Assignment 4: CS 215

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Question 1

(a)

X can take all values inside the unit square of area 4 with equal probability.
Thus, the probability that X takes values inside an arbitrary region of area A (lying inside the unit square) is proportional to A.

$$P(X \in R) = k \cdot area(R)$$

As the probability that X lies inside the unit square is 1, this gives $k = \frac{1}{4}$.
A unit circle has an area of π, thus the probability that X lies inside a unit circle is $\boxed{\frac{\pi}{4}}$

(b)

Estimation of π using X:
Generate n samples of the form (x_1, x_2) where $x_1, x_2 \sim U(-1, 1)$.
Find the number of such points which satisfy $x_1^2 + x_2^2 \leq 1$, let this number be n_u
Thus, $\frac{n_u}{n} \approx \frac{\pi}{4}$, an estimate of π will be $\frac{4n_u}{n}$
More formally, let $I_{||X|| \leq 1}(X)$ be the indicator function that yields 1 if $||X|| = \sqrt{X_1^2 + X_2^2} \leq 1$, and is 0 otherwise.
Then, $n_u = \sum_i^n I_{||x_i|| \leq 1}(x_i)$, thus an estimate for π will be $\frac{4 \sum_i^n I_{||x_i|| \leq 1}(x_i)}{n}$

(c)

Estimates of π obtained for $N = 10, 10^2, 10^3, 10^4, 10^5, 10^6, 10^7, 10^8$ respectively

```
N = 10, pi = 2.0000
N = 100, pi = 3.2400
N = 1000, pi = 3.2080
N = 10000, pi = 3.1316
N = 100000, pi = 3.1425
N = 1000000, pi = 3.1419
N = 10000000, pi = 3.1414
N = 100000000, pi = 3.1416
```

Our code handles $N = 10^9$ by computing n_u using 10 10^8 (Similarly $\frac{n}{10^8}$ iterations for a general n, assuming it's a power of 10) sized arrays, by iterating 10 times, to ensure memory remains in check, although this takes quite a bit of time to execute. We get the output for $N = 10^9$ as `pi = 3.1416`

```
n_fixed = single(10^8);
n_large = single(10^9);
n_iters = single(n_large/n_fixed);

n_u = single(0);
for i = 1:n_iters
    X1 = single(2*rand(n_fixed, 1)-1);
    X2 = single(2*rand(n_fixed, 1)-1);
    n_u = n_u + sum((X1.^2 + X2.^2) <= 1);
end
```

(d)

$n_u = \sum_i^n I_{||x_i|| \leq 1}(x_i)$, also $y_i = I_{||x_i|| \leq 1}(x_i)$ is a Bernoulli random variable with the parameter $p = \frac{\pi}{4}$.
Let $z = \frac{1}{n} \sum_{i=1}^n I_{||x_i|| \leq 1}(x_i)$ (Thus, an estimate of π will be $4z$)
Note that $\{y_i\}_{i=1}^n$ are IID RVs
Let $var(y_i) = \sigma^2$, $E(y_i) = \mu$
Then, by the central limit theorem, the RV $w = \sqrt{n}\frac{(z-\mu)}{\sigma}$ is approximately $\sim \mathcal{N}(0, 1)$
We know that $\mu = \pi/4$ and $\sigma = \sqrt{(\pi/4)(1 - \pi/4)}$ (Mean and variance of a Bernoulli RV) thus, (We know both of these exactly as we are allowed to assume that we know π exactly)
We need the smallest n such that $P(\pi - 0.01 \leq 4z \leq \pi + 0.01) = 0.95$

$$\begin{aligned}
P(\pi - 0.01 \leq 4z \leq \pi + 0.01) &= P((\pi - 0.01)/4 \leq z \leq (\pi + 0.01)/4) \\
&= P(-0.01/4 \leq z - \mu \leq 0.01/4) \\
&= P(-0.01\sqrt{n}/4 \leq \sqrt{n}(z - \mu) \leq 0.01\sqrt{n}/4) \\
&= P\left(-\frac{0.01\sqrt{n}}{4\sigma} \leq w \leq \frac{0.01\sqrt{n}}{4\sigma}\right) = 0.95
\end{aligned}$$

Thus, we need a 95% confidence interval of w
We know that the 95% confidence interval for the standard normal distribution is the interval $(-1.96, 1.96)$
Thus, this gives:

$$\begin{aligned}
\frac{0.01\sqrt{n}}{4\sigma} &= 1.96 \\
n &= (196 \times 4\sigma)^2 \\
&= \left(196 \times 4\sqrt{\frac{\pi(4-\pi)}{16}}\right)^2 \\
&\approx \boxed{103599}
\end{aligned}$$

Instructions for running the code:

1. Unzip and `cd` to `q1/code`, under this find the file named `q1.m`
2. On running, it will print the estimates of π for the given values of n, and it will also print the estimate for π for $n = 10^9$ (This may take some time)