

## Assignment 5: CS 215

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### Question 3

**Prior:**

$$p(\theta) = K \left( \frac{\theta_m}{\theta} \right)^\alpha I(\theta \geq \theta_m)$$

**Likelihood:**

Let  $x_M = \max\{x_i\}_{i=1}^n$

$$L(\{x_i\}_{i=1}^n | \theta) = \frac{1}{\theta^n} I(\theta \geq x_M)$$

For maximizing/computing the mean, we would want Posterior to be non-zero.

This will be possible iff  $\theta \geq \max\{x_M, \theta_m\}$

Let  $M = \max\{x_M, \theta_m\}$

Thus, a constraint on  $\theta$  is  $\theta \geq M$

**Evidence:**

$$\begin{aligned} e(\{x_i\}_{i=1}^n) &= \int_M^\infty L(\{x_i\}_{i=1}^n | \theta) d\theta \\ &= \frac{1}{(n-1)M^{n-1}} \end{aligned}$$

**Posterior:**

$$\begin{aligned} g(\theta | \{x_i\}_{i=1}^n) &= \frac{p(\theta) \cdot L(\{x_i\}_{i=1}^n | \theta)}{e(\{x_i\}_{i=1}^n)} \\ &= K(n-1)(M^{n-1}) \left( \frac{\theta_m}{\theta} \right)^\alpha I(\theta \geq \theta_m) \frac{1}{\theta^n} I(\theta \geq x_M) \\ &= K(n-1)(M^{n-1}) \left( \frac{\theta_m}{\theta} \right)^\alpha \frac{1}{\theta^n} I(\theta \geq M) \\ &= K' \left( \frac{M}{\theta} \right)^{n+\alpha} I(\theta \geq M) \end{aligned}$$

This exactly matches the form of  $\text{Pareto}(M, n + \alpha)$ .

Since both the posterior and  $\text{Pareto}(M, n + \alpha)$  integrate to 1, the normalizing constant is same in both, thus the posterior exactly equals  $\text{Pareto}(M, n + \alpha)$ .

**ML Estimate**

Likelihood is maximized when  $\theta$  takes the least value it is allowed to take,  $= x_M$  (Here, we ignore the prior and thus ignore the constraints arising due to the prior)

$$\theta_{ML} = x_M$$

**MAP Estimate**

Mode of the Posterior Pareto( $M, n + \alpha$ ) distribution occurs at  $M$

$$\theta_{MAP} = M$$

**Posterior Mean Estimate**

This is equal to the mean of the Posterior Pareto( $M, n + \alpha$ ) distribution

$$\theta_{PME} = \left( \frac{\alpha + n - 1}{\alpha + n - 2} \right) M$$

Neither the MAP estimate, nor the PME estimate tends to the ML estimate when  $M \neq x_M$ .

This is not desirable, as ideally more data should improve our estimates.

When  $M = x_M$ , then both MAP and PME estimates tends to the ML estimate as  $n$  increases.