Assignment 2: CS 215

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September 6, 2020

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Question 1

Given that $X_1, \ldots, X_n$ are n Independent Identically distributed random variables with cdf $F_X(x)$ and pdf $f_X(x) = F'_X(x)$.

To determine cdf and pdf for $Y_1 = \max(X_1, \ldots, X_n)$

$$\begin{aligned}
F_{Y_1}(x) &= P(Y_1 \leq x) \\
&= \prod_{i=1}^{n} P(X_i \leq x) && \text{(As } Y_1 = \max(X_i\text{s}),\ Y_1 \geq X_i\ \forall i) \\
&= P(X \leq x)^n && (X_i\text{s are identically distributed})
\end{aligned}$$

$$\boxed{F_{Y_1}(x) = F_X(x)^n}$$

$$\boxed{f_{Y_1}(x) = n \cdot F_X(x)^{n-1} \cdot f_X(x)} \qquad \text{(Differentiation w.r.t } x)$$

To determine cdf and pdf for $Y_2 = \min(X_1, \ldots, X_n)$

$$\begin{aligned}
F_{Y_2}(x) &= P(Y_2 \leq x) = 1 - P(Y_2 \geq x) \\
&= 1 - \prod_{i=1}^{n} P(X_i \geq x) && \text{(As } Y_2 = \min(X_i\text{s}),\ Y_2 \leq X_i\ \forall i) \\
&= 1 - (P(X \geq x))^n && (X_i\text{s are identically distributed}) \\
&= 1 - (1 - P(X \leq x))^n
\end{aligned}$$

$$\boxed{F_{Y_2}(x) = 1 - (1 - F_X(x))^n}$$

$$\boxed{f_{Y_2}(x) = n \cdot (1 - F_X(x))^{n-1} \cdot f_X(x)} \qquad \text{(Differentiation w.r.t } x)$$

The cdf and pdf of $Y_1 = \max(X_1, \ldots, X_n)$ are $F_X(x)^n$ and $n \cdot F_X(x)^{n-1} \cdot f_X(x)$ respectively.
The cdf and pdf of $Y_2 = \min(X_1, \ldots, X_n)$ are $1 - (1 - F_X(x))^n$ and $n \cdot (1 - F_X(x))^{n-1} \cdot f_X(x)$ respectively.

Question 2

Part 1

It's given that X belongs to a Gaussian Mixture Model.

$$f_X(x) = \sum_{i=1}^{k} p_i \mathcal{N}(\mu_i, \sigma_i^2)$$

Throughout the rest of this question, we shall use G_i for denoting $\mathcal{N}(\mu_i, \sigma_i^2)$.

Calculating E(X):

We know that for an arbitrary random variable Y which is distributed as a Gaussian, ($Y \sim G_i$),
$E(Y) = \int_{-\infty}^{\infty} xG_i dx = \mu_i$
Thus,

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f_X(x) dx \\ &= \int_{-\infty}^{\infty} x \sum_{i=1}^{k} p_i G_i dx \\ &= \sum_{i=1}^{k} p_i \int_{-\infty}^{\infty} x G_i dx \\ &= \boxed{\sum_{i=1}^{k} p_i \mu_i} \end{aligned}$$

Calculating Var(X):

We know that:

$$Var(X) = E((X-\mu)^2) = E(X^2) - E(X)^2 = E(X^2) - \mu^2$$

and,

$$Var(Y) = \sigma_i^2 = E(Y^2) - \mu_i^2 \text{ for an arbitrary random variable } Y \text{ distributed as a gaussian } (Y \sim G_i)$$

Thus, $E(Y^2) = \displaystyle\int_{-\infty}^{\infty} x^2 G_i dx = \sigma_i^2 + \mu_i^2$ (Here, $\mu_i = E(Y)$)

We shall now calculate $E(X^2)$

$$\begin{aligned} E(X^2) &= \int_{-\infty}^{\infty} x^2 \sum_{i=1}^{k} p_i G_i dx \\ &= \sum_{i=1}^{k} p_i \int_{-\infty}^{\infty} x^2 G_i dx \\ &= \sum_{i=1}^{k} p_i (\sigma_i^2 + \mu_i^2) \end{aligned}$$

Thus, now we have:

$$Var(X) = E(X^2) - \mu^2 = \boxed{\sum_{i=1}^{k} p_i(\sigma_i^2 + \mu_i^2) - (\sum_{i=1}^{k} p_i\mu_i)^2}$$

Calculating the MGF:

By the definition of the MGF, $\phi_X(t) = E(e^{tX})$
We also know that for an arbitrary random variable Y, which is distributed as a Gaussian G_i ($Y \sim G_i$), we have $\phi_Y(t) = \int_{-\infty}^{\infty} e^{tx} G_i dx = \exp(\mu_i t + \frac{(\sigma_i t)^2}{2})$
Thus for the given X,

$$\begin{aligned}\phi_X(t) &= E(e^{tX}) \\ &= \int_{-\infty}^{\infty} e^{tX} \sum_{i=1}^{k} p_i G_i dx \\ &= \sum_{i=1}^{k} p_i \int_{-\infty}^{\infty} e^{tX} G_i dx \\ &= \boxed{\sum_{i=1}^{k} p_i \exp(\mu_i t + \frac{(\sigma_i t)^2}{2})}\end{aligned}$$

Part II

It is given that $Z = \sum_{i=1}^{k} p_i X_i$ where $X_i \sim G_i$ are independent random variables.
By the properties of Gaussian distribution G_i we know that:

$$E(X_i) = \mu_i$$
$$Var(X_i) = \sigma_i^2$$

Calculating E(Z):

$$\begin{aligned}E(Z) &= E(\sum_{i=1}^{k} p_i X_i) \\ &= \sum_{i=1}^{k} p_i E(X_i) \qquad \text{(Linearity of Expectation)} \\ &= \boxed{\sum_{i=1}^{k} p_i \mu_i}\end{aligned}$$

Calculating Var(Z)

$$\begin{aligned}
Var(Z) &= Var(\sum_{i=1}^{k} p_i X_i) \\
&= \sum_{i=1}^{k} p_i^2 Var(X_i) \qquad \text{(as } \{X_i\}_{i=1}^{k} \text{ are independent and } Var(aX) = a^2 Var(X)) \\
&= \boxed{\sum_{i=1}^{k} p_i^2 \sigma_i^2}
\end{aligned}$$

Calculating the MGF

We know that for a Gaussian $X_i \sim G_i$, we have:

$$\phi_{X_i}(t) = \int_{-\infty}^{\infty} e^{tx} G_i dx = \exp(\mu_i t + \frac{(\sigma_i t)^2}{2})$$

We also know the following properties of $\phi_X(t)$:

$$\phi_{(aX)}(t) = \phi_X(at)$$

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t) \text{ for independent X, Y}$$

Thus, using the above 2 properties, and the fact that $\{X_i\}_{i=1}^{k}$ are independent,

$$\begin{aligned}
\phi_Z(t) &= \prod_{i=1}^{k} \phi_{(p_i X_i)}(t) \\
&= \prod_{i=1}^{k} \phi_{X_i}(p_i t) \\
&= \prod_{i=1}^{k} \exp(\mu_i p_i t + \frac{(\sigma_i p_i t)^2}{2}) \quad \text{(as } X_i \sim G_i) \\
&= \boxed{\exp(t \sum_{i=1}^{k} \mu_i p_i + \frac{t^2}{2} \sum_{i=1}^{k} p_i^2 \sigma_i^2} \quad \text{(as } \exp(a)\exp(b) = \exp(a+b))
\end{aligned}$$

Calculating the PDF

The obtained MGF of Z is same as that of a gaussian of mean $\mu = \sum_{i=1}^{k}(\mu_i p_i)$ and variance $\sigma^2 = \sum_{i=1}^{k} p_i^2 \sigma_i^2$
Thus, $Z \sim \mathcal{N}(\mu, \sigma^2)$. (Using the Moment Generating Function (MGF) Uniqueness Theorem[1])
Therefore, the PDF of Z will be:

$$\boxed{f_Z(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(- \frac{(z-\mu)^2}{2\sigma^2}\right)}$$

[1] For a given random variable, the MGF and PMF **uniquely** determine each other.

Question 3

To prove:

$$P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2} \qquad \text{for } \tau > 0$$

$$P(X-\mu \geq \tau) \geq 1-\frac{\sigma^2}{\sigma^2+\tau^2} \qquad \text{for } \tau < 0$$

We shall prove some lemmas before coming to the actual proof:
(Consider X to be a random variable and $a \geq 0$.)

1. $P(X \geq a) = P(X+b \geq a+b)$
   Note that $(X \geq a) \iff (X+b \geq a+b)$, thus this directly leads to the above statement.

2. $P(X^2 \geq a^2) \geq P(X \geq a)$
   This is because of the fact that:

$$X^2 \geq a^2 \Rightarrow |X| \geq a \Rightarrow (X > a) \vee (X < -a)$$

   Thus, $P(X^2 \geq a^2) = P(X > a) + P(X < -a)$.
   As probabilities are always ≥ 0, $P(X^2 \geq a^2) \geq P(X > a)$

Markov's Inequality states that, for $a > 0$ and a random variable X which always takes positive values, we have:

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Consider a random variable $Y := X - \mu$ and $Z := Y + b$ and consider the case where $\tau > 0$.
Also consider $b \geq 0$, thus we have $(\tau + b) \geq 0$
Thus, as $Z^2 \geq 0$ and $(\tau+b)^2 > 0$, we can apply the Markov's inequality these:

$$P((Y+b)^2 \geq (\tau+b)^2) \leq \frac{E((Y+b)^2)}{(\tau+b)^2}$$

Also, by Lemma 2, we have $P((Y+b)^2 \geq (\tau+b)^2) \geq P(Y+b \geq \tau+b)$.
By Lemma 1, $P((Y+b) \geq (\tau+b)) = P(Y \geq \tau)$
Now, using the linearity of the expectation operator,

$$\begin{aligned}
E((Y+b)^2) &= E(Y^2) + E(2Yb) + E(b^2) \\
&= E(Y^2) + 2bE(Y) + b^2 \qquad \text{(expectation of a constant is the constant itself)} \\
&= \sigma^2 + 0 + b^2 = \sigma^2 + b^2 \qquad (E((X-\mu)^2) = \sigma^2 \text{ and } E(X-\mu) = 0)
\end{aligned}$$

Thus we have,

$$P(Y \geq \tau) \leq \frac{\sigma^2+b^2}{(\tau+b)^2}$$

To further tighten the inequality, we need to minimize the function $g(b) = \frac{\sigma^2+b^2}{(\tau+b)^2}$, solving $g'(b) = 0$ will yield us an extrema b_0:

$$g'(b) = \frac{2b}{(\tau+b)^2} - \frac{2(\sigma^2+b^2)}{(\tau+b)^3}$$

Equating it to zero, we get $b_0 = \frac{\sigma^2}{\tau}$, furthermore as $g''(b) > 0$, this indeed is the minima.
Thus, after putting $Y = X - \mu$ and $b = b_0$, we get:

$$\boxed{P((X-\mu) \geq \tau) \leq \frac{\sigma^2}{\sigma^2 + \tau^2}} \qquad (\text{for } \tau > 0)$$

Now, as $(X-\mu)^2 = (\mu - X)^2$, the inequality $P((Y) \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$ is valid even for $Y = \mu - X$.
Thus for $\tau < 0$ we can still use the above result for $-\tau$.

$$\begin{aligned} P((\mu - X) \geq (-\tau)) &\leq \frac{\sigma^2}{\sigma^2+\tau^2} \\ P((X-\mu) < \tau) &\leq \frac{\sigma^2}{\sigma^2+\tau^2} \\ 1 - P((X-\mu) \geq \tau) &\leq \frac{\sigma^2}{\sigma^2+\tau^2} \quad (P(X < a) = 1 - P(X \geq a)) \\ \boxed{P((X-\mu) \geq \tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}} & \quad (\text{for } \tau < 0) \end{aligned}$$

Question 4

By definition, MGF of a random variable X for parameter t is $\phi_X(t) = \int_{-\infty}^{\infty} e^{tu} f_X(u) du$.

$$\begin{aligned}
e^{-tx}\phi_X(t) &= \int_{-\infty}^{\infty} e^{t(u-x)} f_X(u) du \\
&= \int_{-\infty}^{x} e^{t(u-x)} f_X(u) du + \int_{x}^{\infty} e^{t(u-x)} f_X(u) du
\end{aligned}$$

As the integrand is always non-negative,

$$e^{-tx}\phi_X(t) \geq \int_{-\infty}^{x} e^{t(u-x)} f_X(u) du \quad \textbf{and,} \quad e^{-tx}\phi_X(t) \geq \int_{x}^{\infty} e^{t(u-x)} f_X(u) du \tag{1}$$

For $t \geq 0$,

$$\begin{aligned}
e^{-tx}\phi_X(t) &\geq \int_{x}^{\infty} e^{t(u-x)} f_X(u) du && \text{(From 1)} \\
&\geq \int_{x}^{\infty} (1 + t(u-x)) f_X(u) du && (e^x \geq (1+x)) \\
&\geq \left(\int_{x}^{\infty} f_X(u) du \right) + \left(t \cdot \int_{x}^{\infty} (u-x) f_X(u) du \right) \\
&\geq P(X \geq x) + \left(t \cdot \int_{x}^{\infty} (u-x) f_X(u) du \right)
\end{aligned} \tag{2}$$

As $\quad t \geq 0; \quad u \geq x; \quad f_X(u) \geq 0 \ \forall u \in (-\infty, \infty), \quad$ we get

$$\boxed{P(X \geq x) \leq e^{-tx}\phi_X(t) \quad \forall t \geq 0} \tag{3}$$

For $t \leq 0$,

$$\begin{aligned}
e^{-tx}\phi_X(t) &\geq \int_{-\infty}^{x} e^{t(u-x)} f_X(u) du && \text{(From 1)} \\
&\geq \int_{-\infty}^{x} (1 + t(u-x)) f_X(u) du && (e^x \geq (1+x)) \\
&\geq \left(\int_{-\infty}^{x} f_X(u) du \right) + \left(t \cdot \int_{-\infty}^{x} (u-x) f_X(u) du \right) \\
&\geq P(X \leq x) + \left(t \cdot \int_{-\infty}^{x} (u-x) f_X(u) du \right)
\end{aligned} \tag{4}$$

As $t \leq 0; \quad u \leq x; \quad f_X(u) \geq 0 \ \forall u \in (-\infty, \infty), \quad$ we get

$$\boxed{P(X \leq x) \leq e^{-tx}\phi_X(t) \quad \forall t \leq 0} \tag{5}$$

Now, $X = \sum_{i=1}^{n} X_i$, where X_i are independent Bernoulli random variables with mean p_i.
$E(X) = \mu = \sum_{i=1}^{n} p_i$.

$$\begin{aligned} \phi_X(t) &= \prod_{i=1}^{n} \phi_{X_i}(t) && (X_i\text{s are independent random variables}) \\ &= \prod_{i=1}^{n} (1 - p_i + p_i e^t) && (X_i\text{s are Bernoulli random variables}) \\ &\leq \prod_{i=1}^{n} (e^{p_i(e^t-1)}) && (e^x \geq (1+x)) \\ &\leq exp(\sum_{i=1}^{n} p_i(e^t - 1)) \\ &\leq exp(\mu(e^t - 1)) \end{aligned} \tag{6}$$

$$\begin{aligned} P(X > (1+\delta)\mu) &\leq e^{-t\cdot((1+\delta)\mu)} \phi_X(t) \quad \forall t \geq 0 && \text{(From 3)} \\ &\leq exp((-(1+\delta)t\mu)) + \mu(e^t - 1)) && \text{(From 6)} \end{aligned} \tag{7}$$

Therefore,

$$\boxed{(X > (1+\delta)\mu) \leq \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)t\mu}} \quad \forall t \geq 0,\ \delta \geq 0} \tag{8}$$

To get a tighter bound, we need the minimum attainable value of the RHS of Eqn 8.
As e^x is a monotonically increasing function, we need to minimize $(-(1+\delta)t\mu)) + \mu(e^t - 1)$.
$\frac{d}{dt}((-(1+\delta)t\mu)) + \mu(e^t - 1)) = (\mu e^t - (1+\delta)\mu) = 0 \quad \implies \quad \boxed{t = ln(1+\delta)}$

$$\boxed{P(X > (1+\delta)\mu) \leq \frac{e^{\mu\delta}}{e^{\mu(1+\delta)(ln(1+\delta))}} \quad \forall \delta \geq 0} \tag{9}$$

Question 5

Instructions for running the code:

- After extracting submitted file, look for a directory named `code`.
- Within this, the code for this question is contained in a directory named `q5`.
- Run the file `q5.m`.
- The plots can be found in `./plots/`
- a_*.png, b_*.png and c.png are plots corresponding part (a), (b) and (c) respectively.

We also had to calculate the true mean μ and std. dev. σ for getting the Gaussian curve.

$$\mu_i = E(X_i) = \sum_{k=1}^{5} k \cdot f_{X_i}(X_i = k) = 1 * 0.05 + 2 * 0.4 + 3 * 0.15 + 4 * 0.3 + 5 * 0.1 = 3$$

$$\sigma_i^2 = Var(X_i) = E((X_i - \mu_i)^2) = E((X_i - 3)^2) = \sum_{k=1}^{5} (k-3)^2 \cdot f_{X_i}(X_i = k)$$

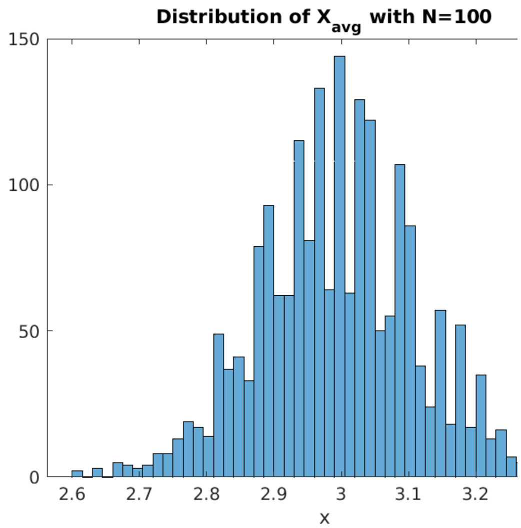
$$= 4 * 0.05 + 1 * 0.4 + 0 * 0.15 + 1 * 0.3 + 4 * 0.1 = 1.3$$

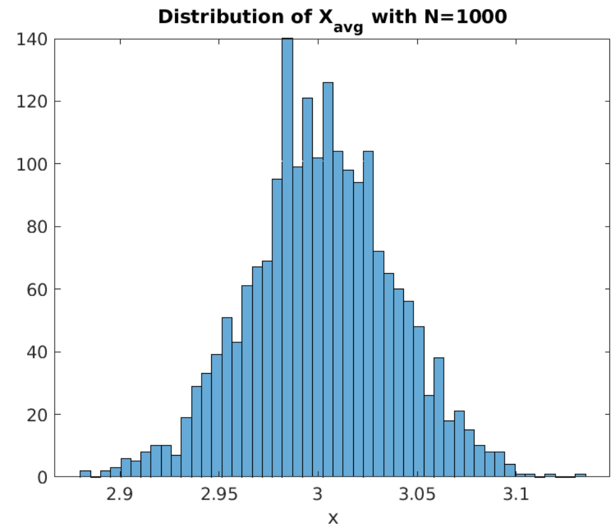
$$\mu = N * \frac{\mu_i}{N} = 3 \qquad (X_{avg} = \sum_{i=1}^{N} \frac{X_i}{N})$$

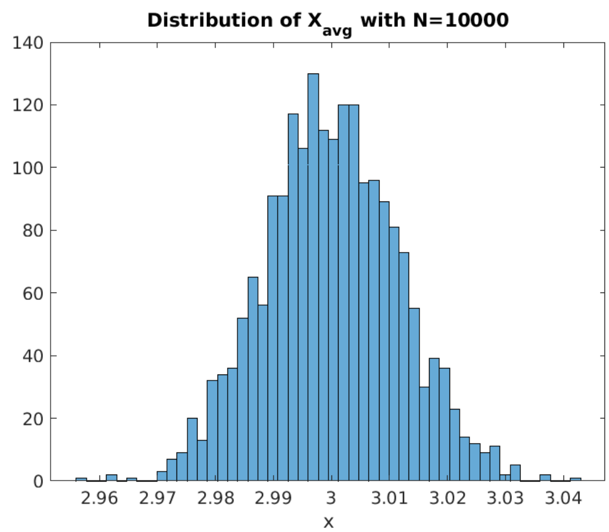
$$\sigma^2 = N * \frac{\sigma_i^2}{N^2} = 1.3/N \qquad (X_{avg} = \sum_{i=1}^{N} \frac{X_i}{N})$$

$$\sigma = \sqrt{\frac{1.3}{N}}$$

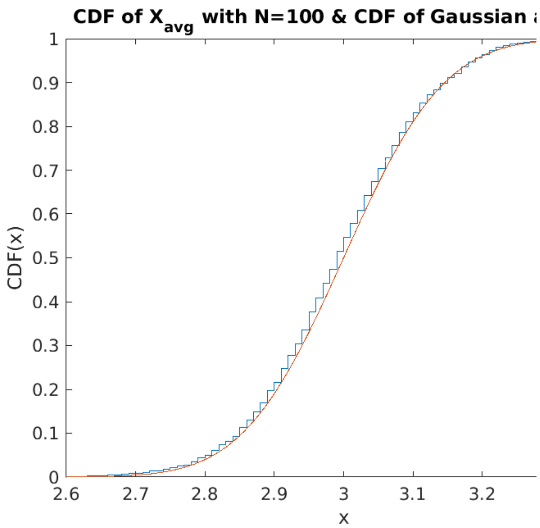
(a)

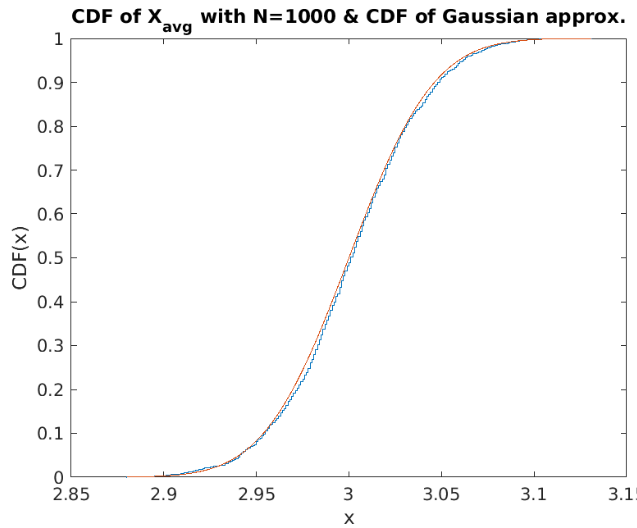

Distribution of X_{avg} with N=100
150
100
50
0
2.6 2.7 2.8 2.9 3 3.1 3.2
x

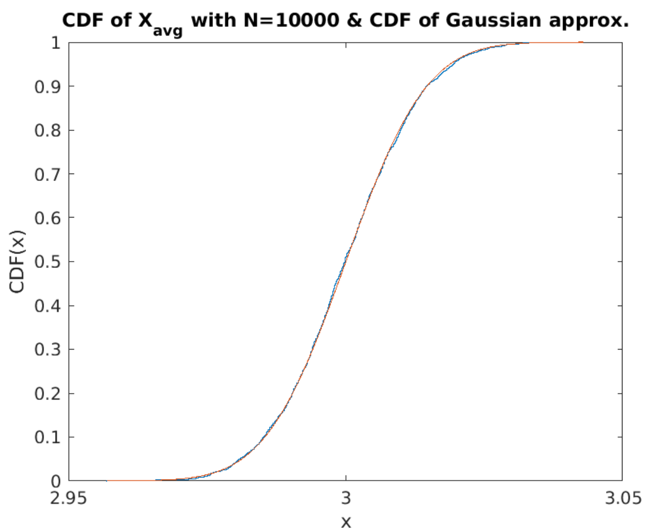

Distribution of X_{avg} with N=1000
140
120
100
80
60
40
20
0
2.9 2.95 3 3.05 3.1
x


Distribution of X_{avg} with N=10000
140
120
100
80
60
40
20
0
2.96 2.97 2.98 2.99 3 3.01 3.02 3.03 3.04
x

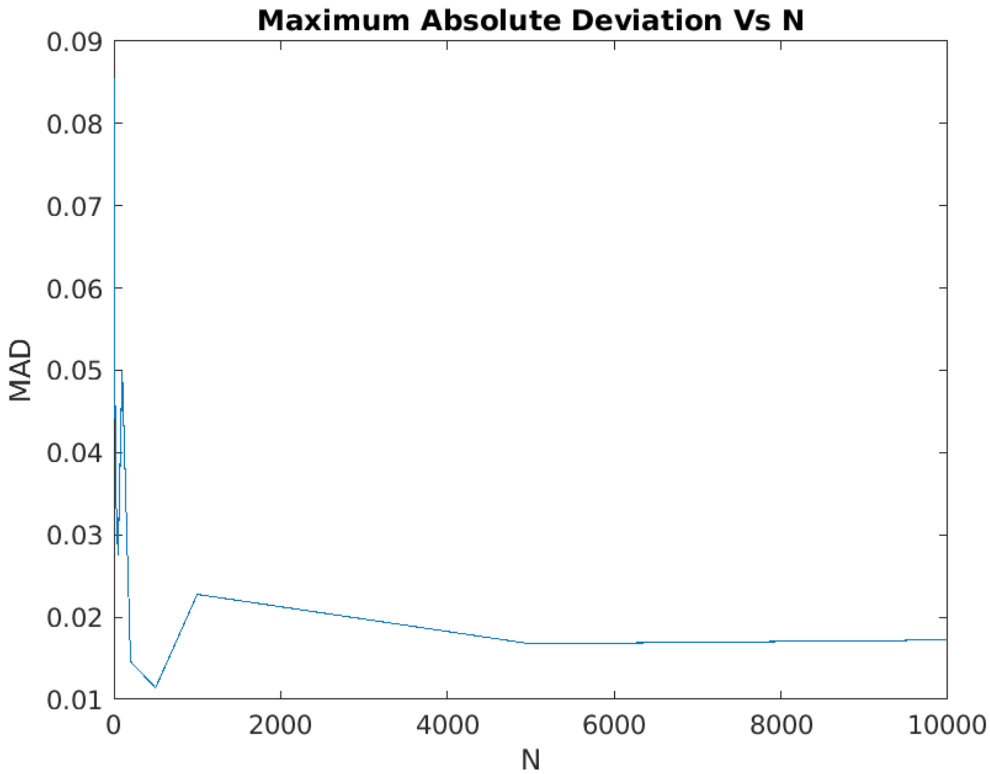
(b)


CDF of X_{avg} with N=100 & CDF of Gaussian a
CDF(x)
x


CDF of X_{avg} with N=1000 & CDF of Gaussian approx.
CDF(x)
x


CDF of X_{avg} with N=10000 & CDF of Gaussian approx.
CDF(x)
x

(c)

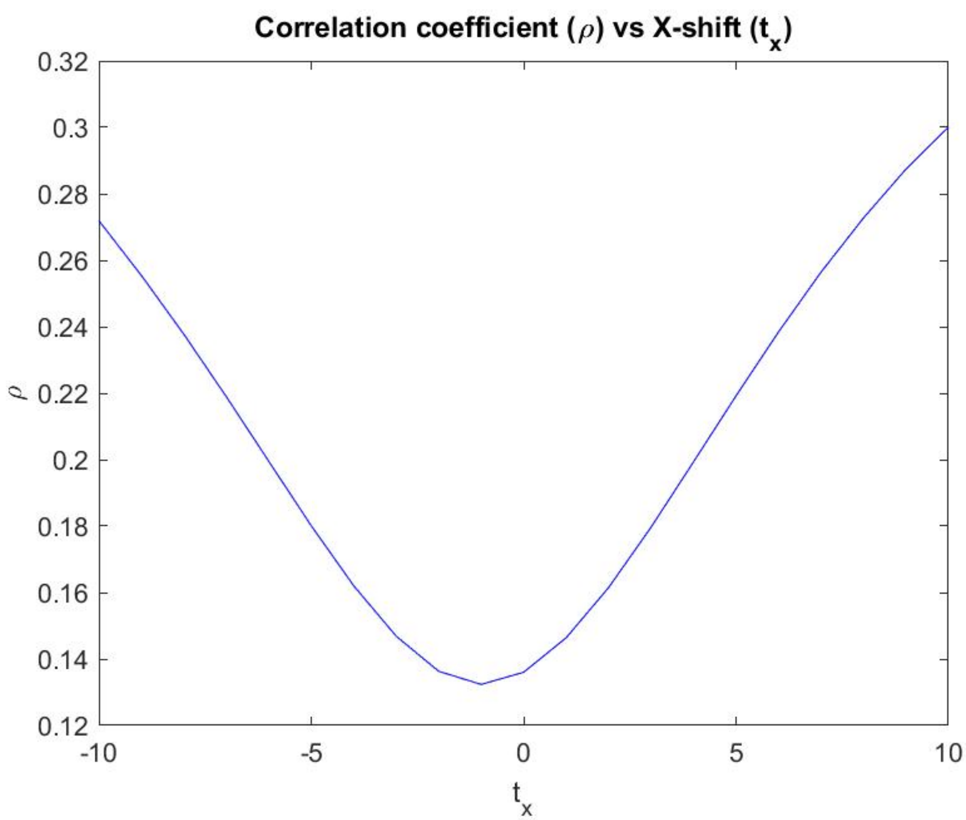

Maximum Absolute Deviation Vs N
MAD
N
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
2000
4000
6000
8000
10000

Question 6

Instructions for running the code:

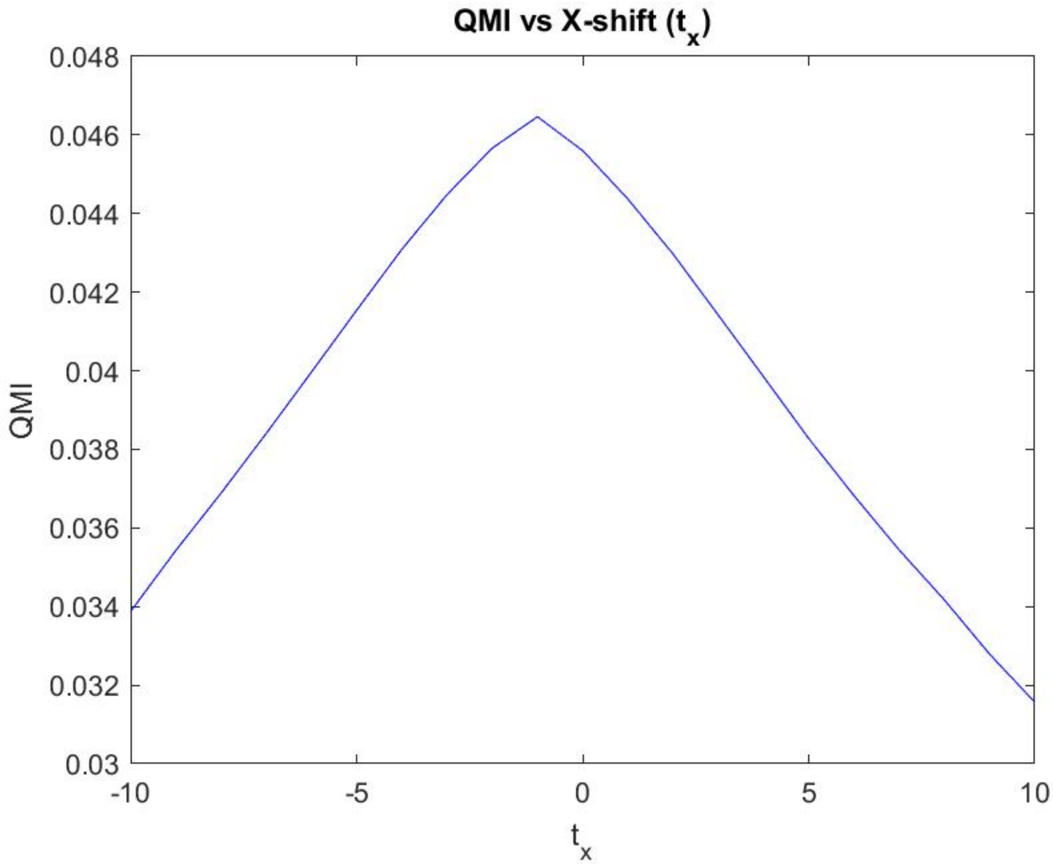
- After extracting submitted file, look for a directory named `code`.
- Within this, the code for this question is contained in a directory named `q6` while the 2 images are contained in the `img` directory.
- Run the file `q6i.m` for plotting the given dependence measures between `T1.jpg` and `T2.jpg` and run `q6ii.m` for plotting the given dependence measures between `T1.jpg` and it's negative.
- Both of these will create two plots and save them at `./plots/`

For T1.jpg and T2.jpg



Comments:

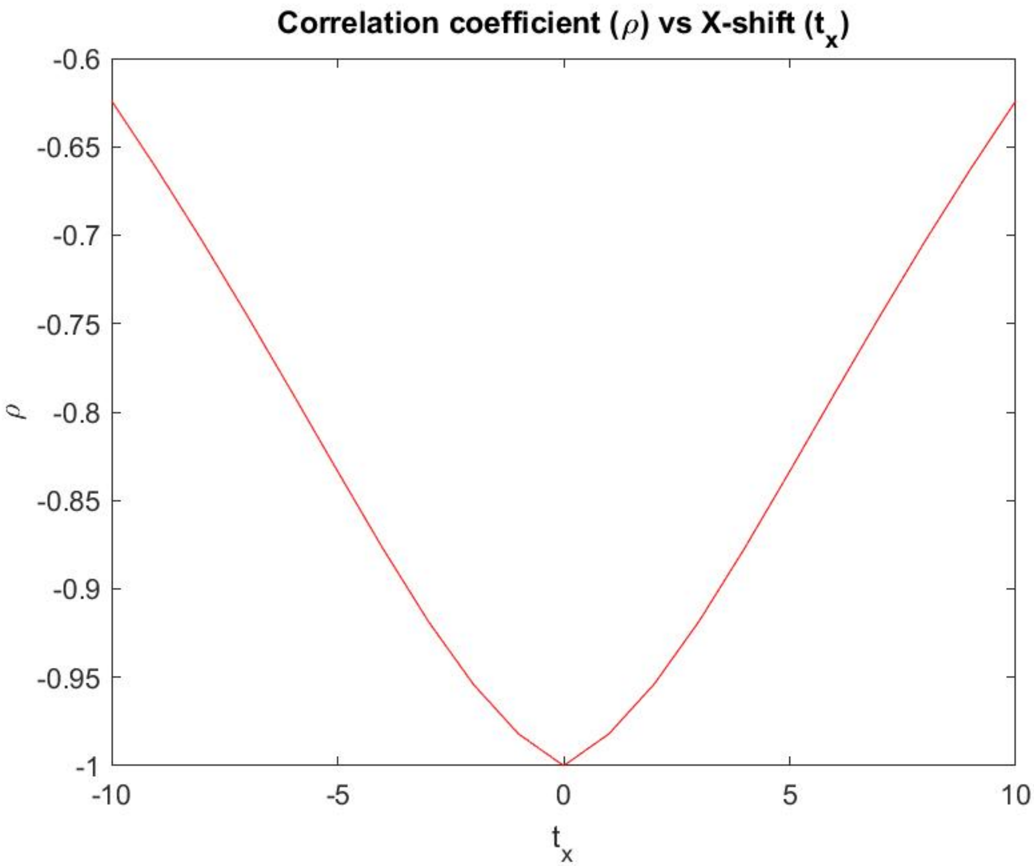
- The correlation is the least when the 2 images are almost exactly aligned ($t_x \approx 0$).
- As the two images don't seem to have any obvious relation w.r.t their intensity values, thus their correlation coefficient values are low.
- As the shift starts to increase in either direction, the images start to become more positively correlated, with the correlation increasing approximately linearly.
- The correlation is always positive in the given range of the shifts.



Comments:

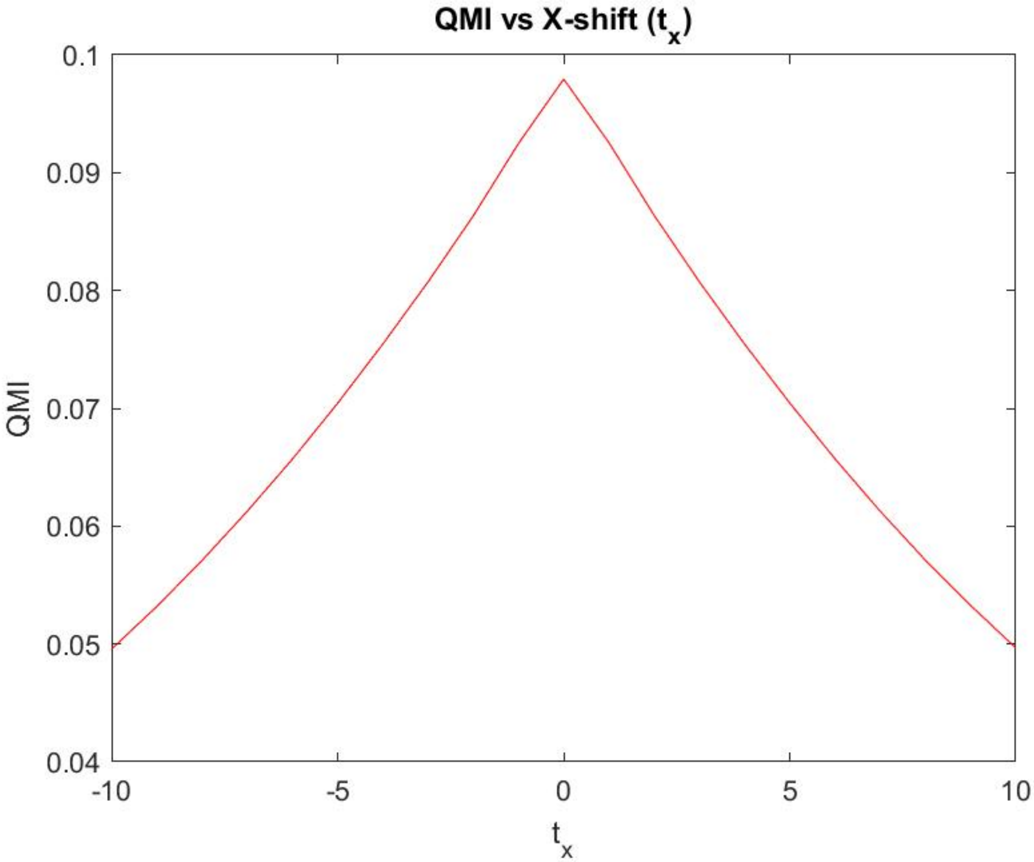
- QMI essentially measures the "dependance" of two random variables on each other. In our case, it is a measure of the mutual information contained between the 2 images.
- In our case, the images seem to have no obvious relation w.r.t their intensity values, thus their QMI values are low.
- The QMI value reaches a maximum at $t_x = -1$ (This is also the point where the correlation coefficient is the least)
- As we further increase the shift in either direction, the QMI value start decreasing approximately linearly.

For T1.jpg and it's negative



Comments:

- As the second image is the negative of the first and the images are exactly aligned at $t_x = 0$, the correlation coefficient is exactly equal to (-1). Thus, at this point $(t_x = 0)$, the two images are completely anti-correlated.
- As the shift starts to increase in either direction, the correlation coefficient also starts to become less negative. The increase is approximately linear.
- The correlation is always negative in the given range of the shifts for the given image and it's negative.



Comments:

- QMI essentially measures the "dependance" of two random variables on each other. In our case, it is a measure of the mutual information contained between the 2 images.
- In our case, the second image is the negative of the first, thus both of them contain the exact same information about each other at $t_x = 0$. ($(255 - I_1 = I_2)$ and $(255 - I_2) = I_1$)
- As we increase the shift in either direction, the QMI values start to decrease, as when we are shifting an image, we are also "deleting" information by blacking out the un-occupied pixels.
- The rate of decrease of QMI is higher in this case as compared to the previous case.
- Although the QMI values appear quite low at $t_x = \pm 10$, these are still higher than the previous case, where the maxima itself was about 0.046. This is because even after deleting some information from the negative, there is still a lot of common information between the 2 images.

Question 7

MGF of Multinomial distribution of $\mathbf{X} = (X_1, \dots, X_k)$, where $\sum_{i=1}^{k} X_i = n$ is $\phi_{\mathbf{X}}(\mathbf{t}) = (p_1 e^{t_1} + \cdots + p_k e^{t_k})^n$, where $\mathbf{t} = (t_1, \dots, t_k)$ and p_i represent probability of X_i.

$$
\begin{aligned}
\frac{\partial}{\partial t_i}\phi_{\mathbf{X}}(\mathbf{t}) &= n \cdot (p_1 e^{t_1} + \cdots + p_k e^{t_k})^{n-1} \cdot (p_i e^{t_i}) \\
\mu_i = E(X_i) &= \frac{\partial}{\partial t_i}\phi_{\mathbf{X}}(\mathbf{t})\Big|_{\mathbf{t}=\mathbf{0}=(0,\dots,0)} \\
&= n \cdot (p_1 + \cdots + p_k)^{n-1} \cdot p_i \\
&= n \cdot p_i \qquad (\sum_{n=1}^{k} p_i = 1)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial t_i^2}\phi_{\mathbf{X}}(\mathbf{t}) &= n(n-1) \cdot (p_1 e^{t_1} + \cdots + p_k e^{t_k})^{n-2} \cdot (p_i e^{t_i})^2 + n \cdot (p_1 e^{t_1} + \cdots + p_k e^{t_k})^{n-1} \cdot (p_i e^{t_i}) \\
E(X_i^2) &= \frac{\partial^2}{\partial t_i^2}\phi_{\mathbf{X}}(\mathbf{t})\Big|_{\mathbf{t}=\mathbf{0}=(0,\dots,0)} \\
&= n(n-1) \cdot (p_1 + \cdots + p_k)^{n-2} \cdot p_i^2 + n \cdot (p_1 + \cdots + p_k)^{n-1} \cdot p_i \\
&= n(n-1) \cdot p_i^2 + n \cdot p_i \qquad (\sum_{n=1}^{k} p_i = 1) \\
Cov(X_i, X_i) &= Var(X_i) \\
&= E(X_i^2) - E(X_i)^2 \\
&= n(n-1) \cdot p_i^2 + n \cdot p_i - (n \cdot p_i)^2 \\
&= n \cdot p_i(1 - p_i)
\end{aligned}
$$

For $i \neq j$,

$$
\begin{aligned}
\frac{\partial^2}{\partial t_j \partial t_i}\phi_{\mathbf{X}}(\mathbf{t}) &= n(n-1) \cdot (p_1 e^{t_1} + \cdots + p_k e^{t_k})^{n-2} \cdot (p_i e^{t_i}) \cdot (p_j e^{t_j}) \\
E(X_i \cdot X_j) &= \frac{\partial^2}{\partial t_j \partial t_i}\phi_{\mathbf{X}}(\mathbf{t})\Big|_{\mathbf{t}=\mathbf{0}=(0,\dots,0)} \\
&= n(n-1) \cdot (p_1 + \cdots + p_k)^{n-2} \cdot p_i \cdot p_j \\
&= n(n-1) \cdot p_i \cdot p_j \qquad (\sum_{n=1}^{k} p_i = 1) \\
Cov(X_i, X_j) &= E[(X_i - \mu_i)(X_j - \mu_j)] \\
&= E[(X_i - E(X_i))(X_j - E(X_j))] \\
&= E(X_i \cdot X_j) - E(X_i)E(X_j) \\
&= n(n-1) \cdot p_i \cdot p_j - (n \cdot p_i)(n \cdot p_j) \\
&= (-n) \cdot p_i \cdot p_j
\end{aligned}
$$

The co-variance matrix **C** of **X** is given by:

$$\boxed{C_{ii} = n \cdot p_i(1 - p_i)}$$

$$\boxed{C_{ij} = (-n) \cdot p_i \cdot p_j \quad \forall\ i \neq j}$$