Epipolar Geometry

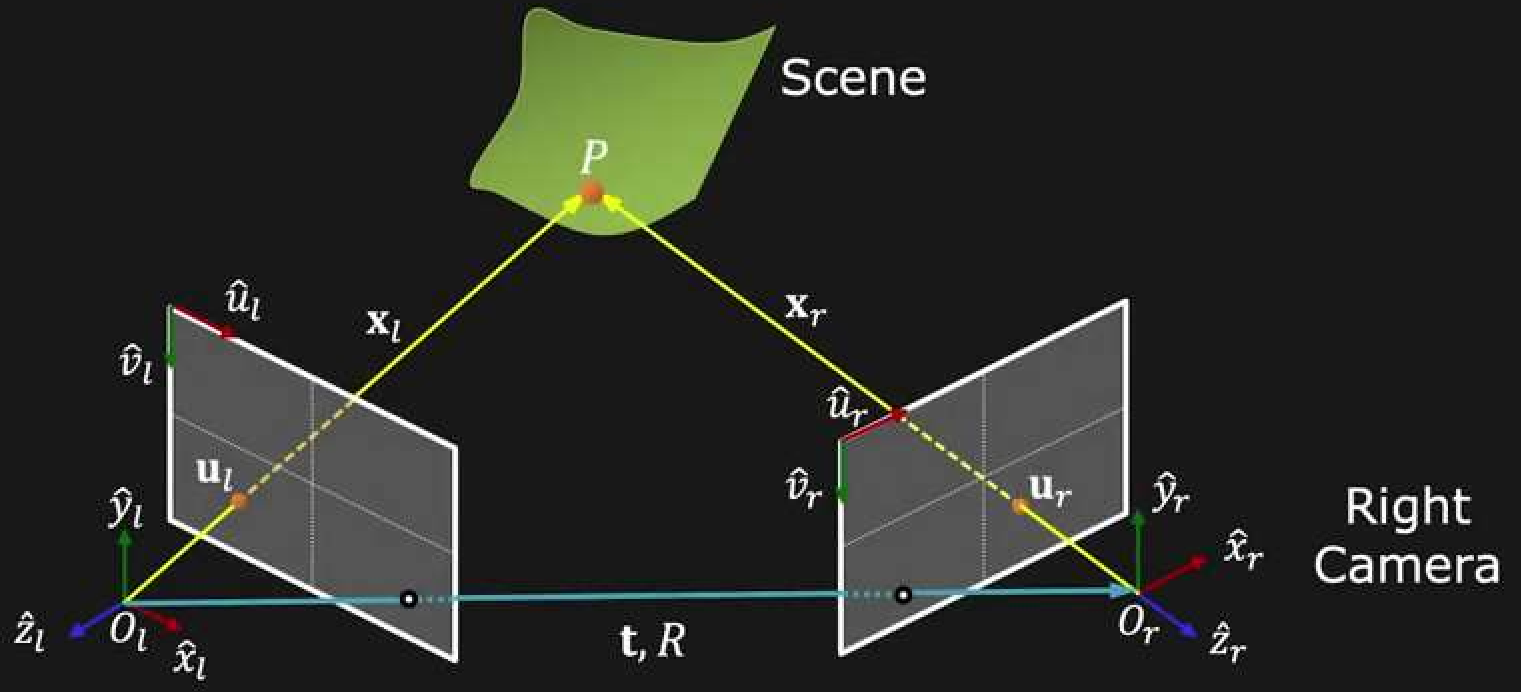
Shree K. Nayar

Columbia University

Topic: Uncalibrated Stereo, Module: Reconstruction II

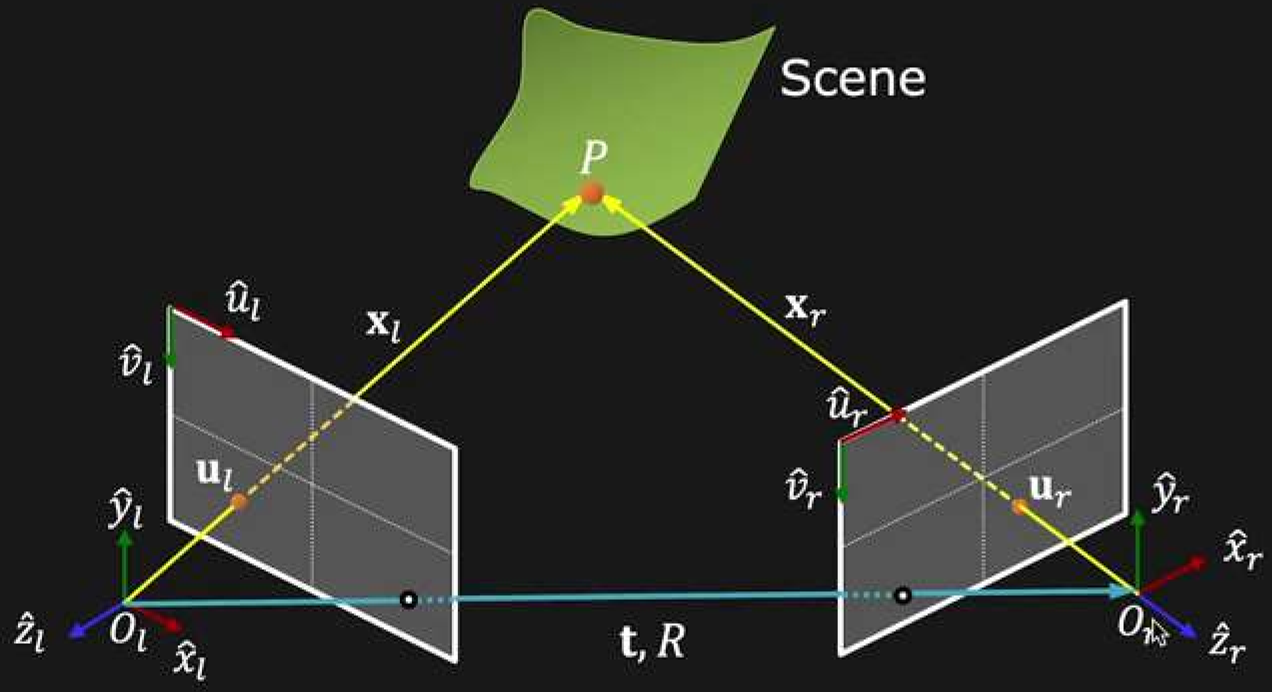
First Principles of Computer Vision

Epipolar Geometry: Epipoles



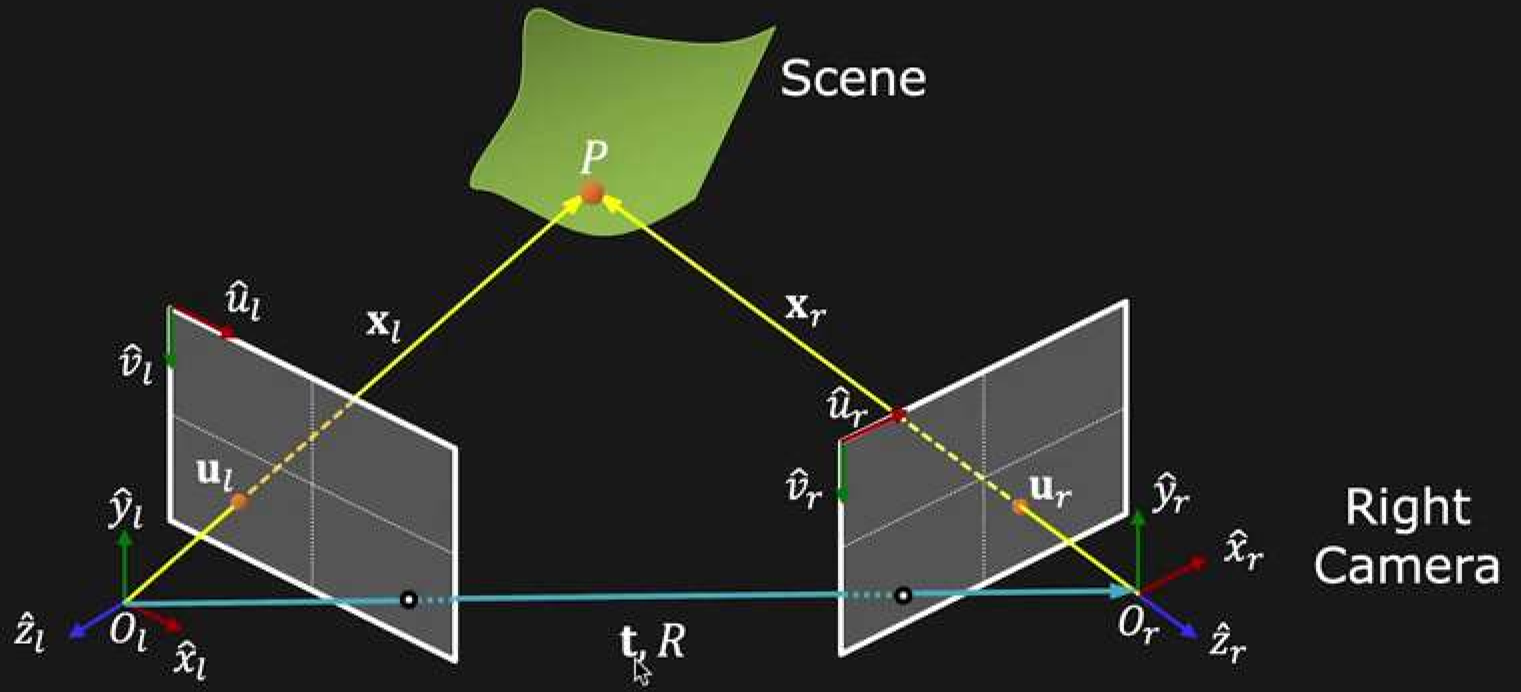
Epipole: Image point of origin/pinhole of one camera as viewed by the other camera.

Epipolar Geometry: Epipoles



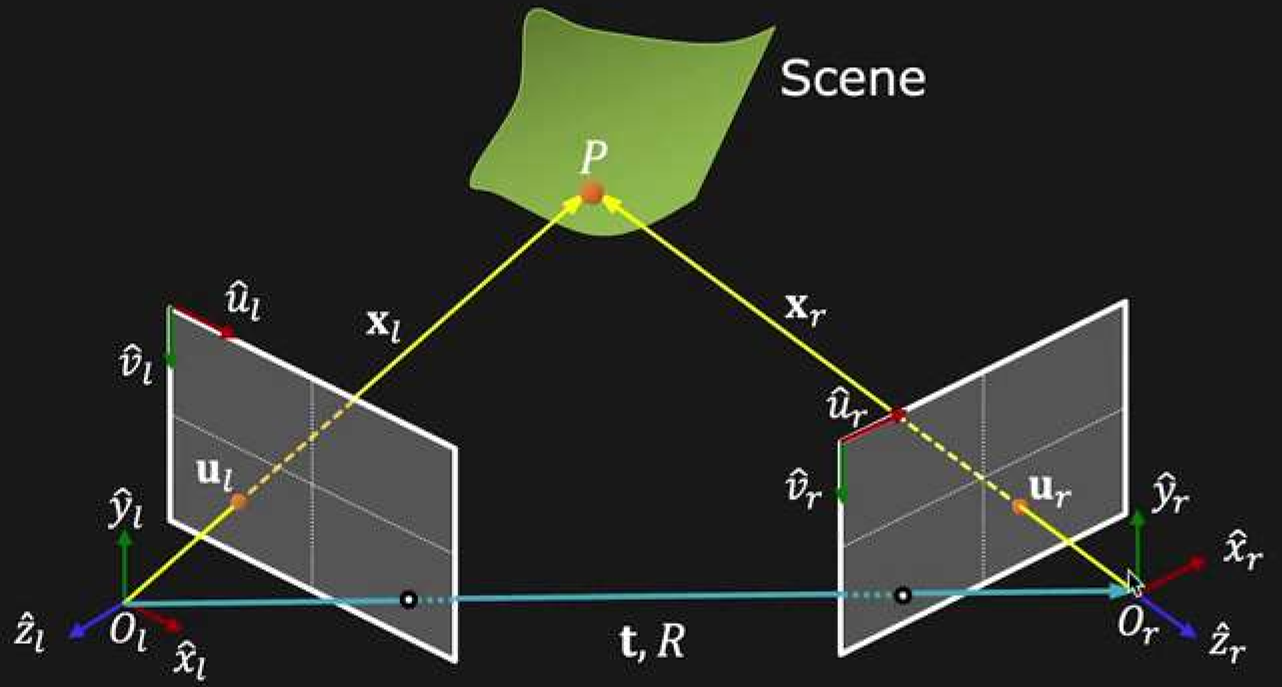
Epipole: Image point of origin/pinhole of one camera as viewed by the other camera.

Epipolar Geometry: Epipoles



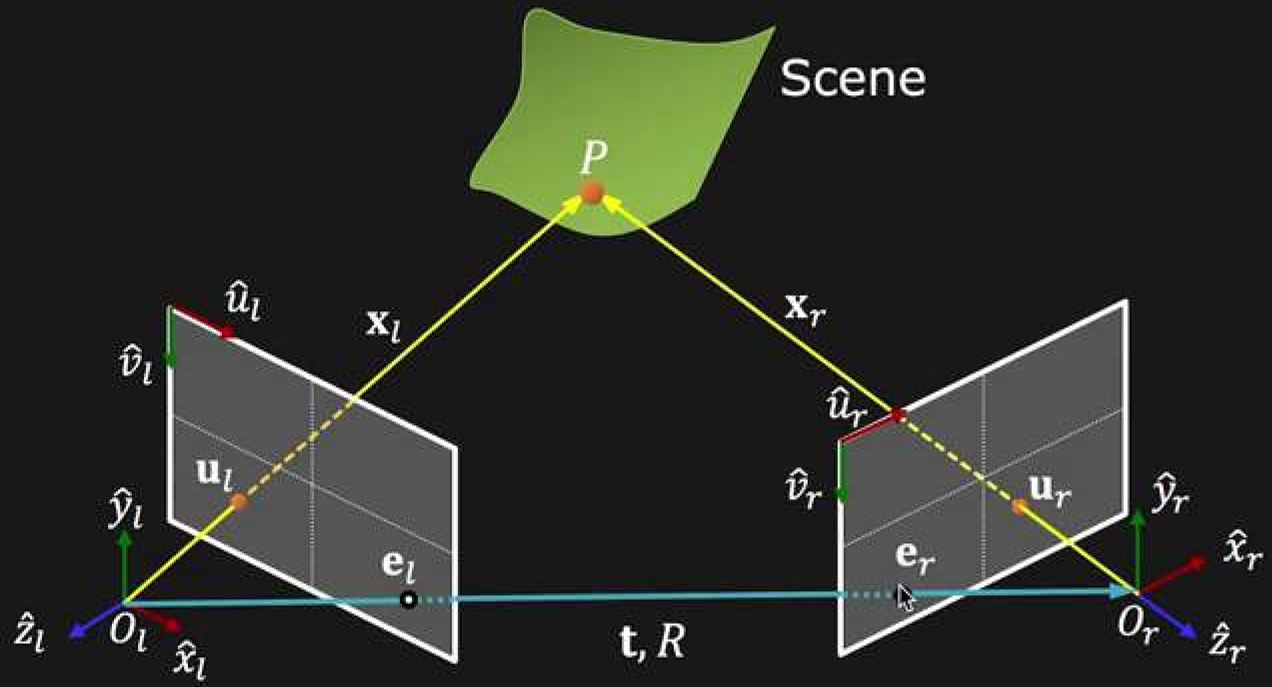
Epipole: Image point of origin/pinhole of one camera as viewed by the other camera.

Epipolar Geometry: Epipoles



Epipole: Image point of origin/pinhole of one camera as viewed by the other camera.

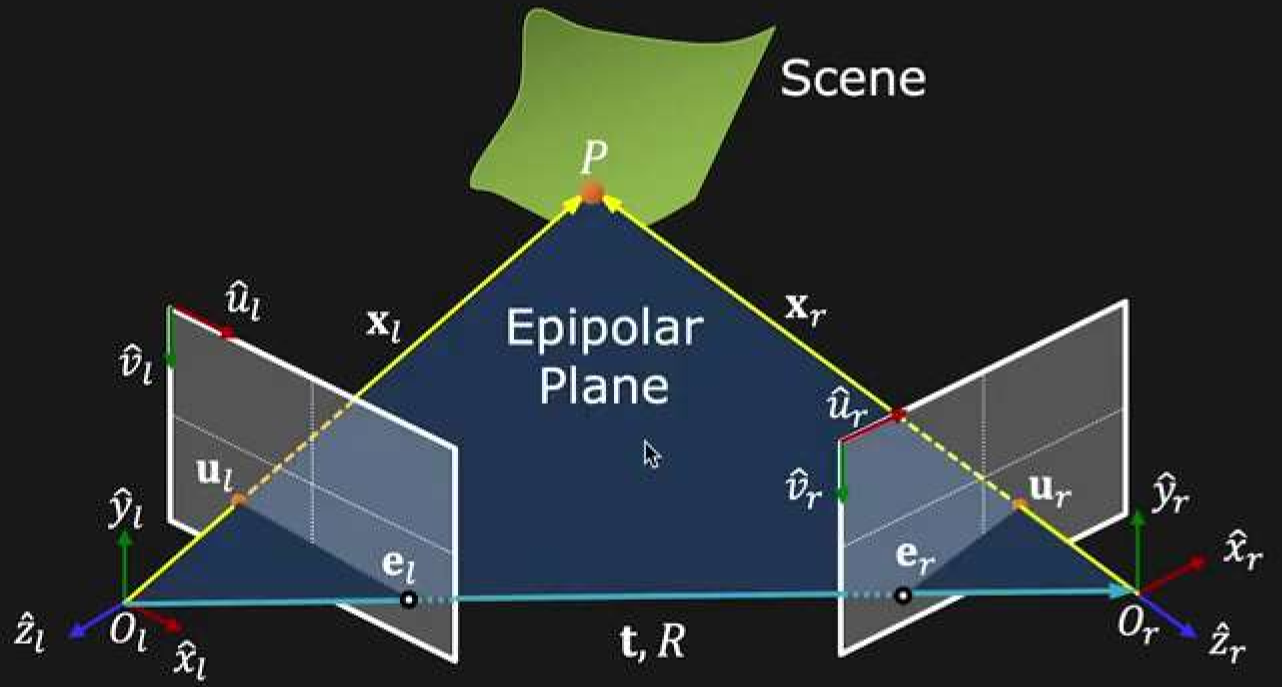
Epipolar Geometry: Epipoles



Epipole: Image point of origin/pinhole of one camera as viewed by the other camera.

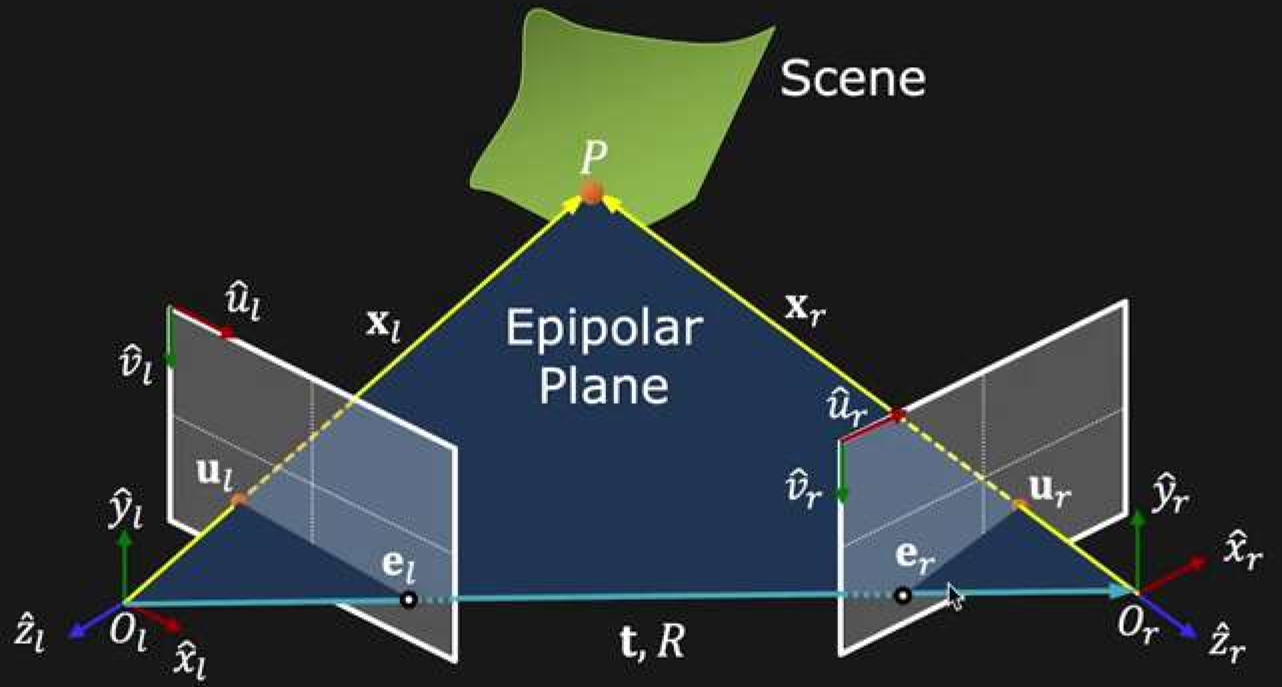
$\mathbf{e}_l$ and $\mathbf{e}_r$ are the epipoles.

Epipolar Geometry: Epipolar Plane



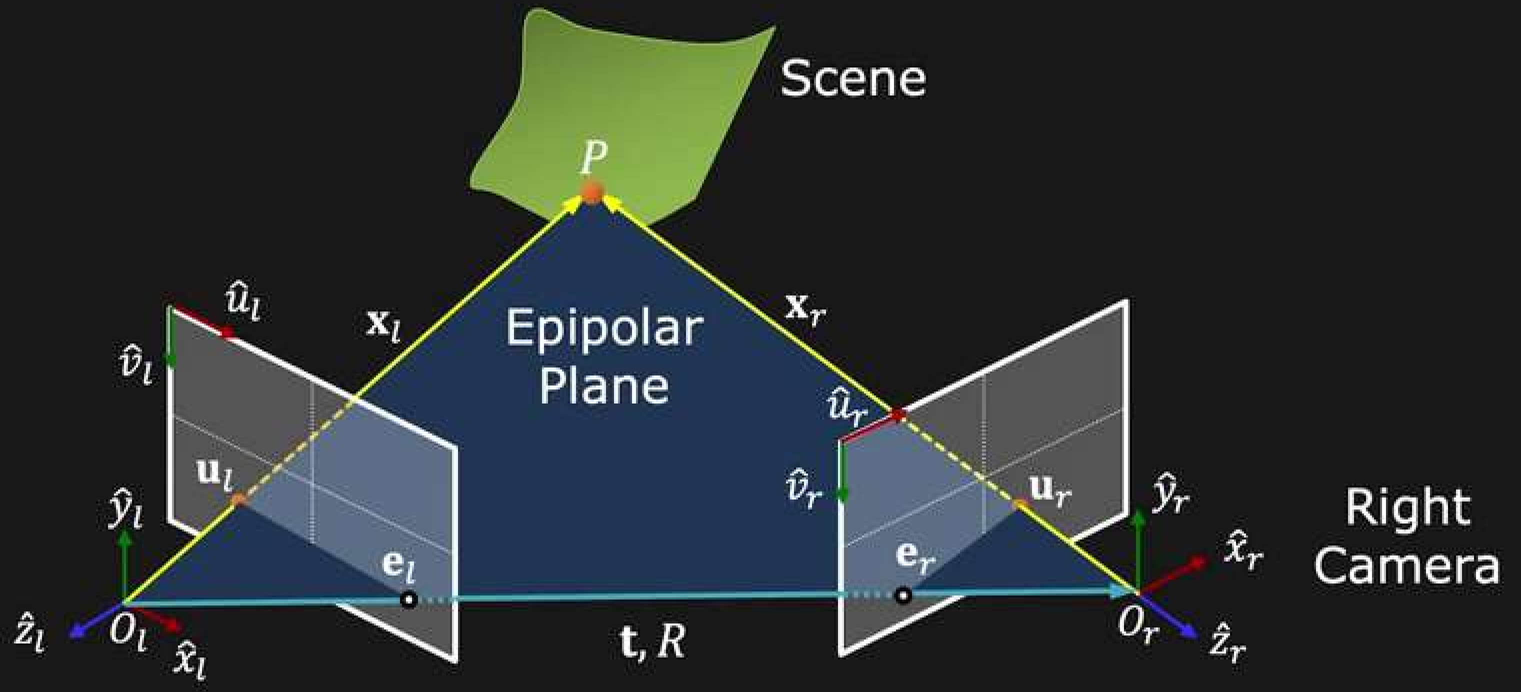
Epipolar Plane of Scene Point P: The plane formed by camera origins (O_l and O_r), epipoles ($\mathbf{e}_l$ and $\mathbf{e}_r$) and scene point P.

Epipolar Geometry: Epipolar Plane



Epipolar Plane of Scene Point P: The plane formed by camera origins (O_l and O_r), epipoles ($\mathbf{e}_l$ and $\mathbf{e}_r$) and scene point P.

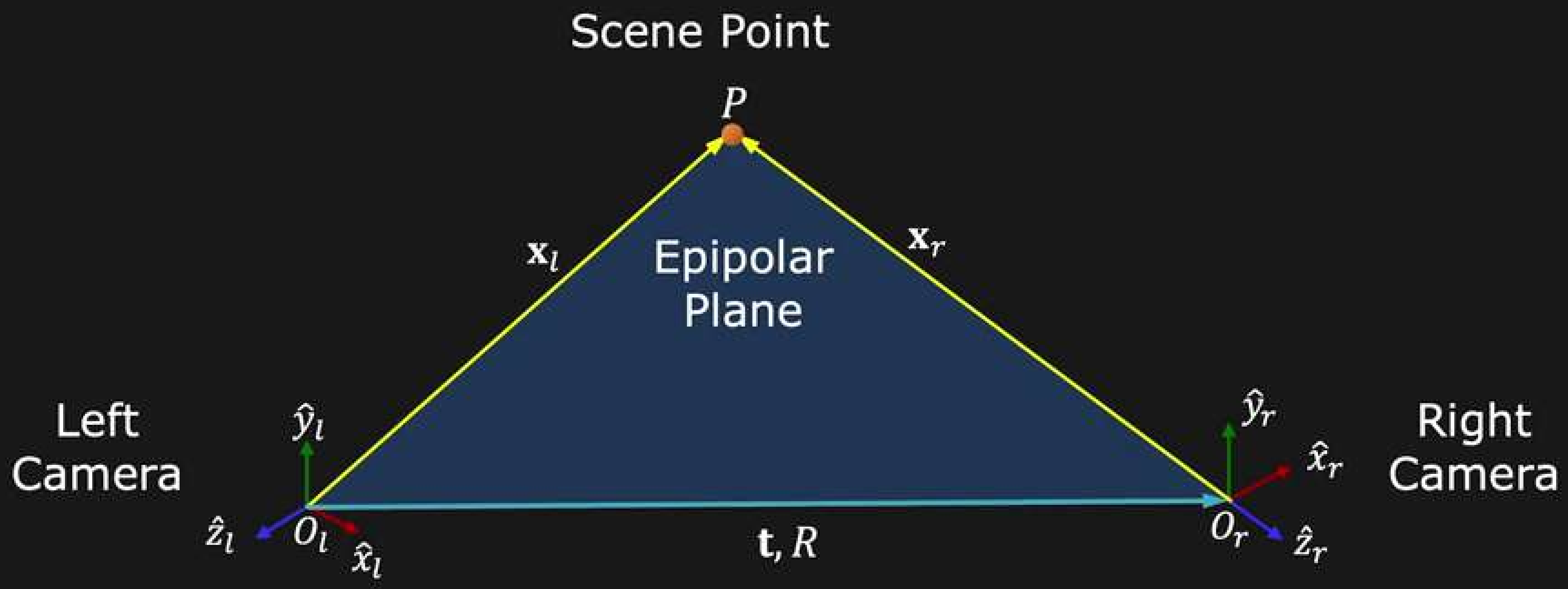
Epipolar Geometry: Epipolar Plane



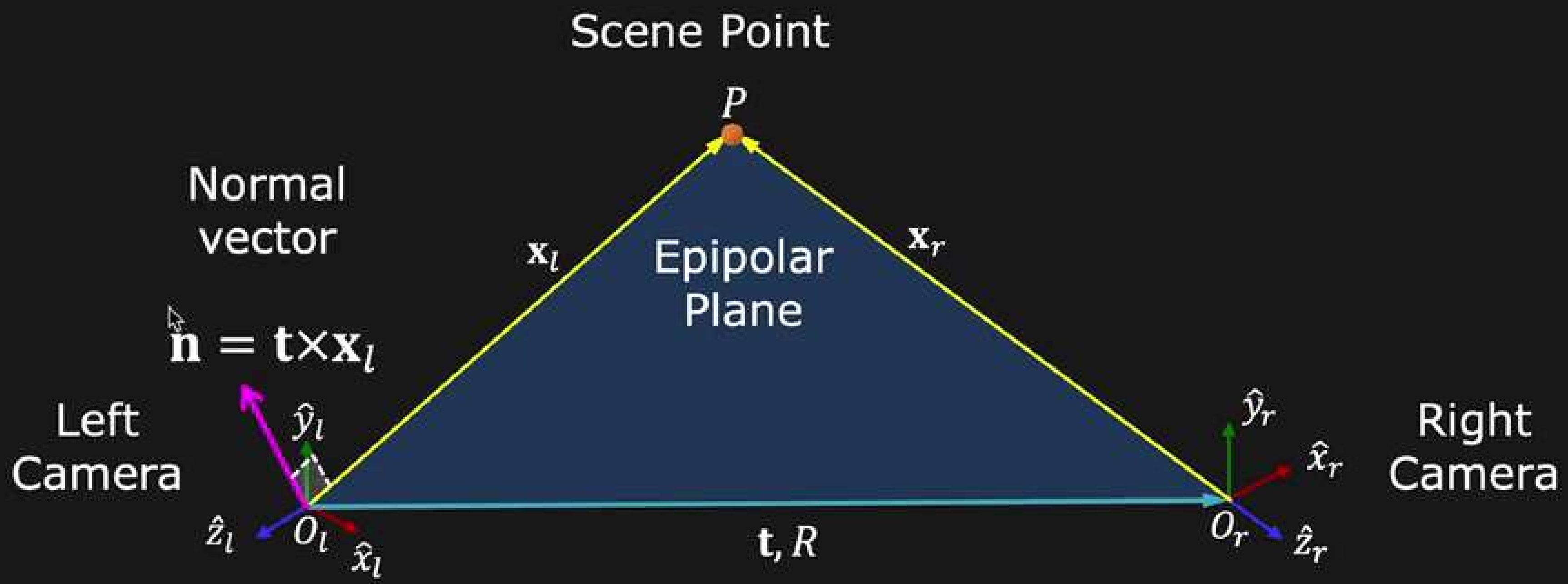
Epipolar Plane of Scene Point P: The plane formed by camera origins (O_l and O_r), epipoles ($\mathbf{e}_l$ and $\mathbf{e}_r$) and scene point P.

Every scene point lies on a unique epipolar plane.

Epipolar Constraint

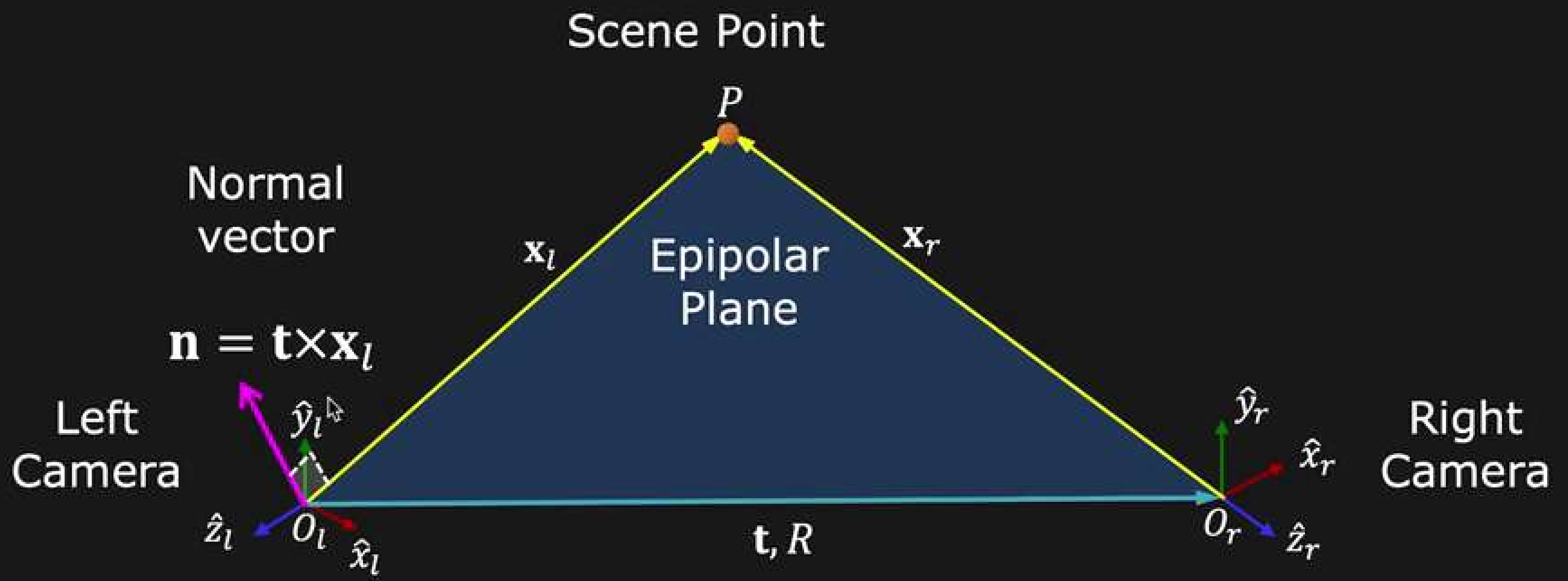


Epipolar Constraint



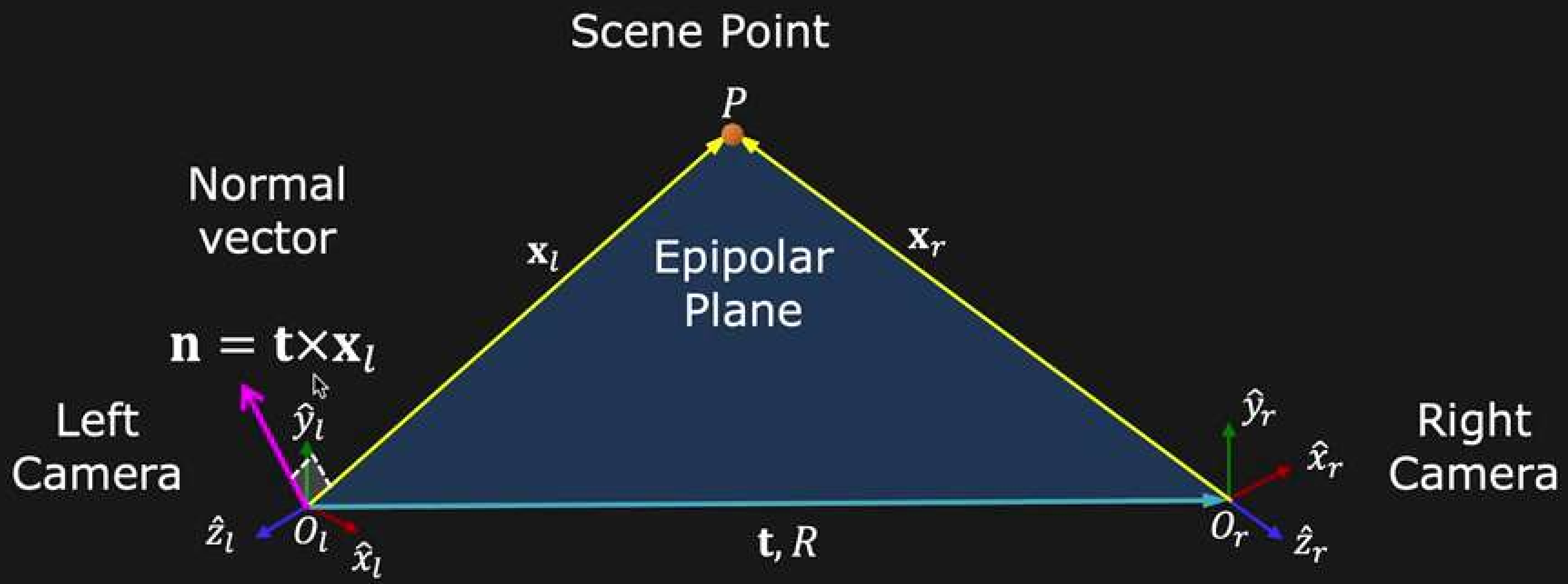
Vector normal to the epipolar plane: $\mathbf{n} = \mathbf{t} \times \mathbf{x}_l$

Epipolar Constraint



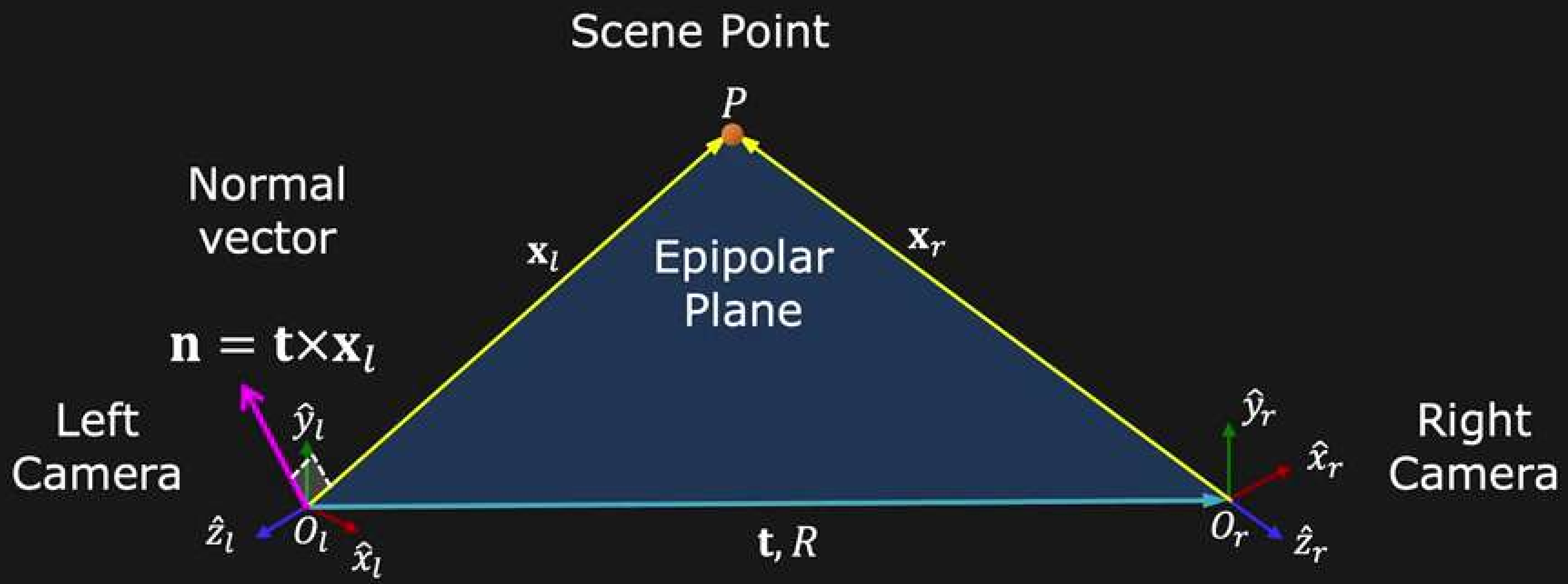
Vector normal to the epipolar plane: $\mathbf{n} = \mathbf{t}\times\mathbf{x}_l$

Epipolar Constraint



Vector normal to the epipolar plane: $\mathbf{n} = \mathbf{t} \times \mathbf{x}_l$

Epipolar Constraint

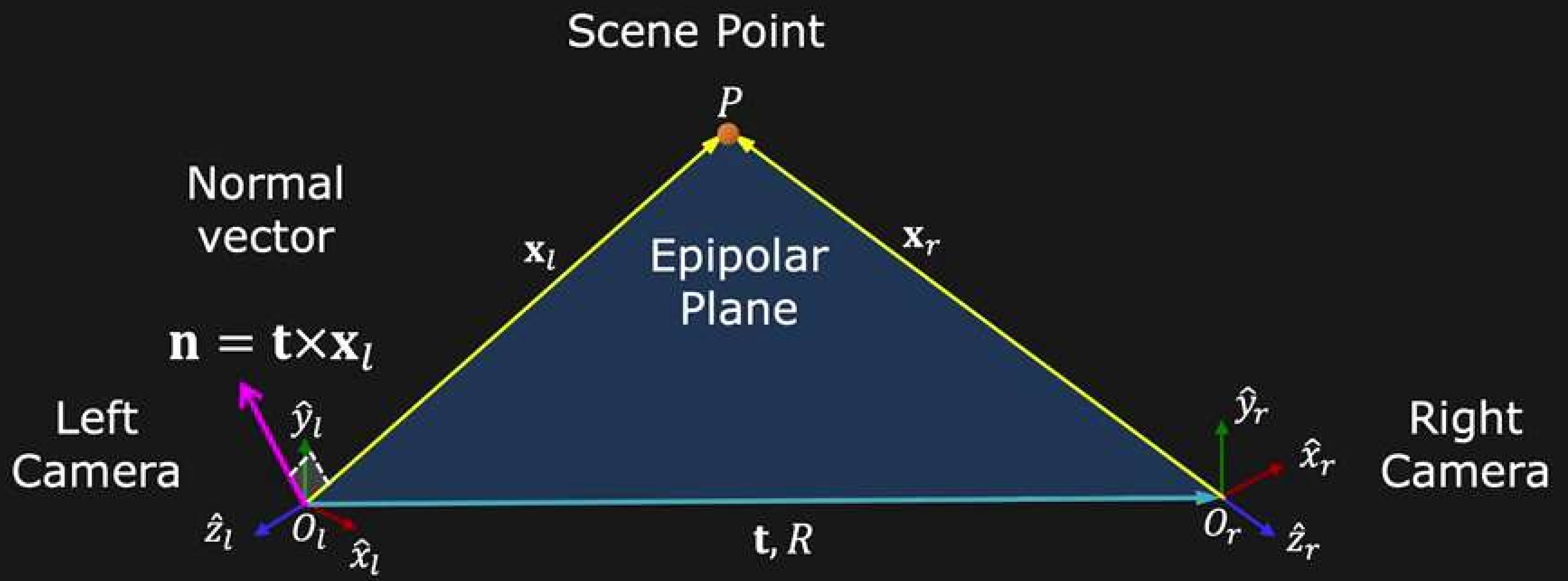


Vector normal to the epipolar plane: $\mathbf{n} = \mathbf{t}\times\mathbf{x}_l$

Dot product of $\mathbf{n}$ and $\mathbf{x}_l$ (perpendicular vectors) is zero:

$$\boxed{\mathbf{x}_l \cdot (\mathbf{t}\times\mathbf{x}_l) = 0}$$

Epipolar Constraint



Vector normal to the epipolar plane: $\mathbf{n} = \mathbf{t} \times \mathbf{x}_l$

Dot product of $\mathbf{n}$ and $\mathbf{x}_l$ (perpendicular vectors) is zero:

$$\boxed{\mathbf{x}_l \cdot (\mathbf{t} \times \mathbf{x}_l) = 0}$$

Epipolar Constraint

Writing the epipolar constraint in matrix form:

$$\mathbf{x}_l \cdot (\mathbf{t} \times \mathbf{x}_l) = 0$$

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} t_y z_l - t_z y_l \\ t_z x_l - t_x z_l \\ t_x y_l - t_y x_l \end{bmatrix} = 0$$

Cross-product definition

Epipolar Constraint

Writing the epipolar constraint in matrix form:

$$\mathbf{x}_l \cdot (\mathbf{t} \times \mathbf{x}_l) = 0$$

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} t_y z_l - t_z y_l \\ t_z x_l - t_x z_l \\ t_x y_l - t_y x_l \end{bmatrix} = 0$$

Cross-product definition

$$\boxed{\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \underbrace{\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}}_{T_\times} \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = 0}$$

Matrix-vector form

Epipolar Constraint

Writing the epipolar constraint in matrix form:

$$\mathbf{x}_l \cdot (\mathbf{t} \times \mathbf{x}_l) = 0$$

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} t_y z_l - t_z y_l \\ t_z x_l - t_x z_l \\ t_x y_l - t_y x_l \end{bmatrix} = 0$$

Cross-product definition

$$\boxed{\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \underbrace{\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}}_{T_\times} \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = 0}$$

Matrix-vector form

Epipolar Constraint

Writing the epipolar constraint in matrix form:

$$\mathbf{x}_l \cdot (\mathbf{t} \times \mathbf{x}_l) = 0$$

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} t_y z_l - t_z y_l \\ t_z x_l - t_x z_l \\ t_x y_l - t_y x_l \end{bmatrix} = 0$$

Cross-product definition

$$\boxed{\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \underbrace{\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}}_{T_\times} \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = 0}$$

Matrix-vector form

$\mathbf{t}_{3\times1}$: Position of Right Camera in Left Camera's Frame

$R_{3\times3}$: Orientation of Left Camera in Right Camera's Frame

Epipolar Constraint

Writing the epipolar constraint in matrix form:

$$\mathbf{x}_l \cdot (\mathbf{t} \times \mathbf{x}_l) = 0$$

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} t_y z_l - t_z y_l \\ t_z x_l - t_x z_l \\ t_x y_l - t_y x_l \end{bmatrix} = 0$$

Cross-product definition

$$\boxed{\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \underbrace{\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}}_{T_\times} \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = 0}$$

Matrix-vector form

$\mathbf{t}_{3\times1}$: Position of Right Camera in Left Camera's Frame

$R_{3\times3}$: Orientation of Left Camera in Right Camera's Frame

$$\boxed{\mathbf{x}_l = R\mathbf{x}_r + \mathbf{t}}$$

$$\boxed{\begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}}$$

Epipolar Constraint

Writing the epipolar constraint in matrix form:

$$\mathbf{x}_l \cdot (\mathbf{t} \times \mathbf{x}_l) = 0$$

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} t_y z_l - t_z y_l \\ t_z x_l - t_x z_l \\ t_x y_l - t_y x_l \end{bmatrix} = 0$$

Cross-product definition

$$\boxed{\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \underbrace{\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}}_{T_\times} \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = 0}$$

Matrix-vector form

$\mathbf{t}_{3\times1}$: Position of Right Camera in Left Camera's Frame

$R_{3\times3}$: Orientation of Left Camera in Right Camera's Frame

$$\boxed{\mathbf{x}_l = R\mathbf{x}_r + \mathbf{t}}$$

$$\boxed{\begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}}$$

Epipolar Constraint

Writing the epipolar constraint in matrix form:

$$\mathbf{x}_l \cdot (\mathbf{t} \times \mathbf{x}_l) = 0$$

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} t_y z_l - t_z y_l \\ t_z x_l - t_x z_l \\ t_x y_l - t_y x_l \end{bmatrix} = 0$$

Cross-product definition

$$\boxed{\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \underbrace{\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}}_{T_\times} \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = 0}$$

Matrix-vector form

$\mathbf{t}_{3\times1}$: Position of Right Camera in Left Camera's Frame

$R_{3\times3}$: Orientation of Left Camera in Right Camera's Frame

$$\boxed{\mathbf{x}_l = R\mathbf{x}_r + \mathbf{t}}$$

$$\boxed{\begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}}$$

Epipolar Constraint

Substituting into the epipolar constraint gives:

$$[x_l \quad y_l \quad z_l]\left(\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} + \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}\right) = 0$$

[Longuet-Higgins 1981]

Epipolar Constraint

Substituting into the epipolar constraint gives:

$$[x_l \quad y_l \quad z_l]\left(\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} + \underbrace{\boxed{\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}}}_{\boldsymbol{t}\times\boldsymbol{t} = \boldsymbol{0}}\right) = 0$$

[Longuet-Higgins 1981]

Epipolar Constraint

Substituting into the epipolar constraint gives:

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \left(\boxed{\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} + \underbrace{\boxed{\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}}}_{\boldsymbol{t} \times \boldsymbol{t} = \boldsymbol{0}} \right) = 0$$

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \boxed{\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Essential Matrix E

[Longuet-Higgins 1981]

Epipolar Constraint

Substituting into the epipolar constraint gives:

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \left(\boxed{\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} + \underbrace{\boxed{\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}}}_{\boldsymbol{t} \times \boldsymbol{t} = \boldsymbol{0}} \right) = 0$$

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \boxed{\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Essential Matrix E

[Longuet-Higgins 1981]

Epipolar Constraint

Substituting into the epipolar constraint gives:

$$[x_l \quad y_l \quad z_l]\left(\boxed{\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}\begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}}_{\boldsymbol{t}\times\boldsymbol{t}=\boldsymbol{0}}\right) = 0$$

$$[x_l \quad y_l \quad z_l]\boxed{\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}}\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Essential Matrix E

$$\boxed{E = T_{\times}R}$$

[Longuet-Higgins 1981]

Essential Matrix E: Decomposition

$$\boxed{E = T_\times R}$$

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Essential Matrix E: Decomposition

$$\boxed{E = T_\times R}$$

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \boxed{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}$$

Given that $T_\times$ is a Skew-Symmetric matrix (a_{ij} = - a_{ji}) and R is an Orthonormal matrix, it is possible to "decouple" $T_\times$ and R from their product using "Singular Value Decomposition".

Essential Matrix E: Decomposition

$$\boxed{E = T_{\times} R}$$

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \boxed{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}$$

Given that $T_{\times}$ is a Skew-Symmetric matrix (a_{ij} = - a_{ji}) and R is an Orthonormal matrix, it is possible to "decouple" $T_{\times}$ and R from their product using "Singular Value Decomposition".

Essential Matrix E: Decomposition

$$\boxed{E = T_\times R}$$

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Given that $T_\times$ is a Skew-Symmetric matrix (a_{ij} = - a_{ji}) and R is an Orthonormal matrix, it is possible to "decouple" $T_\times$ and R from their product using "Singular Value Decomposition".

Essential Matrix E: Decomposition

$$\boxed{E = T_{\times} R}$$

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Given that $T_{\times}$ is a Skew-Symmetric matrix (a_{ij} = - a_{ji}) and R is an Orthonormal matrix, it is possible to "decouple" $T_{\times}$ and R from their product using "Singular Value Decomposition".

Essential Matrix E: Decomposition

$$\boxed{E = T_\times R}$$

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \boxed{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}$$

Given that $T_\times$ is a Skew-Symmetric matrix (a_{ij} = - a_{ji}) and R is an Orthonormal matrix, it is possible to "decouple" $T_\times$ and R from their product using "Singular Value Decomposition".

Take Away: If E is known, we can calculate $\boldsymbol{t}$ and R.

Essential Matrix E: Decomposition

$$\boxed{E = T_\times R}$$

$$\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \boxed{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}$$

Given that $T_\times$ is a Skew-Symmetric matrix (a_{ij} = - a_{ji}) and R is an Orthonormal matrix, it is possible to "decouple" $T_\times$ and R from their product using "Singular Value Decomposition".

Take Away: If E is known, we can calculate $\boldsymbol{t}$ and R.

How do we find E?

Relates 3D position (x_l, y_l, z_l) of scene point w.r.t left camera to its 3D position (x_r, y_r, z_r) w.r.t. right camera

$$\boxed{\mathbf{x}_l^T E \mathbf{x}_r = 0}$$

$$\underbrace{\begin{bmatrix} x_l & y_l & z_l \end{bmatrix}}_{\text{3D position in left camera coordinates}} \underbrace{\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}}_{\text{3x3 Essential Matrix}} \underbrace{\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix}}_{\text{3D position in right camera coordinates}} = 0$$

How do we find E?

Relates 3D position (x_l, y_l, z_l) of scene point w.r.t left camera to its 3D position (x_r, y_r, z_r) w.r.t. right camera

$$\boxed{\mathbf{x}_l^T E \mathbf{x}_r = 0}$$

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

3D position in left camera coordinates

3x3 Essential Matrix

3D position in right camera coordinates

How do we find E?

Relates 3D position (x_l, y_l, z_l) of scene point w.r.t left camera to its 3D position (x_r, y_r, z_r) w.r.t. right camera

$$\boxed{\mathbf{x}_l^T E \mathbf{x}_r = 0}$$

$$\underbrace{\begin{bmatrix} x_l & y_l & z_l \end{bmatrix}}_{\text{3D position in left camera coordinates}} \underbrace{\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}}_{\text{3x3 Essential Matrix}} \underbrace{\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix}}_{\text{3D position in right camera coordinates}} = 0$$

How do we find E?

Relates 3D position (x_l, y_l, z_l) of scene point w.r.t left camera to its 3D position (x_r, y_r, z_r) w.r.t. right camera

$$\boxed{\mathbf{x}_l^T E \mathbf{x}_r = 0}$$

$$\underbrace{[x_l \quad y_l \quad z_l]}_{\text{3D position in left camera coordinates}} \underbrace{\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}}_{\text{3x3 Essential Matrix}} \underbrace{\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix}}_{\text{3D position in right camera coordinates}} = 0$$

Unfortunately, we don't have $\mathbf{x}_l$ and $\mathbf{x}_r$.

But we do know corresponding points in image coordinates.

Incorporating the Image Coordinates

Perspective projection equations for left camera:

$$u_l = f_x^{(l)} \frac{x_l}{z_l} + o_x^{(l)} \qquad v_l = f_y^{(l)} \frac{y_l}{z_l} + o_y^{(l)}$$

$$z_l u_l = f_x^{(l)} x_l + z_l o_x^{(l)} \qquad z_l v_l = f_y^{(l)} y_l + z_l o_y^{(l)}$$

REVIEW

Incorporating the Image Coordinates

Perspective projection equations for left camera:

$$u_l = f_x^{(l)} \frac{x_l}{z_l} + o_x^{(l)} \qquad v_l = f_y^{(l)} \frac{y_l}{z_l} + o_y^{(l)}$$

$$z_l u_l = f_x^{(l)} x_l + z_l o_x^{(l)} \qquad z_l v_l = f_y^{(l)} y_l + z_l o_y^{(l)}$$

Incorporating the Image Coordinates

Perspective projection equations for left camera:

$$u_l = f_x^{(l)} \frac{x_l}{z_l} + o_x^{(l)} \qquad v_l = f_y^{(l)} \frac{y_l}{z_l} + o_y^{(l)}$$

$$z_l u_l = f_x^{(l)} x_l + z_l o_x^{(l)} \qquad z_l v_l = f_y^{(l)} y_l + z_l o_y^{(l)}$$

Representing in matrix form:

$$z_l \begin{bmatrix} u_l \\ v_l \\ 1 \end{bmatrix} = \begin{bmatrix} z_l u_l \\ z_l v_l \\ z_l \end{bmatrix} = \begin{bmatrix} f_x^{(l)} x_l + z_l o_x^{(l)} \\ f_y^{(l)} y_l + z_l o_y^{(l)} \\ z_l \end{bmatrix} = \underbrace{\begin{bmatrix} f_x^{(l)} & 0 & o_x^{(l)} \\ 0 & f_y^{(l)} & o_y^{(l)} \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Known} \\ \text{Camera Matrix } K_l}} \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix}$$

REVIEW

Incorporating the Image Coordinates

Perspective projection equations for left camera:

$$u_l = f_x^{(l)} \frac{x_l}{z_l} + o_x^{(l)} \qquad v_l = f_y^{(l)} \frac{y_l}{z_l} + o_y^{(l)}$$

$$z_l u_l = f_x^{(l)} x_l + z_l o_x^{(l)} \qquad z_l v_l = f_y^{(l)} y_l + z_l o_y^{(l)}$$

Representing in matrix form:

$$z_l \begin{bmatrix} u_l \\ v_l \\ 1 \end{bmatrix} = \begin{bmatrix} z_l u_l \\ z_l v_l \\ z_l \end{bmatrix} = \begin{bmatrix} f_x^{(l)} x_l + z_l o_x^{(l)} \\ f_y^{(l)} y_l + z_l o_y^{(l)} \\ z_l \end{bmatrix} = \underbrace{\begin{bmatrix} f_x^{(l)} & 0 & o_x^{(l)} \\ 0 & f_y^{(l)} & o_y^{(l)} \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Known} \\ \text{Camera Matrix } K_l}} \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix}$$

Incorporating the Image Coordinates

Perspective projection equations for left camera:

$$u_l = f_x^{(l)} \frac{x_l}{z_l} + o_x^{(l)} \qquad v_l = f_y^{(l)} \frac{y_l}{z_l} + o_y^{(l)}$$

$$z_l u_l = f_x^{(l)} x_l + z_l o_x^{(l)} \qquad z_l v_l = f_y^{(l)} y_l + z_l o_y^{(l)}$$

Representing in matrix form:

$$z_l \begin{bmatrix} u_l \\ v_l \\ 1 \end{bmatrix} = \begin{bmatrix} z_l u_l \\ z_l v_l \\ z_l \end{bmatrix} = \begin{bmatrix} f_x^{(l)} x_l + z_l o_x^{(l)} \\ f_y^{(l)} y_l + z_l o_y^{(l)} \\ z_l \end{bmatrix} = \underbrace{\begin{bmatrix} f_x^{(l)} & 0 & o_x^{(l)} \\ 0 & f_y^{(l)} & o_y^{(l)} \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Known} \\ \text{Camera Matrix } K_l}} \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix}$$

Incorporating the Image Coordinates

Left camera

$$z_l \begin{bmatrix} u_l \\ v_l \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f_x^{(l)} & 0 & o_x^{(l)} \\ 0 & f_y^{(l)} & o_y^{(l)} \\ 0 & 0 & 1 \end{bmatrix}}_{K_l} \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix}$$

Right camera

$$z_r \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f_x^{(r)} & 0 & o_x^{(r)} \\ 0 & f_y^{(r)} & o_y^{(r)} \\ 0 & 0 & 1 \end{bmatrix}}_{K_r} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix}$$

Incorporating the Image Coordinates

Left camera

$$z_l \begin{bmatrix} u_l \\ v_l \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f_x^{(l)} & 0 & o_x^{(l)} \\ 0 & f_y^{(l)} & o_y^{(l)} \\ 0 & 0 & 1 \end{bmatrix}}_{K_l} \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix}$$

Right camera

$$z_r \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f_x^{(r)} & 0 & o_x^{(r)} \\ 0 & f_y^{(r)} & o_y^{(r)} \\ 0 & 0 & 1 \end{bmatrix}}_{K_r} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix}$$

Incorporating the Image Coordinates

Left camera

$$z_l \begin{bmatrix} u_l \\ v_l \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f_x^{(l)} & 0 & o_x^{(l)} \\ 0 & f_y^{(l)} & o_y^{(l)} \\ 0 & 0 & 1 \end{bmatrix}}_{K_l} \begin{bmatrix} x_l \\ y_l \\ z_l \end{bmatrix}$$

$$\boxed{\mathbf{x}_l{}^T = [u_l \quad v_l \quad 1] z_l \, K_l^{-1\,T}}$$

Right camera

$$z_r \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f_x^{(r)} & 0 & o_x^{(r)} \\ 0 & f_y^{(r)} & o_y^{(r)} \\ 0 & 0 & 1 \end{bmatrix}}_{K_r} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix}$$

$$\boxed{\mathbf{x}_r = K_r^{-1} z_r \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix}}$$

Incorporating the Image Coordinates

Epipolar constraint:

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Incorporating the Image Coordinates

Epipolar constraint:

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Rewriting in terms of image coordinates:

$$\begin{bmatrix} u_l & v_l & 1 \end{bmatrix} z_l \, {K_l^{-1}}^T \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} K_r^{-1} \, z_r \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = 0$$

Incorporating the Image Coordinates

Epipolar constraint:

$$[x_l \quad y_l \quad z_l] \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Rewriting in terms of image coordinates:

$$[u_l \quad v_l \quad 1] \, z_l \, {K_l^{-1}}^T \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} K_r^{-1} \, z_r \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = 0$$

Incorporating the Image Coordinates

Epipolar constraint:

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Rewriting in terms of image coordinates:

$$\begin{bmatrix} u_l & v_l & 1 \end{bmatrix} z_l \, {K_l^{-1}}^T \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} K_r^{-1} \, z_r \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = 0$$

Incorporating the Image Coordinates

Epipolar constraint:

$$[x_l \quad y_l \quad z_l] \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Rewriting in terms of image coordinates:

$$[u_l \quad v_l \quad 1]\, z_l\, {K_l^{-1}}^T \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} K_r^{-1}\, z_r \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = 0$$

Incorporating the Image Coordinates

Epipolar constraint:

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Rewriting in terms of image coordinates:

$$\begin{bmatrix} u_l & v_l & 1 \end{bmatrix} \cancel{z_l} \, {K_l^{-1}}^T \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} K_r^{-1} \, \cancel{z_r} \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = 0$$

$$\boxed{\begin{array}{c} z_l \neq 0 \\ z_r \neq 0 \end{array}}$$

Incorporating the Image Coordinates

Epipolar constraint:

$$[x_l \quad y_l \quad z_l] \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Rewriting in terms of image coordinates:

$$\boxed{[u_l \quad v_l \quad 1]\, {K_l^{-1}}^T \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} K_r^{-1} \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = 0}$$

Incorporating the Image Coordinates

Epipolar constraint:

$$[x_l \quad y_l \quad z_l] \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Rewriting in terms of image coordinates:

$$\boxed{[u_l \quad v_l \quad 1]\, {K_l^{-1}}^T \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} K_r^{-1} \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = 0}$$

Incorporating the Image Coordinates

Epipolar constraint:

$$
\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0
$$

Rewriting in terms of image coordinates:

$$
\boxed{\begin{bmatrix} u_l & v_l & 1 \end{bmatrix} {K_l^{-1}}^T \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} K_r^{-1} \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = 0}
$$

Fundamental Matrix F

Epipolar constraint:

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Rewriting in terms of image coordinates:

$$\begin{bmatrix} u_l & v_l & 1 \end{bmatrix} \boxed{\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}} \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = 0$$

Fundamental Matrix F

[Fagueras 1992, Luong 1992]

Fundamental Matrix F

Epipolar constraint:

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Rewriting in terms of image coordinates:

$$\begin{bmatrix} u_l & v_l & 1 \end{bmatrix} \boxed{\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}} \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = 0$$

Fundamental Matrix F

[Fagueras 1992, Luong 1992]

Fundamental Matrix F

Epipolar constraint:

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Rewriting in terms of image coordinates:

$$\begin{bmatrix} u_l & v_l & 1 \end{bmatrix} \boxed{\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}} \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = 0$$

Fundamental Matrix F

[Fagueras 1992, Luong 1992]

Fundamental Matrix F

Epipolar constraint:

$$\begin{bmatrix} x_l & y_l & z_l \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Rewriting in terms of image coordinates:

$$\begin{bmatrix} u_l & v_l & 1 \end{bmatrix} \boxed{\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}} \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = 0$$

Fundamental Matrix F

$$\boxed{E = K_l^T F K_r}$$

[Fagueras 1992, Luong 1992]

Fundamental Matrix F

Epipolar constraint:

$$[x_l \quad y_l \quad z_l]\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}\begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Rewriting in terms of image coordinates:

$$[u_l \quad v_l \quad 1]\boxed{\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}}\begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = 0$$

Fundamental Matrix F

$$\boxed{E = K_l^T F K_r}$$

[Fagueras 1992, Luong 1992]

Fundamental Matrix F

Epipolar constraint:

$$[x_l \quad y_l \quad z_l] \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \begin{bmatrix} x_r \\ y_r \\ z_r \end{bmatrix} = 0$$

Rewriting in terms of image coordinates:

$$[u_l \quad v_l \quad 1] \boxed{\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}} \begin{bmatrix} u_r \\ v_r \\ 1 \end{bmatrix} = 0$$

Fundamental Matrix F

$$\boxed{E = K_l^T F K_r} \qquad \boxed{E = T_\times R}$$

[Fagueras 1992, Luong 1992]