

COMPLEX AND HYPERCOMPLEX ITERATIVE METHODS

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OUTLINE

1 Iteration

2 Complex Iterative Methods

3 Quaternions

4 Quaternion Iterative Methods

DEFINITIONS

Definition (Iterative Method)

A sequence where each value in the sequence is defined by the previous value.

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Definition (Dynamical System)

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Definition (Complex Dynamics)

The study of dynamical systems defined by complex functions.

ITERATION

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Definition (Function Iteration)

$$f^0 := I$$

$$f^{k+1} := f \circ f^k$$

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$$f^3(x) = ((x + 1) + 1) + 1$$

⋮

COMPLEX ITERATIVE METHODS

COMPLEX NUMBERS

Definition (Complex Numbers)

$$i^2 = -1$$

$$\{a + bi : a, b \in \mathbb{R}\} \in \mathbb{C}$$

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Let $a, b, x, y \in \mathbb{R}$,

$$\begin{aligned}(a + bi) * (x + yi) &= ax + ayi + bxi + byi^2 \\ &= (ax - by) + (ay + bx)i\end{aligned}$$

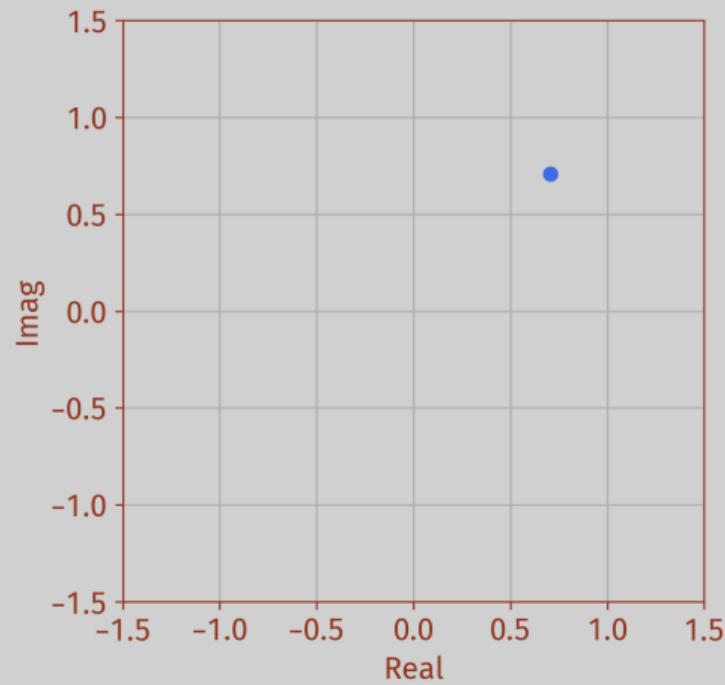
COMPLEX ITERATION

Rules

$$f(z) = z^2$$

$$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

■ $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$



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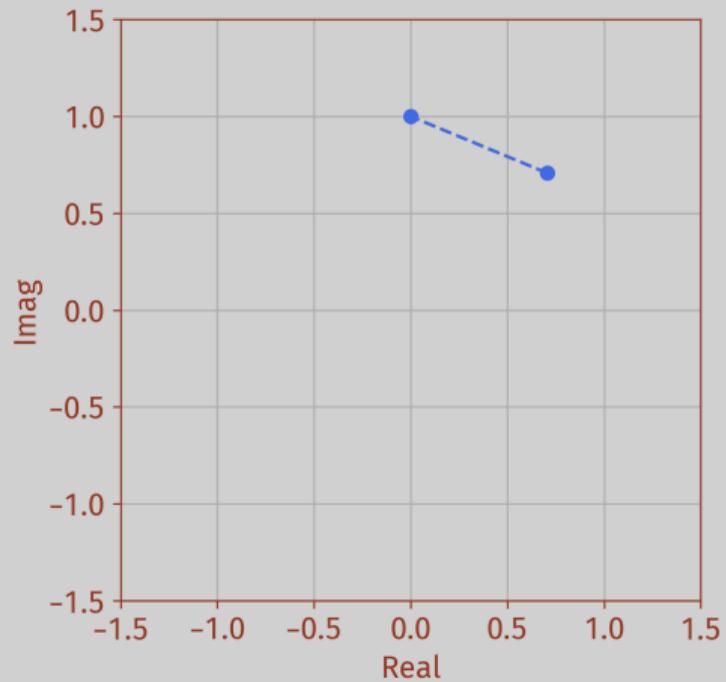
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- $f^1(z) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^2 + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^2 i = i$



COMPLEX ITERATION

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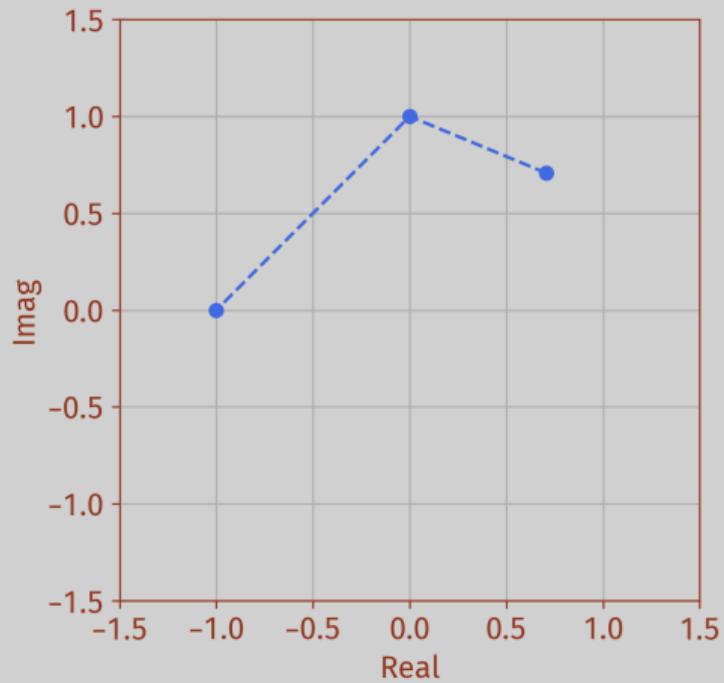
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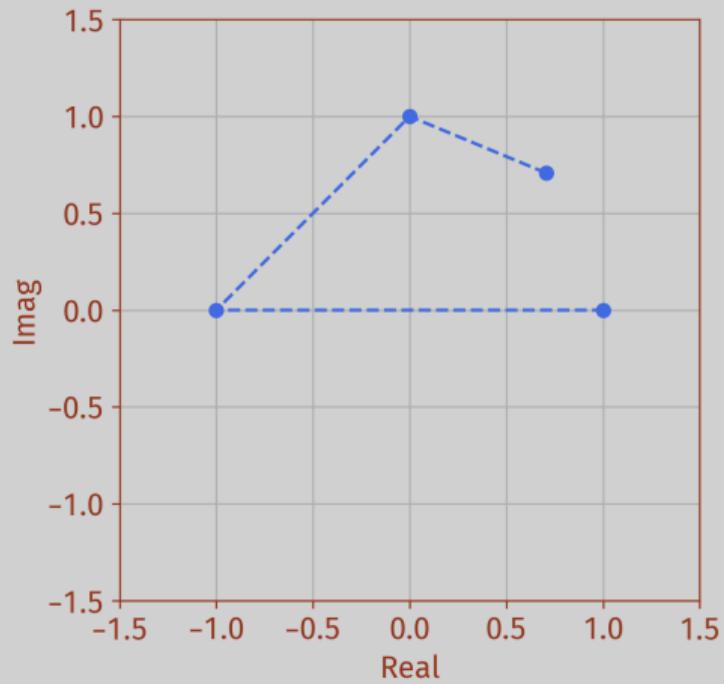
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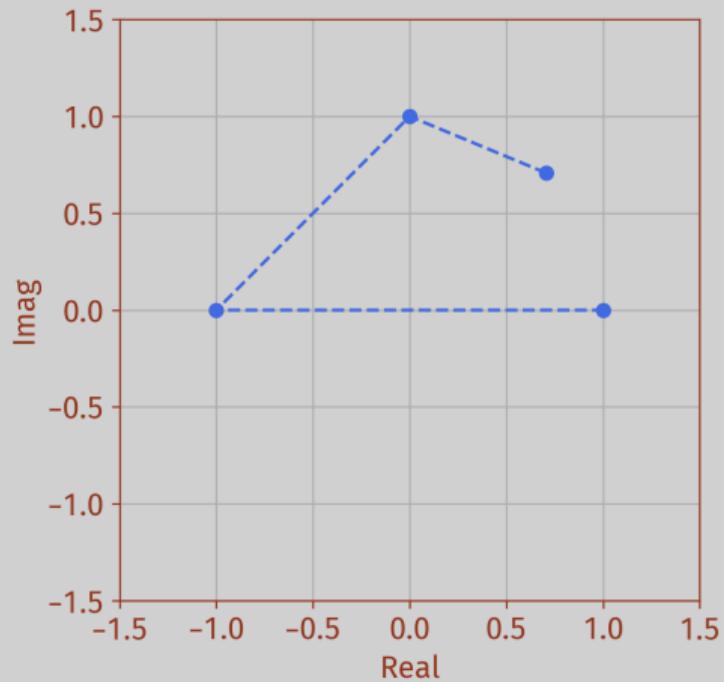
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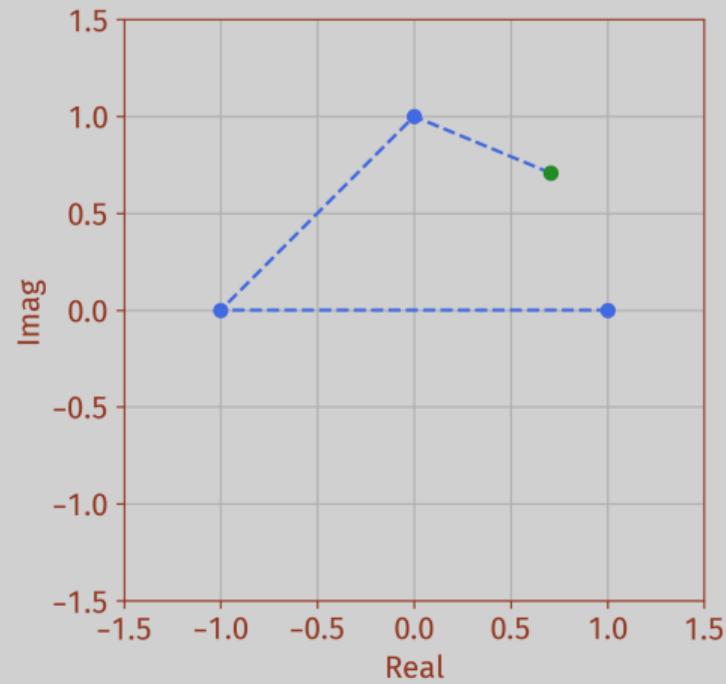
COMPLEX ITERATION

Rules

$$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}i$$

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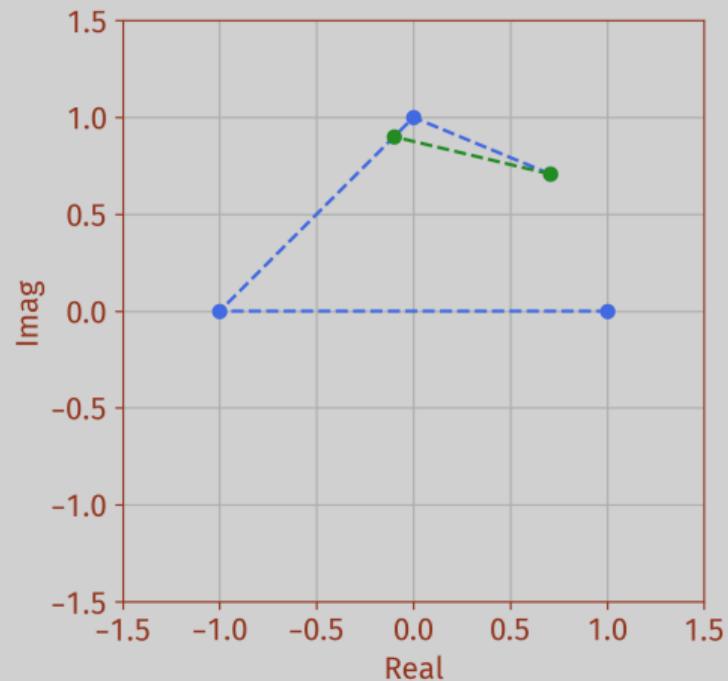
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- $f^1(z) = -\frac{1}{10} + \left(1 - \frac{1}{10}\right)i = -0.1 + 0.9i$



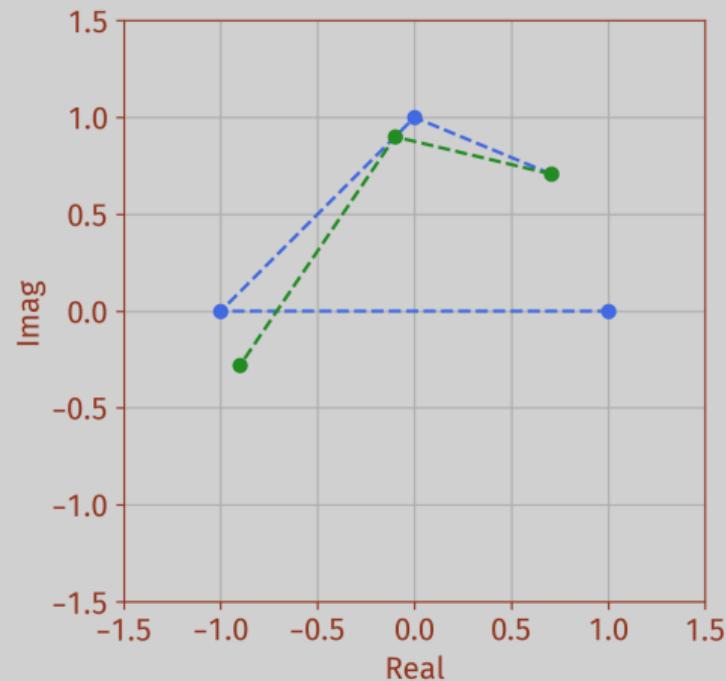
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- $f^2(z) = -0.9 - 0.28i$



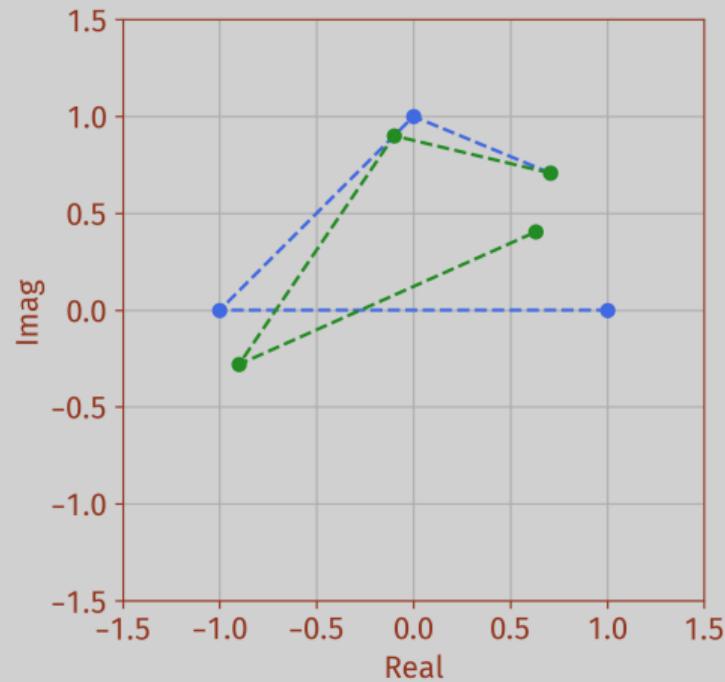
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- $f^3(z) = 0.6316 + 0.404i$



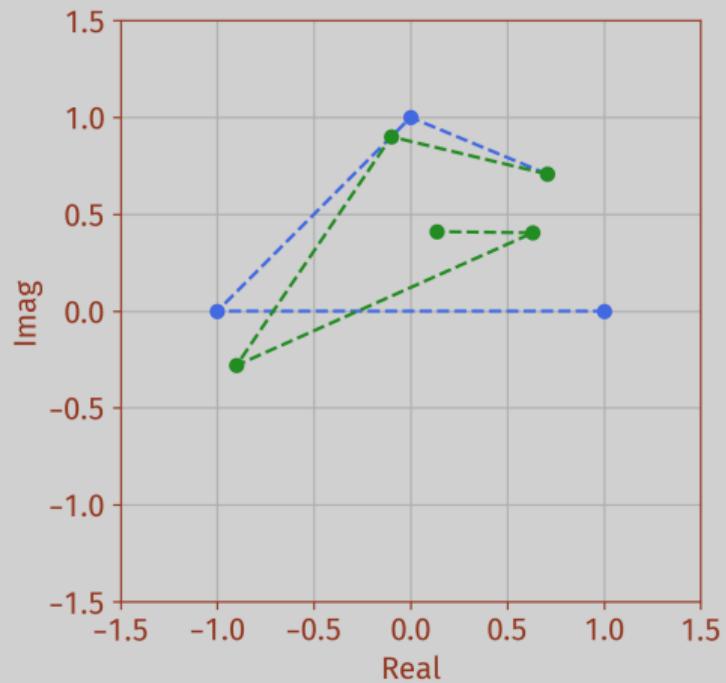
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- $f^4(z) \approx 0.13570256 + 0.4103328i$



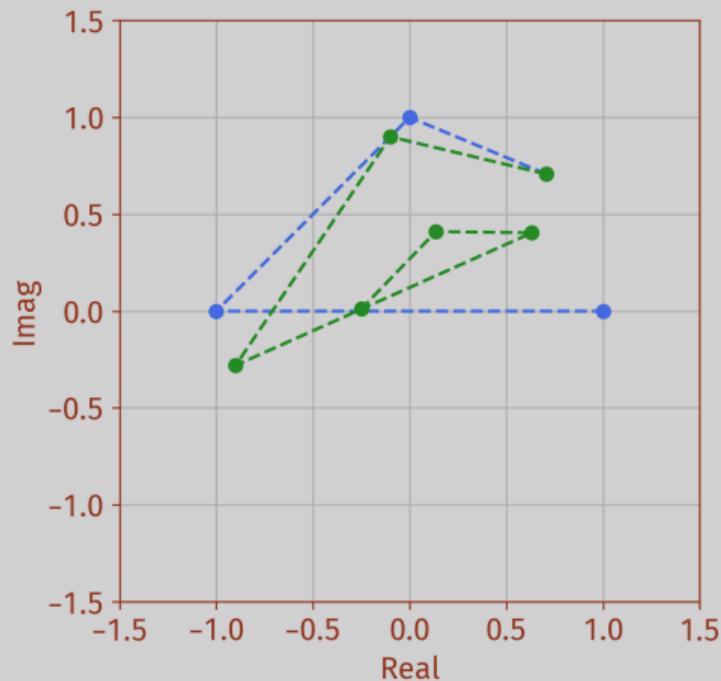
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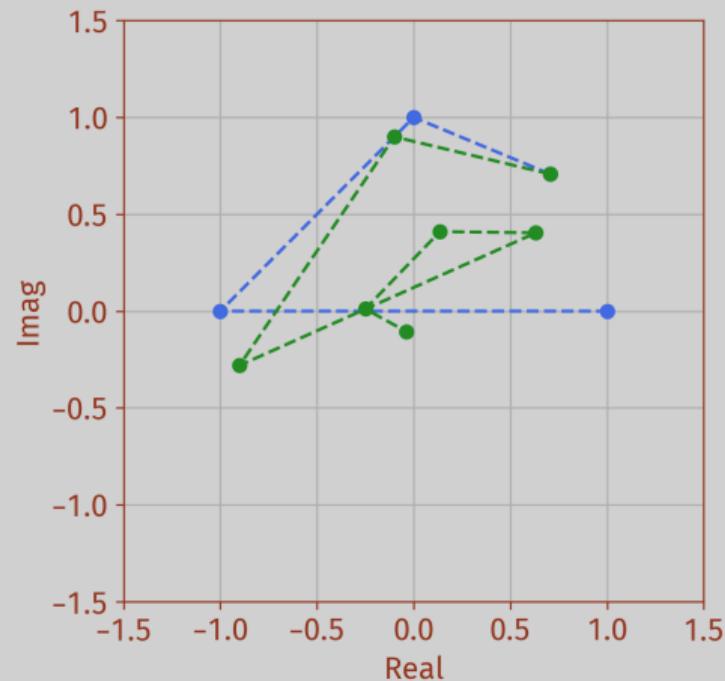
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- $f^6(z) \approx -0.03765028 - 0.10568225i$



GROUP ACTIVITY

Easier

$$f(z) = z^2 - 0.2 + 0i$$

$$z_0 = 0.5 + 0i$$

Harder

$$f(z) = z^2 - 0.2 + 0.4i$$

$$z_0 = 0.5 - 0.5i$$

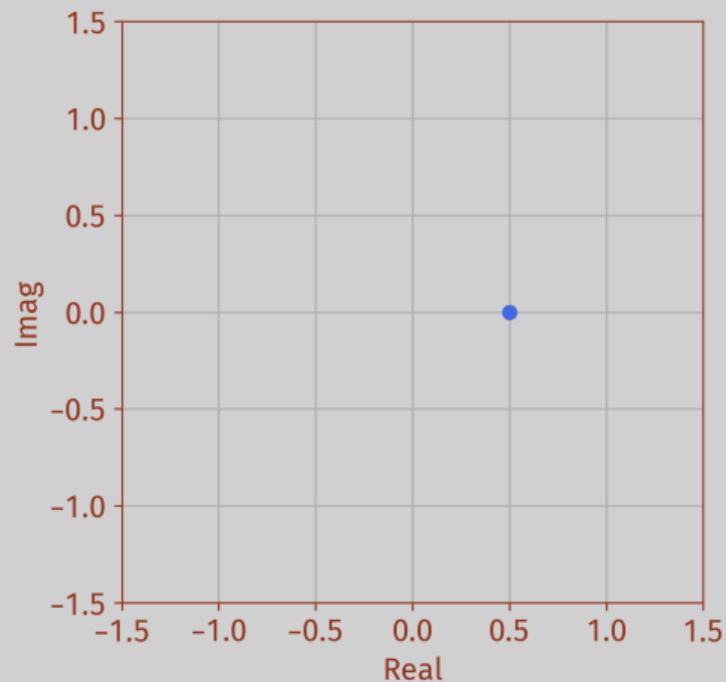
GROUP ACTIVITY (EASIER)

Rules

$$f(z) = z^2 - 0.2 + 0i$$

$$z_0 = 0.5 + 0i$$

■ $f^0(z) = 0.5$



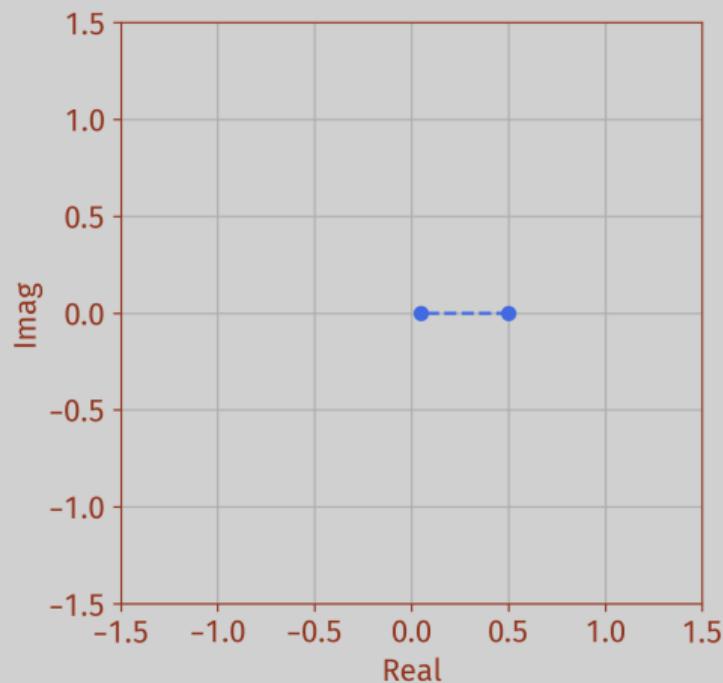
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- $f^1(z) = 0.05$



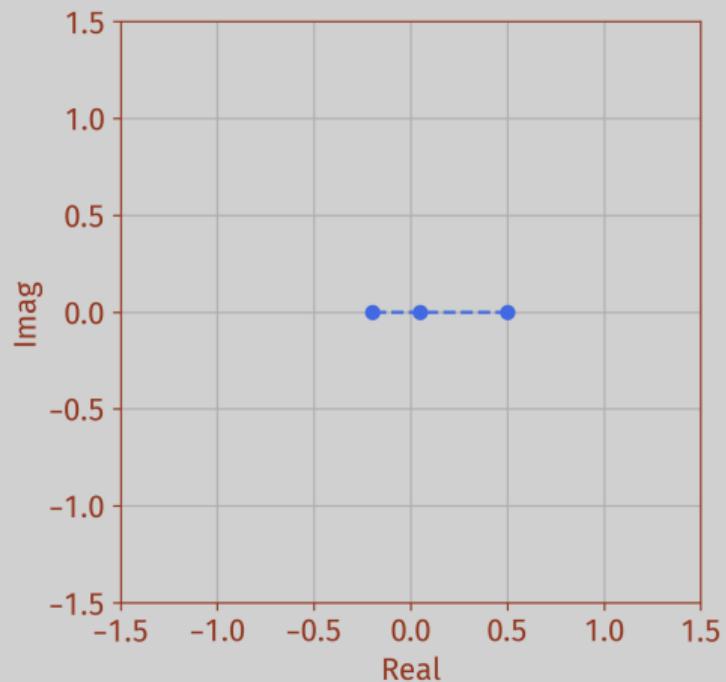
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- $f^2(z) = -0.1975$



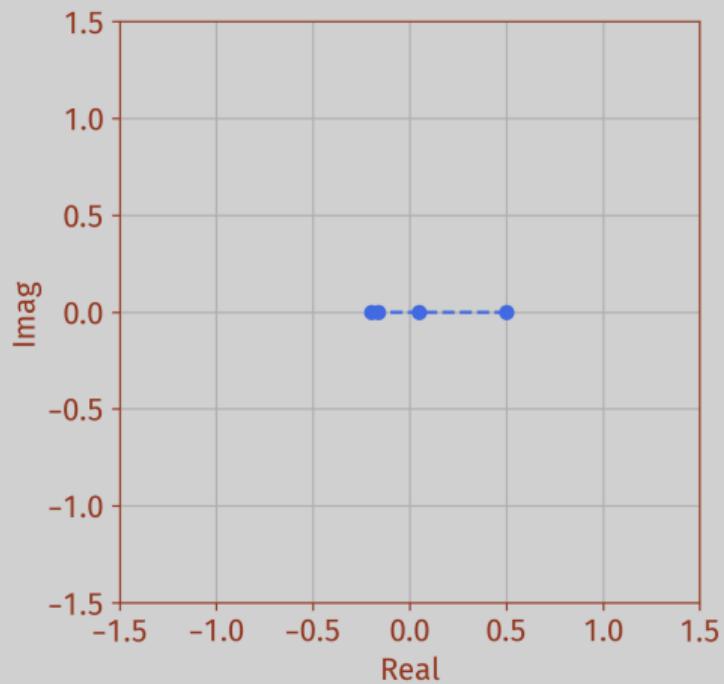
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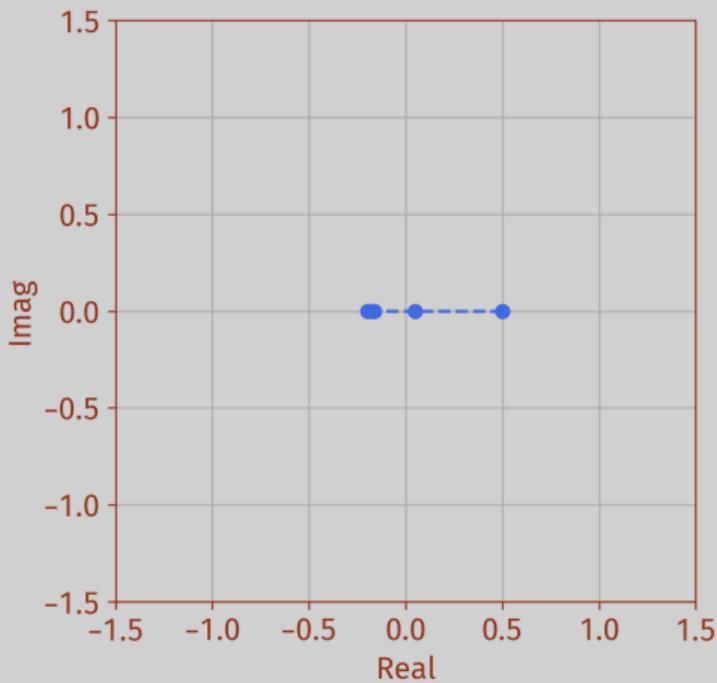
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- $f^4(z) \approx -0.1740810125$



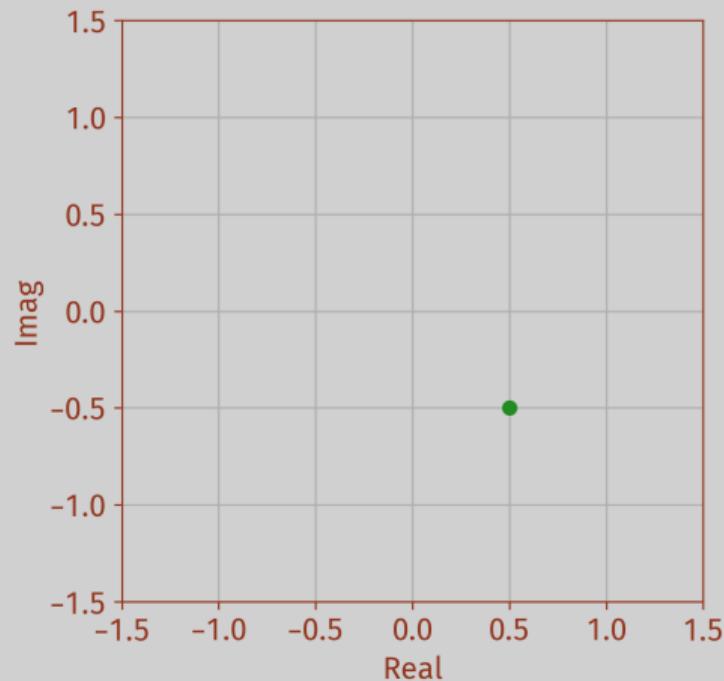
GROUP ACTIVITY (HARDER)

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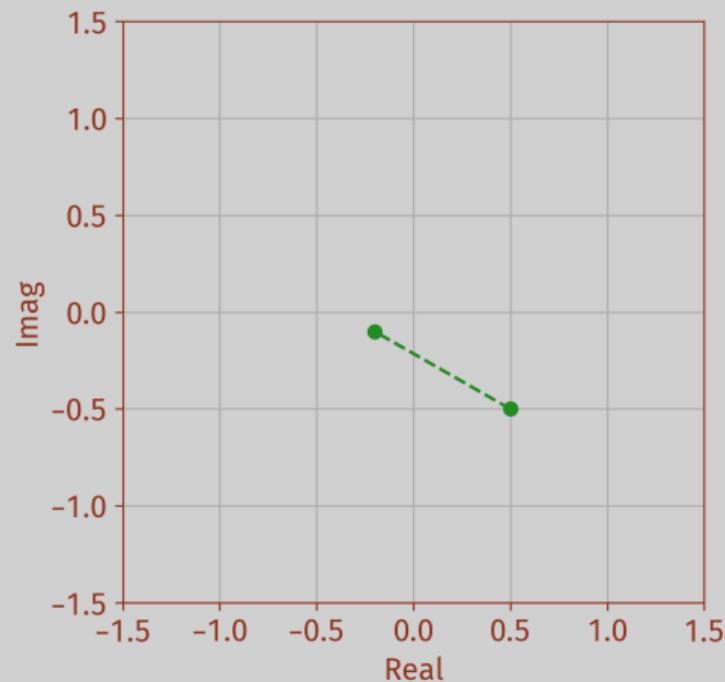
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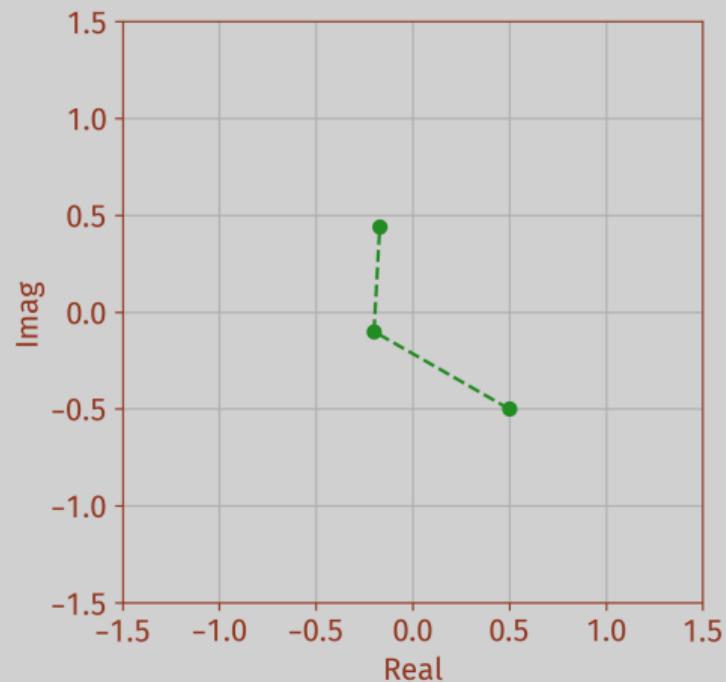
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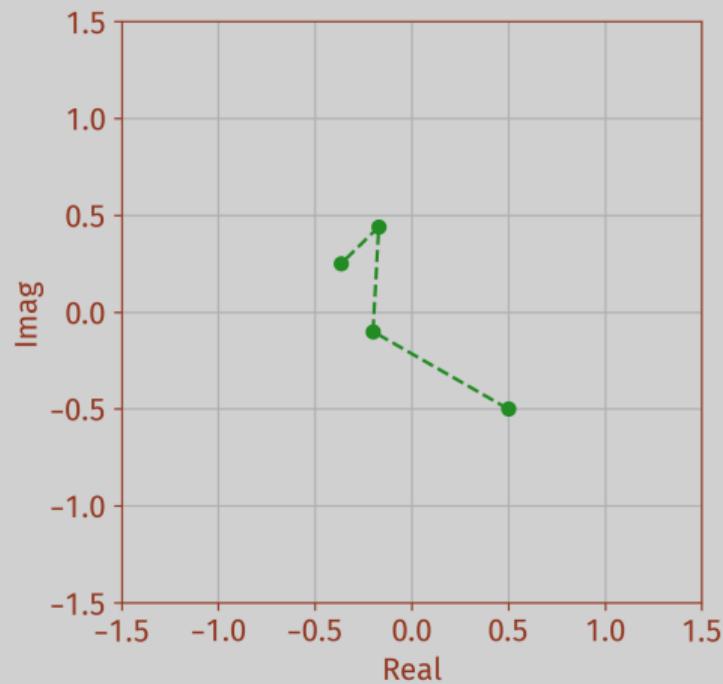
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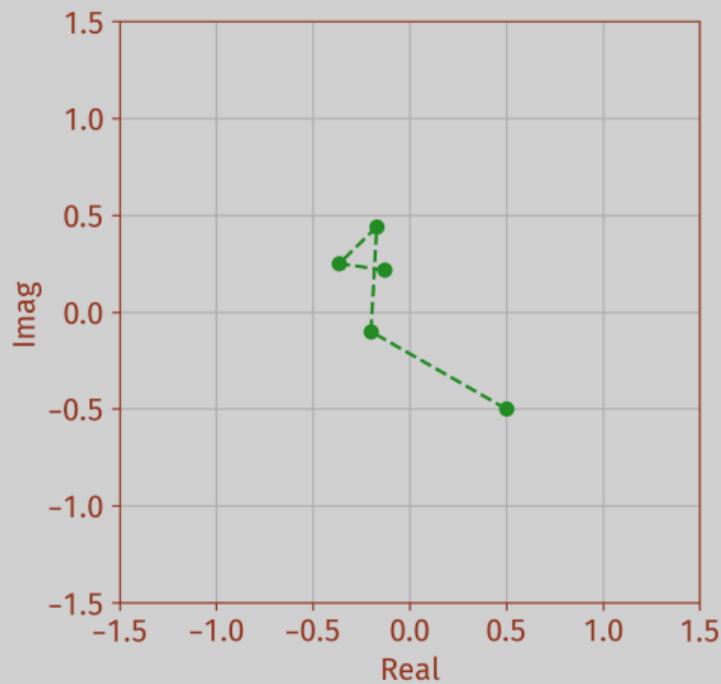
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- $f^4(z) = -0.12969407 + 0.21735824i$



IMPLEMENTATION

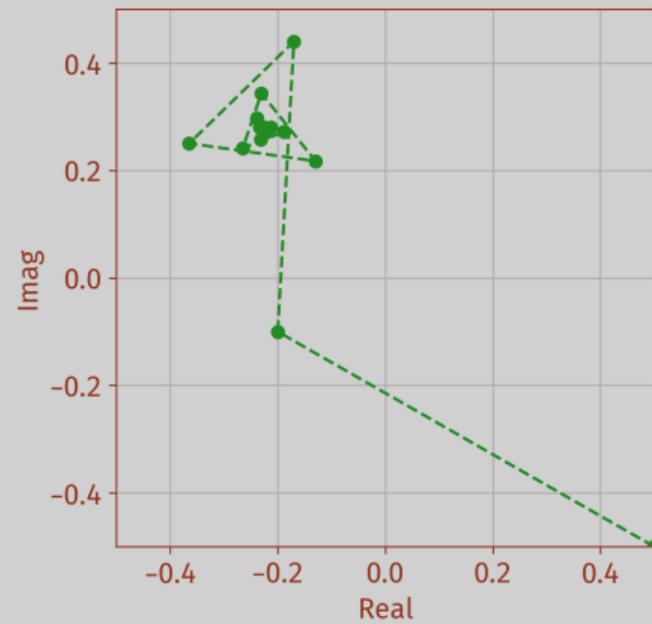
Iteration (Python)

```
1 N = 128
2 B = 4
3 c = complex(-0.2, 0.4)
4 def iterate(z):
5     for n in range(N):
6         z = z*z + c
7         if abs(z) > B*B: break
8     return n
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ITERATIVE FRACTALS

Complex Juila Set Example

Defined by iterative function in complex space

- $f_c(z) = z^2 - 0.675 - 0.112i$
- $K_c = \{z_0 \in \mathbb{C} : |f_c^k(z_0)| > B \text{ as } k \rightarrow \infty\}$

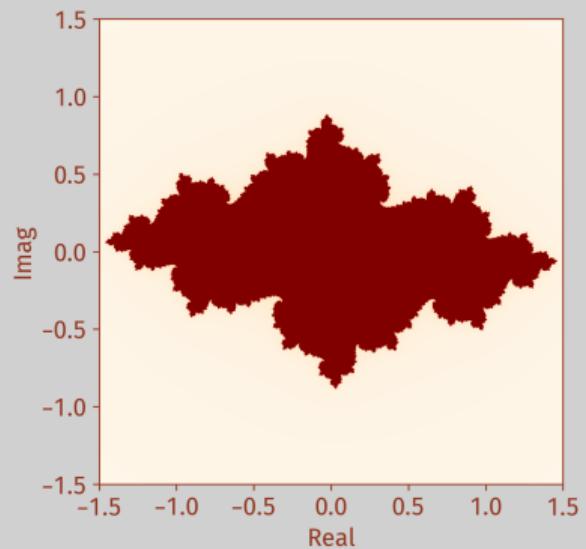


Figure: $f(z) = z^2 - 0.675 - 0.112i$

QUATERNIONS

QUATERNIONS HISTORY



Figure: Quaternion plaque on Brougham Bridge, Dublin
cc BY-SA Wikipedia - Cone83



Figure: Portrait of Sir William Rowan Hamilton
PD Wikipedia - Quibik

QUATERNIONS

Definition (Quaternion)

$$i^2 = j^2 = k^2 = ijk = -1$$

$$\{d + ai + bj + ck : a, b, c, d \in \mathbb{R}\} \in \mathbb{H}$$

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$$i = jk$$

■ $k^2 = ijk$

$$k^2k^{-1} = ijk k^{-1}$$

$$k = ij$$

■ $j = ki$

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$$k = ij$$

■ $j = ki$

■ $i = jk$

$$ji = jjk$$

$$ji = j^2k$$

$$ji = -k$$

$$-k = ji$$

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$$k = ij$$

■ $j = ki$

■ $i = jk$

$$ji = jjk$$

$$ji = j^2k$$

$$ji = -k$$

$$-k = ji$$

■ $-i = kj$

■ $-j = ik$

Let,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\begin{aligned} p * q &= dw + dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \\ &\quad + aw\mathbf{i} + ax\mathbf{i}^2 + ay\mathbf{ij} + az\mathbf{ik} \\ &\quad + bw\mathbf{j} + bx\mathbf{ji} + by\mathbf{j}^2 + bz\mathbf{jk} \\ &\quad + cw\mathbf{k} + cx\mathbf{ki} + cy\mathbf{kj} + cz\mathbf{k}^2 \end{aligned}$$

Let,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + ai + bj + ck$$

$$q = w + xi + yj + zk$$

$$\begin{aligned} p * q &= dw + dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k} \\ &\quad + aw\mathbf{i} + ax\mathbf{i}^2 + ay\mathbf{ij} + az\mathbf{ik} \\ &\quad + bw\mathbf{j} + bx\mathbf{ji} + by\mathbf{j}^2 + bz\mathbf{jk} \\ &\quad + cw\mathbf{k} + cx\mathbf{ki} + cy\mathbf{kj} + cz\mathbf{k}^2 \\ &= dw - ax - by - cz \\ &\quad + dx\mathbf{i} + aw\mathbf{i} + bz\mathbf{i} - cy\mathbf{i} \\ &\quad + dy\mathbf{j} - az\mathbf{j} + bw\mathbf{j} + cx\mathbf{j} \\ &\quad + dz\mathbf{k} + ay\mathbf{k} - bx\mathbf{k} + cw\mathbf{k} \end{aligned}$$

Let,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\begin{aligned} p * q &= dw - ax - by - cz \\ &\quad + dx\mathbf{i} + aw\mathbf{i} + bz\mathbf{i} - cy\mathbf{i} \\ &\quad + dy\mathbf{j} - az\mathbf{j} + bw\mathbf{j} + cx\mathbf{j} \\ &\quad + dz\mathbf{k} + ay\mathbf{k} - bx\mathbf{k} + cw\mathbf{k} \end{aligned}$$

Let,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\begin{aligned} p * q &= dw - (ax + by + cz) \\ &\quad + dx\mathbf{i} + aw\mathbf{i} + bzi - cy\mathbf{i} \\ &\quad + dy\mathbf{j} - az\mathbf{j} + bw\mathbf{j} + cx\mathbf{j} \\ &\quad + dz\mathbf{k} + ay\mathbf{k} - bx\mathbf{k} + cw\mathbf{k} \end{aligned}$$

Let,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\begin{aligned} p * q &= dw - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &\quad + dx\mathbf{i} + aw\mathbf{i} + bz\mathbf{i} - cy\mathbf{i} \\ &\quad + dy\mathbf{j} - az\mathbf{j} + bw\mathbf{j} + cx\mathbf{j} \\ &\quad + dz\mathbf{k} + ay\mathbf{k} - bx\mathbf{k} + cw\mathbf{k} \end{aligned}$$

Let,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\begin{aligned} p * q &= dw - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &\quad + \begin{pmatrix} dx + aw + bz - cy \\ dy - az + bw + cx \\ dz + ay - bx + cw \end{pmatrix} \cdot \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} \end{aligned}$$

Let,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + ai + bj + ck$$

$$q = w + xi + yj + zk$$

$$\begin{aligned} p * q &= dw - \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle \\ &\quad + \left\langle \begin{matrix} dx + aw + bz - cy \\ dy - az + bw + cx \\ dz + ay - bx + cw \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} i \\ j \\ k \end{matrix} \right\rangle \\ &= dw - \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle + \left(d \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle \dots \right) \cdot \left\langle \begin{matrix} i \\ j \\ k \end{matrix} \right\rangle \end{aligned}$$

Let,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\begin{aligned} p * q &= dw - \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle \\ &\quad + \left\langle \begin{matrix} dx + aw + bz - cy \\ dy - az + bw + cx \\ dz + ay - bx + cw \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{matrix} \right\rangle \\ &= dw - \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle + \left(d \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle + w \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \dots \right) \cdot \left\langle \begin{matrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{matrix} \right\rangle \end{aligned}$$

Let,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + ai + bj + ck$$

$$q = w + xi + yj + zk$$

$$\begin{aligned} p * q &= dw - \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle \\ &\quad + \left\langle \begin{matrix} dx + aw + bz - cy \\ dy - az + bw + cx \\ dz + ay - bx + cw \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} i \\ j \\ k \end{matrix} \right\rangle \\ &= dw - \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle + \left(d \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle + w \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle + \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \times \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle \right) \cdot \left\langle \begin{matrix} i \\ j \\ k \end{matrix} \right\rangle \end{aligned}$$

QUATERNION ITERATIVE METHODS

IMPLEMENTATION

Quaternion Multiplication

```
1 def q_mult(p, q):
2     r = Quat(
3         p.w*q.w - p.x*q.x - p.y*q.y - p.z*q.z,
4         p.w*q.x + p.x*q.w + p.y*q.z - p.z*q.y,
5         p.w*q.y - p.x*q.z + p.y*q.w + p.y*q.x,
6         p.w*q.z + p.x*q.y - p.y*q.x + p.z*q.w
7     )
8     return r
```

IMPLEMENTATION

Quaternion Square

```
1 def q_square(q):
2     r = Quat(
3         q.w*q.w - q.x*q.x - q.y*q.y - q.z*q.z,
4         2*q.w*q.x,
5         2*q.w*q.y,
6         2*q.w*q.z
7     )
8     return r
```

IMPLEMENTATION

Quaternion Add

```
1 def q_add(p, q):
2     r = Quat(
3         p.w + q.w,
4         p.x + q.x,
5         p.y + q.y,
6         p.z + q.z
7     )
8     return r
```

IMPLEMENTATION

Quaternion Add

```
1 def q_add(p, q):
2     r = Quat(
3         p.w + q.w,
4         p.x + q.x,
5         p.y + q.y,
6         p.z + q.z
7     )
8     return r
```

Quaternion Magnitude

```
1 def q_abs(q):
2     return q.w*q.w +
3             q.x*q.x +
4             q.y*q.y +
5             q.z*q.z
```

IMPLEMENTATION

Iteration

```
1 N = 12
2 B = 16
3 q = Quat(-0.2, 0.4, -0.4, -0.4)
4 def iterate(z):
5     for n in range(N):
6         z = q_add(q_square(z), q)
7         if q_abs(z) > B*B: break
8     return n
```

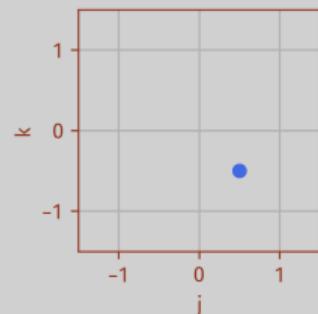
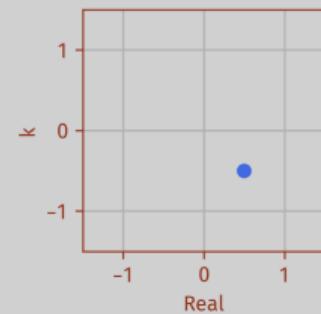
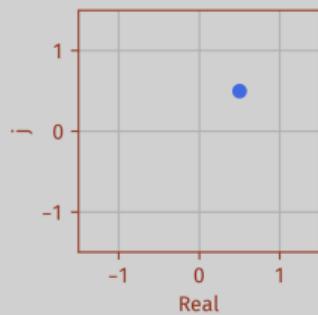
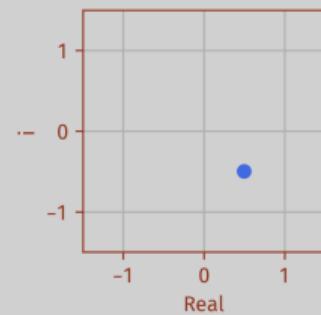
ITERATION

Rules

$$f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$$

$$z_0 = 0.5 - 0.5i + 0.5j - 0.5k$$

■ $f^0(z) = 0.5 - 0.5i + 0.5j - 0.5k$



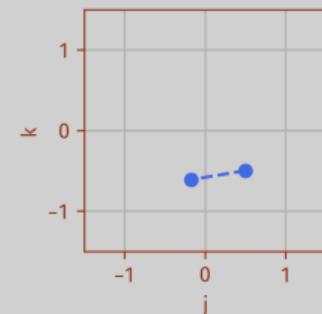
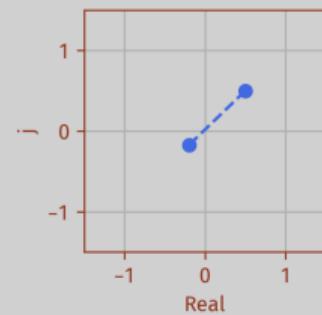
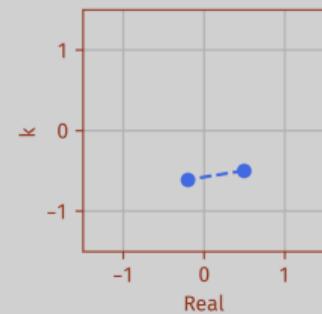
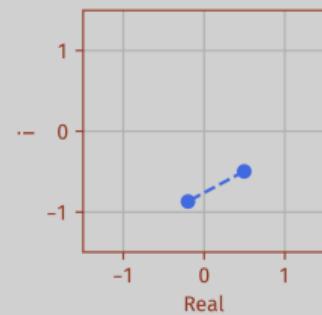
ITERATION

Rules

$$f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$$

$$z_0 = 0.5 - 0.5i + 0.5j - 0.5k$$

- $f^0(z) = 0.5 - 0.5i + 0.5j - 0.5k$
- $f^1(z) = -0.2 - 0.875i - 0.175j - 0.612k$



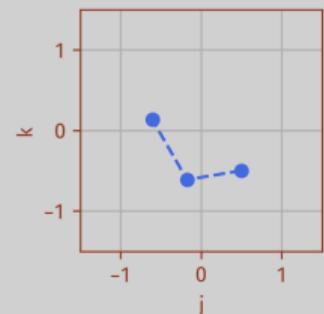
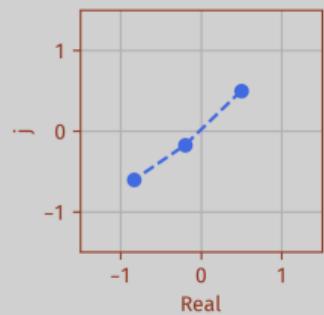
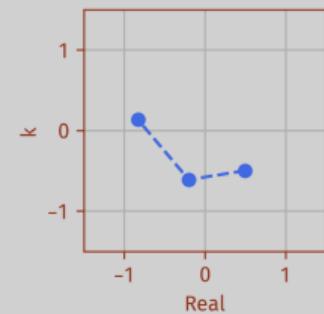
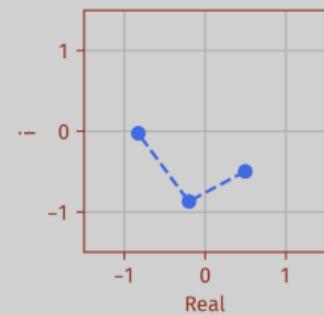
ITERATION

Rules

$$f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$$

$$z_0 = 0.5 - 0.5i + 0.5j - 0.5k$$

- $f^0(z) = 0.5 - 0.5i + 0.5j - 0.5k$
- $f^1(z) = -0.2 - 0.875i - 0.175j - 0.612k$
- $f^2(z) = -0.831 - 0.025i - 0.605j + 0.133k$



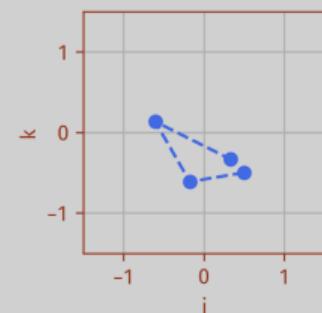
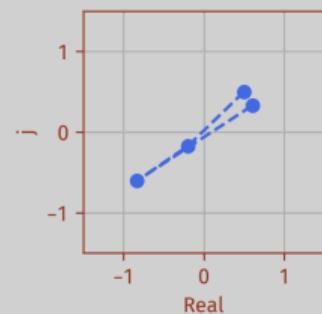
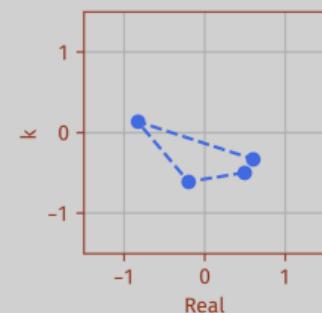
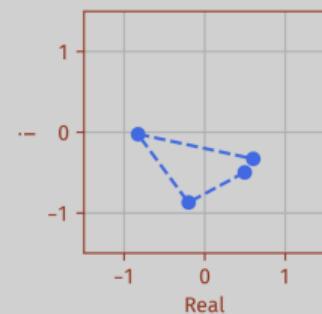
ITERATION

Rules

$$f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$$

$$z_0 = 0.5 - 0.5i + 0.5j - 0.5k$$

- $f^0(z) = 0.5 - 0.5i + 0.5j - 0.5k$
- $f^1(z) = -0.2 - 0.875i - 0.175j - 0.612k$
- $f^2(z) = -0.831 - 0.025i - 0.605j + 0.133k$
- $f^3(z) \approx 0.6066 - 0.333i + 0.330j - 0.333k$



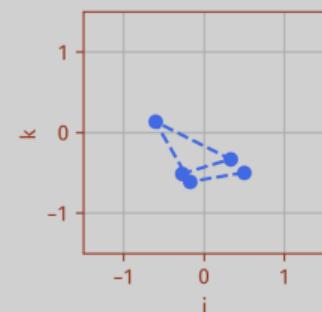
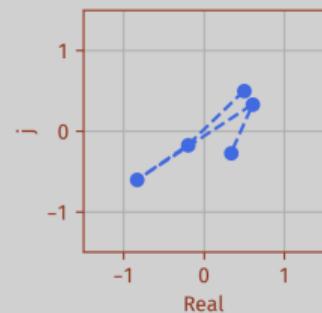
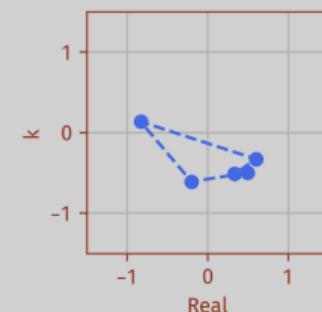
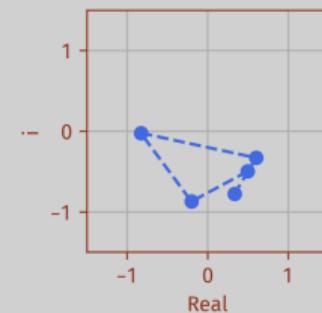
ITERATION

Rules

$$f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$$

$$z_0 = 0.5 - 0.5i + 0.5j - 0.5k$$

- $f^0(z) = 0.5 - 0.5i + 0.5j - 0.5k$
- $f^1(z) = -0.2 - 0.875i - 0.175j - 0.612k$
- $f^2(z) = -0.831 - 0.025i - 0.605j + 0.133k$
- $f^3(z) \approx 0.6066 - 0.333i + 0.330j - 0.333k$
- $f^4(z) \approx 0.336 - 0.779i - 0.274j - 0.515k$



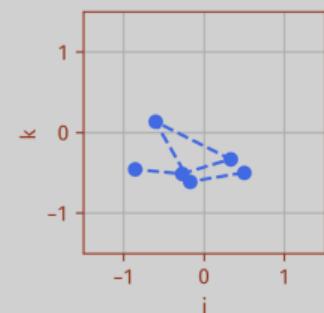
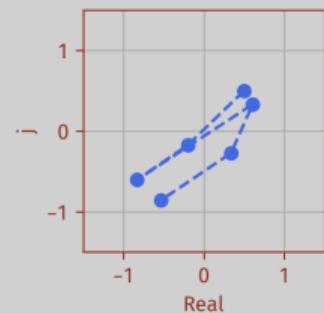
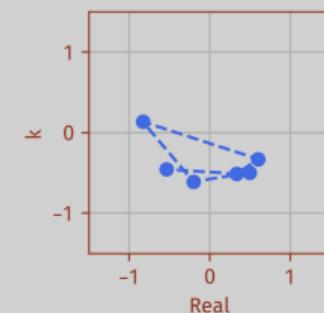
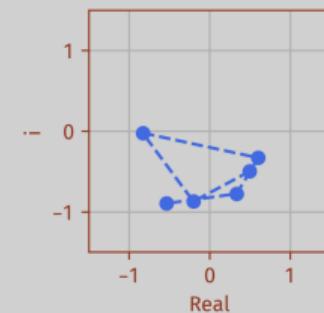
ITERATION

Rules

$$f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$$

$$z_0 = 0.5 - 0.5i + 0.5j - 0.5k$$

- $f^0(z) = 0.5 - 0.5i + 0.5j - 0.5k$
- $f^1(z) = -0.2 - 0.875i - 0.175j - 0.612k$
- $f^2(z) = -0.831 - 0.025i - 0.605j + 0.133k$
- $f^3(z) \approx 0.6066 - 0.333i + 0.330j - 0.333k$
- $f^4(z) \approx 0.336 - 0.779i - 0.274j - 0.515k$
- $f^5(z) \approx -0.535 - 0.899i - 0.860j - 0.458k$



PLOTTING

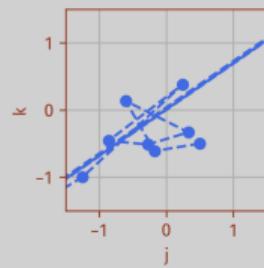
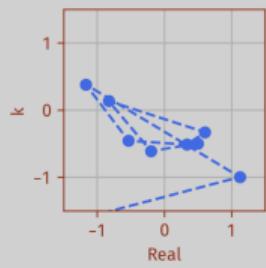
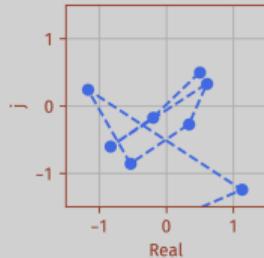
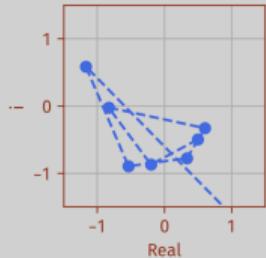


Figure: $f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$
 $f^{12}(0.5 - 0.5i + 0.5j - 0.5k)$

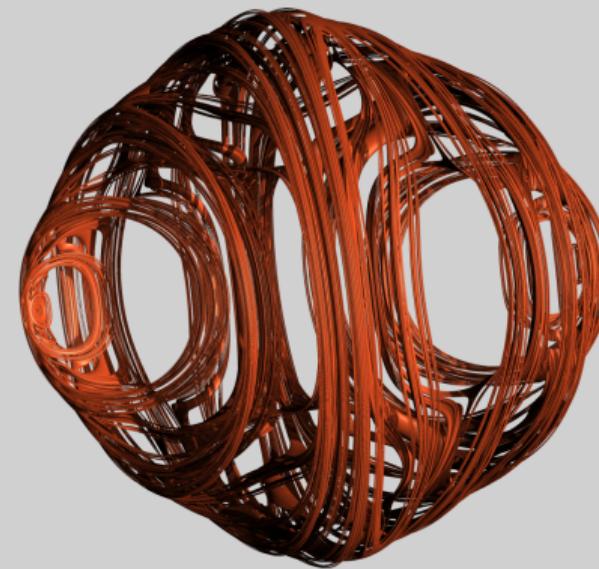


Figure: $f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$
Axis: Real, i, j

RAY TRACING

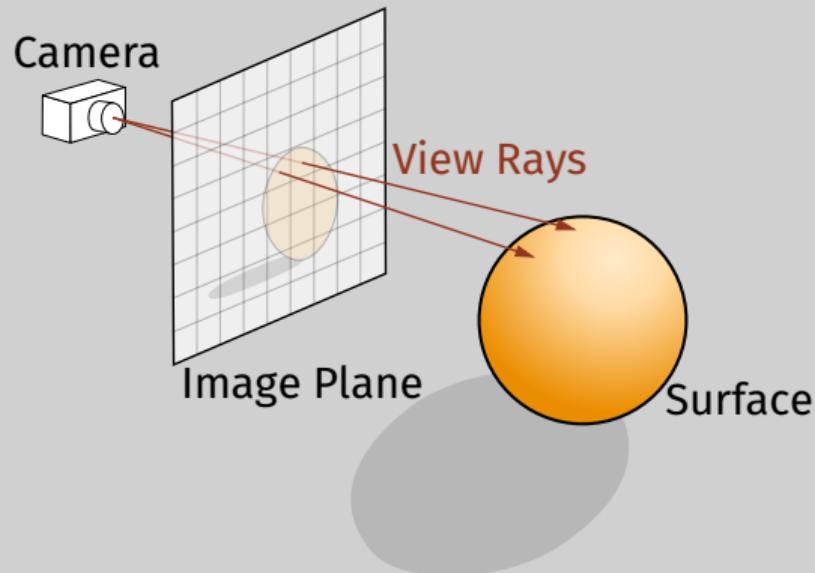


Figure: Ray Tracing Diagram
CC BY-SA Wikipedia - Henrik with modifications

RAY MARCHING

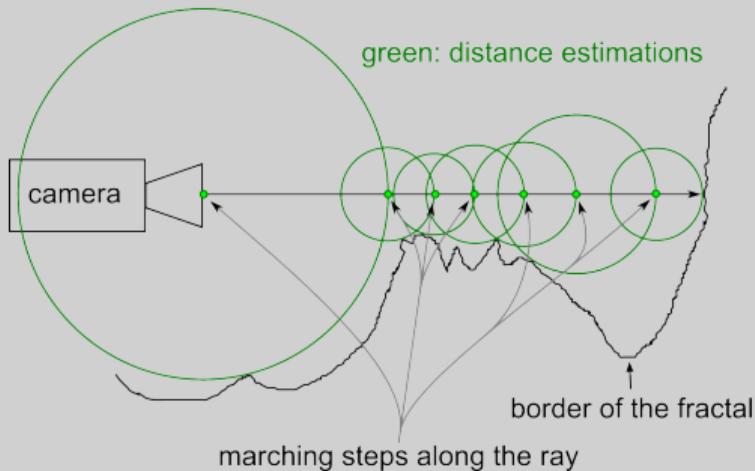


Figure: Ray Marching
©Adam Celarek

Theorem (Distance Estimation)

Let $f(z) = z^m + q$ where $q \in \mathbb{H}, m \in \mathbb{Z}^+$ be the iterative function of a Julia fractal. Then, the distance, δ , to the fractal can be approximated by

$$a \frac{z_n}{z'_n} < \delta$$

where $a \in \mathbb{R}$ is some constant coefficient.

NORMAL ESTIMATION

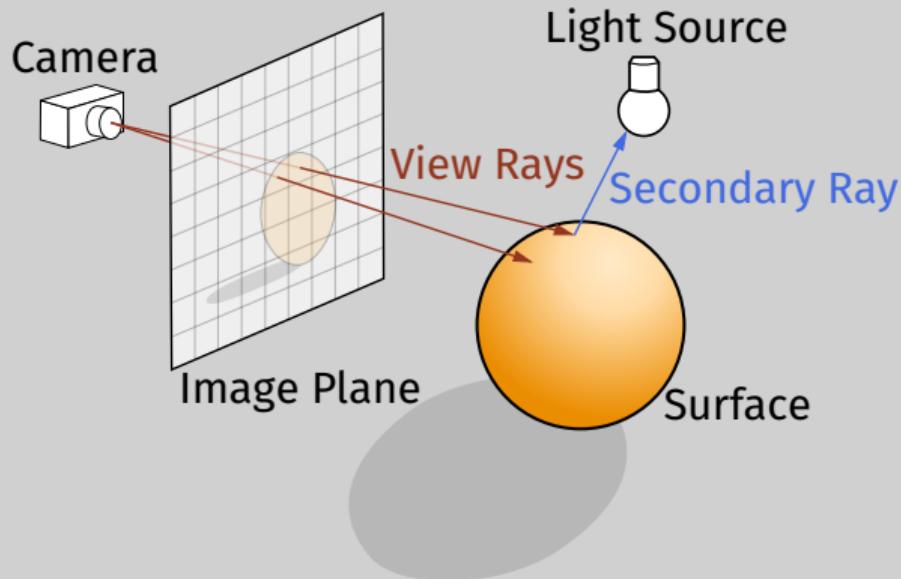


Figure: Ray Tracing Diagram
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QUATERNION ITERATIVE FRACTALS

Quaternion Juila Set Example

- Defined by iterative function in 4D Quaternion space

Figure: $f(z) = z^2 + 0.3 - 0.375\mathbf{i} - 0.675\mathbf{j} - 0.112\mathbf{k}$
Axis: Real, \mathbf{i}, \mathbf{j}
 $N = 1-11$

SUMMARY

1 Iteration

2 Complex Iterative Methods

3 Quaternions

4 Quaternion Iterative Methods

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QUESTIONS?

<https://github.com/scrufulufugus/senior-synthesis>

