Optical Flow Constraint Equation

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Columbia University

Topic: Motion and Optical Flow, Module: Reconstruction II

First Principles of Computer Vision

Optical Flow

t

$t + \delta t$

[Horn 1981]

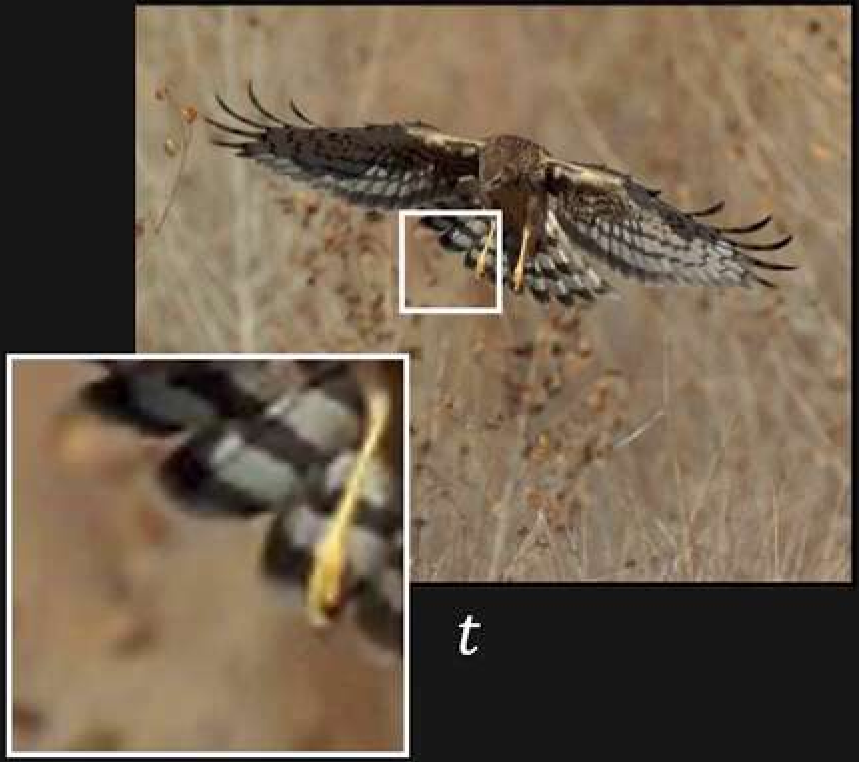
Optical Flow

t

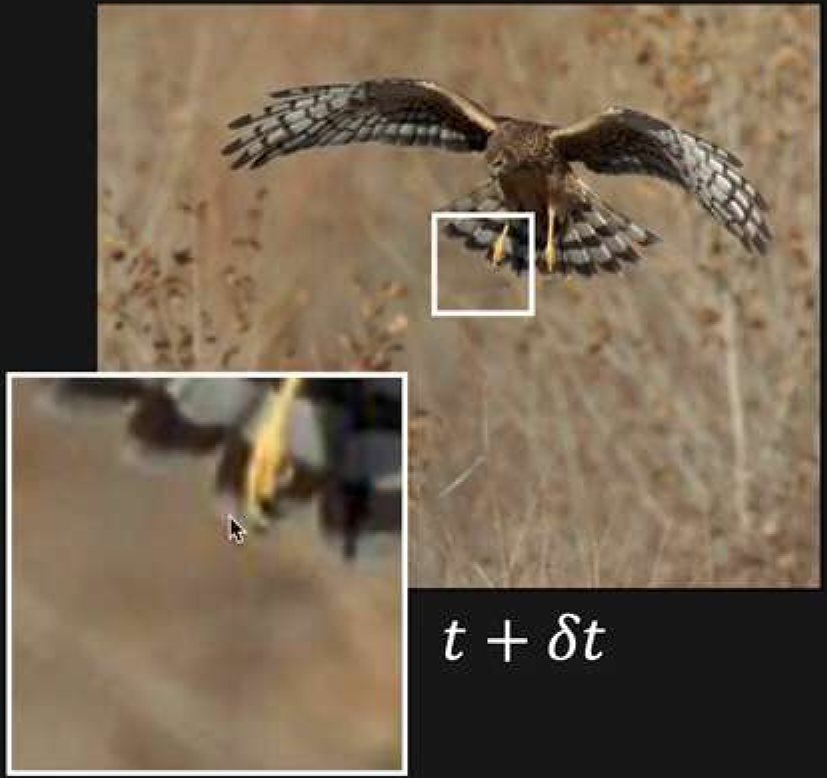
$t + \delta t$

[Horn 1981]

Optical Flow



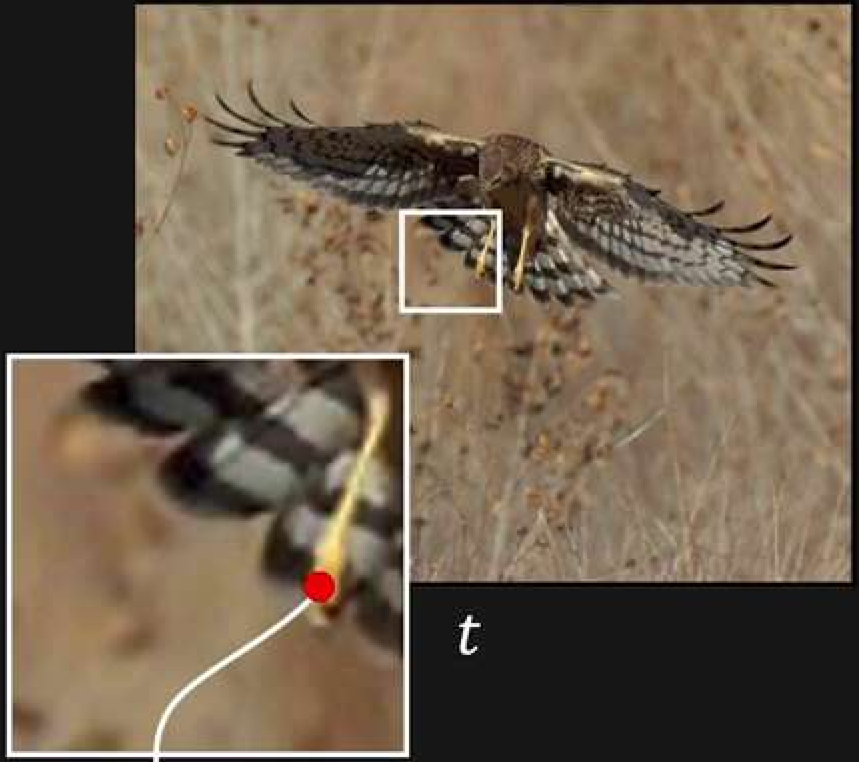
t



$t + \delta t$

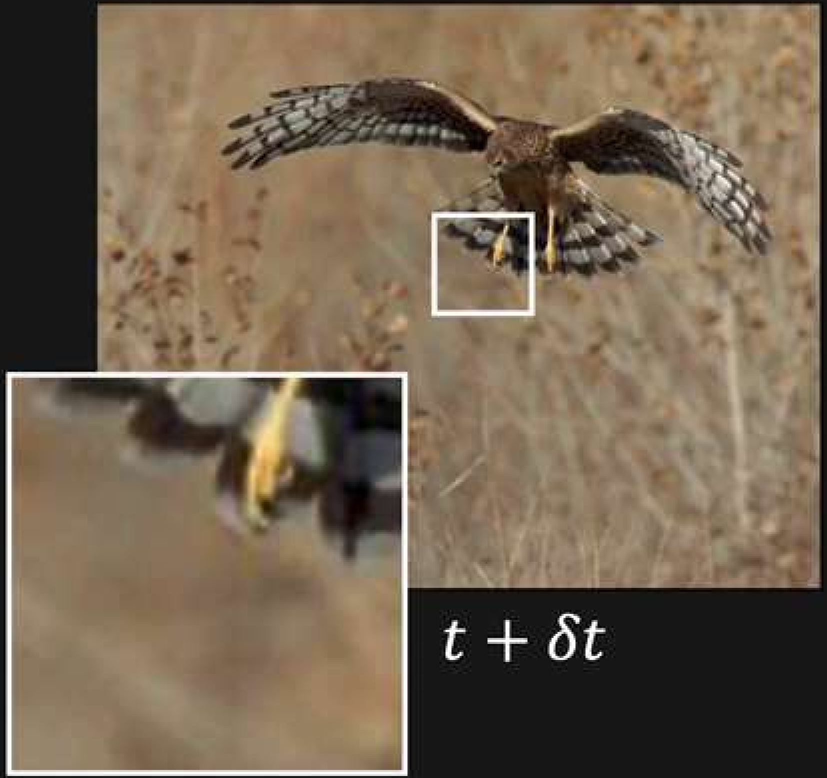
[Horn 1981]

Optical Flow



t

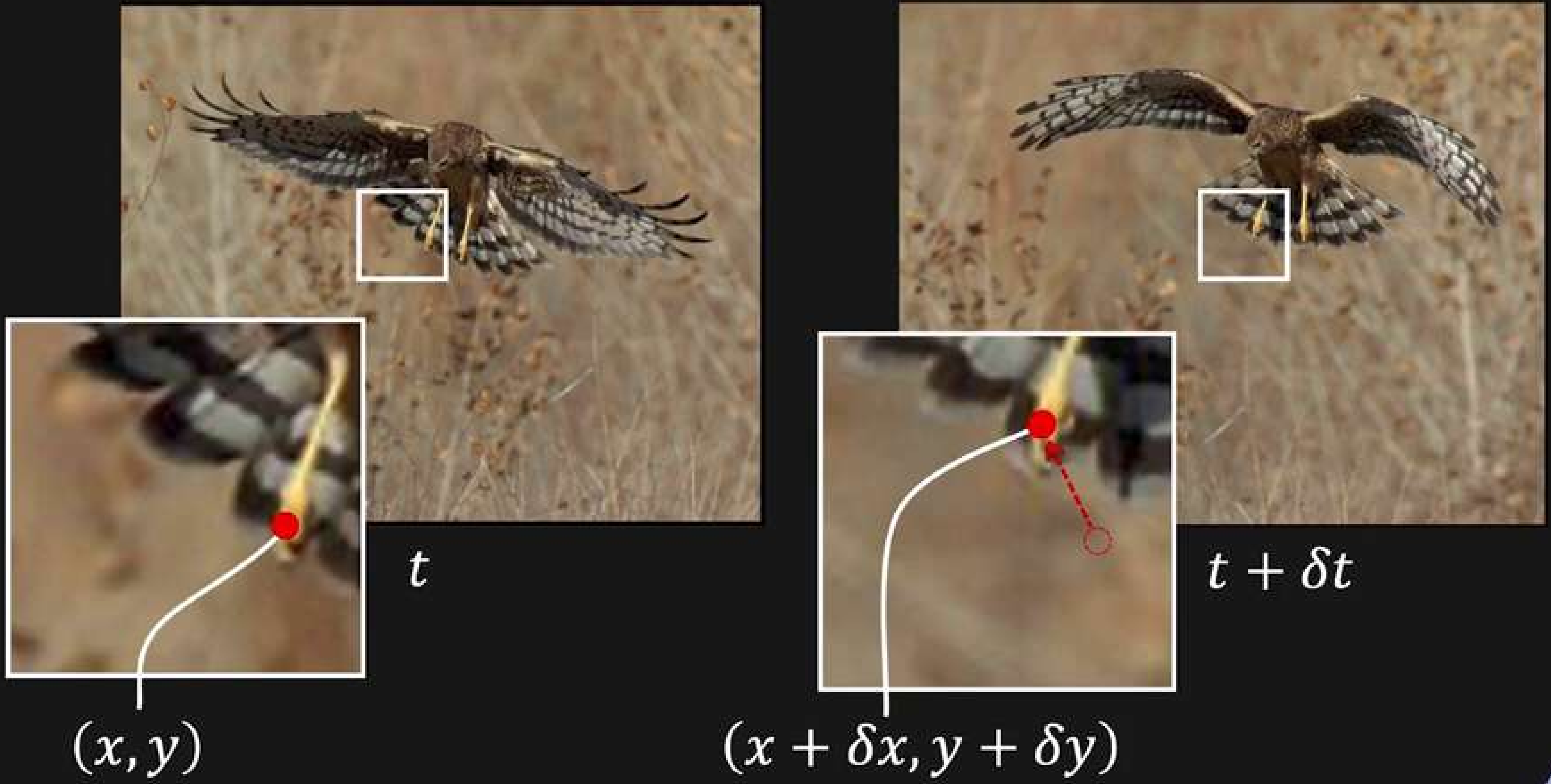
(x, y)



$t + \delta t$

[Horn 1981]

Optical Flow



Displacement: $(\delta x, \delta y)$

Optical Flow: $(u, v) = \left(\frac{\delta x}{\delta t}, \frac{\delta y}{\delta t}\right)$

[Horn 1981]

Optical Flow

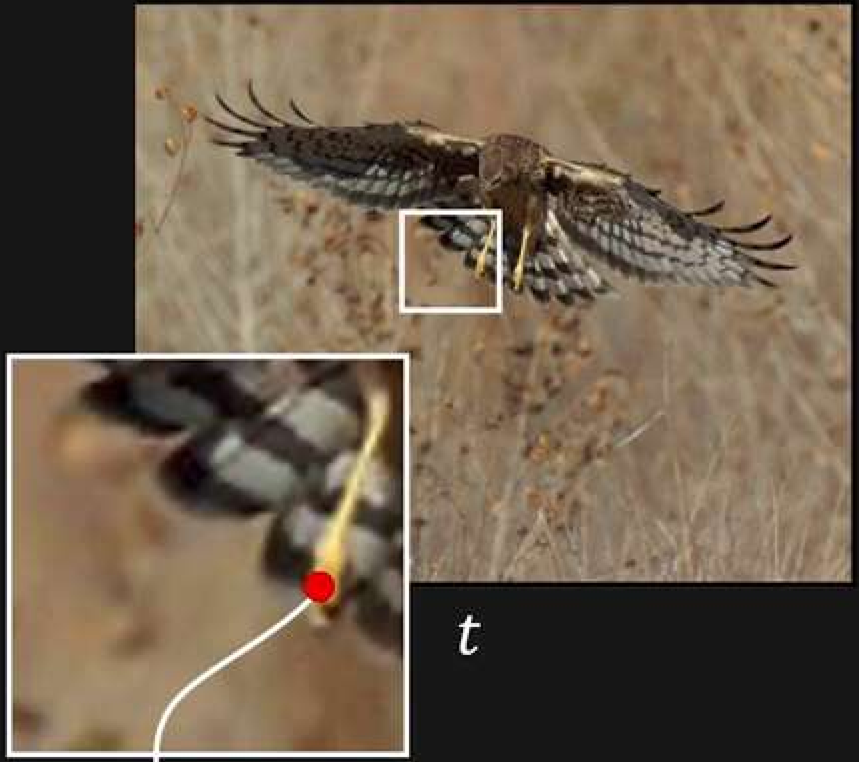
t $\qquad$ $t + \delta t$

(x, y) $\qquad$ $(x + \delta x, y + \delta y)$

Displacement: $(\delta x, \delta y)$ $\qquad$ Optical Flow: $(u, v) = \left(\frac{\delta x}{\delta t}, \frac{\delta y}{\delta t}\right)$

[Horn 1981]

Optical Flow



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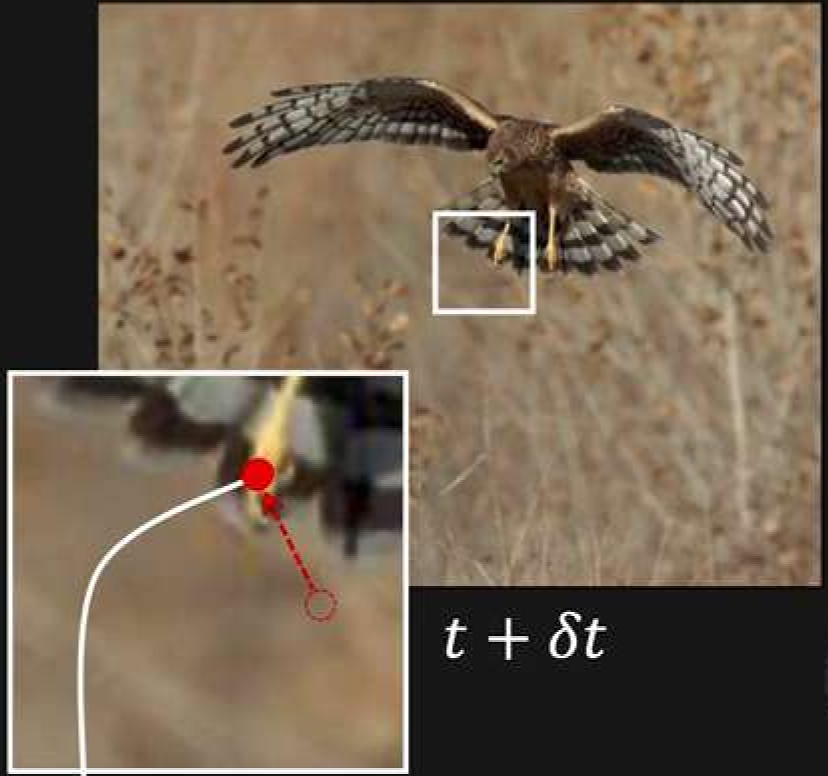

$t + \delta t$

$(x + \delta x, y + \delta y)$

Displacement: $(\delta x, \delta y)$

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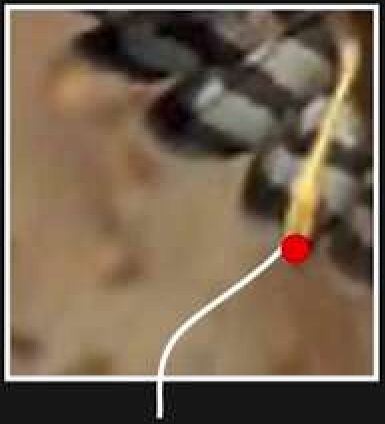
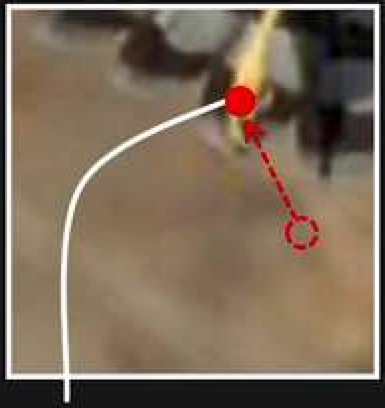
[Horn 1981]

Optical Flow Constraint Equation





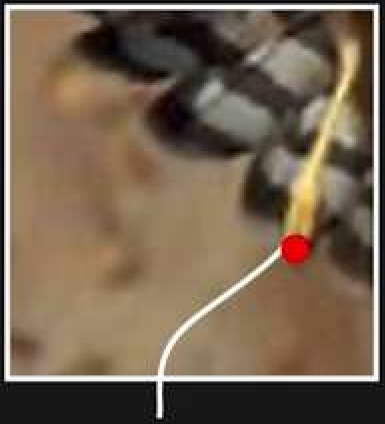
$I(x, y, t)$ $\qquad$ $I(x + \delta x, y + \delta y, t + \delta t)$

Assumption #1:

Brightness of image point remains constant over time

$$\boxed{I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t)}$$

Optical Flow Constraint Equation



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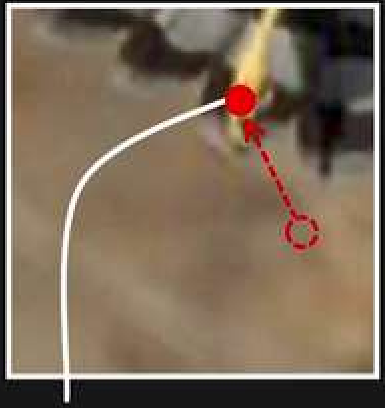

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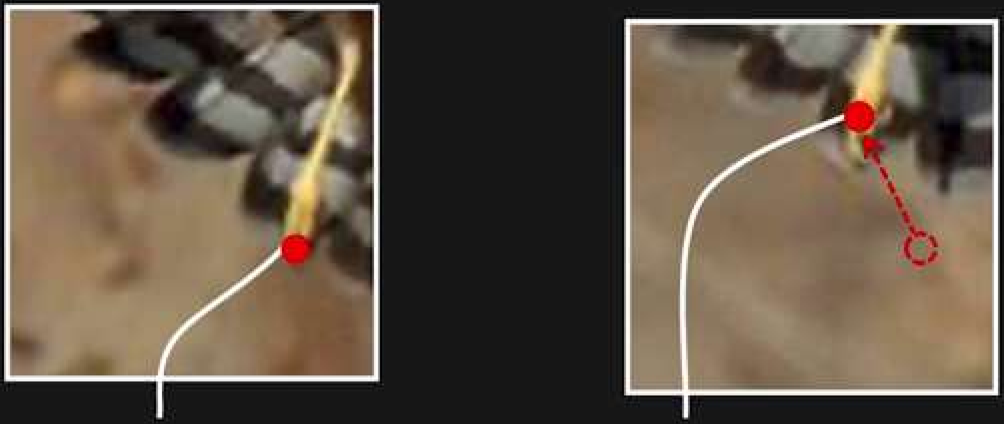

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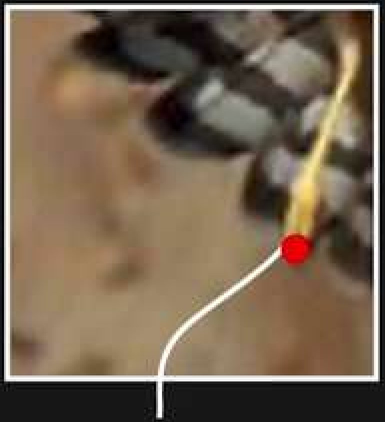

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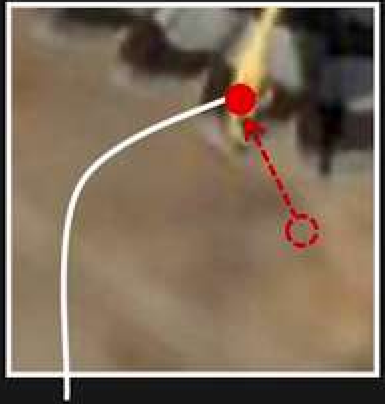

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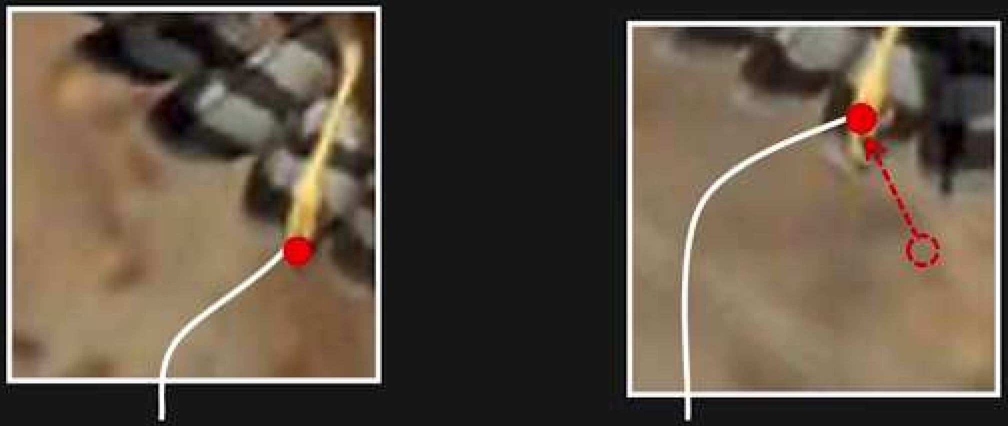

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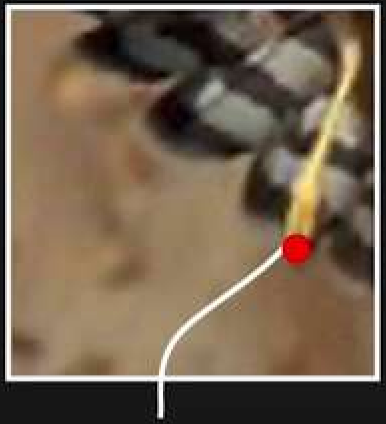

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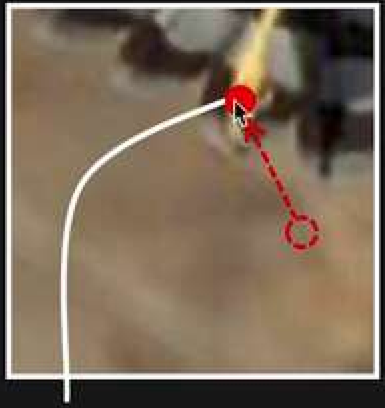

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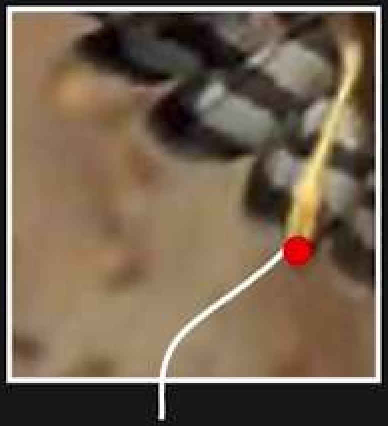

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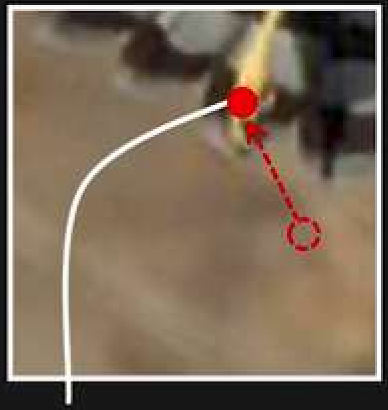

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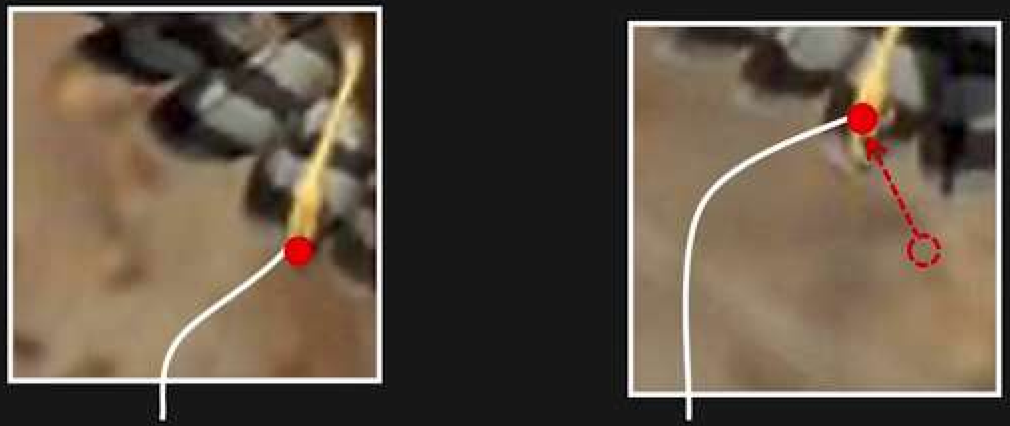

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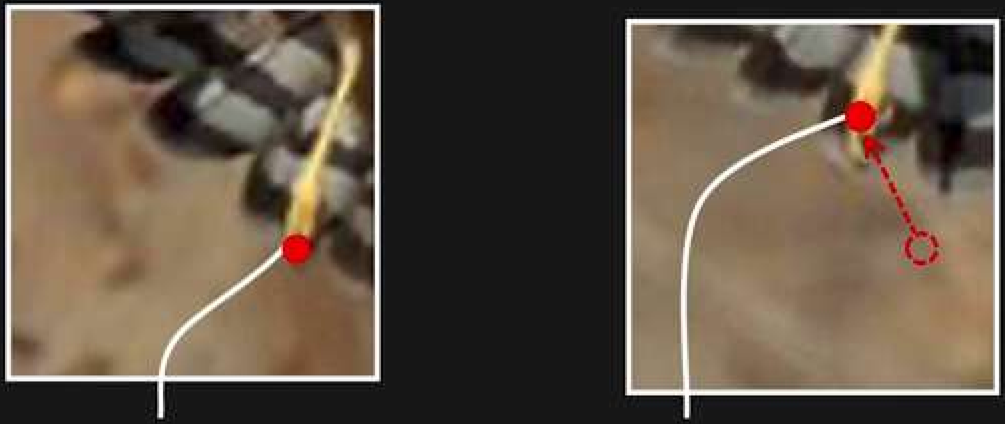
Assumption #2:

Displacement $(\delta x, \delta y)$ and time step δt are small

Optical Flow Constraint Equation



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Taylor Series Expansion

Expand a function as an infinite sum of its derivatives

$$f(x + \delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \frac{\partial^2 f}{\partial x^2}\frac{\delta x^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{\delta x^n}{n!}$$

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If δx is small:

$$f(x+\delta x) = f(x) + \frac{\partial f}{\partial x}\delta x + \boxed{O(\delta x^2)} \rightarrow \text{Almost Zero}$$

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For a function of three variables with small $\delta x, \delta y, \delta t$:

$$f(x+\delta x, y+\delta y, t+\delta t) \approx f(x,y,t) + \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial t}\delta$$

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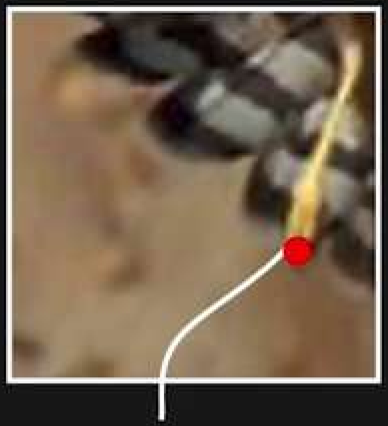
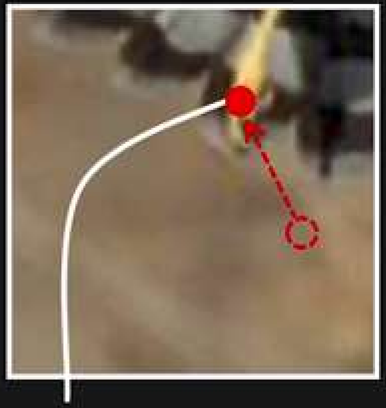
Optical Flow Constraint Equation



$I(x, y, t)$



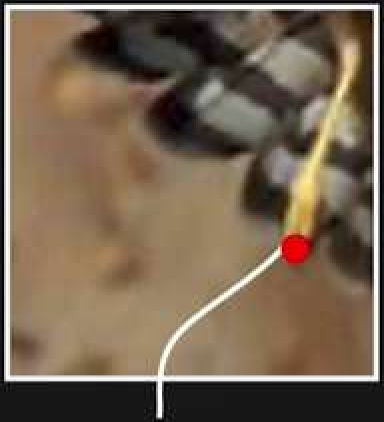
$I(x + \delta x, y + \delta y, t + \delta t)$

Assumption #2:

Displacement $(\delta x, \delta y)$ and time step δt are small

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + \frac{\partial I}{\partial x}\delta x + \frac{\partial I}{\partial y}\delta y + \frac{\partial I}{\partial t}\delta t$$

Optical Flow Constraint Equation



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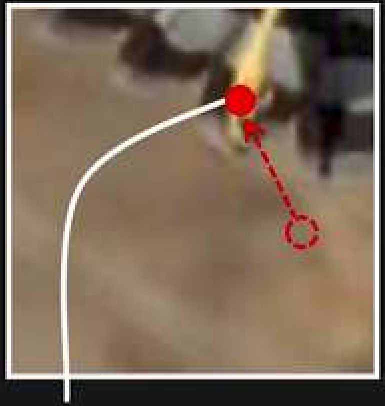

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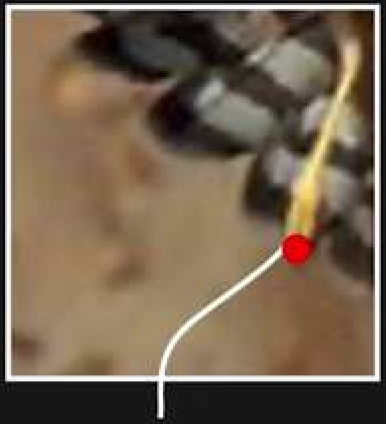

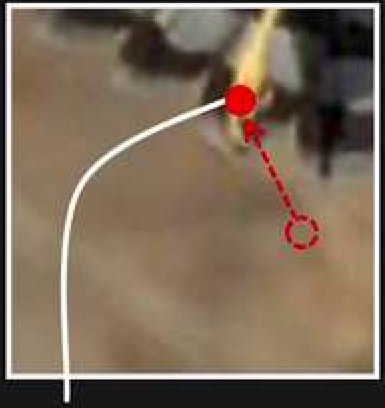

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$$\boxed{I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + I_x \delta x + I_y \delta y + I_t \delta t}$$

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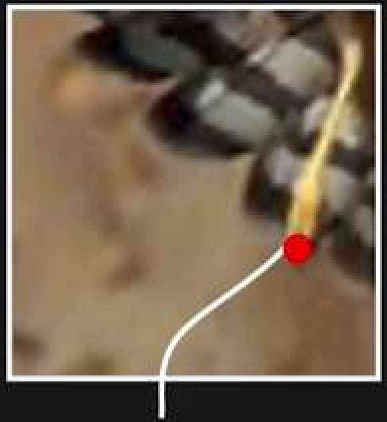

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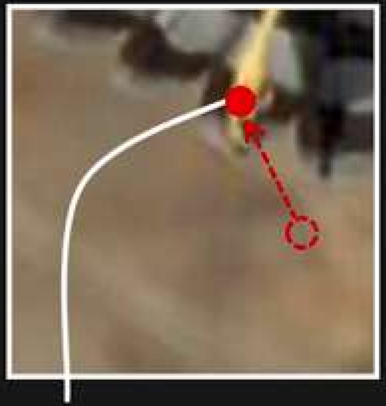

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Optical Flow Constraint Equation

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) \qquad \text{------------ (1)}$$

$$I(x + \delta x, y + \delta y, t + \delta t) = I(x, y, t) + I_x \delta x + I_y \delta y + I_t \delta t$$

------------ (2)

Subtract (1) from (2): $I_x \delta x + I_y \delta y + I_t \delta t = 0$

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Constraint Equation: $\boxed{I_x u + I_y v + I_t = 0}$ (u, v): Optical Flow

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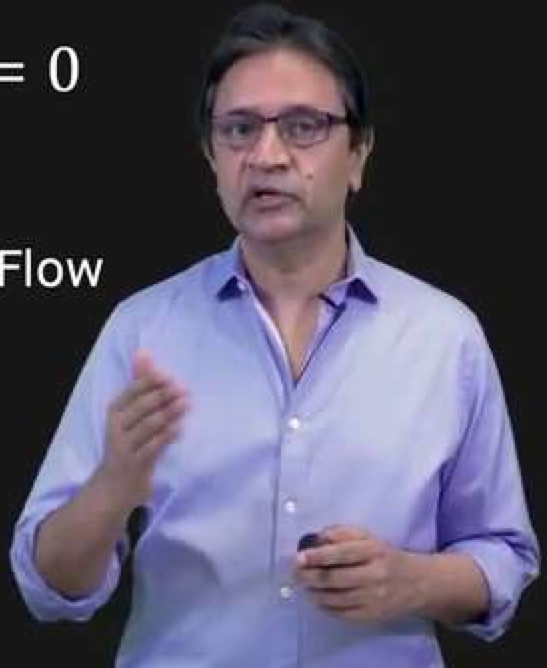
(I_x, I_y, I_t) can be easily computed from two frames



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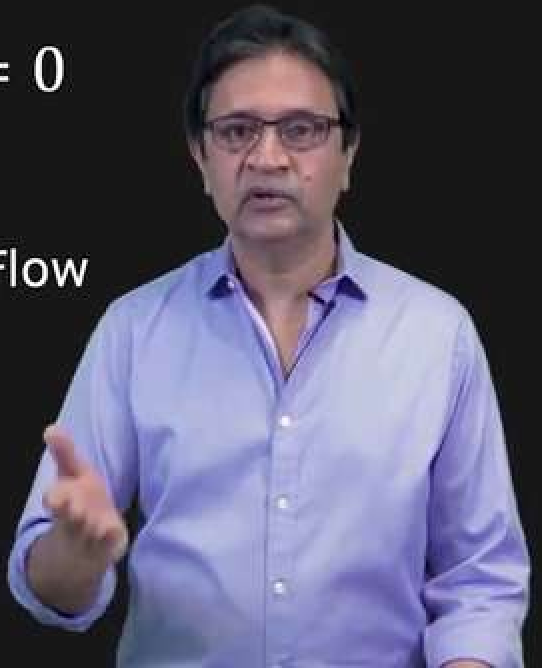
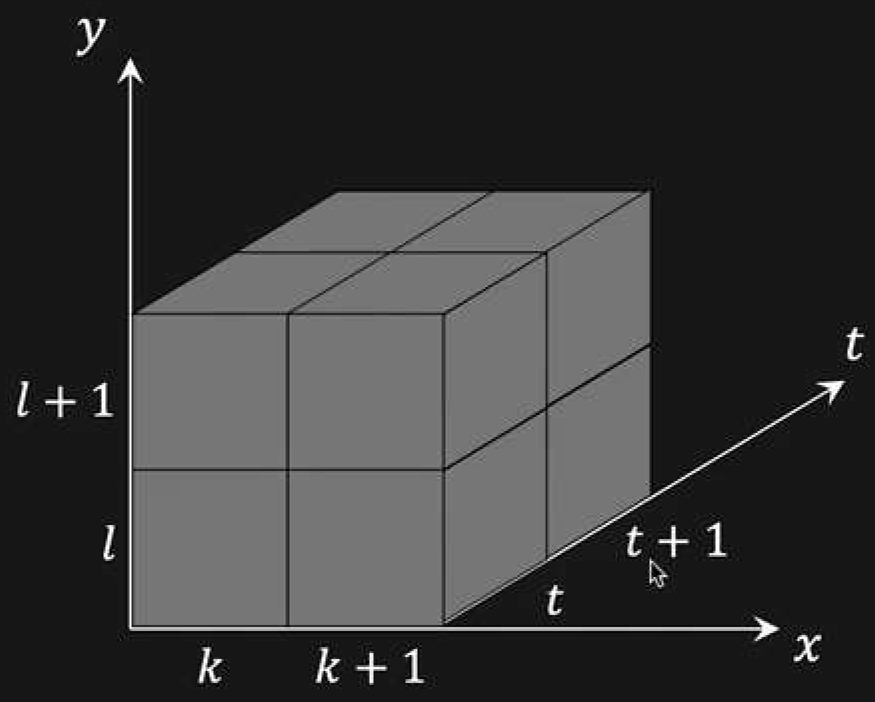
Constraint Equation: $\boxed{I_x u + I_y v + I_t = 0}$ (u,v): Optical Flow

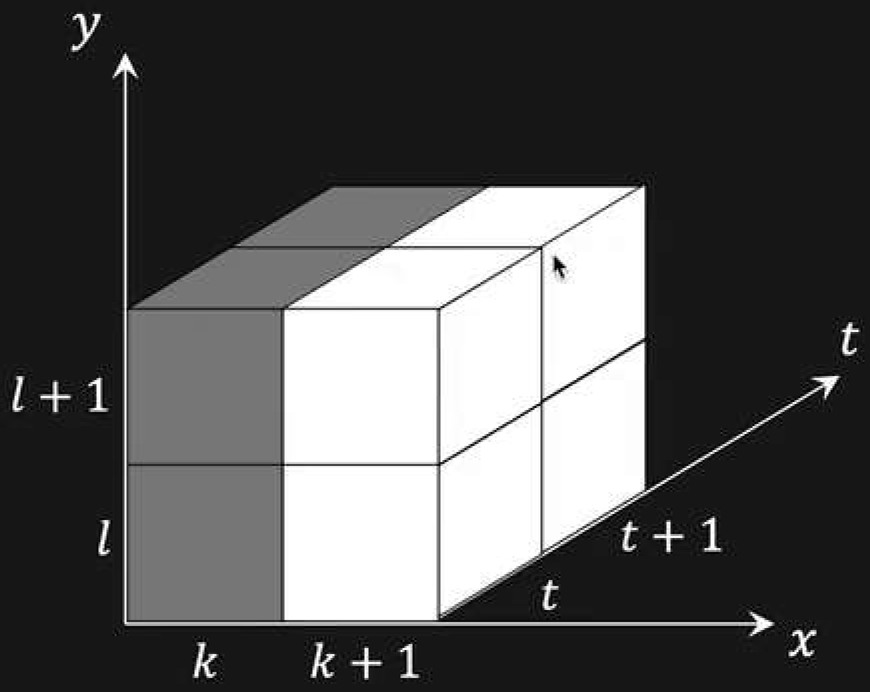
(I_x, I_y, I_t) can be easily computed from two frames



Computing Partial Derivatives I_x, I_y, I_t

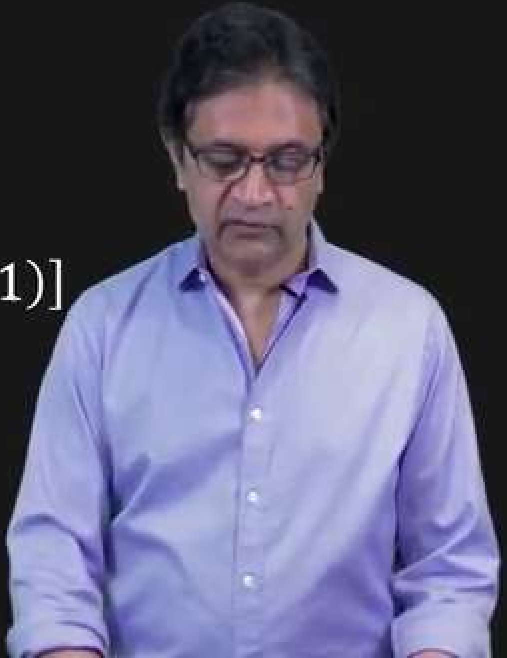


Computing Partial Derivatives I_x, I_y, I_t

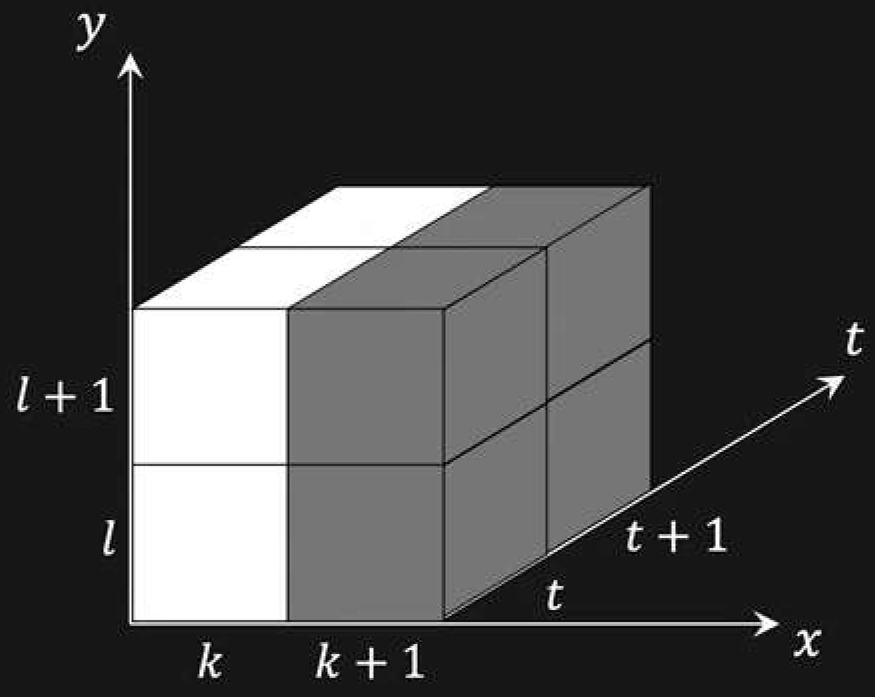


$I_x(k, l, t) =$

$$\frac{1}{4}[I(k+1, l, t) + I(k+1, l, t+1) + I(k+1, l+1, t) + I(k+1, l+1, t+1)]$$



Computing Partial Derivatives I_x, I_y, I_t

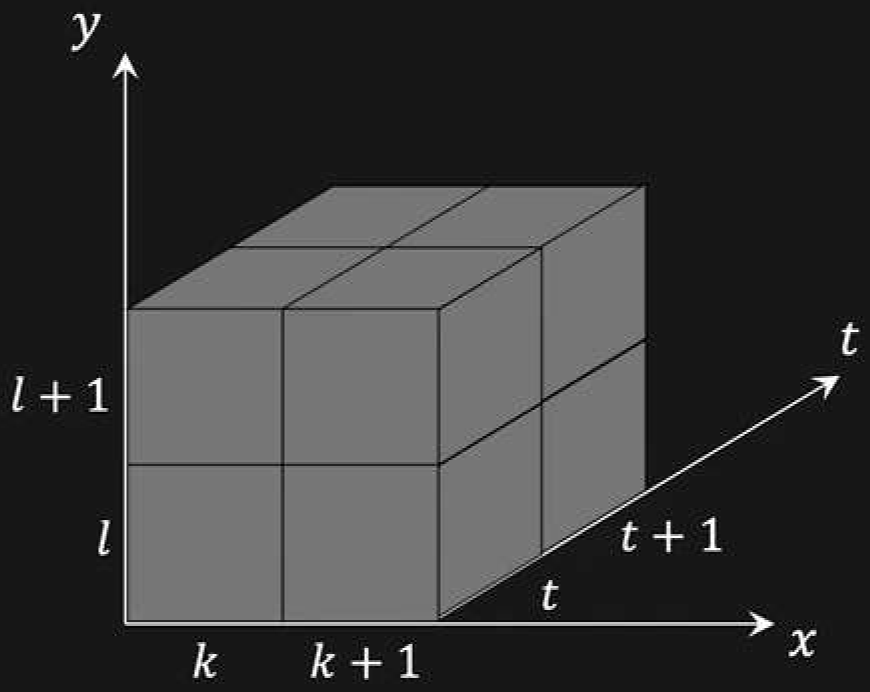


$I_x(k,l,t) =$

$\frac{1}{4}[I(k+1,l,t) + I(k+1,l,t+1) + I(k+1,l+1,t) + I(k+1,l+1,t+1)]$

$-\frac{1}{4}[I(k,l,t) + I(k,l,t+1) + I(k,l+1,t) + I(k,l+1,t+1)]$

Computing Partial Derivatives I_x, I_y, I_t



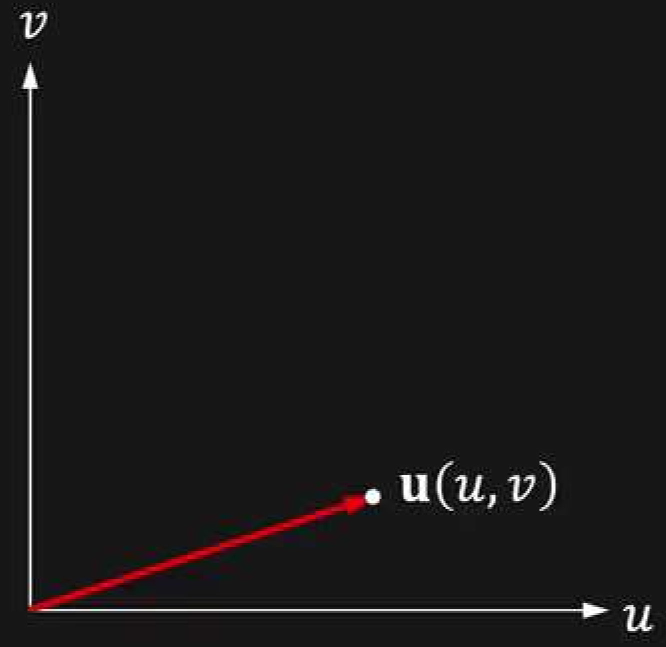
$I_x(k,l,t) =$

$\frac{1}{4}[I(k+1,l,t) + I(k+1,l,t+1) + I(k+1,l+1,t) + I(k+1,l+1,t+1)]$

$-\frac{1}{4}[I(k,l,t) + I(k,l,t+1) + I(k,l+1,t) + I(k,l+1,t+1)]$

Similarly find $I_y(k,l,t)$ and $I_t(k,l,t)$

Geometric Interpretation



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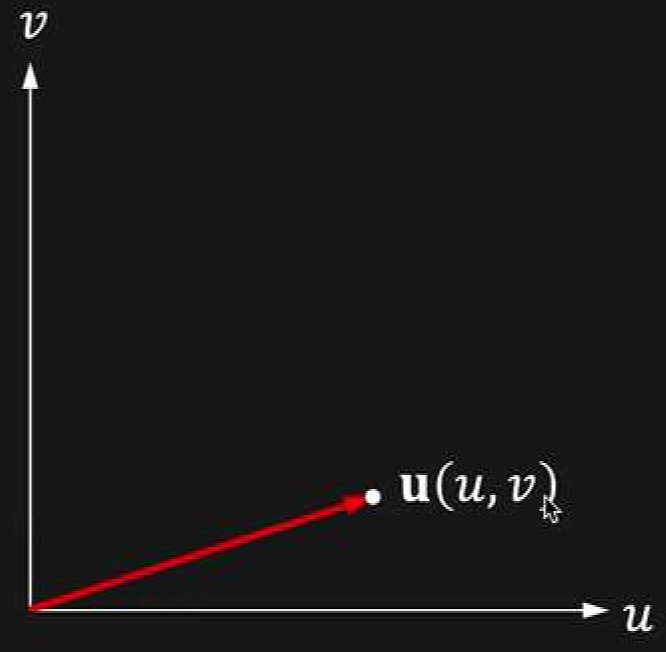
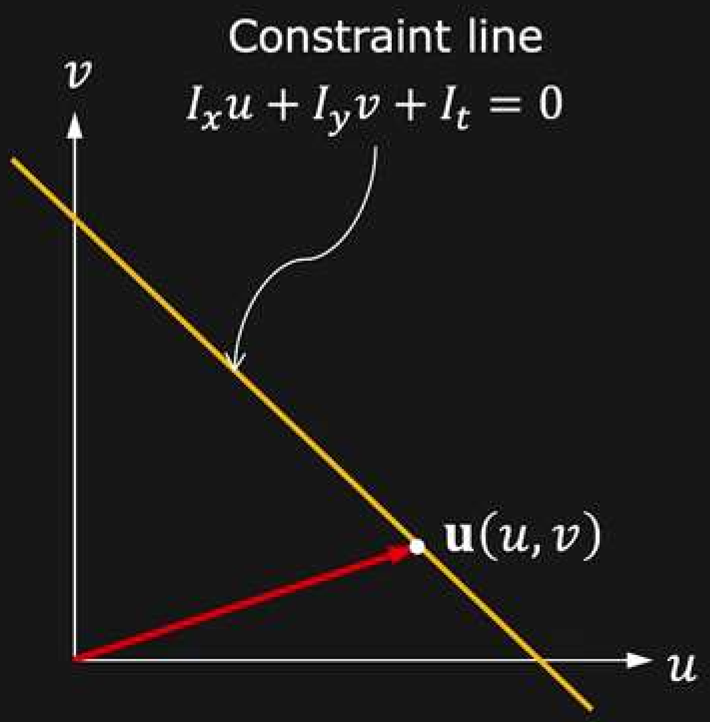


Geometric Interpretation

For any point (x, y) in the image, its optical flow (u, v) lies on the line:

$$I_x u + I_y v + I_t = 0$$



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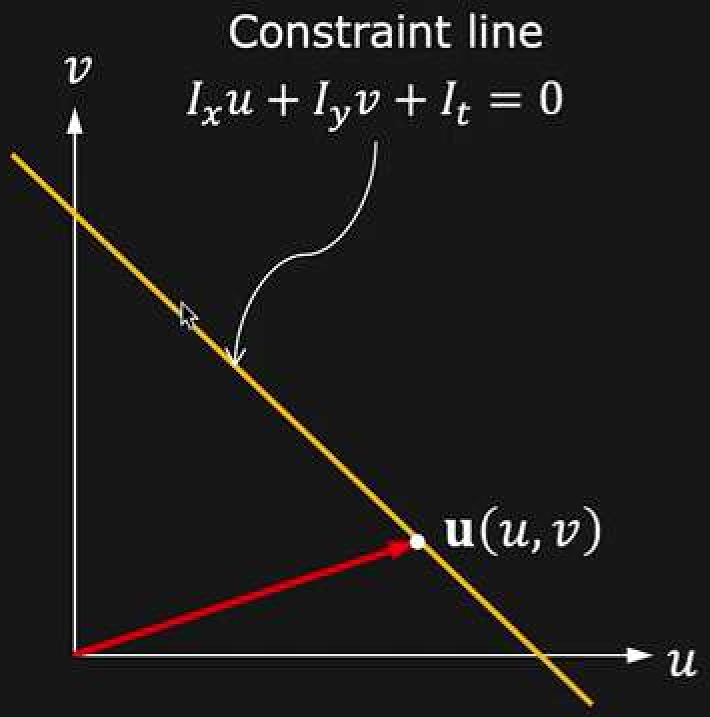

Geometric Interpretation

For any point (x, y) in the image, its optical flow (u, v) lies on the line:

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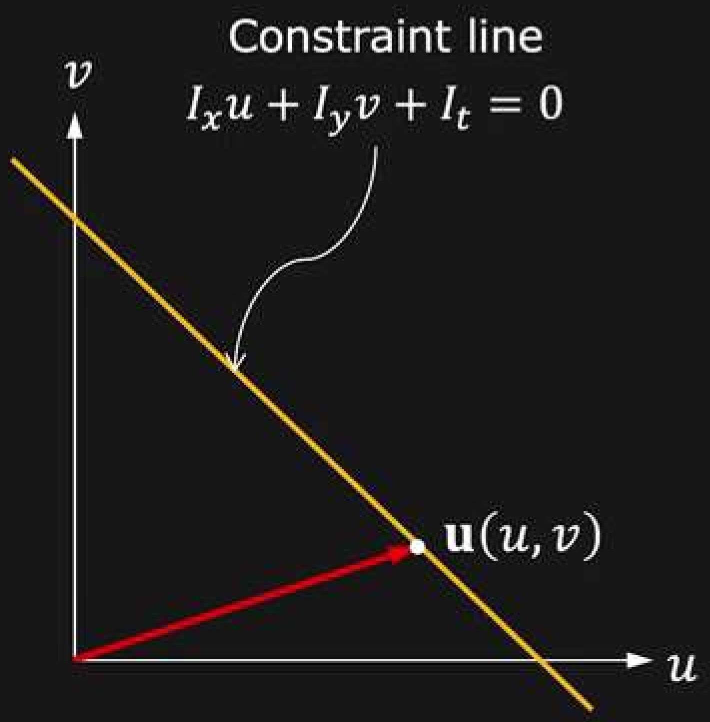
Optical Flow can be split into two components.

$$\mathbf{u} = \mathbf{u}_n + \mathbf{u}_p$$



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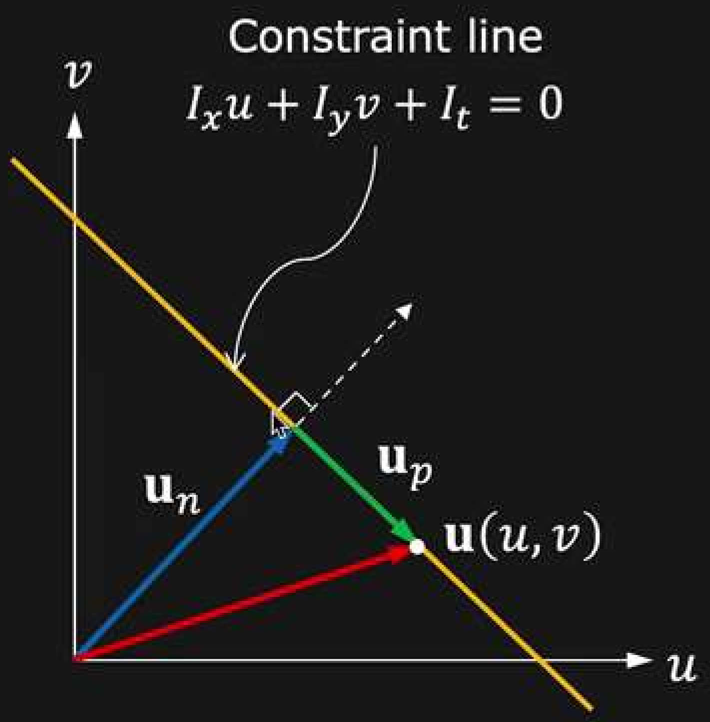

Optical Flow can be split into two components.

$$\mathbf{u} = \mathbf{u}_n + \mathbf{u}_p$$

$\mathbf{u}_n$: Normal Flow

$\mathbf{u}_p$: Parallel Flow

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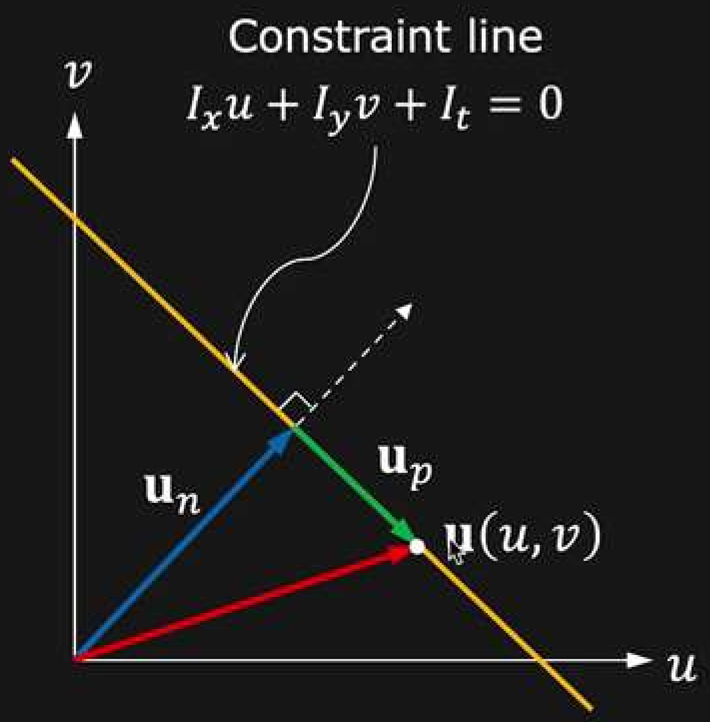

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$$\mathbf{u} = \mathbf{u}_n + \mathbf{u}_p$$

$\mathbf{u}_n$: Normal Flow

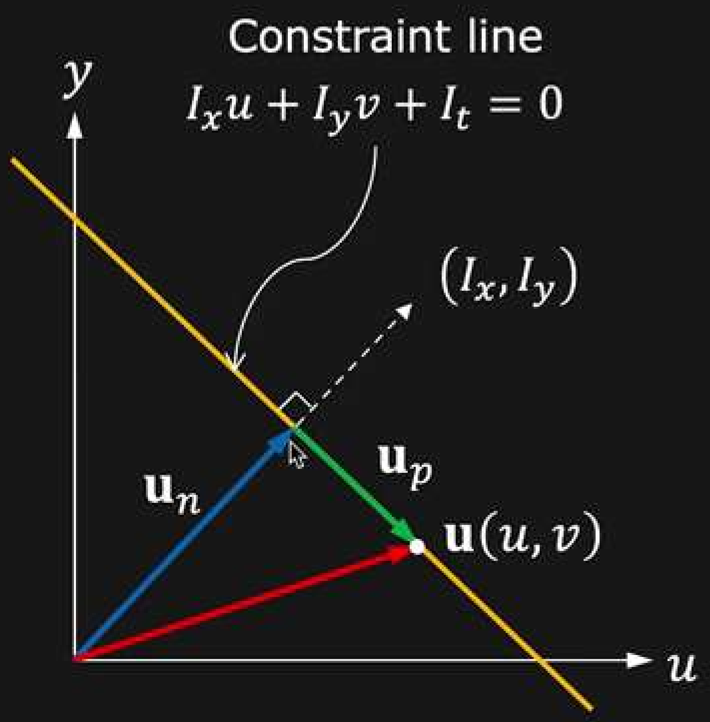
$\mathbf{u}_p$: Parallel Flow

Normal Flow

Direction of Normal Flow:

Unit vector perpendicular to the constraint line:

$$\hat{\mathbf{u}}_n = \frac{(I_x, I_y)}{\sqrt{I_x^2 + I_y^2}}$$



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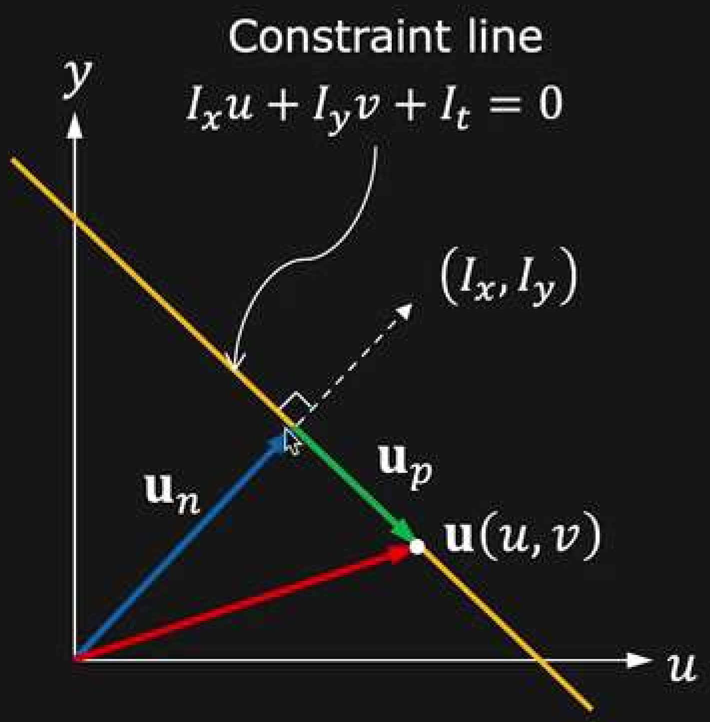

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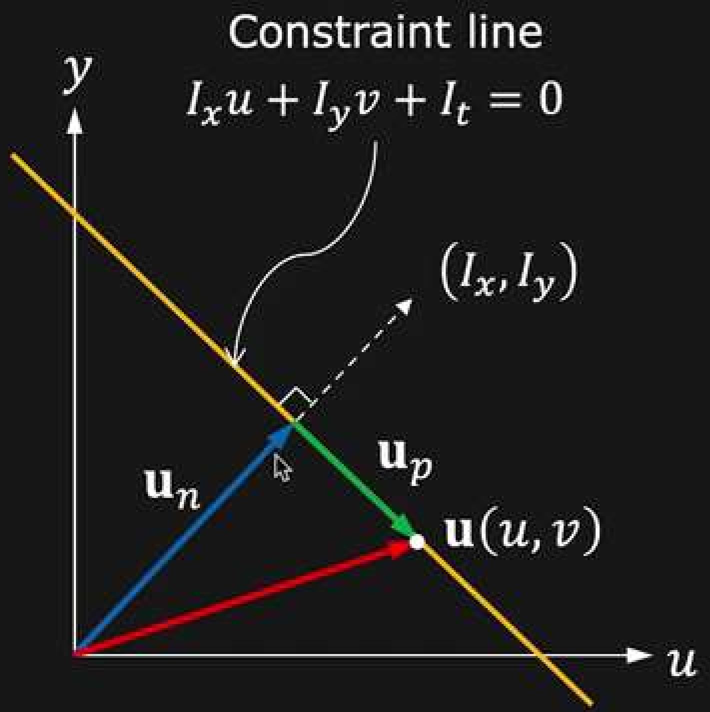


Magnitude of Normal Flow:

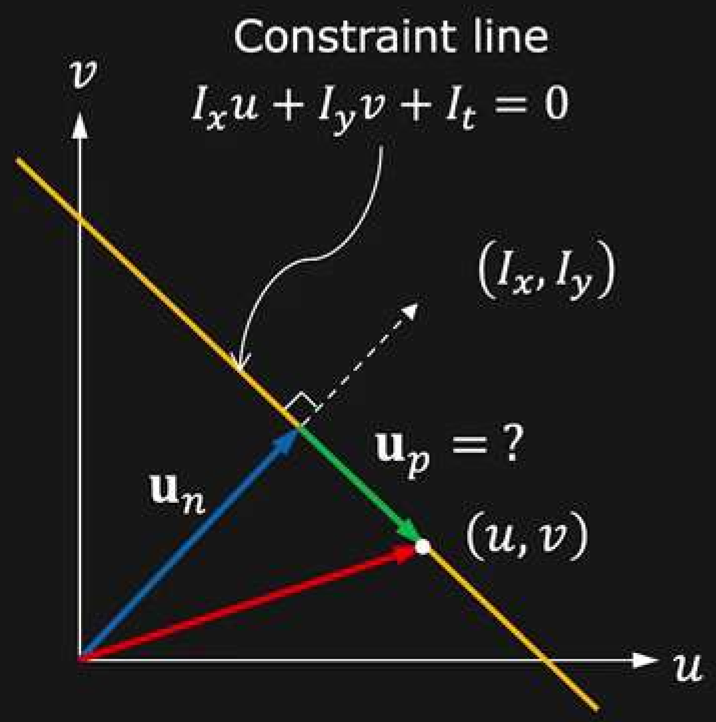
Distance of origin from the constraint line:

$$|\mathbf{u}_n| = \frac{|I_t|}{\sqrt{I_x^2 + I_y^2}}$$

$$\boxed{\mathbf{u}_n = \frac{|I_t|}{(I_x^2 + I_y^2)}(I_x, I_y)}$$

Parallel Flow

We cannot determine $\mathbf{u}_p$, the optical flow component parallel to the constraint line.



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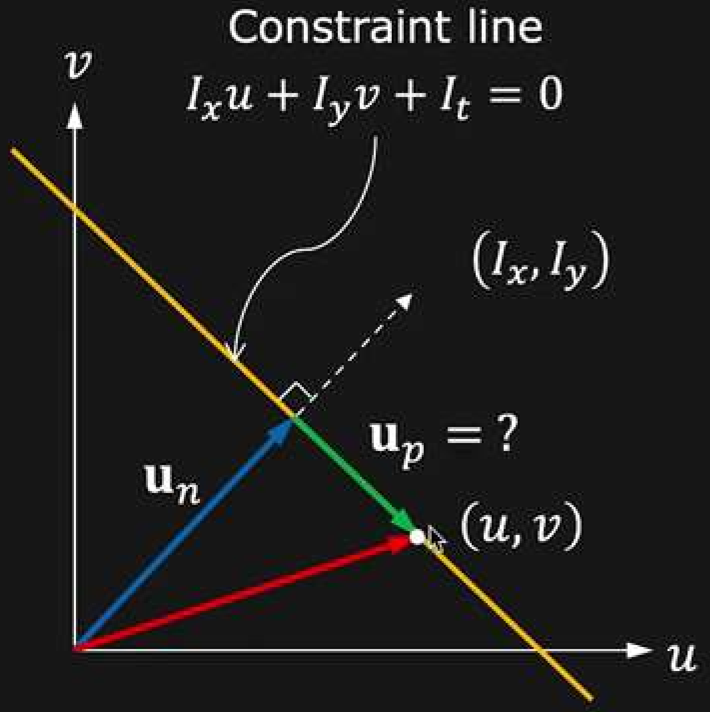
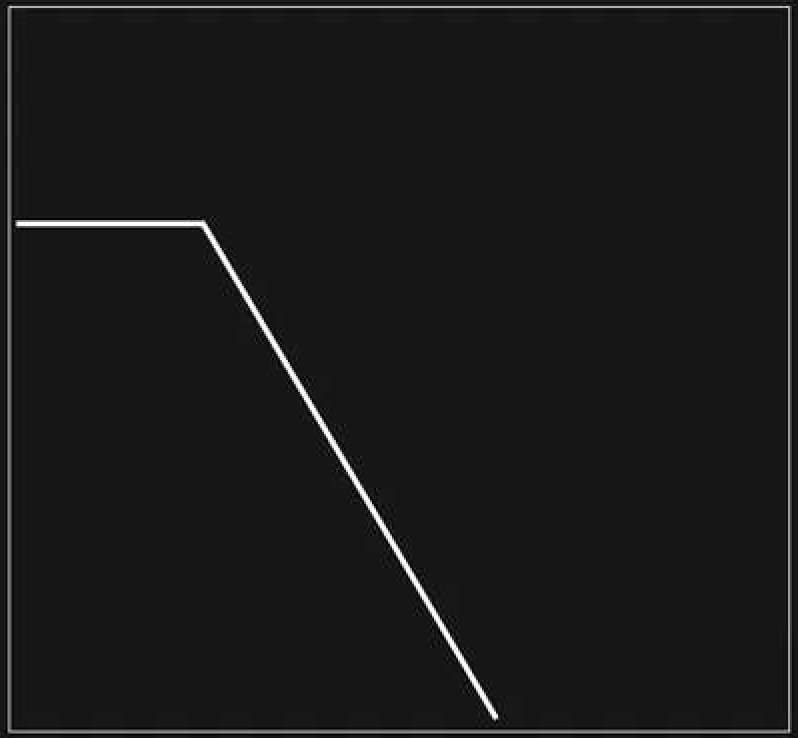
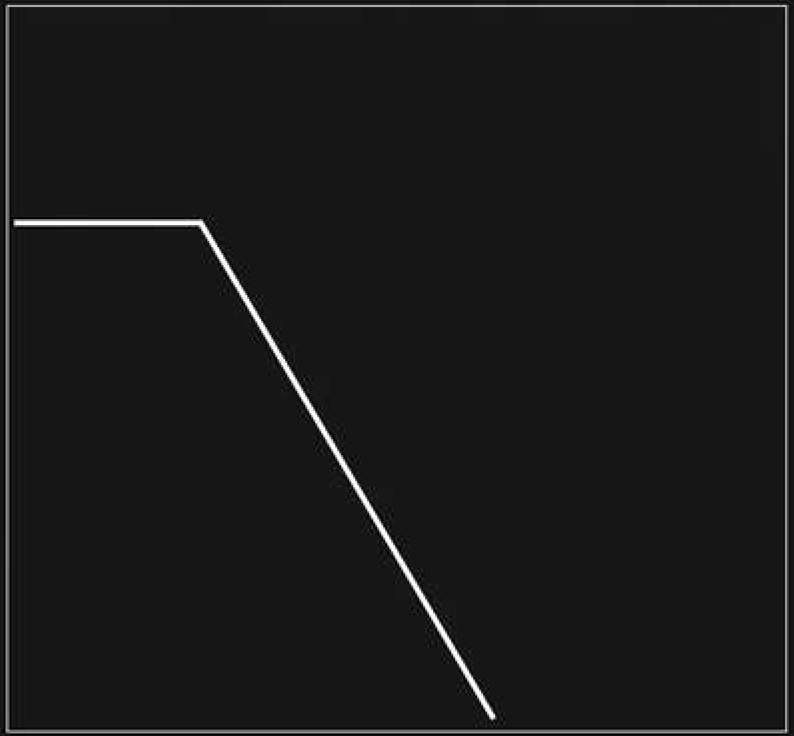


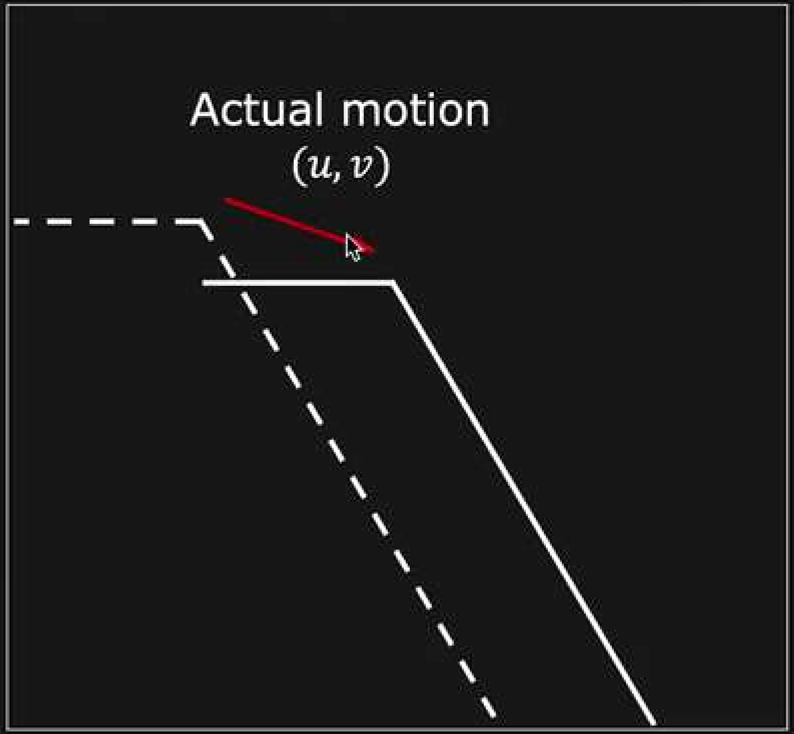
Aperture Problem



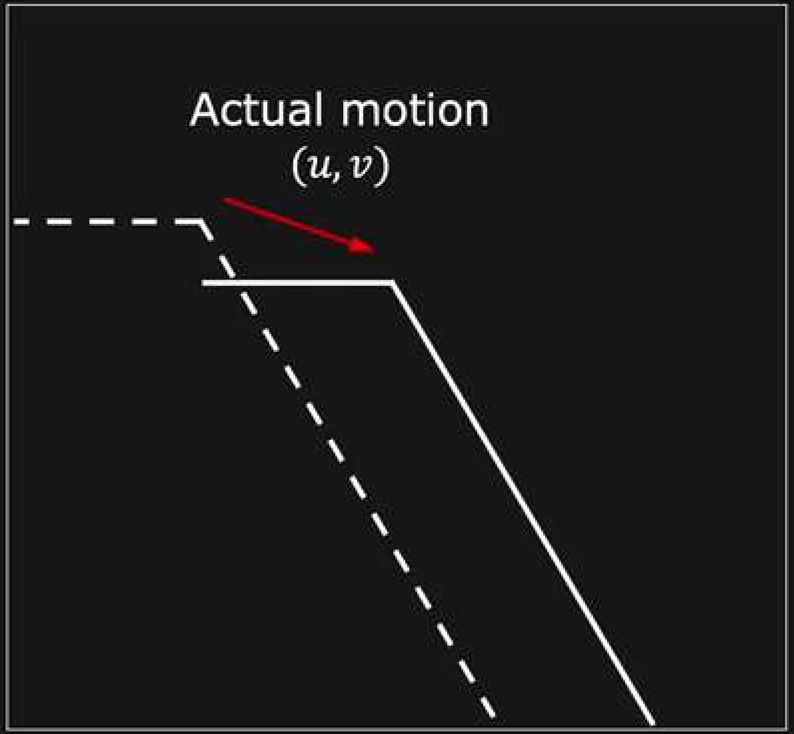
Aperture Problem



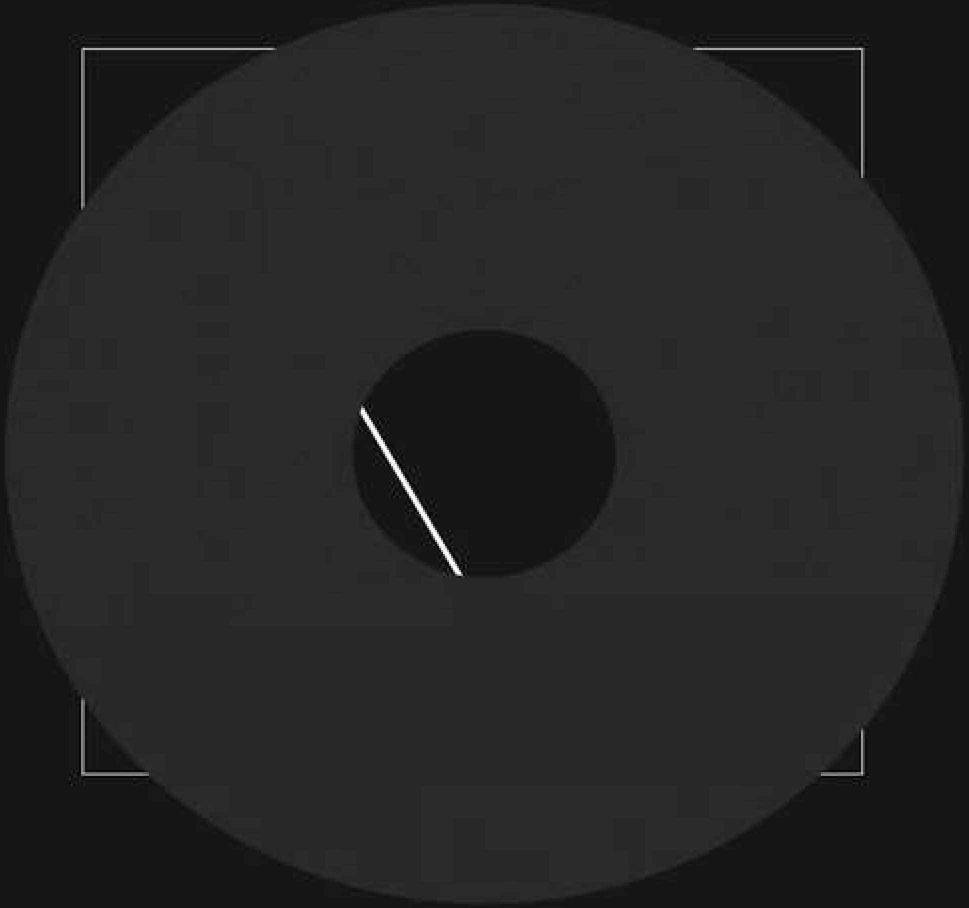
Aperture Problem



Aperture Problem



Aperture Problem



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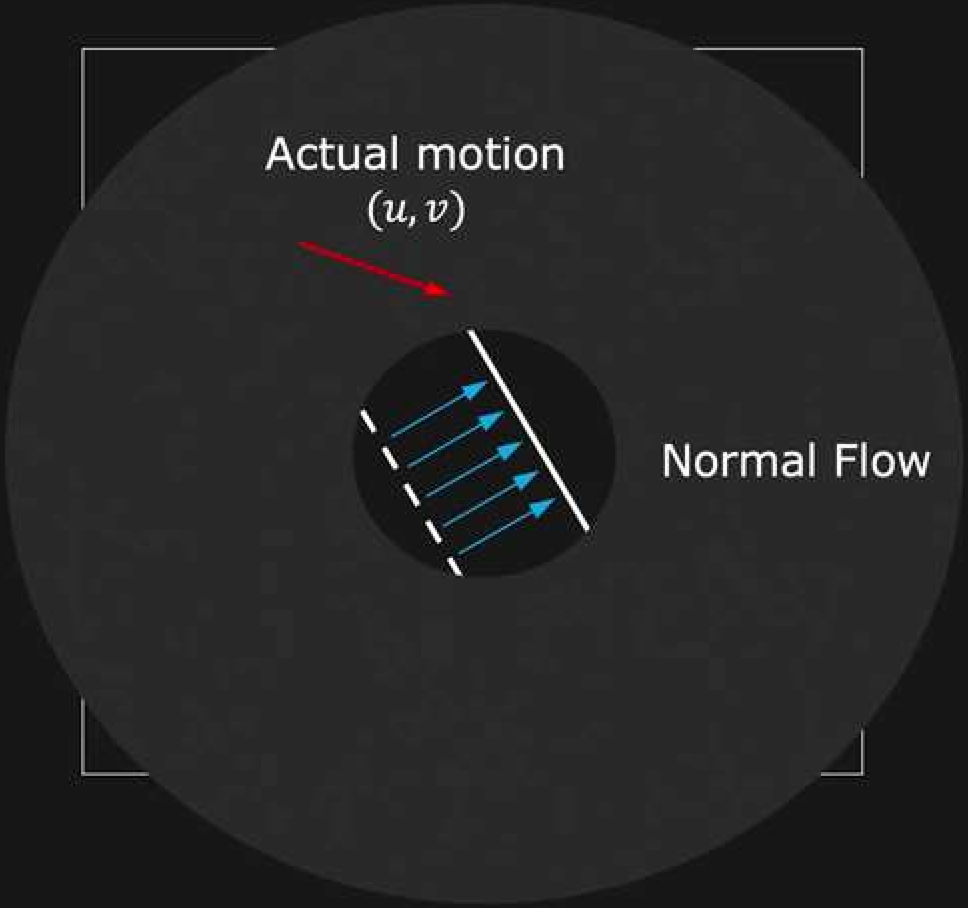
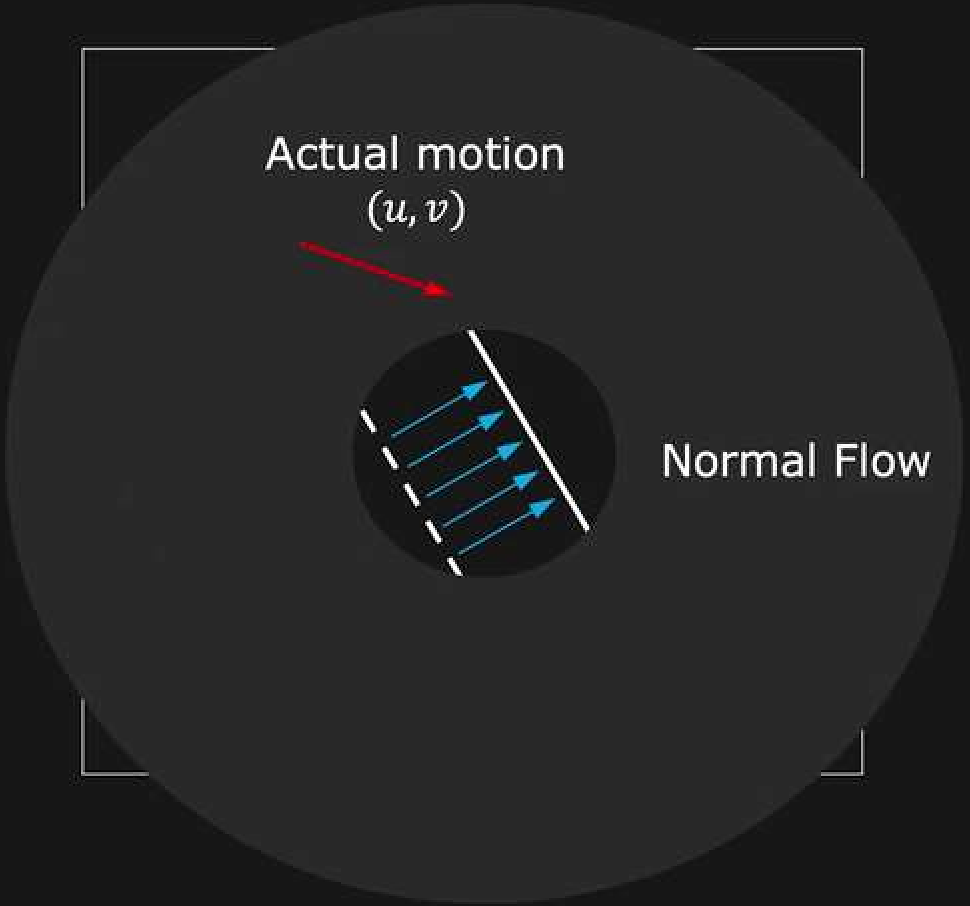


Aperture Problem



Locally, we can only determine normal flow!

Optical Flow is Under Constrained

Constraint Equation: $\boxed{I_x u + I_y v + I_t = 0}$

2 unknowns, 1 equation.