

COMPLEX AND HYPERCOMPLEX ITERATIVE METHODS

SAMUEL J MONSON

2023-03-07

Definition (Function Iteration)

$$\begin{aligned}f^0 &:= I \\ f^{k+1} &:= f \circ f^k\end{aligned}$$

Example

Given $f(x) = x + 1$,

Definition (Function Iteration)

$$f^0 := I$$

$$f^{k+1} := f \circ f^k$$

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Example

Given $f(x) = x + 1$,

$$f^0(x) = x$$

$$f^1(x) = x + 1$$

Definition (Function Iteration)

$$f^0 := I$$

$$f^{k+1} := f \circ f^k$$

Example

Given $f(x) = x + 1$,

$$f^0(x) = x$$

$$f^1(x) = x + 1$$

$$f^2(x) = (x + 1) + 1$$

Definition (Dynamical System)

A system that enacts rules on a set of variables to produce a state.

Definition (Complex Dynamics)

The study of Dynamical Systems defined by complex iterative functions.

COMPLEX NUMBERS

Definition (Complex Numbers)

$$i^2 = -1$$

$$\{a + bi : a, b \in \mathbb{R}\} \in \mathbb{C}$$

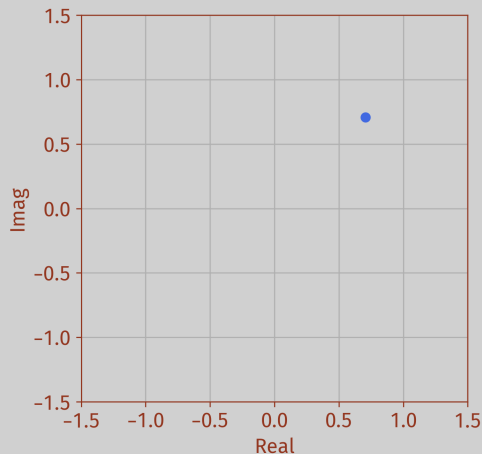
COMPLEX ITERATION

Rules

$$f(z) = z^2$$

$$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\blacksquare f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$



COMPLEX ITERATION

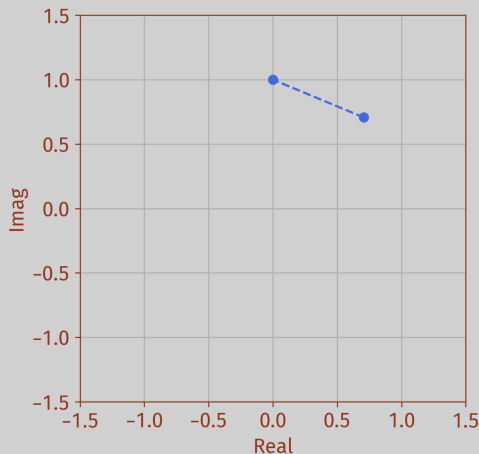
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$$\blacksquare f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\blacksquare f^1(z) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 i = i$$



COMPLEX ITERATION

Rules

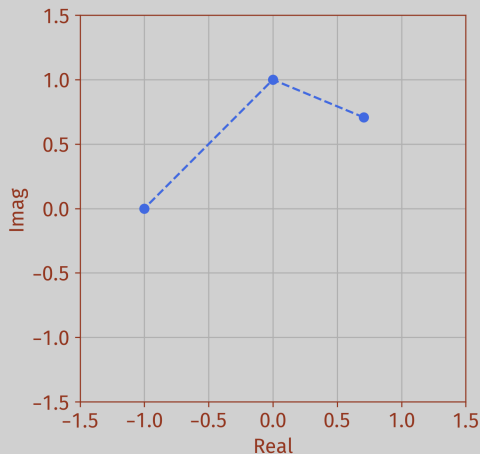
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$$\blacksquare f^2(z) = -1$$



COMPLEX ITERATION

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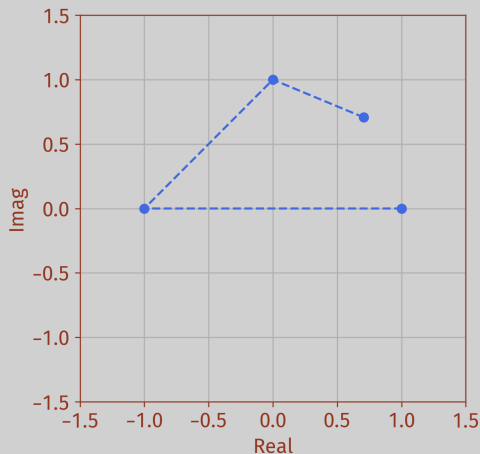
$$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\blacksquare f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\blacksquare f^1(z) = i$$

$$\blacksquare f^2(z) = -1$$

$$\blacksquare f^3(z) = 1$$



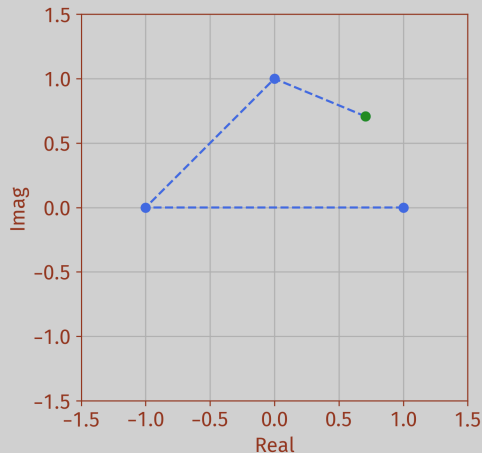
COMPLEX ITERATION

Rules

$$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}i$$

$$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\blacksquare f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$



COMPLEX ITERATION

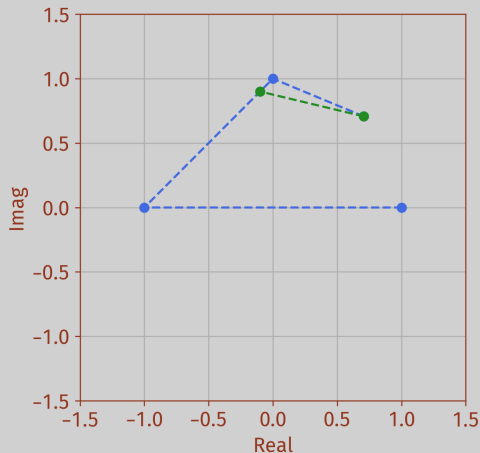
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$$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}i$$

$$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\blacksquare f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\blacksquare f^1(z) = -\frac{1}{10} + \left(1 - \frac{1}{10}\right)i = -0.9 - 0.28i$$



COMPLEX ITERATION

Rules

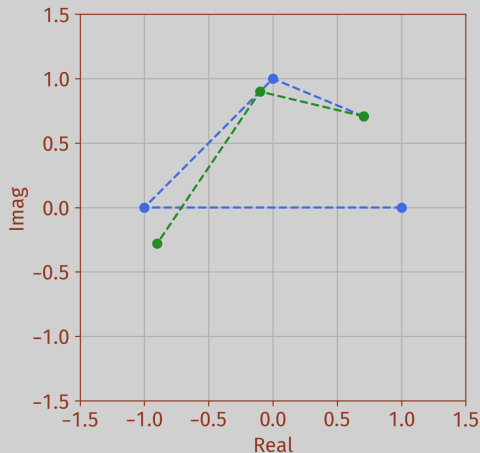
$$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}i$$

$$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\blacksquare f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\blacksquare f^1(z) = -0.9 - 0.28i$$

$$\blacksquare f^2(z) = 0.6316 + 0.404i$$



COMPLEX ITERATION

Rules

$$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}i$$

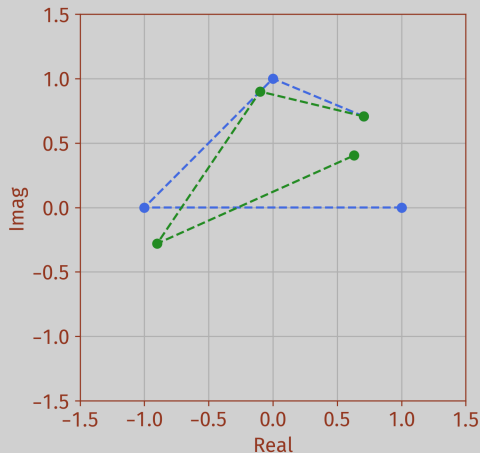
$$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\blacksquare f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$\blacksquare f^1(z) = -0.9 - 0.28i$$

$$\blacksquare f^2(z) = 0.6316 + 0.404i$$

$$\blacksquare f^3(z) \approx 0.13570256 + 0.4103328i$$



GROUP ACTIVITY

$$f(z) = z^2 + c$$

Easier

$$c = -0.2 + 0i$$

$$z_0 = 0.5 + 0i$$

Harder

$$c = -0.2 + 0.4i$$

$$z_0 = 0.5 - 0.5i$$

GROUP ACTIVITY (EASIER)

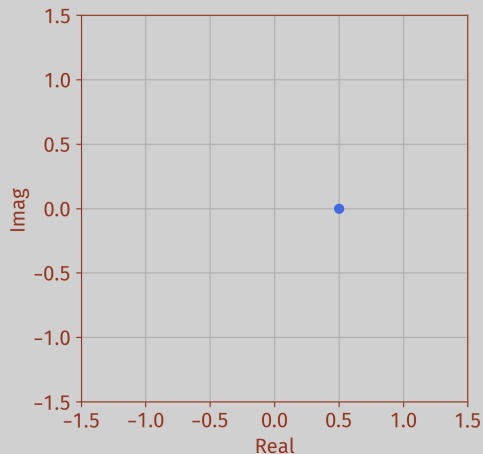
Rules

$$f(z) = z^2 + c$$

$$c = -0.2 + 0i$$

$$z_0 = 0.5 + 0i$$

■ $f^1(z) = 0.05$



GROUP ACTIVITY (EASIER)

Rules

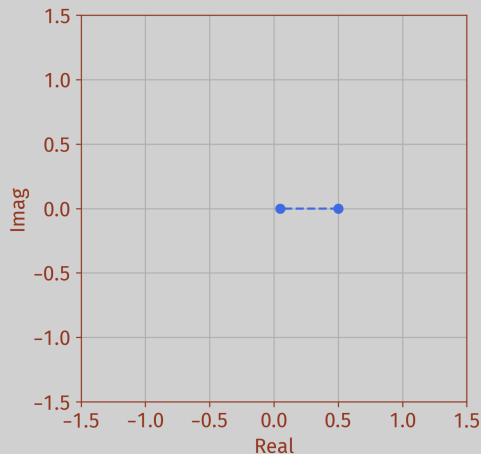
$$f(z) = z^2 + c$$

$$c = -0.2 + 0i$$

$$z_0 = 0.5 + 0i$$

■ $f^1(z) = 0.05$

■ $f^2(z) = -0.1975$



GROUP ACTIVITY (EASIER)

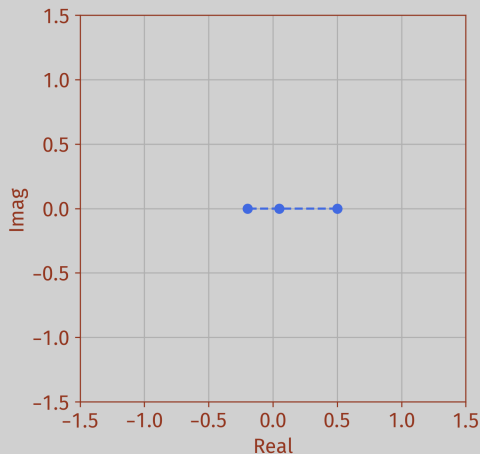
Rules

$$f(z) = z^2 + c$$

$$c = -0.2 + 0i$$

$$z_0 = 0.5 + 0i$$

- $f^1(z) = 0.05$
- $f^2(z) = -0.1975$
- $f^3(z) = -0.16099375$



GROUP ACTIVITY (EASIER)

Rules

$$f(z) = z^2 + c$$

$$c = -0.2 + 0i$$

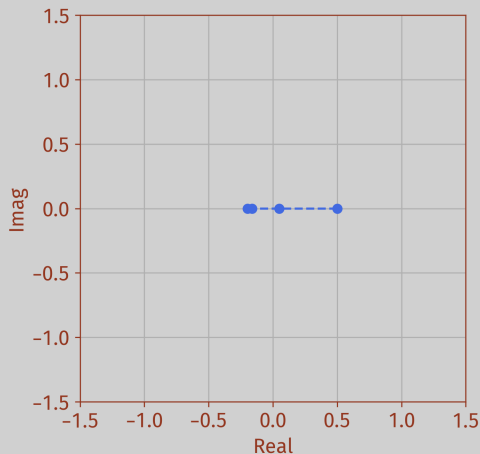
$$z_0 = 0.5 + 0i$$

■ $f^1(z) = 0.05$

■ $f^2(z) = -0.1975$

■ $f^3(z) = -0.16099375$

■ $f^4(z) = -0.1740810125$



GROUP ACTIVITY (HARDER)

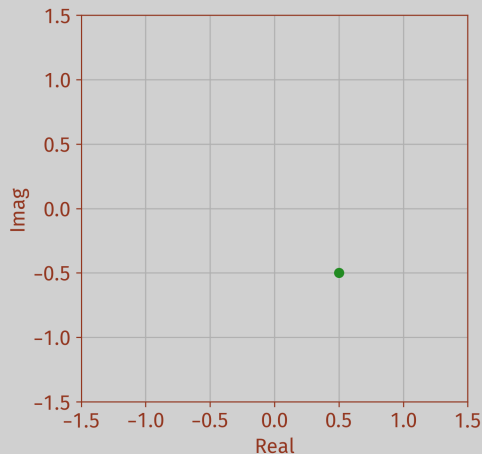
Rules

$$f(z) = z^2 + c$$

$$c = -0.2 + 0.4i$$

$$z_0 = 0.5 - 0.5i$$

■ $f^1(z) = -0.2 - 0.1i$



GROUP ACTIVITY (HARDER)

Rules

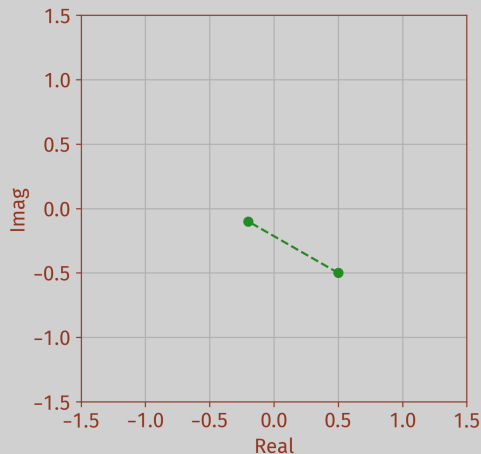
$$f(z) = z^2 + c$$

$$c = -0.2 + 0.4i$$

$$z_0 = 0.5 - 0.5i$$

$$\blacksquare f^1(z) = -0.2 - 0.1i$$

$$\blacksquare f^2(z) = -0.17 + 0.44i$$



GROUP ACTIVITY (HARDER)

Rules

$$f(z) = z^2 + c$$

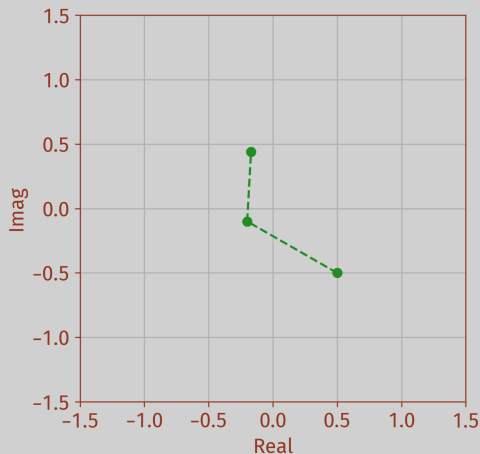
$$c = -0.2 + 0.4i$$

$$z_0 = 0.5 - 0.5i$$

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$$\blacksquare f^3(z) = -0.3647 + 0.2504i$$



GROUP ACTIVITY (HARDER)

Rules

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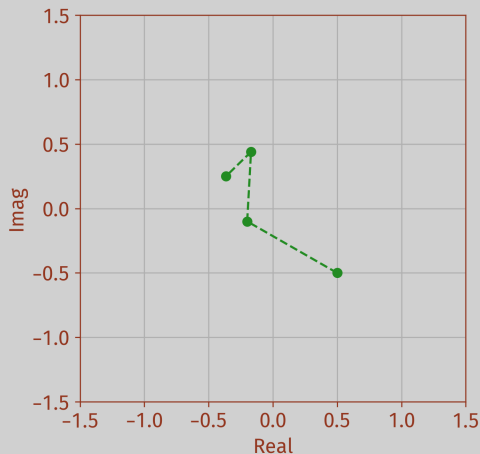
$$z_0 = 0.5 - 0.5i$$

$$\blacksquare f^1(z) = -0.2 - 0.1i$$

$$\blacksquare f^2(z) = -0.17 + 0.44i$$

$$\blacksquare f^3(z) = -0.3647 + 0.2504i$$

$$\blacksquare f^4(z) = -0.12969407 + 0.21735824i$$



IMPLEMENTATION

Iteration

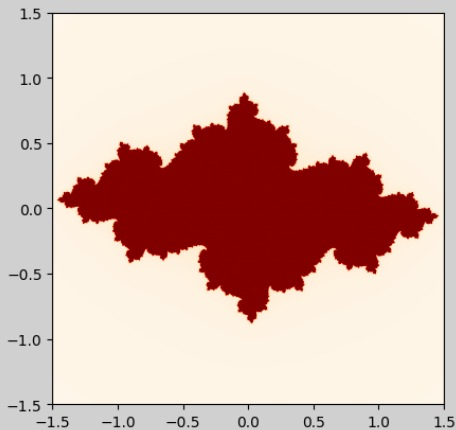
```
1 N = 128
2 B = 16
3 c = complex(-0.675, -0.112)
4 def iterate(z):
5     for n in range(N):
6         z = z*z + c
7         if abs(z) > B: break
8     return n
```

ITERATIVE FRACTALS

Complex Juila Set Example

Defined by iterative function in complex space

- $f_c(z) = z^2 + c$
- $\{z_0 \in \mathbb{C} : |f_c^k(z_0)| \in \mathbb{C} \text{ as } k \rightarrow \infty\} \in K_c$



HYPERCOMPLEX (QUATERNIONS)

Definition (Quaternion)

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$\{d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k} : a, b, c, d \in \mathbb{R}\} \in \mathbb{H}$$

■ $\mathbf{i}^2 = \mathbf{ijk}$

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- $\mathbf{k}^2 = \mathbf{ijk}$

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- $\mathbf{k}^2 = \mathbf{ijk}$

- ▶ $\mathbf{k}^2\mathbf{k}^{-1} = \mathbf{ijkk}^{-1}$

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- ▶ $\mathbf{k}^2\mathbf{k}^{-1} = \mathbf{ijkk}^{-1}$

- ▶ $\mathbf{k} = \mathbf{ij}$

- $\mathbf{j} = \mathbf{ki}$

HYPERCOMPLEX (QUATERNIONS)

Definition (Quaternion)

$$i^2 = j^2 = k^2 = ijk = -1$$

$$\{d + ai + bj + ck : a, b, c, d \in \mathbb{R}\} \in \mathbb{H}$$

$$\blacksquare i^2 = ijk$$

$$\triangleright i^{-1}i^2 = i^{-1}ijk$$

$$\triangleright i = jk$$

$$\blacksquare k^2 = ijk$$

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$$\{d + ai + bj + ck : a, b, c, d \in \mathbb{R}\} \in \mathbb{H}$$

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▶ $i^{-1}i^2 = i^{-1}ijk$

▶ $i = jk$

■ $k^2 = ijk$

▶ $k^2k^{-1} = ijk k^{-1}$

▶ $k = ij$

■ $j = ki$

■ $i = jk$

▶ $ji = jjk$

▶ $ji = j^2k$

▶ $ji = -k$

▶ $-k = ji$

HYPERCOMPLEX (QUATERNIONS)

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$$\triangleright k = ij$$

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$$\triangleright ji = jjk$$

$$\triangleright ji = j^2k$$

$$\triangleright ji = -k$$

$$\triangleright -k = ji$$

$$\blacksquare -i = kj$$

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$$\triangleright ji = jjk$$

$$\triangleright ji = j^2k$$

$$\triangleright ji = -k$$

$$\triangleright -k = ji$$

$$\blacksquare -i = kj$$

$$\blacksquare -j = ik$$

HYPERCOMPLEX (QUATERNIONS)

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\begin{aligned} p \times q = & dw + dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \\ & + aw\mathbf{i} + ax\mathbf{i}^2 + ay\mathbf{ij} + az\mathbf{ik} \\ & + bw\mathbf{j} + bx\mathbf{ji} + by\mathbf{j}^2 + bz\mathbf{jk} \\ & + cw\mathbf{k} + cx\mathbf{ki} + cy\mathbf{kj} + cz\mathbf{k}^2 \end{aligned}$$

HYPERCOMPLEX (QUATERNIONS)

GROUP ACTIVITY

$$f(z) = z^2 + q$$

Easier

$$q = -0.2 + 0.4\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

$$z_0 = 0.5 - 0.5\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

Harder

$$q = -0.2 + 0.4\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

$$z_0 = 0.5 - 0.5\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

Quaternion Multiplication

```
1 def qMult(p, q):  
2     r = Quat(  
3         p.r*q.r - p.i*q.i - p.j*q.j - p.k*q.k,  
4         p.r*q.i + p.i*q.r + p.j*q.k - p.k*q.j,  
5         p.r*q.j - p.i*q.k + p.j*q.r + p.j*q.i,  
6         p.r*q.k + p.i*q.j - p.j*q.i + p.k*q.r  
7     )  
8     return r
```

Quaternion Square

```
1 def qSquare(q):  
2     r = Quat(  
3         q.r*q.r - q.i*q.i - q.j*q.j - q.k*q.k,  
4         2*q.r*q.i,  
5         2*q.r*q.j,  
6         2*q.r*q.k  
7     )  
8     return r
```

Quaternion Add

```
1 def qAdd(p, q):  
2     r = Quat(  
3         p.r + q.r,  
4         p.i + q.i,  
5         p.j + q.j,  
6         p.k + q.k  
7     )  
8     return r
```

IMPLEMENTATION

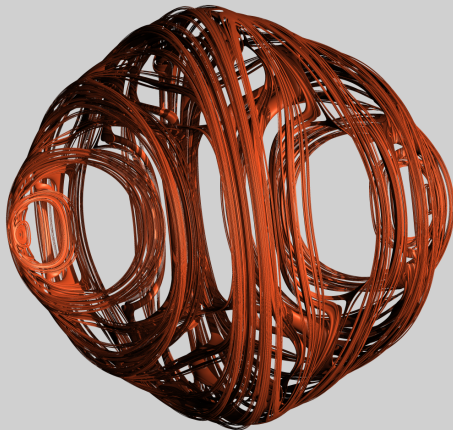
Iteration

```
1 N = 12
2 B = 16
3 c = Quat(-0.2, 0.4, -0.4, -0.4)
4 def iterate(z):
5     for n in range(N):
6         z = z*z + c
7         if abs(z) > B: break
8     return n
```


HYPERCOMPLEX ITERATIVE FRACTALS

Hypercomplex Julia Set Example

- Defined by iterative function in 4D Quaternion space



CONCLUSION

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