar{f}_e(\boldsymbol{r}, \boldsymbol{v} - \boldsymbol{u}(t, \boldsymbol{r})) + h(t, \boldsymbol{r}, \boldsymbol{v})$ and expand in small $\boldsymbol{u}$.

Anticipating that the ponderomotive "potential" will enter on equal footing with the slow potential and that the slow perturbed electron distribution will express the Boltzmann response to the latter modified by the former, we mandate the ordering

$$\frac{\delta \bar{f}_e}{f_{0e}} \sim \frac{e\bar{\varphi}}{T_e} \sim \frac{e^2 \overline{\tilde{E}^2}}{m_e \omega_{\mathrm{pe}}^2 T_e} \sim (k\lambda_{\mathrm{De}})^2 \left(\frac{e\tilde{\varphi}}{T_e}\right)^2 \sim \varepsilon^2. \qquad (10.15)$$

Since the inhomogeneous terms in (10.12) are, thus, $O(\varepsilon^2)$, it follows that $h \sim \varepsilon^2 f_{0e}$ and, since the first term in (10.10) has no density moment, $\tilde{n}_e \sim \varepsilon^2 n_{0e}$.

From (10.12), to lowest order,

$$\frac{\partial h^{(2)}}{\partial t} = \boldsymbol{v} \cdot \boldsymbol{\nabla u} \cdot \frac{\partial f_{0e}}{\partial \boldsymbol{v}} + \frac{\partial \boldsymbol{u}}{\partial t} \cdot \widetilde{\frac{\partial}{\partial \boldsymbol{v}} \boldsymbol{u}} \cdot \frac{\partial f_{0e}}{\partial \boldsymbol{v}}$$

$$= -\frac{2}{v_{\mathrm{the}}^2} \left[v_i v_j \partial_i u_j + \left(\delta_{ij} - \frac{2v_i v_j}{v_{\mathrm{the}}^2}\right) \frac{\partial u_i}{\partial t} u_j \right] f_{0e}, \qquad (10.16)$$

where we have used (10.11) and the fact that f_{0e} is a Maxwellian.

10.3.3. *Ponderomotive response*

With (10.16) in hand, we are now in a position to calculate the last term in (10.5). First, using (10.10) and (10.11) and keeping terms of order ε^2 and ε^3,

$$\frac{e}{m_e} \overline{(\boldsymbol{\nabla}\tilde{\varphi}) \cdot \frac{\partial \tilde{f}_e}{\partial \boldsymbol{v}}} = \overline{\frac{\partial \boldsymbol{u}}{\partial t} \cdot \frac{\partial}{\partial \boldsymbol{v}} \left(-\boldsymbol{u} \cdot \frac{f_{0e}}{\partial \boldsymbol{v}} + h^{(2)}\right)}$$

$$= \frac{2}{v_{\mathrm{the}}^2} \overline{\left(\delta_{ij} - \frac{2v_i v_j}{v_{\mathrm{the}}^2}\right) \frac{\partial u_i}{\partial t} u_j f_{0e}} + \overline{\frac{\partial \boldsymbol{u}}{\partial t} \cdot \frac{\partial h^{(2)}}{\partial \boldsymbol{v}}}$$

$$= \overline{\frac{\partial}{\partial t} \left\{ \frac{2}{v_{\mathrm{the}}^2} \left[\frac{u^2}{2} - \frac{(\boldsymbol{u} \cdot \boldsymbol{v})^2}{v_{\mathrm{the}}^2}\right] f_{0e} + \boldsymbol{u} \cdot \frac{\partial h^{(2)}}{\partial \boldsymbol{v}} \right\}} - \overline{\boldsymbol{u} \cdot \frac{\partial}{\partial \boldsymbol{v}} \frac{\partial h^{(2)}}{\partial t}}. \qquad (10.17)$$

The first term is a full time derivative and so vanishes under averaging, whereas the second term can be calculated using (10.16):

$$-\overline{\boldsymbol{u} \cdot \frac{\partial}{\partial \boldsymbol{v}} \frac{\partial h^{(2)}}{\partial t}} = \frac{2}{v_{\mathrm{the}}^2} \left[\overline{v_j u_l \partial_l u_j + v_i u_l \partial_i u_l - \frac{2v_i v_j v_l}{v_{\mathrm{the}}^2} u_l \partial_i u_j} \right.$$

$$\left. - \frac{2}{v_{\mathrm{the}}^2} \overline{\left(v_l u_l \frac{\partial u_i}{\partial t} u_i + v_j u_l \frac{\partial u_l}{\partial t} u_j + v_i u_l \frac{\partial u_i}{\partial t} u_l - \frac{2v_i v_j v_l}{v_{\mathrm{the}}^2} u_l \frac{\partial u_i}{\partial t} u_j\right)} \right] f_{0e}$$

$$= \frac{2}{v_{\mathrm{the}}^2} \left\{ \overline{\boldsymbol{u} \cdot \boldsymbol{\nabla u} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \left[\frac{u^2}{2} - \frac{(\boldsymbol{u} \cdot \boldsymbol{v})^2}{v_{\mathrm{the}}^2}\right]} \right.$$

$$\left. - \frac{2}{v_{\mathrm{the}}^2} \overline{\frac{\partial}{\partial t} \left[u^2 \boldsymbol{u} \cdot \boldsymbol{v} - \frac{2(\boldsymbol{u} \cdot \boldsymbol{v})^3}{3v_{\mathrm{the}}^2}\right]} \right\} f_{0e}$$

$$= 2\boldsymbol{v} \cdot \boldsymbol{\nabla} \overline{\left[\frac{u^2}{v_{\mathrm{the}}^2} - \frac{(\boldsymbol{u} \cdot \boldsymbol{v})^2}{v_{\mathrm{the}}^4}\right]} f_{0e}. \qquad (10.18)$$

The last expression was obtained after noticing that any full time derivative vanishes under averaging and that, $\boldsymbol{u}$ defined by (10.11) being a potential field, we could rewrite $\boldsymbol{u} \cdot \boldsymbol{\nabla u} = \boldsymbol{\nabla} |\boldsymbol{u}|^2/2$.

Note that (10.18) is $O(\varepsilon^3)$, as are the other two terms in (10.5). Inserting (10.18) into (10.5), we obtain the following solution for the slow part of the perturbed elecron distribution:

$$\delta \bar{f}_e = \left\{ \frac{e\bar{\varphi}}{T_e} - 2 \overline{\left[\frac{u^2}{v_{\mathrm{the}}^2} - \frac{(\boldsymbol{u} \cdot \boldsymbol{v})^2}{v_{\mathrm{the}}^4}\right]} \right\} f_{0e}. \qquad (10.19)$$

The first term is the Boltzmann response, the second the ponderomotive one. The resulting

electron density perturbation is

$$\frac{\delta\bar{n}_e}{n_{0e}} = \frac{e\bar{\varphi}}{T_e} - \frac{\overline{u^2}}{v_{\text{the}}^2}. \tag{10.20}$$

10.3.4. *Electron fluid dynamics*

In order to obtain the evolution equation for $\tilde{\varphi}$, we will need to solve (10.12), coupled to (10.8), to higher order than the lowest, namely, up to ε^4. Rather than solving the kinetic equation (10.12) order by order, it turns out to be a faster procedure to take moments of it exactly and then close the resulting hierarchy by calculating the second moment using $h^{(2)}$ given by (10.16).

The zeroth (density) moment of (10.12) is

$$\frac{\partial\tilde{n}_e}{\partial t} + \boldsymbol{\nabla}\cdot(\bar{n}_e\boldsymbol{u}) + \boldsymbol{\nabla}\cdot\int \mathrm{d}^3\boldsymbol{v}\,\boldsymbol{v}h = 0, \tag{10.21}$$

the continuity equation. The first moment is

$$\frac{\partial}{\partial t}\int \mathrm{d}^3\boldsymbol{v}\,\boldsymbol{v}h = -\boldsymbol{\nabla}\cdot\int \mathrm{d}^3\boldsymbol{v}(\boldsymbol{u}\boldsymbol{v}+\boldsymbol{v}\boldsymbol{u})\bar{f}_e - \boldsymbol{\nabla}\cdot\int \mathrm{d}^3\boldsymbol{v}\,\boldsymbol{v}\boldsymbol{v}h + \frac{e}{m_e}\tilde{n}_e\boldsymbol{\nabla}\bar{\varphi} + \frac{e}{m_e}\widetilde{\bar{n}_e\boldsymbol{\nabla}\tilde{\varphi}}. \tag{10.22}$$

The first term on the right-hand side is zero to all orders up to at least ε^4 because, according to (10.19), $\bar{f}_e$ is even in $\boldsymbol{v}$ up to second order. The remaining terms are $O(\varepsilon^3)$, except the penultimate one, which is $O(\varepsilon^5)$ and can be safely dropped. Combining (10.21) with (10.22) and using (10.11), we have

$$\frac{\partial^2\tilde{n}_e}{\partial t^2} + \boldsymbol{\nabla}\cdot\left(\frac{e}{m_e}\bar{n}_e\boldsymbol{\nabla}\tilde{\varphi}\right) - \boldsymbol{\nabla}\boldsymbol{\nabla}:\int \mathrm{d}^3\boldsymbol{v}\,\boldsymbol{v}\boldsymbol{v}h + \boldsymbol{\nabla}\cdot\left(\frac{e}{m_e}\widetilde{\bar{n}_e\boldsymbol{\nabla}\tilde{\varphi}}\right) = 0. \tag{10.23}$$

This equation is valid up to and including terms of order ε^4.

Note that, in order to maintain this level of precision, we need to keep the lowest-order contribution to h in the second velocity moment. This satisfies (10.16), which it is now convenient to rewrite as

$$\frac{\partial h^{(2)}}{\partial t} = -\frac{2}{v_{\text{the}}^2}\left[v_iv_j\partial_iu_j + \frac{\partial}{\partial t}\left(\frac{u^2}{2} - \frac{u_iu_jv_iv_j}{v_{\text{the}}^2}\right)\right]f_{0e}. \tag{10.24}$$

The stress tensor satisfies

$$\frac{\partial}{\partial t}\int \mathrm{d}^3\boldsymbol{v}\,v_iv_jh = -\frac{n_{0e}v_{\text{the}}^2}{2}\left(\partial_iu_j + \partial_ju_i + \delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) + \frac{\partial}{\partial t}n_{0e}u_iu_j. \tag{10.25}$$

Therefore,

$$\frac{\partial}{\partial t}\left(\boldsymbol{\nabla}\boldsymbol{\nabla}:\int \mathrm{d}^3\boldsymbol{v}\,\boldsymbol{v}\boldsymbol{v}h\right) = -\frac{3}{2}v_{\text{the}}^2\nabla^2\boldsymbol{\nabla}\cdot(n_{0e}\boldsymbol{u}) + \frac{\partial}{\partial t}n_{0e}\boldsymbol{\nabla}\boldsymbol{\nabla}:\boldsymbol{u}\boldsymbol{u}. \tag{10.26}$$

From (10.21), to lowest order, $\boldsymbol{\nabla}\cdot(n_{0e}\boldsymbol{u}) = -\partial\tilde{n}_e/\partial t$ and so the above equation can be integrated in time:

$$\boldsymbol{\nabla}\boldsymbol{\nabla}:\int \mathrm{d}^3\boldsymbol{v}\,\boldsymbol{v}\boldsymbol{v}h = \frac{3}{2}v_{\text{the}}^2\nabla^2\tilde{n}_e + n_{0e}\boldsymbol{\nabla}\boldsymbol{\nabla}:\widetilde{\boldsymbol{u}\boldsymbol{u}}. \tag{10.27}$$

Inserting (10.27) into (10.23) and using also (10.8) to express $\tilde{n}_e$ via $\tilde{\varphi}$, we obtain

$$\frac{\partial^2}{\partial t^2}\nabla^2\tilde{\varphi} + \boldsymbol{\nabla}\cdot\left(\omega_{\text{pe}}^2\frac{\bar{n}_e}{n_{0e}}\boldsymbol{\nabla}\tilde{\varphi}\right) - \frac{3}{2}v_{\text{the}}^2\nabla^4\tilde{\varphi} = 4\pi en_{0e}\boldsymbol{\nabla}\boldsymbol{\nabla}:\widetilde{\boldsymbol{u}\boldsymbol{u}} - \boldsymbol{\nabla}\cdot\left[\frac{e}{m_e}\widetilde{(\nabla^2\tilde{\varphi})\boldsymbol{\nabla}\tilde{\varphi}}\right]. \tag{10.28}$$

The left-hand side of this equation manifestly describes Langmuir waves with the usual dispersion relation $\omega^2 = \omega_{\text{pe}}^2 + 3k^2v_{\text{the}}^2/2$ [see (3.39)]. Note that, since $\bar{n}_e = n_{0e} + \delta\bar{n}_e$, the second term on the left-hand side contains the nonlinear "modulational interaction": the Langmuir waves have the plasma frequency that is locally modified by the slow variation of electron density, given by (10.20) (which depends on the mean energy of the Langmuir waves themselves and also brings in ion dynamics). The terms on the right-hand side of (10.28) are nonlinear interactions between Langmuir waves, which will disappear in a moment.

There are manifestly two frequency scales in (10.28): ω_{pe} and $kv_{\text{the}} \sim \varepsilon\omega_{\text{pe}}$. These can now

separated in the following way. Let

$$\tilde{\varphi} = \frac{1}{2} \left(\psi e^{-i\omega_{\mathrm{pe}}t} + \psi^* e^{i\omega_{\mathrm{pe}}t} \right), \tag{10.29}$$

where ψ varies on the time scale $(kv_{\mathrm{the}})^{-1}$. From (10.11), to lowest order in ε,

$$\boldsymbol{u} = i\frac{e}{2m_e\omega_{\mathrm{pe}}} \boldsymbol{\nabla} \left(\psi e^{-i\omega_{\mathrm{pe}}t} - \psi^* e^{i\omega_{\mathrm{pe}}t} \right). \tag{10.30}$$

Substituting these expressions into (10.28), neglecting $\partial_t^2 \psi \ll i\omega_{\mathrm{pe}}\partial_t\psi$, dividing through by $-\omega_{\mathrm{pe}}e^{-i\omega_{\mathrm{pe}}t}$ and averaging out the oscillatory terms with frequencies ω_{pe} and $2\omega_{\mathrm{pe}}$, we obtain Zakharov's first equation:

$$\boxed{\nabla^2 \left(i\omega_{\mathrm{pe}}^{-1}\frac{\partial\psi}{\partial t} + \frac{3}{2}\lambda_{\mathrm{De}}^2\nabla^2\psi \right) = \frac{1}{2}\boldsymbol{\nabla}\cdot\left(\frac{\delta\bar{n}_e}{n_{0e}}\boldsymbol{\nabla}\psi \right)}. \tag{10.31}$$

Finally, substituting (10.30) into (10.20), we have, for the slow density perturbation,

$$\frac{\delta\bar{n}_e}{n_{0e}} = \frac{e\bar{\varphi}}{T_e} - \frac{\overline{|\boldsymbol{\nabla}\psi|^2}}{16\pi n_{0e}T_e}. \tag{10.32}$$

To get $\bar{\varphi}$, we need to bring in the ions.

10.3.5. *Ion kinetics*

Since the left-hand side of the slow Poisson equation (10.7) is $O(\varepsilon^4)$, while the right-hand side is $O(\varepsilon^2)$, (10.7) predictably turns into the quasineutrality equation

$$\delta\bar{n}_e = Z\delta n_i. \tag{10.33}$$

Combined with (10.32), this becomes

$$\frac{e\bar{\varphi}}{T_e} = \frac{\overline{|\boldsymbol{\nabla}\psi|^2}}{16\pi n_{0e}T_e} + \frac{1}{n_{0i}}\int \mathrm{d}^3\boldsymbol{v}\,\delta f_i, \tag{10.34}$$

where ψ obeys (10.31). The ion distribution function $f_i = f_{0i} + \delta f_i$ is found from the ion Vlasov–Landau equation (5.1) with the slow potential $\bar{\varphi}$:

$$\frac{\partial f_i}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}f_i - \frac{Ze}{m_i}(\boldsymbol{\nabla}\bar{\varphi})\cdot\frac{\partial f_i}{\partial\boldsymbol{v}} = \left(\frac{\partial f_i}{\partial t} \right)_{\mathrm{c}}. \tag{10.35}$$

Together with (10.31), (10.35) and (10.34) make up a closed hybrid system describing kinetic ions and fluid electrons. The electrons affect the ions via the ponderomotive nonlinearity in (10.34), while the ions modulate the plasma frequency and thereby the dynamics of the electrons.

10.3.6. *Ion fluid dynamics*

For completeness, let us show how ions can become fluid, giving rise to the second equation in the classic Zakharov (1972) system.

The zeroth and first moments of (10.35) are

$$\frac{\partial\delta n_i}{\partial t} + \boldsymbol{\nabla}\cdot\int \mathrm{d}^3\boldsymbol{v}\,\boldsymbol{v}\delta f_i = 0, \tag{10.36}$$

$$\frac{\partial}{\partial t}\int \mathrm{d}^3\boldsymbol{v}\,\boldsymbol{v}\delta f_i + \boldsymbol{\nabla}\cdot\int \mathrm{d}^3\boldsymbol{v}\,\boldsymbol{v}\boldsymbol{v}\delta f_i = -\frac{Ze}{m_i}n_i\boldsymbol{\nabla}\bar{\varphi} = -c_{\mathrm{s}}^2 n_i\boldsymbol{\nabla}\left(\frac{\delta n_i}{n_{0i}} + \frac{\overline{|\boldsymbol{\nabla}\psi|^2}}{16\pi n_{0e}T_e} \right), \tag{10.37}$$

where $c_{\mathrm{s}} = (ZT_e/m_i)^{1/2}$ is the sound speed and the last expression was obtained with the aid of (10.34). Combining these two equations and keeping only the lowest-order terms, both in ε and in T_i/T_e, which is now assumed small so as to allow us to neglect the ion pressure (stress) tensor in the left-hand side of (10.37), we get

$$\boxed{\left(\frac{\partial^2}{\partial t^2} - c_{\mathrm{s}}^2\nabla^2 \right)\frac{\delta\bar{n}_e}{n_{0e}} = c_{\mathrm{s}}^2\nabla^2\frac{\overline{|\boldsymbol{\nabla}\psi|^2}}{16\pi n_{0e}T_e}}. \tag{10.38}$$

This is Zakharov's second equation, describing sound waves excited by the ponderomotive force. We have replaced $\delta n_i/n_{0i}$ with $\delta \bar{n}_e/n_{0e}$ (by quasineutrality) to emphasise that the Zakahrov equations (10.31) and (10.38) are a closed system.

When the ions are not cold (T_i/T_e not small), (10.38) regains the ion pressure term, via which it couples to the rest of the moments of δf_i. This is a dissipation channel for the sound waves, via Landau damping, at a typical rate $\sim k v_{\mathrm{th}i}$.

10.4. *Secondary Instability of a Langmuir Wave*

See Thornhill & ter Haar (1978), §3.

10.4.1. *Decay Instability*

10.4.2. *Modulational Instability*

10.5. *Weak Langmuir Turbulence*

See Zakharov (1972); Musher *et al.* (1995); Kingsep (2004).

10.6. *Langmuir Collapse*

See Zakharov (1972).

10.7. *Solitons and Cavitons*

See Thornhill & ter Haar (1978).

10.8. *Kingsep–Rudakov–Sudan Turbulence*

See Kingsep *et al.* (1973).

10.9. *Pelletier's Equilibrium Ensemble*

See Pelletier (1980).

Plasma Kinetics Problem Set

1. Industrialised linear theory with the Z function. Consider a two-species plasma close to Maxwellian equilibrium. Rederive all the results obtained in §§3.4, 3.5, 3.8, 3.9, 3.10 starting from (3.83) and using the asymptotic expansions (3.90) and (3.91) of the plasma dispersion function.

Namely, consider the limits $\zeta_e \gg 1$ or $\zeta_e \ll 1$ and $\zeta_i \gg 1$, find solutions in these limits and establish the conditions on the wave number of the perturbations and on the equilibrium parameters under which these solutions are valid.

In particular, for the case of $\zeta_e \ll 1$ and $\zeta_i \gg 1$, obtain general expressions for the wave frequency and damping without assuming $k\lambda_{\mathrm{D}e}$ to be either small or large. Recover from your solution the cases considered in §§3.8, 3.9 and 3.10.

Find also the ion contribution to the damping of the ion acoustic and Langmuir waves and comment on the circumstances in which it might be important to know what it is.

Convince yourself that you believe the sketch of longitudinal plasma waves in Fig. 14. If you feel computationally inclined, solve the plasma dispersion relation (3.83) numerically [using, e.g., (3.89)] and see if you can reproduce Fig. 14.

You may wish to check your results against some textbook: e.g., Krall & Trivelpiece (1973) and Alexandrov *et al.* (1984) give very thorough treatments of the linear theory (although in rather different styles than I did).

2. Transverse plasma waves. Go back to the Vlasov–Maxwell, rather then Vlasov–Poisson, system and consider *electromagnetic* perturbations in a Maxwellian unmagnetised plasma (unmagnetised in the sense that in equilibrium, $\boldsymbol{B}_0 = 0$):

$$\frac{\partial \delta f_\alpha}{\partial t} + i\boldsymbol{k} \cdot \boldsymbol{v}\, \delta f_\alpha + \frac{q_\alpha}{m_\alpha}\left(\boldsymbol{E} + \frac{\boldsymbol{v} \times \boldsymbol{B}}{c}\right) \cdot \frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}} = 0, \tag{10.39}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfy Maxwell's equations (1.23–1.26) with charge and current densities determined by the perturbed distribution function δf_α.

(a) Consider an initial-value problem for such perturbations and show that the equation for the Laplace transform of $\boldsymbol{E}$ can be written in the form[64]

$$\hat{\epsilon}(p, \boldsymbol{k}) \cdot \hat{\boldsymbol{E}}(p) = \left(\begin{array}{c}\text{terms associated with initial}\\ \text{perturbations of } \delta f_\alpha,\ \boldsymbol{E} \text{ and } \boldsymbol{B}\end{array}\right), \tag{10.40}$$

where the *dielectric tensor* $\hat{\epsilon}(p, \boldsymbol{k})$ is, in tensor notation,

$$\epsilon_{ij}(p, \boldsymbol{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\epsilon_{\mathrm{TT}}(p, k) + \frac{k_i k_j}{k^2}\,\epsilon_{\mathrm{LL}}(p, k) \tag{10.41}$$

and the *longitudinal dielectric function* $\epsilon_{\mathrm{LL}}(p, k)$ is the familiar electrostatic one, given by (3.83), while the *transverse dielectric function* is

$$\boxed{\epsilon_{\mathrm{TT}}(p, k) = 1 + \frac{1}{p^2}\left[k^2 c^2 - \sum_\alpha \omega_{\mathrm{p}\alpha}^2 \zeta_\alpha Z(\zeta_\alpha)\right]}. \tag{10.42}$$

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[64] In Q3, dealing with the Weibel instability, you will have to do essentially the same calculation, but with a non-Maxwellian equilibrium. To avoid doing the work twice, you could do that question first and then specialise to a Maxwellian $f_{0\alpha}$. However, the algebra is a bit hairier for the non-Maxwellian case, so it may be useful to do the simpler case first, to train your hand—and also to have a way to cross-check the later calculation.

(b) Hence solve the transverse dispersion relation, $\epsilon_{TT}(p,k) = 0$, and show that, in the high-frequency limit ($|\zeta_e| \gg 1$), the resulting waves are simply the light waves, which, at long wave lengths, turn into plasma oscillations. What is the wave length above which light can "feel" that it is propagating through plasma?—this is called the *plasma (electron) skin depth, d_e*. Are these waves damped?

(c) In the low-frequency limit ($|\zeta_e| \ll 1$), show that perturbations are aperiodic (have zero frequency) and damped. Find their damping rate and show that this result is valid for perturbations with wave lengths longer than the plasma skin depth ($kd_e \ll 1$). Explain physically why these perturbations fail to propagate.

Do either Q3 or Q4.

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3. Weibel instability. Weibel (1958) realised that transverse plasma perturbations can go unstable if the equilibrium distribution is anisotropic with respect to some special direction $\hat{\boldsymbol{n}}$, namely if $f_{0\alpha} = f_{0\alpha}(v_\perp, v_\parallel)$, where $v_\parallel = \boldsymbol{v} \cdot \hat{\boldsymbol{n}}$, $v_\perp = |\boldsymbol{v}_\perp|$ and $\boldsymbol{v}_\perp = \boldsymbol{v} - v_\parallel \hat{\boldsymbol{n}}$. The anisotropy can be due to some beam or stream of particles injected into the plasma, it also arises in collisionless shocks or, generically, when plasma is sheared or non-isotropically compressed by some external force. The simplest model for an anisotropic distribution of the required type is a *bi-Maxwellian*[65]:

$$f_{0\alpha} = \frac{n_\alpha}{\pi^{3/2} v_{\text{th}\perp\alpha}^2 v_{\text{th}\parallel\alpha}} \exp\left(-\frac{v_\perp^2}{v_{\text{th}\perp\alpha}^2} - \frac{v_\parallel^2}{v_{\text{th}\parallel\alpha}^2}\right), \tag{10.43}$$

where, formally, $v_{\text{th}\perp\alpha} = \sqrt{2T_{\perp\alpha}/m_\alpha}$ and $v_{\text{th}\parallel\alpha} = \sqrt{2T_{\parallel\alpha}/m_\alpha}$ are the two "thermal speeds" in a plasma characterised by two effective temperatures $T_{\perp\alpha}$ and $T_{\parallel\alpha}$ (for each species).

(a) Using exactly the same method as in Q2, consider electromagnetic perturbations in a bi-Maxwellian plasma, assuming their wave vectors to be parallel to the direction of anisotropy, $\boldsymbol{k} \parallel \hat{\boldsymbol{n}}$. Show that the dielectric tensor again has the form (10.41) and the longitudial dielectric function is again given by (3.83), while the transverse dielectric function is

$$\boxed{\epsilon_{TT}(p,k) = 1 + \frac{1}{p^2}\left[k^2 c^2 + \sum_\alpha \omega_{\text{p}\alpha}^2 \left(1 - \frac{T_{\perp\alpha}}{T_{\parallel\alpha}}\left[1 + \zeta_\alpha Z(\zeta_\alpha)\right]\right)\right].} \tag{10.44}$$

(b) Show that in one of the tractable asymptotic limits, this dispersion relation has a zero-frequency, purely growing solution with the growth rate

$$\gamma = \frac{kv_{\text{th}\parallel e}}{\sqrt{\pi}} \frac{T_{\parallel e}}{T_{\perp e}} \left(\Delta_e - k^2 d_e^2\right), \tag{10.45}$$

where $\Delta_e = T_{\perp e}/T_{\parallel e} - 1$ is the fractional temperature anisotropy, which must be positive in order for the instability to occur. Find the maximum growth rate and the corresponding wave number. Under what condition(s) is the asymptotic limit in which you worked indeed a valid approximation for this solution?

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[65]In Exercise 4.8, you need the dielectric tensor in terms of a general equilibrium distribution $f_{0\alpha}(v_x, v_y, v_z)$. If you are planning to do that exercise, it may save time (at the price of a very slight increase in algebra) to do the derivation with a general $f_{0\alpha}$ and then specialise to the bi-Maxwellian (10.43). You can check your algebra by looking up the result in Krall & Trivelpiece (1973) or in Davidson (1983).

(c) Are there any other unstable solutions? (cf. Weibel 1958)

(d) What happens if the electrons are isotropic but ions are not?

(e**) If you want a challenge and a test of stamina, work out the case of perturbations whose wave number is *not* necessarily in the direction of the anisotropy ($\boldsymbol{k} \times \hat{\boldsymbol{n}} \neq 0$). Are the $\boldsymbol{k} \parallel \hat{\boldsymbol{n}}$ or some oblique perturbations the fastest growing? This is a lot of algebra, so only do it if you enjoy this sort of thing. The dispersion relation for this general case appears to be in the Appendix of Ruyer *et al.* (2015), but they only solve it numerically; no one seems to have looked at asymptotic limits. This could be the start of a dissertation.

4. Two-stream instability.[66] Consider one-dimensional, electrostatic perturbations in a two-species (electron-ion) plasma. Let the electron distribution function with respect to velocities in the direction (z) of the spatial variation of perturbations be a "double Lorentzian" consisting of two counterpropagating beams with velocity u_{b} and width v_{b}, viz.,

$$F_e(v_z) = \frac{n_e v_{\mathrm{b}}}{2\pi} \left[\frac{1}{(v_z - u_{\mathrm{b}})^2 + v_{\mathrm{b}}^2} + \frac{1}{(v_z + u_{\mathrm{b}})^2 + v_{\mathrm{b}}^2} \right] \tag{10.46}$$

(see Fig. 12b), while the ions are Maxwellian with thermal speed $v_{\mathrm{th}i} \ll u_{\mathrm{b}}$. Assume also that the phase velocity (p/k) will be of the same order as u_{b} and hence that the ion contribution to the dielectric function (3.26) is negligible.

(a) By integrating by parts and then choosing the integration contour judiciously, or otherwise, calculate the dielectric function $\epsilon(p, k)$ for this plasma and hence show that the dispersion relation is

$$\sigma^4 + (2u_{\mathrm{b}}^2 + v_{\mathrm{p}}^2)\sigma^2 + u_{\mathrm{b}}^2(u_{\mathrm{b}}^2 - v_{\mathrm{p}}^2) = 0, \tag{10.47}$$

where $\sigma = v_{\mathrm{b}} + p/k$ and $v_{\mathrm{p}} = \omega_{\mathrm{p}e}/k$.

(b) In the long-wavelength limit, viz., $k \ll \omega_{\mathrm{p}e}/u_{\mathrm{b}}$, find the condition for an instability to exist and calculate the growth rate of this instability. Is the nature of this instability kinetic (due to Landau resonance) or hydrodynamic?

(c) Consider the case of cold beams, $v_{\mathrm{b}} = 0$. Without making any *a priori* assumptions about k, calculate the maximum growth rate of the instability. Sketch the growth rate as a function of k.

(d) Allowing warm beams, $v_{\mathrm{b}} > 0$, show that the system is unstable provided

$$u_{\mathrm{b}} > v_{\mathrm{b}} \quad \text{and} \quad k < \omega_{\mathrm{p}e} \frac{\sqrt{u_{\mathrm{b}}^2 - v_{\mathrm{b}}^2}}{u_{\mathrm{b}}^2 + v_{\mathrm{b}}^2}. \tag{10.48}$$

What is the effect that a finite beam width has on the stability of the system and on the kind of perturbations that can grow?

In §4.4, you might find it instructive to compare the results that you have just obtained by solving the dispersion relation (10.47) directly with what can be inferred via Penrose's criterion and Nyquist's method.

5. Free energy and stability. (a) Starting from the linearised Vlasov–Poisson system and assuming a Maxwellian equilibrium, show by direct calculation from the equations, rather then via expansion of the entropy function and the use of energy conservation (as

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[66]This is based on the 2019 exam question.

was done in §5.2), that free energy is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3\boldsymbol{r} \left[\sum_\alpha \int \mathrm{d}^3\boldsymbol{v} \, \frac{T_\alpha \delta f_\alpha^2}{2 f_{0\alpha}} + \frac{|\boldsymbol{\nabla}\varphi|^2}{8\pi} \right] = 0. \tag{10.49}$$

This is an exercise in integrating by parts.

(b) Now consider the full Vlasov–Maxwell equations and prove, again for a Maxwellian plasma plus small perturbations,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3\boldsymbol{r} \left[\sum_\alpha \int \mathrm{d}^3\boldsymbol{v} \, \frac{T_\alpha \delta f_\alpha^2}{2 f_{0\alpha}} + \frac{|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2}{8\pi} \right] = 0} \, . \tag{10.50}$$

(c) Consider the same problem, this time with an equilibrium that is not Maxwellian, but merely isotropic, i.e., $f_{0\alpha} = f_{0\alpha}(v)$, or, in what will prove to be a more convenient form,

$$f_{0\alpha} = f_{0\alpha}(\varepsilon_\alpha), \tag{10.51}$$

where $\varepsilon_\alpha = m_\alpha v^2/2$ is the particle energy. Find an integral quantity quadratic in perturbed fields and distributions that is conserved by the Vlasov–Maxwell system under these circumstances and that turns into the free energy (10.50) in the case of a Maxwellian equilibrium (if in difficulty, you will find the answer in, e.g., Davidson 1983 or in Kruskal & Oberman 1958, which appears to be the original source). Argue that

$$\frac{\partial f_{0\alpha}}{\partial \varepsilon_\alpha} < 0 \tag{10.52}$$

is a sufficient condition for stability of small ($\delta f_\alpha \ll f_{0\alpha}$, but not necessarily infinitesimal) perturbations in such a plasma.

6. Fluctuation-dissipation relation for a collisionless plasma. Let us consider a linear kinetic system in which perturbations are stirred up by an external force, which we can think of as an imposed (time-dependent) electric field $\boldsymbol{E}_{\mathrm{ext}} = -\boldsymbol{\nabla}\chi$. The perturbed distribution function then satisfies

$$\frac{\partial \delta f_\alpha}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \delta f_\alpha - \frac{q_\alpha}{m_\alpha} (\boldsymbol{\nabla}\varphi_{\mathrm{tot}}) \cdot \frac{\partial f_{0\alpha}}{\partial \boldsymbol{v}} = 0, \tag{10.53}$$

where $\varphi_{\mathrm{tot}} = \varphi + \chi$ is the total potential, whose self-consistent part, φ, obeys the usual Poisson equation

$$-\nabla^2 \varphi = 4\pi \sum_\alpha q_\alpha \int \mathrm{d}^3\boldsymbol{v} \, \delta f_\alpha \tag{10.54}$$

and the equilibrium $f_{0\alpha}$ is assumed to be Maxwellian.

(a) By considering an initial-value problem for (10.53) and (10.54) with zero initial perturbation, show that the Laplace transforms of φ_{tot} and χ are related by

$$\hat{\varphi}_{\mathrm{tot}}(p) = \frac{\hat{\chi}(p)}{\epsilon(p)}, \tag{10.55}$$

where $\epsilon(p)$ is the dielectric function given by (3.83).

(b) Consider a time-periodic external force,

$$\chi(t) = \chi_0 e^{-i\omega_0 t}. \tag{10.56}$$

Working out the relevant Laplace transforms and their inverses [see (3.14)], show that, after transients have decayed, the total electric field in the system will oscillate at the same frequency as the external force and be given by

$$\varphi_{\text{tot}}(t) = \frac{\chi_0 \, e^{-i\omega_0 t}}{\epsilon(-i\omega_0)}. \tag{10.57}$$

(c) Now consider the *plasma-kinetic Langevin problem:* assume the external force to be a white noise, i.e., a random process with the time-correlation function

$$\langle \chi(t)\chi^*(t')\rangle = 2D\,\delta(t-t'). \tag{10.58}$$

Show that the resulting steady-state mean-square fluctuation level in the plasma will be

$$\boxed{\langle |\varphi_{\text{tot}}(t)|^2\rangle = \frac{D}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{|\epsilon(-i\omega)|^2}}. \tag{10.59}$$

This is a kinetic *fluctuation-dissipation relation:* given a certain level of external stirring, parametrised by D, this formula predicts the fluctuation energy in terms of D and of the internal dissipative properties of the plasma, encoded by its dielectric function.

(d) For a system in which the Landau damping is weak, $|\gamma| \ll kv_{\text{th}\alpha}$, calculate the integral (10.59) using Plemelj's formula (3.25) to show that

$$\langle |\varphi_{\text{tot}}(t)|^2\rangle = D\sum_i \frac{1}{|\gamma_i|}\left[\frac{\partial\,\mathrm{Re}\,\epsilon(-i\omega)}{\partial\omega}\right]^{-2}_{\omega=\omega_i}, \tag{10.60}$$

where $p_i = -i\omega_i + \gamma_i$ are the weak-damping roots of the dispersion relation.

Here is a reminder of how the *standard Langevin problem* can be solved using Laplace transforms. The Langevin equation is

$$\boxed{\frac{\partial\varphi}{\partial t} + \gamma\varphi = \chi(t)}, \tag{10.61}$$

where φ describes some quantity, e.g., the velocity of a Brownian particle, subject to a damping rate γ and an external force χ. In the case of a Brownian particle, χ is assumed to be a white noise, as per (10.58). Assuming $\varphi(t=0) = 0$, the Laplace-trasform solution of (10.61) is

$$\hat{\varphi}(p) = \frac{\hat{\chi}(p)}{p+\gamma}. \tag{10.62}$$

Considering first a non-random oscillatory force (10.56), we have

$$\hat{\chi}(p) = \int_0^\infty \mathrm{d}t\, e^{-pt}\chi(t) = \frac{\chi_0}{p+i\omega_0} \quad\Rightarrow\quad \hat{\varphi}(p) = \frac{\chi_0}{(p+\gamma)(p+i\omega_0)}. \tag{10.63}$$

The inverse Laplace transform of $\hat{\varphi}$ is calculated by shifting the integration contour to large negative $\mathrm{Re}\,p$ while not allowing it to cross the two poles, $p = -\gamma$ and $p = -i\omega_0$, in a manner analogous to that explained in §3.1 (Fig. 5) and shown in Fig. 33. The integral is then dominated by the contributions from the poles:

$$\varphi(t) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma}\mathrm{d}p\, e^{pt}\hat{\varphi}(p) = \chi_0\left(\frac{e^{-i\omega_0 t}}{-i\omega_0+\gamma} + \frac{e^{-\gamma t}}{-\gamma+i\omega_0}\right) \to \frac{\chi_0\, e^{-i\omega_0 t}}{-i\omega_0+\gamma} \quad\text{as}\quad t\to\infty, \tag{10.64}$$

which is quite obviously the right solution of (10.61) with a periodic force (the second term in the brackets is the decaying transient needed to enforce the zero initial condition).
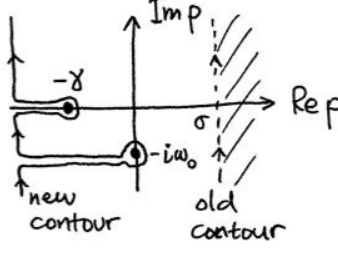
FIGURE 33. Shifting the integration contour in (10.64).

In the more complicated case of a white-noise force [see (10.58)],

$$\langle|\varphi(t)|^2\rangle = \frac{1}{(2\pi)^2}\left\langle\left|\int_{-i\infty+\sigma}^{i\infty+\sigma}\mathrm{d}p\,e^{pt}\frac{\hat{\chi}(p)}{p+\gamma}\right|^2\right\rangle$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\mathrm{d}\omega\int_{-\infty}^{+\infty}\mathrm{d}\omega'e^{[-i(\omega-\omega')+2\sigma]t}\frac{\langle\hat{\chi}(-i\omega+\sigma)\hat{\chi}^*(-i\omega'+\sigma)\rangle}{(-i\omega+\sigma+\gamma)(i\omega'+\sigma+\gamma)}, \quad (10.65)$$

where we have changed variables $p = -i\omega + \sigma$ and similarly for the second integral. The correlation function of the Laplace-transformed force is, using (10.58),

$$\langle\hat{\chi}(p)\hat{\chi}^*(p')\rangle = \int_0^\infty\mathrm{d}t\int_0^\infty\mathrm{d}t'e^{-pt-p'^*t'}\langle\chi(t)\chi^*(t')\rangle = 2D\int_0^\infty\mathrm{d}t\,e^{-(p+p'^*)t} = \frac{2D}{p+p'^*}, \quad (10.66)$$

provided $\mathrm{Re}\,p > 0$ and $\mathrm{Re}\,p' > 0$. Then (10.65) becomes

$$\langle|\varphi(t)|^2\rangle = \frac{D}{2\pi^2}\int_{-\infty}^{+\infty}\mathrm{d}\omega\int_{-\infty}^{+\infty}\mathrm{d}\omega'\frac{e^{[-i(\omega-\omega')+2\sigma]t}}{[-i(\omega-\omega')+2\sigma]\,(-i\omega+\sigma+\gamma)(i\omega'+\sigma+\gamma)}$$

$$= \frac{D}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}\omega'\frac{e^{(i\omega'+\sigma)t}}{(i\omega'+\sigma+\gamma)}\frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma}\mathrm{d}p\frac{e^{pt}}{(p+i\omega'+\sigma)(p+\gamma)}$$

$$= \frac{D}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}\omega'\frac{e^{(i\omega'+\sigma)t}}{(i\omega'+\sigma+\gamma)}\left[\frac{e^{-(i\omega'+\sigma)t}}{-i\omega'-\sigma+\gamma}+\frac{e^{-\gamma t}}{\gamma+i\omega'+\sigma}\right], \quad (10.67)$$

where we have reverted to the p variable in one of the integrals and then performed the integration by the same manipulation of the contour as in (10.64). We now note that, since there are no exponentially growing solutions in this system, $\sigma > 0$ can be chosen arbitrarily small. Taking $\sigma \to +0$ and neglecting the decaying transient in (10.67), we get, in the limit $t \to \infty$,

$$\langle|\varphi(t)|^2\rangle = \frac{D}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{|-i\omega'+\gamma|^2} = \frac{D}{\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega'}{\omega'^2+\gamma^2} = \frac{D}{\gamma}. \quad (10.68)$$

Note that, while the integral in (10.68) is doable exactly, it can, for the case of weak damping, also be computed via Plemelj's formula.

Equation (10.68) is the standard Langevin fluctuation-dissipation relation. It can also be obtained without Laplace transforms, either by directly integrating (10.61) and correlating $\varphi(t)$ with itself or by noticing that

$$\frac{\partial}{\partial t}\frac{\langle\varphi^2\rangle}{2}+\gamma\langle\varphi^2\rangle = \langle\chi(t)\varphi(t)\rangle = \left\langle\chi(t)\int_0^t\mathrm{d}t'\left[-\gamma\varphi(t')+\chi(t')\right]\right\rangle = D, \quad (10.69)$$

where we have used (10.58) and the fact that $\langle\chi(t)\varphi(t')\rangle = 0$ for $t' \leqslant t$, by causality. Equation (10.68) is the steady-state solution to the above, but (10.69) also teaches us that, if we interpret $\langle\varphi^2\rangle/2$ as energy, D is the power that is injected into the system by the external force. Thus, fluctuation-dissipation relations such as (10.68) tells us what fluctuation *energy* will persist in a dissipative system if a certain amount of *power* is pumped in.

7.* Phase-mixing spectrum. Here we study the velocity-space structure of the perturbed distribution function δf derived in Q6.

In order to do this, we need to review the *Hermite transform:*

$$\delta f_m = \frac{1}{n} \int \mathrm{d}v_z \, \frac{H_m(u)\delta f(v_z)}{\sqrt{2^m m!}}, \quad u = \frac{v_z}{v_{\mathrm{th}}}, \quad H_m(u) = (-1)^m e^{u^2} \frac{\mathrm{d}^m}{\mathrm{d}u^m} e^{-u^2}, \tag{10.70}$$

where H_m is the Hermite polynomial of (integer) order m. We are only concerned with the v_z dependence of δf (where z, as always, is along the wave number of the perturbations—in this case set by the wave number of the force); all v_x and v_y dependence is Maxwellian and can be integrated out. The inverse transform is given by

$$\delta f(v_z) = \sum_{m=0}^{\infty} \frac{H_m(u)F(v_z)}{\sqrt{2^m m!}} \, \delta f_m, \quad F(v_z) = \frac{n}{\sqrt{\pi}\, v_{\mathrm{th}}} \, e^{-u^2}. \tag{10.71}$$

Because $H_m(u)$ are orthogonal polynomials, viz.,

$$\frac{1}{n} \int \mathrm{d}v_z \, H_m(u)H_{m'}(u)F(v_z) = 2^m m! \, \delta_{mm'}, \tag{10.72}$$

they have a Parseval theorem and so the contribution of the perturbed distribution function to the free energy [see (5.18)] can be written as

$$\int \mathrm{d}^3\boldsymbol{v} \, \frac{T|\delta f|^2}{2f_0} = \frac{nT}{2} \sum_m |\delta f_m|^2. \tag{10.73}$$

In a plasma where perturbations are constantly stirred up by a force, Landau damping must be operating all the time, removing energy from φ to provide "dissipation" of the injected power. The process of phase mixing that accompanies Landau damping must then lead to a certain fluctuation level $\langle |\delta f_m|^2 \rangle$ in the Hermite moments of δf. Lower m's correspond to "fluid" quantities: density ($m = 0$), flow velocity ($m = 1$), temperature ($m = 2$). Higher m's correspond to finer structure in velocity space: indeed, for $m \gg 1$, the Hermite polynomials can be approximated by trigonometric functions,

$$H_m(u) \approx \sqrt{2}\left(\frac{2m}{e}\right)^{m/2} \cos\left(\sqrt{2m}\, u - \frac{\pi m}{2}\right) e^{u^2/2}, \tag{10.74}$$

and so the Hermite transform is somewhat analogous to a Fourier transform in velocity space with "frequency" $\sqrt{2m}/v_{\mathrm{th}}$.

(a) Show that in the kinetic Langevin problem described in Q6(c), the mean square fluctuation level of the m-th Hermite moment of the perturbed distribution function is given by

$$\langle |\delta f_m(t)|^2 \rangle = \frac{q^2 D}{T^2 \pi 2^m m!} \int_{-\infty}^{+\infty} \mathrm{d}\omega \left| \frac{\zeta Z^{(m)}(\zeta)}{\epsilon(-i\omega)} \right|^2, \quad \zeta = \frac{\omega}{k v_{\mathrm{th}}}, \tag{10.75}$$

where $Z^{(m)}(\zeta)$ is the m-th derivative of the plasma dispersion function [note (3.92)].

(b**) Show that, assuming $m \gg 1$ and $\zeta \ll \sqrt{2m}$,

$$Z^{(m)}(\zeta) \approx \sqrt{2\pi}\, i^{m+1} \left(\frac{2m}{e}\right)^{m/2} e^{i\zeta\sqrt{2m}-\zeta^2/2} \tag{10.76}$$

and, therefore, that

$$\boxed{\langle |\delta f_m(t)|^2 \rangle \approx \frac{\mathrm{const}}{\sqrt{m}}}. \tag{10.77}$$

Thus, the *Hermite spectrum* of the free energy is shallow and, in particular, the total free energy diverges—it has to be regularised by collisions. This is a manifestation of a

copious amount of fine-scale structure in velocity space (note also how this shows that Landau-damped perturbations involve all Hermite moments, not just the "fluid" ones).

Deriving (10.76) is a (reasonably hard) mathematical exercise: it involves using (3.92) and (10.74) and manipulating contours in the complex plane. This is a treat for those who like this sort of thing. Getting to (10.77) will also require the use of Stirling's formula.

The Hermite order at which the spectrum (10.77) must be cut off due to collisions can be quickly deduced as follows. We saw in §5.5 that the typical velocity derivative of δf can be estimated according to (5.26) and the time it takes for this perturbation to be wiped out by collisions is given by (5.31). But, in view of (10.74), the velocity gradients probed by the Hermite moment m are of order $\sqrt{2m}/v_{\text{th}}$. The collisional cut off m_{c} in Hermite space can then be estimated so:

$$m_{\text{c}} \sim v_{\text{th}}^2 \frac{\partial^2}{\partial v^2} \sim (k v_{\text{th}} t_{\text{c}})^2 \sim \left(\frac{k v_{\text{th}}}{\nu} \right)^{2/3}. \tag{10.78}$$

Therefore, the total free energy stored in phase space diverges: using (10.73) and (10.77),

$$\frac{1}{n} \int \mathrm{d}^3 \boldsymbol{v} \frac{\delta f^2}{2 f_0} = \frac{1}{2} \sum_m \left\langle |\delta f_m|^2 \right\rangle \sim \int^{m_{\text{c}}} \mathrm{d}m \frac{\text{const}}{\sqrt{m}} \propto \nu^{-1/3} \to \infty \quad \text{as} \quad \nu \to +0. \tag{10.79}$$

In contrast, the total free-energy dissipation rate is finite, however small is the collision frequency: estimating the right-hand side of (5.18), we get

$$\frac{1}{n} \int \mathrm{d}^3 \boldsymbol{v} \frac{\delta f}{f_0} \left(\frac{\partial \delta f}{\partial t} \right)_{\text{c}} \sim -\nu \sum_m m \left\langle |\delta f_m|^2 \right\rangle \propto \nu \int^{m_{\text{c}}} \mathrm{d}m \sqrt{m} \sim k v_{\text{th}}. \tag{10.80}$$

Thus, the kinetic system can collisionally produce entropy at a rate that is entirely independent of the collision frequency.

If you find phase-space turbulence and generally life in Hermite space as fascinating as I do, you can learn more from Kanekar *et al.* (2015) (on fluctuation-dissipation relations and Hermite spectra) and from Adkins & Schekochihin (2018) (on what happens when nonlinearity strikes).

Do one of Q8, Q9 or Q10.

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8. QL theory of Landau damping. In §7, we discussed the QL theory of an unstable system, in which, whatever the size of the initial electric perturbations, they eventually grow large enough to affect the equilibrium distribution and modify it so as to suppress further growth. In a *stable* equilibrium, any initial perturbations will be Landau-damped, but, if they are sufficiently large to start with, they can also affect f_0 quasilinearly in a way that will slow down this damping.

Consider, in 1D, an initial spectrum $W(0, k)$ of plasma oscillations (waves) excited in the wave-number range $[k_2, k_1] = [\omega_{\text{pe}}/v_2, \omega_{\text{pe}}/v_1] \ll \lambda_{\text{De}}^{-1}$, with total electric energy $\mathcal{E}(0)$. Modify the QL theory of §7 to show the following.

(a) A steady state can be achieved in which the distribution develops a plateau in the velocity interval $[v_1, v_2]$ (Fig. 34). Find F^{plateau} in terms of v_1, v_2 and the initial distribution $F(0, v)$. What is the energy of the waves in this steady state? What is the lower bound on initial electric energy $\mathcal{E}(0)$ below which the perturbations would just decay without forming a fully-fledged plateau?

(b) Derive the evolution equation for the thermal (nonresonant) bulk of the distribution and show that it cools during the QL evolution, with the total thermal energy declining by the same amount as the electric energy of the waves:

$$K_{\text{th}}(t) - K_{\text{th}}(0) = - \left[\mathcal{E}(0) - \mathcal{E}(t) \right]. \tag{10.81}$$

Identify where all the energy lost by the thermal particles and the waves goes and thus
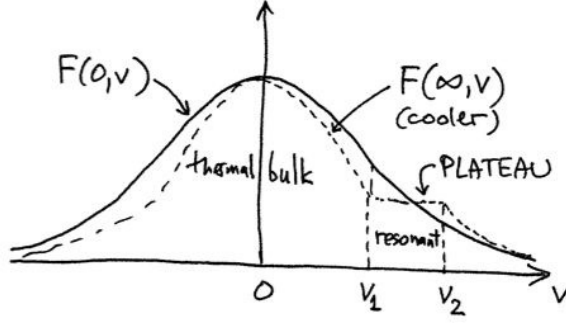
FIGURE 34. Initial stable equilibrium distribution and the final outcome of the QL evolution of a system with Landau-damped electric perturbations.

confirm that the total energy in the system is conserved. Why, physically, do thermal particles lose energy?

(c) Show that we must have

$$\frac{\mathcal{E}(0)}{n_e T_e} \gg \frac{\gamma_k}{\omega_{\mathrm{pe}}} \frac{\delta v}{v} \tag{10.82}$$

in order for the wave energy to change only by a small fraction before saturating and

$$\frac{\mathcal{E}(0)}{n_e T_e} \gg \frac{\gamma_k}{\omega_{\mathrm{pe}}} \left(\frac{\delta v}{v}\right)^3 \frac{1}{(k\lambda_{\mathrm{De}})^2} \tag{10.83}$$

in order for the QL evolution to be faster than the damping. Here $\delta v = v_2 - v_1$ and $v \sim v_1 \sim v_2$.

This question requires some nuance in handling the calculation of the QL diffusion coefficient. In §7.1, we used the expression (7.6) for δf_k in which only the eigenmode-like part was retained, while the phase-mixing terms were dropped on the grounds that we could always just wait long enough for them to be eclipsed by the term containing an exponentially growing factor $e^{\gamma_k t}$. When we are dealing with *damped* perturbations, there is no point in waiting because the exponential term is getting smaller, while the phase-mixing terms do not decay (except by collisions, see §§5.3 and 5.5, but we are not prepared to wait for that).

Let us, therefore, bite the bullet and use the full expression (5.25) for the perturbed distribution function, where we single out the slowest-damped mode and assume that all others, if any, will be damped fast enough never to produce significant QL effects:

$$\delta f_k = \frac{q}{m} \varphi_k \frac{1 - e^{-i(\mathbf{k}\cdot\mathbf{v} - \omega_k)t - \gamma_k t}}{\mathbf{k}\cdot\mathbf{v} - \omega_k - i\gamma_k} \mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{v}} + e^{-i\mathbf{k}\cdot\mathbf{v}t}(g_k + \dots), \tag{10.84}$$

where "$\dots$" stand for any possible undamped, phase-mixing remnants of other modes. When the solution (10.84) is substituted into (7.4), where it is multiplied by φ_k^* and time averaged [according to (2.7)], the second term vanishes because, for resonant particles ($\mathbf{k}\cdot\mathbf{v} \approx \omega_k$), it contains no resonant denominators and so is smaller than the first term, whereas for the nonresonant particles, it is removed by time averaging (check that this works at least for $|\gamma_k|t \lesssim 1$ and indeed beyond that). Keeping only the first term in the expression (10.84), substituting it into (7.4) and going through a calculation analogous to that given in (7.8), we find that the diffusion matrix is (check this)

$$\mathsf{D}(\mathbf{v}) = \frac{q^2}{m^2} \sum_k \frac{\mathbf{k}\mathbf{k}}{k^2} |\mathbf{E}_k|^2 \mathrm{Im} \left\langle \frac{1 - e^{-i(\mathbf{k}\cdot\mathbf{v} - \omega_k)t - \gamma_k t}}{\mathbf{k}\cdot\mathbf{v} - \omega_k - i\gamma_k} \right\rangle, \tag{10.85}$$

which is a generalisation of the penultimate line of (7.8). For nonresonant particles, the phase-mixing term is eliminated by time averaging and we end up with the old result: the last line

of (7.8). For resonant particles, assuming $|\gamma_{\boldsymbol{k}}| \ll |\boldsymbol{k} \cdot \boldsymbol{v} - \omega_{\boldsymbol{k}}| \ll \omega_{\boldsymbol{k}} \sim \boldsymbol{k} \cdot \boldsymbol{v}$ and $|\gamma_{\boldsymbol{k}}| t \ll 1$, we may adopt the approximation (5.39), which we have previously used to analyse the structure of δf. This gives us

$$\mathsf{D}(\boldsymbol{v}) = \frac{q^2}{m^2} \sum_{\boldsymbol{k}} \frac{\boldsymbol{k}\boldsymbol{k}}{k^2} |\boldsymbol{E}_{\boldsymbol{k}}|^2 \pi \delta(\boldsymbol{k} \cdot \boldsymbol{v} - \omega_{\boldsymbol{k}}), \tag{10.86}$$

which is the same result as (7.16)—including, importantly, the sign, which we would have gotten wrong had we just mechanically applied Plemelj's formula to (7.12) with $\gamma_{\boldsymbol{k}} < 0$. This is equivalent to saying that the k integral in (7.16) should be taken along the Landau contour, rather than simply along the real line.

Note that the above construction was done assuming $|\gamma_{\boldsymbol{k}}| t \ll 1$, i.e., all the QL action has to occur before the initial perturbations decay away (which is reasonable). Note also that there is nothing above that would not apply to the case of unstable perturbations ($\gamma_{\boldsymbol{k}} > 0$) and so we conclude that results of §7, derived formally for $\gamma_{\boldsymbol{k}} t \gg 1$, in fact also hold on shorter time scales ($\gamma_{\boldsymbol{k}} t \ll 1$, but, obviously, still $\omega_{\boldsymbol{k}} t \gg 1$).

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9. QL theory of Weibel instability. (a) Starting from the Vlasov equations including magnetic perturbations, show that the slow evolution of the equilibrium distribution function is described by the following diffusion equation:

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}} \cdot \mathsf{D}(\boldsymbol{v}) \cdot \frac{\partial f_0}{\partial \boldsymbol{v}}, \tag{10.87}$$

where the QL diffusion matrix is

$$\mathsf{D}(\boldsymbol{v}) = \frac{q^2}{m^2} \sum_{\boldsymbol{k}} \frac{1}{i(\boldsymbol{k} \cdot \boldsymbol{v} - \omega_{\boldsymbol{k}}) + \gamma_{\boldsymbol{k}}} \left(\boldsymbol{E}_{\boldsymbol{k}}^* + \frac{\boldsymbol{v} \times \boldsymbol{B}_{\boldsymbol{k}}^*}{c} \right) \left(\boldsymbol{E}_{\boldsymbol{k}} + \frac{\boldsymbol{v} \times \boldsymbol{B}_{\boldsymbol{k}}}{c} \right) \tag{10.88}$$

and $\omega_{\boldsymbol{k}}$ and $\gamma_{\boldsymbol{k}}$ are the frequency and the growth rate, respectively, of the fastest-growing mode.

(b) Consider the example of the low-frequency electron Weibel instability with wave numbers $\boldsymbol{k}$ parallel to the anisotropy direction [see (10.45)]. Take $\boldsymbol{k} = k\hat{\boldsymbol{z}}$ and $\boldsymbol{B}_{\boldsymbol{k}} = B_{\boldsymbol{k}}\hat{\boldsymbol{y}}$ and, denoting $\Omega_{\boldsymbol{k}} = eB_{\boldsymbol{k}}/m_e c$ (the Larmor frequency associated with the perturbed magnetic field), show that (10.87) becomes

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v_x} \left(D_{xx} \frac{\partial f_0}{\partial v_x} + D_{xz} \frac{\partial f_0}{\partial v_z} \right) + \frac{\partial}{\partial v_z} D_{zz} \frac{\partial f_0}{\partial v_z}, \tag{10.89}$$

where the coefficients of the QL diffusion tensor are

$$D_{xx} = \sum_{k} \frac{\gamma_k}{k^2} |\Omega_k|^2, \quad D_{xz} = -\sum_{k} \frac{2\gamma_k v_x v_z}{k^2 v_z^2 + \gamma_k^2} |\Omega_k|^2, \quad D_{zz} = \sum_{k} \frac{\gamma_k v_x^2}{k^2 v_z^2 + \gamma_k^2} |\Omega_k|^2. \tag{10.90}$$

(c) Suppose the electron distribution function f_0 is initially the bi-Maxwellian (10.43) with $0 < T_\perp/T_\parallel - 1 \ll 1$ (as should be the case for this instability to work). As QL evolution starts, we may define the temperatures of the evolving distribution according to

$$T_\perp = \frac{1}{n} \int \mathrm{d}^3\boldsymbol{v} \, \frac{m(v_x^2 + v_y^2)}{2} f_0, \quad T_\parallel = \frac{1}{n} \int \mathrm{d}^3\boldsymbol{v} \, m v_z^2 f_0. \tag{10.91}$$

Show that initially, viz., before f_0 has time to change shape significantly so as no longer to be representable as a bi-Maxwellian, the two temperatures will evolve approximately

(using $\gamma_k \ll k v_{\text{th}}$) according to

$$\boxed{\frac{\partial T_\perp}{\partial t} = -\lambda T_\perp, \quad \frac{\partial T_\parallel}{\partial t} = 2\lambda T_\perp}, \quad \text{where} \quad \lambda(T_\perp, T_\parallel) = \sum_k \frac{2\gamma_k |\Omega_k|^2}{k^2 v_{\text{th}\parallel}^2}. \tag{10.92}$$

Thus, QL evolution will lead, at least initially, to the reduction of the temperature anisotropy, thus weaking the instability (these equations should not be used to trace $T_\perp/T_\parallel - 1$ all the way to zero because there is no reason why the QL evolution should preserve the bi-Maxwellian shape of f_0).

Note that, even modulo the caveat about the bi-Maxwellian not being a long-term solution, this does not give us a way to estimate (or even guess) what the saturated fluctuation level will be. The standard Weibel lore is that saturation occurs when the approximations that were used to derive the linear theory (Q3) break down, namely, when magnetic field becomes strong enough to magnetise the plasma, rendering the Larmor scale $\rho_e = v_{\text{the}}/\Omega_k$ associated with the fluctuations small enough to be comparable to the latter's wavelengths $\sim k^{-1}$. Using the typical values of k from (10.45), we can write this condition as follows

$$\Omega_k \sim k v_{\text{the}} \sim \sqrt{\Delta_e} \frac{v_{\text{the}}}{d_e} \quad \Leftrightarrow \quad \boxed{\frac{1}{\beta_e} \equiv \frac{B^2/8\pi}{n_e T_e} \sim \Delta_e}. \tag{10.93}$$

Thus, Weibel instability will produce fluctuations the ratio of whose magnetic-energy density to the electron-thermal-energy density (customarily referred to as the inverse of "plasma beta," $1/\beta_e$) is comparable to the electron pressure anisotropy Δ_e. Because at that point the fluctuations will be relaxing this pressure anisotropy at the same rate as they can grow in the first place [in (10.92), $\lambda \sim \gamma_k$], the QL approach is not valid anymore.

These considerations are, however, usually assumed to be qualitatively sound and lead people to believe that, even in collisionless plasmas, the anisotropy of the electron distribution must be largely self-regulating, with unstable Weibel fluctuations engendered by the anisotropy quickly acting to isotropise the plasma (or at least the electrons).

This is all currently very topical in the part of the plasma-astrophysics world preoccupied with collisionless shocks, origin of the cosmic magnetism, hot weakly collisional environments such as the intergalactic medium (in galaxy clusters) or accretion flows around black holes and many other interesting subjects.

(d) Equations (10.92) say that the total mean kinetic energy,

$$\int \text{d}^3\boldsymbol{v} \, \frac{mv^2}{2} f_0 = n\left(T_\perp + \frac{T_\parallel}{2}\right), \tag{10.94}$$

does not change. But fluctuations are generated and grow at the rate γ_k! Without much further algebra, can you tell whether you should therefore doubt the result (10.92)?

10. QL theory of stochastic acceleration.[67] Consider a population of particles of charge q and mass m. Assume that collisions are entirely negligible. Assume further that an electrostatic fluctuation field $\boldsymbol{E} = -\boldsymbol{\nabla}\varphi$ (with zero spatial mean) is present and that this field is given and externally determined, i.e., it is unaffected by the particles that are under consideration. This might happen physically if, for example, the particles are a low-density admixture in a plasma consisting of some more numerous species of ions and electrons, which dominate the plasma's dielectric response.

As usual, we assume that the distribution function can be represented as $f = f_0(t, \boldsymbol{v}) + \delta f(t, \boldsymbol{r}, \boldsymbol{v})$, where f_0 is spatially homogeneous and changes slowly in time compared to the perturbed distribution $\delta f \ll f_0$. Its evolution is described by (2.11), where angle brackets again denote the time average over the fast variation of the fluctuation field.

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[67]Except for part (d), this is based on the 2018 exam question.

(a) Assume that φ is sufficiently small for it to be possible to determine δf from the linearised kinetic equation. Let $\delta f = 0$ at $t = 0$. Show that f_0 satisfies a QL diffusion equation with the diffusion matrix

$$\mathsf{D}(\boldsymbol{v}) = \frac{q^2}{m^2} \sum_{\boldsymbol{k}} \boldsymbol{k}\boldsymbol{k} \frac{1}{2\pi i} \int \mathrm{d}p \, \frac{1}{p + i\boldsymbol{k}\cdot\boldsymbol{v}} \int_{-\infty}^{t} \mathrm{d}\tau \, e^{p\tau} C_{\boldsymbol{k}}(\tau), \qquad (10.95)$$

where the p integration is along a contour appropriate for an inverse Laplace transform and $C_{\boldsymbol{k}}(t - t') = \langle \varphi_{\boldsymbol{k}}^*(t) \varphi_{\boldsymbol{k}}(t') \rangle$ is the correlation function of the fluctuation field (which is taken to be statistically stationary, so $C_{\boldsymbol{k}}$ depends only on the time difference $t - t'$).

(b) Let the correlation function have the form

$$C_{\boldsymbol{k}}(\tau) = A_{\boldsymbol{k}} e^{-\gamma_{\boldsymbol{k}}|\tau|}, \qquad (10.96)$$

i.e., $\gamma_{\boldsymbol{k}}^{-1}$ is the correlation time of the fluctuation field and $A_{\boldsymbol{k}}$ its spectrum; assume $\gamma_{-\boldsymbol{k}} = \gamma_{\boldsymbol{k}}$. Do the integrals in (10.95) and show that, at $t \gg \gamma_{\boldsymbol{k}}^{-1}$,

$$\mathsf{D}(\boldsymbol{v}) = \frac{q^2}{m^2} \sum_{\boldsymbol{k}} \boldsymbol{k}\boldsymbol{k} \frac{\gamma_{\boldsymbol{k}} A_{\boldsymbol{k}}}{\gamma_{\boldsymbol{k}}^2 + (\boldsymbol{k}\cdot\boldsymbol{v})^2}. \qquad (10.97)$$

(c) Restrict consideration to 1D in space and to the limit in which $\gamma_k \gg kv$ for typical wave numbers of the fluctuations and typical particle velocities (i.e., the fluctuation field is short-time correlated). Assuming that f_0 at $t = 0$ is a Maxwellian with temperature T_0, predict the evolution of f_0 with time. Discuss what physically is happening to the particles. Discuss the validity of the short-correlation-time approximation and of the assumption of slow evolution of f_0. What is, roughly, the condition on the amplitude and the correlation time of the fluctuation field that makes these assumptions compatible?

(d) When the distribution "heats up" sufficiently, the short-correlation-time approximation will be broken. Staying in 1D, consider the opposite limit, $\gamma_k \ll kv$. Show that the resulting QL equation admits a subdiffusive solution, with

$$f_0(t, v) \propto \frac{e^{-v^4/\alpha t}}{t^{1/4}}, \quad \alpha = 16 \frac{q^2}{m^2} \sum_{\boldsymbol{k}} \gamma_{\boldsymbol{k}} A_{\boldsymbol{k}}. \qquad (10.98)$$

In view of this result and of (c), discuss qualitatively how an initially "cold" particle distribution would evolve with time.

The original, classic paper on stochastic acceleration is Sturrock (1966). Note that the velocity dependence of the diffusion matrix (10.97) is determined by the functional form of $C_{\boldsymbol{k}}(\tau)$, so interesting τ dependences of the latter can lead to all kinds of interesting distributions f_0 of the accelerated particles.

IUCUNDI ACTI LABORES.

PART II
Magnetohydrodynamics

11. MHD Equations

Like hydrodynamics from gas kinetics, MHD can be derived systematically from the Vlasov–Maxwell–Landau equations for a plasma in the limit of large collisionality + a number of additional assumptions (see, e.g., Goedbloed & Poedts 2004; Parra 2019*a*). Here I will adopt a purely fluid approach—partly to make these lectures self-consistent and partly because there is a certain beauty in it: we need to know relatively little about the properties of the constituent substance in order to spin out a very sophisticated and complete theory about the way in which it flows. This approach is also more generally applicable because the substance that we will be dealing with need not be gaseous, like plasma—you may also think of liquid metals, various conducting solutions, etc.

So, let us declare an interest in the flow of a conducting fluid and attempt to be guided in our description of it by the very basic things: conservation laws of mass, momentum and energy plus Maxwell's equations for the electric and magnetic fields. This will prove sufficient for most of our purposes. So we consider a fluid characterised by the following quantities:

ρ—mass density,
$\boldsymbol{u}$—flow velocity,
p—pressure,
σ—charge density,
$\boldsymbol{j}$—current density,
$\boldsymbol{E}$—electric field,
$\boldsymbol{B}$—magnetic field.

Our immediate objective is to find a set of closed equations that would allow us to determine all of these quantities as functions of time and space within the fluid.

11.1. *Conservation of Mass*

This is the most standard of all arguments in fluid dynamics (Fig. 35):

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\int_V \mathrm{d}^3\boldsymbol{r}\, \rho}_{\substack{\text{mass inside a} \\ \text{volume of} \\ \text{fluid}}} = -\underbrace{\int_{\partial V} (\rho\boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{S}}_{\substack{\text{mass flux} \\ \text{in/out}}} = -\int_V \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{\nabla} \cdot (\rho\boldsymbol{u}). \tag{11.1}$$

As this equation holds for any V, however small, it can be converted into a differential relation

$$\boxed{\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\boldsymbol{u}) = 0}. \tag{11.2}$$

This is the *continuity equation.*
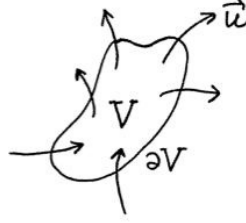
FIGURE 35. Stuff flowing in and out of a volume V enclosed by surface ∂V.

11.2. Conservation of Momentum

A similar approach:

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\int_V \mathrm{d}^3 \boldsymbol{r}\, \rho \boldsymbol{u}}_{\substack{\text{momentum} \\ \text{inside a} \\ \text{volume of} \\ \text{fluid}}} = - \int_{\partial V} \underbrace{(\rho \boldsymbol{u} \boldsymbol{u})}_{\substack{\text{Reynolds} \\ \text{stress}}} \cdot \mathrm{d}\boldsymbol{S} - \underbrace{\int_{\partial V} p\, \mathrm{d}\boldsymbol{S}}_{\substack{\text{pressure on} \\ \text{boundary} \\ \text{(momentum} \\ \text{flux due to} \\ \text{internal} \\ \text{motion)}}} - \underbrace{\int_{\partial V} \boldsymbol{\Pi} \cdot \mathrm{d}\boldsymbol{S}}_{\text{viscous stress}} + \underbrace{\int_V \mathrm{d}^3 \boldsymbol{r}\, \boldsymbol{F}}_{\substack{\text{all other} \\ \text{forces (E\&M} \\ \text{will come in} \\ \text{here)}}}$$

$$= \int_V \mathrm{d}^3 \boldsymbol{r}\, [-\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u} \boldsymbol{u}) - \boldsymbol{\nabla} p - \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} + \boldsymbol{F}]. \tag{11.3}$$

where the Reynolds stress underbrace reads "momentum flux through boundary (fluid flow carrying its own momentum)".

In differential form, this becomes

$$\underbrace{\frac{\partial}{\partial t} \rho \boldsymbol{u}}_{\substack{= \rho \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \frac{\partial \rho}{\partial t} \\ = \rho \frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{u} \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})}} = \underbrace{-\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u} \boldsymbol{u})}_{\substack{-\rho \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \\ -\boldsymbol{u} \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u})}} - \boldsymbol{\nabla} p - \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} + \boldsymbol{F}, \tag{11.4}$$

and so, finally,

$$\boxed{\rho \underbrace{\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \right)}_{\substack{\equiv \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} \\ \text{convective} \\ \text{derivative}}} = -\boldsymbol{\nabla} p - \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} + \boldsymbol{F}} . \tag{11.5}$$

This is the *momentum equation*.

One part of this equation does have to be calculated from some knowledge of the microscopic properties of the constituent fluid or gas—the viscous stress. For a gas, it is done in kinetic theory (e.g., Lifshitz & Pitaevskii 1981; Dellar 2015; Schekochihin 2019):

$$\boldsymbol{\Pi} = -\rho \nu \left[\boldsymbol{\nabla} \boldsymbol{u} + (\boldsymbol{\nabla} \boldsymbol{u})^T - \frac{2}{3} \boldsymbol{\nabla} \cdot \boldsymbol{u}\, \mathsf{I} \right], \tag{11.6}$$

where ν is the kinematic (Newtonian) viscosity. In what follows, we will never require the explicit form of $\boldsymbol{\Pi}$ (except perhaps in §11.10).

In a magnetised plasma (i.e., such that its collision frequency $\ll$ Larmor frequency of the gyrating charges), the viscous stress is much more complicated and anisotropic with respect to the direction of the magnetic field: because of their Larmor motion, charged particles diffuse

differently across and along the field. This gives rise to the so-called *Braginskii (1965) stress* (see, e.g., Helander & Sigmar 2005; Parra 2019*a*).

11.3. *Electromagnetic Fields and Forces*

The fact that the fluid is conducting means that it can have distributed charges (σ) and currents ($\boldsymbol{j}$) and so the electric ($\boldsymbol{E}$) and magnetic ($\boldsymbol{B}$) fields will exert body forces on the fluid. Indeed, for one particle of charge q, the Lorentz force is

$$\boldsymbol{f}_{\mathrm{L}} = q \left(\boldsymbol{E} + \frac{\boldsymbol{v} \times \boldsymbol{B}}{c} \right), \tag{11.7}$$

and if we sum this over all particles (or, to be precise, average over their distribution and sum over species), we will get

$$\boldsymbol{F} = \sigma \boldsymbol{E} + \frac{\boldsymbol{j} \times \boldsymbol{B}}{c}. \tag{11.8}$$

This body force (force density) goes into (11.5) and so we must know $\boldsymbol{E}$, $\boldsymbol{B}$, σ and $\boldsymbol{j}$ in order to compute the fluid flow $\boldsymbol{u}$.

Clearly it is a good idea to bring in Maxwell's equations:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\sigma \quad \text{(Gauss)}, \tag{11.9}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{11.10}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E} \quad \text{(Faraday)}, \tag{11.11}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c} \boldsymbol{j} + \frac{1}{c} \frac{\partial \boldsymbol{E}}{\partial t} \quad \text{(Ampère–Maxwell)}. \tag{11.12}$$

To these, we must append Ohm's law in its simplest form: The electric field in the frame of a fluid element moving with velocity $\boldsymbol{u}$ is

$$\boldsymbol{E}' = \boldsymbol{E} + \frac{\boldsymbol{u} \times \boldsymbol{B}}{c} = \eta\boldsymbol{j}, \tag{11.13}$$

where $\boldsymbol{E}$ is the electric field in the laboratory frame and η is the *Ohmic resistivity*.

Normally, the resistivity, like viscosity, has to be computed from kinetic theory (see, e.g., Helander & Sigmar 2005; Parra 2019*a*) or tabulated by assidiuous experimentalists. In a magnetised plasma, the simple form (11.13) of Ohm's law is only valid at spatial scales longer than the Larmor radii and time scales longer than the Larmor periods of the particles (see, e.g., Goedbloed & Poedts 2004; Parra 2019*b*).

Equations (11.9–11.13) can be reduced somewhat if we assume (quite reasonably for most applications) that our fluid flow is *non-relativistic*. Let us stipulate that all fields evolve on time scales $\sim \tau$, have spatial scales $\sim \ell$ and that the flow velocity is

$$u \sim \frac{\ell}{\tau} \ll c. \tag{11.14}$$

Then, from Ohm's law (11.13),

$$E \sim \frac{u}{c} B \ll B, \tag{11.15}$$

so *electric fields are small compared to magnetic fields.*

In Ampère–Maxwell's law (11.12),

$$\frac{\left|\dfrac{1}{c}\dfrac{\partial \boldsymbol{E}}{\partial t}\right|}{|\boldsymbol{\nabla}\times\boldsymbol{B}|} \sim \frac{\dfrac{1}{c}\dfrac{1}{\tau}\dfrac{u}{c}B}{\dfrac{1}{\ell}B} \sim \frac{u^2}{c^2} \ll 1, \tag{11.16}$$

so *the displacement current is negligible* (note that at this point we have ordered out light waves; see Q2 in Kinetic Theory). This allows us to revert to the pre-Maxwell form of Ampère's law:

$$\boxed{\boldsymbol{j} = \frac{c}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{B}}. \tag{11.17}$$

Thus, the current is no longer an independent field, there is a one-to-one correspondence $\boldsymbol{j} \leftrightarrow \boldsymbol{B}$.

Finally, comparing the electric and magnetic parts of the Lorentz force (11.8), and using Gauss's law (11.9) to estimate $\sigma \sim E/\ell$, we get

$$\frac{|\sigma\boldsymbol{E}|}{\left|\dfrac{1}{c}\boldsymbol{j}\times\boldsymbol{B}\right|} \sim \frac{\dfrac{1}{\ell}E^2}{\dfrac{1}{c}\dfrac{c}{\ell}B^2} \sim \frac{E^2}{B^2} \sim \frac{u^2}{c^2} \ll 1. \tag{11.18}$$

Thus, the MHD body force is

$$\boxed{\boldsymbol{F} = \frac{\boldsymbol{j}\times\boldsymbol{B}}{c} = \frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi}}. \tag{11.19}$$

This goes into (11.5) and we note with relief that σ, $\boldsymbol{j}$ and $\boldsymbol{E}$ have all fallen out of the momentum equation—we only need to know $\boldsymbol{B}$.

11.4. *Maxwell Stress and Magnetic Forces*

Let us take a break from formal derivations to consider what (11.19) teaches us about the sort of new dynamics that our fluid will experience as a result of being conducting. To see this, it is useful to play with the expression (11.19) in a few different ways.

By simple vector algebra,

$$\boldsymbol{F} = \underbrace{\frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B}}{4\pi}}_{\substack{\text{``magnetic}\\\text{tension''}}} - \underbrace{\boldsymbol{\nabla}\frac{B^2}{8\pi}}_{\substack{\text{``magnetic}\\\text{pressure''}}} = -\boldsymbol{\nabla}\cdot\underbrace{\left(\frac{B^2}{8\pi}\mathsf{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}\right)}_{\substack{\text{``Maxwell}\\\text{stress''}}}, \tag{11.20}$$

where the last expression was obtained with the aid of $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$. Thus, the action of the Lorentz force in a conducting fluid amounts to a new form of stress. Mathematically, this *"Maxwell stress"* is somewhat similar to the kind of stress that would arise from a suspension in the fluid of elongated molecules—e.g., polymer chains, or other kinds of "balls on springs" (see, e.g., Dellar 2017; the analogy can be made rigorous: see Ogilvie & Proctor 2003). Thus, we expect that the magnetic field threading the fluid will impart to it a degree of "elasticity."

Exactly what this means dynamically becomes obvious if we rewrite the magnetic tension and pressure forces in (11.20) in the following way. Let $\boldsymbol{b} = \boldsymbol{B}/B$ be the unit vector in the direction of $\boldsymbol{B}$ (the unit tangent to the field line). Then

$$\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B} = B\boldsymbol{b}\cdot\boldsymbol{\nabla}(B\boldsymbol{b}) = B^2\boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b} + \boldsymbol{b}\boldsymbol{b}\cdot\boldsymbol{\nabla}\frac{B^2}{2} \tag{11.21}$$
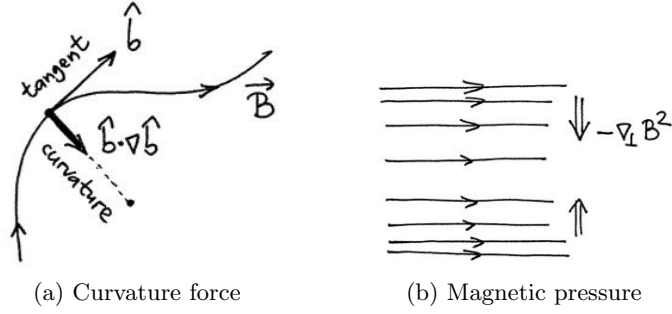
(a) Curvature force   (b) Magnetic pressure

FIGURE 36. Magnetic forces.

and, putting this back into (11.20), we get

$$\boldsymbol{F} = \underbrace{\frac{B^2}{4\pi}\boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b}}_{\substack{\text{"curvature}\\\text{force"}}} - \underbrace{(\mathbf{I}-\boldsymbol{b}\boldsymbol{b})\cdot\boldsymbol{\nabla}}_{\equiv\,\boldsymbol{\nabla}_\perp}\underbrace{\frac{B^2}{8\pi}}_{\substack{\text{magnetic}\\\text{pressure force}}}. \qquad (11.22)$$

Thus, we learn that the Lorentz force consists of two distinct parts (Fig. 36):

• *curvature force*, so called because $\boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{b}$ is the vector curvature of the magnetic field line—the implication being that field lines, if bent, will want to straighten up;
• *magnetic pressure*, whose presence implies that field lines will resist compression or rarefication (the field wants to be uniform in strength).

Note that both forces act perpendicularly to $\boldsymbol{B}$, as they must, since magnetic field never exerts a force along itself on a charged particle [see (11.7)].

So this is the effect of the field on the fluid. What is the effect of the fluid on the field?

11.5. *Evolution of Magnetic Field*

Returning to deriving MHD equations, we use Ohm's law (11.13) to express $\boldsymbol{E}$ in terms of $\boldsymbol{u}$, $\boldsymbol{B}$ and $\boldsymbol{j}$ in the right-hand side of Faraday's law (11.11). We then use Ampere's law (11.17) to express $\boldsymbol{j}$ in terms of $\boldsymbol{B}$. The result is

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times\left(\boldsymbol{u}\times\boldsymbol{B} - \frac{c^2\eta}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{B}\right). \qquad (11.23)$$

After using also $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$ to get $\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B})=-\nabla^2\boldsymbol{B}$ and renaming $c^2\eta/4\pi\to\eta$, the *magnetic diffusivity*, we arrive at the *magnetic induction equation* (due to Hertz):

$$\boxed{\frac{\partial \boldsymbol{B}}{\partial t} = \underbrace{\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})}_{\text{advection}} + \underbrace{\eta\nabla^2\boldsymbol{B}}_{\text{diffusion}}}. \qquad (11.24)$$

Note that if $\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$ is satisfied initially, any solution of (11.24) will remain divergence-free at all times.

11.6. *Magnetic Reynolds Number*

The relative importance of the diffusion term (it is obvious what this does) and the advection term (to be discussed in the next few sections) in (11.24) is measured by a

dimensionless number:

$$\frac{|\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B})|}{|\eta \nabla^2 \boldsymbol{B}|} \sim \frac{\dfrac{u}{\ell} B}{\dfrac{\eta}{\ell^2} B} = \frac{u\ell}{\eta} \equiv \mathrm{Rm}, \tag{11.25}$$

called the *magnetic Reynolds number*. In nature, it can take a very broad range of values:

liquid metals in idustrial contexts (metallurgy): $\mathrm{Rm} \sim 10^{-3} \dots 10^{-1}$,
planet interiors: $\mathrm{Rm} \sim 100 \dots 300$,
solar convective zone: $\mathrm{Rm} \sim 10^6 \dots 10^9$,
interstellar medium ("warm" phase): $\mathrm{Rm} \sim 10^{18}$,
intergalactic medium (cores of galaxy clusters): $\mathrm{Rm} \sim 10^{29}$,
laboratory "dynamo" (§11.10) experiments: $\mathrm{Rm} \sim 1 \dots 10^2$ (and growing).

Generally speaking, when flow velocities are large/distances are large/resistivities are low, $\mathrm{Rm} \gg 1$ and it makes sense to consider *"ideal MHD,"* i.e., the limit $\eta \to 0$. In fact, η often needs to be brought back in to deal with instances of large $\boldsymbol{\nabla}\boldsymbol{B}$, which arise naturally from solutions of ideal MHD equations (see §11.10, §15.2 and Parra 2019a), but let us consider the ideal case for now to understand what the advective part of the induction equation does to $\boldsymbol{B}$.

11.7. Lundquist Theorem

The ideal ($\eta = 0$) version of the induction equation (11.24),

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}), \tag{11.26}$$

implies that *fluid elements that lie on a field line initially will remain on this field line*, i.e., "the magnetic field moves with the flow."

Proof. Unpacking the double vector product in (11.26),

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{u} \cdot \boldsymbol{\nabla} B + \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{B}\boldsymbol{\nabla} \cdot \boldsymbol{u} + \boldsymbol{u}\cancel{\boldsymbol{\nabla} \cdot \boldsymbol{B}}, \tag{11.27}$$

or, using the notation for the "convective derivative" [see (11.5)],

$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} \equiv \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \right) \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{B}\boldsymbol{\nabla} \cdot \boldsymbol{u}. \tag{11.28}$$

The continuity equation (11.2) can be rewritten in a somewhat similar-looking form

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} \equiv \left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \right) \rho = -\rho\boldsymbol{\nabla} \cdot \boldsymbol{u} \quad \Rightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{u} = -\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}. \tag{11.29}$$

The last expression is now used for $\boldsymbol{\nabla} \cdot \boldsymbol{u}$ in (11.28):

$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \frac{\boldsymbol{B}}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}. \tag{11.30}$$

Multiplying this equation by $1/\rho$ and noting that

$$\frac{1}{\rho}\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} - \frac{\boldsymbol{B}}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\boldsymbol{B}}{\rho}, \tag{11.31}$$

we arrive at

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\boldsymbol{B}}{\rho} = \frac{\boldsymbol{B}}{\rho} \cdot \boldsymbol{\nabla}\boldsymbol{u}}. \tag{11.32}$$
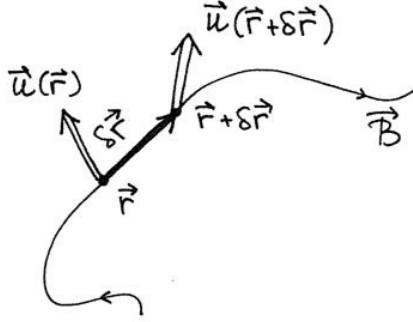
FIGURE 37. Pressure-balanced perturbations.

Let us compare the evolution of the vector $\boldsymbol{B}/\rho$ with the evolution of an infinitesimal Lagrangian separation vector in a moving fluid: the convective derivative is the Lagrangian time derivative, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\boldsymbol{r}(t) = \boldsymbol{u}(\boldsymbol{r}+\delta\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{r}) \approx \delta\boldsymbol{r}\cdot\boldsymbol{\nabla}\boldsymbol{u}. \tag{11.33}$$

Thus, $\delta\boldsymbol{r}$ and $\boldsymbol{B}/\rho$ satisfy the same equation. This means that if two fluid elements are initially on the same field line,

$$\delta\boldsymbol{r} = \mathrm{const}\,\frac{\boldsymbol{B}}{\rho}, \tag{11.34}$$

then they will stay on the same field line, q.e.d.

This means that in MHD, the fluid flow will be entraining the magnetic-field lines with it—and, as we saw in §11.4, the field lines will react back on the fluid:
—when the fluid tries to bend the field, the field will want to spring back,
—when the fluid tries to compress or rarefy the field, the field will resist as if it possessed (perpendicular) pressure.

This is the sense in which MHD fluid is "elastic": it is threaded by magnetic-field lines, which move with it and act as elastic bands.

11.8. *Flux Freezing*

There is an essentially equivalent formulation of the result of §11.7 that highlights the fact that the ideal induction equation (11.26) is a *conservation law*—conservation of *magnetic flux*.

The magnetic flux through a surface S (Fig. 38a) is, by definition,

$$\Phi = \int_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} \tag{11.35}$$

($\mathrm{d}\boldsymbol{S} \equiv \hat{\boldsymbol{n}}\,\mathrm{d}S$, where $\hat{\boldsymbol{n}}$ is a unit normal pointing out of the surface). *The flux Φ depends on the loop ∂S, but not on the choice of the surface spanning it.* Indeed, if we consider two surfaces, S_1 and S_2, spanning the same loop ∂S (Fig. 38b) and define $\Phi_{1,2} = \int_{S_{1,2}}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}$, then the flux out of the volume V enclosed by $S_1 \cup S_2 = \partial V$ is

$$\Phi_2 - \Phi_1 = \int_{\partial V} \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = \int_V \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \quad \text{q.e.d.} \tag{11.36}$$

Alfvén's Theorem. *Flux through any loop moving with the fluid is conserved.*
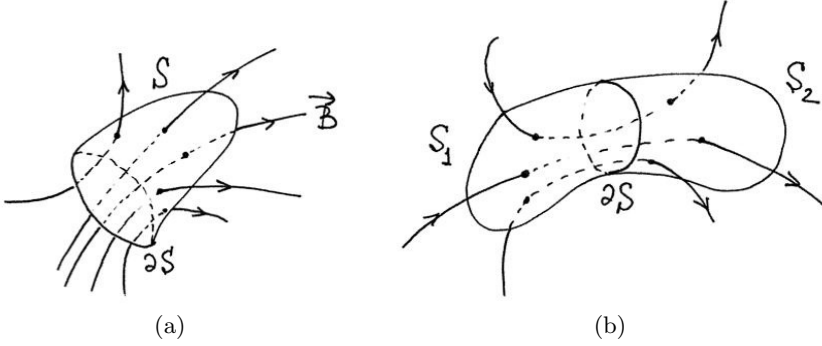
FIGURE 38. Magnetic flux through a loop ∂S (a) is independent of the surface spanning the loop (b).
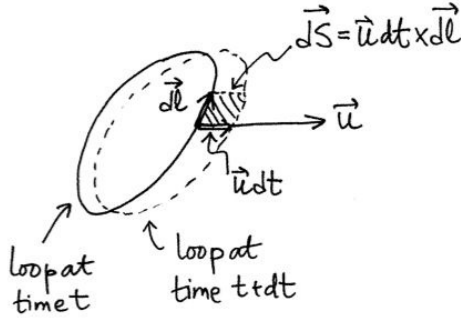
FIGURE 39. A loop moving with the fluid.

Proof. Let $S(t)$ be a surface spanning the loop at time t. If the loop moves with the fluid (Fig. 39), at the slightly later time $t + \mathrm{d}t$ it is spanned (for example) by the surface

$$S(t + \mathrm{d}t) = S(t) \; \cup \; \begin{array}{l} \text{ribbon traced by the loop} \\ \text{as it moves over time } \mathrm{d}t. \end{array} \tag{11.37}$$

Then the flux at time t is

$$\Phi(t) = \int_{S(t)} \boldsymbol{B}(t) \cdot \mathrm{d}\boldsymbol{S} \tag{11.38}$$

and at the later time,

$$\Phi(t + \mathrm{d}t) = \int_{S(t+\mathrm{d}t)} \boldsymbol{B}(t + \mathrm{d}t) \cdot \mathrm{d}\boldsymbol{S}$$

$$= \underbrace{\int_{S(t)} \boldsymbol{B}(t + \mathrm{d}t) \cdot \mathrm{d}\boldsymbol{S}}_{} + \underbrace{\int_{\text{ribbon}} \boldsymbol{B}(t + \mathrm{d}t) \cdot \underbrace{\mathrm{d}\boldsymbol{S}}_{= \boldsymbol{u}\mathrm{d}t \times \mathrm{d}\boldsymbol{l}}}_{}$$

$$\underbrace{= \int_{S(t)} \boldsymbol{B}(t) \cdot \mathrm{d}\boldsymbol{S} + \mathrm{d}t \int_{S(t)} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}}_{= \Phi(t)} \qquad = \mathrm{d}t \int_{\partial S(t)} \boldsymbol{B}(t) \cdot (\boldsymbol{u} \times \mathrm{d}\boldsymbol{l})$$

$$= -\mathrm{d}t \int_{\partial S(t)} (\boldsymbol{u} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{l}$$

$$= -\mathrm{d}t \int_{S(t)} [\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B})] \cdot \mathrm{d}\boldsymbol{S}. \tag{11.39}$$
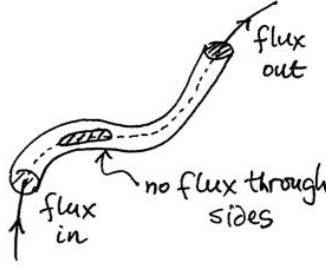
FIGURE 40. Flux tube.

Therefore,

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\Phi(t + \mathrm{d}t) - \Phi(t)}{\mathrm{d}t} = \int_{S(t)} \underbrace{\left[\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B})\right]}_{\text{ideal induction equation (11.26)}} \cdot \mathrm{d}\boldsymbol{S} = 0, \quad \text{q.e.d.} \qquad (11.40)$$

This result means that *field lines are frozen into the flow.* Indeed, consider a flux tube enclosing a field line (Fig. 40). As the tube deforms, the field line stays inside it because fluxes through the ends and sides of the tube cannot change.

Note that *Ohmic diffusion breaks flux freezing*, as is obvious from (11.40) if in the integrand one uses the induction equation (11.24) keeping the resistive term.

11.9. *Amplification of Magnetic Field by Fluid Flow*

An interesting physical consequence of these results is that *flows of conducting fluid can amplify magnetic fields.* For example, consider a flow that stretches an initial cylindrical tube of length l_1 and cross section S_1 into a long thin spaghetto of length l_2 and cross section S_2 (Fig. 41). By conservation of flux,

$$B_1 S_1 = B_2 S_2. \qquad (11.41)$$

By conservation of mass,

$$\rho_1 l_1 S_1 = \rho_2 l_2 S_2. \qquad (11.42)$$

Therefore,

$$\frac{B_2}{\rho_2 l_2} = \frac{B_1}{\rho_1 l_1} \quad \Rightarrow \quad \frac{B_2}{B_1} = \frac{\rho_2 l_2}{\rho_1 l_1}. \qquad (11.43)$$

In an incompressible fluid, $\rho_2 = \rho_1$, and the field is amplified by a factor l_2/l_1. In a compressible fluid, the field can also be amplified by compression.

Going back to the induction equation in the form (11.27),

$$\frac{\partial \boldsymbol{B}}{\partial t} + \underbrace{\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{B}}_{\text{advection}} = \underbrace{\boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{u}}_{\text{stretching}} - \underbrace{\boldsymbol{B} \boldsymbol{\nabla} \cdot \boldsymbol{u}}_{\text{compression}}, \qquad (11.44)$$

the three terms in it are responsible for, in order, advection of the field by the flow (i.e., the flow carrying the field around with it), "stretching" (amplification) of the field by velocity gradients that make fluid elements longer and, finally, compression or rarefication of the field by convergent or divergent flows (unless $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, as it is in an incompressible fluid).

Hence arises the famous problem of *MHD dynamo:* are there fluid flows that lead to
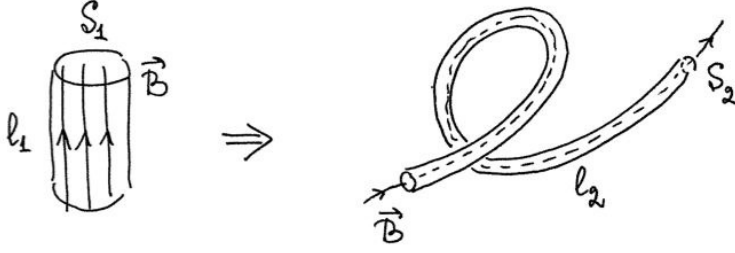
FIGURE 41. Amplification of magnetic field by stretching.

sustained amplification of the magnetic fields? The answer is yes—but the flow must be 3D (the absence of dynamo action in 2D is a theorem, the simplest version of which is due to Zeldovich 1956; see Q4). Magnetic fields of planets, stars, galaxies, etc. are all believed to owe their origin and persistence to this effect. This topic requires (and merits) a more detailed treatment (§11.10), but for now let us flag two important aspects:

• resistivity, however small, turns out to be impossible to neglect because large gradients of $\boldsymbol{B}$ appear as the field is advected by the flow;

• the amplification of the field is checked by the Lorentz force once the field is strong enough that it can act back on the flow, viz., when their energy densities become comparable:

$$\frac{B^2}{8\pi} \sim \frac{\rho u^2}{2}. \tag{11.45}$$

11.10. *MHD Dynamo*

I may fill this in at some point. You will find a (somewhat outdated) preview in my handwritten lecture notes available here: `http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/notes/leshouches07.pdf`. A very short printed review is Schekochihin & Cowley (2007, §3).

A classic (and mostly timeless) text is Moffatt (1978). The new classic is the review by Rincon (2019)—you probably don't need to read anything else.

11.11. *Conservation of Energy*

Let us summarise the equations that we have derived so far, namely (11.2), (11.5) and (11.24), expressing conservation of

$$\text{mass} \quad \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{11.46}$$

$$\text{momentum} \quad \rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \right) = -\boldsymbol{\nabla} p - \boldsymbol{\nabla} \cdot \boldsymbol{\Pi} + \frac{(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi}$$

$$= -\boldsymbol{\nabla} \cdot \underbrace{\left[\left(p + \frac{B^2}{8\pi} \right) \mathbf{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi} + \boldsymbol{\Pi} \right]}_{\text{total stress}}, \tag{11.47}$$

$$\text{and flux} \quad \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}. \tag{11.48}$$

To complete the system, we need an equation for p, which has to come from the one conservation law that we have not yet utilised: *conservation of energy.*

The total *energy density* is

$$\varepsilon = \underbrace{\frac{\rho u^2}{2}}_{\text{kinetic}} + \underbrace{\frac{p}{\gamma - 1}}_{\text{internal}} + \underbrace{\frac{\cancel{E^2}}{8\pi}}_{\text{electric}} + \underbrace{\frac{B^2}{8\pi}}_{\text{magnetic}} , \qquad (11.49)$$

where the electric energy can (and, for consistency with §11.3, must) be neglected because $E^2/B^2 \sim u^2/c^2 \ll 1$. We follow the same logic as we did in §§11.1 and 11.2:

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\int_V \mathrm{d}^3\boldsymbol{r}\, \varepsilon}_{\substack{\text{energy inside} \\ \text{a volume of} \\ \text{fluid}}} = \underbrace{-\int_{\partial V} \left(\frac{\rho u^2}{2} + \frac{p}{\gamma - 1}\right) \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S}}_{\substack{\text{fluid flow carrying its} \\ \text{kinetic and internal energy} \\ \text{through boundary}}} - \underbrace{\int_{\partial V} [(p\,\mathbf{I} + \boldsymbol{\Pi}) \cdot \boldsymbol{u}] \cdot \mathrm{d}\boldsymbol{S}}_{\substack{\text{work done on boundary} \\ \text{by pressure and viscous} \\ \text{stress}}}$$

$$- \underbrace{\int_{\partial V} \boldsymbol{q} \cdot \mathrm{d}\boldsymbol{S}}_{\text{heat flux}} - \underbrace{\int_{\partial V} \frac{c}{4\pi}(\boldsymbol{E} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{S}}_{\text{Poynting flux}} . \qquad (11.50)$$

Like the viscous stress $\boldsymbol{\Pi}$, the heat flux $\boldsymbol{q}$ must be calculated kinetically (in a plasma) or tabulated (in an arbitrary complicated substance). In a gas, $\boldsymbol{q} = -\kappa \boldsymbol{\nabla} T$, but it is more complicated in a magnetised plasma (see, e.g., Braginskii 1965; Helander & Sigmar 2005; Parra 2019*a*).

Note that the magnetic energy and the work done by the Lorentz force are not included in the first two terms on the right-hand side of (11.50) because all of that must already be correctly acounted for by the Poynting flux. Indeed, since $c\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} + \eta \boldsymbol{\nabla} \times \boldsymbol{B}$ [this is (11.13), with η renamed as in (11.24)], we have

$$\int_{\partial V} \frac{c}{4\pi}(\boldsymbol{E} \times \boldsymbol{B}) \cdot \mathrm{d}\boldsymbol{S} = \underbrace{\int_{\partial V} \frac{B^2}{8\pi} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{S}}_{\substack{\text{magnetic energy} \\ \text{flow}}} + \underbrace{\int_{\partial V} \left[\left(\frac{B^2}{8\pi}\mathbf{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi}\right) \cdot \boldsymbol{u}\right] \cdot \mathrm{d}\boldsymbol{S}}_{\text{work done by Maxwell stress}}$$

$$+ \underbrace{\int_{\partial V} \eta \frac{(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi} \cdot \mathrm{d}\boldsymbol{S}}_{\substack{\text{resistive slippage accounting} \\ \text{for field not being precisely} \\ \text{frozen into flow}}} . \qquad (11.51)$$

After application of Gauss's theorem and shrinking of the volume V to infinitesimality, we get the differential form of (11.50):

$$\frac{\partial}{\partial t}\left(\frac{\rho u^2}{2} + \frac{p}{\gamma - 1} + \frac{B^2}{8\pi}\right) = -\boldsymbol{\nabla} \cdot \left[\frac{\rho u^2}{2} \boldsymbol{u} + \frac{\gamma}{\gamma - 1} p\boldsymbol{u} + \boldsymbol{\Pi} \cdot \boldsymbol{u} + \boldsymbol{q}\right.$$

$$\left. + \frac{B^2\mathbf{I} - \boldsymbol{B}\boldsymbol{B}}{4\pi} \cdot \boldsymbol{u} + \eta \frac{(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi}\right]. \qquad (11.52)$$

It remains to separate the evolution equation for p by using the fact that we know the equations for ρ, $\boldsymbol{u}$ and $\boldsymbol{B}$ and so can deduce the rates of change of the kinetic and magnetic energies.

11.11.1. *Kinetic Energy*

Using (11.46) and (11.47),

$$
\frac{\partial}{\partial t}\frac{\rho u^2}{2} = \frac{u^2}{2}\frac{\partial \rho}{\partial t} + \rho \boldsymbol{u} \cdot \frac{\partial \boldsymbol{u}}{\partial t}
$$

$$
= -\frac{u^2}{2}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) - \rho\boldsymbol{u}\cdot\boldsymbol{\nabla}\frac{u^2}{2} - \boldsymbol{u}\cdot\left\{\boldsymbol{\nabla}\cdot\left[\left(p+\frac{B^2}{8\pi}\right)\mathsf{I} - \frac{\boldsymbol{BB}}{4\pi} + \boldsymbol{\Pi}\right]\right\}
$$

$$
= -\boldsymbol{\nabla}\cdot\left[\frac{\rho u^2}{2}\boldsymbol{u} + p\boldsymbol{u} + \left(\frac{B^2}{8\pi}\mathsf{I} - \frac{\boldsymbol{BB}}{4\pi}\right)\cdot\boldsymbol{u} + \boldsymbol{\Pi}\cdot\boldsymbol{u}\right]
$$

$$
+ \underbrace{p\boldsymbol{\nabla}\cdot\boldsymbol{u}}_{\substack{\text{compressional}\\\text{work}}} + \underbrace{\left(\frac{B^2}{8\pi}\mathsf{I} - \frac{\boldsymbol{BB}}{4\pi}\right):\boldsymbol{\nabla}\boldsymbol{u}}_{\substack{\text{energy exchange}\\\text{with magnetic field}}} + \underbrace{\boldsymbol{\Pi}:\boldsymbol{\nabla}\boldsymbol{u}}_{\substack{\text{viscous}\\\text{dissipation}}}. \tag{11.53}
$$

The flux terms (energy flows and work by stresses on boundaries) that have been crossed out cancel with corresponding terms in (11.52) once (11.53) is subtracted from it.

11.11.2. *Magnetic Energy*

Using the induction equation (11.48),

$$
\frac{\partial}{\partial t}\frac{B^2}{8\pi} = \frac{\boldsymbol{B}}{4\pi}\cdot\left[-\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{B} + \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \eta\nabla^2\boldsymbol{B}\right]
$$

$$
= -\boldsymbol{\nabla}\cdot\left[\frac{B^2}{8\pi}\boldsymbol{u} + \eta\frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi}\right] - \underbrace{\left(\frac{B^2}{8\pi}\mathsf{I} - \frac{\boldsymbol{BB}}{4\pi}\right):\boldsymbol{\nabla}\boldsymbol{u}}_{\substack{\text{energy exchange}\\\text{with velocity field}}} - \underbrace{\eta\frac{|\boldsymbol{\nabla}\times\boldsymbol{B}|^2}{4\pi}}_{\substack{\text{Ohmic}\\\text{dissipation}}}.
$$

$$
\tag{11.54}
$$

Again, the crossed out flux terms will cancel with corresponding terms in (11.52). The metamorphosis of the resistive term into a flux term and an Ohmic dissipation term is a piece of vector algebra best checked by expanding the divergence of the flux term. Finally, the $\boldsymbol{u}$-to-$\boldsymbol{B}$ energy exchange term (penultimate on the right-hand side) corresponds precisely to the $\boldsymbol{B}$-to-$\boldsymbol{u}$ exchange term in (11.53) and cancels with it if we add (11.53) and (11.54).

11.11.3. *Thermal Energy*

Subtracting (11.53) and (11.54) from (11.52), consummating the promised cancellations, and mopping up the remaining $\boldsymbol{\nabla}\cdot(p\boldsymbol{u})$ and $p\boldsymbol{\nabla}\cdot\boldsymbol{u}$ terms, we end up with the desired evolution equation for the thermal (internal) energy:

$$
\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\frac{p}{\gamma-1}}_{\substack{\text{advection of}\\\text{internal}\\\text{energy}}} = \underbrace{-\boldsymbol{\nabla}\cdot\boldsymbol{q}}_{\text{heat flux}} \underbrace{-\frac{\gamma}{\gamma-1}p\boldsymbol{\nabla}\cdot\boldsymbol{u}}_{\substack{\text{compressional}\\\text{heating}}} \underbrace{-\boldsymbol{\Pi}:\boldsymbol{\nabla}\boldsymbol{u}}_{\substack{\text{viscous}\\\text{heating}}} + \underbrace{\eta\frac{|\boldsymbol{\nabla}\times\boldsymbol{B}|^2}{4\pi}}_{\substack{\text{Ohmic}\\\text{heating}}}. \tag{11.55}
$$

A further rearrangement and the use of the continuity equation (11.46) to express $\boldsymbol{\nabla} \cdot \boldsymbol{u} = -\mathrm{d} \ln \rho / \mathrm{d}t$ turn (11.55) into

$$\frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{p}{\rho^\gamma} = \frac{\gamma - 1}{p} \left(-\boldsymbol{\nabla} \cdot \boldsymbol{q} - \boldsymbol{\Pi} : \boldsymbol{\nabla}\boldsymbol{u} + \eta \frac{|\boldsymbol{\nabla} \times \boldsymbol{B}|^2}{4\pi} \right). \tag{11.56}$$

This form of the thermal-energy equation has very clear physical content: the left-hand side represents advection of the entropy of the MHD fluid by the flow—each fluid element behaves adiabatically, except for the sundry non-adiabatic effects on the right-hand side. The latter are the heat flux in/out of the fluid element and the dissipative (viscous and resistive) heating, leading to entropy production. Note that the form of the viscous stress $\boldsymbol{\Pi}$ ensures that the viscous heating is always positive [see, e.g., (11.6)]. In these Lectures, we will, for the most part, focus on ideal MHD and so use the adiabatic version of (11.56), with the right-hand side set to zero.

Let us reiterate the equations of *ideal MHD*, now complete:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \tag{11.57}$$

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\boldsymbol{u} \right) = -\boldsymbol{\nabla} p + \frac{(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi}, \tag{11.58}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}), \tag{11.59}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \right) \frac{p}{\rho^\gamma} = 0. \tag{11.60}$$

In what follows, we shall study various solutions and symptotic regimes of these rather nice equations.

11.12. *Virial Theorem*

Are there self-confined states? Coming soon...

11.13. *Lagrangian MHD*

There is a mathematically attractive Lagrangian formulation of MHD, on which there is an excellent classic paper by Newcomb (1962). Read it while this section remains unwritten.

This formalism, besides shedding some conceptual light, turns out to give us some useful analytical tools, e.g., for the treatment of explosive MHD instabilities (Pfirsch & Sudan 1993; Cowley & Artun 1997).

---

12. MHD in a Straight Magnetic Field

Equations (11.57–11.60) have a very simple static, uniform *equilibrium solution:*

$$\rho_0 = \text{const}, \quad p_0 = \text{const}, \quad \boldsymbol{u}_0 = 0, \quad \boldsymbol{B}_0 = B_0 \hat{\boldsymbol{z}} = \text{const}. \tag{12.1}$$

We will turn to more nontrivial equilibria in due course, but first we shall study this one carefully—because it is very generic in the sense that many other, more complicated, equilibria *locally* look just like this.

12.1. *MHD Waves*

If you have an equilibrium solution of any set of equations, your first reflex ought to be to perturb it and see what happens: the system might support waves, instabilities, possibly interesting nonlinear behaviour of small perturbations (e.g., §§7–10).

So we now seek solutions to the MHD equations (11.57–11.60) in the form

$$\rho = \rho_0 + \delta\rho, \quad p = p_0 + \delta p, \quad \boldsymbol{u} = \frac{\partial \boldsymbol{\xi}}{\partial t}, \quad \boldsymbol{B} = B_0\hat{\boldsymbol{z}} + \delta\boldsymbol{B}, \tag{12.2}$$

where we have introduced the fluid *displacement* field $\boldsymbol{\xi}$.[68] To start with, we consider all perturbations to be infinitesimal and so linearise the MHD equations (11.57–11.60) as follows.

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\rho = -\rho\boldsymbol{\nabla}\cdot\boldsymbol{u} \quad \Rightarrow \quad \frac{\partial\delta\rho}{\partial t} = -\rho_0\boldsymbol{\nabla}\cdot\frac{\partial\boldsymbol{\xi}}{\partial t}$$

$$\Rightarrow \quad \boxed{\frac{\delta\rho}{\rho_0} = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}}, \tag{12.3}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\frac{p}{\rho^\gamma} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}\frac{\delta p}{p_0} = \gamma\frac{\partial}{\partial t}\frac{\delta\rho}{\rho_0}$$

$$\Rightarrow \quad \boxed{\frac{\delta p}{p_0} = -\gamma\boldsymbol{\nabla}\cdot\boldsymbol{\xi}}, \tag{12.4}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{u} \quad \Rightarrow \quad \frac{\partial\delta\boldsymbol{B}}{\partial t} = B_0\nabla_\parallel\frac{\partial\boldsymbol{\xi}}{\partial t} - \hat{\boldsymbol{z}}B_0\boldsymbol{\nabla}\cdot\frac{\partial\boldsymbol{\xi}}{\partial t}$$

$$\Rightarrow \quad \frac{\delta\boldsymbol{B}}{B_0} = \nabla_\parallel\boldsymbol{\xi} - \hat{\boldsymbol{z}}\boldsymbol{\nabla}\cdot\boldsymbol{\xi} = \nabla_\parallel\boldsymbol{\xi}_\perp - \hat{\boldsymbol{z}}\boldsymbol{\nabla}_\perp\cdot\boldsymbol{\xi}_\perp$$

$$\Rightarrow \quad \boxed{\frac{\delta\boldsymbol{B}_\perp}{B_0} = \nabla_\parallel\boldsymbol{\xi}_\perp}, \quad \boxed{\frac{\delta B_\parallel}{B_0} = -\boldsymbol{\nabla}_\perp\cdot\boldsymbol{\xi}_\perp}, \tag{12.5}$$

where $\parallel$ and $\perp$ denote projections onto the direction (z) of $\boldsymbol{B}_0$ and onto the plane (x, y) perpendicular to it, respectively. Equations (12.5) tell us that parallel displacements produce no perturbation of the magnetic field—obviously not, because the magnetic field is carried with the fluid flow and nothing will happen if you displace a straight uniform field parallel to itself.

The physics of magnetic-field perturbations becomes clearer if we observe that

$$\frac{\delta\boldsymbol{B}}{B_0} = \frac{\delta(B\boldsymbol{b})}{B_0} = \delta\boldsymbol{b} + \hat{\boldsymbol{z}}\frac{\delta B}{B_0}. \tag{12.6}$$

The perturbed field-direction vector $\delta\boldsymbol{b}$ must be perpendicular to $\hat{\boldsymbol{z}}$ (otherwise the field direction is unperturbed; formally this is shown by perturbing the equation $\boldsymbol{b}^2 = 1$). Therefore, the perpendicular and parallel perturbations of the magnetic field are the

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[68]Thinking in terms of displacements makes sense in MHD but not so much in (homogeneous) hydrodynamics because in the latter case, just displacing a fluid element produces no back reaction, whereas in MHD, because magnetic fields are frozen into the fluid and are elastic, displacing fluid elements causes magnetic restoring forces to switch on. In other words, an (ideal) MHD fluid "remembers" the state from which it has been displaced, whereas neutral (Newtonian) fluids only "know" about velocities at which they flow.
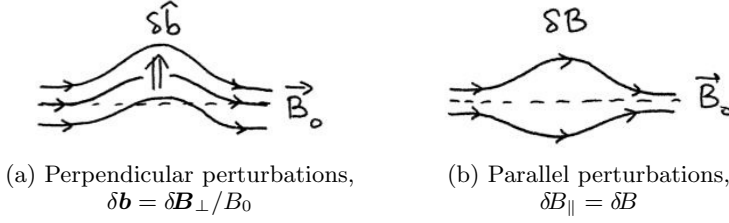
(a) Perpendicular perturbations,
$\delta \boldsymbol{b} = \delta \boldsymbol{B}_\perp / B_0$

(b) Parallel perturbations,
$\delta B_\parallel = \delta B$

FIGURE 42. Magnetic perturbations.

perturbations of its direction and strength, respectively (Fig. 42):

$$\frac{\delta \boldsymbol{B}_\perp}{B_0} = \delta \boldsymbol{b}, \quad \frac{\delta B_\parallel}{B_0} = \frac{\delta B}{B_0}. \tag{12.7}$$

Finally, linearising (11.58) gives us

$$\underbrace{\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \right)}_{= \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2}} = \underbrace{-\boldsymbol{\nabla} p}_{\substack{= -\boldsymbol{\nabla} \delta p \\ = \gamma p_0 \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{\xi}, \\ \text{from (12.4)}}} \underbrace{-\boldsymbol{\nabla} \frac{B^2}{8\pi}}_{= -\frac{B_0^2}{4\pi} \boldsymbol{\nabla} \frac{\delta B}{B_0}} + \underbrace{\frac{\boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{B}}{4\pi}}_{= \frac{B_0^2}{4\pi} \nabla_\parallel \left(\delta \boldsymbol{b} + \hat{\boldsymbol{z}} \frac{\delta B}{B_0} \right)}. \tag{12.8}$$

$$\underbrace{= \frac{B_0^2}{4\pi} \left(-\boldsymbol{\nabla}_\perp \frac{\delta B}{B_0} + \nabla_\parallel \delta \boldsymbol{b} \right)}_{= \frac{B_0^2}{4\pi} \left(\boldsymbol{\nabla}_\perp \boldsymbol{\nabla}_\perp \cdot \boldsymbol{\xi}_\perp + \nabla_\parallel^2 \boldsymbol{\xi}_\perp \right),}$$
$$\text{from (12.7) and (12.5)}$$

Assembling all this, we get

$$\boxed{\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = c_s^2 \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{\xi} + v_A^2 \left(\boldsymbol{\nabla}_\perp \boldsymbol{\nabla}_\perp \cdot \boldsymbol{\xi}_\perp + \nabla_\parallel^2 \boldsymbol{\xi}_\perp \right)}, \tag{12.9}$$

where two special velocities have emerged:

$$c_s = \sqrt{\frac{\gamma p_0}{\rho_0}}, \quad v_A = \frac{B_0}{\sqrt{4\pi \rho_0}}, \tag{12.10}$$

the *sound speed* and the *Alfvén speed*, respectively. The former is familiar from fluid dynamics, while the latter is another speed, arising in MHD, at which perturbations can travel. We shall see momentarily how this happens.

Let us seek wave-like solutions of (12.9), $\boldsymbol{\xi} \propto \exp(-i\omega t + i\boldsymbol{k} \cdot \boldsymbol{r})$. For such perturbations,

$$\omega^2 \boldsymbol{\xi} = c_s^2 \boldsymbol{k} \boldsymbol{k} \cdot \boldsymbol{\xi} + v_A^2 \left(\boldsymbol{k}_\perp \boldsymbol{k}_\perp \cdot \boldsymbol{\xi}_\perp + k_\parallel^2 \boldsymbol{\xi}_\perp \right). \tag{12.11}$$

Without loss of generality, let $\boldsymbol{k} = (k_\perp, 0, k_\parallel)$ (i.e., by definition, x is the direction of $\boldsymbol{k}_\perp$; see Fig. 43). Then (12.11) becomes

$$\omega^2 \xi_x = c_s^2 k_\perp (k_\perp \xi_x + k_\parallel \xi_\parallel) + v_A^2 k^2 \xi_x, \tag{12.12}$$

$$\omega^2 \xi_y = v_A^2 k_\parallel^2 \xi_y, \tag{12.13}$$

$$\omega^2 \xi_\parallel = c_s^2 k_\parallel (k_\perp \xi_x + k_\parallel \xi_\parallel). \tag{12.14}$$
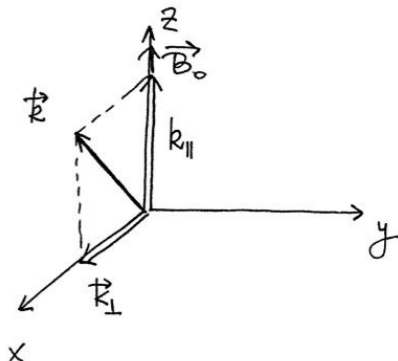
FIGURE 43. Coordinate system for the treatment of MHD waves.
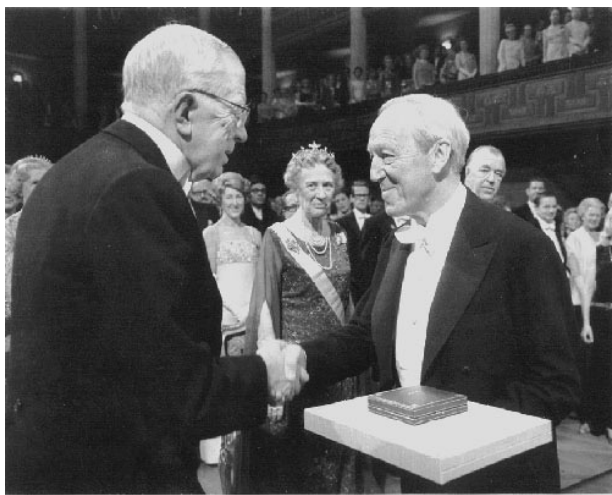
FIGURE 44. Hannes Olof Gösta Alfvén (1908-1995), Swedish electrical engineer and plasma physicist. He was the father of MHD, distrusted religion, computers and Big Bang theory, and got a Nobel Prize "for fundamental work and discoveries in magnetohydrodynamics with fruitful applications in different parts of plasma physics" (1970). In this picture, he is receiving it from King Gustaf VI Adolf of Sweden.

The perturbations of the rest of the fields are

$$\frac{\delta\rho}{\rho_0} = -i\boldsymbol{k}\cdot\boldsymbol{\xi} = -i(k_\perp\xi_x + k_\parallel\xi_\parallel), \tag{12.15}$$

$$\frac{\delta p}{p_0} = \gamma\,\frac{\delta\rho}{\rho_0}, \tag{12.16}$$

$$\delta\boldsymbol{b} = ik_\parallel\boldsymbol{\xi}_\perp = ik_\parallel\begin{pmatrix}\xi_x\\\xi_y\\0\end{pmatrix}, \tag{12.17}$$

$$\frac{\delta B}{B_0} = -ik_\perp\xi_x. \tag{12.18}$$

(a)   (b) $k_\perp \neq 0$

FIGURE 45. Alfvén waves.

12.1.1. *Alfvén Waves*

We start by spotting, instantly, that (12.13) decouples from the rest of the system. Therefore, $\boldsymbol{\xi} = (0, \xi_y, 0)$ is an eigenvector, with two associated eigenvalues
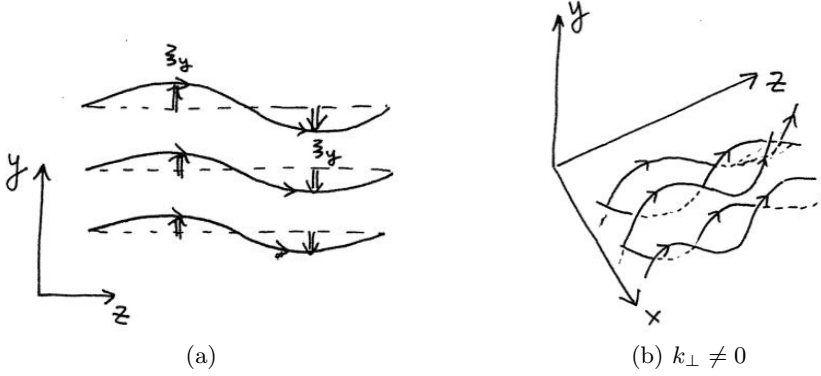
$$\boxed{\omega = \pm k_\parallel v_{\mathrm{A}}}, \tag{12.19}$$

representing *Alfvén waves* propagating parallel and antiparallel to $\boldsymbol{B}_0$. An Alfvénic perturbation is (Fig. 45a)

$$\boldsymbol{\xi} = \xi_y \hat{\boldsymbol{y}}, \quad \delta\rho = 0, \quad \delta p = 0, \quad \delta B = 0, \quad \delta\boldsymbol{b} = i k_\parallel \xi_y \hat{\boldsymbol{y}}, \tag{12.20}$$

i.e., it is incompressible and only involves magnetic field lines behaving as elastic strings, springing back against perturbing motions, due to the restoring curvature force. Note that these waves can have $k_\perp \neq 0$ even though their dispersion relation (12.19) does not depend on $k_\perp$ (Fig. 45b).

12.1.2. *Magnetosonic Waves*

Equations (12.12) and (12.14) form a closed 2D system:

$$\omega^2 \begin{pmatrix} \xi_x \\ \xi_\parallel \end{pmatrix} = \begin{pmatrix} c_{\mathrm{s}}^2 k_\perp^2 + v_{\mathrm{A}}^2 k^2 & c_{\mathrm{s}}^2 k_\parallel k_\perp \\ c_{\mathrm{s}}^2 k_\parallel k_\perp & c_{\mathrm{s}}^2 k_\parallel^2 \end{pmatrix} \cdot \begin{pmatrix} \xi_x \\ \xi_\parallel \end{pmatrix}. \tag{12.21}$$

The resulting dispersion relation is

$$\omega^4 - k^2(c_{\mathrm{s}}^2 + v_{\mathrm{A}}^2)\omega^2 + c_{\mathrm{s}}^2 v_{\mathrm{A}}^2 k^2 k_\parallel^2 = 0. \tag{12.22}$$

This has four solutions:

$$\boxed{\omega^2 = \frac{1}{2} k^2 \left[c_{\mathrm{s}}^2 + v_{\mathrm{A}}^2 \pm \sqrt{(c_{\mathrm{s}}^2 + v_{\mathrm{A}}^2)^2 - 4 c_{\mathrm{s}}^2 v_{\mathrm{A}}^2 \cos^2\theta} \right]}, \quad \cos^2\theta = \frac{k_\parallel^2}{k^2}. \tag{12.23}$$

The two "+" solutions are the *"fast magnetosonic waves"* and the two "−" ones are the *"slow magnetosonic waves"*.

Since both sound and Alfvén speeds are involved, it is obvious that the key parameter demarcating different physical regimes will be their ratio, or, conventionally, the ratio of the thermal to magnetic energies in the MHD medium, known as the *plasma beta:*

$$\beta = \frac{p_0}{B_0^2/8\pi} = \frac{2}{\gamma} \frac{c_{\mathrm{s}}^2}{v_{\mathrm{A}}^2}. \tag{12.24}$$

The magnetosonic waves can be conveniently summarised by the so-called *Friedricks*
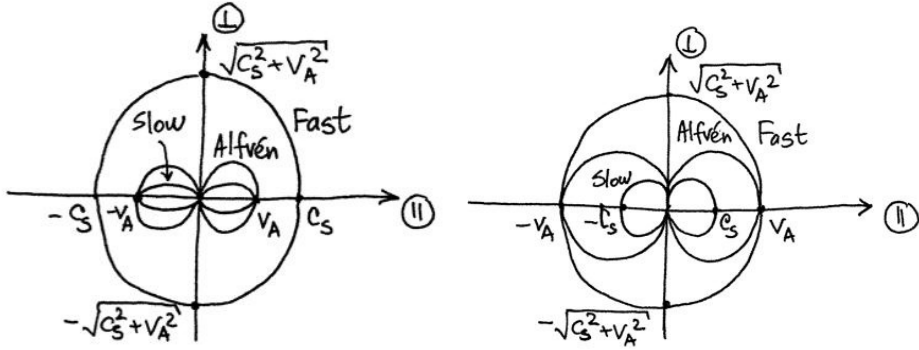
(a) $c_s > v_A$ ("high β")   (b) $c_s < v_A$ ("low β")

FIGURE 46. Friedricks diagram: radius ω/k, angle $\cos\theta = k_\parallel/k$.

diagram, a graph of (12.23) in polar coordinates where the radius is the phase speed ω/k and the angle is θ, the direction of propagation with respect to $\boldsymbol{B}_0$ (Fig. 46).

Clearly, magnetosonic waves contain perturbations of both the magnetic field and of the "hydrodynamic" quantities ρ, p, $\boldsymbol{u}$, but working them all out for the case of general oblique propagation ($\theta \sim 1$) is a bit messy. The physics of what is going on is best understood via a few particular cases.

12.1.3. *Parallel Propagation*

Consider $\boxed{k_\perp = 0}$ ($\theta = 0$). Then $(\xi_x, 0, 0)$ and $(0, 0, \xi_\parallel)$ are eigenvectors of the matrix in (12.21) and the two corresponding waves are

- another *Alfvén wave*, this time with perturbation in the x direction (which, however, is not physically different from the y direction when $k_\perp = 0$):

$$\omega^2 \xi_x = k_\parallel^2 v_A^2 \xi_x \quad \Rightarrow \quad \boxed{\omega = \pm k_\parallel v_A}, \tag{12.25}$$

$$\boldsymbol{\xi} = \xi_x \hat{\boldsymbol{x}}, \quad \delta\rho = 0, \quad \delta p = 0, \quad \delta B = 0, \quad \delta b = i k_\parallel \xi_x \hat{\boldsymbol{x}} \tag{12.26}$$

(at high β, this is the slow wave, at low β, this is the fast wave);

- the parallel-propagating *sound wave* (Fig. 47a):

$$\omega^2 \xi_\parallel = k_\parallel^2 c_s^2 \xi_\parallel \quad \Rightarrow \quad \boxed{\omega = \pm k_\parallel c_s}, \tag{12.27}$$

$$\boldsymbol{\xi} = \xi_\parallel \hat{\boldsymbol{z}}, \quad \frac{\delta\rho}{\rho_0} = -i k_\parallel \xi_\parallel, \quad \frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0}, \quad \delta B = 0, \quad \delta \boldsymbol{b} = 0 \tag{12.28}$$

(at high β, this is the fast wave, at low β, this is the slow wave); the magnetic field does not participate here at all.

12.1.4. *Perpendicular Propagation*

Now consider $\boxed{k_\parallel = 0}$ ($\theta = 90°$). Then $(\xi_x, 0, 0)$ is again an eigenvector of the matrix in (12.21).[69] The resulting fast magnetosonic wave is again a *sound wave*, but because it is perpendicular-propagating, both thermal and magnetic pressures get involved, the

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[69]As is $(0, 0, \xi_\parallel)$, but with $\omega = 0$; we will deal with this mode in §12.3.4.

(a) Parallel propagation, $k_\perp = 0$ (b) Perpendicular propagation, $k_\parallel = 0$
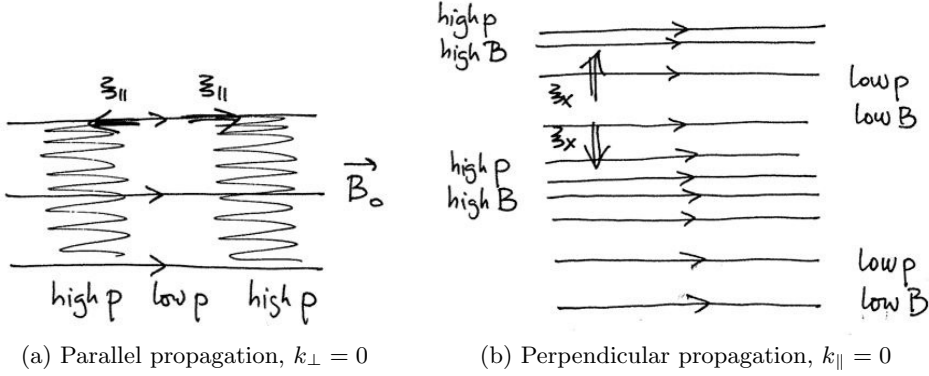
FIGURE 47. Sound waves.

perturbations are compressions/rarefactions in both the fluid and the field, and the speed at which they travel is a combination of the sound and Alfvén speeds (with the latter now representing the magnetic pressure response):

$$\omega^2 \xi_x = k_\perp^2 (c_s^2 + v_A^2)\xi_x \quad \Rightarrow \quad \boxed{\omega = \pm k_\perp \sqrt{c_s^2 + v_A^2}}, \tag{12.29}$$

$$\boldsymbol{\xi} = \xi_x \hat{\boldsymbol{x}}, \quad \frac{\delta\rho}{\rho_0} = -ik_\perp \xi_x, \quad \frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0}, \quad \frac{\delta B}{B_0} = -ik_\perp \xi_x, \quad \delta\boldsymbol{b} = 0. \tag{12.30}$$

Note that the thermal and magnetic compressions are in phase and there is no bending of the magnetic field (Fig. 47b).

12.1.5. *Anisotropic Perturbations:* $k_\parallel \ll k_\perp$

Taking $k_\parallel = 0$ in §12.1.4 was perhaps a little radical as we lost all waves apart from the fast one. As we are about to see, a lot of babies were thrown out with this particular bathwater.

So let us consider MHD waves in the limit $\boxed{k_\parallel \ll k_\perp}$. This turns out to be an extremely relevant regime, because, in a strong magnetic field, realistically excitable perturbations, both linear and nonlinear, tend to be highly elongated in the direction of the field. Going back to (12.23) and enforcing this limit, we get

$$\omega^2 = \frac{1}{2} k^2 (c_s^2 + v_A^2) \left[1 \pm \sqrt{1 - \frac{4c_s^2 v_A^2}{(c_s^2 + v_A^2)^2} \frac{k_\parallel^2}{k^2}} \right]$$

$$\approx \frac{1}{2} k^2 (c_s^2 + v_A^2) \left[1 \pm 1 \mp \frac{2c_s^2 v_A^2}{(c_s^2 + v_A^2)^2} \frac{k_\parallel^2}{k^2} \right]. \tag{12.31}$$

The upper sign gives the familiar *fast wave*

$$\boxed{\omega = \pm k\sqrt{c_s^2 + v_A^2}}. \tag{12.32}$$

This is just the magnetically enhanced sound wave that was considered in §12.1.4. The small corrections to it due to $k_\parallel/k$ are not particularly interesting.

The lower sign in (12.31) gives the *slow wave*

$$\boxed{\omega = \pm k_\parallel \frac{c_s v_A}{\sqrt{c_s^2 + v_A^2}}}, \tag{12.33}$$

which is more interesting. Let us find the corresponding eigenvector: from (12.14),

$$\underbrace{\left(\omega^2 - k_\parallel^2 c_s^2\right)}_{\substack{= -k_\parallel^2 \dfrac{c_s^4}{c_s^2 + v_A^2}, \\ \text{from (12.33)}}} \xi_\parallel = k_\parallel k_\perp c_s^2 \xi_x. \tag{12.34}$$

Therefore, the displacements are mostly parallel:

$$\boxed{\frac{\xi_x}{\xi_\parallel} = -\frac{k_\parallel}{k_\perp} \frac{c_s^2}{c_s^2 + v_A^2} \ll 1}. \tag{12.35}$$

Using this equation together with (12.15–12.18), we find that the perturbations in the remaining fields are

$$\frac{\delta\rho}{\rho_0} = -i(k_\perp \xi_x + k_\parallel \xi_\parallel) = -i \frac{v_A^2}{c_s^2 + v_A^2} k_\parallel \xi_\parallel, \tag{12.36}$$

$$\frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0}, \tag{12.37}$$

$$\delta\boldsymbol{b} = ik_\parallel \xi_x \hat{\boldsymbol{x}} = -i \frac{k_\parallel}{k_\perp} \frac{c_s^2}{c_s^2 + v_A^2} k_\parallel \xi_\parallel \hat{\boldsymbol{x}} \to 0, \tag{12.38}$$

$$\frac{\delta B}{B_0} = -ik_\perp \xi_x = i \frac{c_s^2}{c_s^2 + v_A^2} k_\parallel \xi_\parallel. \tag{12.39}$$

Thus, to lowest order in $k_\parallel/k_\perp$, this wave involves no bending of the magnetic field, but has a pressure/density perturbation and a magnetic-field-strength perturbation—the latter in counter-phase to the former (Fig. 48). To be precise, *the slow-wave perturbations are pressure balanced:*

$$\delta\left(p + \frac{B^2}{8\pi}\right) = p_0 \frac{\delta p}{p_0} + \frac{B_0^2}{4\pi} \frac{\delta B}{B_0} = \rho_0\left(c_s^2 \frac{\delta\rho}{\rho_0} + v_A^2 \frac{\delta B}{B_0}\right) = 0. \tag{12.40}$$

The same is, of course, already obvious from the momentum equation (12.8), where, in the limit $k_\parallel \ll k_\perp$ and $\omega \ll kc_s$ ("incompressible" perturbations; see §12.2), the dominant balance is

$$\boldsymbol{\nabla}_\perp\left(p + \frac{B^2}{8\pi}\right) = 0. \tag{12.41}$$

Finally, the Alfvén waves in the limit of anisotropic propagation are just the same as ever (§12.1.1)—they are unaffected by $k_\perp$, while being perfectly capable of having perpendicular variation (Fig. 45b).

12.1.6. *High-β Limit: $c_s \gg v_A$*

Another limit in which high-frequency acoustic response (fast waves) and low-frequency, pressure-balanced Alfvénic response (slow and Alfvén waves) are separated is $\beta \gg 1 \Leftrightarrow c_s \gg v_A$.[70] In this limit, the approxmate expression (12.31) for the magnetosonic frequencies is still valid, but because v_A/c_s, rather than $k_\parallel/k$, is small.

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[70]This limit is astrophysically very interesting because magnetic fields locally produced by plasma motions in various astrophysical environments (e.g., interstellar and intergalactic media) can only be as strong energetically as the motions that make them [see (11.45) and §11.10] and so, the latter being subsonic, $v_A \sim u \ll c_s$.
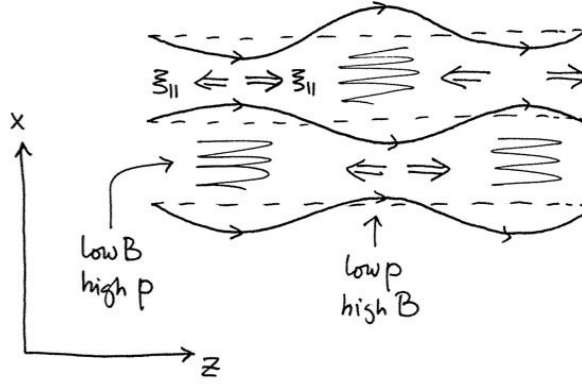
FIGURE 48. Slow wave in the anistropic limit $k_\parallel \ll k_\perp$: pressure balanced, $\xi_x \ll \xi_\parallel$.
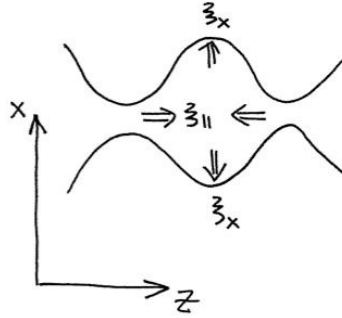
FIGURE 49. Slow wave in the high-β limit: pressure balanced, $\xi_x \sim \xi_\parallel$.

The rest of the calculations in §12.1.5 are also valid, with the following simplifications arising from v_A being negligible compared to c_s.

The upper sign in (12.31) again gives us the *fast wave*, which, this time, is a pure *sound wave*:

$$\boxed{\omega = \pm k c_\mathrm{s}}.\tag{12.42}$$

This is natural because, at high β, the magnetic pressure is negligible compared to thermal pressure and sound can propagate oblivious of the magnetic field.

The lower sign in (12.31) yields the *slow wave*: (12.33) is still valid and becomes, for $v_\mathrm{A} \ll c_\mathrm{s}$,

$$\boxed{\omega = \pm k_\parallel v_\mathrm{A}}.\tag{12.43}$$

Because the slow wave's dispersion relation in this limit looks exactly like the dispersion relation (12.19) of an Alfvén wave, it is called the *pseudo-Alfvén wave*. The similarity is deceptive as the nature of the perturbation (the eigenvector) is completely different. Substituting $\omega^2 = k_\parallel^2 v_\mathrm{A}^2$ into (12.14), we find

$$k_\perp \xi_x + k_\parallel \xi_\parallel = \frac{v_\mathrm{A}^2}{c_\mathrm{s}^2} k_\parallel \xi_\parallel \ll k_\parallel \xi_\parallel.\tag{12.44}$$

This just says that, to lowest order in $1/\beta$, $\boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0$, i.e., the perturbations are incompressible. In contrast to the anisotropic case (12.35), the perpendicular and parallel

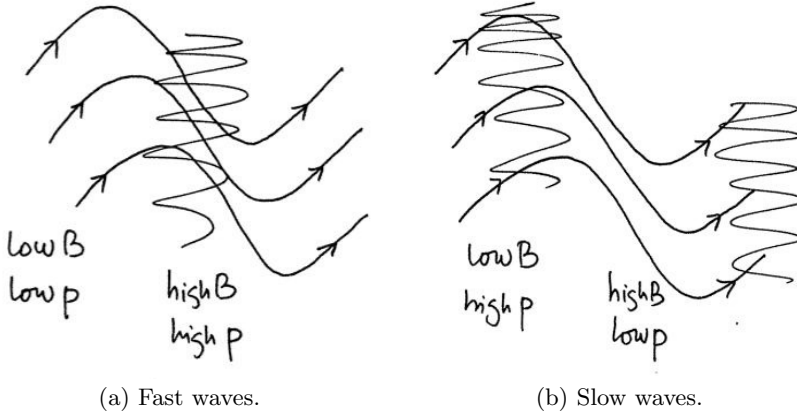(a) Fast waves.   (b) Slow waves.

FIGURE 50. General oblique magnetosonic waves.

displacements are now comparable (assuming, in general, $k_\parallel \sim k_\perp$):

$$\boxed{\frac{\xi_x}{\xi_\parallel} = -\frac{k_\parallel}{k_\perp}}. \tag{12.45}$$

Also in contrast to the anisotropic case, the density and pressure perturbations are now vanishingly small, but the field can be bent as well as compressed:

$$\frac{\delta\rho}{\rho_0} = -i(k_\perp\xi_x + k_\parallel\xi_\parallel) = -i\,\frac{v_A^2}{c_s^2}\,k_\parallel\xi_\parallel \to 0, \tag{12.46}$$

$$\frac{\delta p}{p_0} = \gamma\,\frac{\delta\rho}{\rho_0} \to 0, \tag{12.47}$$

$$\delta\boldsymbol{b} = ik_\parallel\xi_x\hat{\boldsymbol{x}} = -i\,\frac{k_\parallel}{k_\perp}\,k_\parallel\xi_\parallel\hat{\boldsymbol{x}}, \tag{12.48}$$

$$\frac{\delta B}{B_0} = -ik_\perp\xi_x = ik_\parallel\xi_\parallel. \tag{12.49}$$

The δB and $\delta\boldsymbol{b}$ perturbations are in counter-phase, as are $\xi_\parallel$ and ξ_x (Fig. 49). It is easy to check that pressure balance (12.40) is again maintained by these perturbations.

In the more general case of oblique propagation ($k_\parallel \sim k_\perp$) and finite beta ($\beta \sim 1$), the fast and slow magnetosonic waves generally have comparable frequencies and contain perturbations of all relevant fields, with the fast waves tending to have the perturbations of the thermal and magnetic pressure in phase and slow waves in counter-phase (Fig. 50).

12.2. Subsonic Ordering

Enough linear theory! We shall now occupy ourselves with the behaviour of *finite* (although still small) perturbations of a straight-field equilibrium. While we abandon linearisation (i.e., the neglect of nonlinear terms), much of what the linear theory has taught us about the basic responses of an MHD fluid remains true and useful. In particular, the linear relations between the perturbation amplitudes of various fields provide us with a guidance as to the *relative size* of finite perturbations of these fields. This makes sense if, while allowing the nonlinearities back in, we do not assume the linear physics to be completely negligible, i.e., if we allow the linear and nonlinear time scales to compete (§12.2.3). We shall see that solutions for which this is the case satisfy

self-consistent equations, so can be expected to be realisable (and, as we know from experimental, observational and numerical evidence, *are* realised).

I shall start by constructing nonlinear equations that describe the incompressible limit, i.e., fields and motions that are subsonic: both their phase speeds and flow velocities will be assumed small compared to the speed of sound:

$$\frac{\omega/k}{\sqrt{c_\mathrm{s}^2 + v_\mathrm{A}^2}} \ll 1, \quad \mathrm{Ma} \equiv \frac{u}{\sqrt{c_\mathrm{s}^2 + v_\mathrm{A}^2}} \ll 1. \tag{12.50}$$

In this limit, we expect all fast-wave-like perturbations to disappear (in a similar way to the sound waves disappearing in the incompressible Navier–Stokes hydrodynamics) and for the MHD dynamics to contain only Alfvénic and slow-wave-like perturbations. We saw in §§12.1.5 and 12.1.6 that, linearly, fast and slow waves are well separated either in the limit of $k_\parallel/k_\perp \ll 1$ or in the limit of $\beta \gg 1$. Indeed, comparing the Alfvén frequency (12.19) and slow-wave frequency (12.33) to the sound (fast-wave) frequency (12.32), we get

$$\frac{\omega_{\mathrm{Alfvén}}}{\omega_{\mathrm{fast}}} \sim \frac{k_\parallel v_\mathrm{A}}{k\sqrt{c_\mathrm{s}^2 + v_\mathrm{A}^2}} \sim \frac{k_\parallel}{k}\frac{1}{\sqrt{1+\beta}}, \quad \frac{\omega_{\mathrm{slow}}}{\omega_{\mathrm{fast}}} \sim \frac{k_\parallel c_\mathrm{s} v_\mathrm{A}}{k(c_\mathrm{s}^2 + v_\mathrm{A}^2)} \sim \frac{k_\parallel}{k}\frac{\sqrt{\beta}}{1+\beta}, \tag{12.51}$$

both of which are small in either of the two limits, satisfying the first of the conditions (12.50).

The second condition (12.50) involves the *"magnetic Mach number"* Ma (generalised to compare the flow velocity to the speed of sound in a magnetised fluid), which measures the size of the perturbations themselves—in the linear theory, this was arbitrarily small, but now we will need to relate it to our other small parameter(s), $k_\parallel/k$ or $1/\beta$. This means that we would like to construct an asymptotic *ordering* in which there will be some prescription as to how small, or otherwise, various (relative) perturbations and small parameters are—not by themselves, i.e., compared to 1, but compared to each other (compared to 1, the small parameters can all formally be taken to be as small as we desire).

The general strategy for ordering perturbations with respect to each other will be to use the linear relations obtained in the two incompressible limits ($k_\parallel/k \ll 1$ or $\beta \gg 1$). If we do not specifically expect one perturbation to be larger or smaller than another on some physical grounds (like the properties of the linear response), we must order them the same; this does not stop us later from constructing subsidiary expansions in which they might be different. For example, MHD equations themselves were an expansion a number of small parameters, in particular u/c [see (11.14)]. However, at the time of deriving them, we did not want to rule out sonic or supersonic motions and so, effectively, we ordered Ma ~ 1, $k_\parallel/k \sim 1$ and $\beta \sim 1$, as far as the u/c expansion was concerned, i.e., Ma, $k_\parallel/k$, $1/\beta \gg u/c$. Now we are constructing a subsidiary expansion in these other parameters, keeping in mind that they are allowed to be small but not as small as the small parameter already used in the derivation of the MHD equations.[71]

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[71] In principle, you should always feel a little paranoid about the question of whether such "nested" asymptotic expansions commute, i.e., whether it matters in which order they are done. They usually do commute, but you ought to check if you want to be sure. Another formally justified mathematical worry is whether asymptotic solutions of exact equations are the same as exact solutions of asymptotic equations. This will lead you into the world of proofs of existence and uniqueness—where I wish you an enjoyable stay.

12.2.1. *Ordering of Alfvénic Perturbations*

Since the Alfvénic perturbations decouple completely from the rest (§12.1.1), linear theory does not give us a way to relate u_y to $u_\parallel$, so we will exercise the no-prejudice principle stated above and assume

$$u_y \sim u_\parallel, \tag{12.52}$$

i.e., the Mach numbers for the Alfvénic and slow-wave-like motions are comparable. We can, however, relate u_y to $\delta \boldsymbol{b}$, via the curvature-force response (12.20):

$$|\delta \boldsymbol{b}| \sim k_\parallel \xi_y \sim \frac{k_\parallel u_y}{\omega} \sim \frac{u_y}{v_A} \sim \mathrm{Ma}\,\sqrt{1+\beta}. \tag{12.53}$$

12.2.2. *Ordering of Slow-Wave-Like Perturbations*

For slow-wave-like perturbations, in either the anisotropic or the high-β limit, from (12.14) and (12.33),

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} \sim \omega(k_\perp \xi_x + k_\parallel \xi_\parallel) \sim \frac{\omega^2}{k_\parallel c_s^2}\,\omega \xi_\parallel \sim \frac{v_A^2}{c_s^2 + v_A^2}\,k_\parallel u_\parallel \sim \frac{k_\parallel u_\parallel}{1+\beta}. \tag{12.54}$$

Thus, the divergence of the flow velocity is small (the dynamics are incompressible) in all three of our (potentially) small parameters:

$$\frac{\boldsymbol{\nabla} \cdot \boldsymbol{u}}{k\sqrt{c_s^2 + v_A^2}} \sim \frac{k_\parallel}{k}\frac{1}{1+\beta}\,\mathrm{Ma}. \tag{12.55}$$

From this, we can immediately obtain an ordering for the density and pressure perturbations: using (12.3), (12.4), (12.33) and (12.54) [cf. (12.36) and (12.46)],

$$\frac{\delta \rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \boldsymbol{\nabla} \cdot \boldsymbol{\xi} \sim \frac{\boldsymbol{\nabla} \cdot \boldsymbol{u}}{\omega} \sim \frac{\mathrm{Ma}}{\sqrt{\beta}}. \tag{12.56}$$

The magnetic-field-strength (magnetic-pressure) perturbation is, using (12.39) and (12.33) [cf. (12.49)],

$$\frac{\delta B}{B_0} \sim k_\perp \xi_x \sim \frac{c_s^2}{c_s^2 + v_A^2}\frac{k_\parallel u_\parallel}{\omega} \sim \sqrt{\beta}\,\mathrm{Ma}, \tag{12.57}$$

or, perhaps more straightforwardly, from pressure balance (12.40) and using (12.56),

$$\frac{\delta B}{B_0} = -\frac{\beta}{2}\frac{\delta p}{p_0} \sim \sqrt{\beta}\,\mathrm{Ma}. \tag{12.58}$$

Finally, in a similar fashion, using (12.17) and (12.57) [cf. (12.38) and (12.48)], we find

$$|\delta \boldsymbol{b}| \sim k_\parallel \xi_x \sim \frac{k_\parallel}{k_\perp}\,\sqrt{\beta}\,\mathrm{Ma} \tag{12.59}$$

for slow-wave-like perturbations. Note that in all interesting limits this is superceded by the Alfvénic ordering (12.53).

12.2.3. *Ordering of Time Scales*

Let us recall that our motivation for using linear relations between perturbations to determine their relative sizes in a nonlinear regime was that linear response will lose its exclusive sway but remain non-negligible. In formal terms, this means that we must order the linear and nonlinear time scales to be comparable.[72] The nonlinearities in MHD

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[72] In the context of MHD turbulence theory, this principle, applied at each scale, is known as the *critical balance* (see §12.4).

equations are advective, i.e., they are of the form $\boldsymbol{u} \cdot \boldsymbol{\nabla}(\text{stuff})$ and similar, so the rate of nonlinear interaction is $\sim ku$ (in the case of anisotropic perturbations, $\sim k_\perp u_\perp$). Ordering this to be comparable to the frequencies of the Alfvén and slow waves [see (12.51)] gives us

$$\omega_{\text{Alfven}} \sim ku \quad \Rightarrow \quad \text{Ma} \sim \frac{k_\parallel}{k} \frac{1}{\sqrt{1+\beta}}, \tag{12.60}$$

$$\omega_{\text{slow}} \sim ku \quad \Rightarrow \quad \text{Ma} \sim \frac{k_\parallel}{k} \frac{\sqrt{\beta}}{1+\beta}. \tag{12.61}$$

Note that the first of these relations supersedes the second in all interesting limits.

12.2.4. *Summary of Subsonic Ordering*

Thus, the ordering of the time scales determines the size of the perturbations via (12.60). Using this restriction on Ma, we may summarise our subsonic ordering as follows[73]

$$\boxed{\text{Ma} \equiv \frac{u}{\sqrt{c_s^2 + v_A^2}} \sim \frac{|\delta\boldsymbol{b}|}{\sqrt{1+\beta}} \sim \frac{1}{\sqrt{\beta}} \frac{\delta B}{B} \sim \sqrt{\beta} \frac{\delta\rho}{\rho_0} \sim \sqrt{\beta} \frac{\delta p}{p_0} \sim \frac{k_\parallel}{k} \frac{1}{\sqrt{1+\beta}} \ll 1} \tag{12.62}$$

and $\omega \sim ku$. The ordering can be achieved either in the limit of $k_\parallel/k \ll 1$ or $1/\beta \ll 1$, or both. Note that if one of these parameters is small, the other can be order unity or even large (as long as it is not larger than the inverse of the small one).

The case of anisotropic perturbations and arbitrary β applies in a broad range of plasmas, from magnetically confined fusion ones (tokamaks, stellarators) to space (e.g., the solar corona or the solar wind). We shall consider the implications of this ordering in §12.3.

The case of high β applies, e.g., to high-energy galactic and extragalactic plasmas. It is the direct generalisation to MHD of incompressible Navier–Stokes hydrodynamics, i.e., in this case, all one needs to do is solve MHD equations assuming $\rho = \text{const}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$. We shall consider this case now.

12.2.5. *Incompressible MHD Equations*

Assuming $\beta \gg 1$, our ordering becomes

$$\boxed{\frac{u}{c_s} \sim \frac{\omega}{kc_s} \sim \frac{1}{\sqrt{\beta}} \sim \text{Ma}, \quad |\delta\boldsymbol{b}| \sim \frac{\delta B}{B_0} \sim \sqrt{\beta}\,\text{Ma} \sim 1, \quad \frac{\delta\rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \frac{\text{Ma}}{\sqrt{\beta}} \sim \text{Ma}^2}. \tag{12.63}$$

Thus, the density and pressure perturbations are minuscule, while magnetic perturbations are order unity—magnetic fields are relatively easy to bend (i.e., subsonic motions can tangle the field substantially in this regime). Because of this, it will not make sense to split $\boldsymbol{B}$ into $\boldsymbol{B}_0$ and $\delta\boldsymbol{B}$ explicitly, we will treat the magnetic field as a single field, with no need for a strong mean component.

Let us examine the MHD equations (11.57–11.60) under the ordering (12.63).

Since $\omega \sim ku$, the convective derivative $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \boldsymbol{u} \cdot \boldsymbol{\nabla}$ survives intact in all equations, allowing the advective nonlinearity to enter.

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[73]Note that it is not absolutely necessary to work out the detailed linear theory of a set of equations in order to be able to construct such orderings: it is often enough to know roughly where you are going and simply balance terms representing the physics that you wish to keep (or expect to have to keep). An example of this approach is given in §12.2.8.

The continuity equation (11.57) simply reiterates our earlier statement that the velocity field is divergenceless to lowest order:

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = -\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} \sim \omega\,\frac{\delta\rho}{\rho_0} \sim \mathrm{Ma}^3 k c_\mathrm{s} \to 0. \tag{12.64}$$

The momentum equation (11.58) becomes

$$\left(1 + \frac{\delta\rho}{\rho_0}\right)\left(\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right) = -\boldsymbol{\nabla}\underbrace{\left(\frac{c_\mathrm{s}^2}{\gamma}\frac{\delta p}{p_0} + \frac{B^2}{8\pi\rho_0}\right)}_{\equiv\,\tilde{p}} + \frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B}}{4\pi\rho_0}. \tag{12.65}$$

The density perturbation in the left-hand side is $\sim \mathrm{Ma}^2$ and so negligible compared to unity. The remaining terms in this equation are all the same order ($\sim \mathrm{Ma}^2 k c_\mathrm{s}^2$) and so they must all be kept. The total "pressure" $\tilde{p}$ is determined by enforcing $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$ [see (12.64)]. Namely, our equations are

$$\boxed{\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = -\boldsymbol{\nabla}\tilde{p} + \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B}}\,, \tag{12.66}$$

where

$$\nabla^2\tilde{p} = -\boldsymbol{\nabla}\boldsymbol{\nabla} : (\boldsymbol{u}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{B}) \tag{12.67}$$

and we have rescaled the magnetic field to velocity units, $\boldsymbol{B}/\sqrt{4\pi\rho_0} \to \boldsymbol{B}$.

In the induction equation, best written in the form (11.27), all terms are the same order $\sim k u B \sim \mathrm{Ma}\,k c_\mathrm{s} B$ except the one containing $\boldsymbol{\nabla}\cdot\boldsymbol{u}$, which is $\sim \mathrm{Ma}^3 k c_\mathrm{s} B$ and so must be neglected. We are left with

$$\boxed{\frac{\partial\boldsymbol{B}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u}}\,. \tag{12.68}$$

Finally, the internal-energy equation (11.60), which, keeping only the lowest-order terms, becomes

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\right)\left(\frac{\delta p}{p_0} - \gamma\frac{\delta\rho}{\rho_0}\right) = 0, \tag{12.69}$$

can be used to find $\delta\rho/\rho_0$, once $\delta p/p_0 = \gamma(\tilde{p} - B^2/2)/c_\mathrm{s}^2$ is calculated from the solution of (12.66–12.68). Note that $\delta\rho/\rho_0$ is merely a spectator quantity, not required to solve (12.66–12.68), which form a closed set.

Equations (12.66–12.68) are the equations of *incompressible MHD* (let us call it *iMHD*). Note that while they have been obtained in the limit of $\beta \gg 1$, all β dependence has disappeared from them—basically, they describe subsonic dynamics on top of an infinite heat bath. This is how it should be: formally, in any good asymptotic theory, it must be possible to make the small parameter artbitrarily small without changing anything in the equations.

Exercise 12.1. Show that iMHD conserves the sum of kinetic and magnetic energies,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{r}\left(\frac{u^2}{2} + \frac{B^2}{2}\right) = 0. \tag{12.70}$$

Exercise 12.2. Check that you can obtain the right waves, viz., Alfvén (§12.1.1) and pseudo-Alfvén (§12.1.6), directly from iMHD.

12.2.6. *Elsasser MHD*

The iMHD equations possess a remarkable symmetry. Let us introduce *Elsasser (1950) fields*

$$\boldsymbol{Z}^{\pm} = \boldsymbol{u} \pm \boldsymbol{B} \tag{12.71}$$

and rewrite (12.66) and (12.68) as evolution equations for $\boldsymbol{Z}^{\pm}$: after trivial algebra,

$$\boxed{\frac{\partial \boldsymbol{Z}^{\pm}}{\partial t} + \boldsymbol{Z}^{\mp} \cdot \boldsymbol{\nabla} \boldsymbol{Z}^{\pm} = -\boldsymbol{\nabla} \tilde{p}} \tag{12.72}$$

and, since $\boldsymbol{\nabla} \cdot \boldsymbol{Z}^{\pm} = 0$,

$$\nabla^2 \tilde{p} = -\boldsymbol{\nabla}\boldsymbol{\nabla} : \boldsymbol{Z}^{+}\boldsymbol{Z}^{-}. \tag{12.73}$$

Thus, one can think of iMHD as representing the evolution of two incompressible "velocity fields" advecting each other.

Let us restore the separation of the magnetic field into its mean and perturbed parts, $\boldsymbol{B} = \boldsymbol{B}_0 + \delta\boldsymbol{B} = v_{\mathrm{A}}\hat{\boldsymbol{z}} + \delta\boldsymbol{B}$ (recall that $\boldsymbol{B}$ is in velocity units). Then

$$\boldsymbol{Z}^{\pm} = \pm v_{\mathrm{A}}\hat{\boldsymbol{z}} + \delta\boldsymbol{Z}^{\pm} \tag{12.74}$$

and (12.72) becomes

$$\boxed{\frac{\partial \delta\boldsymbol{Z}^{\pm}}{\partial t} \mp v_{\mathrm{A}}\nabla_{\|}\delta\boldsymbol{Z}^{\pm} + \delta\boldsymbol{Z}^{\mp} \cdot \boldsymbol{\nabla}\delta\boldsymbol{Z}^{\pm} = -\boldsymbol{\nabla}\tilde{p}}. \tag{12.75}$$

Thus, $\delta\boldsymbol{Z}^{\pm}$ are finite, counter-propagating (at the Alfvén speed v_{A}) perturbations—and they interact nonlinearly only with each other, not with themselves. If we let, say, $\delta\boldsymbol{Z}^{-} = 0 \Leftrightarrow \boldsymbol{u} = \delta\boldsymbol{B}$, then $\delta\boldsymbol{Z}^{+}$ satisfies

$$\frac{\partial \delta\boldsymbol{Z}^{+}}{\partial t} - v_{\mathrm{A}}\nabla_{\|}\delta\boldsymbol{Z}^{+} = 0, \tag{12.76}$$

and similarly for $\delta\boldsymbol{Z}^{-}$ (propagting at $-v_{\mathrm{A}}$) if $\delta\boldsymbol{Z}^{+} = 0$. Therefore,

$$\delta\boldsymbol{Z}^{\pm} = \boldsymbol{f}(\boldsymbol{r} \pm v_{\mathrm{A}}t\hat{\boldsymbol{z}}), \quad \delta\boldsymbol{Z}^{\mp} = 0, \tag{12.77}$$

where $\boldsymbol{f}$ is an arbitrary function, are *exact nonlinear solutions* of iMHD. They are called *Elsasser states*. Physically, they are isolated Alfvén-wave packets that propagate along the guide field and never interact (because they all travel at the same speed and so can never catch up with or overtake one another). In order to have any interesting *nonlinear* dynamics, the system must have *counter-propagating* Alfvén-wave packets (see §12.4).

12.2.7. *Cross-Helicity*

Equations (12.72) manifestly support *two* conservation laws:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{r}\,\frac{|\boldsymbol{Z}^{\pm}|^2}{2} = 0, \tag{12.78}$$

i.e., the energy of each Elsasser field is individually conserved. This can be reformulated as conservation of the total energy,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{r}\,\frac{1}{2}\left(\frac{|\boldsymbol{Z}^{+}|^2}{2} + \frac{|\boldsymbol{Z}^{-}|^2}{2}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{r}\,\left(\frac{u^2}{2} + \frac{B^2}{2}\right) = 0, \tag{12.79}$$

and of a new quantity, known as the *cross-helicity:*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3\boldsymbol{r}\, \frac{1}{2} \left(\frac{|\boldsymbol{Z}^+|^2}{2} - \frac{|\boldsymbol{Z}^-|^2}{2} \right) = \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{u} \cdot \boldsymbol{B} = 0 \,. \tag{12.80}$$

In the Elsasser formulation, the cross-helicity is a measure of energy *imbalace* between the two Elssasser fields[74]—this is observed, for example, in the solar wind, where there is significantly more energy in the Alfvénic fluctuations propagating away from the Sun than towards it (see, e.g., Wicks *et al.* 2013).

Exercise 12.3. To see why we needed incompressibility to get this new conservation law, work out the time evolution equation for $\int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{u} \cdot \boldsymbol{B}$ from the general (compressible) MHD equations and hence the condition under which the cross-helicity is conserved.

12.2.8. *Stratified MHD*

It is quite instructive to consider a very simple example of non-uniform MHD equilibrium: the case of a *stratified atmosphere.* Let us introduce gravity into MHD equations, viz., the momentum equation (11.58) becomes

$$\rho \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\boldsymbol{\nabla} \left(p + \frac{B^2}{8\pi} \right) + \frac{\boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{B}}{4\pi} - \rho g \hat{\boldsymbol{z}} \tag{12.81}$$

(uniform gravitational acceleration pointing downward, against the z direction). We wish to consider a static equilibrium inhomogeneous in the z direction and threaded by a uniform magnetic field (which may be zero):

$$\rho_0 = \rho_0(z), \quad p_0 = p_0(z), \quad \boldsymbol{u}_0 = 0, \quad \boldsymbol{B}_0 = B_0\boldsymbol{b}_0 = \text{const}, \tag{12.82}$$

where $\boldsymbol{b}_0$ is at some general angle to $\hat{\boldsymbol{z}}$ and $p_0(z)$ and $\rho_0(z)$ are constrained by the vertical force balance:

$$\frac{\mathrm{d}p_0}{\mathrm{d}z} = -\rho_0 g \quad \Rightarrow \quad g = -\frac{p_0}{\rho_0}\frac{\mathrm{d}\ln p_0}{\mathrm{d}z} = \frac{c_\mathrm{s}^2}{\gamma}\frac{1}{H_p}, \tag{12.83}$$

where it has been opportune to define the pressure scale height H_p. We shall now seek time-dependent solutions of the MHD equations for which

$$\rho = \rho_0(z) + \delta\rho, \quad \frac{\delta\rho}{\rho_0} \ll 1, \quad p = p_0(z) + \delta p, \quad \frac{\delta p}{p_0} \ll 1, \tag{12.84}$$

and the spatial variation of all pertubations occurs on scales that are small compared to the pressure scale height H_p or the analogously defined density scale height $H_\rho = -(\mathrm{d}\ln\rho_0/\mathrm{d}z)^{-1}$ (for ordering purposes, we denote them both H):

$$\boxed{kH \gg 1} \,. \tag{12.85}$$

After the equilibrium pressure balance is subtracted from (12.81), this equation becomes, under any ordering in which $\delta\rho \ll \rho_0$,

$$\rho_0 \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\boldsymbol{\nabla} \left(\delta p + \frac{B^2}{8\pi} \right) + \frac{\boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{B}}{4\pi} - \delta\rho g \hat{\boldsymbol{z}} \,. \tag{12.86}$$

The last term is the *buoyancy (Archimedes) force.* In order for this new feature to give rise to any nontrivial new physics, it must be ordered comparable to all the other terms in the equation:

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[74]Cross-helicity can also be interpreted as a topological invariant, counting the linkages between flux tubes and vortex tubes analogously to what magnetic helicity does for the flux tubes alone (see §13.2).

using (12.83) to express $g \sim p_0/\rho_0 H$, we find

$$\delta\rho g \sim k\delta p \quad \Rightarrow \quad \frac{\delta\rho}{\rho_0} \sim kH\frac{\delta p}{p_0} \gg \frac{\delta p}{p_0}, \tag{12.87}$$

$$\delta\rho g \sim \frac{kB^2}{4\pi} \quad \Rightarrow \quad \frac{\delta\rho}{\rho_0} \sim \frac{kH}{\beta} \ll 1 \quad \Rightarrow \quad \beta \gg kH \gg 1. \tag{12.88}$$

So we learn that the density perturbations must now be much larger than the pressure perturbations, but, in order for the former to remain small and for the magnetic field to be in the game, β must be high (it is in anticipation of this that we did not split $\boldsymbol{B}$ into $\boldsymbol{B}_0$ and $\delta\boldsymbol{B}$, expecting them to be of the same order).

Let us now expand the internal-energy equation (11.60) in small density and pressure perturbations. Denoting $s = p/\rho^\gamma = s_0(z) + \delta s$ (entropy density) and introducing the entropy scale height

$$\frac{1}{H_s} \equiv \frac{\mathrm{d}\ln s_0}{\mathrm{d}z} = -\frac{1}{H_p} + \frac{\gamma}{H_\rho} \tag{12.89}$$

(assumed positive), we find[75]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta s}{s_0} = -\frac{u_z}{H_s}, \quad \frac{\delta s}{s_0} = \frac{\delta p}{p_0} - \gamma\frac{\delta\rho}{\rho_0} \approx -\gamma\frac{\delta\rho}{\rho_0}. \tag{12.90}$$

The last, approximate, expression follows from the smallness of pressure perturbations [see (12.87)]. This then gives us

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\rho}{\rho_0} = \frac{u_z}{\gamma H_s}}. \tag{12.91}$$

But, on the other hand, the continuity equation (11.57) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\rho}{\rho_0} = -\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{u_z}{H_\rho} \quad \Rightarrow \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = u_z\left(\frac{1}{H_\rho} - \frac{1}{\gamma H_s}\right) = \frac{u_z}{\gamma H_p} \quad \Rightarrow \quad \frac{\boldsymbol{\nabla}\cdot\boldsymbol{u}}{ku} \sim \frac{1}{kH} \ll 1. \tag{12.92}$$

Thus, the dynamics are incompressible again and the role of the continuity equation is to tell us that we must find δp from the momentum equation (12.86) by enforcing $\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$ to lowest order. The difference with iMHD (§12.2.5) is that $\delta\rho/\rho$ now participates in the dynamics via the buoyancy force and must be found self-consistently from (12.91).

Finally, we rewrite our newly found simplified system of equations for a stratified, high-β atmosphere, in the following neat way:

$$\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = -\boldsymbol{\nabla}\tilde{p} + \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B} + a\hat{\boldsymbol{z}}, \tag{12.93}$$

$$\nabla^2\tilde{p} = -\boldsymbol{\nabla}\boldsymbol{\nabla} : (\boldsymbol{u}\boldsymbol{u} - \boldsymbol{B}\boldsymbol{B}) + \frac{\partial a}{\partial z}, \tag{12.94}$$

$$\frac{\partial a}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}a = -N^2 u_z, \quad N = \frac{c_{\mathrm{s}}}{\gamma\sqrt{H_s H_p}}, \tag{12.95}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{B} = \boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{u}, \tag{12.96}$$

where we have rescaled $\boldsymbol{B}/\sqrt{4\pi\rho_0} \to \boldsymbol{B}$ and denoted the Archimedes acceleration

$$a = -\frac{\delta\rho}{\rho_0}g = -\frac{\delta\rho}{\rho_0}\frac{c_{\mathrm{s}}^2}{\gamma H_p}, \tag{12.97}$$

a quantity also known as *the buoyancy* of the fluid. We shall call (12.93–12.96) the equations of *stratified MHD (SMHD)*.

A new frequency N, known as the *Brunt–Väisälä frequency*, has appeared in our equations.[76]

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[75]We are able to take equilibrium quantities in and out of spatial derivatives because $kH \gg 1$ and the perturbations are small.

[76]N is real because we assumed $H_s > 0$ (a "stably stratified atmosphere"), otherwise

In order for all the linear and nonlinear time scales that are present in our equations to coexist legitimately within our ordering, we must demand that the Alfvén, Brunt–Väisälä and nonlinear time scales all be comparable:

$$kv_{\mathrm{A}} \sim N \sim ku \quad \Rightarrow \quad \boxed{\frac{1}{\sqrt{\beta}} \sim \frac{1}{kH} \sim \mathrm{Ma}} . \tag{12.98}$$

This gives us a relative ordering between all the small parameters that have appeared so far, including the new one, $1/kH$. Using (12.91) and recalling (12.87), let us summarise the ordering of the perturbations:

$$\frac{u}{c_{\mathrm{s}}} \sim \frac{\delta\rho}{\rho_0} \sim \mathrm{Ma}, \quad \frac{\delta p}{p_0} \sim \mathrm{Ma}^2, \quad |\delta\boldsymbol{b}| \sim \frac{\delta B}{B_0} \sim 1. \tag{12.99}$$

The difference with the iMHD high-β ordering (12.63) is that the density perturbations have now been promoted to dynamical relevance, thankfully without jeopardising incompressibility (i.e., still ordering out the sonic perturbations). The ordering (12.99) can be thought of as a generalisation to MHD of the *Boussinesq approximation* in hydrodynamics.

Further investigations of the SMHD equations are undertaken in Q5.

12.3. *Reduced MHD*

We now turn to the anisotropic ordering, $k_\parallel/k \ll 1$ (while $\beta \sim 1$, in general), for which we studied the linear theory in §12.1.5. Specialising to this case from our general ordering (12.62), we have

$$\boxed{\mathrm{Ma} \sim \frac{u_\perp}{c_{\mathrm{s}}} \sim \frac{u_\parallel}{c_{\mathrm{s}}} \sim |\delta\boldsymbol{b}| \sim \frac{\delta B}{B_0} \sim \frac{\delta\rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \frac{\omega}{k_\perp c_{\mathrm{s}}} \sim \frac{k_\parallel}{k_\perp} \ll 1} . \tag{12.100}$$

Starting again with the continuity equation (11.57), dividing through by ρ_0 and ordering all terms, we get

$$\underbrace{\left(\underbrace{\frac{\partial}{\partial t}}_{\mathrm{Ma}} + \underbrace{\boldsymbol{u}_\perp \cdot \boldsymbol{\nabla}_\perp}_{\mathrm{Ma}} + \underbrace{u_\parallel \cancel{\boldsymbol{\nabla}_\parallel}}_{\mathrm{Ma}^2} \right) \underbrace{\frac{\delta\rho}{\rho_0}}_{\mathrm{Ma}}}_{\mathrm{Ma}^2} = - \underbrace{\left(1 + \underbrace{\cancel{\frac{\delta\rho}{\rho_0}}}_{\mathrm{Ma}} \right) \left(\underbrace{\boldsymbol{\nabla}_\perp \cdot \boldsymbol{u}_\perp}_{\mathrm{Ma}} + \underbrace{\nabla_\parallel u_\parallel}_{\mathrm{Ma}^2} \right)}_{\mathrm{Ma}} . \tag{12.101}$$

Thus, to lowest order, the perpendicular velocity field is 2D-incompressible:

$$\mathcal{O}(\mathrm{Ma}) : \quad \boxed{\boldsymbol{\nabla}_\perp \cdot \boldsymbol{u}_\perp = 0} . \tag{12.102}$$

In the next order (which we will need in §12.3.2),

$$\mathcal{O}(\mathrm{Ma}^2) : \quad (\boldsymbol{\nabla} \cdot \boldsymbol{u})_2 = - \left(\frac{\partial}{\partial t} + \boldsymbol{u}_\perp \cdot \boldsymbol{\nabla}_\perp \right) \frac{\delta\rho}{\rho_0} = -\frac{\mathrm{d}}{\mathrm{d}t} \frac{\delta\rho}{\rho_0}, \tag{12.103}$$

where, to leading order, the convective derivative now involves only perpendicular advection.

Equation (12.102) implies that $\boldsymbol{u}_\perp$ can be written in terms of a *stream function:*

$$\boxed{\boldsymbol{u}_\perp = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp \Phi} . \tag{12.104}$$

the atmosphere becomes convectively unstable—this happens when the equilibrium entropy decreases upwards (cf. §14.3, Q9 and Q5c).

Similarly, for the magnetic field, we have

$$0 = \boldsymbol{\nabla} \cdot \boldsymbol{B} = \underbrace{\boldsymbol{\nabla}_\perp \cdot \delta \boldsymbol{B}_\perp}_{\mathrm{Ma}} + \underbrace{\cancel{\boldsymbol{\nabla}_\parallel \delta B_\parallel}}_{\mathrm{Ma}^2} \approx \boldsymbol{\nabla}_\perp \cdot \delta \boldsymbol{B}_\perp, \tag{12.105}$$

so $\delta \boldsymbol{B}_\perp$ is also 2D-solenoidal and can be written in terms of a *flux function:*

$$\boxed{\frac{\delta \boldsymbol{B}_\perp}{\sqrt{4\pi\rho_0}} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp \Psi}. \tag{12.106}$$

Note that $\Psi = -A_\parallel/\sqrt{4\pi\rho_0}$, the parallel component of the vector potential.

Thus, Alfvénically polarised perturbations, $\boldsymbol{u}_\perp$ and $\delta \boldsymbol{B}_\perp$ (see §12.1.1), can be described by two scalar functions, Φ and Ψ. Let us work out the evolution equations for them.

12.3.1. *Alfvénic Perturbations*

We start with the induction equation, again most useful in the form (11.27). Dividing through by B_0, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\delta \boldsymbol{B}}{B_0} = \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{b} \boldsymbol{\nabla} \cdot \boldsymbol{u}. \tag{12.107}$$

Throwing out the obviously subdominant $\delta \boldsymbol{b}$ contribution in the last term on the right-hand side (i.e., approximating $\boldsymbol{b} \approx \hat{\boldsymbol{z}}$ in that term), then taking the perpendicular part of the remaining equation, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\delta \boldsymbol{B}_\perp}{B_0} = \boldsymbol{b} \cdot \boldsymbol{\nabla} \boldsymbol{u}_\perp. \tag{12.108}$$

As we saw above, the convective derivative is with respect to the perpendicular velocity only and, in view of the stream-function representation (12.104) of the latter, for any function f, we have, to leading order,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \boldsymbol{u}_\perp \cdot \boldsymbol{\nabla}_\perp f = \frac{\partial f}{\partial t} + \hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla}_\perp \Phi \times \boldsymbol{\nabla}_\perp f) = \frac{\partial f}{\partial t} + \{\Phi, f\}, \tag{12.109}$$

where the "Poisson bracket" is

$$\{\Phi, f\} = \frac{\partial \Phi}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial \Phi}{\partial y} \frac{\partial f}{\partial x}. \tag{12.110}$$

Similarly, to leading order,

$$\boldsymbol{b} \cdot \boldsymbol{\nabla} f = \frac{\partial f}{\partial z} + \delta \boldsymbol{b} \cdot \boldsymbol{\nabla}_\perp f = \frac{\partial f}{\partial z} + \frac{1}{v_\mathrm{A}} \hat{\boldsymbol{z}} \cdot (\boldsymbol{\nabla}_\perp \Psi \times \boldsymbol{\nabla}_\perp f) = \frac{\partial f}{\partial z} + \frac{1}{v_\mathrm{A}} \{\Psi, f\}. \tag{12.111}$$

Finally, using (12.109) and (12.111) in (12.108) and expressing $\delta \boldsymbol{B}_\perp$ in terms of Ψ [see (12.106)] and $\boldsymbol{u}_\perp$ in terms of Φ [see (12.104)], it is a straightforward exercise to show, after "uncurling" (12.108), that[77]

$$\boxed{\frac{\partial \Psi}{\partial t} + \{\Phi, \Psi\} = v_\mathrm{A} \frac{\partial \Phi}{\partial z}}. \tag{12.112}$$

Turning now to the momentum equation (11.58), taking its perpendicular part and

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[77] Another easy route to this equation is to start from the induction equation in the form (11.59), let $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, "uncurl" (11.59) and take the z component of the resulting evolution equation for $\boldsymbol{A}$.

dividing through by $\rho \approx \rho_0$, we get

$$\underbrace{\frac{d\boldsymbol{u}_\perp}{dt}}_{\text{Ma}^2} = \frac{1}{\rho_0}\left[-\boldsymbol{\nabla}_\perp\left(p + \frac{B^2}{8\pi}\right) + \frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\delta\boldsymbol{B}_\perp}{4\pi}\right] = \underbrace{-\boldsymbol{\nabla}_\perp\left(\frac{c_s^2}{\gamma}\frac{\delta p}{p_0} + v_A^2\frac{\delta B}{B_0}\right)}_{\text{Ma}} + \underbrace{v_A^2\boldsymbol{b}\cdot\boldsymbol{\nabla}\frac{\delta\boldsymbol{B}_\perp}{B_0}}_{\text{Ma}^2}.$$

$$(12.113)$$

To lowest order,

$$\mathcal{O}(\text{Ma}): \quad \boldsymbol{\nabla}_\perp\left(\frac{c_s^2}{\gamma}\frac{\delta p}{p_0} + v_A^2\frac{\delta B}{B_0}\right) = 0 \quad \Rightarrow \quad \boxed{\frac{\delta p}{p_0} = -\gamma\frac{v_A^2}{c_s^2}\frac{\delta B}{B_0}}. \qquad (12.114)$$

This is a statement of pressure balance, which is physically what has been expected [see (12.41)] and which will be useful in §12.3.2. In the next order, (12.113) contains the perpendicular gradient of the second-order contribution to the total pressure. To avoid having to calculate it, we take the curl of (12.113) and thus obtain

$$\mathcal{O}(\text{Ma}^2): \quad \boldsymbol{\nabla}_\perp \times \frac{d\boldsymbol{u}_\perp}{dt} = v_A^2\boldsymbol{\nabla}_\perp \times \left(\boldsymbol{b}\cdot\boldsymbol{\nabla}\frac{\delta\boldsymbol{B}_\perp}{B_0}\right). \qquad (12.115)$$

Finally, using again (12.104), (12.106), (12.109) and (12.111) in (12.115), some slightly tedious algebra leads us to

$$\boxed{\frac{\partial}{\partial t}\nabla_\perp^2\Phi + \{\Phi, \nabla_\perp^2\Phi\} = v_A\frac{\partial}{\partial z}\nabla_\perp^2\Psi + \{\Psi, \nabla_\perp^2\Psi\}}. \qquad (12.116)$$

Note that $\nabla_\perp^2\Phi$ is the vorticity of the flow $\boldsymbol{u}_\perp$ and so the above equation is the MHD generalisation of the 2D Euler equation.

To summarise the equations (12.116) and (12.112) in their most compact form, we have

$$\frac{d}{dt}\nabla_\perp^2\Phi = v_A\boldsymbol{b}\cdot\boldsymbol{\nabla}\nabla_\perp^2\Psi, \qquad (12.117)$$

$$\frac{d\Psi}{dt} = v_A\frac{\partial\Phi}{\partial z}, \qquad (12.118)$$

where the convective time derivative d/dt and the parallel spatial derivative $\boldsymbol{b}\cdot\boldsymbol{\nabla}$ are given by (12.109) and (12.111), respectively. Beautifully, these *nonlinear* equations describing Alfvénic perturbations have decoupled completely from everything else: we do not need to know $\delta\rho$, δp, $u_\parallel$ or δB in order to solve for $\boldsymbol{u}_\perp$ and $\delta\boldsymbol{B}_\perp$. *Alfvénic dynamics are self-contained.*

Equations (12.117) and (12.118) are called the Equations of *Reduced MHD (RMHD)*. They were originally derived in the context of tokamak plasmas (Kadomtsev & Pogutse 1974; Strauss 1976) and are extremely popular as a simple paradigm for MHD is a strong guide field—not just in tokamaks, but also in space.[78]

12.3.2. *Compressive Perturbations*

What about the rest of our fields—in the linear language, the slow-wave-like perturbations (§12.1.5)? While we do not need them to compute the Alfvénic perturbations, we might still wish to know them for their own sake.

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[78]In the latter context, they are used most prominently as a description of Alfvénic turbulence at small scales (see §12.4), for which the RMHD equations can be shown to be the correct description even if the plasma is collisionless and in general requires kinetic treatment (Schekochihin *et al.* 2009; Kunz *et al.* 2015, 2018).

Returning to the induction equation (12.107) and taking its z component, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta B_\parallel}{B_0} = \boldsymbol{b}\cdot\boldsymbol{\nabla}u_\parallel - \boldsymbol{\nabla}\cdot\boldsymbol{u} \quad\Rightarrow\quad \boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta B}{B_0} - \frac{\delta\rho}{\rho_0}\right) = \boldsymbol{b}\cdot\boldsymbol{\nabla}u_\parallel}, \tag{12.119}$$

where all terms are $\mathcal{O}(\mathrm{Ma}^2)$, $\delta B_\parallel \approx \delta B$ to leading order and we used (12.103) to express $\boldsymbol{\nabla}\cdot\boldsymbol{u}$. The derivatives $\mathrm{d}/\mathrm{d}t$ and $\boldsymbol{b}\cdot\boldsymbol{\nabla}$ contain the nonlinearities involving Φ and Ψ, which we already know from (12.117) and (12.118).

To find an equation for $u_\parallel$, we take the z component of the momentum equation (11.58):

$$\underbrace{\frac{\mathrm{d}u_\parallel}{\mathrm{d}t}}_{\mathrm{Ma}^2} = \frac{1}{\rho_0}\left[\underbrace{-\frac{\partial}{\partial z}\left(p + \frac{B^2}{8\pi}\right)}_{\mathrm{Ma}^3} + \underbrace{\frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\delta B_\parallel}{4\pi}}_{\mathrm{Ma}^2}\right] \quad\Rightarrow\quad \boxed{\frac{\mathrm{d}u_\parallel}{\mathrm{d}t} = v_\mathrm{A}^2\,\boldsymbol{b}\cdot\boldsymbol{\nabla}\frac{\delta B}{B_0}}. \tag{12.120}$$

The parallel pressure gradient is $\mathcal{O}(\mathrm{Ma}^3)$ because there is pressure balance (12.114) to lowest order.

Finally, let us bring in the energy equation (11.60), as yet unused. To leading order, it is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta s}{s_0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta p}{p_0} - \gamma\frac{\delta\rho}{\rho_0}\right) = 0 \quad\Rightarrow\quad \boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta\rho}{\rho_0} + \frac{v_\mathrm{A}^2}{c_\mathrm{s}^2}\frac{\delta B}{B_0}\right) = 0}, \tag{12.121}$$

where, to obtain the final version of the equation, we substituted (12.114) for $\delta p/p_0$.

Equations (12.119–12.121) are a complete set of equations for δB, $u_\parallel$ and $\delta\rho$, given Φ and Ψ. These equations are linear in the Lagrangian frame associated with the Alfvénic perturbations, provided the parallel distances are measured along perturbed field lines. Physically, they tell us that *slow waves propagate along perturbed field lines and are passively (i.e., without acting back) advected by the perpendicular Alfvénic flows.*

Exercise 12.4. Check that the linear relationships between various perturbations in a slow wave derived in §12.1.5 are manifest in (12.119–12.121).

In what follows, when we refer to RMHD, we will mean all five equations (12.117–12.118) and (12.119–12.121).

Exercise 12.5. Show that RMHD equations possess the following exact symmetry: $\forall\,\epsilon$ and a, one can simultaneously scale all perturbation amplitudes by ϵ, perpendicular distances by a, parallel distances and times by a/ϵ. This means that parallel and perpendicular distances in RMHD are effectively measured in different units. It also means that the small parameter Ma in RMHD can be made arbitrarily small, without any change in the form of the equations, so RMHD is a *bona fide* asymptotic theory (see remark at the end of §12.2.5).

12.3.3. *Elsasser Fields and the Energetics of RMHD*

The Elsasser approach (§12.2.6) can be adapted to the RMHD system. Defining *Elsasser potentials*

$$\boxed{\zeta^\pm = \Phi \pm \Psi} \quad\Leftrightarrow\quad \delta\boldsymbol{Z}_\perp^\pm = \boldsymbol{u}_\perp \pm \frac{\delta\boldsymbol{B}_\perp}{\sqrt{4\pi\rho_0}} = \hat{\boldsymbol{z}}\times\boldsymbol{\nabla}_\perp\zeta^\pm, \tag{12.122}$$

it is a straighforward exercise to show that the "vorticities" of the the two Elsasser fields,

$$\omega^\pm = \hat{\boldsymbol{z}}\cdot(\boldsymbol{\nabla}_\perp\times\delta\boldsymbol{Z}_\perp^\pm) = \nabla_\perp^2\zeta^\pm \tag{12.123}$$

(fluid vorticities $\pm$ electric currents), satisfy the following evolution equation

$$\boxed{\frac{\partial \omega^\pm}{\partial t} \mp v_\mathrm{A} \frac{\partial \omega^\pm}{\partial z} + \{\zeta^\mp, \omega^\pm\} = \{\partial_j \zeta^\pm, \partial_j \zeta^\mp\}}, \qquad (12.124)$$

where summation over the repeated index j is implied. The main corrolary of this equation is the same as in §12.2.6, although here it applies to perpendicular perturbations only: *only counter-propagating Alfvénic perturbations can interact* and *any finite-amplitude perturbation composed of just one Elsasser field is a nonlinear solution*.

Some light is perhaps shed on the nature of the interaction between Elsasser fields if we notice that the left-hand side of (12.124) tells us that the Elsasser vorticity $\omega^\pm$ is propagated along the mean field at the speed v_A and advected across the field by the Elsasser field $\delta \boldsymbol{Z}_\perp^\mp$. The right-hand side of (12.124) is a kind of vortex-stretching term, implying a tendency for vortices and current layers to be produced in the (x, y) plane. There is a preference for current layers, as it turns out. The term in the right-hand side of (12.124) has opposite signs for the two Elsasser fields. Therefore, arguably, nonlinear dynamics favour $\omega^+ \omega^- < 0$, i.e., $|\nabla_\perp^2 \Psi|^2 > |\nabla_\perp^2 \Phi|^2$ (larger currents than vorticities). This is, indeed, what is seen in numerical simulations of MHD turbulence (see Zhdankin *et al.* 2016 and §12.4).

The energies of the two Elsasser fields are individually conserved (cf. §12.2.7),

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3 \boldsymbol{r} \, |\boldsymbol{\nabla}_\perp \zeta^\pm|^2 = \frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3 \boldsymbol{r} \, |\delta \boldsymbol{Z}_\perp^\pm|^2 = 0, \qquad (12.125)$$

i.e., when the two fields do interact, they scatter each other nonlinearly, but do *not* exchange energy.

There is an Elsasser-like formulation for the slow waves as well:[79]

$$\boxed{\delta Z_\parallel^\pm = u_\parallel \pm \frac{\delta B}{\sqrt{4\pi\rho_0}} \sqrt{1 + \frac{v_\mathrm{A}^2}{c_\mathrm{s}^2}}}. \qquad (12.126)$$

Then, from (12.119–12.121), one gets, after more algebra,

$$\frac{\partial \delta Z_\parallel^\pm}{\partial t} \mp \frac{c_\mathrm{s} v_\mathrm{A}}{\sqrt{c_\mathrm{s}^2 + v_\mathrm{A}^2}} \frac{\partial \delta Z_\parallel^\pm}{\partial z} =$$
$$-\frac{1}{2} \left[\left(1 \mp \frac{1}{\sqrt{1 + v_\mathrm{A}^2/c_\mathrm{s}^2}} \right) \{\zeta^+, \delta Z_\parallel^\pm\} + \left(1 \pm \frac{1}{\sqrt{1 + v_\mathrm{A}^2/c_\mathrm{s}^2}} \right) \{\zeta^-, \delta Z_\parallel^\pm\} \right].$$
$$(12.127)$$

Note the (expected) appearance of the slow-wave phase speed [cf. (12.33)] in the left-hand side. Thus, slow waves interact only with Alfvénic perturbations—when $v_\mathrm{A} \ll c_\mathrm{s}$, only with the counterpropagating ones, but at finite β, because the slow waves are slower, a co-propagating Alfvénic perturbation can catch up with a slow one, have its way with it in passing and speed on (it's a tough world).

There is no energy exchange in these interactions: the "+" and "−" slow-wave energies are individually conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3 \boldsymbol{r} \, |\delta Z_\parallel^\pm|^2 = 0. \qquad (12.128)$$

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[79]At high β, $v_\mathrm{A} \ll c_\mathrm{s}$, so we recover from (12.126) and (12.122) the Elsasser fields as defined for iMHD in (12.71).

12.3.4. *Entropy Mode*

There are only two equations in (12.127), whereas we had three equations (12.119–12.121) for our three compressive fields δB, $u_\parallel$ and $\delta\rho$. The third equation, (12.121), was in fact for the entropy perturbation:

$$\boxed{\frac{\mathrm{d}\delta s}{\mathrm{d}t} = 0}, \quad \frac{\delta s}{s_0} = -\gamma\left(\frac{\delta\rho}{\rho_0} + \frac{v_{\mathrm{A}}^2}{c_{\mathrm{s}}^2}\frac{\delta B}{B_0}\right). \tag{12.129}$$

We see that δs is a decoupled variable, independent from $\zeta^\pm$ or $\delta Z_\parallel^\pm$ (because it is the only one that involves $\delta\rho/\rho_0$). Equation (12.129) says that δs is a *passive scalar field*, simply carried around by the Alfvénic velocity $\boldsymbol{u}_\perp$ (via d/dt). At high β, this is just a density perturbation.

The associated linear mode is not a wave: its dispersion relation is

$$\boxed{\omega = 0}. \tag{12.130}$$

This is the (famously often forgotten) 7th MHD mode, known as the *entropy mode* (there are 7 equations in MHD, so there must be 7 linear modes: two fast waves, two Alfvén waves, two slow waves and one entropy mode).

Exercise 12.6. Go back to §12.1 and find where we overlooked this mode.

Since the entropy mode is decoupled, its "energy" (variance) is individually conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3\boldsymbol{r}\,|\delta s|^2 = 0. \tag{12.131}$$

Thus, *in RMHD, the (nonlinear) evolution of all perturbations is constrained by 5 separate conservation laws*: $\int\mathrm{d}^3\boldsymbol{r}\,|\delta\boldsymbol{Z}_\perp^\pm|^2$, $\int\mathrm{d}^3\boldsymbol{r}\,|\delta Z_\parallel^\pm|^2$ *and* $\int\mathrm{d}^3\boldsymbol{r}\,|\delta s|^2$ *are all invariants.*

12.3.5. *Discussion*

Such are the simplifications allowed by anisotropy. Besides greater mathematical simplicity, what is the moral of this story, physically? Let me leave you with two observations.

• In a strong magnetic field, linear propagation is a parallel effect, whilst nonlinearity is a perpendicular effect (advection by $\boldsymbol{u}_\perp$, adjustment of propagation direction by $\delta\boldsymbol{B}_\perp$). RMHD equations express the idea that linear and nonlinear physics play equally important role—this becomes the fundamental guiding principle in the theory of MHD turbulence (§12.4). The idea is that complicated nonlinear dynamics that emerge in the perpendicular plane get teased out along the field because propagating waves enforce a degree of parallel spatial coherence. The distances over which this happens are determined by equating linear and nonlinear time scales, $k_\parallel v_{\mathrm{A}} \sim k_\perp u_\perp$. Dynamics cannot stay coherent over distances longer than $\sim k_\parallel^{-1}$ determined by this balance because of causality: points separated by longer parallel distances cannot exchange information quickly enough to catch up with perpendicular nonlinearities acting locally at each of these points. This principle is called *critical balance*.

• Restricting the size of perturbations to be small made the RMHD system, in a certain sense, "less nonlinear" than the full MHD (or than iMHD, where $\delta B/B_0 \sim 1$ was allowed). This led to the system's dynamics being constrained by more invariants: the MHD energy invariant got split into 5 individually conserved quadratic quantities.

Exercise 12.7. You might find it an interesting excercise to think about properties of the RMHD system in 2D, in the light of the two observations above. How many invariants are there? In what kind of physical circumstances *can* we use 2D RMHD without necessarily expecting parallel coherence of the system to break down by the causality argument?

12.4. *MHD Turbulence*

RMHD is a good starting point for developing the theory of *MHD turbulence*—a phenomenon observed with great precision in the solar wind and believed ubiquitous in the Universe. I am writing a tutorial review of this topic—a reasonably advanced draft can be found here: http://www-thphys.physics.ox.ac.uk/research/plasma/JPP/papers17/schekochihin2a.pdf.

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13. MHD Relaxation

So far, we have only considered MHD in a straight field against the background of constant density and pressure (except in §12.2.8, where this was generalised slightly). As any more complicated (static) equilibrium will locally look like this, what we have done has considerable universal significance. Now we shall occupy ourselves with a somewhat less universal (i.e., dependent on the circumstances of a particular problem) and more "large-scale" (compared to the dynamics of wavy perturbations) question: *what kind of (static) equilibrium states are there and into which of those states will an MHD fluid normally relax?*

13.1. *Static MHD Equilibria*

Let us go back to the MHD equations (11.57–11.60) and seek static equilibria, i.e., set $\boldsymbol{u} = 0$ and $\partial/\partial t = 0$. The remaining equations are

$$\boxed{-\boldsymbol{\nabla}p + \frac{\boldsymbol{j} \times \boldsymbol{B}}{c} = 0, \quad \boldsymbol{j} = \frac{c}{4\pi}\boldsymbol{\nabla} \times \boldsymbol{B}, \quad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0} \tag{13.1}$$

(the force balance, Ampère's law and the solenoidality-of-$\boldsymbol{B}$ constraint). These are 7 equations for 7 unknowns (p, $\boldsymbol{B}$, $\boldsymbol{j}$), so a complete set. Density is irrelevant because nothing moves and so inertia does not matter.

The force-balance equation has two immediate general consequences:

$$\boldsymbol{B} \cdot \boldsymbol{\nabla}p = 0, \tag{13.2}$$

so *magnetic surfaces are surfaces of constant pressure*, and

$$\boldsymbol{j} \cdot \boldsymbol{\nabla}p = 0, \tag{13.3}$$

so *currents flow along those surfaces*.

Equation (13.2) implies that if magnetic field lines are stochastic and fill the volume of the system, then $p = \text{const}$ across the system and so the force balance becomes
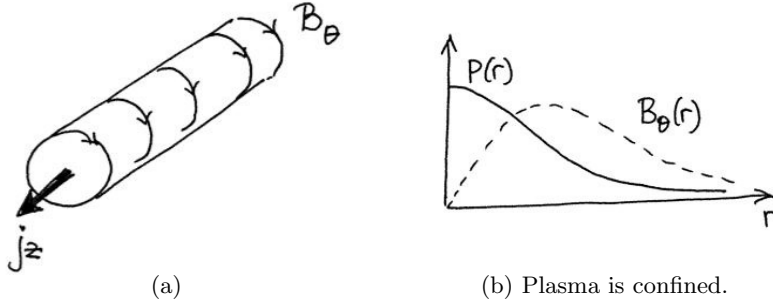
$$\boldsymbol{j} \times \boldsymbol{B} = 0. \tag{13.4}$$

Such equilibria are called *force-free* and turn out to be very interesting, as we shall discover soon (from §13.1.2 onwards).

13.1.1. *MHD Equilibria in Cylindrical Geometry*

As the simplest example of an inhomogeneous equilibrium, let us consider the case of cylindrical and axial symmetry:

$$\frac{\partial}{\partial\theta} = 0, \quad \frac{\partial}{\partial z} = 0. \tag{13.5}$$

(a)　　　　　　　　　　　　　(b) Plasma is confined.

FIGURE 51. z pinch.

Solenoidality of the magnetic field then rules out it having a radial component:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \frac{1}{r}\frac{\partial}{\partial r} rB_r = 0 \quad \Rightarrow \quad rB_r = \text{const} \quad \Rightarrow \quad B_r = 0. \tag{13.6}$$

Ampère's law tells us that currents do not flow radially either:

$$\boldsymbol{j} = \frac{c}{4\pi}\boldsymbol{\nabla} \times \boldsymbol{B} \quad \Rightarrow \quad \begin{cases} j_r = 0, \\[2mm] j_\theta = -\dfrac{c}{4\pi}\dfrac{\partial B_z}{\partial r}, \\[2mm] j_z = \dfrac{c}{4\pi}\dfrac{1}{r}\dfrac{\partial}{\partial r} rB_\theta. \end{cases} \tag{13.7}$$

Finally, the radial pressure balance gives us

$$\frac{\partial p}{\partial r} = \frac{(\boldsymbol{j} \times \boldsymbol{B})_r}{c} = \frac{j_\theta B_z - j_z B_\theta}{c} = \frac{1}{4\pi}\left(-B_z\frac{\partial B_z}{\partial r} - \frac{B_\theta}{r}\frac{\partial}{\partial r} rB_\theta\right)$$

$$= -\frac{\partial}{\partial r}\frac{B_z^2}{8\pi} - \frac{B_\theta^2}{4\pi r} - \frac{\partial}{\partial r}\frac{B_\theta^2}{8\pi} \quad \Rightarrow \quad \boxed{\frac{\partial}{\partial r}\left(p + \frac{B^2}{8\pi}\right) = -\frac{B_\theta^2}{4\pi r}}. \tag{13.8}$$

This simply says that the total pressure gradient is balanced by the tension force. A general equilibrium for which this is satisfied is called a *screw pinch*.

One simple particular case of this is the *z pinch* (Fig. 51a). This is achieved by letting a current flow along the z axis, giving rise to an azimuthal field:

$$j_\theta = 0, \quad j_z = \frac{c}{4\pi}\frac{1}{r}\frac{\partial}{\partial r} rB_\theta \quad \Rightarrow \quad B_\theta = \frac{4\pi}{c}\frac{1}{r}\int_0^r \mathrm{d}r' r' j_z(r'), \quad B_z = 0. \tag{13.9}$$

Equation (13.8) becomes

$$\boxed{\frac{\partial p}{\partial r} = -\frac{1}{c}j_z B_\theta}. \tag{13.10}$$

The "pinch" comes from magnetic loops and is due to the curvature force: the loops want to contract inwards, the pressure gradient opposes this and so *plasma is confined* (Fig. 51b). This configuration will, however, prove to be very badly unstable (§14.4)—which does not stop it from being a popular laboratory set up for short-term confinement experiments (see, e.g., review by Haines 2011).

Another simple particular case is the θ *pinch* (Fig. 52a). This is achieved by imposing a straight but radially non-uniform magnetic field in the z direction and, therefore,

(a)

(b) Coronal loop.

(c) Plasma is confined.

(d) Magnetic field is confined.
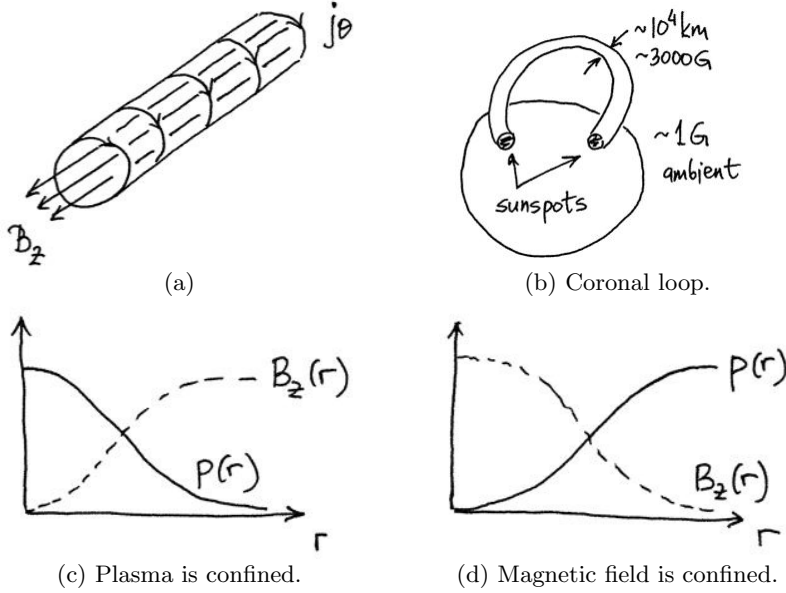
FIGURE 52. θ pinch.

azimuthal currents:

$$B_\theta = 0, \quad j_z = 0, \quad j_\theta = -\frac{c}{4\pi}\frac{\partial B_z}{\partial r}. \tag{13.11}$$

Equation (13.8) is then just a pressure balance, pure and simple:

$$\boxed{\frac{\partial}{\partial r}\left(p + \frac{B_z^2}{8\pi}\right) = 0}. \tag{13.12}$$

In this configuration, we can confine the plasma (Fig. 52c) or the magnetic flux (Fig. 52d). The latter is what happens, for example, in flux tubes that link sunspots (Fig. 52b). The θ pinch is a stable configuration (Q10).

The more general case of a screw pinch (13.8) is a superposition of z and θ pinches, with both magnetic fields and currents wrapping themselves around cylindrical flux surfaces.

The next step in complexity is to assume axial, but not cylindrical symmetry ($\partial/\partial\theta = 0$, $\partial/\partial z \neq 0$). This is explored in Q8.

For a much more thorough treatment of MHD equilibria, the classic textbook is Freidberg (2014).

13.1.2. *Force-Free Equilibria*

Another interesting and elegant class of equilibria arises if we consider situations in which $\boldsymbol{\nabla}p$ is negligible and can be completely omitted from the force balance. This can happen in two possible sets of circumstances:

—pressure is the same across the system, e.g., because the field lines are stochastic [a previously mentioned consequence of (13.2)];

—$\beta = p/(B^2/8\pi) \ll 1$, so thermal energy is negligible compared to magnetic energy and so p is irrelevant.

A good example of the latter situation is the *solar corona*, where $\beta \sim 1{-}10^{-6}$ (assuming

$n \sim 10^9$ cm^{-3}, $T \sim 10^2$ eV and $B \sim 1 - 10^3$ G, the lower value applying in the photosphere, the upper one in the coronal loops; see Fig. 52b)

In such situations, the equilibrium is purely magnetic, i.e., the magnetic field is *"force-free,"* which implies that the current must be parallel to the magnetic field:

$$\boxed{\boldsymbol{j} \times \boldsymbol{B} = 0} \quad \Rightarrow \quad \boldsymbol{j} \parallel \boldsymbol{B} \quad \Rightarrow \quad \frac{4\pi}{c}\boldsymbol{j} = \boldsymbol{\nabla} \times \boldsymbol{B} = \alpha(\boldsymbol{r})\boldsymbol{B}, \qquad (13.13)$$

where $\alpha(\boldsymbol{r})$ is an arbitrary scalar function. Taking the divergence of the last equation tells us that

$$\boldsymbol{B} \cdot \boldsymbol{\nabla}\alpha = 0, \qquad (13.14)$$

so the function $\alpha(\boldsymbol{r})$ is constant on magnetic surfaces. If $\boldsymbol{B}$ is chaotic and volume-filling, then $\alpha = $ const across the system.

The case of $\alpha = $ const is called the *linear force-free field*. In this case, taking the curl of (13.13) and then iterating it once gives us

$$-\nabla^2 \boldsymbol{B} = \alpha \boldsymbol{\nabla} \times \boldsymbol{B} = \alpha^2 \boldsymbol{B} \quad \Rightarrow \quad \boxed{\left(\nabla^2 + \alpha^2\right)\boldsymbol{B} = 0}, \qquad (13.15)$$

so the magnetic field satisfies a Helmholtz equation (to solve which, one must, of course, specify some boundary conditions).

Thus, there is, potentially, a large zoo of MHD equilibria. Some of them are stable, some are not, and, therefore, some are more interesting and/or more relevant than others. How does one tell? A good question to ask is as follows. Suppose we set up some initial configuration of magnetic field (by, say, switching on some current-carrying coils, driving currents inside plasma, etc.)—*to what (stable) equilibrium will this system eventually relax?*

In general, any initially arranged magnetic configuration will exert forces on the plasma, these will drive flows, which in turn will move the magnetic fields around; eventually, everything will settle into some static equilibrium. We expect that, normally, some amount of the energy contained in the initial field will be lost in such a relaxation process because the flows will be dissipating, the fields diffusing and/or reconnecting, etc.—the losses occur due to the resistive and viscous terms in the non-ideal MHD equations derived in §11. Thus, one expects that *the final relaxed static state will be a minimum-energy state* and so we must be able to find it by minimising magnetic energy:

$$\int \mathrm{d}^3\boldsymbol{r}\, \frac{B^2}{8\pi} \to \min. \qquad (13.16)$$

Clearly, if the relaxation occured without any constraints, the solution would just be $\boldsymbol{B} = 0$. In fact, there are constraints. These constraints are *topological:* if you think of magnetic field lines as a tangled mess, you will realise that, while you can change this tangle by moving field lines around, you cannot easily undo linkages, knots, etc.—anything that, to be undone, would require the field lines to have "ends". This intuition can be turned into a quantitative theory once we discover that the induction equation (11.59) has an invariant that involves the magnetic field only and is, in a certain sense, "better conserved" than energy.

13.2. *Helicity*

Magnetic helicity in a volume V is defined as

$$\boxed{H = \int_V \mathrm{d}^3 \boldsymbol{r} \, \boldsymbol{A} \cdot \boldsymbol{B}}, \tag{13.17}$$

where $\boldsymbol{A}$ is the vector potential, $\boldsymbol{\nabla} \times \boldsymbol{A} = \boldsymbol{B}$.

13.2.1. *Helicity Is Well Defined*

This is not obvious because $\boldsymbol{A}$ is not unique: a gauge transformation

$$\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\chi, \tag{13.18}$$

with χ an arbitrary scalar function, leaves $\boldsymbol{B}$ unchanged and so does not affect physics. Under this transformation, helicity stays invariant:

$$H \to H + \int_V \mathrm{d}^3 \boldsymbol{r} \, \boldsymbol{B} \cdot \boldsymbol{\nabla}\chi = H + \int_{\partial V} \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{B} \, \chi = H, \tag{13.19}$$

provided $\boldsymbol{B}$ at the boundary is parallel to the boundary, i.e., provided the volume V encloses the field (nothing sticks out).

13.2.2. *Helicity Is Conserved*

Let us go back to the induction equation (11.23) (in which we retain resistivity to keep track of non-ideal effects, i.e., of the breaking of flux conservation):

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B} - \eta \boldsymbol{\nabla} \times \boldsymbol{B}). \tag{13.20}$$

"Uncurling" this equation, we get

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{u} \times \boldsymbol{B} - \eta \boldsymbol{\nabla} \times \boldsymbol{B} + \boldsymbol{\nabla}\chi. \tag{13.21}$$

Using (13.20) and (13.21), we have

$$\frac{\partial}{\partial t} \boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot (\cancel{\boldsymbol{u} \times \boldsymbol{B}} - \eta \boldsymbol{\nabla} \times \boldsymbol{B} + \boldsymbol{\nabla}\chi) + \boldsymbol{A} \cdot [\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B} - \eta \boldsymbol{\nabla} \times \boldsymbol{B})]$$

$$= -\eta \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) + \boldsymbol{\nabla} \cdot (\boldsymbol{B}\chi)$$

$$- \boldsymbol{\nabla} \cdot [\boldsymbol{A} \times (\boldsymbol{u} \times \boldsymbol{B} - \eta \boldsymbol{\nabla} \times \boldsymbol{B})] + (\cancel{\boldsymbol{u} \times \boldsymbol{B}} - \eta \boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \underbrace{(\boldsymbol{\nabla} \times \boldsymbol{A})}_{= \boldsymbol{B}}$$

$$= \boldsymbol{\nabla} \cdot [\boldsymbol{B}\chi - \boldsymbol{u}\boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{B}\boldsymbol{A} \cdot \boldsymbol{u} + \eta \boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{B})] - 2\eta \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}). \tag{13.22}$$

Integrating this and using Gauss's theorem, we get

$$\frac{\partial}{\partial t} \int_V \mathrm{d}^3 \boldsymbol{r} \, \boldsymbol{A} \cdot \boldsymbol{B} = \int_{\partial V} \mathrm{d}\boldsymbol{S} \cdot [\boldsymbol{B}\chi - \boldsymbol{u}\boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{B}\boldsymbol{A} \cdot \boldsymbol{u} + \eta \boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{B})]$$

$$- 2\eta \int_V \mathrm{d}^3 \boldsymbol{r} \, \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}). \tag{13.23}$$

The surface integral vanishes provided both $\boldsymbol{u}$ and $\boldsymbol{B}$ are parallel to the boundary (no fields stick out and no flows cross). The resistive term in the surface integral can also be ignored either by arranging V appropriately or simply by taking it large enough so $\boldsymbol{B} \to 0$ on ∂V, or, indeed, by taking $\eta \to +0$. Thus,

$$\boxed{\frac{\mathrm{d}H}{\mathrm{d}t} = -2\eta \int \mathrm{d}^3 \boldsymbol{r} \, \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B})}, \tag{13.24}$$

magnetic helicity is conserved in ideal MHD.[80]

Furthermore, it turns out that even in resistive MHD, helicity is "better conserved" than energy, in the following sense. As we saw in §11.11.2, the magnetic energy evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3 r\, \frac{B^2}{8\pi} = \left(\begin{array}{c}\text{energy exchange terms}\\\text{and fluxes}\end{array}\right) - 2\eta \int \mathrm{d}^3 r\, |\boldsymbol{\nabla}\times\boldsymbol{B}|^2. \tag{13.25}$$

The first term on the right-hand side contains various fluxes and energy exchanges with the velocity field [see (11.54)], all of which eventually decay as the system relaxes (flows decay by viscosity). The second term represents Ohmic heating. If η is small but the Ohmic heating is finite, it is finite because magnetic field develops fine-scale gradients: $\boldsymbol{\nabla} \sim \eta^{-1/2}$, so

$$-2\eta \int \mathrm{d}^3 r\, |\boldsymbol{\nabla}\times\boldsymbol{B}|^2 \to \text{const} \quad \text{as} \quad \eta \to +0. \tag{13.26}$$

But then the right-hand side of (13.24) is

$$-2\eta \int \mathrm{d}^3 r\, \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) = \mathcal{O}(\eta^{1/2}) \to 0 \quad \text{as} \quad \eta \to +0. \tag{13.27}$$

Thus, as an initial magnetic configuration relaxes, while its energy can change quickly (on dynamical times), its helicity changes only very slowly in the limit of small η. *The constancy of H (as $\eta \to +0$) provides us with the constraint subject to which the energy will need to be minimised.*

Before we use this idea, let us discuss what the conservation of helicity means physically, or, rather, topologically.

13.2.3. *Helicity Is a Topological Invariant*

Consider two linked flux tubes, T_1 and T_2 (Fig. 53). The helicity of T_1 is the product of the fluxes through T_1 and T_2:

$$H_1 = \int_{T_1} \mathrm{d}^3 r\, \boldsymbol{A}\cdot\boldsymbol{B} = \int_{T_1} \underbrace{\mathrm{d}\boldsymbol{l}}_{\boldsymbol{b}\mathrm{d}l}\cdot\underbrace{\mathrm{d}\boldsymbol{S}}_{\boldsymbol{b}\mathrm{d}S}\,\underbrace{\boldsymbol{A}\cdot\boldsymbol{B}}_{\boldsymbol{A}\cdot\boldsymbol{b}B}$$

$$= \int_{T_1} \boldsymbol{A}\cdot\boldsymbol{b}\mathrm{d}l\, B\boldsymbol{b}\cdot\boldsymbol{b}\mathrm{d}S = \int_{T_1} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}\, \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = \Phi_1 \underbrace{\int_{T_1} \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}}_{\substack{= \int_{S_1}\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}\\ = \Phi_{\text{through hole in } T_1}}} = \Phi_1\Phi_2. \tag{13.28}$$

By the same token, in general, in a system of many linked tubes, the helicity of tube i is

$$H_i = \Phi_i \Phi_{\text{through hole in tube } i} = \Phi_i \sum_j \Phi_j N_{ij}, \tag{13.29}$$

where N_{ij} is the number of times tube j passes through the hole in tube i. The total helicity of the this entire assemblage of flux tubes is then

$$\boxed{H = \sum_{ij} \Phi_i \Phi_j N_{ij}}. \tag{13.30}$$

Thus, H is the number of linkages of the flux tubes weighted by the field strength in them. It is in this sense that helicity is a topological invariant.

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[80]The resistive term in the right-hand side of (13.24) is $\propto \int \mathrm{d}^3 r\, \boldsymbol{B}\cdot\boldsymbol{j}$, a quantity known as the *current helicity*.
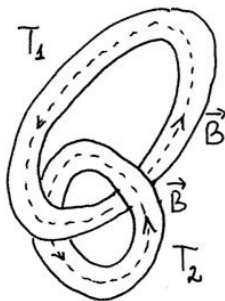
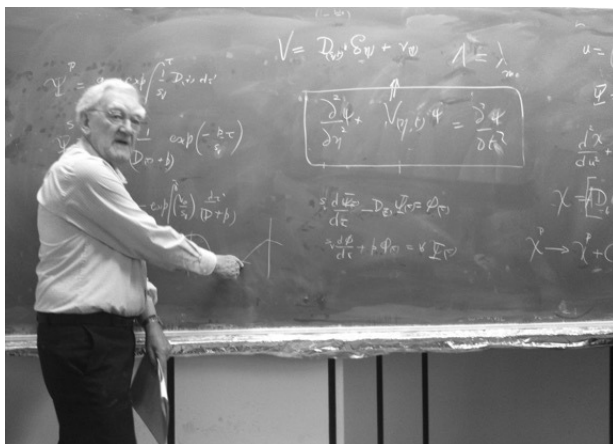FIGURE 53. Linked flux tubes.

FIGURE 54. John Bryan Taylor (born 1929), one of the founding fathers of modern plasma physics, author of the Taylor relaxation (§13.3), Taylor constraint (in dynamo theory), Chirikov–Taylor map (in chaos theory), the ballooning theory (in tokamak MHD), and many other clever things, including the design of the UK's first hydrogen bomb (1957). This picture was taken in 2012 at the Wolfgang Pauli Institute in Vienna.

Note that the cross-helicity $\int \mathrm{d}^3 r\, \boldsymbol{u} \cdot \boldsymbol{B}$ (§12.2.7) can similarly be interpreted as counting the linkages between flux tubes ($\boldsymbol{B}$) and vortex tubes ($\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$). The current helicity $\int \mathrm{d}^3 r\, \boldsymbol{B} \cdot \boldsymbol{j}$ [appearing in the right-hand side of (13.24)] counts the number of linkages between current loops. The latter is not an MHD invariant though.

13.3. *J. B. Taylor Relaxation*

Let us now work out the equilibrium to which an MHD system will relax by *minimising magnetic energy subject to constant helicity*:

$$\delta \int_V \mathrm{d}^3 \boldsymbol{r} \left(B^2 - \alpha\, \boldsymbol{A} \cdot \boldsymbol{B} \right) = 0, \tag{13.31}$$

where α is the Lagrange multiplier introduced to enforce the constant-helicity constraint. Let us work out the two terms:

$$\delta \int_V \mathrm{d}^3\boldsymbol{r}\, B^2 = 2 \int_V \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{B} \cdot \delta\boldsymbol{B} = 2 \int_V \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \delta\boldsymbol{A})$$

$$= 2 \int_V \mathrm{d}^3\boldsymbol{r}\, [-\boldsymbol{\nabla} \cdot (\boldsymbol{B} \times \delta\boldsymbol{A}) + (\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \delta\boldsymbol{A}]$$

$$= -2 \int_{\partial V} \mathrm{d}\boldsymbol{S} \cdot (\boldsymbol{B} \times \delta\boldsymbol{A}) + 2 \int_V \mathrm{d}^3\boldsymbol{r}\, (\boldsymbol{\nabla} \times \boldsymbol{B}) \cdot \delta\boldsymbol{A}, \qquad (13.32)$$

$$\delta H = \delta \int_V \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{A} \cdot \boldsymbol{B} = \int_V \mathrm{d}^3\boldsymbol{r}\, (\boldsymbol{B} \cdot \delta\boldsymbol{A} + \boldsymbol{A} \cdot \delta\boldsymbol{B}) = \int_V \mathrm{d}^3\boldsymbol{r}\, [\boldsymbol{B} \cdot \delta\boldsymbol{A} + \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \delta\boldsymbol{A})]$$

$$= \int_V \mathrm{d}^3 \big[\boldsymbol{B} \cdot \delta\boldsymbol{A} - \boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \delta\boldsymbol{A}) + \underbrace{(\boldsymbol{\nabla} \times \boldsymbol{A})}_{=\,\boldsymbol{B}} \cdot \delta\boldsymbol{A}\big]$$

$$= -\int_{\partial V} \mathrm{d}\boldsymbol{S} \cdot (\boldsymbol{A} \times \delta\boldsymbol{A}) + 2 \int_V \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{B} \cdot \delta\boldsymbol{A}. \qquad (13.33)$$

Now, since

$$\frac{\partial \delta\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{\nabla} \times \left(\frac{\partial \boldsymbol{\xi}}{\partial t} \times \boldsymbol{B}\right) \qquad (13.34)$$

for small displacements, we have $\delta\boldsymbol{A} = \boldsymbol{\xi} \times \boldsymbol{B}$, whence

$$\boldsymbol{B} \times \delta\boldsymbol{A} = B^2\boldsymbol{\xi} - \boldsymbol{B} \cdot \boldsymbol{\xi}\, \boldsymbol{B}, \qquad (13.35)$$

$$\boldsymbol{A} \times \delta\boldsymbol{A} = \boldsymbol{A} \cdot \boldsymbol{B}\, \boldsymbol{\xi} - \boldsymbol{A} \cdot \boldsymbol{\xi}\, \boldsymbol{B}. \qquad (13.36)$$

Therefore, the surface terms in (13.32) and (13.33) vanish if $\boldsymbol{B}$ and $\boldsymbol{\xi}$ are parallel to the boundary ∂V, i.e., if the volume V encloses both $\boldsymbol{B}$ and the plasma—there are no displacements through the boundary.

This leaves us with

$$\delta \int_V \mathrm{d}^3\boldsymbol{r}\, (B^2 - \alpha\, \boldsymbol{A} \cdot \boldsymbol{B}) = 2 \int_V \mathrm{d}^3\boldsymbol{r}\, (\boldsymbol{\nabla} \times \boldsymbol{B} - \alpha\boldsymbol{B}) \cdot \delta\boldsymbol{A} = 0, \qquad (13.37)$$

which instantly implies that $\boldsymbol{B}$ *is a linear force-free field:*

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha\boldsymbol{B} \quad \Rightarrow \quad \boxed{\nabla^2\boldsymbol{B} = -\alpha^2\boldsymbol{B}}. \qquad (13.38)$$

Thus, our system will relax to a linear force-free state determined by (13.38) and system-specific boundary conditions. Here $\alpha = \alpha(H)$ depends on the (fixed by initial conditions) value of H via the equation

$$H(\alpha) = \int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{A} \cdot \boldsymbol{B} = \frac{1}{\alpha} \int \mathrm{d}^3\boldsymbol{r}\, B^2, \qquad (13.39)$$

where $\boldsymbol{B}$ is the solution of (13.38) (since $\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha\boldsymbol{B} = \alpha\boldsymbol{\nabla} \times \boldsymbol{A}$, we have $\boldsymbol{B} = \alpha\boldsymbol{A} + \boldsymbol{\nabla}\chi$ and the χ term vanishes under volume integration).

Thus, the prescription for finding force-free equilibria is

—solve (13.38), get $\boldsymbol{B} = \boldsymbol{B}(\alpha)$, parametrically dependent on α,
—calculate $H(\alpha)$ according to (13.39),
—set $H(\alpha) = H_0$, where H_0 is the initial value of helicity, hence calculate $\alpha = \alpha(H_0)$ and complete the solution by using this α in $\boldsymbol{B} = \boldsymbol{B}(\alpha)$.

Note that it is possible for this procedure to return multiple solutions. In that case, the
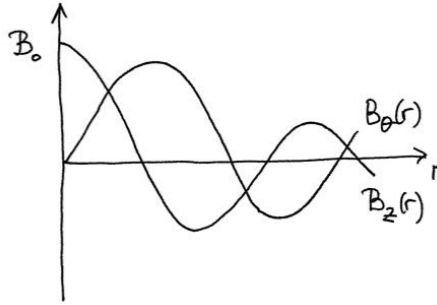
FIGURE 55. Relaxed cylindrical pinch.

solution with the smallest energy must be the right one (if a system relaxed to a local minimum, one can always imagine it being knocked out of it by some perturbation and falling to a lower energy).

Exercise 13.1. Force-free fields in 2D. Show that for MHD confined to the 2D plane (x, y), the quantity $\int d^2 r\, A_z^2$ is conserved. Work out the 2D version of J. B. Taylor relaxation and show that the resulting equilibrium field is a linear force-free field.

13.4. *Relaxed Force-Free State of a Cylindrical Pinch*

Let us illustrate how the procedure derived in §13.3 works by considering again the case of cylindrical and axial symmetry [see (13.5)]. The z component of (13.38) gives us the following equation for $B_z(r)$:

$$B_z'' + \frac{1}{r}\, B_z' + \alpha^2 B_z = 0. \tag{13.40}$$

This is a Bessel equation, whose solution, subject to $B_z(0) = B_0$ and $B_z(\infty) = 0$, is

$$\boxed{B_z(r) = B_0 J_0(\alpha r)}. \tag{13.41}$$

We can now calculate the azimuthal field as follows

$$\alpha B_\theta = (\boldsymbol{\nabla} \times \boldsymbol{B})_\theta = -B_z' \quad \Rightarrow \quad \boxed{B_\theta(r) = B_0 J_1(\alpha r)}. \tag{13.42}$$

This gives us an interesting twisted field (Fig. 55), able to maintain itself in equilibrium without help from pressure gradients.

Finally, we calculate its helicity according to (13.39): assuming that the length of the cylinder is L, its radius R and so its volume $V = \pi R^2 L$, we have

$$H = \frac{1}{\alpha} \int d^3 r\, B^2 = \frac{2\pi L B_0^2}{\alpha} \int_0^R dr\, r \left[J_0^2(\alpha r) + J_1^2(\alpha r) \right]$$
$$= \frac{B_0^2 V}{\alpha^2} \left[J_0^2(\alpha R) + 2 J_1^2(\alpha R) + J_2^2(\alpha R) - \frac{2}{\alpha R}\, J_1(\alpha R) J_2(\alpha R) \right]. \tag{13.43}$$

If we solve this for $\alpha = \alpha(H)$, our solution is complete.

Exercise 13.2. Work out what happens in the general case of $\partial/\partial\theta \neq 0$ and $\partial/\partial z \neq 0$ and whether the simple symmetric solution obtained above is the correct relaxed, minimum-energy state (not always, it turns out). This is not a trivial exercise. The solution is in Taylor & Newton (2015, §9), where you will also find much more on the subject of J. B. Taylor relaxation, relaxed states and much besides—all from the original source.

There are other useful variational principles—other in the sense that the constraints that are imposed are different from helicity conservation. The need for them arises when one considers magnetic equilibria in domains that do not completely enclose the field lines, i.e., when $\mathrm{d}\boldsymbol{S}\cdot\boldsymbol{B} \neq 0$ at the boundary. One example of such a variational principle, also yielding a force-free field (although not necessarily a linear one), is given in Q1(e). A specific example of such a field arises in Q8(f).

13.5. *Parker's Problem and Topological MHD*

Coming soon... On topology in MHD, a very mathematically minded student might enjoy the book by Arnold & Khesin (1999).

14. MHD Stability and Instabilities

We now wish to take a more general view of the MHD stability problem: given some static[81] equilibrium (some ρ_0, p_0, $\boldsymbol{B}_0$ and $\boldsymbol{u}_0 = 0$), will this equilibrium be stable to small perturbations of it, i.e., will these perturbations grow or decay?

There are two ways to answer this question:

1) Carry out the normal-mode analysis, i.e., linearise the MHD equations around the given equilibrium, just as we did when we studied MHD waves in §12.1, and see if any of the frequencies (solutions of the dispersion relation) turn out to be complex, with positive imaginary parts (growth rates). This approach has the advantage of being direct and also of yielding specific information about rates of growth or decay, the character of the growing and decaying modes, etc. However, for spatially complicated equilibria, this is often quite difficult to do and one might be willing to settle for less: just being able to prove that some configuration is stable or that certain types of perturbations might grow. Hence the the second approach:

2) Check whether, for a given equilibrium, all possible perturbations will lead to the energy of the system *increasing*. If so, then the equilibrium is stable—this is called *the energy principle* and we shall prove it shortly. If, on the other hand, certain perturbations lead to the energy *decreasing*, that equilibrium is unstable. The advantage of this second approach is that we do not need to solve the (linearised) MHD equations in order to pronounce on stability, just to examine the properties of the perturbed energy functional.

It should be already quite clear how to do the normal-mode analysis, at least conceptually, so I shall focus on the second approach.

14.1. *Energy Principle*

Recall what the total energy in MHD is (§11.11)

$$\mathcal{E} = \int \mathrm{d}^3\boldsymbol{r} \left(\frac{\rho u^2}{2} + \frac{B^2}{8\pi} + \frac{p}{\gamma - 1} \right) \equiv \int \mathrm{d}^3\boldsymbol{r}\, \frac{\rho u^2}{2} + W. \tag{14.1}$$

As we saw in §12.1, all perturbations of an MHD system away from equilibrium can be expressed in terms of small displacements $\boldsymbol{\xi}$—we will work this out shortly for a general

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[81] A treatment of the more general case of a dynamic equilibrium, $\boldsymbol{u}_0 \neq 0$, can be found in the excellent textbook by Davidson (2016).

equilibrium, but for now, let us accept that this will be true.[82] As $\boldsymbol{u} = \partial\boldsymbol{\xi}/\partial t$ by definition of $\boldsymbol{\xi}$, we have

$$\mathcal{E} = \int \mathrm{d}^3\boldsymbol{r} \, \frac{1}{2} \rho_0 \left|\frac{\partial\boldsymbol{\xi}}{\partial t}\right|^2 + W_0 + \delta W_1[\boldsymbol{\xi}] + \delta W_2[\boldsymbol{\xi},\boldsymbol{\xi}] + \dots, \qquad (14.2)$$

where we have kept terms up to second order in ξ and so W_0 is the equilibrium part of W (i.e., its value for $\boldsymbol{\xi} = 0$), $\delta W_1[\boldsymbol{\xi}]$ is linear in $\boldsymbol{\xi}$, $\delta W_2[\boldsymbol{\xi},\boldsymbol{\xi}]$ is bilinear (quadratic), etc. Energy must be conserved to all orders, so

$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \int \mathrm{d}^3\boldsymbol{r} \, \underbrace{\rho_0 \frac{\partial^2\boldsymbol{\xi}}{\partial t^2}}_{\equiv \, \boldsymbol{F}[\boldsymbol{\xi}]} \cdot \frac{\partial\boldsymbol{\xi}}{\partial t} + \delta W_1\left[\frac{\partial\boldsymbol{\xi}}{\partial t}\right] + \delta W_2\left[\frac{\partial\boldsymbol{\xi}}{\partial t},\boldsymbol{\xi}\right] + \delta W_2\left[\boldsymbol{\xi},\frac{\partial\boldsymbol{\xi}}{\partial t}\right] + \dots = 0. \quad (14.3)$$

This must be true at all times, including at $t = 0$, when $\boldsymbol{\xi}$ and $\partial\boldsymbol{\xi}/\partial t$ can be chosen independently (MHD equations are second-order in time if written in terms of displacements). Therefore, for arbitrary functions $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$,

$$\int \mathrm{d}^3\boldsymbol{r} \, \boldsymbol{\eta} \cdot \boldsymbol{F}[\boldsymbol{\xi}] + \delta W_1[\boldsymbol{\eta}] + \delta W_2[\boldsymbol{\eta},\boldsymbol{\xi}] + \delta W_2[\boldsymbol{\xi},\boldsymbol{\eta}] + \dots = 0. \qquad (14.4)$$

In the first order, this tells us that

$$\delta W_1[\boldsymbol{\eta}] = 0, \qquad (14.5)$$

which is good to know because it means that δW_1 disappears from (14.2) (there are no first-order energy perturbations). In the second order, we get

$$\int \mathrm{d}^3\boldsymbol{r} \, \boldsymbol{\eta} \cdot \boldsymbol{F}[\boldsymbol{\xi}] = -\delta W_2[\boldsymbol{\eta},\boldsymbol{\xi}] - \delta W_2[\boldsymbol{\xi},\boldsymbol{\eta}]. \qquad (14.6)$$

Let $\boldsymbol{\eta} = \boldsymbol{\xi}$. Then (14.6) implies

$$\boxed{\delta W_2[\boldsymbol{\xi},\boldsymbol{\xi}] = -\frac{1}{2}\int \mathrm{d}^3\boldsymbol{r} \, \boldsymbol{\xi} \cdot \boldsymbol{F}[\boldsymbol{\xi}]} \, . \qquad (14.7)$$

This is the part of the perturbed energy in (14.2) that can be both positive and negative. The *Energy Principle* is

$$\boxed{\delta W_2[\boldsymbol{\xi},\boldsymbol{\xi}] > 0 \text{ for any } \boldsymbol{\xi} \quad \Leftrightarrow \quad \text{equilibrium is stable}} \qquad (14.8)$$

(Bernstein *et al.* 1958). Before we are in a position to prove this, we must do some preparatory work.

14.1.1. *Properties of the Force Operator* $\boldsymbol{F}[\boldsymbol{\xi}]$

Since the right-hand side of (14.6) is symmetric with respect to swapping $\boldsymbol{\xi} \leftrightarrow \boldsymbol{\eta}$, so must be the left-hand side:

$$\int \mathrm{d}^3\boldsymbol{r} \, \boldsymbol{\eta} \cdot \boldsymbol{F}[\boldsymbol{\xi}] = \int \mathrm{d}^3\boldsymbol{r} \, \boldsymbol{\xi} \cdot \boldsymbol{F}[\boldsymbol{\eta}]. \qquad (14.9)$$

Therefore, *operator* $\boldsymbol{F}[\boldsymbol{\xi}]$ *is self-adjoint*. Since, by definition,

$$\boldsymbol{F}[\boldsymbol{\xi}] = \rho_0 \frac{\partial^2\boldsymbol{\xi}}{\partial t^2}, \qquad (14.10)$$

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[82] In fact, also the fully nonlinear dynamics can be completely expressed in terms of displacements if the MHD equations are written in Lagrangian coordinates (see §11.13).
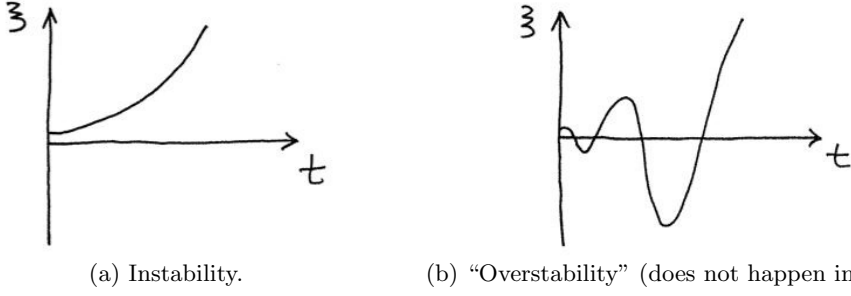
(a) Instability.   (b) "Overstability" (does not happen in MHD).

FIGURE 56. MHD instabilities.

the eigenmodes of this operator satisfy

$$\boldsymbol{\xi}(t, \boldsymbol{r}) = \boldsymbol{\xi}_n(\boldsymbol{r}) e^{-i\omega_n t} \quad \Rightarrow \quad \boldsymbol{F}[\boldsymbol{\xi}_n] = -\rho_0 \omega_n^2 \boldsymbol{\xi}_n. \tag{14.11}$$

As always for self-adjoint operators, we can prove a number of useful statements.

1) The eigenvalues $\{\omega_n^2\}$ are real.

Proof. If (14.11) holds, so must

$$\boldsymbol{F}[\boldsymbol{\xi}_n^*] = -\rho_0 (\omega_n^2)^* \boldsymbol{\xi}_n^*, \tag{14.12}$$

provided $\boldsymbol{F}$ has no complex coefficients (we shall confirm this explicitly in §14.2.1). Taking the full scalar products (including integarting over space) of (14.11) with $\boldsymbol{\xi}_n^*$ and of (14.12) with $\boldsymbol{\xi}_n$ and subtracting one from the other, we get

$$- \left[\omega_n^2 - (\omega_n^2)^* \right] \underbrace{\int \mathrm{d}^3\boldsymbol{r}\, \rho_0 |\boldsymbol{\xi}_n|^2}_{> 0} = \int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{\xi}_n^* \cdot \boldsymbol{F}[\boldsymbol{\xi}_n] - \int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{\xi}_n \cdot \boldsymbol{F}[\boldsymbol{\xi}_n^*] = 0$$

$$\Rightarrow \quad \boxed{\omega_n^2 = (\omega_n^2)^*}, \quad \text{q.e.d.} \tag{14.13}$$

This result implies that, if any MHD equilibrium is *unstable*, at least one of the eigenvalues must be $\omega_n^2 < 0$ and, since it is guaranteed to be real, any MHD instability will give rise to purely growing modes (Fig. 56a), rather than growing oscillations (also known as "overstabilities"; see Fig. 56b).

2) The eigenmodes $\{\boldsymbol{\xi}_n\}$ are orthogonal.

Proof. Taking the full scalar products of (14.11) with $\boldsymbol{\xi}_m$ (assuming $m \neq n$ and non-degeneracy of $\omega_{m,n}^2$), and of the analogous equation

$$\boldsymbol{F}[\boldsymbol{\xi}_m] = -\rho_0 \omega_m^2 \boldsymbol{\xi}_m \tag{14.14}$$

with $\boldsymbol{\xi}_n$ and subtracting them, we get[83]

$$- \underbrace{(\omega_n^2 - \omega_m^2)}_{\neq 0} \int \mathrm{d}^3\boldsymbol{r}\, \rho_0 \boldsymbol{\xi}_n \cdot \boldsymbol{\xi}_m = \int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{\xi}_m \cdot \boldsymbol{F}[\boldsymbol{\xi}_n] - \int \mathrm{d}^3\boldsymbol{r}\, \boldsymbol{\xi}_n \cdot \boldsymbol{F}[\boldsymbol{\xi}_m] = 0$$

$$\Rightarrow \quad \boxed{\int \mathrm{d}^3\boldsymbol{r}\, \rho_0 \boldsymbol{\xi}_n \cdot \boldsymbol{\xi}_m = \delta_{nm} \int \mathrm{d}^3\boldsymbol{r}\, \rho_0 |\boldsymbol{\xi}_n|^2}, \quad \text{q.e.d.}$$

$$\tag{14.15}$$

---

[83] Note that in view of (14.13), we can take $\{\boldsymbol{\xi}_n\}$ to be real.

14.1.2. *Proof of the Energy Principle (14.8)*

Let us assume completeness of the set of eigenmodes $\{\boldsymbol{\xi}_n\}$ (not, in fact, an indispensable assumption, but we shall not worry about this nuance here; see Kulsrud 2005, §7.2). Then any displacement at any given time t can be decomposed as

$$\boldsymbol{\xi}(t,\boldsymbol{r}) = \sum_n a_n(t)\boldsymbol{\xi}_n(\boldsymbol{r}). \tag{14.16}$$

The energy perturbation (14.7) is

$$\delta W_2[\boldsymbol{\xi},\boldsymbol{\xi}] = -\frac{1}{2}\int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{\xi}\cdot\boldsymbol{F}[\boldsymbol{\xi}] = -\frac{1}{2}\sum_{nm} a_n a_m \int \mathrm{d}^3\boldsymbol{r}\,\boldsymbol{\xi}_n\cdot\boldsymbol{F}[\boldsymbol{\xi}_m]$$

$$= \frac{1}{2}\sum_{nm} a_n a_m \omega_m^2 \underbrace{\int \mathrm{d}^3\boldsymbol{r}\,\rho_0\boldsymbol{\xi}_n\cdot\boldsymbol{\xi}_m}_{\text{use (14.15)}} = \frac{1}{2}\sum_n a_n^2\omega_n^2\int \mathrm{d}^3\boldsymbol{r}\,\rho_0|\boldsymbol{\xi}_n|^2. \tag{14.17}$$

By the same token,

$$K[\boldsymbol{\xi},\boldsymbol{\xi}] \equiv \frac{1}{2}\int \mathrm{d}^3\boldsymbol{r}\,\rho_0|\boldsymbol{\xi}|^2 = \frac{1}{2}\sum_n a_n^2\int \mathrm{d}^3\boldsymbol{r}\,\rho_0|\boldsymbol{\xi}_n|^2. \tag{14.18}$$

Then, if we arrange $\omega_1^2 \leqslant \omega_2^2 \leqslant \ldots$, the smallest eigenvalue is

$$\omega_1^2 = \min_{\boldsymbol{\xi}} \frac{\delta W_2[\boldsymbol{\xi},\boldsymbol{\xi}]}{K[\boldsymbol{\xi},\boldsymbol{\xi}]}. \tag{14.19}$$

Therefore,

• condition (14.8) is *sufficient* for stability because, if $\delta W_2[\boldsymbol{\xi},\boldsymbol{\xi}] > 0$ for all possible $\boldsymbol{\xi}$, then the smallest eigenvalue $\omega_1^2 > 0$, and so all eigenvalues are positive, $\omega_n^2 \geqslant \omega_1^2 > 0$;

• condition (14.8) is *necessary* for stability because, if the equilibrium is stable, then all eigenvalues are positive, $\omega_n^2 > 0$, whence $\delta W_2[\boldsymbol{\xi},\boldsymbol{\xi}] > 0$ in view of (14.17), q.e.d.

14.2. *Explicit Calculation of δW_2*

Now that we know that we need the sign of δW_2 to ascertain stability (or otherwise), it is worth working out δW_2 as an explicit function of $\boldsymbol{\xi}$. It is a second-order quantity, but (14.7) tells us that all we need to calculate is $\boldsymbol{F}[\boldsymbol{\xi}]$ to first order in $\boldsymbol{\xi}$, i.e., we just need to linearise the MHD equations around an arbitrary static equilibrium. The procedure is the same as in §12.1, but without assuming ρ_0, p_0 and $\boldsymbol{B}_0$ to be spatially homogeneous.

14.2.1. *Linearised MHD Equations*

Thus, generalising somewhat the procedure adopted in (12.3–12.5), we have

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0 \quad \Rightarrow \quad \frac{\partial \delta\rho}{\partial t} = -\boldsymbol{\nabla} \cdot \left(\rho_0 \frac{\partial \boldsymbol{\xi}}{\partial t} \right)$$

$$\Rightarrow \quad \boxed{\delta\rho = -\boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{\xi})}, \qquad (14.20)$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \right) p = -\gamma p \boldsymbol{\nabla} \cdot \boldsymbol{u} \quad \Rightarrow \quad \frac{\partial \delta p}{\partial t} = -\frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \boldsymbol{\nabla} p_0 - \gamma p_0 \boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{\xi}}{\partial t}$$

$$\Rightarrow \quad \boxed{\delta p = -\boldsymbol{\xi} \cdot \boldsymbol{\nabla} p_0 - \gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}}, \qquad (14.21)$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) \quad \Rightarrow \quad \frac{\partial \delta\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times \left(\frac{\partial \boldsymbol{\xi}}{\partial t} \times \boldsymbol{B}_0 \right)$$

$$\Rightarrow \quad \boxed{\delta\boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}_0)}. \qquad (14.22)$$

Note that again $\delta\rho$, δp and $\delta\boldsymbol{B}$ are all expressed as linear operators on $\boldsymbol{\xi}$—and so $\delta W = \delta \int \mathrm{d}^3\boldsymbol{r} \left[B^2/8\pi + p/(\gamma-1) \right]$ must also be some operator involving $\boldsymbol{\xi}$ and its gradients but not $\partial\boldsymbol{\xi}/\partial t$ (as we assumed in §14.1).

Finally, we deal with the momentum equation (to which we add gravity as this will give some interesting instabilities):

$$\rho \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} \right) = -\boldsymbol{\nabla} p + \frac{(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi} + \rho \boldsymbol{g}. \qquad (14.23)$$

This gives us

$$\boldsymbol{F}[\boldsymbol{\xi}] = \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\boldsymbol{\nabla} \delta p + \frac{(\boldsymbol{\nabla} \times \boldsymbol{B}_0) \times \delta\boldsymbol{B}}{4\pi} + \frac{(\boldsymbol{\nabla} \times \delta\boldsymbol{B}) \times \boldsymbol{B}_0}{4\pi} + \delta\rho\, \boldsymbol{g}$$

$$= \boldsymbol{\nabla} \left(\boldsymbol{\xi} \cdot \boldsymbol{\nabla} p_0 + \gamma p_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi} \right) - \boldsymbol{g} \boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{\xi}) + \frac{\boldsymbol{j}_0 \times \delta\boldsymbol{B}}{c} + \frac{(\boldsymbol{\nabla} \times \delta\boldsymbol{B}) \times \boldsymbol{B}_0}{4\pi}, \qquad (14.24)$$

where $\boldsymbol{j}_0 = c(\boldsymbol{\nabla} \times \boldsymbol{B}_0)/4\pi$, we have used (14.20) and (14.21) for $\delta\rho$ and δp, respectively, and $\delta\boldsymbol{B}$ is given by (14.22).

14.2.2. *Energy Perturbation*

Now we can use (14.24) in (14.7) to calculate explicitly

$$\delta W_2 = \frac{1}{2}\int \mathrm{d}^3 r \left[\underbrace{-\boldsymbol{\xi}\cdot\boldsymbol{\nabla}\left(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p_0 + \gamma p_0 \boldsymbol{\nabla}\cdot\boldsymbol{\xi}\right)}_{\substack{= (\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p_0)\,\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \gamma p_0(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 \\ \text{after integration by parts}}} + (\boldsymbol{g}\cdot\boldsymbol{\xi})\,\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{\xi}) \right.$$

$$\left. \underbrace{-\frac{(\boldsymbol{j}_0\times\delta\boldsymbol{B})\cdot\boldsymbol{\xi}}{c}}_{\substack{= \dfrac{\boldsymbol{j}_0\cdot(\boldsymbol{\xi}\times\delta\boldsymbol{B})}{c}}} \quad \underbrace{-\frac{(\boldsymbol{\nabla}\times\delta\boldsymbol{B})\times\boldsymbol{B}_0}{4\pi}\cdot\boldsymbol{\xi}}_{\substack{= \dfrac{(\boldsymbol{\nabla}\times\delta\boldsymbol{B})\cdot(\boldsymbol{\xi}\times\boldsymbol{B}_0)}{4\pi} \\ \text{by parts} \\ = \dfrac{(\delta\boldsymbol{B}\times\boldsymbol{\nabla})\cdot(\boldsymbol{\xi}\times\boldsymbol{B}_0)}{4\pi} \\ = \dfrac{\delta\boldsymbol{B}\cdot[\boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)]}{4\pi} \\ = \dfrac{|\delta\boldsymbol{B}|^2}{4\pi},\ \text{using (14.22)}}} \right]. \qquad (14.25)$$

Thus, we have arrived at a standard textbook (e.g., Kulsrud 2005) expression for the energy perturbation (this expression is non-unique because one can do various integrations by parts):

$$\delta W_2 = \frac{1}{2}\int \mathrm{d}^3 r \left[(\boldsymbol{\xi}\cdot\boldsymbol{\nabla}p_0)\,\boldsymbol{\nabla}\cdot\boldsymbol{\xi} + \gamma p_0(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})^2 + (\boldsymbol{g}\cdot\boldsymbol{\xi})\,\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{\xi}) \right.$$

$$\left. + \frac{\boldsymbol{j}_0\cdot(\boldsymbol{\xi}\times\delta\boldsymbol{B})}{c} + \frac{|\delta\boldsymbol{B}|^2}{4\pi} \right], \qquad (14.26)$$

where $\delta\boldsymbol{B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}_0)$. Note that two of the terms inside the integral (the second and the fifth) are positive-definite and so always stabilising. The terms that are not sign-definite and so potentially destabilising involve equilibrium gradients of pressure, density and magnetic field (currents). It is perhaps not a surprise to learn that Nature, with its fundamental yearning for thermal equilibrium, might dislike gradients—while it is of course not a rule that all such inhomogeneities render the system unstable, we will see that they often do, usually when gradients exceed certain critical thresholds.

All we need to do now is calculate δW_2 according to (14.26) for any equilibrium that interests us and see if it can be negative for any class of perturbations (or show that it is positive for all perturbations).

14.3. *Interchange Instabilities*

As the first and simplest example of how one does stability calculations using the Energy Principle, we will (perhaps disappointingly) consider a purely hydrodynamic situation: the stability of a simple hydrostatic equilibrium describing a generic *stratified atmosphere:*

$$\rho_0 = \rho_0(z) \quad \text{and} \quad p_0 = p_0(z) \quad \text{satisfying} \quad \boxed{\frac{\mathrm{d}p_0}{\mathrm{d}z} = -\rho_0 g} \qquad (14.27)$$

(gravity acts downward, against the z direction, $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$).

14.3.1. *Formal Derivation of the Schwarzschild Criterion*

With $\boldsymbol{B}_0 = 0$ and the hydrostatic equilibrium (14.27), (14.26) becomes

$$\delta W_2 = \frac{1}{2} \int \mathrm{d}^3 \boldsymbol{r} \left[\xi_z p_0' \boldsymbol{\nabla} \cdot \boldsymbol{\xi} + \gamma p_0 (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})^2 - g \xi_z (\rho_0' \xi_z + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{\xi}) \right]$$

$$= \frac{1}{2} \int \mathrm{d}^3 \boldsymbol{r} \left[2 p_0' \xi_z \boldsymbol{\nabla} \cdot \boldsymbol{\xi} + \gamma p_0 (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})^2 - \rho_0' g \xi_z^2 \right], \tag{14.28}$$

where we have used $\rho_0 g = -p_0'$. We see that δW_2 depends on ξ_z and $\boldsymbol{\nabla} \cdot \boldsymbol{\xi}$. Let us treat them as independent variables and minimise δW_2 with respect to them (i.e., *seek the most unstable possible situation*):

$$\frac{\partial}{\partial (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})} \left[\begin{array}{c} \text{integrand} \\ \text{of (14.28)} \end{array} \right] = 2 p_0' \xi_z + 2 \gamma p_0 (\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) = 0 \quad \Rightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{\xi} = -\frac{p_0'}{\gamma p_0} \xi_z. \tag{14.29}$$

Substituting this back into (14.28), we get

$$\delta W_2 = \frac{1}{2} \int \mathrm{d}^3 \boldsymbol{r} \left(-\frac{p_0'^2}{\gamma p_0} - \rho_0' g \right) \xi_z^2 = \frac{1}{2} \int \mathrm{d}^3 \boldsymbol{r} \frac{\rho_0 g}{\gamma} \underbrace{\left(\frac{p_0'}{p_0} - \gamma \frac{\rho_0'}{\rho_0} \right)}_{= \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{p_0}{\rho_0^\gamma}} \xi_z^2. \tag{14.30}$$

By the Energy Principle, the system is stable iff

$$\delta W_2 > 0 \quad \Leftrightarrow \quad \boxed{\frac{\mathrm{d} \ln s_0}{\mathrm{d}z} > 0}, \tag{14.31}$$

where $s_0 = p_0 / \rho_0^\gamma$ is the entropy function. The inequality (14.31) is the *Schwarzschild criterion* for convective stability.[84] If this criterion is broken, there will be an instability, called the *interchange instability.*

This calculation illustrates both the power and the weakness of the method:

—on the one hand, we have obtained a stability criterion quite quickly and without having to solve the underlying equations,

—on the other hand, while we have established the condition for instability, we have as yet absolutely no idea what is going on physically.

14.3.2. *Physical Picture*

We can remedy the latter problem by examining what type of displacements give rise to $\delta W_2 < 0$ when the Schwarzschild criterion is broken. Recalling (14.20) and (14.21) and specialising to the displacements given by (14.29) (as they are the ones that minimise δW_2), we get

$$\frac{\delta p}{p_0} = -\frac{\xi \cdot \boldsymbol{\nabla} p_0}{p_0} - \gamma \boldsymbol{\nabla} \cdot \boldsymbol{\xi} = -\frac{p_0'}{p_0} \xi_z - \gamma \boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0, \tag{14.32}$$

$$\frac{\delta \rho}{\rho_0} = -\frac{1}{\rho_0} \boldsymbol{\nabla} \cdot (\rho_0 \boldsymbol{\xi}) = -\frac{\rho_0'}{\rho_0} \xi_z - \boldsymbol{\nabla} \cdot \boldsymbol{\xi} = \frac{1}{\gamma} \left(-\gamma \frac{\rho_0'}{\rho_0} + \frac{p_0'}{p_0} \right) = \frac{1}{\gamma} \underbrace{\frac{\mathrm{d} \ln s_0}{\mathrm{d}z}}_{\substack{< 0 \\ (\text{unstable})}} \xi_z. \tag{14.33}$$

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[84] We studied perturbations of a stably stratified atmosphere in §12.2.8 and Q5, where we saw that these perturbations indeed did not grow provided the entropy scale length $1/H_s = \mathrm{d} \ln s_0 / \mathrm{d}z$ was positive.
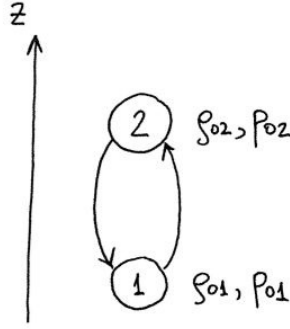
FIGURE 57. Interchange instability.

Thus, the offending perturbations maintain themselves in pressure balance (i.e., they are *not* sound waves) and locally increase or decrease density for blobs of fluid that fall ($\xi_z < 0$) or rise ($\xi_z > 0$), respectively.

This gives us some handle on the situation: if we imagine a blob of fluid slowly rising (slowly, so $\delta p = 0$) from the denser nether regions of the atmosphere to the less dense upper ones, then we can ask whether staying in pressure balance with its surroundings will require the blob to expand ($\delta \rho < 0$) or contract ($\delta \rho > 0$). If it is the latter, it will fall back down, pulled by gravity; if the former, then it will keep rising (buoyantly) and the system will be unstable. The direction of the entropy gradient determines which of these two scenarios is realised.

14.3.3. *Intuitive Rederivation of the Schwarzschild Criterion*

We can use this physical intuition to derive the Schwarzschild criterion directly. Consider two blobs, at two different vertical locations, lower (1) and upper (2), where the equilibrium densities and pressures are ρ_{01}, p_{01} and ρ_{02}, p_{02}. Now interchange these two blobs (Fig. 57). Inside the blobs, the new densities and pressures are ρ_1, p_1 and ρ_2, p_2.

Requiring the blobs to stay in pressure balance with their local surroundings gives

$$p_1 = p_{02}, \quad p_2 = p_{01}. \tag{14.34}$$

Requiring the blobs to rise or fall adiabatically, i.e., to satisfy $p/\rho^\gamma = \text{const}$, and then using pressure balance (14.34) gives

$$\frac{p_{01}}{\rho_{01}^\gamma} = \frac{p_1}{\rho_1^\gamma} = \frac{p_{02}}{\rho_1^\gamma} \quad \Rightarrow \quad \frac{\rho_1}{\rho_{01}} = \left(\frac{p_{02}}{p_{01}}\right)^{1/\gamma}. \tag{14.35}$$

Requiring that the buoyancy of the rising blob overcome gravity, i.e., that the weight of the displaced fluid be larger than the weight of the blob,

$$\rho_{02} g > \rho_1 g, \tag{14.36}$$

gives the condition for instability:

$$\rho_1 < \rho_{02} \quad \Leftrightarrow \quad \frac{\rho_1}{\rho_{02}} = \frac{\rho_{01}}{\rho_{02}} \left(\frac{p_{02}}{p_{01}}\right)^{1/\gamma} < 1 \quad \Leftrightarrow \quad \boxed{\frac{p_{02}}{\rho_{02}^\gamma} < \frac{p_{01}}{\rho_{01}^\gamma}}. \tag{14.37}$$

This is exactly the same as the Schwarzschild condition (14.31) for the interchange instability (and this is why the instability is called that).

Note that, while this is of course a much simpler and more intuitive agrument than the application of the Energy Principle, it only gives us a particular example of the

kind of perturbation that would be *unstable* under particular conditions, not any general criterion of what equilibria might be guaranteed to be stable.

In Q9, we will explore how the above considerations can be generalised to an equilibrium that also features a non-zero magnetic field.

14.4. *Instabilities of a Pinch*

As our second (also classic) example, we consider the stability of a z-pinch equilibrium (§13.1.1, Fig. 51):

$$\boldsymbol{B}_0 = B_0(r)\hat{\boldsymbol{\theta}}, \quad \boldsymbol{j}_0 = j_0(r)\hat{\boldsymbol{z}} = \frac{c}{4\pi r}(rB_0)'\hat{\boldsymbol{z}}, \quad p_0'(r) = -\frac{1}{c}j_0B_0 = -\frac{B_0(rB_0)'}{4\pi r}. \quad (14.38)$$

Since we are going to have to work in cylindrical coordinates, we must first write all the terms in (14.26) in these coordinates and with the equilibrium (14.38):

$$(\boldsymbol{\xi} \cdot \boldsymbol{\nabla} p_0)(\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) = \xi_r p_0' \left(\frac{1}{r}\frac{\partial}{\partial r} r\xi_r + \cancel{\frac{1}{r}\frac{\partial \xi_\theta}{\partial \theta}} + \frac{\partial \xi_z}{\partial z} \right)$$
$$= p_0'\frac{\xi_r^2}{r} + p_0'\xi_r \left(\frac{\partial \xi_r}{\partial r} + \frac{\partial \xi_z}{\partial z} \right), \quad (14.39)$$

$$\gamma p_0(\boldsymbol{\nabla} \cdot \boldsymbol{\xi})^2 = \gamma p_0 \left(\frac{1}{r}\frac{\partial}{\partial r} r\xi_r + \frac{1}{r}\frac{\partial \xi_\theta}{\partial \theta} + \frac{\partial \xi_z}{\partial z} \right)^2, \quad (14.40)$$

$$\delta\boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}_0) = \hat{\boldsymbol{r}} \left(\frac{1}{r}\frac{\partial}{\partial \theta}\xi_r B_0 \right) + \hat{\boldsymbol{\theta}} \left(-\frac{\partial}{\partial z}\xi_z B_0 - \frac{\partial}{\partial r}\xi_r B_0 \right) + \hat{\boldsymbol{z}} \left(\frac{1}{r}\frac{\partial}{\partial \theta}\xi_z B_0 \right),$$
$$(14.41)$$

$$\frac{\boldsymbol{j}_0 \cdot (\boldsymbol{\xi} \times \delta\boldsymbol{B})}{c} = \underbrace{\frac{j_0}{c}}_{= -\frac{p_0'}{B_0}} (\xi_r \delta B_\theta - \xi_\theta \delta B_r) = p_0' \left[\xi_r \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} + \xi_r \frac{B_0'}{B_0} \right) + \cancel{\xi_\theta \frac{1}{r}\frac{\partial \xi_r}{\partial \theta}} \right],$$
$$= p_0'\xi_r \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right) + \frac{p_0'B_0'}{B_0}\xi_r^2, \quad (14.42)$$

$$\frac{|\delta\boldsymbol{B}|^2}{4\pi} = \frac{B_0^2}{4\pi r^2} \left[\left(\frac{\partial \xi_r}{\partial \theta} \right)^2 + \left(\frac{\partial \xi_z}{\partial \theta} \right)^2 \right] + \underbrace{\frac{B_0^2}{4\pi} \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} + \xi_r \frac{B_0'}{B_0} \right)^2}_{\begin{subarray}{l} = \frac{B_0^2}{4\pi} \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right)^2 + \\ \frac{2B_0B_0'}{4\pi}\xi_r \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right) + \frac{B_0'^2}{4\pi}\xi_r^2 \end{subarray}}. \quad (14.43)$$

The terms that are crossed out have been dropped because they combine into a full derivative with respect to θ and so, upon substitution into (14.26), vanish under integra-

tion. Assembling all this together, we have

$$\delta W_2 = \frac{1}{2} \int \mathrm{d}^3 \boldsymbol{r} \left\{ \underbrace{\left(p_0' + \frac{p_0' r B_0'}{B_0} + \frac{r B_0'^2}{4\pi} \right)}_{= 2p_0' + \frac{B_0^2}{4\pi r}} \frac{\xi_r^2}{r} + 2 \underbrace{\left(p_0' + \frac{B_0 B_0'}{4\pi} \right)}_{= -\frac{B_0^2}{4\pi r}} \xi_r \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right) \right.$$

$$\left. + \gamma p_0 (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})^2 + \frac{B_0^2}{4\pi r^2} \left[\left(\frac{\partial \xi_r}{\partial \theta} \right)^2 + \left(\frac{\partial \xi_z}{\partial \theta} \right)^2 \right] + \frac{B_0^2}{4\pi} \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} \right)^2 \right\}$$

$$= \frac{1}{2} \int \mathrm{d}^3 \boldsymbol{r} \left\{ 2p_0' \frac{\xi_r^2}{r} + \frac{B_0^2}{4\pi} \left(\frac{\partial \xi_z}{\partial z} + \frac{\partial \xi_r}{\partial r} - \frac{\xi_r}{r} \right)^2 \right.$$

$$\left. + \gamma p_0 (\boldsymbol{\nabla} \cdot \boldsymbol{\xi})^2 + \frac{B_0^2}{4\pi r^2} \left[\left(\frac{\partial \xi_r}{\partial \theta} \right)^2 + \left(\frac{\partial \xi_z}{\partial \theta} \right)^2 \right] \right\}, \tag{14.44}$$

where, in simplifying the first two terms in the integrand, we used the equilibrium equation (14.38):

$$p_0' = -\frac{B_0^2}{4\pi r} - \frac{B_0 B_0'}{4\pi} \quad \Rightarrow \quad \frac{r B_0'^2}{4\pi} = -\frac{p_0' r B_0'}{B_0} - \frac{B_0 B_0'}{4\pi} = -\frac{p_0' r B_0'}{B_0} + p_0' + \frac{B_0^2}{4\pi r}. \tag{14.45}$$

Finally, after a little further tiding up,

$$\delta W_2 = \frac{1}{2} \int \mathrm{d}^3 \boldsymbol{r} \left\{ 2p_0' \frac{\xi_r^2}{r} + \frac{B_0^2}{4\pi} \left(r \frac{\partial}{\partial r} \frac{\xi_r}{r} + \frac{\partial \xi_z}{\partial z} \right)^2 + \gamma p_0 \left(\frac{1}{r} \frac{\partial}{\partial r} r \xi_r + \frac{1}{r} \frac{\partial \xi_\theta}{\partial \theta} + \frac{\partial \xi_z}{\partial z} \right)^2 \right.$$

$$\left. + \frac{B_0^2}{4\pi r^2} \left[\left(\frac{\partial \xi_r}{\partial \theta} \right)^2 + \left(\frac{\partial \xi_z}{\partial \theta} \right)^2 \right] \right\}. \tag{14.46}$$

14.4.1. *Sausage Instability*

Let us first consider axisymmetric perturbations: $\partial/\partial\theta = 0$. Then δW_2 depends on two variables only:

$$\xi_r \quad \text{and} \quad \eta \equiv \frac{\partial \xi_r}{\partial r} + \frac{\partial \xi_z}{\partial z}. \tag{14.47}$$

Indeed, unpacking all the r derivatives in (14.46), we get

$$\boxed{\delta W_2 = \frac{1}{2} \int \mathrm{d}^3 \boldsymbol{r} \left[2p_0' \frac{\xi_r^2}{r} + \frac{B_0^2}{4\pi} \left(\eta - \frac{\xi_r}{r} \right)^2 + \gamma p_0 \left(\eta + \frac{\xi_r}{r} \right)^2 \right].} \tag{14.48}$$

We shall treat ξ_r and η as independent variables and minimise δW_2 with respect to η:

$$\frac{\partial}{\partial \eta} \left[\begin{array}{c} \text{integrand} \\ \text{of (14.48)} \end{array} \right] = 2 \frac{B_0^2}{4\pi} \left(\eta - \frac{\xi_r}{r} \right) + 2\gamma p_0 \left(\eta + \frac{\xi_r}{r} \right) = 0 \quad \Rightarrow \quad \eta = \frac{1 - \gamma\beta/2}{1 + \gamma\beta/2} \frac{\xi_r}{r}, \tag{14.49}$$

where, as usual, $\beta = 8\pi p_0/B_0^2$. Putting this back into (14.48), we get

$$\delta W_2 = \int \mathrm{d}^3 \boldsymbol{r} \, p_0 \left[\frac{r p_0'}{p_0} + \frac{1}{\beta} \left(\frac{\gamma\beta}{1 + \gamma\beta/2} \right)^2 + \frac{\gamma}{2} \left(\frac{2}{1 + \gamma\beta/2} \right)^2 \right] \frac{\xi_r^2}{r^2}$$

$$= \int \mathrm{d}^3 \boldsymbol{r} \, p_0 \left(r \frac{\mathrm{d}\ln p_0}{\mathrm{d}r} + \frac{2\gamma}{1 + \gamma\beta/2} \right) \frac{\xi_r^2}{r^2}. \tag{14.50}$$
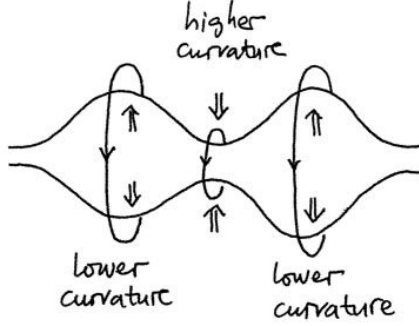
FIGURE 58. Sausage instability.

There will be an instability ($\delta W_2 < 0$) if (but not only if, because we are considering the restricted set of axisymmetric displacements)

$$\boxed{-r\,\frac{\mathrm{d}\ln p_0}{\mathrm{d}r} > \frac{2\gamma}{1+\gamma\beta/2}}, \tag{14.51}$$

i.e., when the pressure gradient is too steep, the equilibrium is unstable.

What sort of instability is this? Recall that the perturbations that we have identified as making $\delta W_2 < 0$ are axisymmetric, have some radial and axial displacements and are *compressible:* from (14.49),

$$\boldsymbol{\nabla} \cdot \boldsymbol{\xi} = \eta + \frac{\xi_r}{r} = \frac{2}{1+\gamma\beta/2}\frac{\xi_r}{r}. \tag{14.52}$$

They are illustrated in Fig. 58. The mechanism of this aptly named *sausage instability* is clear: squeezing the flux surfaces inwards increases the curvature of the azimuthal field lines, this exerts stronger curvature force, leading to further squeezing; conversely, expanding outwards weakens curvature and the plasma can expand further.

Exercise 14.1. Convince yourself that the displacements that have been identified cause magnetic perturbations that are consistent with the cartoon in Fig. 58.

14.4.2. *Kink Instability*

Now consider non-axisymmetric perturbations ($\partial/\partial\theta \neq 0$) to see what other instabilities might be there. First of all, since we now have θ variation, δW_2 depends on ξ_θ. However, in (14.46), ξ_θ only appears in the third term, where it is part of $\boldsymbol{\nabla} \cdot \boldsymbol{\xi}$, which enters quadratically and with a positive coefficient γp_0. We can treat $\boldsymbol{\nabla} \cdot \boldsymbol{\xi}$ as an independent variable, alongside ξ_r and ξ_z, and minimise δW_2 with respect to it. Obviously, the energy perturbation is minimal when

$$\boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0, \tag{14.53}$$

i.e., the most dangerous non-axisymmetric perturbations are incompressible (unlike for the case of the axisymmetric sausage mode in §14.4.1: there we could not—and did not—have such incompressible perturbations because we did not have ξ_θ at our disposal, to be chosen in such a way as to enforce incompressibility).

To carry out further minimisation of δW_2, it is convenient to Fourier transform our displacements in the θ and z directions—both are directions of symmetry (i.e., the

equilibrium profiles do not vary in these directions), so this can be done with impunity:

$$\boldsymbol{\xi} = \sum_{m,k} \boldsymbol{\xi}_{mk}(r)\, e^{i(m\theta + kz)}. \tag{14.54}$$

Then (14.46) (with $\boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0$) becomes, by Parseval's theorem (the operator $\boldsymbol{F}[\boldsymbol{\xi}]$ being self-adjoint; see §14.1.1),

$$\delta W_2 = \frac{1}{2} \sum_{m,k} 2\pi L_z \int_0^\infty \mathrm{d}r\, r \left\{ 2p_0' \frac{|\xi_r|^2}{r} + \frac{B_0^2}{4\pi} \left[\left| r \frac{\partial}{\partial r} \frac{\xi_r}{r} + ik\xi_z \right|^2 + \frac{m^2}{r^2} \left(|\xi_r|^2 + |\xi_z|^2 \right) \right] \right\}. \tag{14.55}$$

As ξ_z and ξ_z^* only appear algebraically in (14.55) (no r derivatives), it is easy to minimise δW_2 with respect to them: setting the derivative of the integrand with respect to either ξ_z or ξ_z^* to zero, we get

$$-ik \left(r \frac{\partial}{\partial r} \frac{\xi_r}{r} + ik\xi_z \right) + \frac{m^2}{r^2} \xi_z = 0 \quad \Rightarrow \quad \xi_z = \frac{ikr^3}{m^2 + k^2 r^2} \frac{\partial}{\partial r} \frac{\xi_r}{r}. \tag{14.56}$$

Putting this back into (14.55) and assembling terms, we get

$$\delta W_2 = \sum_{m,k} \pi L_z \int_0^\infty \mathrm{d}r\, r \left\{ 2p_0 \left(\frac{rp_0'}{p_0} + \frac{m^2}{\beta} \right) \frac{|\xi_r|^2}{r^2} \right. $$
$$\left. + \frac{B_0^2}{4\pi} \underbrace{\left[\left(1 - \frac{k^2 r^2}{m^2 + k^2 r^2} \right)^2 + \frac{m^2 k^2 r^2}{(m^2 + k^2 r^2)^2} \right]}_{= \frac{m^2}{m^2 + k^2 r^2}} \left| r \frac{\partial}{\partial r} \frac{\xi_r}{r} \right|^2 \right\}. \tag{14.57}$$

The second term here is always stabilising. The most unstable modes will be ones with $k \to \infty$, for which the stabilising term is as small as possible. The remaining term will allow $\delta W_2 < 0$ and, therefore, an instability, if

$$\boxed{-r \frac{\mathrm{d} \ln p_0}{\mathrm{d}r} > \frac{m^2}{\beta}}. \tag{14.58}$$

Again, the equilibrium is unstable if the pressure gradient is too steep. The most unstable modes are ones with the smallest m, viz., $m = 1$.

Note that another way of writing the instability condition (14.58) is

$$- rp_0' = \frac{B_0^2}{4\pi} + \frac{rB_0 B_0'}{4\pi} > m^2 \frac{B_0^2}{8\pi} \quad \Rightarrow \quad \boxed{r \frac{\mathrm{d} \ln B_0}{\mathrm{d}r} > \frac{m^2}{2} - 1}, \tag{14.59}$$

where we have used the equilibrium equation (14.38).

What does this instability look like? The unstable perturbations are incompressible:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\xi} = 0 \quad \Rightarrow \quad \frac{1}{r} \frac{\partial}{\partial r} r\xi_r + \frac{im}{r} \xi_\theta + ik\xi_z = 0. \tag{14.60}$$
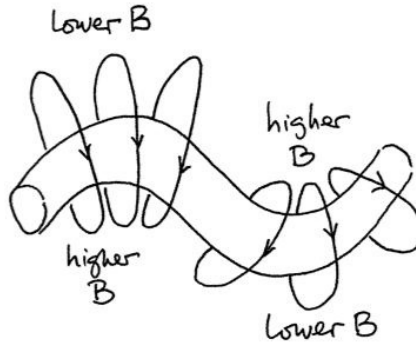
FIGURE 59. Kink instability.

Setting $m = 1$ and using (14.56), we find

$$i\xi_\theta = -\frac{\partial}{\partial r} r\xi_r + \underbrace{\frac{k^2 r^4}{m^2 + k^2 r^2}}_{\substack{\approx r^2 \\ \text{as } k \to \infty}} \frac{\partial}{\partial r} \frac{\xi_r}{r} \approx -2\xi_r \quad \text{and} \quad \xi_z \ll \xi_r. \tag{14.61}$$

The basic cartoon (Fig. 59) is as follows: the flux surfaces are bent, with a twist (to remain uncompressed). The bending pushes the magnetic loops closer together and thus increases magnetic pressure in concave parts and, conversely, decreases it in the convex ones. Plasma is pushed from the areas of higher B to those with lower B, thermal pressure in the latter (convex) areas becomes uncompensated, the field lines open up further, etc. This is called the *kink instability.*

Similar methodology can be used to show that, unlike the z pinch, the θ pinch (§13.1.1, Fig. 52) is always stable: see Q10.

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15. Further Reading

What follows is *not* a literature survey, but rather just a few pointers for the keen and the curious.

15.1. *MHD Instabilities*

There are very many of these, easily a whole course's worth. They are an interesting topic. A founding text is the old, classic, super-meticulous monograph by Chandrasekhar (2003). In the context of **toroidal (fusion) plasmas**, you want to learn the so-called **ballooning theory**, a tour de force of theoretical plasma physics, which, like the relaxation theory, is associated with J. B. Taylor's name (so his lectures, Taylor & Newton 2015, are a good starting point; the original paper on the subject is Connor *et al.* 1979). In the unlikely event that you have an appetite for more **energy-principle calculations** in the style of §14.4, the book by Freidberg (2014) will teach you more than you ever wanted to know. In **astrophysics**, MHD instabilities have been a hot topic since the early 1990s, not least due the realisation by Balbus & Hawley (1991) that the **magnetorotational instability (MRI)** is responsible for triggering turbulence and, therefore, maintaining momentum transport in accretion flows—so the lecture notes by Balbus (2015) are an excellent place to start learning about this subject (this is also an opportunity to learn

how to handle equilibria that are *not static*,e.g., most interestingly, featuring **rotating and shear flows**).[85]

As with everything in physics, the frontier in this subject is nonlinear phenomena. One very attractive theoretical topic has been the theory of **explosive instabilities and erupting flux tubes** by S. C. Cowley and his co-workers: the founding (quite pedagogically written) paper was Cowley & Artun (1997), the key recent one is Cowley *et al.* (2015); follow the paper trail from there for various refinements and applications (from space to tokamaks).

15.2. *Resistive MHD*

Most of our discussion revolved around properties of ideal MHD equations. It is, in fact, quite essential to study resistive effects, even when resistivity is very small, because many ideal solutions have a natural tendency to develop ever smaller spatial gradients, which can only be regularised by resistivity (we touched on this, e.g., in §13.2.2). The key linear result here is the **tearing mode**, a resistive instability associated with the propensity of magnetic-field lines to reconnect—change their topology in such a way as to release some of their energy. This is covered in the lectures by Parra (2019*a*); other good places to read about it are Taylor & Newton (2015) again, the original paper by Furth *et al.* (1963), or standard textbooks (e.g., Sturrock 1994, §17).

Here again the frontier is nonlinear: the theory of **magnetic reconnection**: tearing modes, in their nonlinear stage, tend to lead to formation of *current sheets* (which is, in fact, a general tendency of X-point solutions in MHD), and how reconnection happens after that has been a subject of active research since mid-20th century. Magnetic reconnection is believed to be a key player in a host of plasma phenomena, from solar flares to the so-called "sawtooth crash" in tokamaks, to MHD turbulence. Kulsrud (2005, §14) has a good introduction to the history and the basics of the subject from a live witness and key contributor. There has been much going on in it in the last decade, many of the advances occurring on the collisionless reconnection front requiring kinetic theory (some key names to search for in the extensive recent literature are W. Daughton, J. Drake, J. Egedal), but even within MHD, the discovery of the **plasmoid instability** (amounting to the realisation that current sheets are tearing unstable; see Loureiro *et al.* 2007) has led to a new theory of resistive MHD reconnection (Uzdensky *et al.* 2010), a development that I (obviously) find important.

Even more recently, magnetic reconnection became intimately intertwined with the theory of MHD turbulence (§12.4)—you will find an account of this in my (hopefully pedagogical) review, a draft of which is here: `http://www-thphys.physics.ox.ac.uk/research/plasma/JPP/papers17/schekochihin2a.pdf`. Appendix C of this document also contains a "reconnection primer" covering tearing modes, current sheets and related topics in the most straightforward non-rigorous way that I could manage.

15.3. *Dynamo Theory and MHD Turbulence*

These are topics of active research, which one can have full access to with the education provided by these notes, and indeed it is to an extent with these topics in mind (or, at any rate, on my mind) that some of these notes were written. Hence §§11.10 and 12.4, where further pointers are provided.

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[85]Another excellent set of lecture notes on **astrophysical fluid dynamics** is Ogilvie (2016), this one originating from Cambridge Part III.

15.4. *Hall MHD, Electron MHD, Braginskii MHD*

These and other "two-fluid" approximations of plasma dynamics have to do with with (i) what happens at scales where different species (ions and electrons) cannot be considered to move together (**Hall/Electron MHD**; see, e.g., Q6) and (ii) how momentum transport (viscosity) and energy transport (heat conduction) operate in a *magnetised* plasma, i.e., a plasma where the Larmor motion of particles dominates over their Coulomb collisions, even though the latter might be faster than the fluid motions (**Braginskii 1965 MHD**). In general, this is a kinetic subject, although certain limits can be treated by fluid approximations. An introduction to these topics is given in Parra (2019*b*) and Parra (2019*a*) (see also Goedbloed & Poedts 2004, §3 and the excellent monograph by Helander & Sigmar 2005).

15.5. *Double-Adiabatic MHD and Onwards to Kinetics*

A conceptually interesting and important paradigm is the so-called **double-adiabatic MHD** (or **CGL equations**, after the orginal authors Chew *et al.* 1956; see also Kulsrud 1983). This deals with a situation in a magnetised plasma (in the sense defined in §15.4) when pressure becomes anisotropic, with pressures perpendicular and parallel to the local direction of the magnetic field evolving each according to its own, separate equation, replacing the adiabatic law (11.60) and based on the conservation of the adiabatic invariants of the Larmor-gyrating particles. The **dynamics of pressure-anisotropic plasma**, based on CGL equations or, which is usually more correct physically, on the full kinetic description (and its reduced versions, e.g., **Kinetic MHD**; see Parra 2019*b*, also Kulsrud 1983), are another current frontier, with applications to weakly collisional astrophysical plasmas (from interplanetary to intergalactic). A key feature that makes this topic both interesting and difficult is that pressure anisotropies in high-β plasmas trigger small-scale instabilities (in particular, the Alfvén wave becomes unstable—the so-called **firehose instability**), which break the fluid approximation and leave us without a good mean-field theory for the description of macroscopic motions in such environments (for a short introduction to these issues, see Schekochihin *et al.* 2010, although this subject is developing so fast that anything written 10 years ago is at least partially obsolete; you can read Squire *et al.* 2017 for a taste of how hairy things become in what concerns even such staples as Alfvén waves).

Magnetohydrodynamics Problem Set

1. Clebsch Coordinates. As $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, it is always possible to find two scalar functions $\alpha(\boldsymbol{r})$ and $\beta(\boldsymbol{r})$ such that

$$\boldsymbol{B} = \boldsymbol{\nabla}\alpha \times \boldsymbol{\nabla}\beta. \qquad (15.1)$$

(a) Argue that any magnetic field line can be described by the equations

$$\alpha = \text{const}, \quad \beta = \text{const}. \qquad (15.2)$$

This means that (α, β, ℓ), where ℓ is the distance (arc length) along the field line, are a good set of curvilinear coordinates, known as the *Clebsch coordinates*.

(b) Show that the magnetic flux through any area S in the (x, y) plane is

$$\Phi = \int_{\tilde{S}} \mathrm{d}\alpha\, \mathrm{d}\beta, \qquad (15.3)$$

where $\tilde{S}$ is the area S in new coordinates after transforming $(x, y) \to (\alpha(x, y, 0), \beta(x, y, 0))$.

(c) Show that if (15.1) holds at time $t = 0$ and α and β are evolved in time according to

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\beta}{\mathrm{d}t} = 0, \qquad (15.4)$$

where $\mathrm{d}/\mathrm{d}t$ is the convective derivative, then (15.1) correctly describes the magnetic field at all $t > 0$.

(d) Argue from the above that magnetic flux is frozen into the flow and magnetic field lines move with the flow.

(e) Show that the field that minimises the magnetic energy within some domain subject to the constraint that the values of α and β are fixed at the boundary of this domain (i.e., that the "footpoints" of the field lines are fixed) is a force-free field.[86]

A prototypical example of the kind of fields that arise from the variational principle in (e) is the "arcade" fields describing magnetic loops sticking out of the Sun's surface, with footpoints anchored at the surface. One such field will be considered in Q8(f) and more can be found in Sturrock (1994, §13).

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2. Uniform Collapse. A simple model of star formation envisions a sphere of galactic plasma with number density $n_{\text{gal}} = 1 \text{ cm}^{-3}$ undergoing a gravitational collapse to a spherical star with number density $n_{\text{star}} = 10^{26} \text{ cm}^{-3}$. The magnetic field in the galactic plasma is $B_{\text{gal}} \sim 3 \times 10^{-6}$ G. Assuming that flux is frozen, estimate the magnetic field in a star. Find out if this is a good estimate. If not, how, in your view, could we account for the discrepancy?

3. Flux Concentration. Consider a simple 2D model of incompressible convective motion (Fig. 60):

$$\boldsymbol{u} = U \left(-\sin\frac{\pi x}{L} \cos\frac{\pi z}{L}, \ 0, \ \cos\frac{\pi x}{L} \sin\frac{\pi z}{L} \right). \qquad (15.5)$$

(a) In the neighbourhood of the stagnation point $(0, 0, 0)$, linearise the flow, assume vertical magnetic field, $\boldsymbol{B} = (0, 0, B(t, x))$ and derive an evolution equation for $B(t, x)$,

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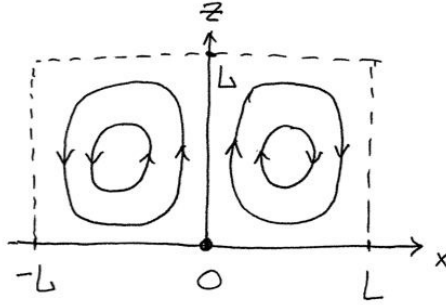[86]This is based on the 2017 exam question.

FIGURE 60. Convective cells from Q3.

including both advection by the flow and Ohmic diffusion. Suppose the field is initially uniform, $B(t = 0, x) = B_0 = $ const. It should be clear to you from your equation that magnetic field is being swept towards $x = 0$. What is the time scale of this sweeping? Given the magnetic Reynolds number $\mathrm{Rm} = UL/\eta \gg 1$, show that flux conservation holds on this time scale.

(b) Find the steady-state solution of your equation. Assume $B(x) = B(-x)$ and use flux conservation to determine the constants of integration (in terms of B_0 and Rm). What is the width of the region around $x = 0$ where the flux is concentrated? What is the magnitude of the field there?

(c*) Obtain the time-dependent solution of your equation for B and confirm that it indeed converges to your steady-state solution. Find the time scale on which this happens.

Hint. The following changes of variables may prove useful: $\xi = \sqrt{\pi \mathrm{Rm}}\, x/L$, $\tau = \pi U t/L$, $X = \xi e^\tau$, $s = (e^{2\tau} - 1)/2$.

(d) Can you think of a quick heuristic argument based on the induction equation that would tell you that all these answers were to be expected?

4. Zeldovich's Antidynamo Theorem. Consider an arbitrary 2D velocity field: $\boldsymbol{u} = (u_x, u_y, 0)$. Assume incompressibility. Show that, in a finite system (i.e., in a system that can be enclosed within some volume outside which there are no fields or flows), this velocity field cannot be a dynamo, i.e., any initial magnetic field will always eventually decay.

Hint. Consider separately the evolution equations for B_z and for the magnetic field in the (x, y)-plane. Show that B_z decays by working out the time evolution of the volume integral of B_z^2. Then write B_x, B_y in terms of one scalar function (which must be possible because $\partial B_x/\partial x + \partial B_y/\partial y = 0$) and show that it decays as well.

5. MHD Waves in a Stratified Atmosphere. The generalisation of iMHD to the case of a stratified atmosphere is explained in §12.2.8. Convince yourself that you understand how the SMHD equations and the SMHD ordering arise and then study them as follows.

(a) Work out all SMHD waves (both their frequencies and the corresponding eigenvectors). It is convenient to choose the coordinate system in such a way that $\boldsymbol{k} = (k_x, 0, k_z)$, where z is the vertical direction (the direction of gravity). The mean magnetic field $\boldsymbol{B}_0 = B_0 \boldsymbol{b}_0$ is assumed to be straight and uniform, at a general angle to z. We continue referring to the projection of the wave number onto the magnetic-field direction as $k_\parallel = \boldsymbol{k} \cdot \boldsymbol{b}_0 = k_x b_{0x} + k_z b_{0z}$. Note that in the case of $B_0 = 0$, you are dealing with

stratified hydrodynamics, not MHD—the waves that you obtain in this case are the well known *gravity waves*, or *"g-modes"*.

(b) Explain the physical nature of the perturbations (what makes the fluid oscillate) in the special cases (i) $k_z = 0$ and $\boldsymbol{b}_0 = \hat{\boldsymbol{z}}$, (ii) $k_z = 0$ and $\boldsymbol{b}_0 = \hat{\boldsymbol{x}}$, (iii) $k_x = 0$, (iv) $k_z \neq 0$, $k_x \neq 0$ and $\boldsymbol{b}_0 = \hat{\boldsymbol{z}}$.

(c) Under what conditions are the perturbations you have found unstable? What is the physical mechanism for the instability? What role does the magnetic field play (stabilising or destabilising) and why? Cross-check your answers with §14.3 and Q9.

(d) Find the conserved energy (a quadratic quantity whose integral over space stays constant) for the full nonlinear SMHD equations (12.93–12.96). Give a physical interpretation of the quantity that you have obtained—why should it be conserved?

Do either Q6 or Q7.

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6. Electron MHD. In certain physical regimes (roughly realised, for example, in the solar-wind and other kinds of astrophysical turbulence at scales smaller than the ion Larmor radius; see Schekochihin *et al.* 2009, Boldyrev *et al.* 2013), plasma turbulence can be described by an approximation in which the magnetic field is frozen into the electron flow $\boldsymbol{u}_e$, while ions are considered motionless, $\boldsymbol{u}_i = 0$. In this approximation, Ohm's law becomes[87]

$$\boldsymbol{E} = -\frac{\boldsymbol{u}_e \times \boldsymbol{B}}{c}. \tag{15.6}$$

Here $\boldsymbol{u}_e$ can be expressed directly in terms of $\boldsymbol{B}$ because the current density in a plasma consisting of motionless hydrogen ions ($n_i = n_e$) and moving electrons is

$$\boldsymbol{j} = en_e(\boldsymbol{u}_i - \boldsymbol{u}_e) = -en_e\boldsymbol{u}_e, \tag{15.7}$$

but, on the other hand, $\boldsymbol{j}$ is known via Ampère's law. Here n_e is the electron number density and e the electron charge.

(a) Using this and Faraday's law, show that the evolution equation for the magnetic field in this approximation is

$$\boxed{\frac{\partial \boldsymbol{B}}{\partial t} = -d_i \boldsymbol{\nabla} \times [(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}]}, \tag{15.8}$$

where the magnetic field has been rescaled to Alfvénic velocity units, $\boldsymbol{B}/\sqrt{4\pi m_i n_i} \rightarrow \boldsymbol{B}$, and $d_i = c/\omega_{\mathrm{pi}}$ is the ion inertial scale ("ion skin depth"), $\omega_{\mathrm{pi}} = \sqrt{4\pi e^2 n_i/m_i}$. Equation (15.8) is the equation of *Electron MHD (EMHD)*, completely self-consistent for $\boldsymbol{B}$.

(b) Show that magnetic energy is conserved by (15.8). Is magnetic helicity conserved? Does J. B. Taylor relaxation work and what kind of field will be featured in the relaxed state? Is it obvious that this field is a good steady-state solution of (15.8)?

(c) Consider infinitesimal perturbations of a straight-field equilibrium, $\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + \delta\boldsymbol{B}$,

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[87]Strictly speaking, the generalised Ohm's law in this approximation also contains an electron-pressure gradient (see, e.g., Goedbloed & Poedts 2004), but that vanishes upon substitution of $\boldsymbol{E}$ into Faraday's law.

and show that they are helical waves with the dispersion relation

$$\boxed{\omega = \pm k_\parallel v_\text{A} k d_i} \,. \tag{15.9}$$

These are called *Kinetic Alfvén Waves (KAW)*.

(d) Now consider finite perturbations and argue that the appropriate ordering in which linear and nonlinear physics can coexist while perturbations remain small is

$$|\delta \boldsymbol{b}| \sim \frac{\delta B}{B} \sim \frac{k_\parallel}{k} \ll 1. \tag{15.10}$$

Under this ordering, show that the magnetic field can be represented as

$$\frac{\delta \boldsymbol{B}}{B_0} = \frac{1}{v_\text{A}} \, \hat{\boldsymbol{z}} \times \boldsymbol{\nabla}_\perp \Psi + \hat{\boldsymbol{z}} \, \frac{\delta B}{B} \tag{15.11}$$

and the evolution equations for Ψ and $\delta B / B_0$ are

$$\boxed{\frac{\partial \Psi}{\partial t} = v_\text{A}^2 d_i \boldsymbol{b} \cdot \boldsymbol{\nabla} \frac{\delta B}{B_0}, \quad \frac{\partial}{\partial t} \frac{\delta B}{B_0} = -d_i \boldsymbol{b} \cdot \boldsymbol{\nabla} \nabla_\perp^2 \Psi} \,, \tag{15.12}$$

where $\boldsymbol{b} \cdot \boldsymbol{\nabla}$ is given by (12.111). These are the equations of *Reduced Electron MHD*.

(e) Check that the conservation of magnetic energy and the KAW dispersion relation (15.9) are recovered from (15.12). Is there any other conservation law?

7. Hydrodynamics of Rotating Fluid.[88] Most of this question is not on MHD, but deals with equations describing a somewhat analogous system: also embedded into an external field and supporting anisotropic wave-like perturbations. It is an incompressible fluid rotating at angular velocity $\boldsymbol{\Omega} = \Omega \hat{\boldsymbol{z}}$, where $\hat{\boldsymbol{z}}$ is the unit vector in the direction of the z axis. The velocity field $\boldsymbol{u}$ in such a fluid satisfies the following equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + 2 \boldsymbol{u} \times \boldsymbol{\Omega}, \tag{15.13}$$

where pressure p is found from the incompressibility condition $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, the last term on the right-hand side is the Coriolis force, the centrifugal force has been absorbed into p, and viscosity has been ignored.

(a) Consider infinitesimal perturbations of a static ($\boldsymbol{u}_0 = 0$), homogeneous equilibrium of (15.13). Show that the system supports waves with the dispersion relation

$$\boxed{\omega = \pm 2\Omega \frac{k_\parallel}{k}} \,. \tag{15.14}$$

These are called *inertial waves*. Here $\boldsymbol{k} = (k_\perp, 0, k_\parallel)$ (without loss of generality); the subscripts refer to directions perpendicular and parallel to the axis of rotation.

(b) In the case $k_\parallel \ll k_\perp$, determine the direction of propagation of the inertial waves. Determine also the relationship between the components of the velocity vector $\boldsymbol{u}$ associated with the wave. Comment on the polarisation of the wave.

(c) When rotation is strong, i.e., when $\Omega \gg ku$, perturbations in a rotating system are anisotropic with $\epsilon = k_\parallel / k_\perp \ll 1$. Order the linear and nonlinear time scales to be similar to each other and work out the ordering of all relevant quantities, namely, $\boldsymbol{u}_\perp$

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[88]This is based on the 2018 exam question.

(horizontal velocity), $u_\parallel$ (vertical velocity), δp (perturbed pressure), ω, Ω, $k_\parallel$, $k_\perp$ with respect to each other and to ϵ. Using this ordering, show that the motions of a rotating fluid satisfy the following reduced equations

$$\frac{\partial}{\partial t}\nabla_\perp^2 \Phi + \{\Phi, \nabla_\perp^2 \Phi\} = 2\Omega\frac{\partial u_\parallel}{\partial z}, \quad \frac{\partial u_\parallel}{\partial t} + \{\Phi, u_\parallel\} = -2\Omega\frac{\partial \Phi}{\partial z}, \tag{15.15}$$

where the "Poisson bracket" is defined by (12.110) and Φ is the stream function of the perpendicular velocity, i.e., to the lowest order in ϵ, $\boldsymbol{u}_\perp^{(0)} = \hat{\boldsymbol{z}} \times \nabla_\perp \Phi$. Note that, in order to obtain the above equations, you will need to work out $\nabla_\perp \cdot \boldsymbol{u}_\perp$ to both the lowest and next order in ϵ, i.e., both $\nabla_\perp \cdot \boldsymbol{u}_\perp^{(0)}$ and $\nabla_\perp \cdot \boldsymbol{u}_\perp^{(1)}$.

(d) Show that any purely horizontal flows in a strongly rotating fluid must be exactly two-dimensional (i.e., constant along the axis of rotation).

(e) For a strongly rotating, incompressible, highly electrically conducting fluid embedded in a strong uniform magnetic field $\boldsymbol{B}_0$ parallel to the axis of rotation, discuss qualitatively under what conditions you would expect anisotropic ($k_\parallel \ll k_\perp$) Alfvénic and slow-wave-like (pseudo-Alfvénic) perturbations to be decoupled from each other?

There are certain interesting similarities between MHD turbulence and turbulence in rotating fluid systems described by (15.15) and, indeed, also turbulence in stratified environments that we dealt with in §12.2.8 and Q5. If you would like to know more, see Nazarenko & Schekochihin (2011) and follow the paper trail from there.

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8. Grad–Shafranov Equation. Consider static MHD equilibria (13.1) in cylindrical coordinates (r, θ, z) and assume axisymmetry, $\partial/\partial\theta = 0$.

(a) Using the solenoidality of the magnetic field, show that any axisymmetric such field can be expressed in the form

$$\boldsymbol{B} = I\nabla\theta + \nabla\psi \times \nabla\theta, \tag{15.16}$$

where I and ψ are functions of r and z and $\nabla\theta = \hat{\boldsymbol{\theta}}/r$ ($\hat{\boldsymbol{\theta}}$ is the unit basis vector in the θ direction). Show that magnetic surfaces are surfaces of $\psi = $ const.

(b) Using the force balance, show that $\nabla I \times \nabla\psi = 0$ and $\nabla p \times \nabla\psi = 0$ and hence argue that

$$I = I(\psi) \quad \text{and} \quad p = p(\psi) \tag{15.17}$$

are functions of ψ only (i.e., they are constant on magnetic surfaces).

(c) Again from the force balance, show that $\psi(r, z)$ satisfies the *Grad–Shafranov equation*

$$-\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{\partial^2\psi}{\partial z^2}\right) = 4\pi r^2\frac{\mathrm{d}p}{\mathrm{d}\psi} + I\frac{\mathrm{d}I}{\mathrm{d}\psi}. \tag{15.18}$$

This defines the shape of an axisymmetric equilibrium, given the profiles $p(\psi)$ and $I(\psi)$.

(d) Show that in cylindrical symmetry ($\partial/\partial\theta = 0$, $\partial/\partial z = 0$), (15.18) reduces to (13.8).

(e) Assume $I(\psi) = $ const (so the azimuthal field $B_\theta = I/r$ is similar to the magnetic field from a central current) and $p(\psi) = a\psi$, where a is some constant. Find a solution of (15.18) that gives rise to magnetic surfaces that resemble nested tori, but with "D-shaped" cross section (Fig. 61; this looks a bit like the modern tokamaks). If you stipulate that p must vanish at $r = 0$ and at $r = R$ along the $z = 0$ axis and also at $z = \pm L$
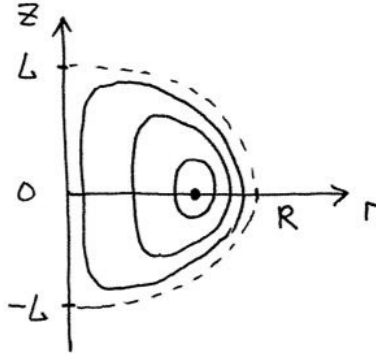
FIGURE 61. A simple equilibrium from Q8, superficially resembling the poloidal cross-section of a tokamak.

along the $r = 0$ axis and that the maximum pressure at $r < R$ is p_0, show that the corresponding magnetic surfaces are described by

$$\psi = 2\sqrt{\frac{2\pi p_0}{1 + R^2/4L^2}}\, r^2 \left(1 - \frac{r^2}{R^2} - \frac{z^2}{L^2} \right). \tag{15.19}$$

Where is the (azimuthal) magnetic axis of these surfaces? What is the value of a?

(f) Seek solutions to (15.18) that are linear force-free fields. Show that in this case, (15.18) reduces to the Bessel equation (a substitution $\psi = rf(r, z)$ will prove useful). Set $B_z(0, 0) = B_0$. Find solutions of two kinds: (i) ones in a semi-infinite domain $z \geqslant 0$, with the field vanishing exponentially at $z \to \infty$; (ii) ones periodic in z. If you also impose the boundary condition $B_r = 0$ at $r = R$, how can this be achieved? Can either of these solutions be the result of J. B. Taylor relaxation of an MHD system? If so, how would one decide whether it is more or less likely to be the correct relaxed state than the solution derived in §13.4?

You will find the solution of the type (i) in Sturrock (1994, §13) (who also shows how to construct many other force-free fields, useful in various physical and astrophysical contexts). Think of this solution in the context of Q1(e). The solution of type (ii) is a particular case of the general $(\partial/\partial\theta \neq 0)$ equilibrium solution derived and discussed in Taylor & Newton (2015, §9). However, the axisymmteric solution is not very useful because, as they show, depending on the values of helicity and of R, the true relaxed state is either the cylidrically and axially symmetric solution derived in §13.4 or one which also has variation in the θ direction.

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9. Magnetised Interchange Instability. Consider the same set up as in §14.3, but now the stratified atmosphere is threaded by straight horizontal magnetic field (Fig. 62):

$$\rho_0 = \rho_0(z), \quad p_0 = p_0(z), \quad \boldsymbol{B}_0 = B_0(z)\hat{\boldsymbol{x}}, \quad \frac{\mathrm{d}}{\mathrm{d}z}\left(p_0 + \frac{B_0^2}{8\pi} \right) = -\rho_0 g. \tag{15.20}$$

We shall be concerned with the stability of this equilibrium.

(a) For simplicity, assume $\partial\boldsymbol{\xi}/\partial x = 0$. This rules out any perturbations of the magnetic-field direction, $\delta\boldsymbol{b} = 0$, so there will be no field-line bending, no restoring curvature forces. For this restricted set of perturbations, work out δW_2 and observe that, like in the unmagnetised case considered in §14.3, it depends only on $\boldsymbol{\nabla} \cdot \boldsymbol{\xi}$ and ξ_z. Minimise
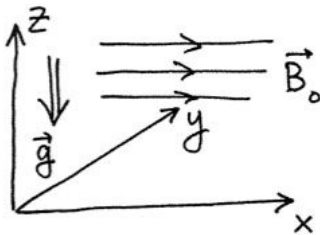
FIGURE 62. Magnetised atmosphere.

δW_2 with respect to $\boldsymbol{\nabla} \cdot \boldsymbol{\xi}$ and show that

$$\frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{p_0}{\rho_0^\gamma} + \frac{2}{\beta} \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{B_0}{\rho_0} < 0 \tag{15.21}$$

is a sufficient condition for instability (the *magnetised interchange instability*). Would you be justified in expecting stability if the condition (15.21) were not satisfied?

(b) Explain how this instability operates and rederive the condition for instability by considering interchanging blobs (or, rather, flux tubes), in the spirit of §14.3.3.

If field-line bending is allowed ($\partial \xi / \partial x \neq 0$), another instability emerges, the *Parker (1966) instability*. Do investigate.

10. Stability of the θ Pinch. Consider the following cylindrically and axially symmetric equilibrium:

$$\boldsymbol{B}_0 = B_0(r)\hat{\boldsymbol{z}}, \quad \boldsymbol{j}_0 = j_0(r)\hat{\boldsymbol{\theta}} = -\frac{c}{4\pi} B_0'(r)\hat{\boldsymbol{\theta}}, \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(p_0 + \frac{B_0^2}{8\pi}\right) = 0 \tag{15.22}$$

(a θ pinch; see §13.1.1, Fig. 52). Consider general displacements of the form

$$\boldsymbol{\xi} = \boldsymbol{\xi}_{mk}(r)e^{im\theta + ikz}. \tag{15.23}$$

Show that the θ pinch is always stable. Specifically, you should be able to show that

$$\delta W_2 = \pi L_z \int_0^\infty \mathrm{d}r\, r \left\{ \gamma p_0 |\boldsymbol{\nabla} \cdot \boldsymbol{\xi}|^2 + \frac{B_0^2}{4\pi}\left[k^2 \left(|\xi_r|^2 + |\xi_\theta|^2\right) + \left|\frac{\xi_r}{r} + \frac{\partial \xi_r}{\partial r} + \frac{im\xi_\theta}{r}\right|^2 \right] \right\} > 0, \tag{15.24}$$

where L_z is the length of the cylinder.

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REFERENCES

ADAMS, D. 1979 *The Hitch-Hiker's Guide to the Galaxy*. London: Pan Books.

ADKINS, T. & SCHEKOCHIHIN, A. A. 2018 A solvable model of Vlasov-kinetic plasma turbulence in Fourier-Hermite phase space. *J. Plasma Phys.* **84**, 905840107.

ADKINS, T. & SCHEKOCHIHIN, A. A. 2019 Relaxation and nonlinearity: the phase-space dynamics of Vlasov-kinetic plasmas. in preparation.

ALEXANDROV, A. F., BOGDANKEVICH, L. S. & RUKHADZE, A. A. 1984 *Principles of Plasma Electrodynamics*. Berlin: Springer.

ARNOLD, V. I. & KHESIN, B. A. 1999 *Topological Methods in Hydrodynamics*. Berlin: Springer.

BALBUS, S. A. 2015 *Astrophysical Fluid Dynamics*. Lecture Notes for the Oxford MMathPhys course; URL: https://www2.physics.ox.ac.uk/contacts/people/balbus.

BALBUS, S. A. & HAWLEY, J. F. 1991 A powerful local shear instability in weakly magnetized disks. I. Linear analysis. *Astrophys. J.* **376**, 214.

BALESCU, R. 1960 Irreversible processes in ionized gases. *Phys. Fluids* **3**, 52.

BEDROSSIAN, J. 2016 Nonlinear echoes and Landau damping with insufficient regularity. *arXiv:1605.06841*.

BERNSTEIN, I. B. 1958 Waves in a plasma in a magnetic field. *Phys. Rev.* **109**, 10.

BERNSTEIN, I. B., FRIEMAN, E. A., KRUSKAL, M. D. & KULSRUD, R. M. 1958 An energy principle for hydromagnetic stability problems. *Proc. R. Soc. London A* **244**, 17.

BESSE, N., ELSKENS, Y., ESCANDE, D. F. & BERTRAND, P. 2011 Validity of quasilinear theory: refutations and new numerical confirmation. *Plasma Phys. Control. Fusion* **53**, 025012.

BINNEY, J. J. 2016 *Kinetic Theory of Self-Gravitating Systems*. Lecture Notes for the Oxford MMathPhys course on Kinetic Theory; URL: http://www-thphys.physics.ox.ac.uk/user/JamesBinney/kt.pdf.

BOHM, D. & GROSS, E. P. 1949*a* Theory of plasma oscillations. A. Origin of medium-like behavior. *Phys. Rev.* **75**, 1851.

BOHM, D. & GROSS, E. P. 1949*b* Theory of plasma oscillations. B. Excitation and damping of oscillations. *Phys. Rev.* **75**, 1864.

Boldyrev, S., Horaites, K., Xia, Q. & Perez, J. C. 2013 Toward a theory of astrophysical plasma turbulence at subproton scales. *Astrophys. J.* **777**, 41.

Braginskii, S. I. 1965 Transport processes in a plasma. *Rev. Plasma Phys.* **1**, 205.

Buneman, O. 1958 Instability, turbulence, and conductivity in current-carrying plasma. *Phys. Rev. Lett.* **1**, 8.

Case, K. M. 1959 Plasma oscillations. *Ann. Phys. (N.Y.)* **7**, 349.

Chandrasekhar, S. 2003 *Hydrodynamic and Hydromagnetic Stability*. Dover.

Chapman, S. & Cowling, T. G. 1991 *The Mathematical Theory of Non-Uniform Gases*. Cambridge: Cambridge University Press, 3$^{\rm rd}$ Edition.

Chew, G. F., Goldberger, M. L. & Low, F. E. 1956 The Boltzmann equation and the one-fluid hydromagnetic equations in the absence of particle collisions. *Proc. R. Soc. London A* **236**, 112.

Connor, J. W., Hastie, R. J. & Taylor, J. B. 1979 High mode number stability of an axisymmetric toroidal plasma. *Proc. R. Soc. London A* **365**, 1.

Cowley, S. C. 2004 *Plasma Physics*. Lecture Notes for the UCLA course; URL: http://www-thphys.physics.ox.ac.uk/research/plasma/CowleyLectures/.

Cowley, S. C. & Artun, M. 1997 Explosive instabilities and detonation in magnetohydrodynamics. *Phys. Rep.* **283**, 185.

Cowley, S. C., Cowley, B., Henneberg, S. A. & Wilson, H. R. 2015 Explosive instability and erupting flux tubes in a magnetized plasma. *Proc. R. Soc. London A* **471**, 20140913.

Davidson, P. A. 2016 *Introduction to Magnetohydrodynamics* (2nd edition). Cambridge: Cambridge University Press.

Davidson, R. C. 1983 Kinetic waves and instabilities in a uniform plasma. In *Handbook of Plasma Physics, Volume 1* (ed. A. A. Galeev & R. N. Sudan), p. 519. Amsterdam: North-Holland.

Dellar, P. J. 2015 *Kinetic Theory of Gases*. Lecture Notes for the Oxford MMathPhys course on Kinetic Theory; URL: http://people.maths.ox.ac.uk/dellar/MMPkinetic.html.

Dellar, P. J. 2017 *Hydrodynamics of Complex Fluids*. Lecture Notes for the Oxford MMathPhys course on Advanced Fluid Dynamics; URL: http://people.maths.ox.ac.uk/dellar/MMP_Non_Newt_fluids.html.

Diamond, P. H., Itoh, S.-I. & Itoh, K. 2010 *Modern Plasma Physics. Volume 1: Physical Kinetics of Turbulent Plasmas*. Cambridge: Cambridge University Press.

Dodin, I. Y. & Fisch, N. J. 2005 Variational formulation of Gardner's restacking algorithm. *Phys. Lett. A* **341**, 187.

Drummond, W. E. & Pines, D. 1962 Nonlinear stability of plasma oscillations. *Nucl. Fusion Suppl., Part 3* p. 1049.

DuBois, D. F., Russell, D. & Rose, H. A. 1995 Reduced description of strong Langmuir turbulence from kinetic theory. *Phys. Plasmas* **2**, 76.

Dupree, T. H. 1972 Theory of phase space density granulation in plasma. *Phys. Fluids* **15**, 334.

Elsasser, W. M. 1950 The hydromagnetic equations. *Phys. Rev.* **79**, 183.

Fouvry, J.-B. 2019 *Kinetic Theory of Self-Gravitating Systems*. Lecture Notes for the Oxford MMathPhys course on Kinetic Theory; URL: http://www-thphys.physics.ox.ac.uk/user/JamesBinney/kt.pdf.

Fowler, T. K. 1963 Lyapunov's stability criteria for plasmas. *J. Math. Phys.* **4**, 559.

Fowler, T. K. 1964 Bounds on plasma instability growth rates. *Phys. Fluids* **7**, 249.

Fowler, T. K. 1968 Thermodynamics of unstable plasmas. In *Advances in Plasma Physics* (ed. A. Simon & W. B. Thompson), , vol. 1, p. 201. New York: Interscience Press.

Fowler, T. K. 2016 Speculations about plasma free energy, 50 years later. *J. Plasma Phys.* **82**, 925820501.

Freidberg, J. 2014 *Ideal MHD*. Cambridge: Cambridge University Press.

Fried, B. D. & Conte, S. D. 1961 *The Plasma Dispersion Function*. New York: Academic Press.

Furth, H. P., Killeen, J. & Rosenbluth, M. N. 1963 Finite-resistivity instabilities of a sheet pinch. *Phys. Fluids* **6**, 459.

Gardner, C. S. 1963 Bound on the energy available from a plasma. *Phys. Fluids* **6**, 839.

GOEDBLOED, H. & POEDTS, S. 2004 *Principles of Magnetohydrodynamics*. Cambridge: Cambridge University Press.

GOLDMAN, M. V. 1984 Strong turbulence of plasma waves. *Rev. Mod. Phys.* **56**, 709.

GOULD, R. W., O'NEIL, T. M. & MALMBERG, J. H. 1967 Plasma wave echo. *Phys. Rev. Lett.* **19**, 219.

HAEFF, A. 1949 The electron-wave tube—a novel method of generation and amplification of microwave energy. *Proc. IRE* **37**, 4.

HAINES, M. G. 2011 A review of the dense Z-pinch. *Plasma Phys. Control. Fusion* **53**, 093001.

HELANDER, P. 2017 Available energy and ground states of collisionless plasmas. *J. Plasma Phys.* **83**, 715830401.

HELANDER, P. & SIGMAR, D. J. 2005 *Collisional Transport in Magnetized Plasmas*. Cambridge: Cambridge University Press.

HENINGER, J. M. & MORRISON, P. J. 2018 An integral transform technique for kinetic systems with collisions. *Phys. Plasmas* **25**, 082118.

JACKSON, J. D. 1960 Longitudinal plasma oscillations. *J. Nuclear Energy C* **1**, 171.

KADOMTSEV, B. B. 1965 *Plasma Turbulence*. Academic Press.

KADOMTSEV, B. B. & POGUTSE, O. P. 1970 Collisionless relaxation in systems with Coulomb interactions. *Phys. Rev. Lett.* **25**, 1155.

KADOMTSEV, B. B. & POGUTSE, O. P. 1971 Theory of beam-plasma interaction. *Phys. Fluids* **14**, 2470.

KADOMTSEV, B. B. & POGUTSE, O. P. 1974 Nonlinear helical perturbations of a plasma in the tokamak. *Sov. Phys.–JETP* **38**, 575.

KANEKAR, A., SCHEKOCHIHIN, A. A., DORLAND, W. & LOUREIRO, N. F. 2015 Fluctuation-dissipation relations for a plasma-kinetic Langevin equation. *J. Plasma Phys.* **81**, 305810104.

KINGSEP, A. S. 2004 *Introduction to Nonlinear Plasma Physics*. Moscow: MZ Press (in Russian).

KINGSEP, A. S., RUDAKOV, L. I. & SUDAN, R. N. 1973 Spectra of strong Langmuir turbulence. *Phys. Rev. Lett.* **31**, 1482.

KLIMONTOVICH, YU. L. 1967 *The Statistical Theory of Non-Equilibrium Processes in a Plasma*. MIT Press.

KOLMOGOROV, A. N. 1941 The local structure of turbulence in incompressible viscous fluid at very large Reynolds numbers. *Dokl. Acad. Nauk SSSR* **30**, 299.

KRALL, N. A. & TRIVELPIECE, A. W. 1973 *Principles of Plasma Physics*. New York: McGraw-Hill.

KROMMES, J. A. 2015 A tutorial introduction to the statistical theory of turbulent plasmas, a half-century after Kadomtsev's *Plasma Turbulence* and the resonance-broadening theory of Dupree and Weinstock. *J. Plasma Phys.* **81**, 205810601.

KROMMES, J. A. 2018 Projection-operator methods for classical transport in magnetized plasmas. Part I. Linear response, the Braginskii equations and fluctuating hydrodynamics. *J. Plasma Phys.* **84**, 925840401.

KRUSKAL, M. D. & OBERMAN, C. R. 1958 On the stability of plasma in static equilibrium. *Phys. Fluids* **1**, 275.

KULSRUD, R. M. 1983 MHD description of plasma. In *Handbook of Plasma Physics, Volume 1* (ed. A. A. Galeev & R. N. Sudan), p. 1. Amsterdam: North-Holland.

KULSRUD, R. M. 2005 *Plasma Physics for Astrophysics*. Princeton: Princeton University Press.

KUNZ, M. W., ABEL, I. G., KLEIN, K. G. & SCHEKOCHIHIN, A. A. 2018 Astrophysical gyrokinetics: turbulence in pressure-anisotropic plasmas at ion scales and beyond. *J. Plasma Phys.,* in press; E-print arXiv:1712.02269.

KUNZ, M. W., SCHEKOCHIHIN, A. A., CHEN, C. H. K., ABEL, I. G. & COWLEY, S. C. 2015 Inertial-range kinetic turbulence in pressure-anisotropic astrophysical plasmas. *J. Plasma Phys.* **81**, 325810501.

LANDAU, L. 1936 Transport equation in the case of Coulomb interaction. *Zh. Eksp. Teor. Fiz.* **7**, 203.

LANDAU, L. 1946 On the vibration of the electronic plasma. *Zh. Eksp. Teor. Fiz.* **16**, 574, English translation: *J. Phys. USSR* **10**, 25 (1946).

LANDAU, L. D. & LIFSHITZ, E. M. 1976 *Mechanics (L. D. Landau and E. M. Lifshitz's Course of Theoretical Physics, Volume 1)*. Butterworth–Heinemann.

Lenard, A. 1960 On Bogoliubov's kinetic equation for a spatially homogeneous plasma. *Ann. Phys. (New York)* **10**, 390.

Lifshitz, E. M. & Pitaevskii, L. P. 1981 *Physical Kinetics (L. D. Landau and E. M. Lifshitz's Course of Theoretical Physics, Volume 10)*. Elsevier.

Loureiro, N. F., Schekochihin, A. A. & Cowley, S. C. 2007 Instability of current sheets and formation of plasmoid chains. *Phys. Plasmas* **14**, 100703.

Lynden-Bell, D. 1967 Statistical mechanics of violent relaxation in stellar systems. *Mon. Not. R. Astron. Soc.* **136**, 101.

Malmberg, J. H. & Wharton, C. B. 1964 Collisionless damping of electrostatic plasma waves. *Phys. Rev. Lett.* **13**, 184.

Malmberg, J. H., Wharton, C. B., Gould, R. W. & O'Neil, T. M. 1968 Plasma wave echo experiment. *Phys. Rev. Lett.* **20**, 95.

Mazitov, R. K. 1965 Damping of plasma waves. *J. Appl. Mech. Tech. Phys.* **6**, 22.

Moffatt, H. K. 1978 *Magnetic Field Generation in Electrically Conducting Fluids*. Cambridge: Cambridge University Press.

Morrison, P. 2000 Hamiltonian description of Vlasov dynamics: Action-angle variables for the continuous spectrum. *Transport Theory Stat. Phys.* **29**, 397.

Morrison, P. J. 1994 The energy of perturbations for Vlasov plasmas. *Phys. Plasmas* **1**, 1447.

Musher, S. L., Rubenchik, A. M. & Zakharov, V. E. 1995 Weak Langmuir turbulence. *Phys. Rep.* **252**, 177.

Nazarenko, S. 2011 *Wave Turbulence*. Berlin: Springer.

Nazarenko, S. V. & Schekochihin, A. A. 2011 Critical balance in magnetohydrodynamic, rotating and stratified turbulence: towards a universal scaling conjecture. *J. Fluid Mech.* **677**, 134.

Newcomb, W. A. 1962 Lagrangian and Hamiltonian methods in magnetohydrodynamics. *Nuclear Fusion: Supplement, Part 2* p. 451.

Nyquist, H. 1932 Regeneration theory. *Bell Syst. Tech. J.* **11**, 126.

Ogilvie, G. I. 2016 Astrophysical fluid dynamics. *J. Plasma Phys.* **82**, 205820301.

Ogilvie, G. I. & Proctor, M. R. E. 2003 On the relation between viscoelastic and magnetohydrodynamic flows and their instabilities. *J. Fluid Mech.* **476**, 389.

O'Neil, T. 1965 Collisionless damping of nonlinear plasma oscillations. *Phys. Fluids* **8**, 2255.

Parker, E. N. 1966 The dynamical state of the interstellar gas and field. *Astrophys. J.* **145**, 811.

Parra, F. I. 2019*a* *Collisional Plasma Physics*. Lecture Notes for an Oxford MMathPhys course; URL: http://www-thphys.physics.ox.ac.uk/people/FelixParra/\CollisionalPlasmaPhysics/CollisionalPlasmaPhysics.html.

Parra, F. I. 2019*b* *Collisionless Plasma Physics*. Lecture Notes for an Oxford MMathPhys course; URL: http://www-thphys.physics.ox.ac.uk/people/FelixParra/\CollisionlessPlasmaPhysics/CollisionlessPlasmaPhysics.html.

Peierls, R. E. 1955 *Quantum Theory of Solids*. Oxford: Clarendon Press.

Pelletier, G. 1980 Langmuir turbulence as a critical phenomenon. Part 1. Destruction of the statistical equilibrium of an interacting-modes ensemble. *J. Plasma Phys.* **24**, 287.

Penrose, O. 1960 Electrostatic instabilities of a uniform non-Maxwellian plasma. *Phys. Fluids* **3**, 258.

Pfirsch, D. & Sudan, R. N. 1993 Nonlinear ideal magnetohydrodynamics instabilities. *Phys. Fluids B* **5**, 2052.

Pierce, J. R. & Hebenstreit, W. B. 1949 A new type of high-frequency amplifier. *Bell Syst. Tech. J.* **28**, 33.

Ramos, J. J. & White, R. L. 2018 Normal-mode-based analysis of electron plasma waves with second-order Hermitian formalism. *Phys. Plasmas* **25**, 034501.

Rincon, F. 2019 Dynamo theories. *J. Plasma Phys.* **85**, 205850401.

Robinson, P. A. 1997 Nonlinear wave collapse and strong turbulence. *Rev. Mod. Phys.* **69**, 507.

Rudakov, L. I. & Tsytovich, V. N. 1978 Strong Langmuir turbulence. *Phys. Rep.* **40**, 1.

Ruyer, C., Gremillet, L., Debayle, A. & Bonnaud, G. 2015 Nonlinear dynamics of the ion Weibel-filamentation instability: An analytical model for the evolution of the plasma and spectral properties. *Phys. Plasmas* **22**, 032102.

SAGDEEV, R. Z. & GALEEV, A. A. 1969 *Nonlinear Plasma Theory*. W. A. Benjamin.

SCHEKOCHIHIN, A. A. 2019 *Kinetic Theory and Statistical Physics*. Lecture Notes for the Oxford Physics course, Paper A1; URL: http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/A1/2014/A1LectureNotes.pdf.

SCHEKOCHIHIN, A. A. & COWLEY, S. C. 2007 Turbulence and magnetic fields in astrophysical plasmas. In *Magnetohydrodynamics: Historical Evolution and Trends* (ed. S. Molokov, R. Moreau & H. K. Moffatt), p. 85. Berlin: Springer, reprint: URL: http://www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/mhdbook.pdf.

SCHEKOCHIHIN, A. A., COWLEY, S. C., DORLAND, W., HAMMETT, G. W., HOWES, G. G., QUATAERT, E. & TATSUNO, T. 2009 Astrophysical gyrokinetics: kinetic and fluid turbulent cascades in magnetized weakly collisional plasmas. *Astrophys. J. Suppl.* **182**, 310.

SCHEKOCHIHIN, A. A., COWLEY, S. C., RINCON, F. & ROSIN, M. S. 2010 Magnetofluid dynamics of magnetized cosmic plasma: firehose and gyrothermal instabilities. *Mon. Not. R. Astron. Soc.* **405**, 291.

SCHEKOCHIHIN, A. A., PARKER, J. T., HIGHCOCK, E. G., DELLAR, P. J., DORLAND, W. & HAMMETT, G. W. 2016 Phase mixing versus nonlinear advection in drift-kinetic plasma turbulence. *J. Plasma Phys.* **82**, 905820212.

SKOUTNEV, V., HAKIM, A., JUNO, J. & TENBARGE, J. M. 2019 Temperature-dependent saturation of Weibel-type instabilities in counter-streaming plasmas. *Astrophys. J.* **872**, L28.

SQUIRE, J., SCHEKOCHIHIN, A. A. & QUATAERT, E. 2017 Amplitude limits and nonlinear damping of shear-Alfvén waves in high-beta low-collisionality plasmas. *New J. Phys.* **19**, 055005.

STRAUSS, H. R. 1976 Nonlinear, three-dimensional magnetohydrodynamics of noncircular tokamaks. *Phys. Fluids* **19**, 134.

STURROCK, P. A. 1966 Stochastic acceleration. *Phys. Rev.* **141**, 186.

STURROCK, P. A. 1994 *Plasma Physics. An Introduction to the Theory of Astrophysical, Geophysical and Laboratory Plasmas*. Cambridge: Cambridge University Press.

SWANSON, D. G. 2008 *Plasma Kinetic Theory*. CRC Press.

TAYLOR, J. B. & NEWTON, S. L. 2015 Special topics in plasma confinement. *J. Plasma Phys.* **81**, 205810501.

THORNHILL, S. G. & TER HAAR, D. 1978 Langmuir turbulence and modulational instability. *Phys. Rep.* **43**, 43.

TONKS, L. & LANGMUIR, I. 1929 Oscillations in ionized gases. *Phys. Rev.* **33**, 195.

TSYTOVICH, V. N. 1995 *Lectures on Non-linear Plasma Kinetics*. Berlin: Springer.

UZDENSKY, D. A., LOUREIRO, N. F. & SCHEKOCHIHIN, A. A. 2010 Fast magnetic reconnection in the plasmoid-dominated regime. *Phys. Rev. Lett.* **105**, 235002.

VAN KAMPEN, N. G. 1955 On the theory of stationary waves in plasmas. *Physica* **9**, 949.

VAN KAMPEN, N. G. & FELDERHOF, B. U. 1967 *Theoretical Methods in Plasma Physics*. Amsterdam: North-Holland.

VEDENOV, A. A. 1963 Quasilinear plasma theory (theory of a weakly turbulent plasma). *Plasma Phys. (J. Nucl. Energy C)* **5**, 169.

VEDENOV, A. A., VELIKHOV, E. P. & SAGDEEV, R. Z. 1962 Quasilinear theory of plasma oscillations. *Nucl. Fusion Suppl., Part 2* p. 465 (in Russian).

VILLANI, C. 2014 Particle systems and nonlinear Landau damping. *Phys. Plasmas* **21**, 030901.

WEIBEL, E. S 1958 Spontaneously growing transverse waves in a plasma due to an anisotropic velocity distribution. *Phys. Rev. Lett.* **2**, 83.

WICKS, R. T., ROBERTS, D. A., MALLET, A., SCHEKOCHIHIN, A. A., HORBURY, T. S. & CHEN, C. H. K. 2013 Correlations at large scales and the onset of turbulence in the fast solar wind. *Astrophys. J.* **778**, 177.

ZAKHAROV, V. E. 1972 Collapse of Langmuir waves. *Sov. Phys.–JETP* **35**, 908.

ZAKHAROV, V. E., LVOV, V. S. & FALKOVICH, G. 1992 *Kolmogorov Spectra of Turbulence I: Wave Turbulence*. Berlin: Springer (updated online version URL: https://www.weizmann.ac.il/chemphys/lvov/research/springer-series-nonlinear-dynamics).

ZAKHAROV, V. E., MUSHER, S. L. & RUBENCHIK, A. M. 1985 Hamiltonian approach to the description of non-linear plasma phenomena. *Phys. Rep.* **129**, 285.

ZELDOVICH, YA. B. 1956 The magnetic field in the two-dimensional motion of a conducting turbulent liquid. *Zh. Eksp. Teor. Fiz.* **31**, 154, English translation: *Sov. Phys. JETP* **4**, 460 (1957).

ZHDANKIN, V., BOLDYREV, S. & UZDENSKY, D. A. 2016 Scalings of intermittent structures in magnetohydrodynamic turbulence. *Phys. Plasmas* **23**, 055705.

ZIMAN, J. M. 1960 *Electrons and Phonons: The Theory of Transport Phenomena in Solids*. Oxford: Clarendon Press.