# Rank of Observation Matrix

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Columbia University

Topic: Structure from Motion, Module: Reconstruction II

First Principles of Computer Vision

# Linear Independence of Vectors

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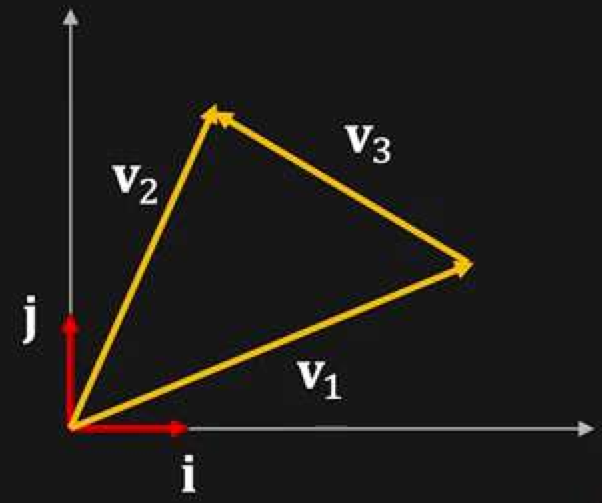

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$\{\mathbf{i}, \mathbf{j}, \mathbf{v}_1\}$ is linearly dependent.

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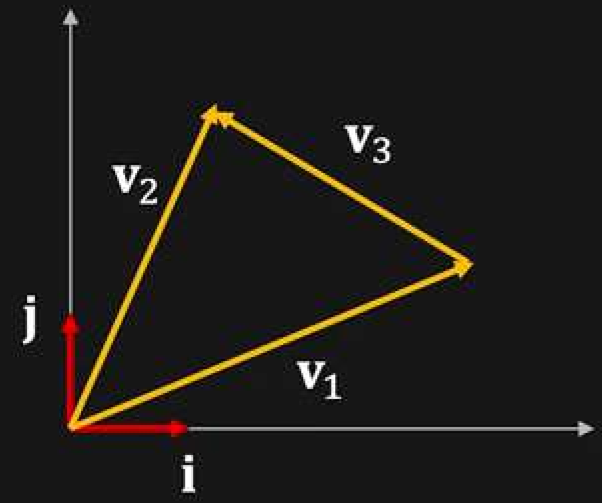


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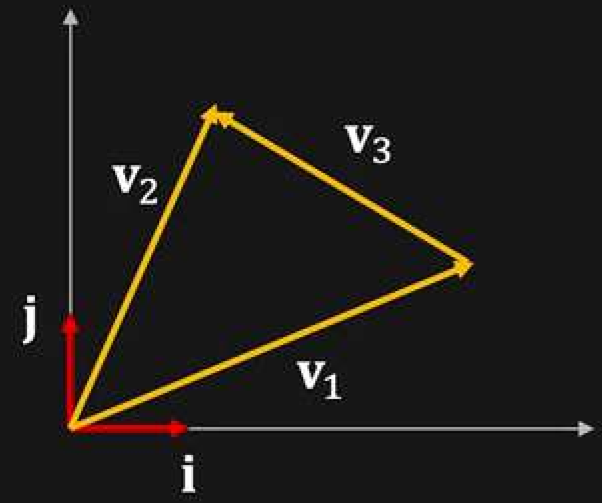


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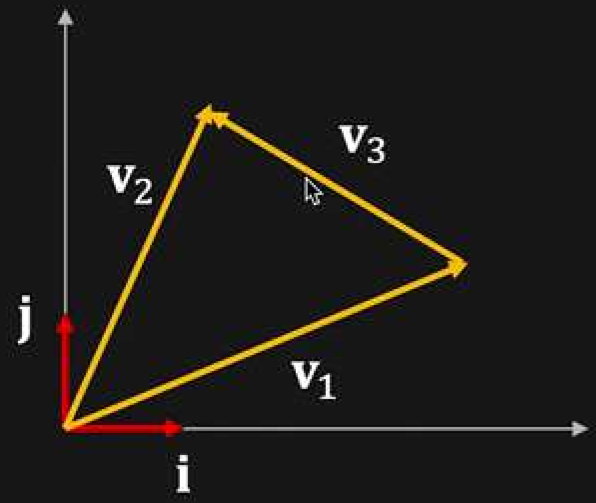

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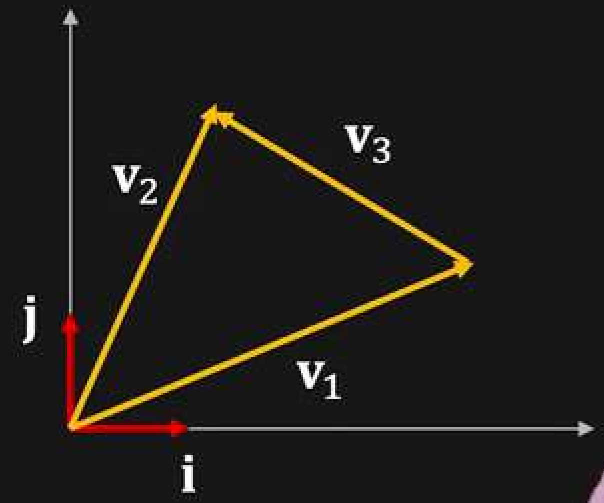


# Rank of a Matrix

**Column Rank**: The number of linearly independent columns of the matrix.

**Row Rank**: The number of linearly independent rows of the matrix.

$$\begin{array}{cc} m & \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \\ & n \end{array} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \vdots \\ \mathbf{r}_m^T \end{bmatrix}$$

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$$ColumnRank(A) \leq n \qquad\qquad RowRank(A) \leq m$$

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$$ColumnRank(A) \leq n \qquad RowRank(A) \leq m$$

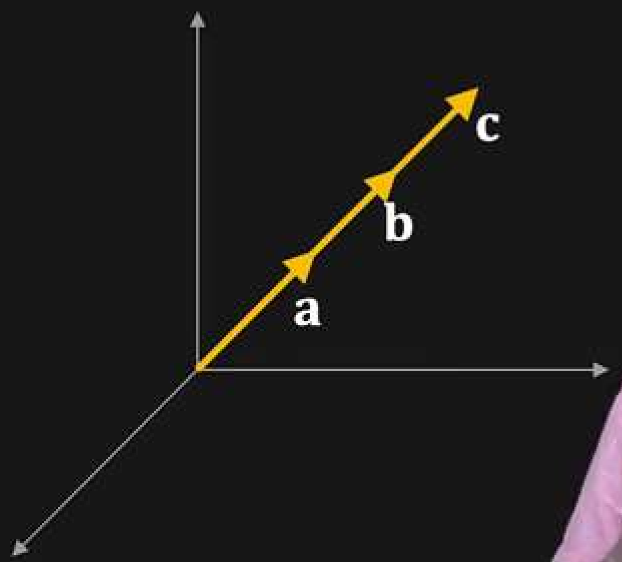
$$\boxed{\begin{array}{c} ColumnRank(A) = RowRank(A) = Rank(A) \\ \\ Rank(A) \leq \min(m, n) \end{array}}$$

# Geometric Meaning of Matrix Rank

Rank is the dimensionality of the space spanned by column or row vectors of the matrix.

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = [\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}]$$

$$Rank(A) = 1 \quad \Longleftrightarrow$$

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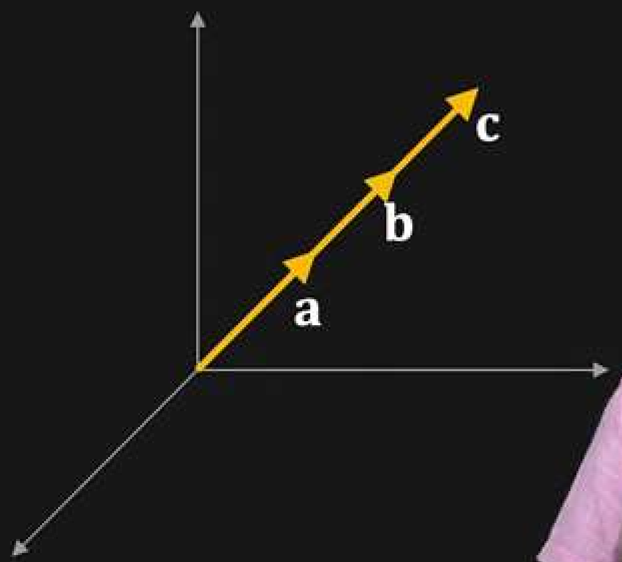


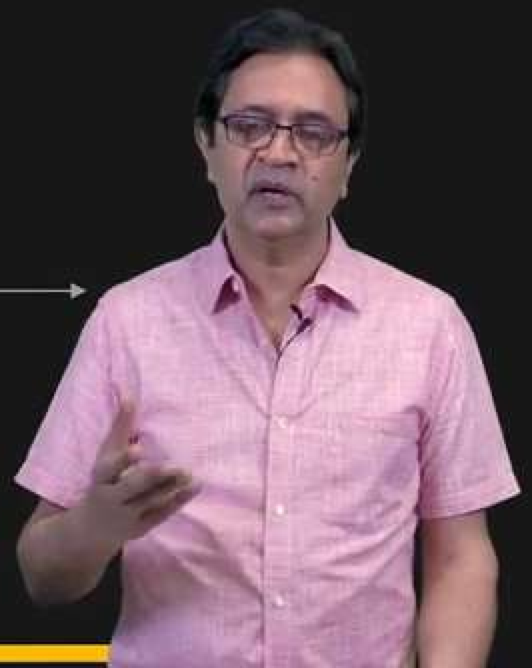

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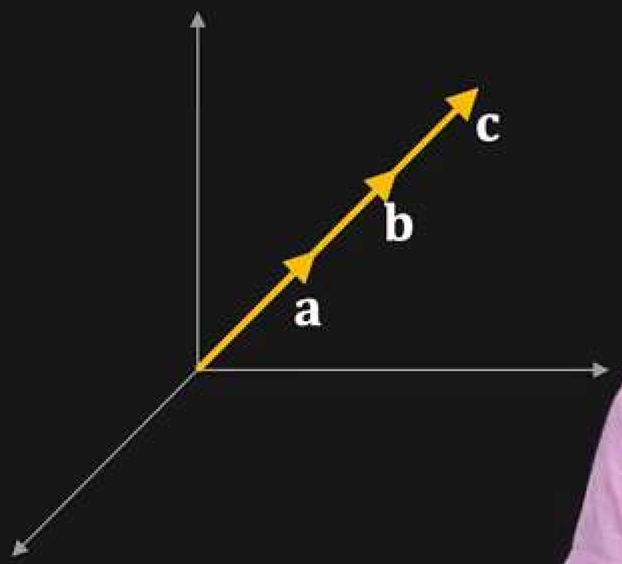


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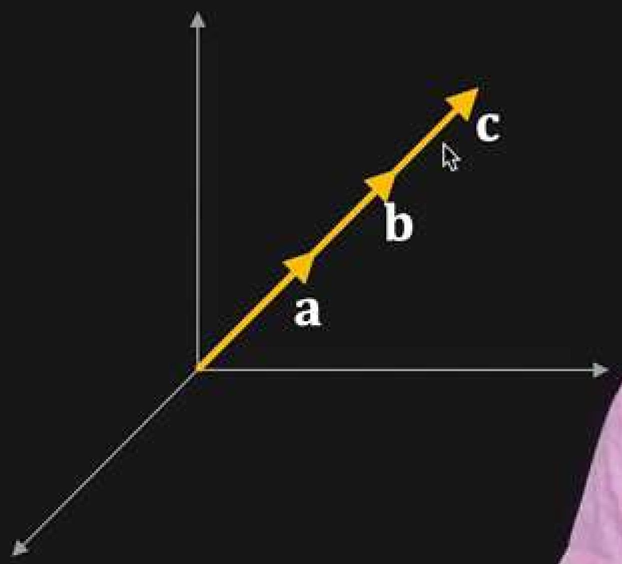

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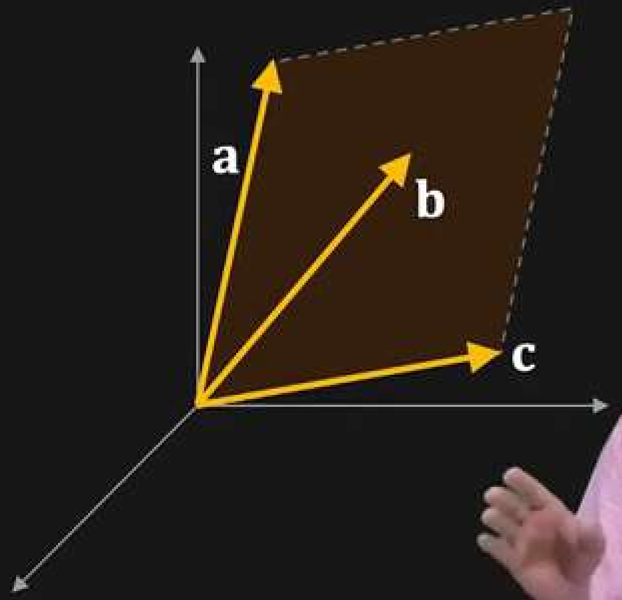
$$Rank(A) = 2 \quad \Leftrightarrow$$

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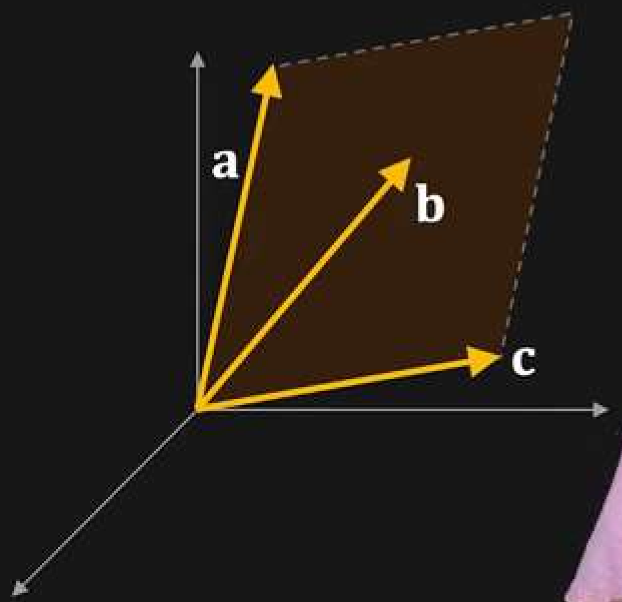


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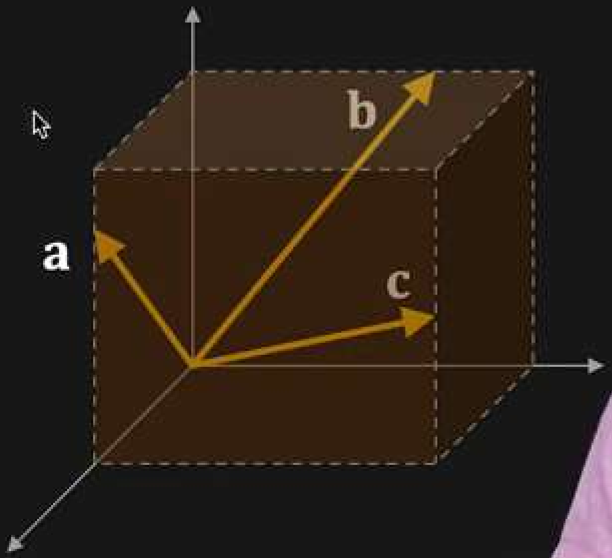
$$Rank(A) = 3 \quad \Longleftrightarrow$$

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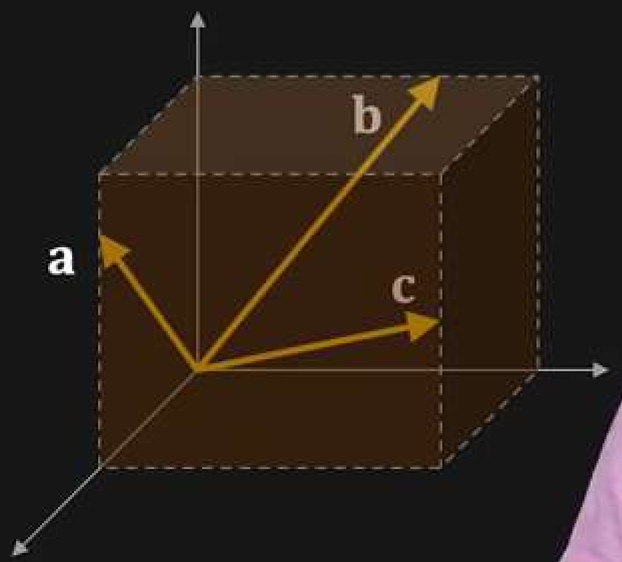


# Important Properties of Matrix Rank

- $Rank(A^T) = Rank(A)$

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- $Rank\left(A_{m\times n}B_{n\times p}\right) = \min\left(Rank(A_{m\times n}), Rank\left(B_{n\times p}\right)\right)$
  $\leq \min(m, n, p)$

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  $\leq \min(m, n, p)$

- $Rank(AA^T) = Rank(A^TA) = Rank(A^T) = Rank(A)$

- $A_{m\times m}$ is invertible iff $Rank(A_{m\times m}) = m$

# ...Back to Observation Matrix $W$

$$
\begin{array}{rc}
 & \begin{array}{cccc} \text{Point 1} & \text{Point 2} & & \text{Point N} \end{array} \\
\begin{array}{r} \text{Image 1} \\ \text{Image 2} \\ \\ \text{Image F} \\ \text{Image 1} \\ \text{Image 2} \\ \\ \text{Image F} \end{array} &
\begin{bmatrix}
\tilde{u}_{1,1} & \tilde{u}_{1,2} & \cdots & \tilde{u}_{1,N} \\
\tilde{u}_{2,1} & \tilde{u}_{2,2} & \cdots & \tilde{u}_{2,N} \\
\vdots & \vdots & \vdots & \vdots \\
\tilde{u}_{F,1} & \tilde{u}_{F,2} & \cdots & \tilde{u}_{F,N} \\
\tilde{v}_{1,1} & \tilde{v}_{1,2} & \cdots & \tilde{v}_{1,N} \\
\tilde{u}_{2,1} & \tilde{u}_{2,2} & \cdots & \tilde{v}_{2,N} \\
\vdots & \vdots & \vdots & \vdots \\
\tilde{v}_{F,1} & \tilde{v}_{F,2} & \cdots & \tilde{v}_{F,N}
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{i}_1^T \\ \mathbf{i}_2^T \\ \vdots \\ \mathbf{i}_F^T \\ \mathbf{j}_1^T \\ \mathbf{j}_2^T \\ \vdots \\ \mathbf{j}_F^T
\end{bmatrix}
\begin{array}{c}
\begin{array}{cccc} \text{Point 1} & \text{Point 2} & & \text{Point N} \end{array} \\
[P_1 \quad P_2 \quad \cdots \quad P_N]
\end{array}
$$

$W_{2F\times N}$

Centroid-Subtracted Feature Points (Known)

$M_{2F\times 3}$

Camera Motion (Unknown)

$S_{3\times N}$

Scene Structure (Unknown)

# ...Back to Observation Matrix $W$

$$
\begin{array}{c c}
 & \begin{array}{cccc} \text{Point 1} & \text{Point 2} & & \text{Point N} \end{array} \\
\begin{array}{c} \text{Image 1} \\ \text{Image 2} \\ \\ \text{Image F} \\ \text{Image 1} \\ \text{Image 2} \\ \\ \text{Image F} \end{array} &
\begin{bmatrix}
\tilde{u}_{1,1} & \tilde{u}_{1,2} & \cdots & \tilde{u}_{1,N} \\
\tilde{u}_{2,1} & \tilde{u}_{2,2} & \cdots & \tilde{u}_{2,N} \\
\vdots & \vdots & \vdots & \vdots \\
\tilde{u}_{F,1} & \tilde{u}_{F,2} & \cdots & \tilde{u}_{F,N} \\
\tilde{v}_{1,1} & \tilde{v}_{1,2} & \cdots & \tilde{v}_{1,N} \\
\tilde{u}_{2,1} & \tilde{u}_{2,2} & \cdots & \tilde{v}_{2,N} \\
\vdots & \vdots & \vdots & \vdots \\
\tilde{v}_{F,1} & \tilde{v}_{F,2} & \cdots & \tilde{v}_{F,N}
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{i}_1^T \\ \mathbf{i}_2^T \\ \vdots \\ \mathbf{i}_F^T \\ \mathbf{j}_1^T \\ \mathbf{j}_2^T \\ \vdots \\ \mathbf{j}_F^T
\end{bmatrix}
\begin{array}{c}
\begin{array}{cccc} \text{Point 1} & \text{Point 2} & & \text{Point N} \end{array} \\
\begin{bmatrix} P_1 & P_2 & \cdots & P_N \end{bmatrix}
\end{array}
$$

$W_{2F\times N}$
Centroid-Subtracted Feature Points (Known)

$M_{2F\times 3}$
Camera Motion
(Unknown)

$S_{3\times N}$
Scene Structure
(Unknown)

# ...Back to Observation Matrix $W$

$$
\begin{array}{l}
\\
\text{Image 1} \\
\text{Image 2} \\
\\
\text{Image F} \\
\text{Image 1} \\
\text{Image 2} \\
\\
\text{Image F}
\end{array}
\begin{array}{c}
\begin{array}{cccc} \text{Point 1} & \text{Point 2} & & \text{Point N} \end{array} \\
\begin{bmatrix}
\tilde{u}_{1,1} & \tilde{u}_{1,2} & \cdots & \tilde{u}_{1,N} \\
\tilde{u}_{2,1} & \tilde{u}_{2,2} & \cdots & \tilde{u}_{2,N} \\
\vdots & \vdots & \vdots & \vdots \\
\tilde{u}_{F,1} & \tilde{u}_{F,2} & \cdots & \tilde{u}_{F,N} \\
\tilde{v}_{1,1} & \tilde{v}_{1,2} & \cdots & \tilde{v}_{1,N} \\
\tilde{u}_{2,1} & \tilde{u}_{2,2} & \cdots & \tilde{v}_{2,N} \\
\vdots & \vdots & \vdots & \vdots \\
\tilde{v}_{F,1} & \tilde{v}_{F,2} & \cdots & \tilde{v}_{F,N}
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{i}_1^T \\ \mathbf{i}_2^T \\ \vdots \\ \mathbf{i}_F^T \\ \mathbf{j}_1^T \\ \mathbf{j}_2^T \\ \vdots \\ \mathbf{j}_F^T
\end{bmatrix}
\begin{array}{c}
\begin{array}{cccc} \text{Point 1} & \text{Point 2} & & \text{Point N} \end{array} \\
\begin{bmatrix} P_1 & P_2 & \cdots & P_N \end{bmatrix}
\end{array}
$$

$S_{3\times N}$

Scene Structure
(Unknown)

$W_{2F\times N}$

Centroid-Subtracted
Feature Points (Known)

$M_{2F\times 3}$

Camera Motion
(Unknown)

# ...Back to Observation Matrix $W$

$$
\begin{array}{rc}
 & \begin{array}{cccc} \text{Point 1} & \text{Point 2} & & \text{Point N} \end{array} \\
\begin{array}{r} \text{Image 1} \\ \text{Image 2} \\ \\ \text{Image F} \\ \text{Image 1} \\ \text{Image 2} \\ \\ \text{Image F} \end{array} &
\begin{bmatrix}
\tilde{u}_{1,1} & \tilde{u}_{1,2} & \cdots & \tilde{u}_{1,N} \\
\tilde{u}_{2,1} & \tilde{u}_{2,2} & \cdots & \tilde{u}_{2,N} \\
\vdots & \vdots & \vdots & \vdots \\
\tilde{u}_{F,1} & \tilde{u}_{F,2} & \cdots & \tilde{u}_{F,N} \\
\tilde{v}_{1,1} & \tilde{v}_{1,2} & \cdots & \tilde{v}_{1,N} \\
\tilde{u}_{2,1} & \tilde{u}_{2,2} & \cdots & \tilde{v}_{2,N} \\
\vdots & \vdots & \vdots & \vdots \\
\tilde{v}_{F,1} & \tilde{v}_{F,2} & \cdots & \tilde{v}_{F,N}
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{i}_1^T \\ \mathbf{i}_2^T \\ \vdots \\ \mathbf{i}_F^T \\ \mathbf{j}_1^T \\ \mathbf{j}_2^T \\ \vdots \\ \mathbf{j}_F^T
\end{bmatrix}
\begin{array}{c}
\begin{array}{cccc} \text{Point 1} & \text{Point 2} & & \text{Point N} \end{array} \\
[P_1 \quad P_2 \quad \cdots \quad P_N]
\end{array}
$$

$W_{2F\times N}$
Centroid-Subtracted Feature Points (Known)

$M_{2F\times 3}$
Camera Motion (Unknown)

$S_{3\times N}$
Scene Structure (Unknown)

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$$
\begin{array}{cc}
 & \begin{array}{cccc} \text{Point 1} & \text{Point 2} & & \text{Point N} \end{array} \\
\begin{array}{c} \text{Image 1} \\ \text{Image 2} \\ \\ \text{Image F} \\ \text{Image 1} \\ \text{Image 2} \\ \\ \text{Image F} \end{array} &
\begin{bmatrix}
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\tilde{u}_{2,1} & \tilde{u}_{2,2} & \cdots & \tilde{u}_{2,N} \\
\vdots & \vdots & \vdots & \vdots \\
\tilde{u}_{F,1} & \tilde{u}_{F,2} & \cdots & \tilde{u}_{F,N} \\
\tilde{v}_{1,1} & \tilde{v}_{1,2} & \cdots & \tilde{v}_{1,N} \\
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\vdots & \vdots & \vdots & \vdots \\
\tilde{v}_{F,1} & \tilde{v}_{F,2} & \cdots & \tilde{v}_{F,N}
\end{bmatrix}
\end{array}
=
\begin{bmatrix}
\mathbf{i}_1^T \\ \mathbf{i}_2^T \\ \vdots \\ \mathbf{i}_F^T \\ \mathbf{j}_1^T \\ \mathbf{j}_2^T \\ \vdots \\ \mathbf{j}_F^T
\end{bmatrix}
\begin{array}{c}
\begin{array}{cccc} \text{Point 1} & \text{Point 2} & & \text{Point N} \end{array} \\
[P_1 \quad P_2 \quad \cdots \quad P_N]
\end{array}
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Centroid-Subtracted Feature Points (Known)

$M_{2F\times 3}$

Camera Motion (Unknown)

$S_{3\times N}$

Scene Structure (Unknown)

# Rank of Observation Matrix

$$\underset{2F\times N}{W} = \underset{2F\times 3}{M} \times \underset{3\times N}{S}$$

[Tomasi 1992]

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[Tomasi 1992]

# Rank of Observation Matrix

$$\underset{2F\times N}{W} = \underset{2F\times 3}{M} \times \underset{3\times N}{S}$$

We know:

$$Rank(MS) \leq Rank(M) \qquad Rank(MS) \leq Rank(S)$$

$$\Rightarrow \quad Rank(MS) \leq \min(3,2F) \qquad Rank(MS) \leq \min(3,N)$$

$$\Rightarrow \quad Rank(W) = Rank(MS) \leq \min(3,N,2F)$$

[Tomasi 1992]

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$$\underset{2F\times N}{W} = \underset{2F\times 3}{M} \times \underset{3\times N}{S}$$

We know:

$$Rank(MS) \leq Rank(M) \qquad Rank(MS) \leq Rank(S)$$

$$\Rightarrow \quad Rank(MS) \leq \min(3,2F) \qquad Rank(MS) \leq \min(3,N)$$

$$\Rightarrow \quad Rank(W) = Rank(MS) \leq \min(3,N,2F)$$

[Tomasi 1992]

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$$\underset{2F\times N}{W} = \underset{2F\times 3}{M} \times \underset{3\times N}{S}$$

We know:

$$Rank(MS) \le Rank(M) \qquad Rank(MS) \le Rank(S)$$

$$\Rightarrow \quad Rank(MS) \le \min(3,2F) \qquad Rank(MS) \le \min(3,N)$$

$$\Rightarrow \quad Rank(W) = Rank(MS) \le \min(3,N,2F)$$

[Tomasi 1992]

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$$\boxed{\text{Rank Theorem: } Rank(W) \le 3}$$

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