Complex and Hypercomplex Iterative Methods

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Definition (Function Iteration)

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$$f^{k+1} := f \circ f^k$$

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Given $f(x) = x + 1$,

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Given $f(x) = x + 1$,

$$f^0(x) = x$$

Definition (Function Iteration)

$$
\begin{aligned}
f^{0} &:= \boldsymbol{I} \\
f^{k+1} &:= f \circ f^{k}
\end{aligned}
$$

Example

Given $f(x) = x + 1$,

$$
\begin{aligned}
f^{0}(x) &= x \\
f^{1}(x) &= x + 1
\end{aligned}
$$

Definition (Function Iteration)

$$f^0 := \boldsymbol{I}$$
$$f^{k+1} := f \circ f^k$$

Example

Given $f(x) = x + 1$,

$$f^0(x) = x$$
$$f^1(x) = x + 1$$
$$f^2(x) = (x + 1) + 1$$

Definition (Dynamical System)

A system that enacts rules on a set of variables to produce a state.

Definition (Complex Dynamics)

The study of Dynamical Systems defined by complex iterative functions.

Definition (Complex Numbers)

$\boldsymbol{i}^2 = -1$

$\{a + b\boldsymbol{i} \,:\, a, b \in \mathbb{R}\} \in \mathbb{C}$

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Addition

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- Let $a, b, x, y \in \mathbb{R}$,

$$(a + b\boldsymbol{i}) + (x + y\boldsymbol{i})$$

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- Let $a, b, x, y \in \mathbb{R}$,

  $$(a + b\boldsymbol{i}) + (x + y\boldsymbol{i}) = (a + x) + (b + y)\boldsymbol{i}$$

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$$(a + b\boldsymbol{i}) + (x + y\boldsymbol{i}) = (a + x) + (b + y)\boldsymbol{i}$$

Multiplication

- Let $a, b, x, y \in \mathbb{R}$,

$$(a + b\boldsymbol{i}) \times (x + y\boldsymbol{i})$$

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- Let $a, b, x, y \in \mathbb{R}$,

$$(a + b\boldsymbol{i}) + (x + y\boldsymbol{i}) = (a + x) + (b + y)\boldsymbol{i}$$

Multiplication

- Let $a, b, x, y \in \mathbb{R}$,

$$(a + b\boldsymbol{i}) \times (x + y\boldsymbol{i}) = ax + ay\boldsymbol{i} + bx\boldsymbol{i} + by\boldsymbol{i}^2$$

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Addition

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$$(a + b\boldsymbol{i}) + (x + y\boldsymbol{i}) = (a + x) + (b + y)\boldsymbol{i}$$

Multiplication

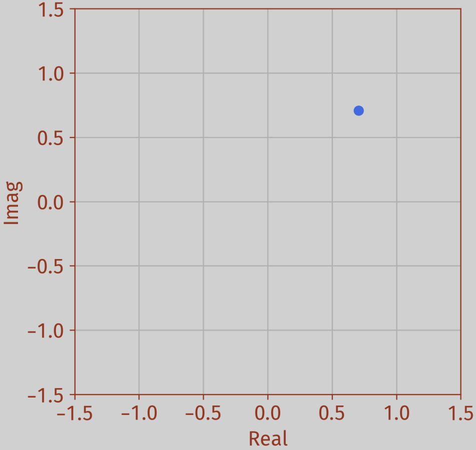
- Let $a, b, x, y \in \mathbb{R}$,

$$\begin{aligned}(a + b\boldsymbol{i}) \times (x + y\boldsymbol{i}) &= ax + ay\boldsymbol{i} + bx\boldsymbol{i} + by\boldsymbol{i}^2 \\ &= (ax - by) + (ay + bx)\boldsymbol{i}\end{aligned}$$

Rules

$f(z) = z^2$
$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

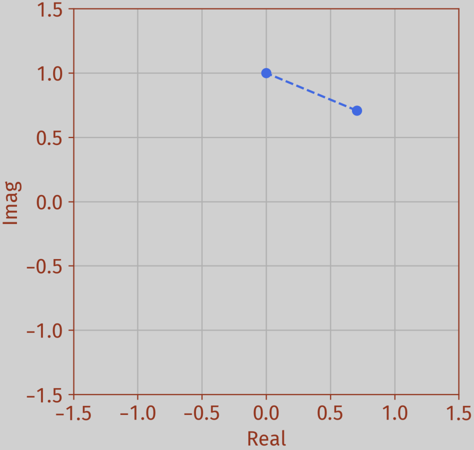


Rules

$f(z) = z^2$

$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$
- $f^1(z) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}\right)^2 =$
  $\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 \boldsymbol{i} = \boldsymbol{i}$



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- $f^1(z) = \boldsymbol{i}$
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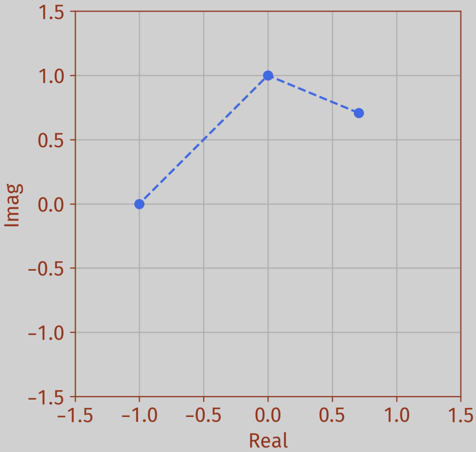

Rules

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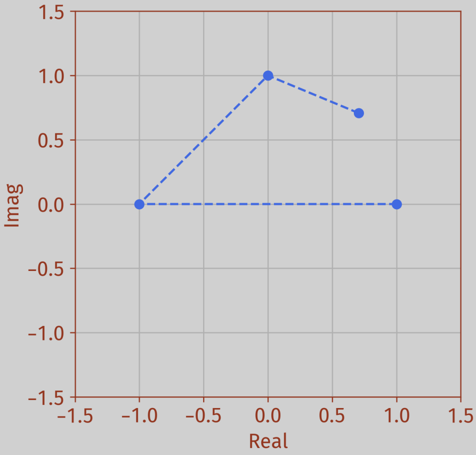
- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$
- $f^1(z) = \boldsymbol{i}$
- $f^2(z) = -1$
- $f^3(z) = 1$

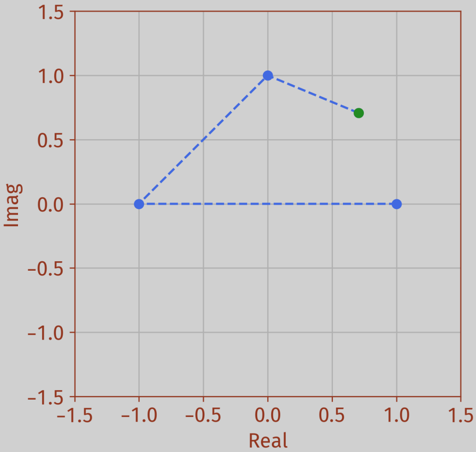


Rules

$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}\boldsymbol{i}$

$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

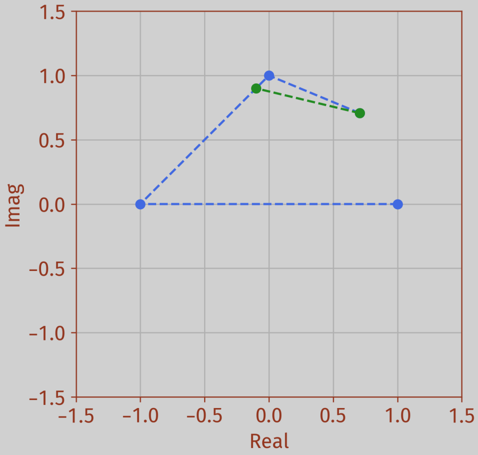


Rules

$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}\boldsymbol{i}$

$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

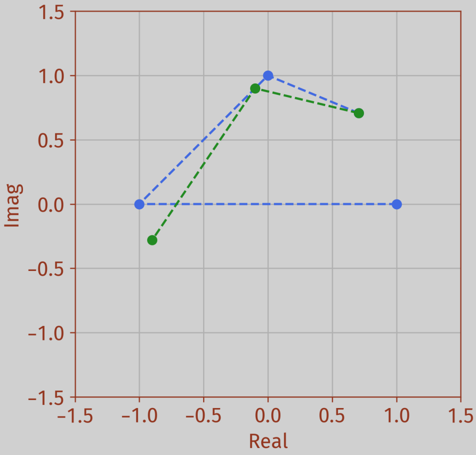
- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$
- $f^1(z) = -\frac{1}{10} + \left(1 - \frac{1}{10}\right)\boldsymbol{i} = -0.9 - 0.28\boldsymbol{i}$



Rules

$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}\boldsymbol{i}$
$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$
- $f^1(z) = -0.9 - 0.28\boldsymbol{i}$
- $f^2(z) = 0.6316 + 0.404\boldsymbol{i}$



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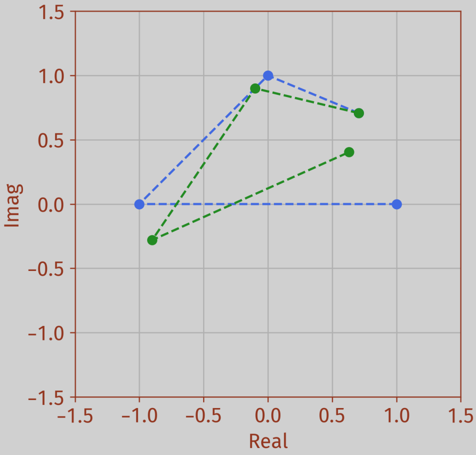
- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$
- $f^1(z) = -0.9 - 0.28\boldsymbol{i}$
- $f^2(z) = 0.6316 + 0.404\boldsymbol{i}$
- $f^3(z) \approx 0.13570256 + 0.4103328\boldsymbol{i}$



$f(z) = z^2 + c$

Easier

$c = -0.2 + 0\boldsymbol{i}$
$z_0 = 0.5 + 0\boldsymbol{i}$

Harder

$c = -0.2 + 0.4\boldsymbol{i}$
$z_0 = 0.5 - 0.5\boldsymbol{i}$

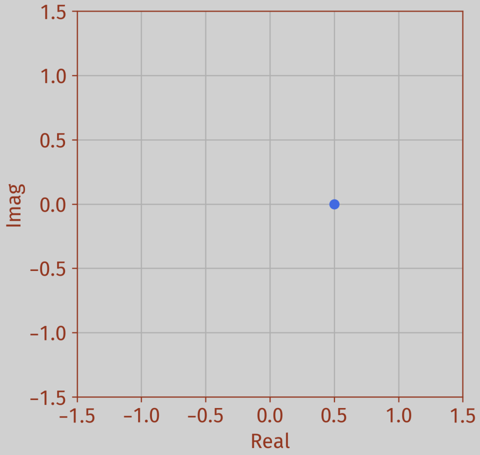
Rules

$f(z) = z^2 + c$

$c = -0.2 + 0\boldsymbol{i}$

$z_0 = 0.5 + 0\boldsymbol{i}$

- $f^1(z) = 0.05$



Rules

$f(z) = z^2 + c$

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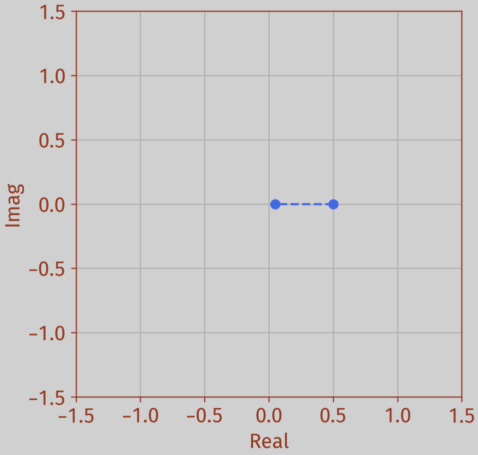
- $f^1(z) = 0.05$
- $f^2(z) = -0.1975$



Group Activity (Easier)

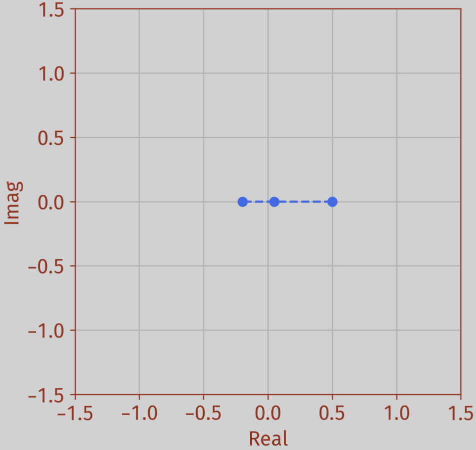
Rules

$f(z) = z^2 + c$

$c = -0.2 + 0\boldsymbol{i}$

$z_0 = 0.5 + 0\boldsymbol{i}$

- $f^1(z) = 0.05$
- $f^2(z) = -0.1975$
- $f^3(z) = -0.16099375$



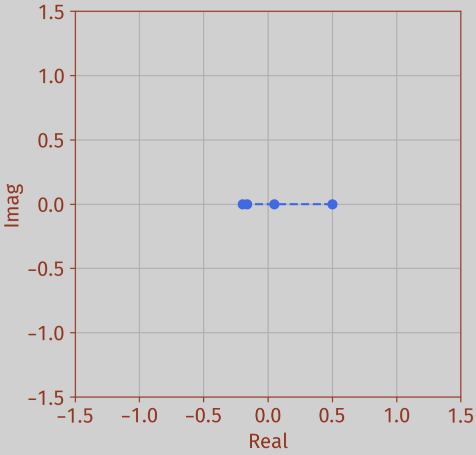
Rules

$f(z) = z^2 + c$

$c = -0.2 + 0\boldsymbol{i}$

$z_0 = 0.5 + 0\boldsymbol{i}$

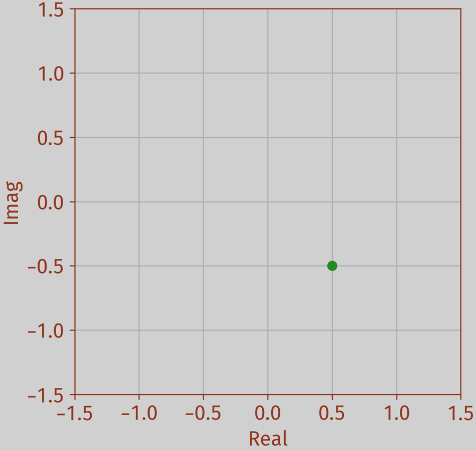
- $f^1(z) = 0.05$
- $f^2(z) = -0.1975$
- $f^3(z) = -0.16099375$
- $f^4(z) = -0.1740810125$



Rules

$f(z) = z^2 + c$
$c = -0.2 + 0.4\boldsymbol{i}$
$z_0 = 0.5 - 0.5\boldsymbol{i}$

- $f^1(z) = -0.2 - 0.1\boldsymbol{i}$

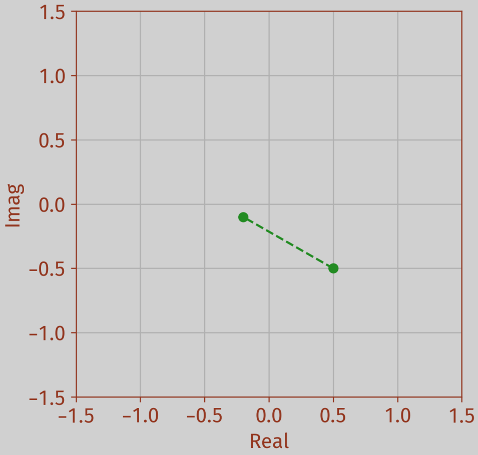


Group Activity (Harder)

Rules

$f(z) = z^2 + c$
$c = -0.2 + 0.4\boldsymbol{i}$
$z_0 = 0.5 - 0.5\boldsymbol{i}$

- $f^1(z) = -0.2 - 0.1\boldsymbol{i}$
- $f^2(z) = -0.17 + 0.44\boldsymbol{i}$



Group Activity (Harder)

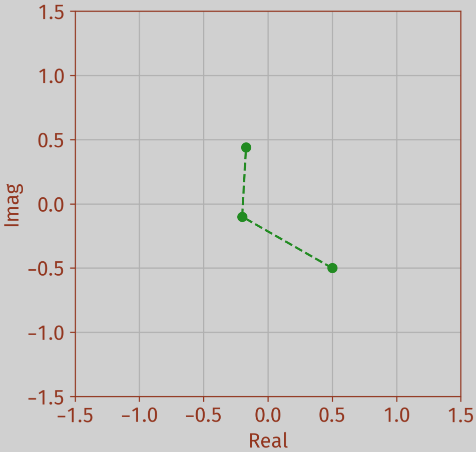
Rules

$f(z) = z^2 + c$

$c = -0.2 + 0.4\boldsymbol{i}$

$z_0 = 0.5 - 0.5\boldsymbol{i}$

- $f^1(z) = -0.2 - 0.1\boldsymbol{i}$
- $f^2(z) = -0.17 + 0.44\boldsymbol{i}$
- $f^3(z) = -0.3647 + 0.2504\boldsymbol{i}$



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$z_0 = 0.5 - 0.5\boldsymbol{i}$

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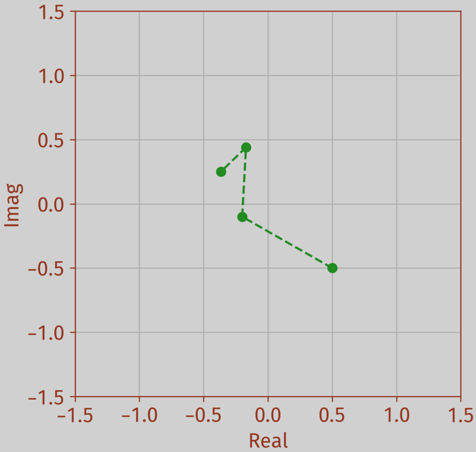
- $f^2(z) = -0.17 + 0.44\boldsymbol{i}$
- $f^3(z) = -0.3647 + 0.2504\boldsymbol{i}$
- $f^4(z) = -0.12969407 + 0.21735824\boldsymbol{i}$



Iteration

```
N = 128
B = 16
c = complex(-0.675, -0.112)
def iterate(z):
    for n in range(N):
        z = z*z + c
        if abs(z) > B: break
    return n
```

Complex Juila Set Example

Defined by iterative function in complex space

- $f_c(z) = z^2 + c$
- $\left\{z_0 \in \mathbb{C} : \left|f_c^k\left(z_0\right)\right| \in \mathbb{C} \text{ as } k \to \infty\right\} \in K_c$

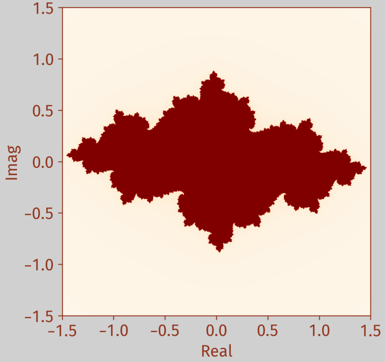


Figure: $f(z) = z^2 - 0.675 - 0.112\boldsymbol{i}$

Definition (Quaternion)

$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$

$\{d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} : a, b, c, d \in \mathbb{R}\} \in \mathbb{H}$

- $\boldsymbol{i}^2 = \boldsymbol{ijk}$

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Hypercomplex (Quaternions)

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  - $\boldsymbol{k}^2\boldsymbol{k}^{-1} = \boldsymbol{ijkk}^{-1}$
  - $\boldsymbol{k} = \boldsymbol{ij}$
- $\boldsymbol{j} = \boldsymbol{ki}$

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- $\boldsymbol{i} = \boldsymbol{jk}$
  - $\boldsymbol{ji} = \boldsymbol{jjk}$

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  - $\boldsymbol{k}^2\boldsymbol{k}^{-1} = \boldsymbol{ijkk}^{-1}$
  - $\boldsymbol{k} = \boldsymbol{ij}$
- $\boldsymbol{j} = \boldsymbol{ki}$
- $\boldsymbol{i} = \boldsymbol{jk}$
  - $\boldsymbol{ji} = \boldsymbol{jjk}$
  - $\boldsymbol{ji} = \boldsymbol{j}^2\boldsymbol{k}$

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  - $\boldsymbol{k} = \boldsymbol{ij}$
- $\boldsymbol{j} = \boldsymbol{ki}$
- $\boldsymbol{i} = \boldsymbol{jk}$
  - $\boldsymbol{ji} = \boldsymbol{jjk}$
  - $\boldsymbol{ji} = \boldsymbol{j}^2\boldsymbol{k}$
  - $\boldsymbol{ji} = -\boldsymbol{k}$

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  - $\boldsymbol{k}^2\boldsymbol{k}^{-1} = \boldsymbol{ijkk}^{-1}$
  - $\boldsymbol{k} = \boldsymbol{ij}$
- $\boldsymbol{j} = \boldsymbol{ki}$
- $\boldsymbol{i} = \boldsymbol{jk}$
  - $\boldsymbol{ji} = \boldsymbol{jjk}$
  - $\boldsymbol{ji} = \boldsymbol{j}^2\boldsymbol{k}$
  - $\boldsymbol{ji} = -\boldsymbol{k}$
  - $-\boldsymbol{k} = \boldsymbol{ji}$

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  - $\boldsymbol{i} = \boldsymbol{jk}$
- $\boldsymbol{k}^2 = \boldsymbol{ijk}$
  - $\boldsymbol{k}^2\boldsymbol{k}^{-1} = \boldsymbol{ijkk}^{-1}$
  - $\boldsymbol{k} = \boldsymbol{ij}$
- $\boldsymbol{j} = \boldsymbol{ki}$
- $\boldsymbol{i} = \boldsymbol{jk}$
  - $\boldsymbol{ji} = \boldsymbol{jjk}$
  - $\boldsymbol{ji} = \boldsymbol{j}^2\boldsymbol{k}$
  - $\boldsymbol{ji} = -\boldsymbol{k}$
  - $-\boldsymbol{k} = \boldsymbol{ji}$
- $-\boldsymbol{i} = \boldsymbol{kj}$

Definition (Quaternion)

$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$

$\{d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k} : a, b, c, d \in \mathbb{R}\} \in \mathbb{H}$

- $\mathbf{i}^2 = \mathbf{ijk}$
  - $\mathbf{i}^{-1}\mathbf{i}^2 = \mathbf{i}^{-1}\mathbf{ijk}$
  - $\mathbf{i} = \mathbf{jk}$
- $\mathbf{k}^2 = \mathbf{ijk}$
  - $\mathbf{k}^2\mathbf{k}^{-1} = \mathbf{ijkk}^{-1}$
  - $\mathbf{k} = \mathbf{ij}$
- $\mathbf{j} = \mathbf{ki}$
- $\mathbf{i} = \mathbf{jk}$
  - $\mathbf{ji} = \mathbf{jjk}$
  - $\mathbf{ji} = \mathbf{j}^2\mathbf{k}$
  - $\mathbf{ji} = -\mathbf{k}$
  - $-\mathbf{k} = \mathbf{ji}$
- $-\mathbf{i} = \mathbf{kj}$
- $-\mathbf{j} = \mathbf{ik}$

Hypercomplex (Quaternions)

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$
$$p = d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$
$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\begin{aligned} p \times q = {} & dw + dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \\ & + aw\mathbf{i} + ax\mathbf{i}^2 + ay\mathbf{ij} + az\mathbf{ik} \\ & + bw\mathbf{j} + bx\mathbf{ji} + by\mathbf{j}^2 + bz\mathbf{jk} \\ & + cw\mathbf{k} + cx\mathbf{ki} + cy\mathbf{kj} + cz\mathbf{k}^2 \end{aligned}$$

Hypercomplex (Quaternions)

$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$
$p = d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$
$q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

$$\begin{aligned}
p \times q = {} & dw + dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k} \\
& + aw\boldsymbol{i} + ax\boldsymbol{i}^2 + ay\boldsymbol{ij} + az\boldsymbol{ik} \\
& + bw\boldsymbol{j} + bx\boldsymbol{ji} + by\boldsymbol{j}^2 + bz\boldsymbol{jk} \\
& + cw\boldsymbol{k} + cx\boldsymbol{ki} + cy\boldsymbol{kj} + cz\boldsymbol{k}^2 \\
& = dw - ax - by - cz \\
& + dx\boldsymbol{i} + aw\boldsymbol{i} + bz\boldsymbol{i} - cy\boldsymbol{i} \\
& + dy\boldsymbol{j} - az\boldsymbol{j} + bw\boldsymbol{j} + cx\boldsymbol{j} \\
& + dz\boldsymbol{k} + ay\boldsymbol{k} - bx\boldsymbol{k} + cw\boldsymbol{k}
\end{aligned}$$

Group Activity

$f(z) = z^2 + q$

Easier

$q = -0.2 + 0.4\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$

$z_0 = 0.5 - 0.5\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$

Harder

$q = -0.2 + 0.4\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$

$z_0 = 0.5 - 0.5\boldsymbol{i} + 0\boldsymbol{j} + 0\boldsymbol{k}$

Implementation

Quaternion Multiplication

```
def qMult(p, q):
    r = Quat(
        p.r*q.r - p.i*q.i - p.j*q.j - p.k*q.k,
        p.r*q.i + p.i*q.r + p.j*q.k - p.k*q.j,
        p.r*q.j - p.i*q.k + p.j*q.r + p.j*q.i,
        p.r*q.k + p.i*q.j - p.j*q.i + p.k*q.r
    )
    return r
```

Quaternion Square

```
def qSquare(q):
    r = Quat(
        q.r*q.r - q.i*q.i - q.j*q.j - q.k*q.k,
        2*q.r*q.i,
        2*q.r*q.j,
        2*q.r*q.k
    )
    return r
```

Quaternion Add

```
def qAdd(p, q):
    r = Quat(
        p.r + q.r,
        p.i + q.i,
        p.j + q.j,
        p.k + q.k
    )
    return r
```

Iteration

```
N = 12
B = 16
c = Quat(-0.2, 0.4, -0.4, -0.4)
def iterate(z):
    for n in range(N):
        z = z*z + c
        if abs(z) > B: break
    return n
```

Ray Marching

Hypercomplex Juila Set Example

- Defined by iterative function in 4D Quaternion space

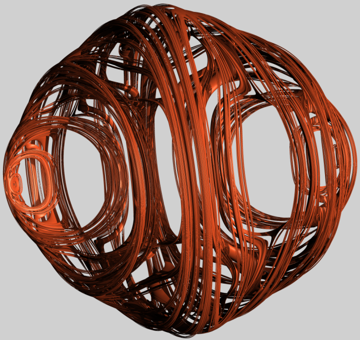

Figure: $f(z) = z^2 + 0.3 - 0.375\boldsymbol{i} - 0.675\boldsymbol{j} - 0.112\boldsymbol{k}$

Conclusion

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Link to Slides

https://github.com/scrufulufugus/senior-synthesis