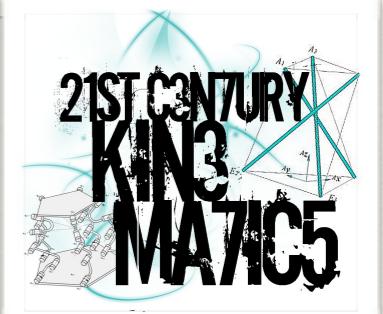


J. Michael McCarthy

Type Synthesis

Gruebler's Equation, Assur
Groups, Baranov Trusses,
Graph Theory and Rigidity
Theory

Simple Machines



- ❖ The classification of machines was introduced during the Renaissance and was based on Hero of Alexandria's Mechanics which identified five lift mechanisms (Usher, Abbott Payson (1988). A History of Mechanical Inventions. USA: Courier Dover Publications.):

the lever, windlass, pulley, wedge and screw.

- ❖ These five simple machines are taught in elementary school, with the idea that more complicated machines, often called compound machines, are constructed from these elements.

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D.3 The Six Types of Simple Machines

Wheel & Axle - Makes work easier by moving objects across distances. The wheel (or round end) turns with the axle (or cylindrical post) causing movement. On a wagon, for example, a container rests on top of the axle.

Inclined Plane - A flat surface (or plane) that is slanted, or inclined, so it can help move objects across distances. A common inclined plane is a ramp.

Wedge - Instead of using the smooth side of the inclined plane to make work easier, you can also use the pointed edges to do other kinds of work. When you use the edge to push things apart, this movable inclined plane is called a wedge. An ax blade is one example of a wedge.

Lever - Any tool that prys something loose is a lever. Levers can also lift objects. A lever is an arm that "pivots" (or turns) against a fulcrum (the point or support on which a lever pivots). Think of the claw end of a hammer that you use to pry nails loose; it's a lever. A see-saw is also a lever.

Pulley - Instead of an axle, a wheel could also rotate a rope, cord, or belt. This variation of the wheel and axle is the pulley. In a pulley, a cord wraps around a wheel. As the wheel rotates, the cord moves in either direction. Attach a hook to the cord, and now you can use the wheel's rotation to raise and lower objects, making work easier. On a flagpole, for example, a rope is attached to a pulley to raise and lower the flag more easily.

Screw - When you wrap an inclined plane around a cylinder, its sharp edge becomes another simple tool: a screw. If you put a metal screw beside a ramp, it may be hard to see similarities, but a screw is actually just another kind of inclined plane. One example of how a screw helps you do work is that it can be easily turned to move itself through a solid space like a block of wood.

Wheel and Axle

Inclined Plane

Wedge

Lever

Pulley

Screw

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Machine Catalogs



- The inclined plan was added to this list during the Renaissance, and since that time catalogs of special machines were generated through the 1800's. (The figure shows a page from Chambers' Cyclopaedia published in 1728.)
- Franz Reuleaux assembled over 800 models that represent elementary machines. He defined machines as assemblies of links and joints. (F. Realeaux, Lehrbuch der Kinematik, Vol 1, 11, Braunschweig: F. Vieweg und sohn)

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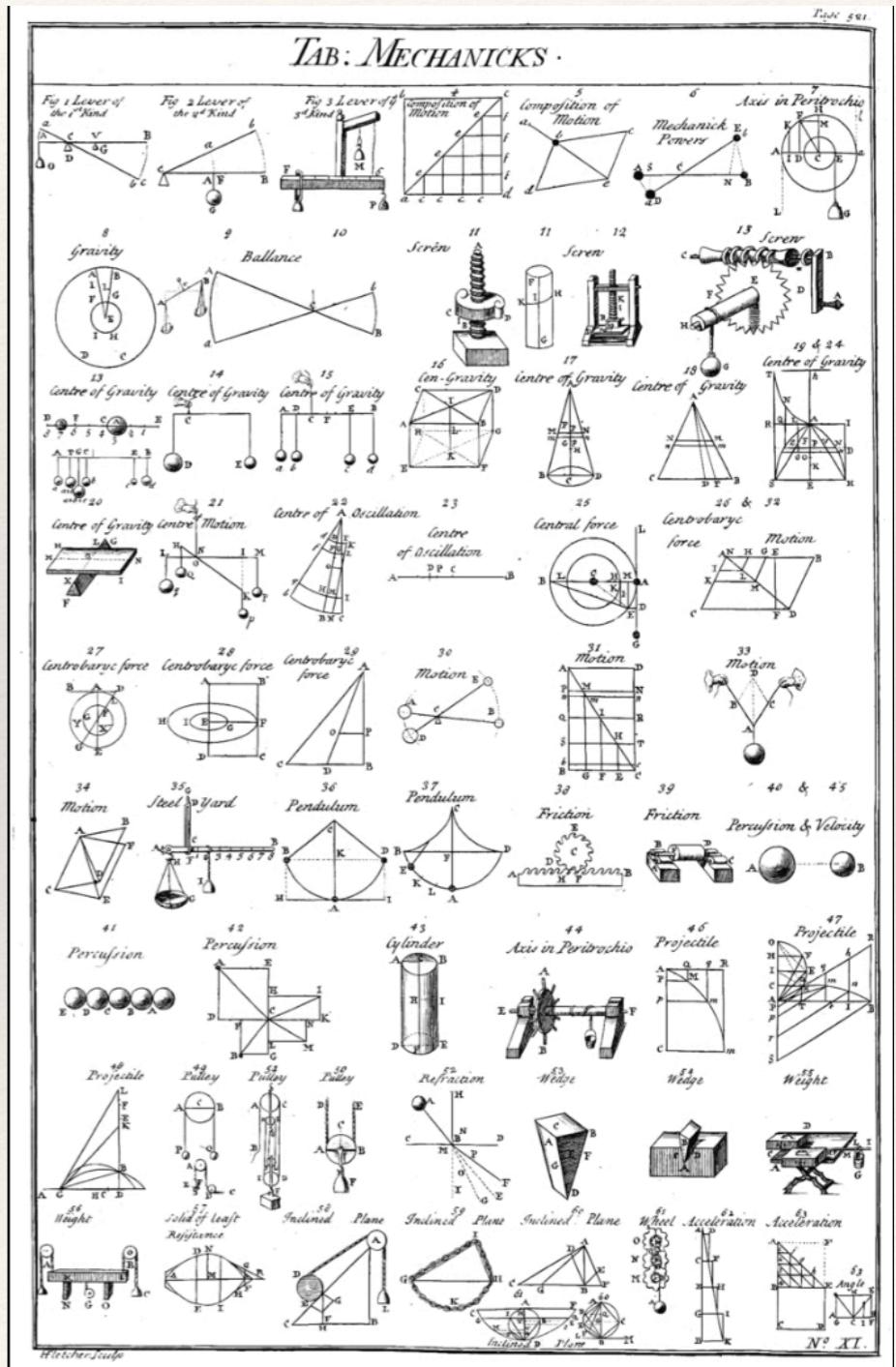
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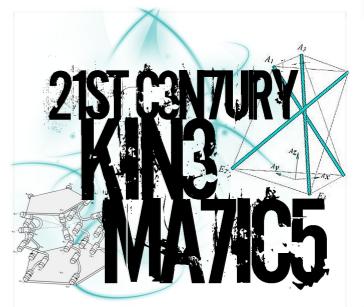
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Gruebler's Equation



- ❖ Abstraction of a machine as an assembly of links and joints provides a formula for its degrees of freedom, or mobility, known as Gruebler's equation:

links : $N = n_2 + n_3 + n_4 + \dots = \sum_{p=2} n_p,$

joints : $2J = 2n_2 + 3n_3 + 4n_4 + \dots = \sum_{p=2} pn_p,$

mobility : $F = 3(N - 1) - 2J.$

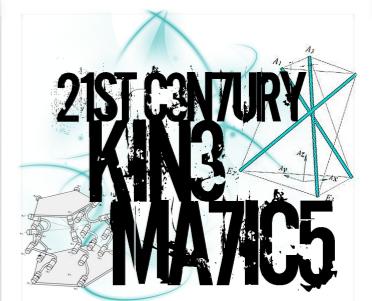
- ❖ Crossley used this formula to identify one degree of freedom linkages by setting $F=1$ and solving to obtain:

$$J = \frac{3}{2}N - 2$$

- ❖ He used this to identify that one degree of freedom systems must have an even number of links,
- ❖ And that two ternary links can be added to any system of N links without changing the degrees of freedom,

Group	Table 1 $X = 1$ mechanisms	N	n_2	n_3	n_4	n_5	n_6
	
I	4	4
II	6	4	2
		5	0	1
III	8	4	4
		5	2	1
		6	0	2
		6	1	0	1
IV	10	4	6
		5	4	1
		6	3	0	1
		6	2	2
		7	1	1	1
		7	0	3	0
		8	0	0	2
		8	0	1	0	1	..
V	12	4	8
		5	6	1
		6	4	2
	etc.						

F. R. E. Crossley, Contribution to Gruebler's Theory in the Number Synthesis of Plane Mechanisms, J. Eng. Ind 86(1), 1-5 (Feb 01, 1964)



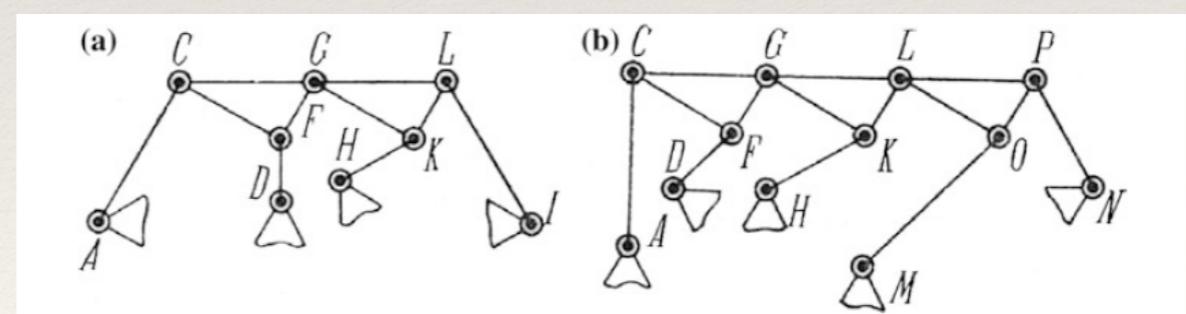
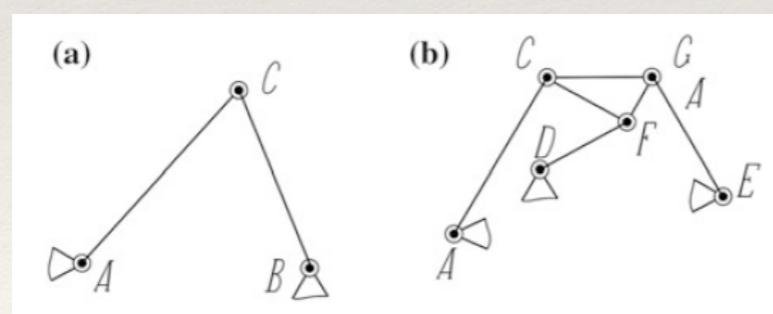
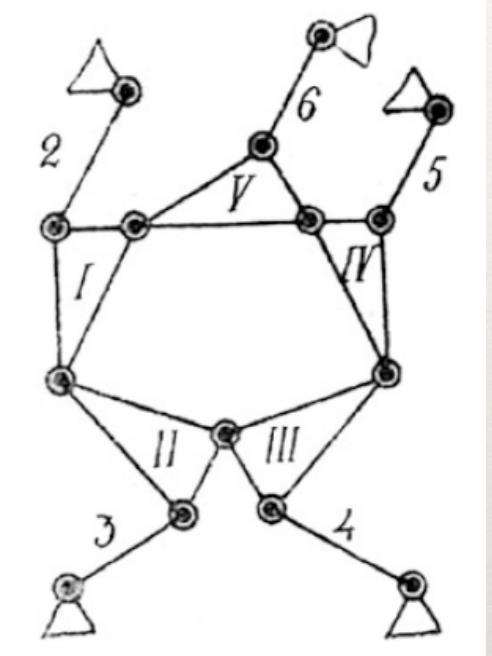
Assur Groups

- ❖ Leonid Assur (1878-1920) noticed that systems of links that satisfy the equation

$$3N - 2J = 0$$

will have zero degrees of freedom when attached to a fixed ground link. It is called an Assur Group.

- ❖ In fact, an Assur group can be added to any mechanism without changing its degrees of freedom.
- ❖ Wohlhart states that a class 3 order 6 Assur group has the greatest closed polygon with three sides and six external joints for input. Therefore, the closed chain shown is a class 5, order 5 Assur Group. (K. Wohlhart, Position analyses of normal quadrilateral Assur groups, Mechanism and Machine Theory 45 (2010) 1367–1384)
- ❖ When the external joints of an Assur Group are attach to a single ground link, the system becomes a statically determinate Baranov Truss.





Baranov Truss

- ❖ A Baranov truss is a statically determinate truss, also a mechanism with zero degrees of freedom, so

$$F = 3N - 2J - 3 = 0 \quad \text{or} \quad J = \frac{3}{2}(N - 1)$$

- ❖ This shows that a Baranov truss must have an odd number of bodies.
- ❖ A given Baranov truss can be transformed into an Assur group by removing any one link.
- ❖ The joints exposed by removing the link become the external joints available for attachment to a mechanism.

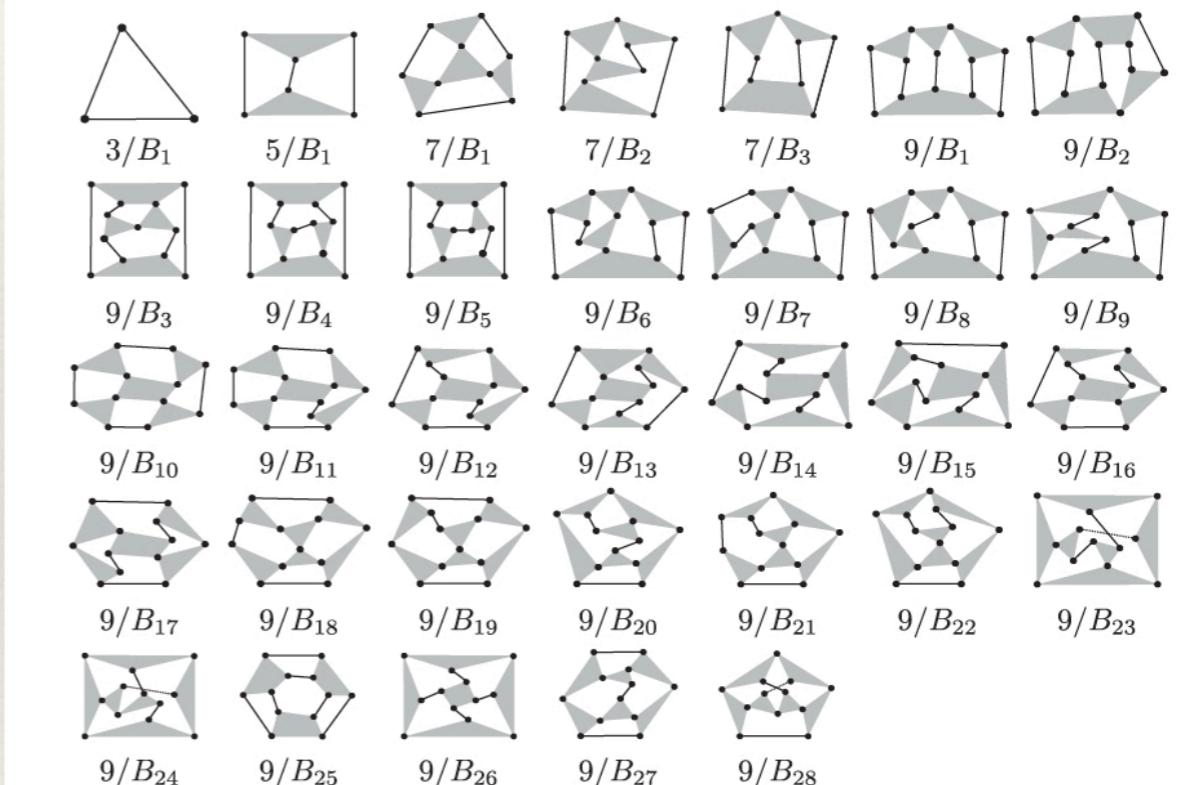


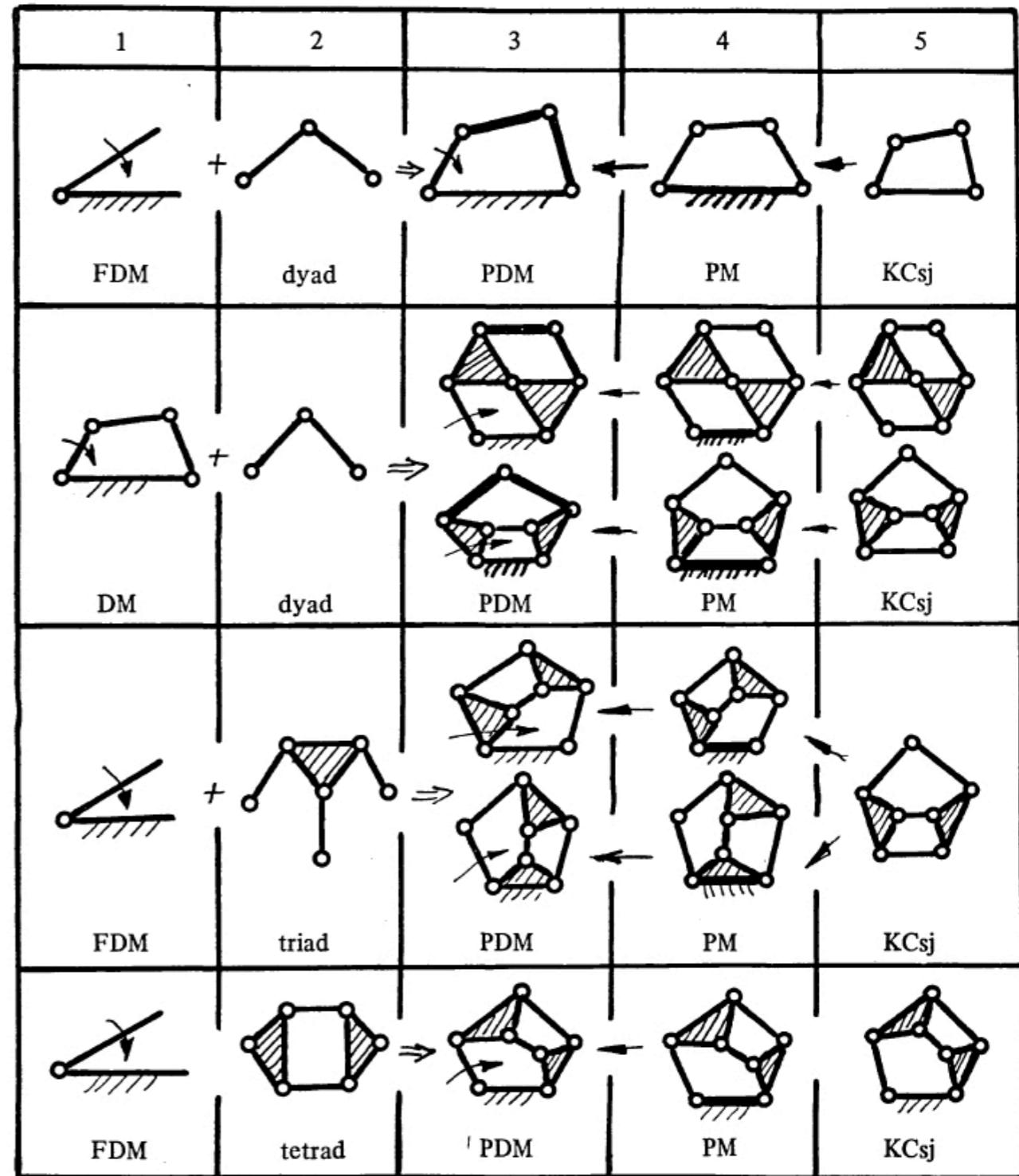
Fig. 1. The cataloged Baranov trusses.

N. Rojas and F. Thomas, On closed-form solutions to the position analysis of Baranov trusses, Mechanism and Machine Theory 50 (2012) 179–196



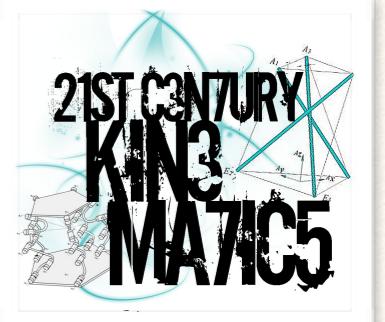
Assembling Linkages

- ❖ Manolescu illustrates how a driving mechanism (DM) and Assur group are combined to form a new mechanism.
- ❖ He also shows how a linkage topology is specialized by selecting ground and input links.



N. I. Manolescu, A unified method for the formation of all planar jointed kinematic chains and Baranov trusses, Environment and Planning B, 1979, volume 6, pages and 447-454

Graph Theory



- ❖ A graph can be associated with a mechanism by identifying the links as vertices and the joints as edges.
- ❖ A contracted graph collects a serial chain of joints into a single edge.

Table D.1 Planar One-dof Four-Bar Linkage.

No. of Loops	Vertex Degree	Contracted Graph	Conventional Graph	Kinematic Chain
1	4000	Not applicable		

Table D.2 Planar One-dof Six-Bar Linkages.

No. of Loops	Vertex Degree	Contracted Graph	Conventional Graph	Kinematic Chain
2	4200			

L. W. Tsai, Mechanism Design: Enumeration of Kinematic Structures According to Function, CRC Press, Boca Raton, 2000.

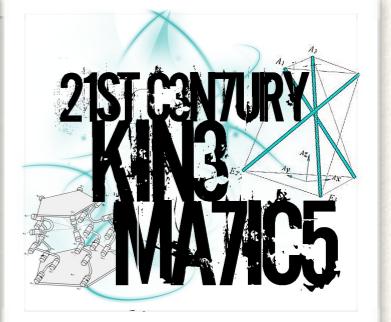
Table D.3 Planar One-dof Eight-Bar Linkages: Part 1.

No. of Loops	Vertex Degree	Contracted Graph	Conventional Graph	Kinematic Chain
3	4400			

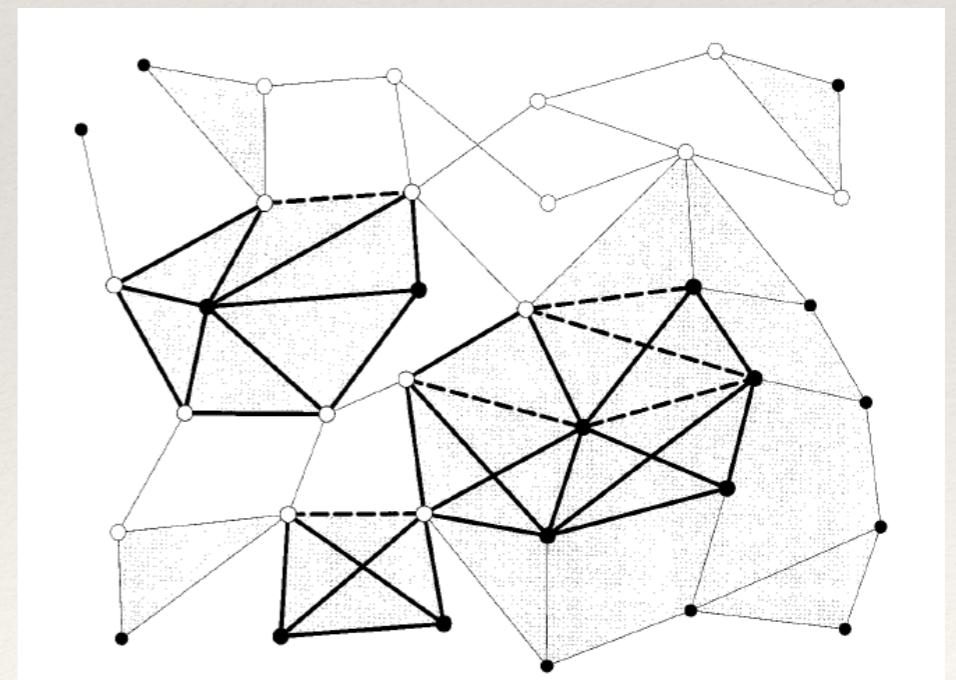
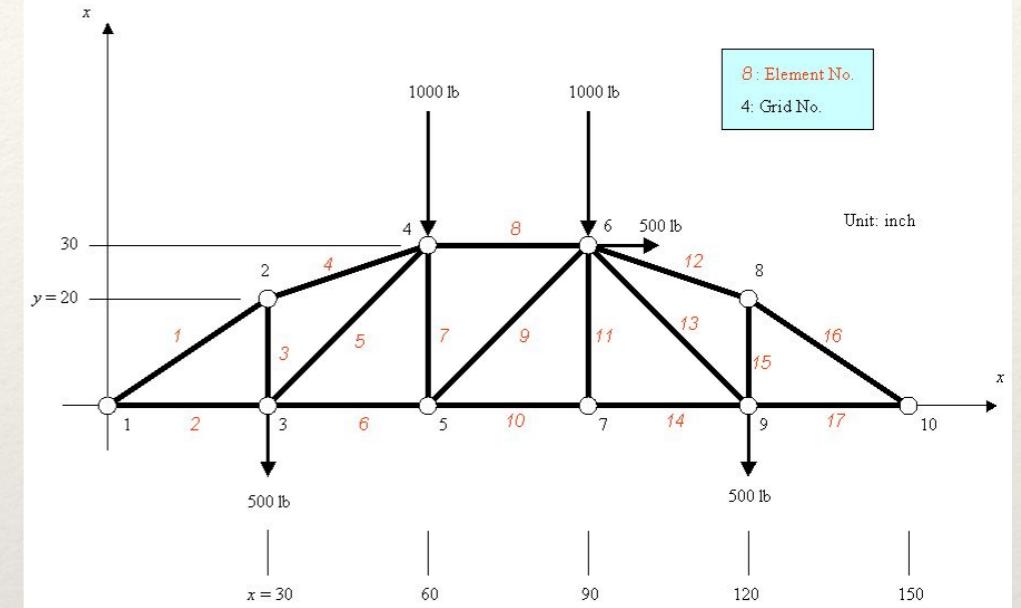
Table D.4 Planar One-dof Eight-Bar Linkages: Part 2.

No. of Loops	Vertex Degree	Contracted Graph	Conventional Graph	Kinematic Chain
3	4400			

Rigidity Theory



- ❖ Civil engineers define a truss to be an assembly of two force members connected by pin joints.
- ❖ For mechanical engineers this requires all polygons to be triangulated. Thus, the truss becomes a graph with joints called vertices, v , and binary links called edges, e .
- ❖ A truss is statically determinate, or isostatic, if the forces along each of the edges can be determined using the equilibrium equations at each joint.
- ❖ There are $2v$ joint equilibrium equations and e link forces and 3 ground reaction forces, so $2v = e+3$, or $2v-3-e=0$.
- ❖ A graph is defined to be *minimally rigid* if $e=2v-3$, and all subgraphs have at least $e'=2v'-3$. Minimally rigid is the same as statically determinate and isostatic. (Laman Theorem)

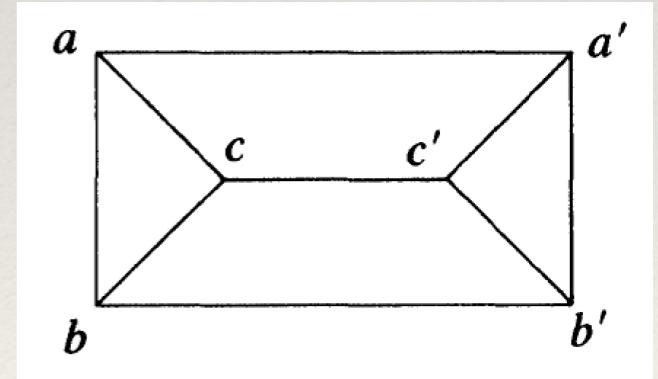
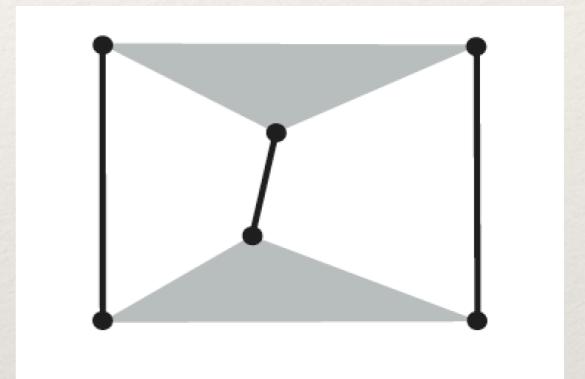


Baranov Graphs



- ❖ We can revise the definition of Baranov truss to reduce the polygons to two force members, or binary links.
- ❖ A ternary link yields 3 binary links so $e_3 = 3n_3$, a quaternary link becomes two triangles with a shared edge, or $e_4 = (3 \cdot 2 - 1)n_4$, a p-gon becomes $e_p = (3 \cdot (p-2) - (p-3))n_p$, $e_p = (2p-3)n_p$. Thus, the rigid links become minimally rigid subgraphs.
- ❖ Compute the mobility of the Baranov graph constructed from binary links. Let the number of binary links that meet at the vertex v be defined as its valence k. In Gruebler's equations joints that have a valence k are counted as $k-1$ joints.
- ❖ Let e be the number of links and J_k be the number of vertices of valence k, then Gruebler's formula becomes

$$\begin{aligned}
 F &= 3(e - 1) - 2 \sum_{k=2} (k - 1)J_k = 0, \\
 &= 3e - 3 - 2 \sum_{k=2} kJ_k + 2 \sum_{k=2} J_k = 0, \\
 &= 3e - 3 - 2(2e) + 2v = 0, \\
 &= 2v - e - 3 = 0.
 \end{aligned}$$



- ❖ Thus, Gruebler's formula is equivalent to the condition for minimal rigidity for graphs.



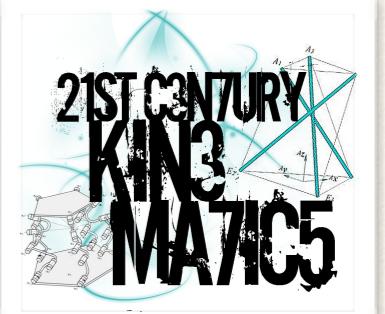
Rigidity Matrix

- ❖ If a graph with v vertices are assigned coordinates \mathbf{p}_i , $i=1,\dots,v$, in the plane, then it is called a *framework*
- ❖ The rigidity of a framework is studied using the e length equations $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}_i - \mathbf{p}_j) = L_{ij}^2$
- ❖ The framework moves then the velocities of each vertex must satisfy the equation $(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{v}_i - \mathbf{v}_j) = 0$, where \mathbf{v}_i is the velocity of \mathbf{p}_i . This yields the e by $2v$ matrix equation, $[R]\mathbf{m} = \mathbf{0}$, where R is called the Rigidity matrix.
- ❖ For example a graph with four vertices and six edges has the rigidity matrix

$$\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4) = (p_{11}, p_{12}, p_{21}, p_{22}, p_{31}, p_{32}, p_{41}, p_{42});$$

$$\begin{bmatrix} p_{11}-p_{21} & p_{12}-p_{22} & p_{21}-p_{11} & p_{22}-p_{12} & 0 & 0 & 0 & 0 \\ p_{11}-p_{31} & p_{12}-p_{32} & 0 & 0 & p_{31}-p_{11} & p_{32}-p_{12} & 0 & 0 \\ p_{11}-p_{41} & p_{12}-p_{42} & 0 & 0 & 0 & 0 & p_{41}-p_{11} & p_{42}-p_{12} \\ 0 & 0 & p_{21}-p_{31} & p_{22}-p_{32} & p_{31}-p_{21} & p_{32}-p_{22} & 0 & 0 \\ 0 & 0 & p_{21}-p_{41} & p_{22}-p_{42} & 0 & 0 & p_{41}-p_{21} & p_{42}-p_{22} \\ 0 & 0 & 0 & 0 & p_{31}-p_{41} & p_{32}-p_{42} & p_{41}-p_{31} & p_{42}-p_{32} \end{bmatrix}$$

- ❖ A framework is rigid if the range of the rigidity matrix is $\text{rank}(R)=2v-3$.



Henneberg Method

- ❖ Henneberg constructions add vertices and edges without changing rigidity.

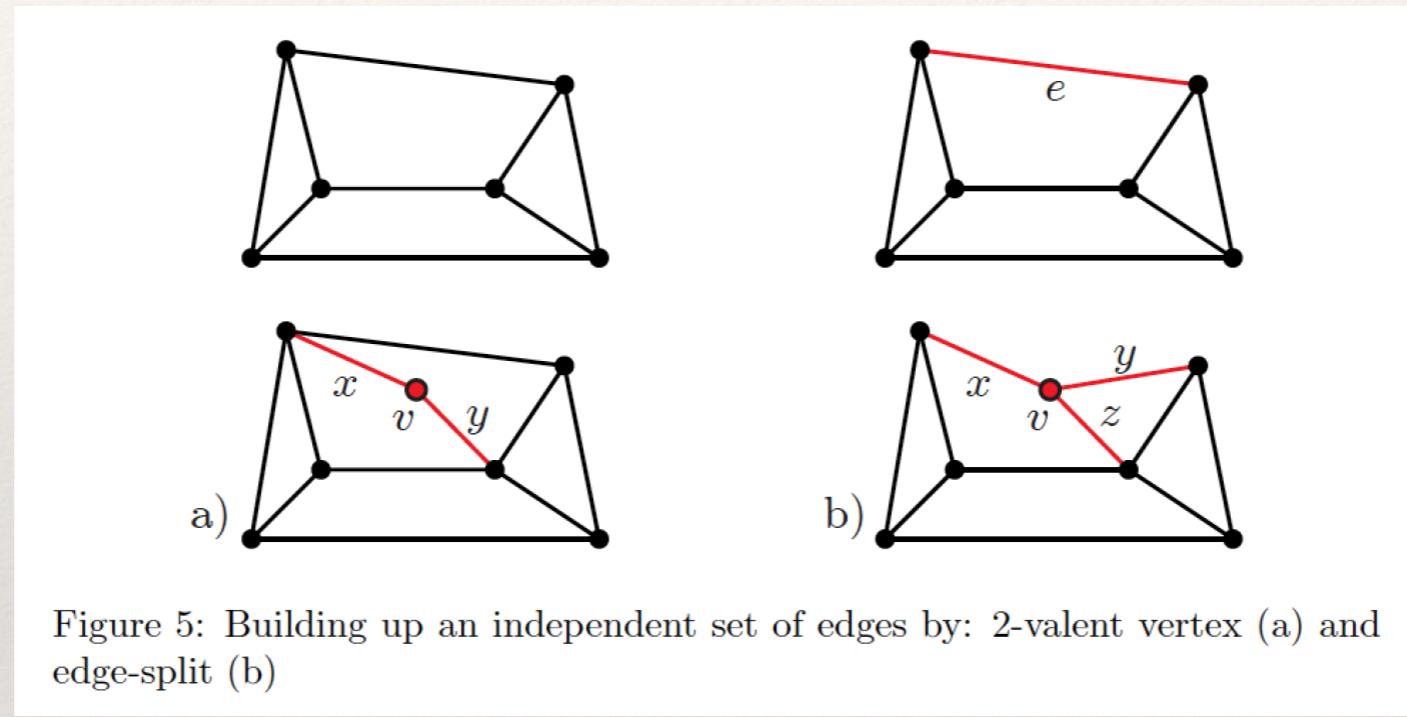


Figure 5: Building up an independent set of edges by: 2-valent vertex (a) and edge-split (b)

- ❖ Removing one bar transforms a statistically determinate structure into a one degree of freedom mechanism.



Pebble Game

- ❖ Linear dependence of the rows of the rigidity matrix defines a *matroid*.
- ❖ The evaluation of the rigidity of an arbitrary graph can be calculated using an algorithm known as the Pebble Game.
- ❖ D. J. Jacobs and B. Hendrickson, An Algorithm for Two-Dimensional Rigidity Percolation: The Pebble Game, JOURNAL OF COMPUTATIONAL PHYSICS 137, 346–365 (1997)

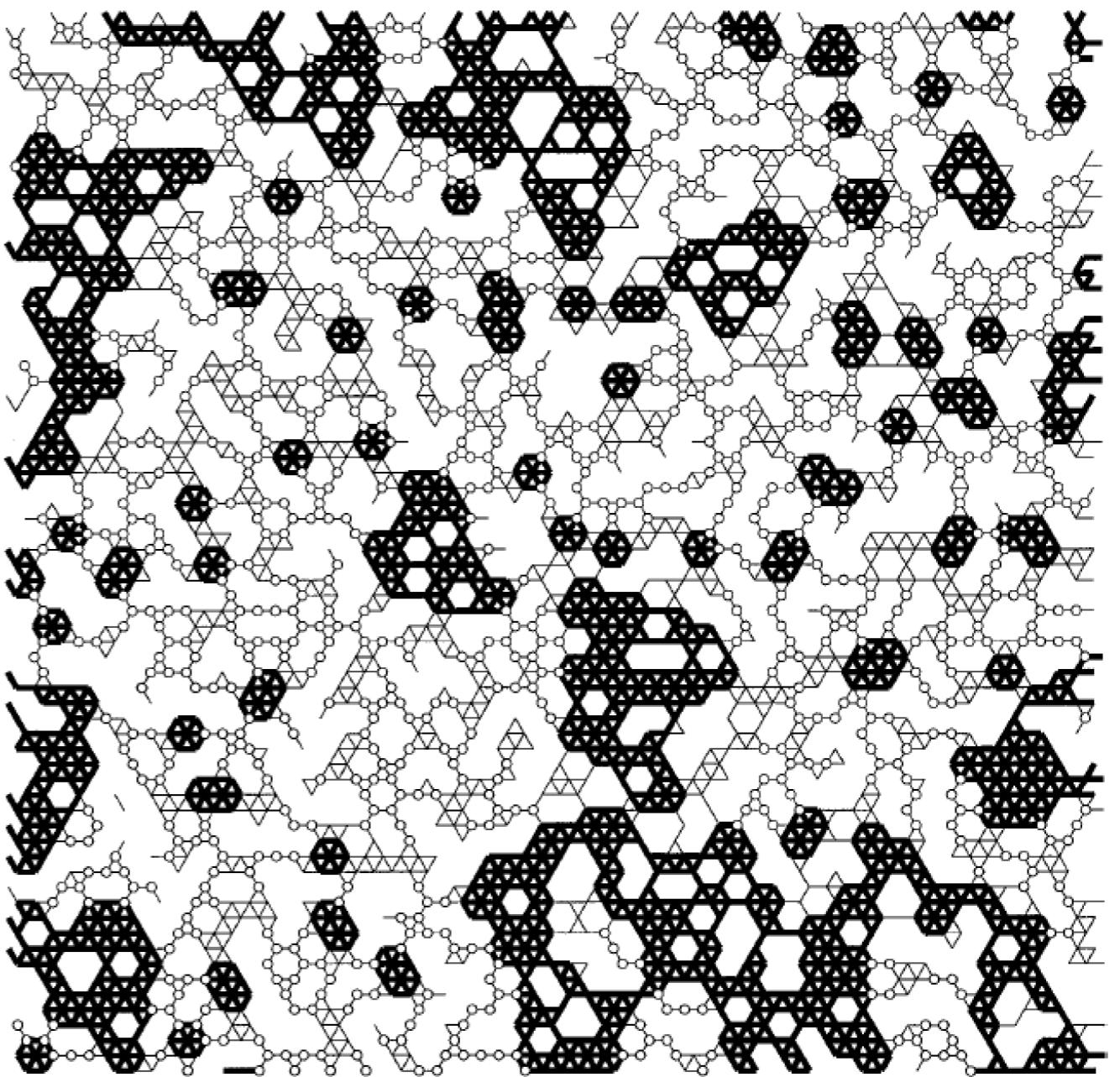


FIG. 7. The *topology* of a section of a site-diluted generic triangular network below the percolation threshold ($q = 0.6825$). Any particular realization will have distortions (not shown). The thin (thick) lines denote isostatic (overconstrained) bonds. The open circles denote pivot sites.