Complex and Hypercomplex Iterative Methods

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Outline

Definition (Iterative Method)

A sequence where each value in the sequence is defined by the previous value.

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Definition (Dynamical System)

A system that enacts rules on a set of variables to produce a state.

Definitions

Definition (Iterative Method)

A sequence where each value in the sequence is defined by the previous value.

Definition (Dynamical System)

A system that enacts rules on a set of variables to produce a state.

Definition (Complex Dynamics)

The study of dynamical systems defined by complex functions.

Iteration

Definition (Function Iteration)

$$
\begin{aligned}
f^0 &:= \boldsymbol{I} \\
f^{k+1} &:= f \circ f^k
\end{aligned}
$$

Definition (Function Iteration)

$$f^0 := \boldsymbol{I}$$
$$f^{k+1} := f \circ f^k$$

Example

Given $f(x) = x + 1$,

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Example

Given $f(x) = x + 1$,

$$f^0(x) = x$$

Definition (Function Iteration)

$$f^0 := \boldsymbol{I}$$
$$f^{k+1} := f \circ f^k$$

Example

Given $f(x) = x + 1$,

$$f^0(x) = x$$
$$f^1(x) = x + 1$$

Definition (Function Iteration)

$$f^0 := \mathbf{I}$$
$$f^{k+1} := f \circ f^k$$

Example

Given $f(x) = x + 1$,

$$f^0(x) = x$$
$$f^1(x) = x + 1$$
$$f^2(x) = (x + 1) + 1$$

Definition (Function Iteration)

$f^0 := I$
$f^{k+1} := f \circ f^k$

Example

Given $f(x) = x + 1$,

$$
\begin{aligned}
f^0(x) &= x \\
f^1(x) &= x + 1 \\
f^2(x) &= (x + 1) + 1 \\
f^3(x) &= ((x + 1) + 1) + 1 \\
&\vdots
\end{aligned}
$$

Complex Iterative Methods

Definition (Complex Numbers)

$\boldsymbol{i}^2 = -1$

$\{a + b\boldsymbol{i} \;:\; a, b \in \mathbb{R}\} \in \mathbb{C}$

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Addition

Let $a, b, x, y \in \mathbb{R}$,

Definition (Complex Numbers)

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$\{a + b\boldsymbol{i} \; : \; a, b \in \mathbb{R}\} \in \mathbb{C}$

Addition

Let $a, b, x, y \in \mathbb{R}$,

$$(a + b\boldsymbol{i}) + (x + y\boldsymbol{i})$$

Definition (Complex Numbers)

$\boldsymbol{i}^2 = -1$

$\{a + b\boldsymbol{i} \ : \ a, b \in \mathbb{R}\} \in \mathbb{C}$

Addition

Let $a, b, x, y \in \mathbb{R}$,

$$(a + b\boldsymbol{i}) + (x + y\boldsymbol{i}) = (a + x) + (b + y)\boldsymbol{i}$$

Definition (Complex Numbers)

$\boldsymbol{i}^2 = -1$

$\{a + b\boldsymbol{i} \;:\; a, b \in \mathbb{R}\} \in \mathbb{C}$

Addition

Let $a, b, x, y \in \mathbb{R}$,

$$(a + b\boldsymbol{i}) + (x + y\boldsymbol{i}) = (a + x) + (b + y)\boldsymbol{i}$$

Multiplication

Let $a, b, x, y \in \mathbb{R}$,

Definition (Complex Numbers)

$\boldsymbol{i}^2 = -1$

$\{a + b\boldsymbol{i} \ : \ a, b \in \mathbb{R}\} \in \mathbb{C}$

Addition

Let $a, b, x, y \in \mathbb{R}$,

$$(a + b\boldsymbol{i}) + (x + y\boldsymbol{i}) = (a + x) + (b + y)\boldsymbol{i}$$

Multiplication

Let $a, b, x, y \in \mathbb{R}$,

$$(a + b\boldsymbol{i}) * (x + y\boldsymbol{i})$$

Definition (Complex Numbers)

$\boldsymbol{i}^2 = -1$

$\{a + b\boldsymbol{i} \ : \ a, b \in \mathbb{R}\} \in \mathbb{C}$

Addition

Let $a, b, x, y \in \mathbb{R}$,

$$(a + b\boldsymbol{i}) + (x + y\boldsymbol{i}) = (a + x) + (b + y)\boldsymbol{i}$$

Multiplication

Let $a, b, x, y \in \mathbb{R}$,

$$(a + b\boldsymbol{i}) * (x + y\boldsymbol{i}) = ax + ay\boldsymbol{i} + bx\boldsymbol{i} + by\boldsymbol{i}^2$$

Definition (Complex Numbers)

$\boldsymbol{i}^2 = -1$

$\{a + b\boldsymbol{i} \ : \ a, b \in \mathbb{R}\} \in \mathbb{C}$

Addition

Let $a, b, x, y \in \mathbb{R}$,

$$(a + b\boldsymbol{i}) + (x + y\boldsymbol{i}) = (a + x) + (b + y)\boldsymbol{i}$$

Multiplication

Let $a, b, x, y \in \mathbb{R}$,

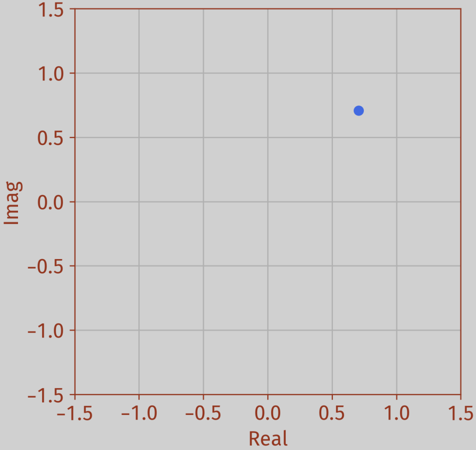
$$\begin{aligned}(a + b\boldsymbol{i}) * (x + y\boldsymbol{i}) &= ax + ay\boldsymbol{i} + bx\boldsymbol{i} + by\boldsymbol{i}^2 \\ &= (ax - by) + (ay + bx)\boldsymbol{i}\end{aligned}$$

Rules

$f(z) = z^2$

$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

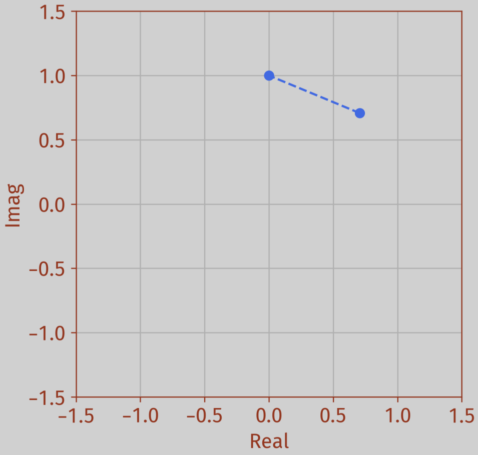
- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$



Rules

$f(z) = z^2$
$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

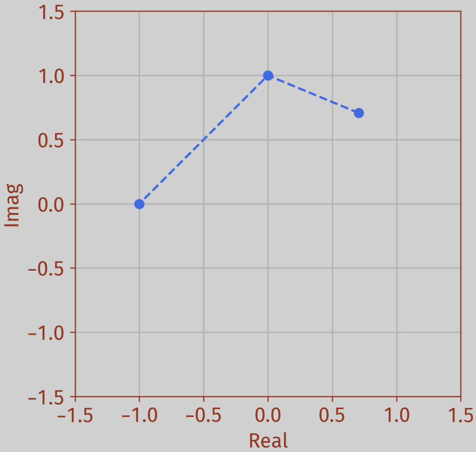
- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$
- $f^1(z) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}\right)^2 =$
  $\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 \boldsymbol{i} = \boldsymbol{i}$



Rules

$f(z) = z^2$
$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$
- $f^1(z) = \boldsymbol{i}$
- $f^2(z) = -1$



Rules

$f(z) = z^2$

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- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$
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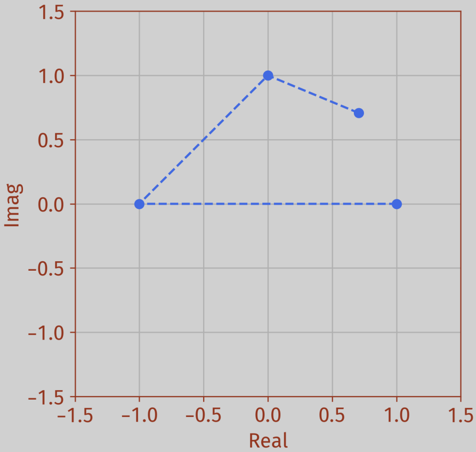
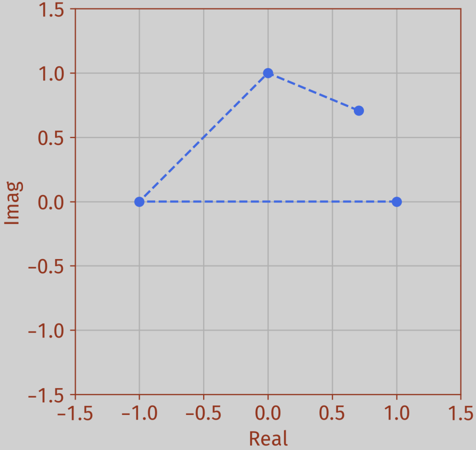
- $f^2(z) = -1$
- $f^3(z) = 1$



Rules

$f(z) = z^2$
$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$
- $f^1(z) = \boldsymbol{i}$
- $f^2(z) = -1$
- $f^3(z) = 1$
- $f^4(z) = f^5(z) = f^6(z) = 1$

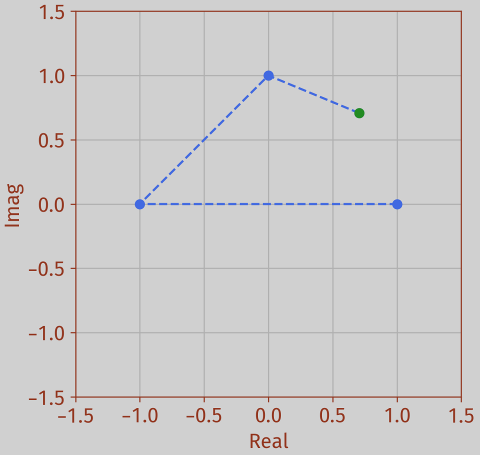


Rules

$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}\boldsymbol{i}$

$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

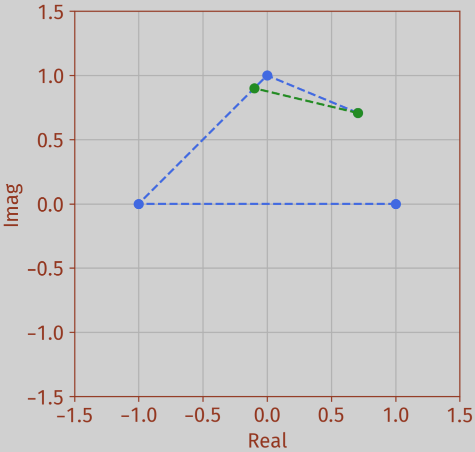


Rules

$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}\boldsymbol{i}$

$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$
- $f^1(z) = -\frac{1}{10} + \left(1 - \frac{1}{10}\right)\boldsymbol{i} = -0.1 + 0.9\boldsymbol{i}$

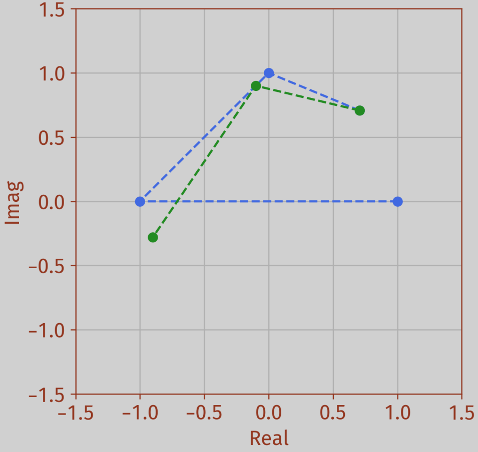


Rules

$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}\boldsymbol{i}$

$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$

- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$
- $f^1(z) = -0.1 + 0.9\boldsymbol{i}$
- $f^2(z) = -0.9 - 0.28\boldsymbol{i}$



Rules

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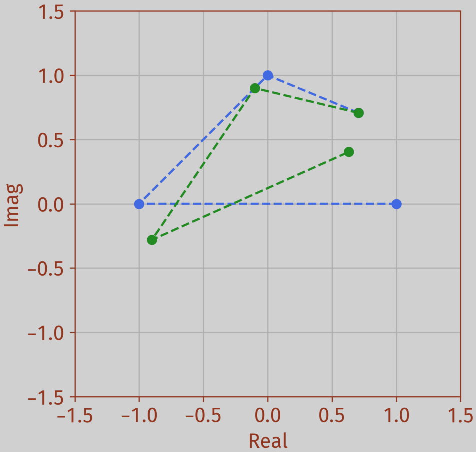
- $f^1(z) = -0.1 + 0.9\boldsymbol{i}$
- $f^2(z) = -0.9 - 0.28\boldsymbol{i}$
- $f^3(z) = 0.6316 + 0.404\boldsymbol{i}$



Rules

$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}\boldsymbol{i}$

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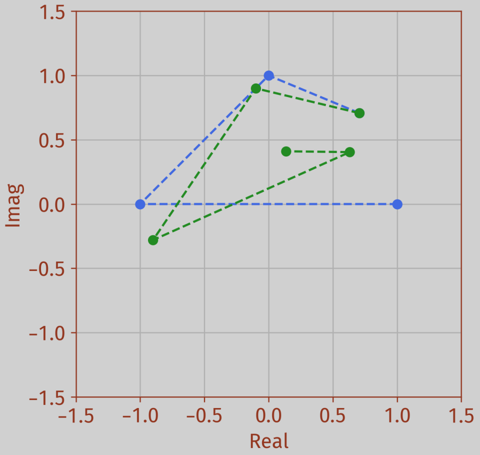
- $f^2(z) = -0.9 - 0.28\boldsymbol{i}$
- $f^3(z) = 0.6316 + 0.404\boldsymbol{i}$
- $f^4(z) \approx 0.13570256 + 0.4103328\boldsymbol{i}$



Rules

$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}\boldsymbol{i}$

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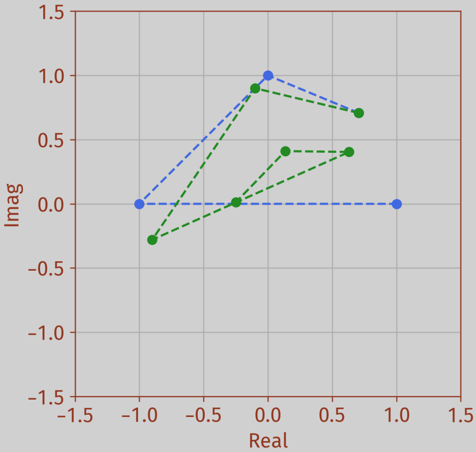
- $f^3(z) = 0.6316 + 0.404\boldsymbol{i}$
- $f^4(z) \approx 0.13570256 + 0.4103328\boldsymbol{i}$
- $f^5(z) \approx -0.24995782 + 0.01136642\boldsymbol{i}$



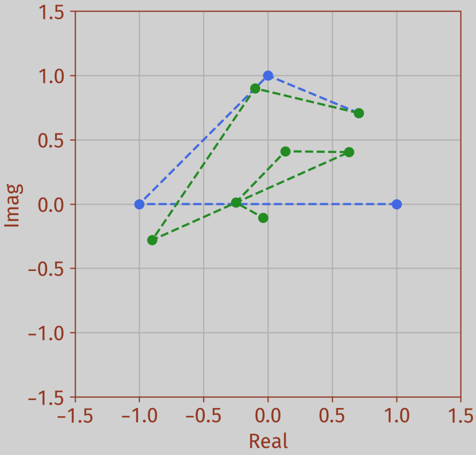
Complex Iteration

Rules

$$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}\boldsymbol{i}$$

$$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$$

- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\boldsymbol{i}$
- $f^1(z) = -0.1 + 0.9\boldsymbol{i}$
- $f^2(z) = -0.9 - 0.28\boldsymbol{i}$
- $f^3(z) = 0.6316 + 0.404\boldsymbol{i}$
- $f^4(z) \approx 0.13570256 + 0.4103328\boldsymbol{i}$
- $f^5(z) \approx -0.24995782 + 0.01136642\boldsymbol{i}$
- $f^6(z) \approx -0.03765028 - 0.10568225\boldsymbol{i}$



Group Activity

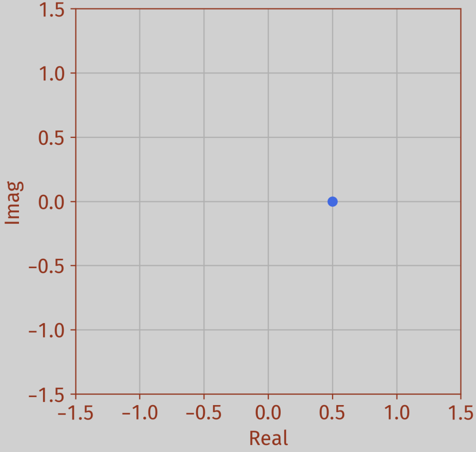
Easier	Harder
$f(z) = z^2 - 0.2 + 0\boldsymbol{i}$ $z_0 = 0.5 + 0\boldsymbol{i}$	$f(z) = z^2 - 0.2 + 0.4\boldsymbol{i}$ $z_0 = 0.5 - 0.5\boldsymbol{i}$

Rules

$f(z) = z^2 - 0.2 + 0\boldsymbol{i}$

$z_0 = 0.5 + 0\boldsymbol{i}$

- $f^0(z) = 0.5$

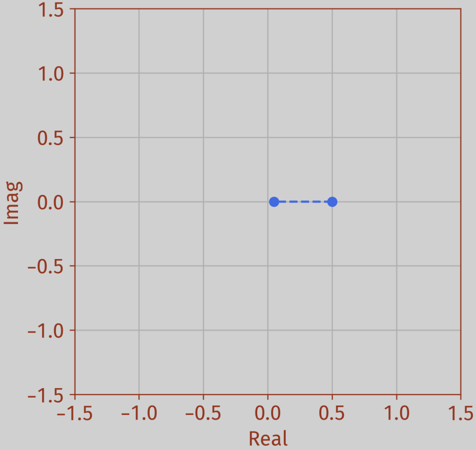


Rules

$f(z) = z^2 - 0.2 + 0\boldsymbol{i}$

$z_0 = 0.5 + 0\boldsymbol{i}$

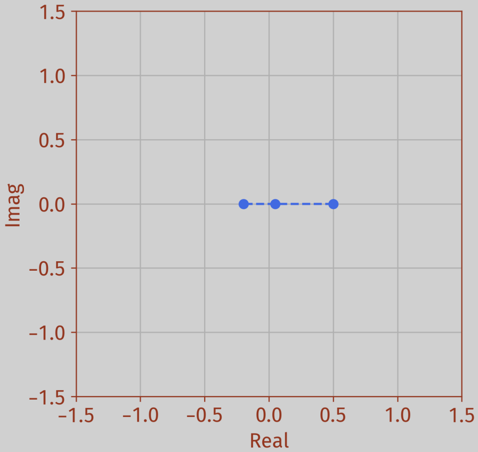
- $f^0(z) = 0.5$
- $f^1(z) = 0.05$



Rules

$f(z) = z^2 - 0.2 + 0\boldsymbol{i}$
$z_0 = 0.5 + 0\boldsymbol{i}$

- $f^0(z) = 0.5$
- $f^1(z) = 0.05$
- $f^2(z) = -0.1975$



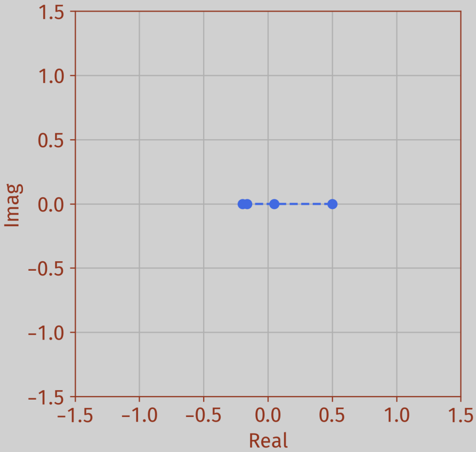
Group Activity (Easier)

Rules

$f(z) = z^2 - 0.2 + 0\boldsymbol{i}$

$z_0 = 0.5 + 0\boldsymbol{i}$

- $f^0(z) = 0.5$
- $f^1(z) = 0.05$
- $f^2(z) = -0.1975$
- $f^3(z) = -0.16099375$

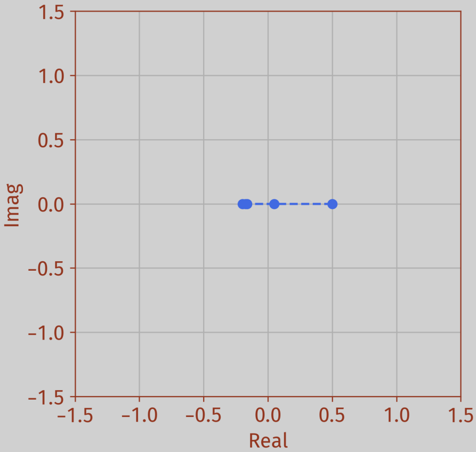


Group Activity (Easier)

Rules

$f(z) = z^2 - 0.2 + 0\boldsymbol{i}$
$z_0 = 0.5 + 0\boldsymbol{i}$

- $f^0(z) = 0.5$
- $f^1(z) = 0.05$
- $f^2(z) = -0.1975$
- $f^3(z) = -0.16099375$
- $f^4(z) \approx -0.1740810125$

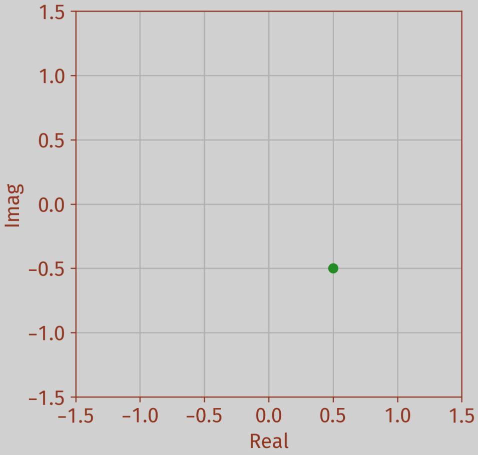


Rules

$f(z) = z^2 - 0.2 + 0.4\boldsymbol{i}$

$z_0 = 0.5 - 0.5\boldsymbol{i}$

- $f^0(z) = 0.5 - 0.5\boldsymbol{i}$



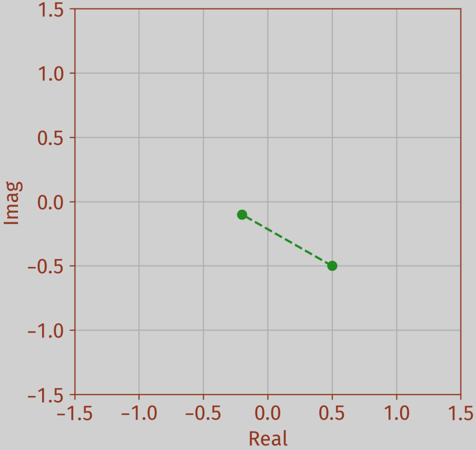
Group Activity (Harder)

Rules

$f(z) = z^2 - 0.2 + 0.4\boldsymbol{i}$

$z_0 = 0.5 - 0.5\boldsymbol{i}$

- $f^0(z) = 0.5 - 0.5\boldsymbol{i}$
- $f^1(z) = -0.2 - 0.1\boldsymbol{i}$

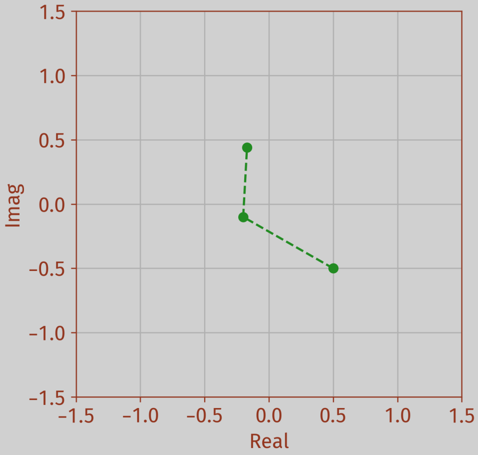


Rules

$f(z) = z^2 - 0.2 + 0.4\boldsymbol{i}$

$z_0 = 0.5 - 0.5\boldsymbol{i}$

- $f^0(z) = 0.5 - 0.5\boldsymbol{i}$
- $f^1(z) = -0.2 - 0.1\boldsymbol{i}$
- $f^2(z) = -0.17 + 0.44\boldsymbol{i}$



Rules

$f(z) = z^2 - 0.2 + 0.4\boldsymbol{i}$
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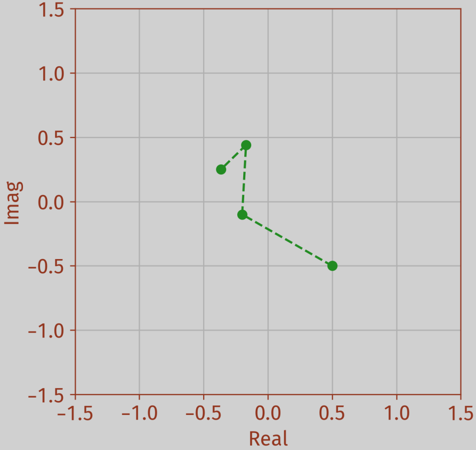
- $f^0(z) = 0.5 - 0.5\boldsymbol{i}$
- $f^1(z) = -0.2 - 0.1\boldsymbol{i}$
- $f^2(z) = -0.17 + 0.44\boldsymbol{i}$
- $f^3(z) = -0.3647 + 0.2504\boldsymbol{i}$



Group Activity (Harder)

Rules

$f(z) = z^2 - 0.2 + 0.4\boldsymbol{i}$
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- $f^0(z) = 0.5 - 0.5\boldsymbol{i}$
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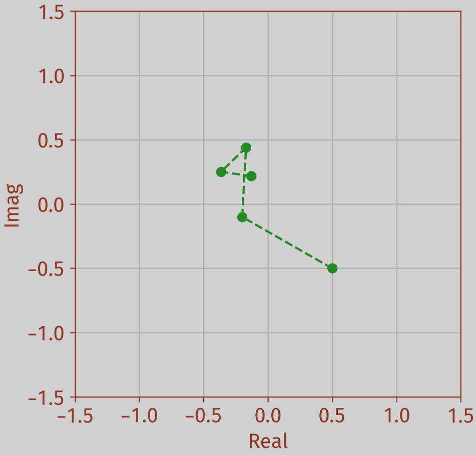
- $f^2(z) = -0.17 + 0.44\boldsymbol{i}$
- $f^3(z) = -0.3647 + 0.2504\boldsymbol{i}$
- $f^4(z) = -0.12969407 + 0.21735824\boldsymbol{i}$

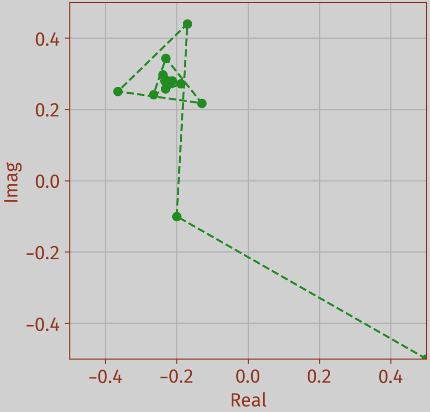


Iteration (Python)

```
N = 128
B = 4
c = complex(-0.2, 0.4)
def iterate(z):
    for n in range(N):
        z = z*z + c
        if abs(z) > B*B: break
    return n
```

Iteration (Python)

```python
N = 128
B = 4
c = complex(-0.2, 0.4)
def iterate(z):
    for n in range(N):
        z = z*z + c
        if abs(z) > B*B: break
    return n
```



Complex Juila Set Example

Defined by iterative function in complex space

- $f_c(z) = z^2 - 0.675 - 0.112\boldsymbol{i}$
- $K_c = \left\{z_0 \in \mathbb{C} : |f_c^k(z_0)| > B \text{ as } k \to \infty\right\}$

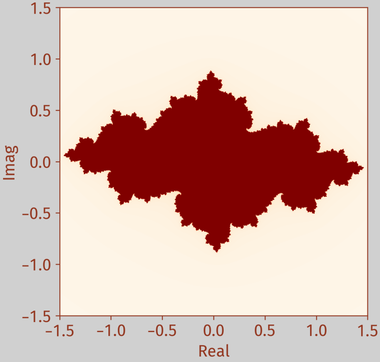


Figure: $f(z) = z^2 - 0.675 - 0.112\boldsymbol{i}$

Quaternions

Quaternions History

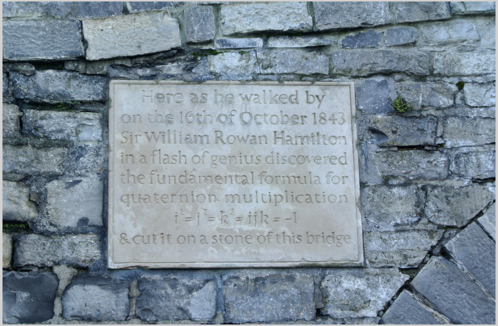


Figure: Quaternion plaque on Brougham Bridge, Dublin
🅭🅯🄎 Wikipedia - Cone83

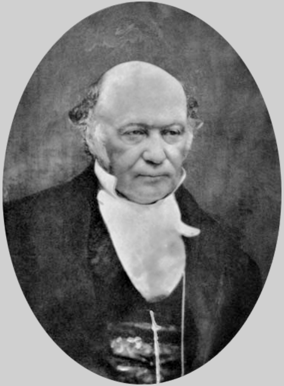


Figure: Portrait of Sir William Rowan Hamilton
🅮 Wikipedia - Quibik

Definition (Quaternion)

$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$

$\{d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} : a, b, c, d \in \mathbb{R}\} \in \mathbb{H}$

Definition (Quaternion)

$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$

$\{d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} : a, b, c, d \in \mathbb{R}\} \in \mathbb{H}$

- $\boldsymbol{i}^2 = \boldsymbol{ijk}$

$$\boldsymbol{i}^{-1}\boldsymbol{i}^2 = \boldsymbol{i}^{-1}\boldsymbol{ijk}$$
$$\boldsymbol{i} = \boldsymbol{jk}$$

Definition (Quaternion)

$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$

$\{d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} : a, b, c, d \in \mathbb{R}\} \in \mathbb{H}$

- $\boldsymbol{i}^2 = \boldsymbol{ijk}$

$$\boldsymbol{i}^{-1}\boldsymbol{i}^2 = \boldsymbol{i}^{-1}\boldsymbol{ijk}$$
$$\boldsymbol{i} = \boldsymbol{jk}$$

- $\boldsymbol{k}^2 = \boldsymbol{ijk}$

$$\boldsymbol{k}^2\boldsymbol{k}^{-1} = \boldsymbol{ijk}\boldsymbol{k}^{-1}$$
$$\boldsymbol{k} = \boldsymbol{ij}$$

- $\boldsymbol{j} = \boldsymbol{ki}$

Definition (Quaternion)

$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$

$\{d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} : a, b, c, d \in \mathbb{R}\} \in \mathbb{H}$

- $\boldsymbol{i}^2 = \boldsymbol{ijk}$

$$\boldsymbol{i}^{-1}\boldsymbol{i}^2 = \boldsymbol{i}^{-1}\boldsymbol{ijk}$$
$$\boldsymbol{i} = \boldsymbol{jk}$$

- $\boldsymbol{k}^2 = \boldsymbol{ijk}$

$$\boldsymbol{k}^2\boldsymbol{k}^{-1} = \boldsymbol{ijk}\boldsymbol{k}^{-1}$$
$$\boldsymbol{k} = \boldsymbol{ij}$$

- $\boldsymbol{j} = \boldsymbol{ki}$

- $\boldsymbol{i} = \boldsymbol{jk}$

$$\boldsymbol{ji} = \boldsymbol{jjk}$$
$$\boldsymbol{ji} = \boldsymbol{j}^2\boldsymbol{k}$$
$$\boldsymbol{ji} = -\boldsymbol{k}$$
$$-\boldsymbol{k} = \boldsymbol{ji}$$

Definition (Quaternion)

$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$

$\{d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k} : a, b, c, d \in \mathbb{R}\} \in \mathbb{H}$

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$$\boldsymbol{i} = \boldsymbol{jk}$$
- $\boldsymbol{k}^2 = \boldsymbol{ijk}$
$$\boldsymbol{k}^2\boldsymbol{k}^{-1} = \boldsymbol{ijk}\boldsymbol{k}^{-1}$$
$$\boldsymbol{k} = \boldsymbol{ij}$$
- $\boldsymbol{j} = \boldsymbol{ki}$
- $\boldsymbol{i} = \boldsymbol{jk}$
$$\boldsymbol{ji} = \boldsymbol{jjk}$$
$$\boldsymbol{ji} = \boldsymbol{j}^2\boldsymbol{k}$$
$$\boldsymbol{ji} = -\boldsymbol{k}$$
$$-\boldsymbol{k} = \boldsymbol{ji}$$
- $-\boldsymbol{i} = \boldsymbol{kj}$
- $-\boldsymbol{j} = \boldsymbol{ik}$

Let,

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$
$$p = d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$$
$$q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\begin{aligned} p * q = {} & dw + dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k} \\ & + aw\boldsymbol{i} + ax\boldsymbol{i}^2 + ay\boldsymbol{ij} + az\boldsymbol{ik} \\ & + bw\boldsymbol{j} + bx\boldsymbol{ji} + by\boldsymbol{j}^2 + bz\boldsymbol{jk} \\ & + cw\boldsymbol{k} + cx\boldsymbol{ki} + cy\boldsymbol{kj} + cz\boldsymbol{k}^2 \end{aligned}$$

Let,

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$
$$p = d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$$
$$q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\begin{aligned}
p * q = {} & dw + dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k} \\
& + aw\boldsymbol{i} + ax\boldsymbol{i}^2 + ay\boldsymbol{ij} + az\boldsymbol{ik} \\
& + bw\boldsymbol{j} + bx\boldsymbol{ji} + by\boldsymbol{j}^2 + bz\boldsymbol{jk} \\
& + cw\boldsymbol{k} + cx\boldsymbol{ki} + cy\boldsymbol{kj} + cz\boldsymbol{k}^2 \\
= {} & dw - ax - by - cz \\
& + dx\boldsymbol{i} + aw\boldsymbol{i} + bz\boldsymbol{i} - cy\boldsymbol{i} \\
& + dy\boldsymbol{j} - az\boldsymbol{j} + bw\boldsymbol{j} + cx\boldsymbol{j} \\
& + dz\boldsymbol{k} + ay\boldsymbol{k} - bx\boldsymbol{k} + cw\boldsymbol{k}
\end{aligned}$$

Let,

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$
$$p = d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$$
$$q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\begin{aligned} p * q = {} & dw - ax - by - cz \\ & + dx\boldsymbol{i} + aw\boldsymbol{i} + bz\boldsymbol{i} - cy\boldsymbol{i} \\ & + dy\boldsymbol{j} - az\boldsymbol{j} + bw\boldsymbol{j} + cx\boldsymbol{j} \\ & + dz\boldsymbol{k} + ay\boldsymbol{k} - bx\boldsymbol{k} + cw\boldsymbol{k} \end{aligned}$$

Let,

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$
$$p = d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$$
$$q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\begin{aligned} p * q = dw &- (ax + by + cz) \\ &+ dx\boldsymbol{i} + aw\boldsymbol{i} + bz\boldsymbol{i} - cy\boldsymbol{i} \\ &+ dy\boldsymbol{j} - az\boldsymbol{j} + bw\boldsymbol{j} + cx\boldsymbol{j} \\ &+ dz\boldsymbol{k} + ay\boldsymbol{k} - bx\boldsymbol{k} + cw\boldsymbol{k} \end{aligned}$$

Let,

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$
$$p = d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$$
$$q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\begin{aligned} p * q = dw - \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle \\ + dx\boldsymbol{i} + aw\boldsymbol{i} + bz\boldsymbol{i} - cy\boldsymbol{i} \\ + dy\boldsymbol{j} - az\boldsymbol{j} + bw\boldsymbol{j} + cx\boldsymbol{j} \\ + dz\boldsymbol{k} + ay\boldsymbol{k} - bx\boldsymbol{k} + cw\boldsymbol{k} \end{aligned}$$

Let,

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$
$$p = d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$$
$$q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$p * q = dw - \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle + \left\langle \begin{matrix} dx + aw + bz - cy \\ dy - az + bw + cx \\ dz + ay - bx + cw \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{matrix} \right\rangle$$

Let,

$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$

$p = d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$

$q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

$$\begin{aligned} p * q &= dw - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &+ \begin{pmatrix} dx + aw + bz - cy \\ dy - az + bw + cx \\ dz + ay - bx + cw \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} \\ &= dw - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \left(d \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdots \right) \cdot \begin{pmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{pmatrix} \end{aligned}$$

Let,

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$
$$p = d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$$
$$q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\begin{aligned} p * q &= dw - \left\langle \begin{smallmatrix} a \\ b \\ c \end{smallmatrix} \right\rangle \cdot \left\langle \begin{smallmatrix} x \\ y \\ z \end{smallmatrix} \right\rangle \\ &+ \left\langle \begin{matrix} dx + aw + bz - cy \\ dy - az + bw + cx \\ dz + ay - bx + cw \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{matrix} \right\rangle \\ &= dw - \left\langle \begin{smallmatrix} a \\ b \\ c \end{smallmatrix} \right\rangle \cdot \left\langle \begin{smallmatrix} x \\ y \\ z \end{smallmatrix} \right\rangle + \left(d \left\langle \begin{smallmatrix} x \\ y \\ z \end{smallmatrix} \right\rangle + w \left\langle \begin{smallmatrix} a \\ b \\ c \end{smallmatrix} \right\rangle \cdots \right) \cdot \left\langle \begin{smallmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{smallmatrix} \right\rangle \end{aligned}$$

Let,

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$
$$p = d + a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$$
$$q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

$$\begin{aligned} p * q &= dw - \left\langle \begin{smallmatrix} a \\ b \\ c \end{smallmatrix} \right\rangle \cdot \left\langle \begin{smallmatrix} x \\ y \\ z \end{smallmatrix} \right\rangle \\ &\quad + \left\langle \begin{matrix} dx + aw + bz - cy \\ dy - az + bw + cx \\ dz + ay - bx + cw \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{matrix} \right\rangle \\ &= dw - \left\langle \begin{smallmatrix} a \\ b \\ c \end{smallmatrix} \right\rangle \cdot \left\langle \begin{smallmatrix} x \\ y \\ z \end{smallmatrix} \right\rangle + \left(d \left\langle \begin{smallmatrix} x \\ y \\ z \end{smallmatrix} \right\rangle + w \left\langle \begin{smallmatrix} a \\ b \\ c \end{smallmatrix} \right\rangle + \left\langle \begin{smallmatrix} a \\ b \\ c \end{smallmatrix} \right\rangle \times \left\langle \begin{smallmatrix} x \\ y \\ z \end{smallmatrix} \right\rangle \right) \cdot \left\langle \begin{smallmatrix} \boldsymbol{i} \\ \boldsymbol{j} \\ \boldsymbol{k} \end{smallmatrix} \right\rangle \end{aligned}$$

Quaternion Iterative Methods

Implementation

Quaternion Multiplication

```
def q_mult(p, q):
    r = Quat(
        p.r*q.r - p.i*q.i - p.j*q.j - p.k*q.k,
        p.r*q.i + p.i*q.r + p.j*q.k - p.k*q.j,
        p.r*q.j - p.i*q.k + p.j*q.r + p.j*q.i,
        p.r*q.k + p.i*q.j - p.j*q.i + p.k*q.r
    )
    return r
```

Quaternion Square

```
def q_square(q):
    r = Quat(
        q.r*q.r - q.i*q.i - q.j*q.j - q.k*q.k,
        2*q.r*q.i,
        2*q.r*q.j,
        2*q.r*q.k
    )
    return r
```

Quaternion Add

```
def q_add(p, q):
    r = Quat(
        p.r + q.r,
        p.i + q.i,
        p.j + q.j,
        p.k + q.k
    )
    return r
```

Iteration

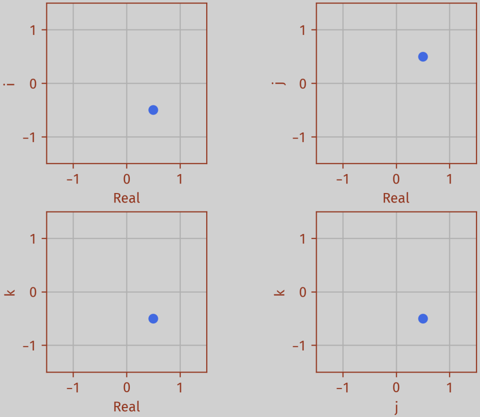
```
N = 12
B = 16
q = Quat(-0.2, 0.4, -0.4, -0.4)
def iterate(z):
    for n in range(N):
        z = q_add(q_square(z), q)
        if q_abs(z) > B*B: break
    return n
```

Rules

$f(z) = z^2 + 0.3 - 0.375\boldsymbol{i} - 0.675\boldsymbol{j} - 0.112\boldsymbol{k}$

$z_0 = 0.5 - 0.5\boldsymbol{i} + 0.5\boldsymbol{j} - 0.5\boldsymbol{k}$

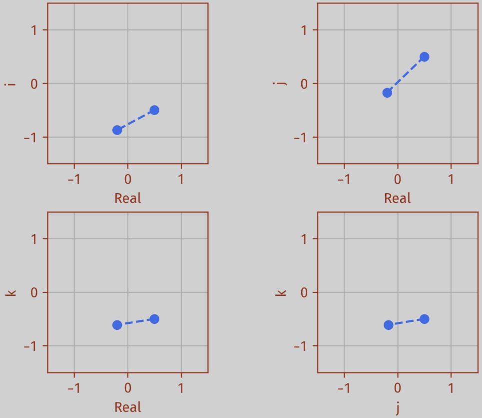
- $f^0(z) = 0.5 - 0.5\boldsymbol{i} + 0.5\boldsymbol{j} - 0.5\boldsymbol{k}$



Rules

$f(z) = z^2 + 0.3 - 0.375\boldsymbol{i} - 0.675\boldsymbol{j} - 0.112\boldsymbol{k}$
$z_0 = 0.5 - 0.5\boldsymbol{i} + 0.5\boldsymbol{j} - 0.5\boldsymbol{k}$

- $f^0(z) = 0.5 - 0.5\boldsymbol{i} + 0.5\boldsymbol{j} - 0.5\boldsymbol{k}$
- $f^1(z) = -0.2 - 0.875\boldsymbol{i} - 0.175\boldsymbol{j} - 0.612\boldsymbol{k}$

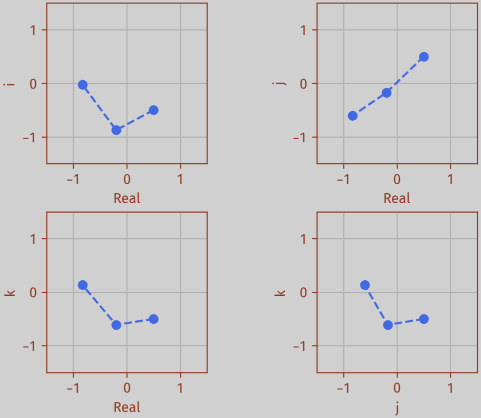


Rules

$f(z) = z^2 + 0.3 - 0.375\boldsymbol{i} - 0.675\boldsymbol{j} - 0.112\boldsymbol{k}$

$z_0 = 0.5 - 0.5\boldsymbol{i} + 0.5\boldsymbol{j} - 0.5\boldsymbol{k}$

- $f^0(z) = 0.5 - 0.5\boldsymbol{i} + 0.5\boldsymbol{j} - 0.5\boldsymbol{k}$
- $f^1(z) = -0.2 - 0.875\boldsymbol{i} - 0.175\boldsymbol{j} - 0.612\boldsymbol{k}$
- $f^2(z) = -0.831 - 0.025\boldsymbol{i} - 0.605\boldsymbol{j} + 0.133\boldsymbol{k}$



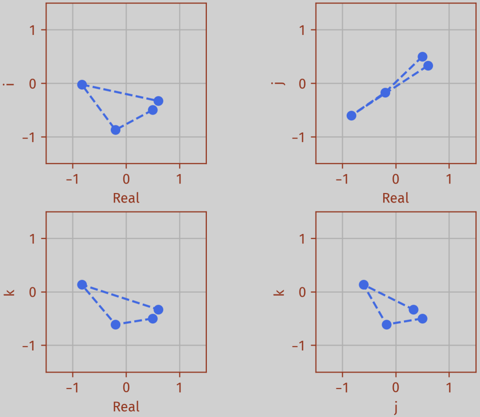
Iteration

Rules

$f(z) = z^2 + 0.3 - 0.375\boldsymbol{i} - 0.675\boldsymbol{j} - 0.112\boldsymbol{k}$

$z_0 = 0.5 - 0.5\boldsymbol{i} + 0.5\boldsymbol{j} - 0.5\boldsymbol{k}$

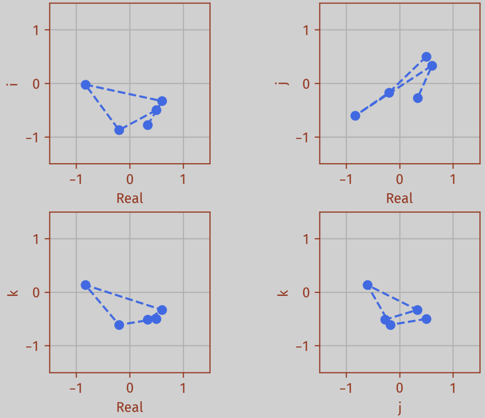
- $f^0(z) = 0.5 - 0.5\boldsymbol{i} + 0.5\boldsymbol{j} - 0.5\boldsymbol{k}$
- $f^1(z) = -0.2 - 0.875\boldsymbol{i} - 0.175\boldsymbol{j} - 0.612\boldsymbol{k}$
- $f^2(z) = -0.831 - 0.025\boldsymbol{i} - 0.605\boldsymbol{j} + 0.133\boldsymbol{k}$
- $f^3(z) \approx 0.6066 - 0.333\boldsymbol{i} + 0.330\boldsymbol{j} - 0.333\boldsymbol{k}$



Rules

$$f(z) = z^2 + 0.3 - 0.375\boldsymbol{i} - 0.675\boldsymbol{j} - 0.112\boldsymbol{k}$$
$$z_0 = 0.5 - 0.5\boldsymbol{i} + 0.5\boldsymbol{j} - 0.5\boldsymbol{k}$$

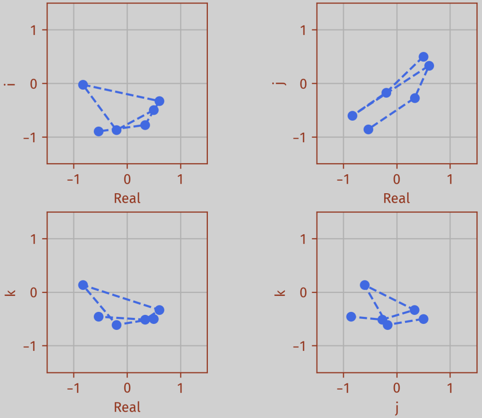
- $f^0(z) = 0.5 - 0.5\boldsymbol{i} + 0.5\boldsymbol{j} - 0.5\boldsymbol{k}$
- $f^1(z) = -0.2 - 0.875\boldsymbol{i} - 0.175\boldsymbol{j} - 0.612\boldsymbol{k}$
- $f^2(z) = -0.831 - 0.025\boldsymbol{i} - 0.605\boldsymbol{j} + 0.133\boldsymbol{k}$
- $f^3(z) \approx 0.6066 - 0.333\boldsymbol{i} + 0.330\boldsymbol{j} - 0.333\boldsymbol{k}$
- $f^4(z) \approx 0.336 - 0.779\boldsymbol{i} - 0.274\boldsymbol{j} - 0.515\boldsymbol{k}$



Rules

$f(z) = z^2 + 0.3 - 0.375\boldsymbol{i} - 0.675\boldsymbol{j} - 0.112\boldsymbol{k}$
$z_0 = 0.5 - 0.5\boldsymbol{i} + 0.5\boldsymbol{j} - 0.5\boldsymbol{k}$

- $f^0(z) = 0.5 - 0.5\boldsymbol{i} + 0.5\boldsymbol{j} - 0.5\boldsymbol{k}$
- $f^1(z) = -0.2 - 0.875\boldsymbol{i} - 0.175\boldsymbol{j} - 0.612\boldsymbol{k}$
- $f^2(z) = -0.831 - 0.025\boldsymbol{i} - 0.605\boldsymbol{j} + 0.133\boldsymbol{k}$
- $f^3(z) \approx 0.6066 - 0.333\boldsymbol{i} + 0.330\boldsymbol{j} - 0.333\boldsymbol{k}$
- $f^4(z) \approx 0.336 - 0.779\boldsymbol{i} - 0.274\boldsymbol{j} - 0.515\boldsymbol{k}$
- $f^5(z) \approx -0.535 - 0.899\boldsymbol{i} - 0.860\boldsymbol{j} - 0.458\boldsymbol{k}$



Plotting

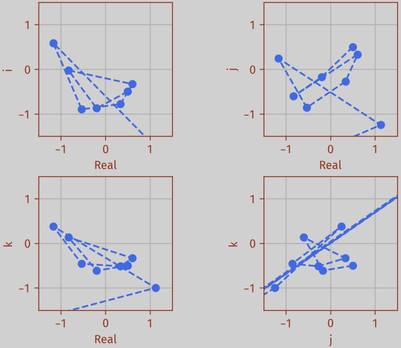


Figure: $f(z) = z^2 + 0.3 - 0.375\boldsymbol{i} - 0.675\boldsymbol{j} - 0.112\boldsymbol{k}$
$f^{12}(0.5 - 0.5\boldsymbol{i} + 0.5\boldsymbol{j} - 0.5\boldsymbol{k})$

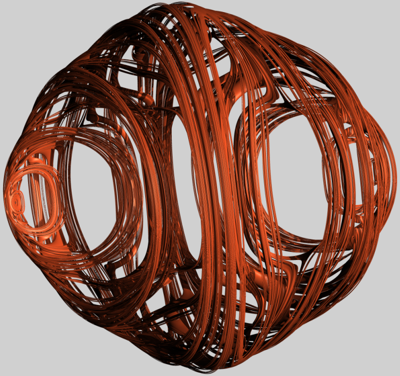


Figure: $f(z) = z^2 + 0.3 - 0.375\boldsymbol{i} - 0.675\boldsymbol{j} - 0.112\boldsymbol{k}$
Axis: Real, $\boldsymbol{i}, \boldsymbol{j}$

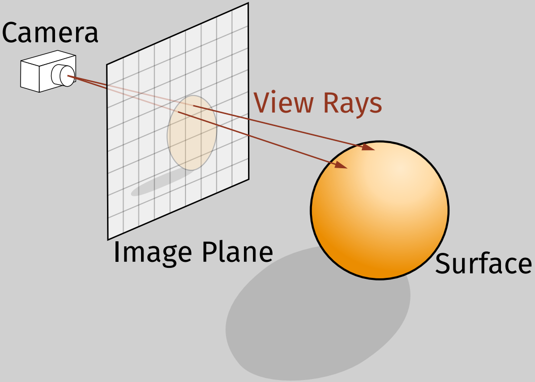


Figure: Ray Tracing Diagram
Wikipedia - Henrik with modifications

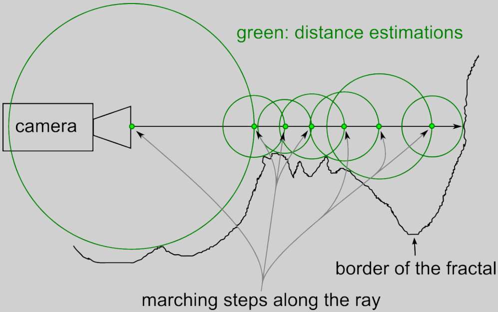


Figure: Ray Marching
©Adam Celarek

Theorem (Distance Estimation)

Let $f(z) = z^m + q$ where $q \in \mathbb{H}, m \in \mathbb{Z}^+$ be the iterative function of a Julia fractal. Then, the distance, δ, to the fractal can be approximated by

$$a\frac{z_n}{z'_n} < \delta$$

where $a \in \mathbb{R}$ is some constant coefficient.

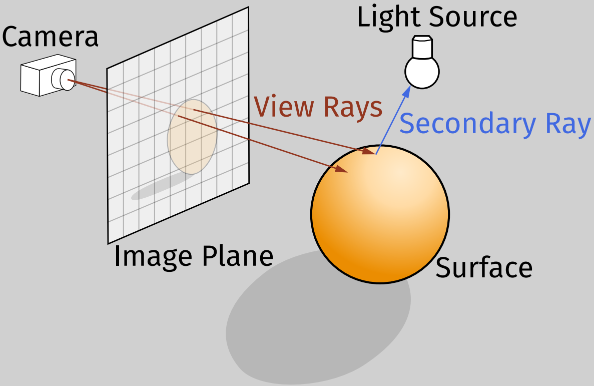


Figure: Ray Tracing Diagram
Wikipedia - Henrik with modifications

Quaternion Juila Set Example

- Defined by iterative function in 4D Quaternion space

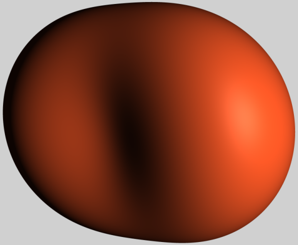


Figure: $f(z) = z^2 + 0.3 - 0.375\boldsymbol{i} - 0.675\boldsymbol{j} - 0.112\boldsymbol{k}$
Axis: Real, $\boldsymbol{i}, \boldsymbol{j}$
N = 1-11

Summary

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Accessed: 2023-02-10.

Questions?

https://github.com/scrufulufugus/senior-synthesis