Alberto Paoluzzi and Giorgio Scorzelli

BIM geometry with Julia Plasm

Functional language for CAD programming

June 6, 2024

Springer Nature

Contents

Chapter 4
Geometric models

Julia `Plasm` is the best choice to write symbolic geometric models for Building Information Modeling (BIM) and Computer-Aided Design (CAD). Geometric models specify the physical appearance of Architecture, Engineering, and Construction products at any scale, from structural and envelope components to whole buildings and built environments, and are used for design, tender, contract, and collaboration. We ported the functional language `Plasm` to Julia for better supporting design, model generation, and visualization of geometric objects. In this chapter we introduce the great expressive power of `Plasm` geometric types and parametric functions, as well the simple methods used to build parametric assemblies, where objects itself can be used as actual parameters. We show also that `Plasm` offers a general mechanism (Julia dictionaries) to export models characterized by colors, textures, materials, and so on. `Plasm` can be even embedded in the `Jupiter` platform in order to document the design choices step-by-step in digital notebooks.

4.1 Plasm geometric types

Even if Julia does not pretend the user specifies the type of data objects, which are inferred at compile time, it may always be useful to annotate with their type the parameters and the returned value from function applications in order to get faster codes from the Julia compiler. The best reason concerns program documentation, making it easier to understand the Julia's sources.

Let's remember that `Plasm` derives from three founts: (1) the classic `PLaSM` set up on `FL` functional combinators; (2) the porting to Python (object-oriented language), and finally (3) the embedding into Julia (functional and multi-paradigm), after ten years of algebraic research finalized to understand the role of topology in elaborating digital geometric models.

This development defined different data types and user structures, which the current version of the language proudly unifies by scheduling them to different roles and uses.

1. The Hierarchical Polyhedral Complex, now denoted in Julia `Plasm` as the `Hpc` datatype, was characterized by models defined as aggregation of multidimensional convex cells, described only by their vertices and by multidimensional affine matrices.
2. Our research about algebraic topology of geometric design directed us to design the Linear Algebraic Representation, currently the `Lar` datatype, used to work with chain complexes, and able to fully specify the geometry and topology of the *solid objects* under consideration, even non-manifold.
3. Finally, a third Julia's user-defined `struct`, named `Geo` for Geometry, is being used as container of huge datasets for `Plasm`-coded generation of `BIM` objects and 3D point clouds from surveys.

Hpc Type

This recursive type is mainly used for geometric object definition, including the hierarchical values generated by the `STRUCT` function, and interactive graphics visualization on the display device.

An object of `Hpc` type has three fields: a multidimensional matrix `T::MatrixNd`; a vector `childs` (i.e., children) either of elements `Hpc` or of elements `Geometry`, and a `Properties` field of dictionary type, i.e., `Dict{Any, Any}`.

The `mutable struct Hpc` is a typical recursive data structure to represent dynamically a data object of tree type, where the `children` nodes of a node may be in any number since stored into a Julia's `Vector`.

```
mutable struct Hpc
T::MatrixNd
childs::Union{Vector{Hpc}, Vector{Geometry}}
properties::Dict{Any, Any}
# constructor
   function Hpc(T::MatrixNd=MatrixNd(0), childs:: Union{Vector{
    Hpc}, Vector{Geometry}}=[], properties=Dict())
      self = new()
      self.childs = childs
      self.properties = properties
      if length(childs) ) 0
         Tdim = maximum([dim(child) for child in childs]) + 1
         self.T = embed(T, Tdim)
      else
         self.T = T
      end
      return self
   end
end
```

Lar Type

The `mutable struct Lar` is used to represent synthetically a generic cellular or chain complex, together with some of its properties. Given `obj::Lar`, it represents with `obj.d`, `obj.m`, `obj.n`, `obj.V`, and `obj.C`, respectively, the intrinsic dimension (1 for curves, 2 for surfaces, 3 for solids), the number of its coordinates, the number of vertices, and a dictionary of of chain bases or chain operators, stored when already available throughout a computation.

```
mutable struct Lar
   d::Int # intrinsic dimension
   m::Int # embedding dimension (rows of V)
   n::Int # number of vertices  (columns of V)
   V::Matrix{Float64} # object geometry
   C::Dict{Symbol, AbstractArray} # object topology (C for cells
    )
   # inner constructors
   Lar() = new( -1, 0, 0, Matrix{Float64}(undef,0,0), Dict{
    Symbol, AbstractArray}() )
   Lar(m::Int,n::Int) = new( m,m,n, Matrix(undef,m,n), Dict{
    Symbol,AbstractArray}() )
   Lar(d::Int,m::Int,n::Int) = new( d,m,n, Matrix(undef,m,n),
    Dict{Symbol,AbstractArray}() )
   Lar(V::Matrix) = begin m, n = size(V); new( m,m,n, V, Dict{
    Symbol,AbstractArray}() ) end
   Lar(V::Matrix,C::Dict) = begin m,n = size(V); new( m,m,n, V,
    C )  end
   Lar(d::Int,V::Matrix,C::Dict) = begin m,n = size(V); new( d,m
    ,n, V, C )  end
   Lar(d,m,n, V,C) = new( d,m,n, V,C )
end
```

Geo type

A `mutable struct Geometry` is used as a container for single whole geometric objects, allowing to store the various dimensional cellular subcomplexes that partition the geometric value. Conversely, any hierarchical assembly is stored in `Plasm` within a mixture of `Hpc` and `Geometry` nodes. The `Geometry` data structure contains arrays of integers, denoting the ordered bases of the topological chains of different dimensions that decompose the represented geometric value. It also contains a numeric `db` (data base), implemented as a

Julia dictionary with key the (suitably rounded) coordinate vector sof vertex `point` and with value the corresponding integer index.

```
mutable struct Geometry
   db::Dict{Vector{Float64}, Int}
   points::Vector{Vector{Float64}}
   edges::Vector{Vector{Int}}
   faces::Vector{Vector{Int}}
   hulls::Vector{Vector{Int}}
   # constructor
   function Geometry()
   self = new(
      Dict{Vector{Float64}, Int}(),
      Vector{Vector{Float64}}(),
      Vector{Vector{Int}}(),
      Vector{Vector{Int}}(),
      Vector{Vector{Int}}(),
   )
   return self
   end
end
```

Topological types

Some abstract types are defined in `Plasm` in order to characterize and document the type of variables and/or parameters within complicated definitions and function codes. They are mainly used within the structures of `Lar` type, to document the topology, and are defined as global `const` symbols.

```
const Points = Matrix{number}
const Cells = Vector{Vector{Int}}
const Cell = SparseVector{Int8, Int}
const Chain = SparseVector{Int8,Int}
const ChainOp = SparseMatrixCSC{Int8,Int}
const ChainComplex = Vector{ChainOp}
```

The `Cells` type stores the cellular bases and some subsets of cells as `Vector` of `Vector` of integers. The `Cell` type is utilized to memorize a single cell's (sparse) representation. The `Chain` type item equals the cell type, and is used only for documentation aims. The `ChainOp` type allows the storage of the topological operators, including boundary and coboundary, and other higher degree operators, e.g., `FV`. The `ChainComplex` type is employed as the multidimensional `Vector` store of chain complexes, by now only in 2D and 3D, with two and three `ChainOp` sparse matrices, respectively.

Coding 4.1.1 (3-cube topology is a ChainOp object) The expression `cube.C[:FV]` returns a dictionary value of type `Cells`, which contains the basis of our cubic cellular complex. `KFV` is of type `ChainOp`:

```
cube = LAR(CUBE(3))          #=
Lar(3, 3, 8, [3.0 0.0 … 3.0 0.0; 3.0 3.0 … 3.0 3.0; 0.0 0.0 …
    3.0 3.0], Dict{Symbol, AbstractArray}(:CV =) [[1, 2, 3, 4,
    5, 6, 7, 8]], :FV =) [[1, 2, 3, 4], [3, 4, 5, 6], [1, 3, 5,
    7], [2, 4, 6, 8], [1, 2, 7, 8], [5, 6, 7, 8]], :EV =) [[3,
    4], [2, 4], [1, 2], [1, 3], [5, 6], [4, 6], [3, 5], [5, 7],
    [1, 7], [6, 8], [2, 8], [7, 8]])) =#

FV = cube.C[:FV]             #=
6-element Vector{Vector{Int64}}:
 [1, 2, 3, 4]
 [3, 4, 5, 6]
 [1, 3, 5, 7]
 [2, 4, 6, 8]
 [1, 2, 7, 8]
 [5, 6, 7, 8]           =#

KFV = lar2cop(FV::Cells)::ChainOp       #=
6×8 SparseArrays.SparseMatrixCSC{Int8, Int64} with 24 stored
    entries:
 1  1  1  1  ⋅  ⋅  ⋅  ⋅
 ⋅  ⋅  1  1  1  1  ⋅  ⋅
 1  ⋅  1  ⋅  1  ⋅  1  ⋅
 ⋅  1  ⋅  1  ⋅  1  ⋅  1
 1  1  ⋅  ⋅  ⋅  ⋅  1  1
 ⋅  ⋅  ⋅  ⋅  1  1  1  1         =#
```

Of course, `(typeof(FV)==Cells) && (typeof(KFV)==ChainOp) # =) true`, since `lar2cop` : Cells → `ChainOp`. □

Remark 4.1 (About sparsity) While in this small case, the `KFV` matrix is not very sparse, the sparsity overgrows as the cellular complex proliferates, since the non-zero elements grow linearly with the number of cells. In contrast, the zero elements grow quadratically (with the matrix elements).

Remark 4.2 (Global view of incidencies) See, by inspection of `KFV`, that each face is incident to 4 cube vertices and each vertex is incident to 3 faces.

4.2 Plasm parametric primitives

In this section, we introduce and exemplify several features of the working of `Plasm` with geometric objects. The `Plasm` package is quite different from most geometric and graphics `API`s since it is based on `FL`-style combinators.

4.2.1 Geometric Transformations

The user interface to affine coordinate transformation of geometric objects is given through standard Julia functions and matrices. Internally, `Plasm` implements such mapping of local coordinates using its own multidimensional matrix type, called `MatrixNd`, and the use of the field `T::MatrixNd` within the recursive datatype `Hpc`.

Definition 4.1 (Geometric transformation) A *geometric transformation* is a bijective function, i.e., a one-to-one (injective) and onto (surjective) mapping $\mathbb{E}^d \to \mathbb{E}^d$.

By definition, geometric transformations of plane or space are invertible, and hence represented by invertible square matrices. We will see that *rotation*, *scaling*, and *shearing* are linear transformations; *translation* is affine.

Homogeneous Coordinates

In computer graphics the *homogeneous coordinates* are often used instead of Cartesian coordinates. In the homogeneous plane or space, lines are mapped to lines, but parallel lines are not conserved parallel. The main reason for this change is the ability to treat affine maps (translation) as linear, and combine smoothly with linear maps (rotation, scaling, etc.)

In homogeneous coordinates the Euclidean plane $\mathbb{E}^2 \setminus \{0\}$ is considered in bijective correspondence with the bundle of lines in $\mathbb{E}^3 \setminus \mathbb{E}^2$ (a model for the projective plane) so that each point $(x, y) \in \mathbb{E}^2$ corresponds to a line $\lambda(W, X, Y)$ such that $(x, y) \equiv \frac{W}{W}, \frac{X}{W}, \frac{Y}{W}) = (1, x, y)$. Same for each $\mathbb{E}^d$, $d \geq 2$. After the division, homogeneous coordinates are said *normalized*.

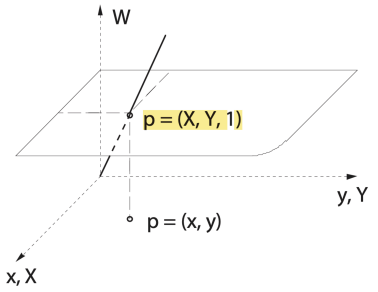


Fig. 4.1 The homogeneous plane is a model of a projective plane, where all finite points have a homogeneous coordinate equal to one, and the points at infinity have it equal to zero. All the points at infinity form the line at infinity, and all the lines at infinity form the plane at infinity.

In `Plasm`, by design choice to make the multidimensional approach to geometric design more accessible, the added homogeneous coordinate is the first, not the last, as we may see in many computer graphics books.

Even more, for the sake of clarity, we can use the `HOMO` operator to transform a $d \times d$ matrix in a $(d+1) \times (d+1)$ matrix, i.e., a 3×3 on the 2D plane and 4×4 on 3D space. The type of returned matrix is `MatrixNd`, which is used for dimension-independent programming.

Homogeneous coordinates allow to combine linearly all transformations, using products of their matrices in homogeneous coordinates. In the remainder of this section we describe the geometric effect of each transformation and the structure of the corresponding matrices.

Remark 4.3 The reader should note that our maps or transformations are invertible functions of a space into itself (automorphisms), represented (even translation, as we will see) by square matrices, i.e. are *rank 2* tensors. Since they are also `Plasm` functions, can be *applied* to geometric objects; as matrices, they multiply the object coordinates.

2D rotation

In a *planar rotation* all points of the 2D plane move along an arc of circle, with same angle at center, while the center is the only fixed point. In a *space rotation* there is a straight line of fixed points (the axis) passing for the origin. All the other 3D points describe a circle arc with the same angle along the plane (orthogonal to rotation axis) which they belong to.

Let us show (see Figure **??**) how unit vectors $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, columns of the matrix $\begin{pmatrix} e_1 & e_2 \end{pmatrix}$ are transformed by the (yet unknown) $R(\alpha)$ rotation matrix into the columns of the matrix at right-hand side:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = R(\alpha) \begin{pmatrix} e_1 & e_2 \end{pmatrix}. \quad \text{Hence we have:} \quad R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

There is only one class of planar rotations, parameterized by α, the *rotation angle* about the origin. Conversely, we will see three classes of elementary space rotations, parameterized by $\alpha_x, \alpha_y, \alpha_z$, the rotation angles about each coordinate axis.

Coding 4.2.1 (Plasm notation for rotation) The plane rotation function in `Plasm` is: `R([1,2])(α)` because its effect is to change the first and second coordinates of the 2D model it is applied to. They are applied to a planar geometric object of `Hpc` type, by using the `STRUCT` operator (see Section **??**) that contains `Hpc` values and transformation tensors:

```
SQUARE(d) = CUBOID([d,d])       #=
SQUARE (generic function with 1 method)     =#

obj = R(1,2)(π/4)(SQUARE(1))        #=
```

```
Hpc(MatrixNd([[1.0, 0.0, 0.0], [0.0, 0.7071067811865476,
    -0.7071067811865475], [0.0, 0.7071067811865475,
    0.7071067811865476]]), Hpc(MatrixNd(3), Hpc(MatrixNd(3),
    Geometry([[0.0, 0.0], [1.0, 0.0], [1.0, 1.0], [0.0, 1.0]],
    hulls=[[1, 2, 3, 4]])))) =#

VIEW(obj)
```

Remark 4.4 `CUBOID(shape)::Hpc` is the generator of multidimensional hyperparallelopipeds, depending on length and content of `shape` vector.

`CUBOID([1,1])` is the unit square; `CUBOID([1,2,3])` is the parallelopiped of sides 1, 2, and 3; `CUBOID([1,1,1,1])` is the 4D unit hypercube.

Elementary rotations

The multidimensional `Plasm` language has the following definition of elementary rotation, that allows to rotate a r-model ($r \leq d$) in any dimension $d \geq 2$.

Definition 4.2 (Elementary rotation) The reader should note that the *elementary* rotation is defined in any dimension d such that only 2 coordinates are changed by the rotation.

Remark 4.5 It is easy to see that in any dimension d there are `binomial(d,2)` elementary rotations, how many are the ways to choose 2 coordinates over d. Hence we have 1 for $d = 2$, 3 for $d = 3$, 6 for $d = 3$, and so on.

Assume that the rotation axes are e_1, e_2, e_3, with rotation angles α, β, γ respectively. The corresponding elementary matrices, derivable as before by change of coordinates, are:

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \; R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}, \; R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.1}$$

In `Plasm` the elementary rotations are represented, respectively, by the tensors: `R([2,3])(`α`)`, `R([1,3])(`β`)`, and `R([1,2])(`γ`)`.

Coding 4.2.2 (Elementary rotation) We use an interesting 3D polyhedron, called permutahedron, to show the application of the tensor `R([1,2])(pi/2)` to it:

```
obj = R([1,2])(pi/2)( PERMUTAHEDRON(3) );
VIEW( obj )
```

Remark 4.6 (Reduction of visual noice) Just note that in all `Plasm` geometric operators, the constraint of using functions as unary has been relaxed, in order to make possible to write, e.g., `obj = R(1,2)(pi/2)(obj)` instead than `obj = R([1,2])(pi/2)(obj)`. In the remainder we use always this new style.

Coding 4.2.3 (Permutahedron) The reader might be curious to see how such important and beautiful polyhedron [39] whose vertex coordinates are the permutations of the first d natural numbers. Iy is generated in `Plasm`:

```
function PERMUTAHEDRON(d)
    vertices = ToFloat64(PERMUTATIONS(collect(1:d+1)))
    center = MEANPOINT(vertices)
    cells = [collect(1:length(vertices))]
    object = MKPOL(vertices, cells, [[1]])
    object = T(INTSTO(d))(-center)(object)
    for i in 1:d
        object = R(i,d+1)(pi/4)(object)
    end
    object = PROJECT(1)(object)
    return object
end
```

The `Plasm` function `INTSTO(`d`)` (integers to d) is used to generate the sequence `[1,2,...,d]`, extremes included. The other functions are easy to understand.□

General rotation in 3D

A rotation of 3D space has a fixed line of points (the rotation axis) passing through the origin. We may compute the corresponding matrix as a function of a direction vector for the axis and a real value for the rotation angle. For this purpose we can compone three linear transformations by multiplication of their matrices. Therefore we have:

Definition 4.3 (General 3D rotation with axis d and angle α) Clearly, the ordering of transformations is from right to left:

$$R(d,\alpha) = Q^{-1}(d)\, R_z(\alpha)\, Q(d)$$

First, a space rotation that brings the vector d on a coordinate axis, say e_3; second, a space rotation $R_z(\alpha)$ about the z-axis; third, the inverse of the first transformation, so to bring the rotation axis in its original direction.

$Q(d)$ must transform the unit vector d to the e_3 unit vector. So, we may compute the coordinate transformation that brings three orthonormal vectors (u_1, u_2, u_3) to become the standard basis (e_1, e_2, e_3). We can choose the triple:

$$\begin{aligned} u_3 &= d/\|d\|, \\ u_2 &= (u_3 \times e_3)/\|u_3 \times e_3\|, \\ u_1 &= u_2 \times u_3, \end{aligned} \tag{4.2}$$

to write the transformation of coordinates:

$$(e_1, e_2, e_3) = Q(d)\,(u_1, u_2, u_3)$$

so that we have:

$$Q(d) = (u_1, u_2, u_3)^{-1}$$

But $Q(d)$ maps orthonormal vectors to orthonormal vectors, hence it is a normal transformation, so its inverse is equal to its transpose. So, we can write:

$$R(d, \alpha) = Q^t(d)\, R_z(\alpha)\, Q(d). \tag{4.3}$$

Coding 4.2.4 (3D General Rotation matrix) Let's use a test-driven programming style, with parameter values easy to test Rotate of 45 degrees about the diagonal axis the unit cube with a vertex on the origin, i.e. the model generated by the `CUBE(1)` expression in `Plasm`. We may follow this procedure using a functional approach:

```
using Plasm, LinearAlgebra
d = [1,1,1];
u₃ = normalize(d);
u₂ = normalize(u₃ × [0,0,1]);
u₁ = u₂ × u₃;
```

and write the following matrix for the transformation of coordinates that maps the e_3 axis to the direction of the `d` vector.

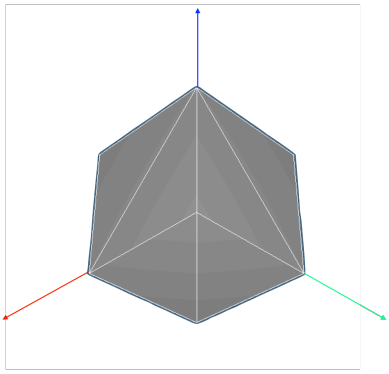
```
Q(d) = [u₁ u₂ u₃]'
```

The single quote stands for the Julia `transpose` of a matrix. □

Coding 4.2.5 (General 3D rotation tensor) In what follows, `MAT` transforms a Julia `Matrix` into a `Plasm` tensor applicable to `Hpc` values. The `HOMO` function apply to a square matrix, adding new unitary first row and column, for homogeneous coordinates (see Section **??**).

```
GR(d,α) = MAT(HOMO(Q(d)')) ∘ R(1,2)(α) ∘ MAT(HOMO(Q(d)))
```

The `GR` (general rotation) is a `Plasm` tensor depending on the axis `d` and the angle α. Our geometric model is therefore rotated and viewed as follows. □



```
rotated = GR([1,1,1],π/3)(CUBE(1))
VIEW(rotated)
```

Scaling

In a scaling transformation, all points are moved along the line passing for the origin they belong to. The scaling is said *elementary* when only one of the coordinates changes. There are two scaling parameters s_x, s_y in 2D geometry and three scaling parameters s_x, s_y, s_z in 3D, to be used in scalar products by the point coordinates. The transformation can be a *dilatation* of space when scaling parameters are greter than one, or a *contraction* of space when scaling parameters are lesser than one.

$$S(s_x,s_y,s_z) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix},\ S_x = \begin{pmatrix} s_x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ S_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix},\ S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s_z \end{pmatrix}$$

The scaling matrices are diagonal. The origin remains fixed. In fact:

$$S\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^t = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^t.$$

Hence, a scaling transformation is linear. It is easy to see that scale transformations are multiplicative, commutative, and associative because the matrix is diagonal:

$$S(s_x,s_y,s_z) = S_x(s_x)\, S_y(s_y)\, S_z(s_z)$$

.

Coding 4.2.6 (How to scale a Plasm model?) As in the previous coding example, let's go to use the `cube(1)` as our model object.

```
scaledcube1 = S(1,2,3)(.1,.1,10)(CUBE(1))
scaledcube2 = S(3)(100)(CUBE(1))
```

Coding 4.2.7 (How to scale a Plasm model?) Note that the effect of transformations impacts only the homogeneous matrices ahead of `Hpc` values.

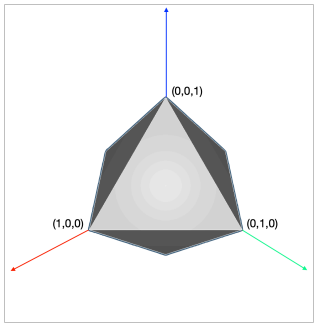
```
scaledcube1 = S(1,2,3)(.1,.1,10)(CUBE(1))   #=
Hpc(MatrixNd([[1.0, 0.0, 0.0, 0.0], [0.0, 0.1, 0.0, 0.0], [0.0,
    0.0, 0.1, 0.0], [0.0, 0.0, 0.0, 10.0]]), Hpc(MatrixNd(4),
    Hpc(MatrixNd(4), Geometry([[0.0, 0.0, 0.0], [1.0, 0.0, 0.0],
     [0.0, 1.0, 0.0], [1.0, 1.0, 0.0], [0.0, 0.0, 1.0], [1.0,
    0.0, 1.0], [0.0, 1.0, 1.0], [1.0, 1.0, 1.0]], hulls=[[1, 2,
    3, 4, 5, 6, 7, 8]])))) =#
scaledcube2 = S(2)(100)(SQUARE(1))  #=
Hpc(MatrixNd([[1.0, 0.0, 0.0], [0.0, 1.0, 0.0], [0.0, 0.0,
    100.0]]), Hpc(MatrixNd(3), Hpc(MatrixNd(3), Geometry([[0.0,
    0.0], [1.0, 0.0], [1.0, 1.0], [0.0, 1.0]], hulls=[[1, 2, 3,
    4]])))) =#
```

Of course, `S(1,2,3)(.1,.1,10)` and `S(2)(100)` are tensor objects. □

Coding 4.2.8 (Construction of octahedron model) As an exciting coding example, we show a simple construction of an octahedron model, starting from the 3D SIMPLEX model.

```
tetra = SIMPLEX(3);
twotetra = STRUCT( tetra, S(1)(-1), tetra );
fourtetra = STRUCT( twotetra, S(2)(-1), twotetra );
octahedron = STRUCT( fourtetra, S(3)(-1), fourtetra );
```

Looking at the whole cellular complex corresponding to the solid model `octahedron::Hpc` is worthwhile. For this purpose, we transform it into an object of `Lar` type:




```
VIEW(octahedron::Hpc)
```

Fig. 4.2 Plasm viewing generator expression. Remember that VIEW applies to Hpc values.

Coding 4.2.9 (The cellular complex) Let's note that the `octahedron::Hpc` is viewed, and that the `octahedron::Lar` is explored for stored data:

```
LAR(Octahedron).V   #=
3×7 Matrix{Float64}:
  0.0  -1.0  0.0   0.0  1.0  0.0  0.0
 -1.0   0.0  0.0   0.0  0.0  1.0  0.0
  0.0   0.0  0.0  -1.0  0.0  0.0  1.0   =#
LAR(Octahedron).C   #=
Dict{Symbol, AbstractArray} with 3 entries:
  :CV =) [[1, 2, 3, 4], [1, 3, 4, 5], [2, 3, 4, 6], [3, 4, 5,…
  :FV =) [[1, 2, 3], [2, 3, 4], [1, 3, 4], [1, 2, 4], [1, 3, …
  :EV =) [[2, 3], [1, 3], [1, 2], [3, 4], [2, 4], [1, 4], [3,…=#
```

For `#C, #F, #E, #V`, we see, looking at `.V` and `.C` above:

```
AA(LEN)(values(octahedron.C))'  #=
1×3 adjoint(::Vector{Int64}) with eltype Int64:
 8  20  18      =#
```

Therefore, we may see that the combinatorial (simplicial) complex corresponding to the 3D octahedron is made by $8 + 20 + 18 + 7 = 53$ cells of dimension 3, 2, 1, and 0, respectively (see Figure 4.2). □

Shearing

In a 2D elementary *shearing tranformation* all points of each line (plane in 3D) orthogonal to a coordinate axis move by summing one (fixed) vector. The coordinate line (plane in 3D) remain fixed, and the translation vector change linearly with the distance of its line (plane) from the origin. Each of two elementary planar shearing depends on a single scalar parameter (the translation of the line at unit distance from the coordinate line).

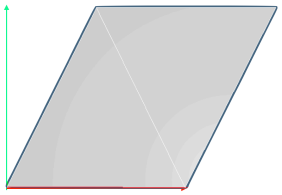
Each of the three elementary space shearing in 3D conversely depends on two scalar parameters (the coordinates of the planar translation vector of the plane at unit distance from the coordinate plane. Their 2D and 3D matrices are as follows.

$$H_x(h_x) = \begin{pmatrix} 1 & 0 \\ h_x & 1 \end{pmatrix}, \qquad H_y(h_y) = \begin{pmatrix} 1 & h_y \\ 0 & 1 \end{pmatrix};$$

$$H_x(h_y, h_z) = \begin{pmatrix} 1 & 0 & 0 \\ h_y & 1 & 0 \\ h_z & 0 & 1 \end{pmatrix}, \; H_y(h_x, h_z) = \begin{pmatrix} 1 & h_x & 0 \\ 0 & 1 & 0 \\ 0 & h_z & 1 \end{pmatrix}, \; H_z(h_x, h_y) = \begin{pmatrix} 1 & 0 & h_x \\ 0 & 1 & h_y \\ 0 & 0 & 1 \end{pmatrix}.$$

An elementary shearing differs from the identity matrix only for the elements of a single column, both in 2D and in 3D, and also in homogeneous 4D coordinates. In `Plasm`, the shearing tensor is named `H` and has the following semantics: first, indicate the column index; then give the $d-1$ ordered trans-

formation parameters, i.e., one in 2D and two in 3D, some of which possibly zeros. Therefore we have `H(col)(pars)`.

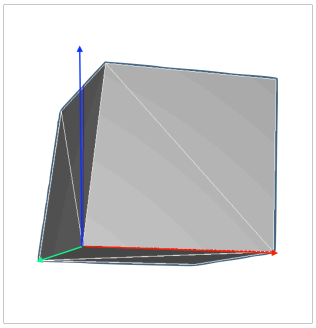


```
SQUARE(d) = CUBOID([d,d])
shearedsquare = H(2)(.5)(SQUARE(1))

VIEW(shearedsquare)
```

Fig. 4.3 Unit square sheared on the (second) coordinate y. The y of model points does not change.

Typically, `shearing` is used by typesetting systems of computerized typography to get *italic* versions of character fonts. Note also above how we define a parametric square (with a vertex on the origin).



```
shearedcube = H(3)(.2,.3)(CUBE(1))

VIEW(shearedcube)
```

Fig. 4.4 Unit cube sheared on the (third) coordinate z. The z of points do not change.

It is worthwhile to remark that the `H` mapping, as `R`, `GR`, `S`, `T`, `MAT`, and `HOMO` are dimension-independent, so can be applied to models of whatever embedding dimension d of geometric models. Homogeneous *normalized* matrices are used for implementation purpose.

Translation

Definition 4.4 (Translation transformation) a translation is an invertible transformation of Euclidean space $\mathbb{E}^d$ generated by summing a fixed vector to all points.

A translation of plane $\mathbb{E}^2$ (or space $\mathbb{E}^3$) is not a linear transformation, since it moves the origin, but it is an *affine* transformation, since all $\mathbb{E}^2$ (or $\mathbb{E}^3$) mapped points change by sum with a fixed vector (an *affine action*). □

Hence, a 2D or 3D translation depends on two (or three) scalar parameters, i.e., by the coordinates of the *translation vector*. We may therefore translate, using coordinates, a generic vector $v = (v_i) \in \mathbb{E}^d$:

$$v' = v + t \tag{4.4}$$

where $t = (t_i)$ is the translation vector, applied to all points in $\mathbb{E}^d$.

Translation in homogeneous coordinates

The translation 4.4 is reduced to a linear transformation and hence is representable by a product with a square matrix when using normalized homogeneous coordinates. Let's remind our choice to use as homogeneous the first coordinate. For example, a translation of $\mathbb{E}^3$ is representable as

$$T(t_x, t_y, t_z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_x & 1 & 0 & 0 \\ t_y & 0 & 1 & 0 \\ t_z & 0 & 0 & 1 \end{pmatrix} \tag{4.5}$$

We can see the equivalence between translation with Cartesian coordinates and homogenous normalized (affine) coordinates. Let $v = (x, y, z)$ be a point in 3D Euclidean space $\mathbb{E}^3$, and $v' = (w = 1, x, y, z)$ the same point in $\mathbb{R}^4$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ z + t_z \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_x & 1 & 0 & 0 \\ t_y & 0 & 1 & 0 \\ t_z & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ x + t_x \\ y + t_y \\ z + t_z \end{pmatrix}$$

Just notice that a translation in 3D is actually a shearing $H_w(t_x, t_y, t_z)$ orthogonal to the added component in normalized homogeneous coordinates.

Coding 4.2.10 (Translation of 3D geometric object) In `Plasm` we translate a geometric object of `Hpc` type via tensor application:

```
t_cube1 = T(1,2,3)(.5,.5,.5)(CUBE(1))
t_cube2 = T(3)(1)(CUBE(1))

VIEW(t_cube1);  VIEW(t_cube2)
```

A triple application of `T` function is needed: first to indices, then to translation paramenters, finally to the object of `Hpc` type to be translated. □

Coding 4.2.11 (Parametric linear ladder stair) The `step` is a `Hpc` solid obtained by product of three line segments of given sizes. An array of n pairs `[move, step]` is generated and concatenated by the `CAT` operator. Finally,

the semantics of STRUCT aggregator combinator (see Section 4.3) produces the whole parametric object, shown in Figure 4.5.

```
function Ladder(lx,ly,lz, n)::Hpc
    step   = QUOTE(lx) * QUOTE(ly) * QUOTE(lz)
    move   = T(2,3)(0.8*ly, 0.8*lz)
    ramp   = STRUCT( CAT([[step, move] for k=1:n]) )
end #=
Ladder (generic function with 1 method) =#
stair = Ladder(.8, .22, .18, 15);

VIEW(stair)
```

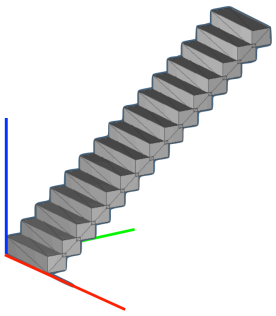


Fig. 4.5 Simple linear scale, demostrating an iterative use of tensors in STRUCT. Of course, not only the number, but also the size and even the shape of step model can be parametrized, as arguments of a geometric function returning Hpc objects.

4.3 Assembly of geometric objects

Complex shapes are usually defined as hierarchical assemblies of either geometric primitives or more complex shapes, each defined in a local coordinate system.

Most graphics and modeling systems implement this semantics as a tree or *hierarchical graph*, where affine geometry within the *nodes* is defined in local coordinate frames, and arcs are associated to affine transformations that move the whole subgraph rooted in the ending node onto the coordinate system of the first node of the arc. The very first node is called *root* of the data structure.

4.3.1 Hierarchical graphs

Acyclic graphs/multigraphs are also called *hierarchical graphs*, because can be associated to a tree, generated at run-time by visiting the graph with some standard traversal algorithm [10], e.g., with a depth-first-search (DFS). The ordered sequence of nodes produced by the traversal is sometimes called a *linearized graph*. Each node in this sequence is suitably transformed from local coordinates to *world coordinates*, i.e. to the coordinates of the root, by the traversal algorithm.

The main ideas concerning *scene graphs* can be summarized as follows. Nodes are *containers* of geometrical datasets stored in *local* coordinates. Nodes are also used and implemented as root of subgraphs, whose data are transformed to the node coordinates by a traversal algorithm. Arcs (a, b) are associated with affine transformations, which map the data contained in b from their local coordinates to the coordinates of a. More than one arc may exist between the same node pair. This allows storage in memory only of *one copy* of each container. The composite transformations of coordinates applied to the linearized graph generated at traversal time are collectively known as the *modeling transformation*.

Object Transform

Any `Plasm` geometric value of `Hpc` type can be affinely transformed by direct application of an affine tensor to it.

Coding 4.3.1 (Direct use of Tensor) Let's combine with other language tensors, while generating the translated 1-skeleton of a cube:

```
SK = SKELETON;
translatedcube = (SK(1) ∘ T(1)(1) ∘ CUBE)(1);
```

Coding 4.3.2 (Example.2) Then, aggregate two objects into a single object within the *same* coordinate frame:

```
singleframe = STRUCT(cube, tetra);
VIEW(singleframe)
```

Coding 4.3.3 (Example.3) *Direct* application of tensor $T_z(1)$, followed by application of $R_z(-\pi/2)$ to `tetra` value, which changes accordingly:

```
doubleframes = STRUCT(cube, (R(1,2)(-π/2) ∘ T(3)(1.0))(tetra)
VIEW(doubleframes)
```

Coding 4.3.4 (Example.4) Exactly the same effect could be obtained by the following expression, because *a transformation tensor is implicitly applied to every geometric value following it* in the parameter sequence of a `STRUCT` combinator. Evaluation is right-to-left, according to math composition rule:

```
doubleframe = STRUCT(cube, R(1,2)(-π/2), T(3)(1), tetra )
```

Assembly of components

Hierarchical models of complex assemblies are generated by aggregation of cellular complexes, each one defined in a local coordinate system, and possibly relocated by affine transformations of coordinates. This operation may be repeated hierarchically, with subassemblies defined by aggregation of simpler parts, and so on, until to have a set of leaves holding primitive models, which cannot be further decomposed.

Two main advantages can be found in a hierarchical modeling approach. Each component complex and each partial assembly, at every hierarchical level, are defined independently from each other, using their `PROPERTIES` and local coordinate frame, suitably chosen to make the definition easier. Furthermore, only one copy of each component is stored in memory, and may be instanced in different locations and orientations how many times it is needed.

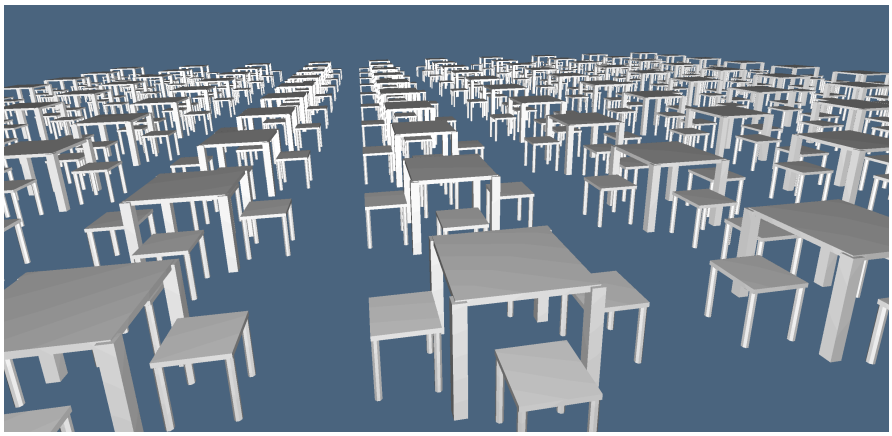


Fig. 4.6 Hierarchical assembly of cellular 3-complexes.

Directed Acyclic Graph (DAG)

A *hierarchical model*, defined inductively as an assembly of component parts, is described by an *acyclic directed multigraph*, often called a *scene graph* or *hierarchical structure* in computer graphics and modeling. The main algorithm with hierarchical assemblies is the *traversal* function, which transforms every component of the assembly from *local coordinates* to global coordinates, often called *world coordinates*.

Definition 4.5 (Directed graph) A directed graph G is a pair (N,A), where N is a set of *nodes* and A is a set of directed *arcs*, given as ordered pairs of nodes. □

Such a definition is not sufficient when more than one arc must be considered between the same pair of nodes. The notion of *multigraph* is hence introduced. In a multigraph, the same pair of nodes can be connected by multiple arcs.

Definition 4.6 (Directed multigraph) A directed multigraph is a triplet $G := (N,A,f)$ where N and A are sets of nodes and arcs, respectively, and $f : A \to \mathbf{N}^2$ is a mapping from arcs to node pairs. □

Directed graphs or multigraphs are said to be *acyclic* when they do not contain cycles, i.e. when no path starts and ends at the same vertex. *Trees* are common examples of acyclic graphs. A tree, where each non-leaf node is the root of a subtree, is the best model of the concept of *hierarchy*. Nodes in a tree can be associated with their integer *distance* from the root, defined by the number of edges on the unique path from the root to the node. A tree can be layered by *levels*, by putting in the same subset (level) all the nodes with equal distance from the root.

Hierarchical structures in Plasm

A *container* of geometrical objects is defined in `Plasm` by applying the combinator `STRUCT` to the contained objects' sequence (or array). The value returned from the function application is of type *hierarchical polyhedral complex* `Hpc`. The coordinate system used by the returned value is the one associated with the first geometric object of the argument sequence.

The resulting geometrical value is often associated with a variable used as the container's name, as in

```
obj = STRUCT( obj_1, obj_2, ..., obj_n ); VIEW(obj)
```

The `obj` geometry can be pictorially described, using the previously discussed graph model of hierarchical structures, as shown in Figure 4.7a. Clearly, each component object may in turn be defined as a container of other objects, i.e. as the root of a subgraph, as shown as shown below:

```
obj2 = STRUCT( obj21, ..., obj2m )
```

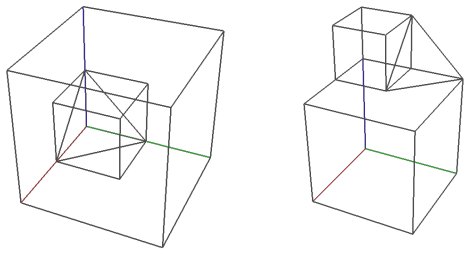


Fig. 4.7 Assembly by STRUCT: (a) without coordinate transformations. All three objects have the same origin; (b) with coordinate transformations.

Exactly the same geometric result would be generated by direct nesting of `STRUCT` sub-expressions:

```
obj = STRUCT(obj1, STRUCT(obj21, ..., obj2m) ..., objn)
```

The sequence argument of the `STRUCT` operator may either contain or not affine transformations, together with polyhedral complexes. This fact results in generating an assembly either by using the same (global) coordinates for the various components or by using different (local) coordinate systems. The two cases are discussed in the two following subsections, respectively.

Assembly with global vs local coordinates

Let's assume that the sequence argument of a `STRUCT` expression does not contain affine transformations. In other words, we assume that the evaluations of the Plasm expressions in the argument sequence only return polyhedral values. In this case, the output polyhedral complex is returned within the coordinate frame of the first element of the input sequence, and no transformations of coordinates are applied to the assembly components, which are only aggregated in the same space, as shown by the the following example.

Coding 4.3.5 (STRUCT assembly (1)) The expression given below returns the object of Figure 4.7a. Note that the three component shapes' local origin and coordinate axes coincide. The `SK(1)` operator (extraction of 1-skeleton) was `@1`, not available in Julia, in classic `PlaSM`).

```
cube2, cube1, simplex = CUBE(2), CUBE(1), SIMPLEX(3)
obj1 = (SK(1) ∘ STRUCT)( cube2, cube1, simplex )
```

Coding 4.3.6 (STRUCT assembly (2)) Here we aggregate the same geometric components used in Coding 4.3.5, but also add up some transformations of coordinates to the sequence of parameters of resulting assembly.

```
obj2 = (SK(1) ∘ STRUCT)(cube2, T(3)(2), cube1, T(2)(1), simplex)
```

The resulting geometric assembly is shown in 4.7b. □

STRUCT semantics

We assume that in Plasm, the word *tensor* stands for affine transformation. Let's suppose that tensors T_k are contained within the sequence argument of a STRUCT expression. Each tensor in a STRUCT is applied to all polyhedral complexes that follow it. The subsequent expressions are equivalent:

$$\begin{aligned}&\mathtt{STRUCT(\ pol_1,\ T_1,\ pol_2,\ T_2,\ pol_3,\ \dots\ ,\ T_{n-1},\ pol_n\ )} \equiv\\ &\mathtt{STRUCT(\ pol_1,\ T_1(pol_2),\ (T_1{\sim}T_2)(pol_3),\ \dots,\ (T_1{\circ}T_2{\circ}\cdots{\circ}T_{n-1})(pol_n)}\end{aligned}$$

Looking at the internal behavior of the geometric kernel of the language, the following maps are applied to the STRUCT application at evaluation time:

$$\begin{aligned}&\mathtt{STRUCT(\ pol_1,\ (T_1\ \circ\ STRUCT)(\ pol_2,\ (T_2\ \circ\ STRUCT)(\ pol_3,\ \dots}\\ &\quad\mathtt{(T_{n-2}\ \circ\ STRUCT)(\ pol_{n-1},\ T_{n-1}(pol_n)\ )\ \dots\ )\ )\ )}\end{aligned}$$

The above kind of evaluation (in DFS postorder) has inspired the actual implementation of the Hpc data structure. Note, looking at the geometric result shown in Figure 4.7b, that, according to Coding 4.3.6:

1. the output assembly is represented in the coordinate system of first cube;
2. the second cube is translated in z direction;
3. the unit tetrahedron is translated both in z and in y directions.

Traversal algorithms

There are many ways to visit (or walking) a graph or multigraph, traversing at least once every node or arc. The *traversal* of a hierarchical structure consists of a modified *Depth First Search* (DFS) of its acyclic multigraph,[1] where each arc — and not each node — is traversed only once. In particular, each node is traversed a number of times equal to the number of different paths that reach it from the root node.

[1] Notice that the standard *dfs* graph traversal (see e.g. [**?**]) visits all the nodes once, since it works by recursively visiting those sons of each node that it has not already visited.

The aim of a traversal algorithm is to "linearize" a structure network, by transforming all its substructures (i.e. all the subgraphs) from their *local coordinates* to the coordinates of the root node, assumed as *world coordinates*.

For this purpose, a matrix denoted as the *current transformation matrix* (CTM) is maintained. Such a CTM is equal to the product of matrices associated with the arcs of the current path from the root to the current node. For the sake of efficiency, the traversal algorithm is implemented by using a stack of CTMs. When a new arc is traversed, the old CTM is pushed on the stack, and a new CTM is computed by (right) multiplication of the old one times the matrix of the arc. When unfolding from the recursive visit of the subgraph appended to the arc,[2] the CTM is substituted by the one popped from the stack. The TRAVERSAL algorithm is specified in pseudo-language below.

Script 4.3.1 (Traversal of a multigraph)

```
algorithm TRAVERSAL ((N,A,f) : multigraph)
    CTM := identity matrix;
    TraverseNode (root)

proc TRAVERSENODE (n : node)
    foreach a ∈ A outgoing from n do
        TraverseArc (a);
        ProcessNode (n)

proc TRAVERSEARC (a = (n,m) : arc)
    Stack.push (CTM);
    CTM := CTM * a.mat;
    TraverseNode (m);
    CTM := Stack.pop()

proc PROCESSNODE (n : node)
    foreach object ∈ n do
        Process( CTM*object )
```

□

The CTM is normally used to (left) multiply the vertices of geometric objects stored in the traversed containers. But the reader should remember that equations of hyperplanes and normal vectors must be conversely (right) multiplied for the inverse of the applied transformation, according to the mapping of covectors discussed in Section **??**. A double stack of matrices, where to push/pop both the CTM and its current inverse, may therefore speed up the traversal. As a result of the algorithm, a linearized model in world coordinates is produced, which may be used, e.g., for rendering purposes, as discussed in the next chapter.

---

[2] Using a pictorial image, we could say: when the arc is traversed in the opposite direction.

Assembly examples

The two examples given in this section are finalized to introduce some powerful Plasm higher-level functions and combinators. In particular, we use STRUCT and MAP, CAT, DIESIS, PROPERTIES, and THINSOLID. The geometric primitives CUBE, CYLINDER, and SIMPLEX are also used.

Refectory room model

Some coordinated coding examples are given here, to show the development bottom-up of the geometric model of a refectory room with its main appliances, tables and user chairs. The room is visualized in Figure 4.6.

Coding 4.3.7 (Table) Both the tableTop and tablelegs with 4 tableLeg instances are generated starting from 3D cube of side 1:

```
cube = T(1,2)(-.5,-.5)(CUBE(1));
tableTop  = STRUCT( T(3)(.85), S(1,2,3)(1,1,.05), cube );
tableLeg  = STRUCT( T(1,2)(-.475,-.475), S(1,2,3)(.1,.1,.89),
    cube );
tablelegs = STRUCT( CAT(DIESIS(4)([tableLeg, R(1,2)(π/2)])) );
table = STRUCT( tableTop, tablelegs );
VIEW( table )
```

The operator DIESIS (#) in old Plasm repeats 4 times its second argument.□

Coding 4.3.8 (Chair) The chair object is made by a chairTop and 4 cylindrical chairLeg with radius 0.06 and height 0.5:

```
cylinder  = CYLINDER([.06, .5])(8)
chairTop  = STRUCT( T(3)(0.5), S(1,2,3)(0.5,0.5,0.04), cube );
chairLeg  = STRUCT( T(1,2)(-.22,-.22), S(1,2)(.5,.5), R(1,2)(π
    /8), cylinder );
chairlegs = STRUCT( CAT(DIESIS(4)([chairLeg, R(1,2)(π/2)]) );
chair = STRUCT( chairTop, chairlegs );
VIEW( chair )
```

Coding 4.3.9 (Four sits) Four sits are produced by alternating chairs with local rotations in object fourChairs:

```
theChair   = STRUCT( T(1)(-.8), chair )
fourChairs = STRUCT( CAT(DIESIS(10)([R(1,2)(π/2), theChair]) );
fourSit    = STRUCT( fourChairs, table );
VIEW( fourSit )
```

Coding 4.3.10 (Single row of tables and chairs) The single 4-sit's row is generated by alternating 10 instances of `fourSit` and T(2)(2.5) tensors:

```
singleRow = STRUCT( CAT(DIESIS(10)([fourSit, T(2)(2.5)])) );
VIEW( singleRow )
```

Coding 4.3.11 (Whole refectory) Analogously, the refectory room is furnished by juxtaposing 10 instances of `singleRow` and x-translations:

```
refectory = STRUCT( CAT(DIESIS(10)([singleRow, T(1)(3)])) );
VIEW( refectory )
```

Plasm design of a turbo pump

This section discusses the generation of a highly parameterized family of geometric objects stepwise. First, a code template `SOLIDHELICOID`, is given, and then a definitive model, `TURBOPUMP`, is produced. This is impossible to definie or build with the any `GUI`-based CAD system.

Coding 4.3.12 (Solid helicoid) A parametric helicoid surface and solid follow. The source code represents a whole family of ∞^6 different shapes:

```
function SOLIDHELICOID(; nturns=3,R=1.,r=0.0,shape=[36*nturns,
    8], pitch=2, thickness=0.1)
   totalangle = nturns*2*pi
   grid2D = INTERVALS(36*nturns)(36*nturns)*INTERVALS(4)(8)
   Domain2D=T(2)(r)(S(1,2)([totalangle/shape[1],R-r])(grid2D))
   surface = p-)((u,v)=p;[v*cos(u);v*sin(u);u*(pitch/(2*pi))])
   solidMapping = THINSOLID(surface)
   Domain3D = Domain2D * INTERVALS(thickness)(1);
   view = Dict("background_color"=)[1.,1,1])
   VIEW(MAP(surface)(Domain2D), view)
   VIEW(MAP(solidMapping)(Domain3D), view)
end;

julia) SOLIDHELICOID( r=0.4, thickness=0.2 )
```

An atypical method was used in `SOLIDHELICOID` template, in order to `VIEW` both the first step (a parametric surface), and the definitive parametric solid.

Coding 4.3.13 (Turbo pump) Starting The main difference with `SOLIDHELICOID` template is the piecewise-linear `Dom2D` mapped from the rectangular `grid2D`.

```
function TURBOPUMP(; nturns=3, R=1., r=0.0, shape=[36*nturns,8],
     pitch=2, thickness=0.1)
```

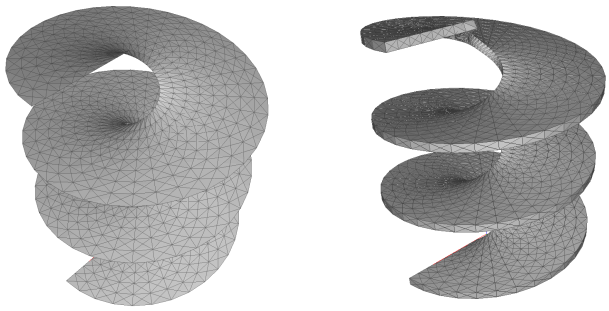


Fig. 4.8 Parametrized helicoid values: (a) surface; (b) thin solid.

```
    totalangle = nturns*2*pi;  xM = 36*nturns
    (x0,xm1,xm2,xM) = (0.0, xM/2nturns, 6xM/2nturns, xM)
    G = (x-) x<xm1 ? x/xm1 : (x)xm2 ? 1+(xm2-x)/(xM-xm2) : 1))
    grid2D = INTERVALS(xM)(xM) * INTERVALS(1)(8)
    dom = MAP( p-)begin (u,v)=p; [u, (G(u)+0.1)*v] end )(grid2D)
    Dom2D= T([2])([r])(S([1,2])([totalangle/shape[1],R-r])(dom))
    surface = p-)((u,v)=p;[v*cos(u); v*sin(u); u*(pitch/(2*pi))])
    solidMapping = THINSOLID(surface)
    Domain3D = Dom2D * INTERVALS(thickness)(1)
    return MAP(solidMapping)(Domain3D)
end
```

The piecewise linear `dom` shape is produced by the Julia function assigned to symbol `G`, implemented as a nested conditional triple statement. The three subdomain of piecewise `G` domain are defined by `(x0,xm1,xm2,xM)`.

Coding 4.3.14 (Turbo pump's object viewing)

```
obj = TURBOPUMP( pitch=0.15, thickness=0.015, nturns=5, r=1/2 );
view = Dict("background_color"=)[1,1,1]); VIEW(obj, view)
```

The other parameters have default value, given in the definition head. □

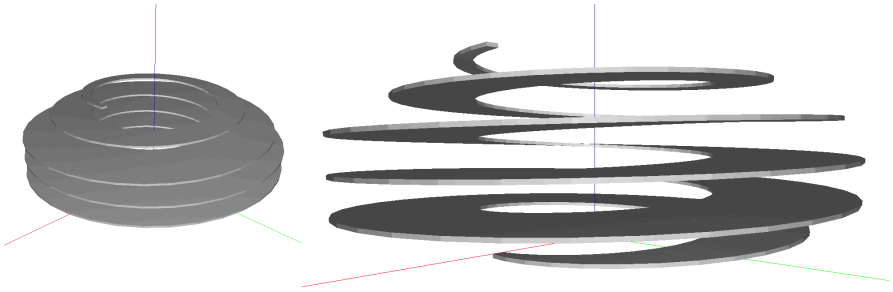


Fig. 4.9 Two prospective images of the turbo pump component: (a) from top; (b) from side.

4.4 Attach properties to geometry

4.5 Design documentation (Jupyter notebooks)

4.6 Export geometry

There are several ways to export either produced or in-progress geometric `Plasm` models, starting from directly exporting the Julia sources. Depending on the size of the building project, the digital project evaluated within a short number of seconds of computing time could easily amount to datasets many hundreds of thousands of times more significant.

Of course, the `Plasm` platform may also export/import Computer Graphics standard formats for polyhedral modeling, particularly using the boundary representation of the Boolean algebras' atoms associated with the building design. The simplest options for file transport of geometry are boundary triangulations or polygons, directly representable through the `PolygonSet` entity primitives of the IFC file standard.

Direct export/import of sources

Julia has two mechanisms for loading source code: *code inclusion* and *package loading* [**?**]. Code inclusion is relatively straightforward: it evaluates the given source file in the context of the caller process. It is good practice when writing a source file scheduled to export a geometric model (1) to choose a possibly long but semantically significant name and (2) to conclude each such file with one or more model `VIEW()`s of the exported building parts/aggregates. Inclusion also allows you to split a single program across multiple source files.

The expression `include("source.jl")` causes the contents of the file `"source.jl"` to be evaluated in the global scope of the module where the

include call occurs. The expression `include()` can be called multiple times with the same file, and the file is evaluated every time.

The second mechanism, *package loading*, is more general and powerful. The `import <name>` or `using <name>` allows you to load a package—i.e., an independent, reusable collection of Julia code, wrapped in one or more module, and makes the resulting software library available by the name `<name>` inside the importing module [**?**].

Remark 4.7 Creating a specialized Julia module or package for a specific architectural or engineering project may be of great professional interest and economic satisfaction. E.g., the project could import Julia packages for the building typology under consideration, if any. In architecture, typology refers to identifying and grouping buildings according to their kind of use and similarity of essential characteristics, such as Residential, Educational, Institutional, Commercial, Business, or Industrial Buildings. □

Remark 4.8 You can have subtypes in each type with characteristic morphology: for example, in residential higher education, or in various types of specialty hospital establishments, which must follow strict legal requirements, and so on. In addition, the specific design package might import parametric `Plasm` functions already developed for similar constructions and enrich them with new function methods finalized to the singular, unique design structure and shape under development. □

This kind of digital innovation process, on top of BIM and IFC standards (see Chapter 10), could be of significant interest in case of development programs launched as sectorial construction interventions by governments (say housing, schools, etc.) or by charitable foundations in developing countries, where some kind of standardization can greatly reduce the program costs.

Unevaluated/Evaluated/Binary LAR format

Linear Algebraic Representation (`LAR`) is the name of the algebraic geometrical data structures and computational pipeline developed by the authors and others in the last decade, and embedded recently in `Plasm`. This data structure has various formats, depending of various advancement of computation. In particular, three LAR states may be enuciated: unevaluated, evaluated and binary.

Unevaluated LAR Sparse binary matrices that encode the cells of a cellular complex as incidence relations between d-cellls and 0-cells ($0 \leq d \leq 3$). All the binary relations between cells of different dimensions can be easily generated. The occupied storage space is less or equal to that of common incidence structures in graphics and solid modeling.

Evaluated LAR is a pair (*geometry*, t*opology*) where *geometry* is the matrix `Float64` of coordinates of 0-cells (`VE`), and *topology* is a triple (in 3D)

of sparse binary matrices, encoding the coboundary operators $\delta_0, \delta_1, \delta_2$ corresponding to incidence relations `VE`, `EF`, and `FC`. These result by the 3D algorithmic *arrangement* pipeline that partitions the Euclidean 3s-pace on the boundary surfaces of any given collection of `Brep` solid models.

Binary LAR is the result of computation of the binary algebra induced by the spatial collocation and orientation of the whole collection of B-rep primitives and solids considered by the designer at the beginning of the computational pipeline, i.e., when all the geometric elements were collocated within the geometric design of the considered building portion. This binary matrix has number of columns equal to the number of computed atoms of the space arrangement and number of rows equal to the number of primitive shapes (of uniform material) used in a geometric design.

Every noteworthy subset of design atoms, selected by any reason—in particular, to assign material properties, catalog products, or compute volumes or cost—will be easily extracted as a subset of corresponding columns and rows.

IFC PolygonSet file format

Our preliminary tests showed that all the `Plasm` geometric shapes we defined or selected are exportable to transport file `IFC` as `PolygonSet` entities and their associated entities concerning polygons or triangles, vertex indices, and vertex coordinates. This point will be treated extensively in Chapter 10.

References

1. Arakaki., T.: Julia Data Parallel Computing. URL https://juliafolds.github.io/data-parallelism/tutorials/quick-introduction/. [retrieved june 27, 2023]
2. Arnold, D.N.: Finite Element Exterior Calculus, *CBMS-NSF Regional Conference Series in Applied Mathematics*, vol. 93. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (2018)
3. Aubanel, E.: Elements of Parallel Computing. Chapman Hall/CRC Press (2016)
4. Backus, J.: Can Programming Be Liberated from the von Neumann Style? A Functional Style and Its Algebra of Programs. Commun. ACM **21**(8), 613641 (1978). DOI 10.1145/359576.359579. URL https://doi.org/10.1145/359576.359579
5. Backus, J., Williams, J., Wimmers, E.: An introduction to the programming language FL. In: D. Turner (ed.) Research Topics in Functional Programming. Addison-Wesley, Reading, MA (1990)
6. Backus, J., Williams, J.H., Wimmers, E.L., Lucas, P., Aiken, A.: FL LANGUAGE MANUAL. PARTS l AND 2. Tech. Rep. RJ 7100 (67163), IBM Almaden Research Center (1989)
7. Bezanson, J., Edelman, A., Karpinski, S., Shah, V.B.: Julia: A fresh approach to numerical computing. SIAM Review **59**(1), 65–98 (2017). URL https://doi.org/10.1137/141000671
8. Carlsson, C.: Syntax highlighting – OhMyREPL. URL https://kristofferc.github.io/OhMyREPL.jl/latest/. [retrieved may 22, 2024]
9. Cimrman, R.: Sparse matrices in scipy. In: G. Varoquaux, E. Gouillart, O. Vahtras, P. deBuyl (eds.) Scipy lecture notes, release: 2022.1 edn., p. Section 2.5. Zenodo (2015). DOI 10.5281/zenodo.594102. URL https://scipy-lectures.org/advanced/scipy_sparse/index.html
10. Cormen, T.H., Leiserson, C.E., Rivest, R.L., Stein, C.: Introduction to Algorithms, Third Edition, 3rd edn. The MIT Press (2009)
11. Danisc, S., Kavalar, M., Dombrowski, M., Markovics, P.: An Introduction to GPU Programming in Julia. URL https://nextjournal.com/sdanisch/julia-gpu-programming. [retrieved june 29, 2023]
12. Delfinado, C., Edelsbrunner, H.: An incremental algorithm for betti numbers of sinplicial complexes on the 3-sphere. Computer Aided Geometric Design **12**, 771–784 (1995)
13. DiCarlo, A., Milicchio, F., Paoluzzi, A., Shapiro, V.: Chain-based representations for solid and physical modeling. Automation Science and Engineering, IEEE Transactions on **6**(3), 454 –467 (2009). DOI 10.1109/giorgio
14. DiCarlo, A., Paoluzzi, A., Shapiro, V.: Linear algebraic representation for topological structures. Computer-Aided Design **46**, 269–274 (2014). DOI 10.1016/j.cad.2013.08.044. URL https://doi.org/10.1016/j.cad.2013.08.044
15. DiCarlo, A., Paoluzzi, A., Shapiro, V.: Linear algebraic representation for topological structures. Comput. Aided Des. **46**, 269–274 (2014). DOI 10.1016/j.cad.2013.08.044. URL http://dx.doi.org/10.1016/j.cad.2013.08.044
16. Ferrucci, V.: Generalised extrusion of polyhedra. In: Proceedings on the Second ACM Symposium on Solid Modeling and Applications, SMA '93, p. 3542. Association for Computing Machinery, New York, NY, USA (1993). DOI 10.1145/164360.164376. URL https://doi.org/10.1145/164360.164376
17. Ferrucci, V., Paoluzzi, A.: Extrusion and boundary evaluation for multidimensional polyhedra. Comput. Aided Des. **23**(1), 4050 (1991). DOI 10.1016/0010-4485(91)90080-G. URL https://doi.org/10.1016/0010-4485(91)90080-G
18. Hatcher, A.: Algebraic topology. Cambridge University Press (2002)

19. Julia: Manual: Base/lib-collections/iteration. URL https://docs.julialang.org/en/v1/base/collections/#lib-collections-iteration. [retrieved june 25, 2023]
20. Julia: Manual: Distributed-Computing. URL https://docs.julialang.org/en/v1/manual/distributed-computing/#Multi-processing-and-Distributed-Computing. [retrieved june 29, 2023]
21. Julia: Manual: Linear Algebra. URL https://docs.julialang.org/en/v1/stdlib/LinearAlgebra/#man-linalg. [retrieved july 3, 2023]
22. Julia: Manual: LinearAlgebra.factorize. URL https://docs.julialang.org/en/v1/stdlib/LinearAlgebra/#LinearAlgebra.factorize/. [retrieved june 25, 2023]
23. Julia: Manual: Metaprogramming. URL https://docs.julialang.org/en/v1/manual/metaprogramming/. [retrieved june 21, 2023]
24. Julia: Manual: Modules. URL https://docs.julialang.org/en/v1/manual/modules/#modules. [retrieved june 29, 2023]
25. Julia: Manual: Multi-Threading. URL https://docs.julialang.org/en/v1/manual/multi-threading/#man-multithreading. [retrieved june 29, 2023]
26. Julia: Manual: Parallel Computing. URL https://docs.julialang.org/en/v1/manual/parallel-computing/#Parallel-Computing. [retrieved june 26, 2023]
27. Julia: Manual: Parallel Computing. URL https://docs.julialang.org/en/v1/base/parallel/#Tasks. [retrieved june 27, 2023]
28. Julia: Manual: stdlib/SparseArrays. URL https://docs.julialang.org/en/v1/stdlib/SparseArrays/#man-csc/. [retrieved june 25, 2023]
29. Julia: Manual: Types. URL https://docs.julialang.org/en/v1/manual/types/#man-types. [retrieved june 30, 2023]
30. Julia Community: How to develop a Julia package. URL https://julialang.org/contribute/developing_package/. [retrieved june 29, 2023]
31. JuliaGPU: A gentle introduction to parallelization and GPU programming in Julia. URL https://cuda.juliagpu.org/stable/tutorials/introduction/#Introduction. [retrieved june 29, 2023]
32. Kamiski, B.: Julia for Data Analysis. Manning (2023)
33. Mainon, P.: Writing type-stable julia code (2021). URL https://blog.sintef.com/industry-en/writing-type-stable-julia-code/
34. Paoluzzi, A.: Geometric Programming for Computer Aided Design. John Wiley Sons, Chichester, UK (2003). URL https://onlinelibrary.wiley.com/doi/book/10.1002/0470013885
35. Paoluzzi, A., Pascucci, V., Vicentino, M.: Geometric programming: a programming approach to geometric design. ACM Trans. Graph. **14**(3), 266–306 (1995). DOI 10.1145/212332.212349. URL http://doi.acm.org/10.1145/212332.212349
36. Paoluzzi, A., Scorzelli, G., Vicentino, M.: Securing the cultural heritage via geometric programming and modeling. Tech. rep., Dept of Computer Science and Engineering, Roma Tre University (2009). URL https://www.academia.edu/47017676
37. Paoluzzi, A., Shapiro, V., DiCarlo, A., Furiani, F., Martella, G., Scorzelli, G.: Topological computing of arrangements with (co)chains. ACM Trans. Spatial Algorithms Syst. **7**(1) (2020). DOI 10.1145/3401988. URL https://doi.org/10.1145/3401988
38. Paoluzzi, A., Shapiro, V., DiCarlo, A., Scorzelli, G., Onofri, E.: Finite algebras for solid modeling using julias sparse arrays. Computer-Aided Design **155**, 103436 (2023). DOI https://doi.org/10.1016/j.cad.2022.103436. URL https://www.sciencedirect.com/science/article/pii/S0010448522001695
39. Permutohedron: Permutohedron — Wikipedia, the free encyclopedia (2023). URL https://en.wikipedia.org/wiki/Permutohedron. [Online; accessed 31-May-2024]

40. Roth, A., Weisstein, E.W.: Standard basis. In: MathWorld, p. Algebra > Linear Algebra > Linear Systems of Equations. A Wolfram Web Resource (2005). URL https://mathworld.wolfram.com/StandardBasis.html
41. Rowland, T.: Characteristic function. In: From MathWorld, p. Foundations of Mathematics > Set Theory > Sets. A Wolfram Web Resource (2005). URL ttps://mathworld.wolfram.com/CharacteristicFunction.html
42. Scorzelli, G.: Pyplasm library (2023). URL https://libraries.io/pypi/pyplasm
43. Scorzelli, G., Paoluzzi, A.: Plasm.jl: v0.1.0 (2023). URL https://github.com/scrgiorgio/Plasm.jl
44. Weisstein, E.W.: Julia set. From MathWorld–A Wolfram Web Resource URL https://mathworld.wolfram.com/JuliaSet.html
45. Whitaker, S.: Mastering the Julia REPL. URL https://blog.glcs.io/julia-repl#heading-starting-the-julia-repl. [retrieved may 22, 2024]
46. Whitney, H.: Geometric Integration Theory. Princeton University Press, Princeton (1957). DOI doi:10.1515/9781400877577. URL https://doi.org/10.1515/9781400877577
47. Wikibooks: Introducing julia/dictionaries and sets — wikibooks, the free textbook project (2020). URL https://en.wikibooks.org/. [Online; accessed 20-June-2023]
48. Williams, J.H., Wimmers, E.L.: Sacrificing Simplicity for Convenience: Where Do You Draw the Line? In: Proceedings of the 15th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL '88, p. 169179. Association for Computing Machinery, New York, NY, USA (1988). DOI 10.1145/73560.73575. URL https://doi.org/10.1145/73560.73575