Kinematic Synthesis 2

October 8, 2015

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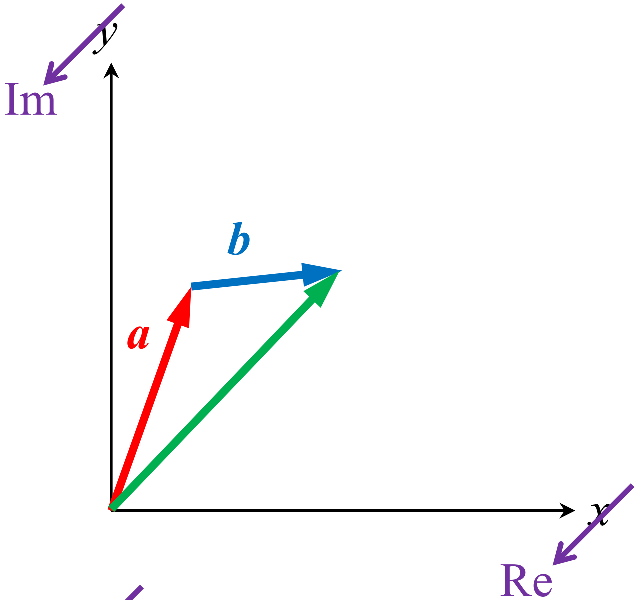
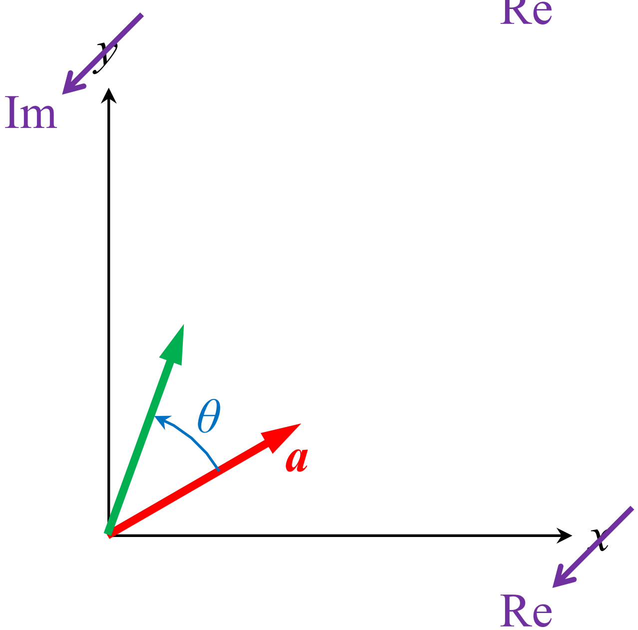
Planar Kinematics With Complex Numbers



$$\begin{Bmatrix} a_x \\ a_y \end{Bmatrix} + \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} = \begin{Bmatrix} a_x + b_x \\ a_y + b_y \end{Bmatrix}$$

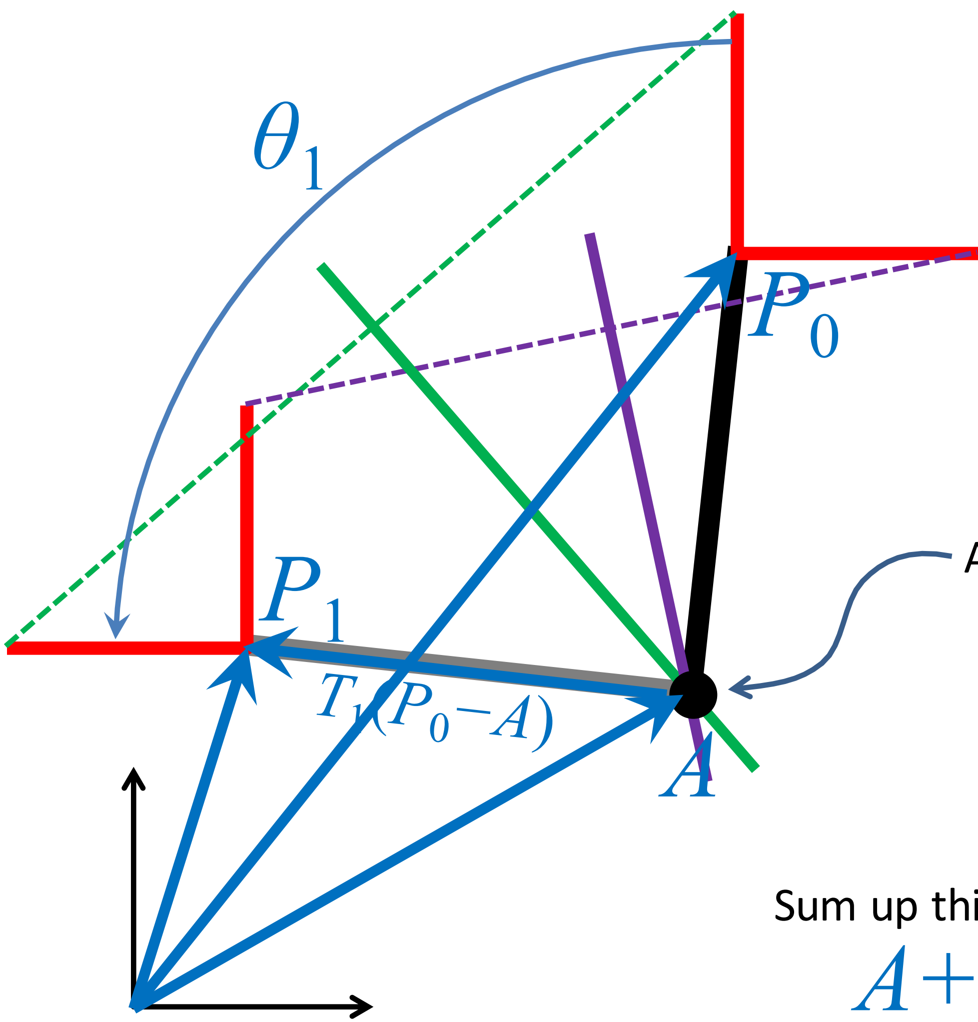
$$(a_x + ia_y) + (b_x + ib_y) = (a_x+b_x) + i(a_y+b_y)$$



$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} a_x \\ a_y \end{Bmatrix} = \begin{Bmatrix} a_x\cos\theta - a_y\sin\theta \\ a_x\sin\theta + a_y\cos\theta \end{Bmatrix}$$

$$\begin{aligned} e^{i\theta}(a_x + ia_y) &= (\cos\theta + i\sin\theta)(a_x + ia_y) \\ &= (a_x\cos\theta - a_y\sin\theta) + i(a_x\sin\theta + a_y\cos\theta) \end{aligned}$$

Motion Generation of a Crank



Any two planar positions define a pole

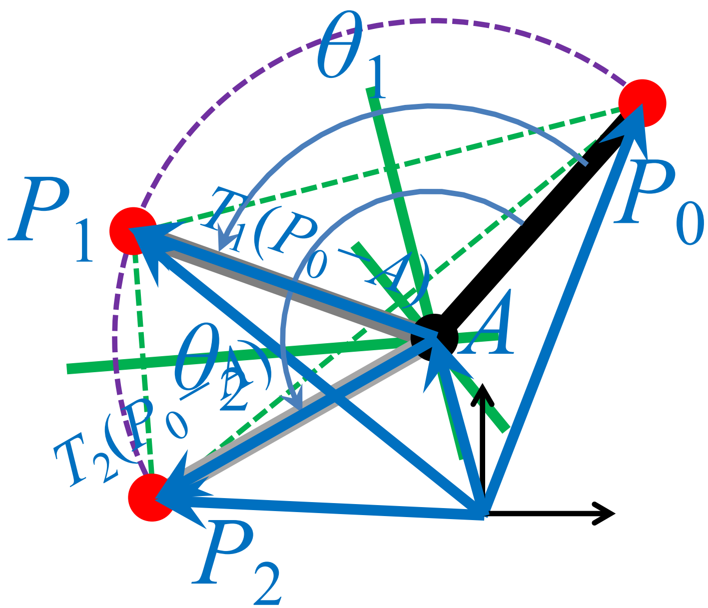
$$T_1 = e^{i\theta_1}$$

Sum up this vector loop:

$$A + T_1(P_0 - A) = P_1$$

A is the only unknown and this equation is linear

Path Generation of a Crank



$$T_1 = e^{i\theta_1}, \quad T_2 = e^{i\theta_2}$$

Now we can sum 2 loop equations:

$$A + T_1(P_0 - A) = P_1$$

$$A + T_2(P_0 - A) = P_2$$

This time as unknowns, we don't know A, T_1, or T_2

Since T_1 and T_2 are rotation operators, we know they must satisfy constraints

$T_1\bar{T}_1 = 1, \quad T_2\bar{T}_2 = 1,$ because

$$T_j\bar{T}_j = e^{i\theta_j}e^{-i\theta_j} = (\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta) = \sin^2\theta + \cos^2\theta = 1,$$

This introduces new unknowns $\bar{T}_1$ *and* $\bar{T}_2$.

4 equations & 5 unknowns

Path Generation of a Crank -2

In order to accommodate these new conjugate unknowns, we append new conjugate loop equations.

$$\bar{A} + \bar{T}_j(\bar{P}_0 + \bar{A}) = \bar{P}_j, \qquad j = 1,2$$

In all we have 6 equations in 6 unknowns:

$$\begin{gathered} A + T_j(P_0 + A) = P_j \\ \bar{A} + \bar{T}_j(\bar{P}_0 + \bar{A}) = \bar{P}_j \\ T_j\bar{T}_j = 1, \qquad j = 1,2 \end{gathered}$$

Unknowns:

$$A, \bar{A}, T_1, \bar{T}_1, T_2, \bar{T}_2$$

We consider $\bar{A}$, $\bar{T}_1$, and $\bar{T}_2$ to be unknowns, which makes the system polynomials

Bézout's Theorem: the maximum number of roots of a polynomial system is the product of the degrees of each equation

In this case, the degree is $2^6 = 64$. Meaning there are 64 solutions. Which means there are 64 cranks which travel through 3 points. That can't be true!

Path Generation of a Crank -3

$$A + T_j(P_0 + A) = P_j \qquad\qquad \bar{A} + \bar{T}_j(\bar{P}_0 + \bar{A}) = \bar{P}_j$$

solve for T_j solve for $\bar{T}_j$

substitute into

$$T_j \bar{T}_j = 1$$

to eliminate T_j and $\bar{T}_j$, and obtain

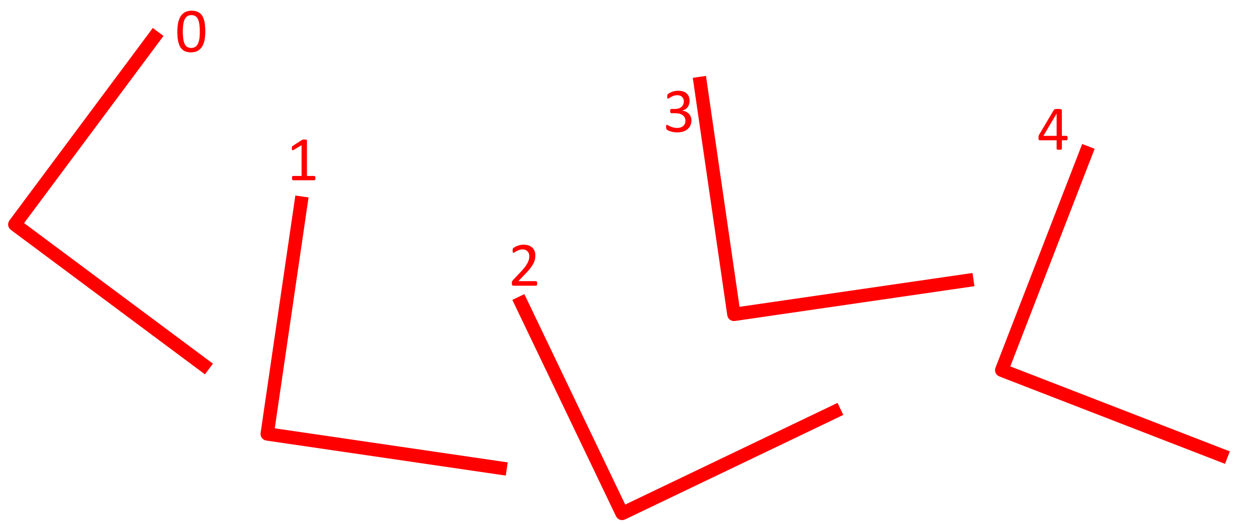
$$P_j \bar{P}_j - P_0 \bar{P}_0 - A\left(\bar{P}_j - \bar{P}_0\right) - \bar{A}\left(P_j - P_0\right) = 0, \qquad j = 1,2$$

which are 2 linear equations in 2 unknowns, therefore there is 1 solution

Phew! That's good because we know there is only 1 circle that goes through 3 points, and its coordinates will be at $A = A_x + iA_y$.

Looks like our original estimate of 64 was a vast overestimation!

Motion Generation of a Four-bar

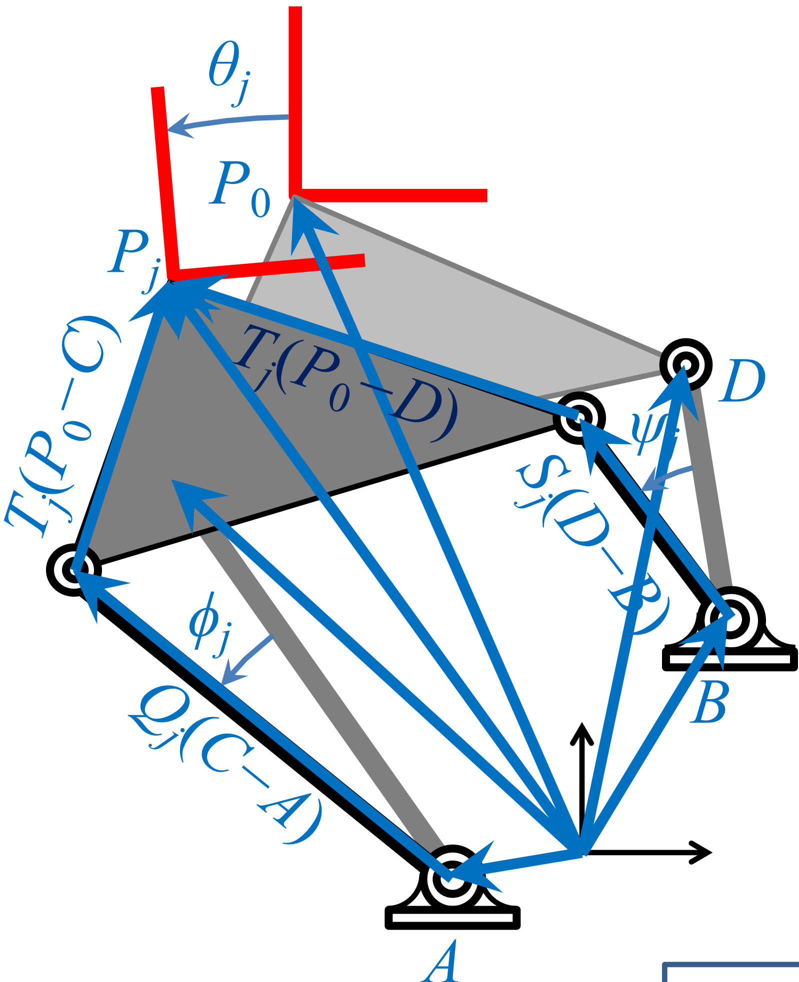


Graphical method: It exists but it's cumbersome

See *Geometric Design of Linkages* by McCarthy and Soh

…and where does the number 5 come from?

Motion Generation of a Four-bar -2



$$Q_j = e^{i\phi_j}, \quad S_j = e^{i\psi_j}, \quad T_j = e^{i\theta_j}$$

$$A + Q_j(C - A) + T_j(P_0 - C) = P_j$$

$$B + S_j(D - B) + T_j(P_0 - D) = P_j$$

Then append conjugate loop equations:

$$\bar{A} + \bar{Q}_j(\bar{C} - \bar{A}) + \bar{T}_j(\bar{P}_0 - \bar{C}) = \bar{P}_j$$

$$\bar{B} + \bar{S}_j(\bar{D} - \bar{B}) + \bar{T}_j(\bar{P}_0 - \bar{D}) = \bar{P}_j$$

and unit magnitude equations for the unknown rotation operators:

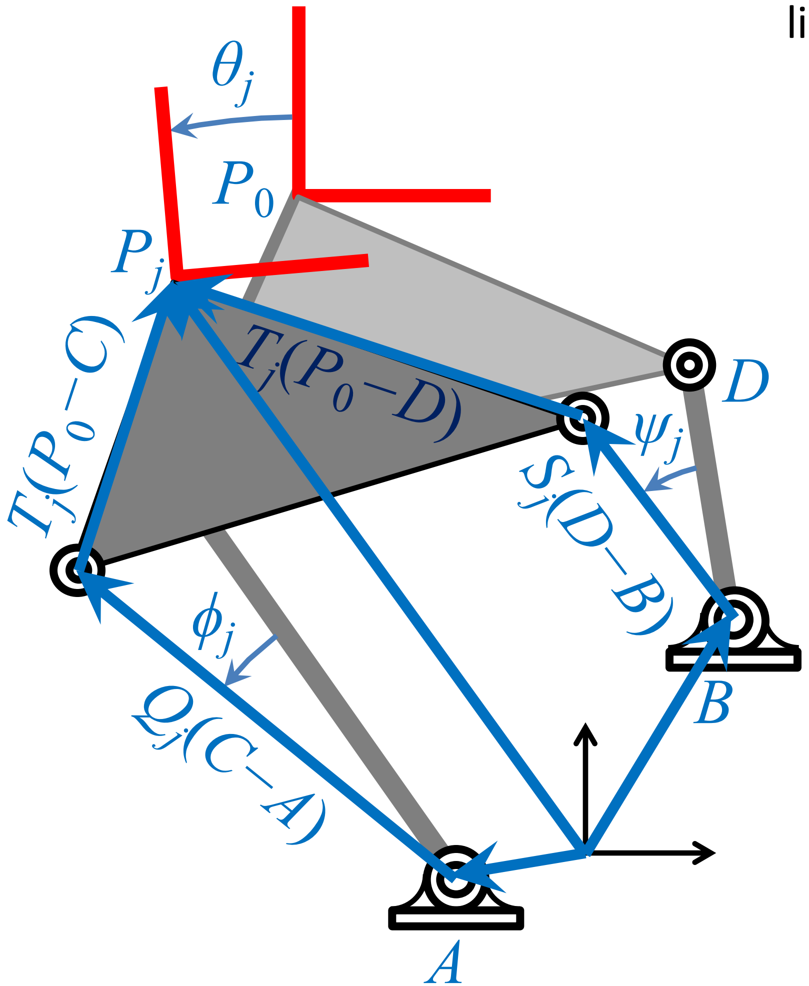
$$Q_j\bar{Q}_j = 1 \qquad S_j\bar{S}_j = 1$$

N is no. of positions

Knowns: $P_j, \bar{P}_j, T_j, \bar{T}_j$

Unknowns: $A, \bar{A}, B, \bar{B}, C, \bar{C}, D, \bar{D},$
$Q_j, \bar{Q}_j, S_j, \bar{S}_j, \qquad j = 1, \ldots, N-1$

Motion Generation of a Four-bar -3



The equations corresponding to the 2 halves of the linkage are symmetric:

$$A + Q_j(C-A) + T_j(P_0-C) = P_j$$

$$B + S_j(D-B) + T_j(P_0-D) = P_j$$

As well, the unknowns can be partitioned so that we can focus on one half of the linkage at a time

$$\boxed{A, \bar{A}, C, \bar{C}, Q_j, \bar{Q}_j,} \quad \boxed{B, \bar{B}, D, \bar{D}, S_j, \bar{S}_j,}$$

$$j = 1, \dots, N-1$$

Each half is called an RR chain. If we can find at least two RR chains, we can combine them to create a four-bar linkage.

Solving for an RR Chain

Equations to solve:

$$A + Q_j(C - A) + T_j(P_0 - C) = P_j$$
$$\bar{A} + \bar{Q}_j(\bar{C} - \bar{A}) + \bar{T}_j(\bar{P}_0 - \bar{C}) = \bar{P}_j$$
$$Q_j\bar{Q}_j = 1, \qquad j = 1, \dots, N-1$$

N is no. of positions

Find N such that there is an equal number of equations and unknowns: $N - 1 = 4$

This system consists of 12 quadratic equations. According to Bézout's theorem, a maximum of $2^{12} = 4096$ roots exist. This is an overestimation, and by eliminating some variables we can compute a lower Bézout degree.

$$A + Q_j(C - A) + T_j(P_0 - C) = P_j \qquad \bar{A} + \bar{Q}_j(\bar{C} - \bar{A}) + \bar{T}_j(\bar{P}_0 - \bar{C}) = \bar{P}_j$$

solve for Q_j solve for $\bar{Q}_j$

substitute into

$$Q_j\bar{Q}_j = 1$$

to eliminate Q_j and $\bar{Q}_j$, and obtain

That is 4 quadratic equations in 4 unknowns $A, \bar{A}, C, \bar{C}$, with a Bézout degree of $2^4 = 16$

$$(C - A)(\bar{C} - \bar{A}) = \big(A + T_j(P_0 - C) - P_j\big)\big(\bar{A} + \bar{T}_j(\bar{P}_0 - \bar{C}) - \bar{P}_j\big), \qquad j = 1, \dots, 4$$

Solving for an RR Chain -2

However, it turns out that 16 is also an overestimation to number of roots. So how do we find the real number of roots?

We must analyze the monomial structure of the polynomial system. If we were to expand

$$(C - A)(\bar{C} - \bar{A}) = \left(A + T_j(P_0 - C) - P_j\right)\left(\bar{A} + \bar{T}_j(\bar{P}_0 - \bar{C}) - \bar{P}_j\right), \qquad j = 1, \ldots, 4$$

We would find the following list of monomials in each polynomial

$$\langle A\bar{C}, \bar{A}C, A, \bar{A}, C, \bar{C}, 1\rangle$$

Note that a general quadratic that contains the variables $A, \bar{A}, C,$ and $\bar{C}$ would have monomials

$$\langle A^2, \bar{A}^2, C^2, \bar{C}^2, A\bar{A}, C\bar{C}, AC, \bar{A}\bar{C}, A\bar{C}, \bar{A}C, A, \bar{A}, C, \bar{C}, 1\rangle$$

It is because of this sparse monomial structure that the system does not have 16 roots.

So how do we figure out how many roots we have? Say we piece together another dummy system that looks like this:

$$\begin{aligned}
(\blacksquare A + \blacksquare C + 1)(\blacksquare \bar{A} + \blacksquare \bar{C} + 1) &= 0 \\
(\blacksquare A + \blacksquare C + 1)(\blacksquare \bar{A} + \blacksquare \bar{C} + 1) &= 0 \\
(\blacksquare A + \blacksquare C + 1)(\blacksquare \bar{A} + \blacksquare \bar{C} + 1) &= 0 \\
(\blacksquare A + \blacksquare C + 1)(\blacksquare \bar{A} + \blacksquare \bar{C} + 1) &= 0
\end{aligned}$$

Counting the Number of Roots

…another dummy system that looks like this:

$$
\begin{aligned}
&1\colon\ (\blacksquare A + \blacksquare C + 1)(\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0\\
&2\colon\ (\blacksquare A + \blacksquare C + 1)(\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0\\
&3\colon\ (\blacksquare A + \blacksquare C + 1)(\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0\\
&4\colon\ (\blacksquare A + \blacksquare C + 1)(\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0
\end{aligned}
$$

where $\blacksquare$s are stand-ins for random coefficients. It has a monomial structure which looks like this:

$$\langle A\bar{A}, C\bar{C}, A\bar{C}, \bar{A}C, A, \bar{A}, C, \bar{C}, 1\rangle$$

where the monomials in red are extra from our original system. Therefore, if we can figure out the number of roots of our dummy system (which is more general than our original system), we can conjecture that our original system will not have more roots than that.

Our dummy system has an easy solution. We can simply solve a sequence of linear systems.

Solving the Dummy System

The dummy system can be solved by solving a sequence of linear systems:

$$
\begin{aligned}
1&: (\blacksquare A + \blacksquare C + 1)(\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0\\
2&: (\blacksquare A + \blacksquare C + 1)(\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0\\
3&: (\blacksquare A + \blacksquare C + 1)(\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0\\
4&: (\blacksquare A + \blacksquare C + 1)(\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0
\end{aligned}
$$

There can be no more than 6 solutions to our original system!

$$
\begin{aligned}
1&: (\blacksquare A + \blacksquare C + 1) = 0\\
2&: (\blacksquare A + \blacksquare C + 1) = 0\\
3&: (\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0\\
4&: (\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0
\end{aligned}
$$

$$
\begin{aligned}
1&: (\blacksquare A + \blacksquare C + 1) = 0\\
3&: (\blacksquare A + \blacksquare C + 1) = 0\\
2&: (\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0\\
4&: (\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0
\end{aligned}
$$

$$
\begin{aligned}
1&: (\blacksquare A + \blacksquare C + 1) = 0\\
4&: (\blacksquare A + \blacksquare C + 1) = 0\\
2&: (\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0\\
3&: (\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0
\end{aligned}
$$

$$
\begin{aligned}
2&: (\blacksquare A + \blacksquare C + 1) = 0\\
3&: (\blacksquare A + \blacksquare C + 1) = 0\\
1&: (\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0\\
4&: (\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0
\end{aligned}
$$

$$
\begin{aligned}
2&: (\blacksquare A + \blacksquare C + 1) = 0\\
4&: (\blacksquare A + \blacksquare C + 1) = 0\\
1&: (\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0\\
3&: (\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0
\end{aligned}
$$

$$
\begin{aligned}
3&: (\blacksquare A + \blacksquare C + 1) = 0\\
4&: (\blacksquare A + \blacksquare C + 1) = 0\\
1&: (\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0\\
2&: (\blacksquare \bar{A} + \blacksquare \bar{C} + 1) = 0
\end{aligned}
$$

The number of combinations is $4 \text{ choose } 2 = 6$.

Solving for RR Chains

Now that we know the max number of roots to our system is 6. Let's solve it!

$$(C - A)(\bar{C} - \bar{A}) = \big(A + T_j(P_0 - C) - P_j\big)\big(\bar{A} + \bar{T}_j(\bar{P}_0 - \bar{C}) - \bar{P}_j\big), \qquad j = 1, \dots, 4$$

GOAL: Turn this system into a single univariate polynomial in terms of A. We begin solving it by expanding the equations and writing them in this form:

$$k_{1j}\bar{A}C + k_{2j}\bar{A} + k_{3j}C + k_{4j}\bar{C} + k_{5j} = 0, \qquad j = 1, \dots, 4$$

where
$$\begin{aligned}
k_{1j} &= T_j - 1 \\
k_{2j} &= P_j - T_j P_0 \\
k_{3j} &= \bar{P}_0 - T_j \bar{P}_j \\
k_{4j} &= \big(\bar{T}_j - 1\big)A + P_0 - \bar{T}_j P_j \\
k_{5j} &= \big(\bar{P}_j - \bar{T}_j \bar{P}_0\big)A + T_j \bar{P}_j P_0 + \bar{T}_j P_j \bar{P}_0 - P_j \bar{P}_j - P_0 \bar{P}_0
\end{aligned}$$

where A is the suppressed variable which is only found inside of the k coefficients
Written this way, the system is linear in terms of monomial unknowns

$$\begin{bmatrix} k_{11} & k_{21} & k_{31} & k_{41} & k_{51} \\ k_{12} & k_{22} & k_{32} & k_{42} & k_{52} \\ k_{13} & k_{23} & k_{33} & k_{43} & k_{53} \\ k_{14} & k_{24} & k_{34} & k_{44} & k_{54} \end{bmatrix} \begin{Bmatrix} \bar{A}C \\ \bar{A} \\ C \\ \bar{C} \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Creating a Resultant Matrix

For the time being, we are going to pretend each monomial is an independent unknown

$$\begin{bmatrix} k_{11} & k_{21} & k_{31} & k_{41} & k_{51} \\ k_{12} & k_{22} & k_{32} & k_{42} & k_{52} \\ k_{13} & k_{23} & k_{33} & k_{43} & k_{53} \\ k_{14} & k_{24} & k_{34} & k_{44} & k_{54} \end{bmatrix} \begin{Bmatrix} \bar{A}C \\ \bar{A} \\ C \\ \bar{C} \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Matrix is rank 4

5 monomial quantities are considered unknowns

System is underdetermined

Now for the trick: multiply all the equations by C and append them to the system

$$\begin{bmatrix} 0 & k_{11} & 0 & 0 & k_{21} & k_{31} & k_{41} & k_{51} \\ 0 & k_{12} & 0 & 0 & k_{22} & k_{32} & k_{42} & k_{52} \\ 0 & k_{13} & 0 & & & k_{33} & k_{43} & k_{53} \\ 0 & k_{14} & 0 & & & k_{34} & k_{44} & k_{54} \\ k_{11} & k_{21} & k_{31} & & & k_{51} & 0 & 0 \\ k_{12} & k_{22} & k_{32} & & & k_{52} & 0 & 0 \\ k_{13} & k_{23} & k_{33} & k_{43} & 0 & k_{53} & 0 & 0 \\ k_{14} & k_{24} & k_{34} & k_{44} & 0 & k_{54} & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{A}C^2 \\ \bar{A}C \\ C^2 \\ C\bar{C} \\ \bar{A} \\ C \\ \bar{C} \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

A is the only variable suppressed inside here

must have nullity > 0,
therefore the determinant must equal 0

vector must live in the null space

Building Four-bars from RR Chains

The determinant of the matrix is a degree 4 univariate polynomial f(A) = 0. Which can be solved by standard approaches e.g. the quartic equation or numpy.roots

So it ends up that the real number of solutions is even less than our estimate of 6, this is because our dummy system used for counting had a more general monomial structure

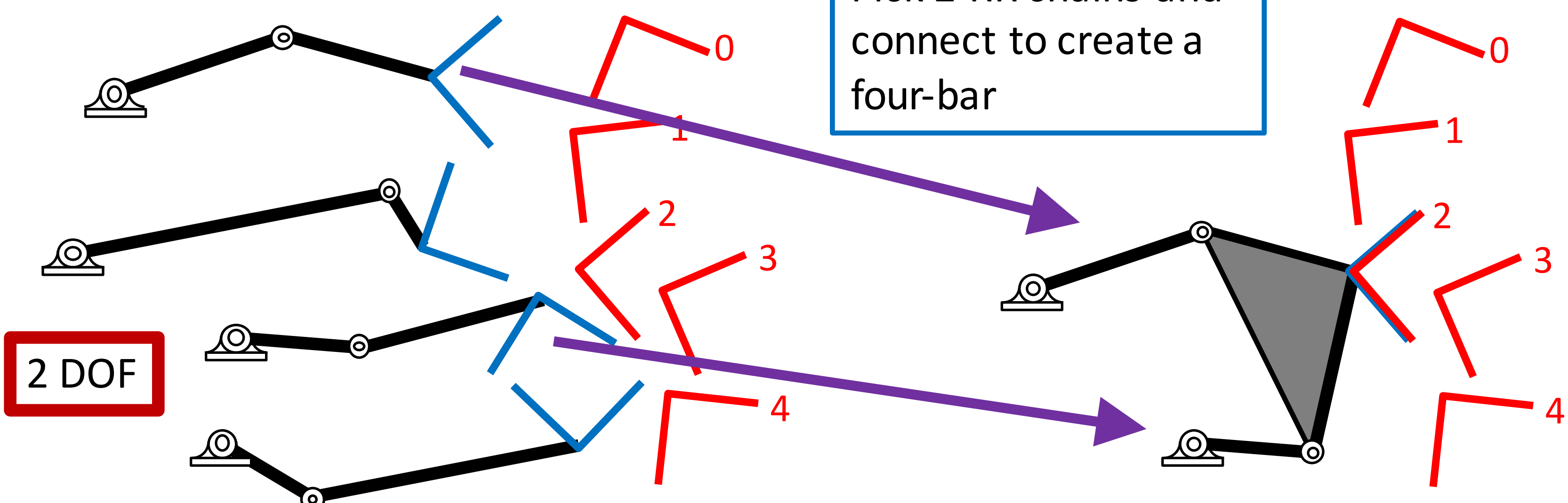
Our system:
$\langle A\bar{C}, \bar{A}C, A, \bar{A}, C, \bar{C}, 1\rangle$

Dummy system:
$\langle A\bar{A}, C\bar{C}, A\bar{C}, \bar{A}C, A, \bar{A}, C, \bar{C}, 1\rangle$

Once 4 values of A are found, back-substitution can find corresponding values for $\bar{A}$, C, and $\bar{C}$

Each solution represents an RR chain:

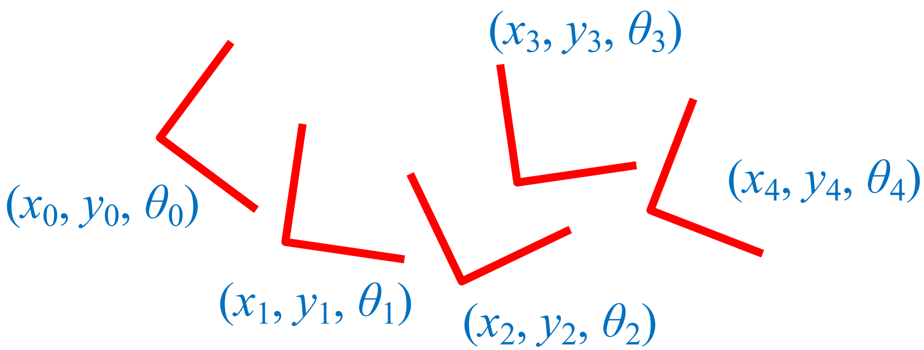
Pick 2 RR chains and connect to create a four-bar



Total of 4 choose $2 = 6$ four-bars

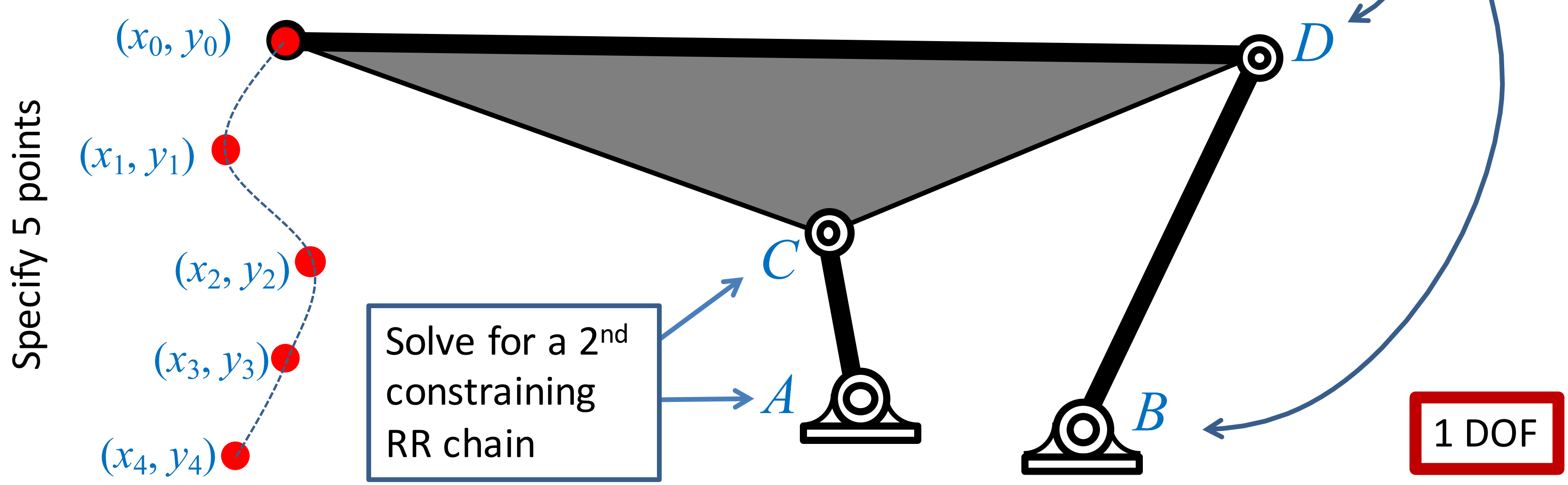
Another Approach: Four-bar as a Constrained RR chain for Path Generation

Last time we started with 5 task positions

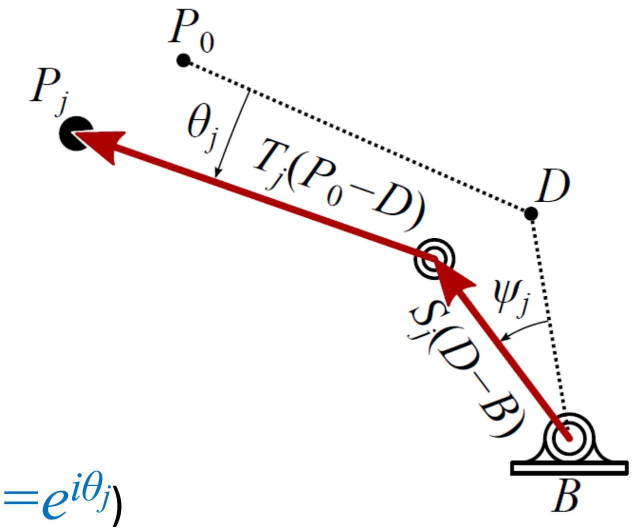


You pick an RR chain!

Let’s start with a different problem statement



Inverse Kinematics of an RR chain



Write the loop equations for the known RR chain:

$$B + S_j(D - B) + T_j(P_0 - D) = P_j, \quad j = 1, \ldots 4$$

In this case we know dimensions: B and D

We know the end effector position: $P = x_j + iy_j$

We want to find joint angle parameters: ψ_j and θ_j

(or equivalently $S_j = e^{i\psi_j}$ and $T_j = e^{i\theta_j}$)

This is inverse kinematics!

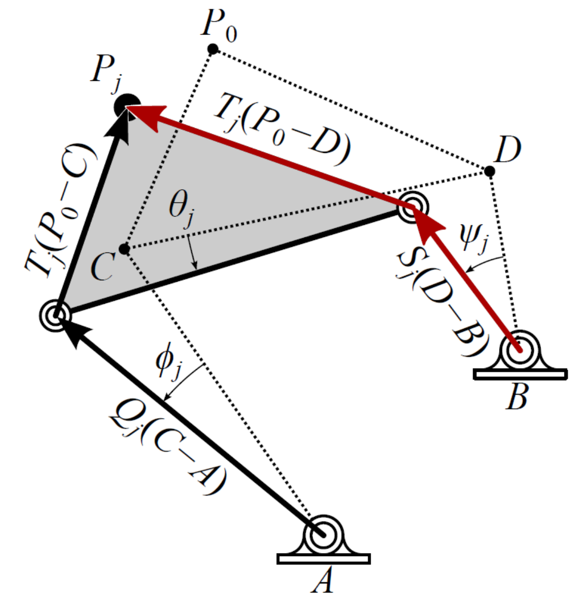
The loop equation can be rewritten as

$$a_j T_j + b_j + c_j \bar{T}_j = 0,$$

$$\text{where} \quad \begin{aligned} a_j &= (P_0 - D)(\bar{P}_j - \bar{B}), \\ b_j &= (D - B)(\bar{D} - \bar{B}) - (P_0 - D)(\bar{P}_0 - \bar{D}) - (P_j - B)(\bar{P}_j - \bar{B}), \\ c_j &= (\bar{P}_0 - \bar{D}_0)(P_j - B), \qquad j = 1, \ldots, 4, \end{aligned}$$

and solved for all T_j using the quadratic equation.

Solutions for a Constrained RR Chain



Now let's model the unknown side of the four-bar linkage:

$$A + Q_j(C - A) + T_j(P_0 - C) = P_j$$
$$\bar{A} + \bar{Q}_j(\bar{C} - \bar{A}) + \bar{T}_j(\bar{P}_0 - \bar{C}) = \bar{P}_j$$
$$Q_j\bar{Q}_j = 1, \qquad j = 1, \dots, N-1$$

Since we've already solved for T_j in the previous step, these are the exact same synthesis equations of an RR constraint. They can be reduced to a univariate quartic, and will return 4 solutions for $\{A, \bar{A}, C, \bar{C}\}$.

One of these solutions will be the original RR chain BDP_0.

It is possible that a solution will just be numbers that solve the equations and not correspond to physical joint coordinates i.e. $\bar{A} \neq A_x - iA_y$ or $\bar{C} \neq C_x - iC_y$

These type of solutions always occur in pairs. This means of the 4 solutions:

- 1 is BDP_0, 1 is a physical design, 2 are not physical designs
- 1 is BDP_0, 3 are physical designs