Lucas-Kanade Method

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Columbia University

Topic: Motion and Optical Flow, Module: Reconstruction II

First Principles of Computer Vision

Lucas-Kanade Solution

Assumption: For each pixel, assume Motion Field, and hence Optical Flow (u, v), is constant within a small neighborhood W.

[Lucas 1981]

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W

That is for all points $(k, l) \in W$:

$$I_x(k,l)u + I_y(k,l)v + I_t(k,l) = 0$$

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For all points $(k,l) \in W$: $I_x(k,l)u + I_y(k,l)v + I_t(k,l) = 0$

Let the size of window W be $n \times n$

In matrix form:

$$\begin{bmatrix} I_x(1,1) & I_y(1,1) \\ I_x(k,l) & I_y(k,l) \\ \vdots & \vdots \\ I_x(n,n) & I_y(n,n) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} I_t(1,1) \\ I_t(k,l) \\ \vdots \\ I_t(n,n) \end{bmatrix}$$

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n^2 Equations, 2 Unknowns: Find Least Squares Solution

Least Squares Solution

Solve linear system: $A\mathbf{u} = B$

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Indices (k, l) not written for simplicity

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When Does Optical Flow Estimation Work?

$\boxed{A\mathbf{u} = B}$ $\boxed{A^T A\mathbf{u} = A^T B}$

- $A^T A$ must be invertible. That is $\det(A^T A) \neq 0$
- $A^T A$ must be well-conditioned.

If λ_1 and λ_2 are eigen values of $A^T A$, then

$\lambda_1 > \epsilon$ and $\lambda_2 > \epsilon$

$\lambda_1 \geq \lambda_2$ but not $\lambda_1 \ggg \lambda_2$

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Smooth Regions (Bad)

$\lambda_1 \sim \lambda_2$
Both are Small

Equations for all pixels in window are more or less the sam

Cannot reliably compute flow!

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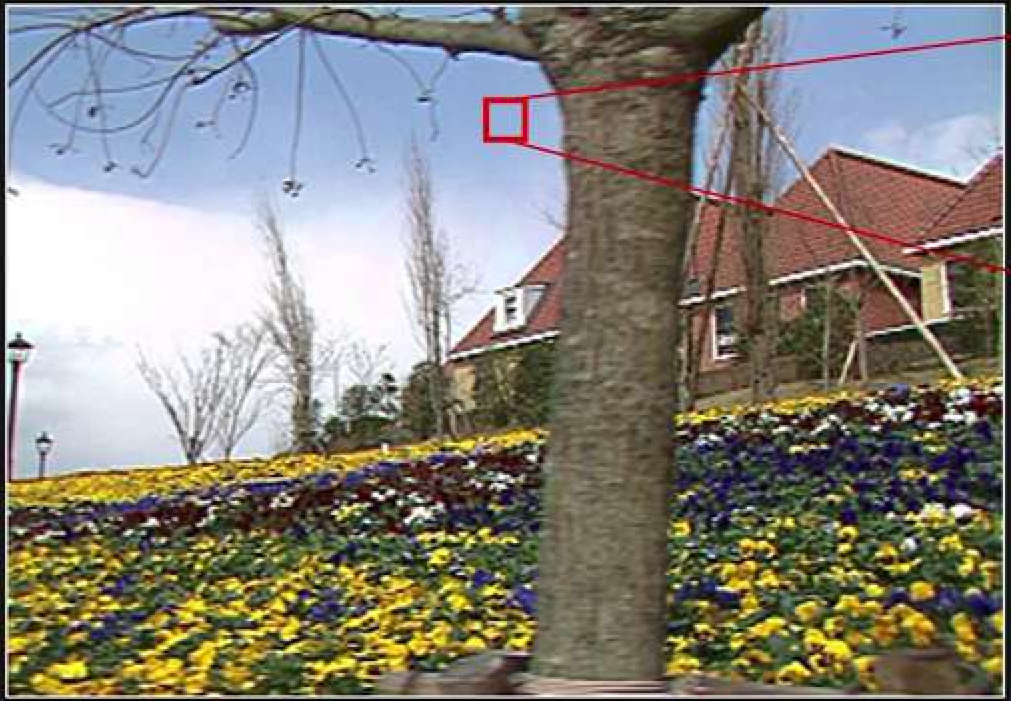

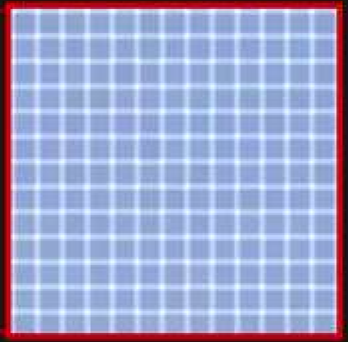

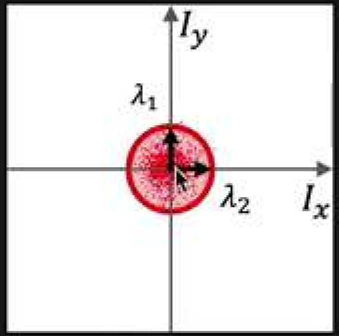

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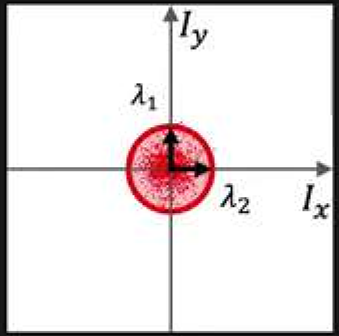

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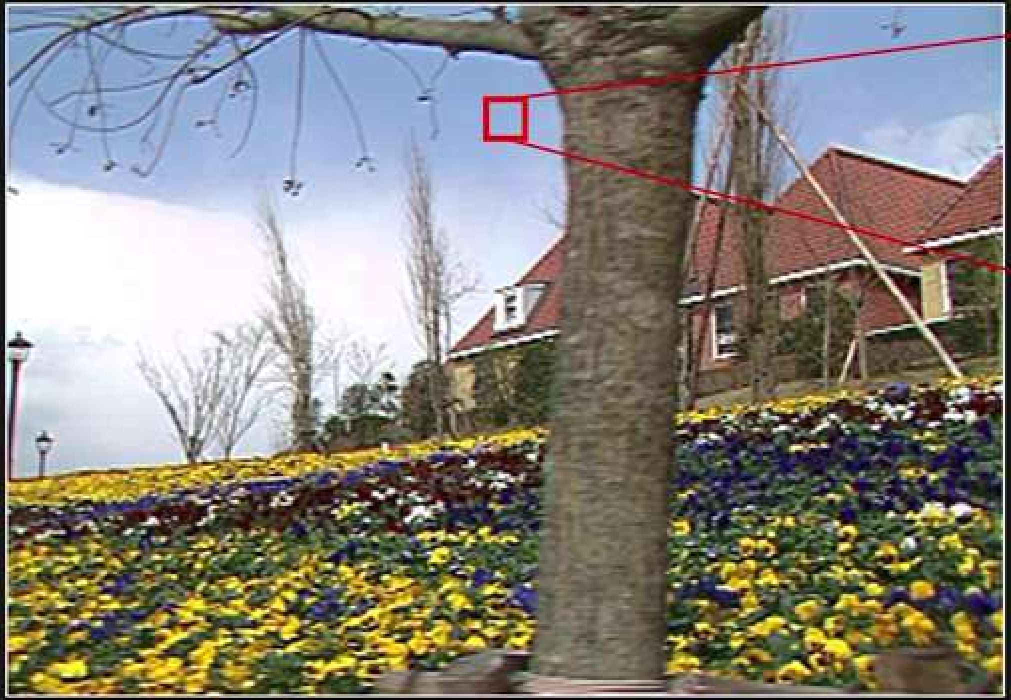

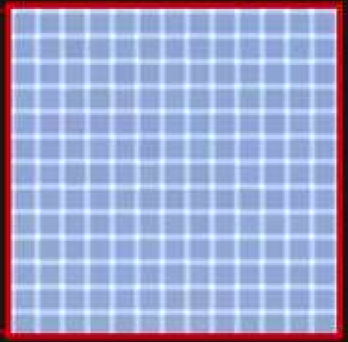

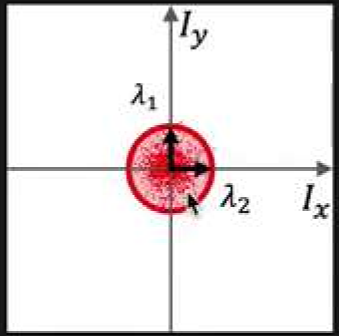

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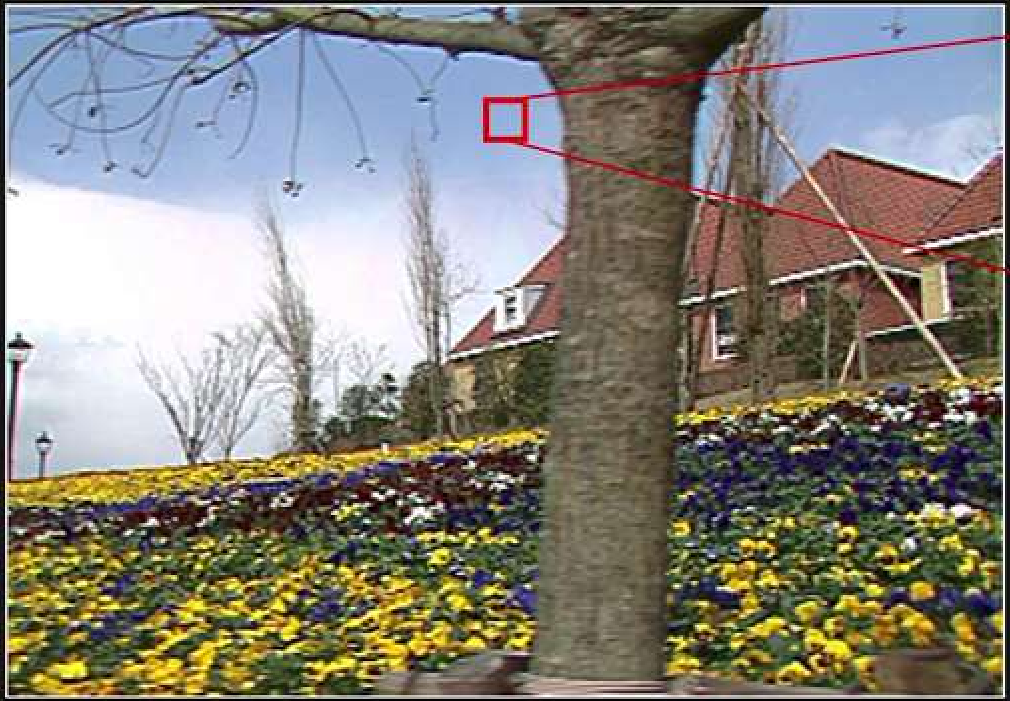

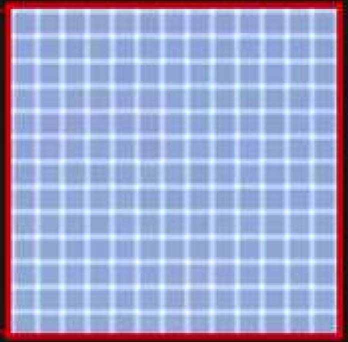

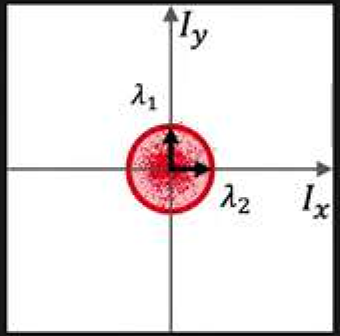

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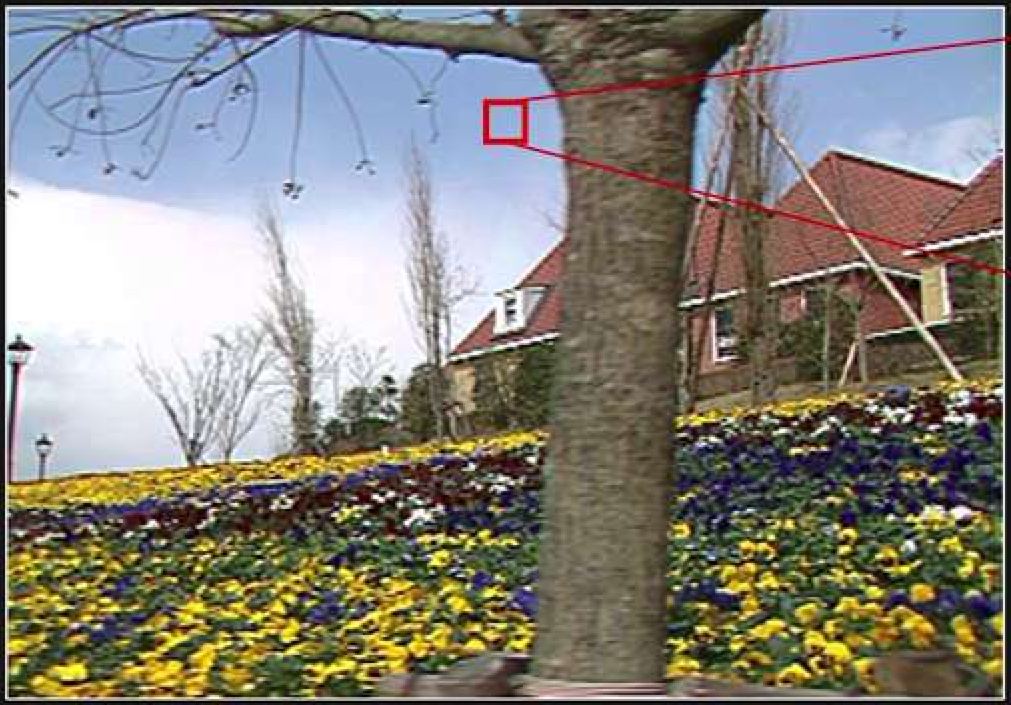

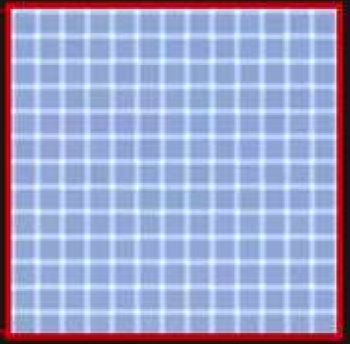

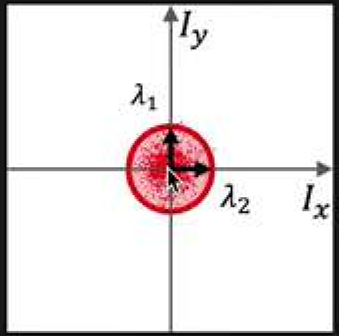

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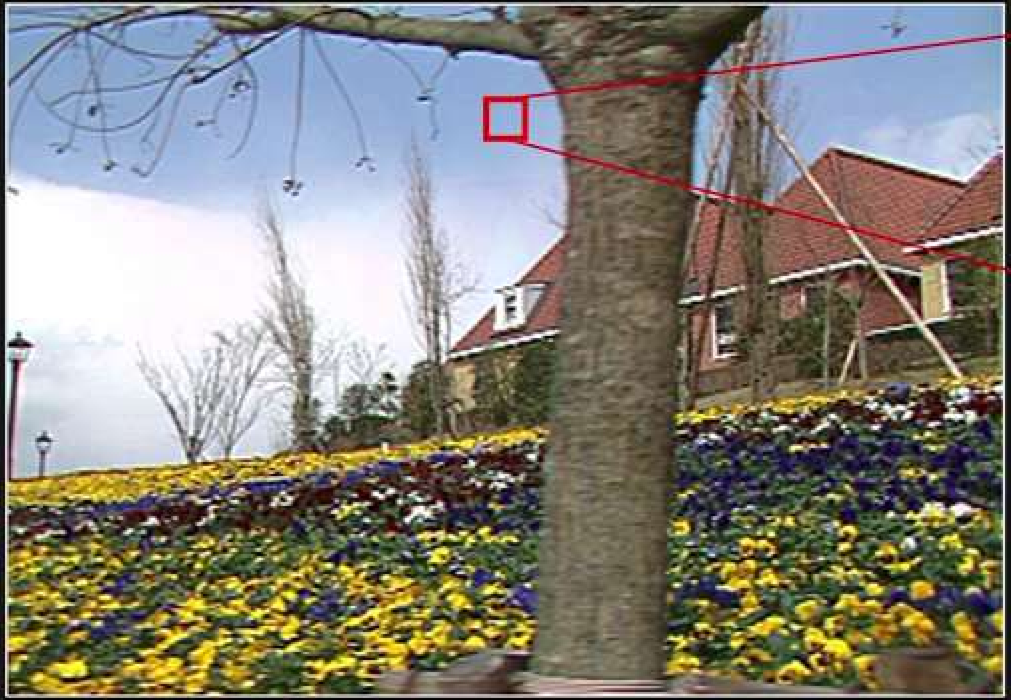

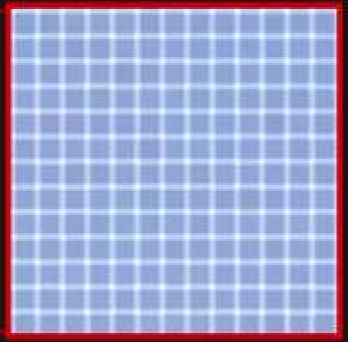

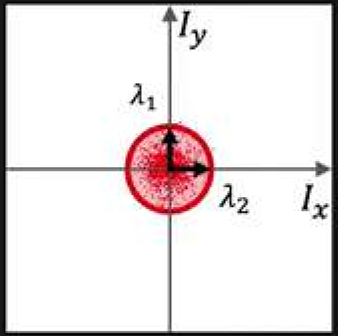

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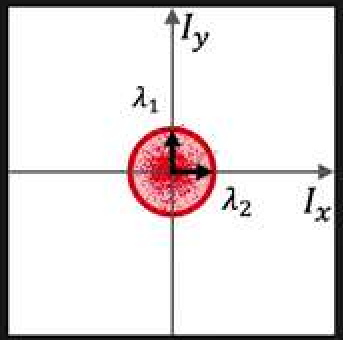

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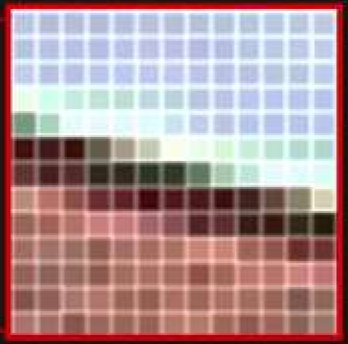
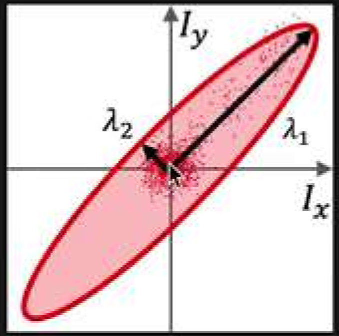




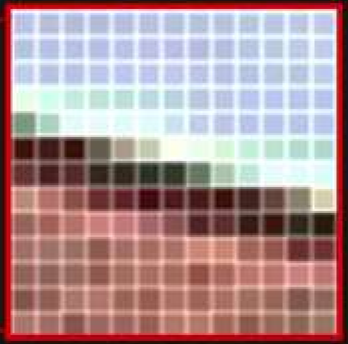
$\lambda_1 \ggg \lambda_2$

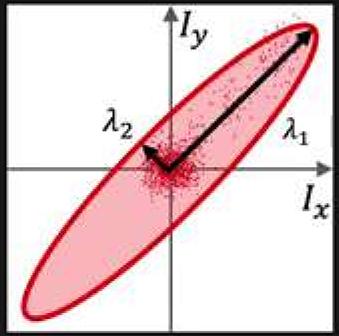
Badly conditioned. Prominent gradient in one direction.

Cannot reliably compute flow!

Same as Aperture Problem.

Edges (Bad)





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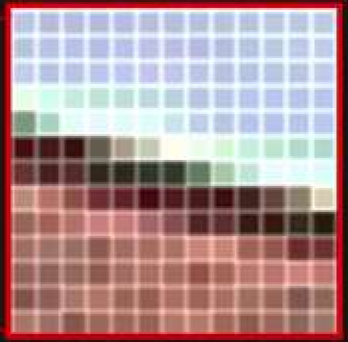

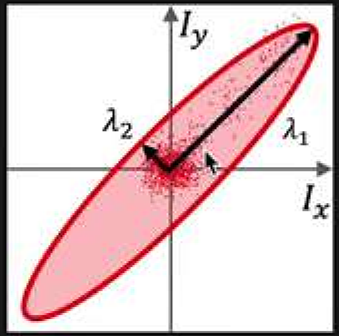

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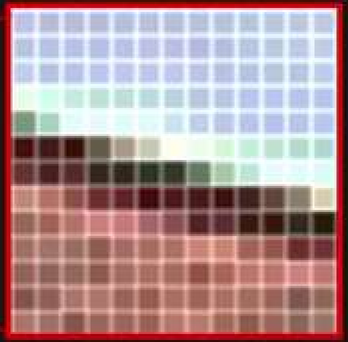

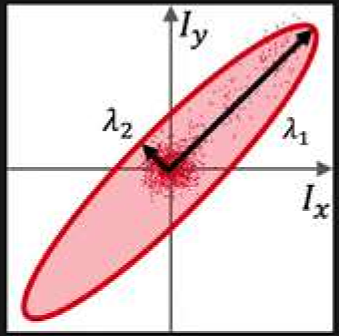

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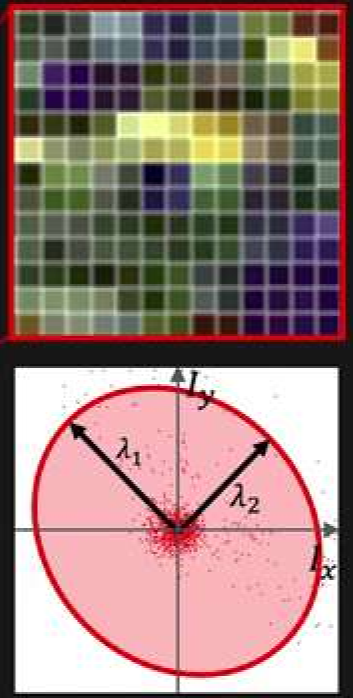
Textured Regions (Good)

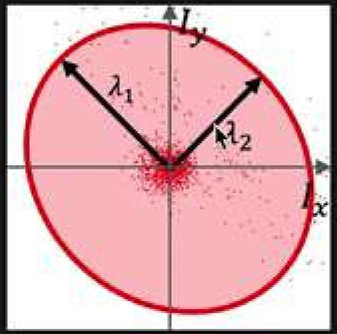


$\lambda_1 \sim \lambda_2$
Both are Large

Well conditioned. Large and diverse gradient magnitudes.

Can reliably compute optical flow.

Textured Regions (Good)



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