AN ALGORITHM FOR PLANAR LINKAGE SIMULATION

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Abstract. Planar linkage performance is described by sets of nonlinear equations due to the geometric nonlinearity. Existing methods lack in generality because each mechanism with different configuration requires that the mechanism equations need to be written explicitely and no general purpose algorithm exist for a symbolic description of the mechanism. On the other hand, the above problem leads to the need for programming the solution for each mechanism separately which excludes the possibility of a general purpose computer program. To overcome these problems, a general purpose algorithm was devised for the analysis of planar linkages. The linkage consists of links and pins and it is described in symbolic form by a matrix with binary elements 0-1 which describe the element interconnections. The algorithm then proceeds with determination of a linear system for the derivatives of the output angles in respect to the input. Intergration is then performed numerically to yield the output angles. Determination of angular velocities is also included. Application to four-bar and eight-bar linkages showed excellent numerical characteristics. The algorithm, once programmed, can be used for any planar linkage using as input the initial position of the linkage and the symbolic matrix describing the linkage in binary form. In particular the algorithm is suitable for real time control applications such as for robotics arms.

Keywords. Mechanical Variables Control, Robots, Computer aided design.

INTRODUCTION

Several methods have been utilized for the analysis of planar linkages. One of the first and most used is the vector-algebraic method described by Hartenberg [1]. Roth, Freudenstein and Sandor [2], among others, reported on the use of complex arithmetic for analysis and synthesis of planar linkages. This method enables the more systematic analysis and the development of general algorithms. However, the method suffers from the same limitation, though to a lesser extend, as the vector-algebraic methods: for every different form of linkage an appropriate algorithm has to be devised.

The idea of a more general and systematic approach was introduced by Hartenberg [1] with the matrix chain representation which still requires the development of new algorithms for different forms of linkages.

Dimarogonas, Sandor and Erdman [3] introduced an algorithm based on a matrix representation on the complex plane for the analysis and synthesis of geared linkages of one degree of freedom with any number of links and gears. The method was applied by Pafelias and Sandor [4] for design of geared path generators.

Dimarogonas and Sandor [5] introduced a general method for representing a linkage in symbolic form by way of a symbolic matrix with elements 0,1,i which describes the interconnection of the links and a vector with the initial conditions. In the present work this symbolism will be utilized for the development of a general algorithm for analysis of planar linkages without limitation to their form and purpose.

METHOD OF ANALYSIS

Consider a linkage on the complex plane consisting of bars connected with pins. Each bar is represented by a vector. A closed loop with k+1 pins is thus represented with a polygone, fig 1, which has equation of closure

$$Z_m+Z_{m+1}+Z_{m+2}+\dots+Z_{m+k}=0 \tag{1}$$

Further, consider that this is the starting position and the linkage moves to a new position. The links-vectors will rotate respectively by angles ϑ_j where j takes the values corresponding to the polygon vectors. The equation of closure in the displaced position will be

$$e^{i\vartheta_m} Z_m + e^{i\vartheta_{m+1}} Z_{m+1} + \ldots + e^{i\vartheta_{m+k}} Z_{m+k} = 0 \qquad (2)$$

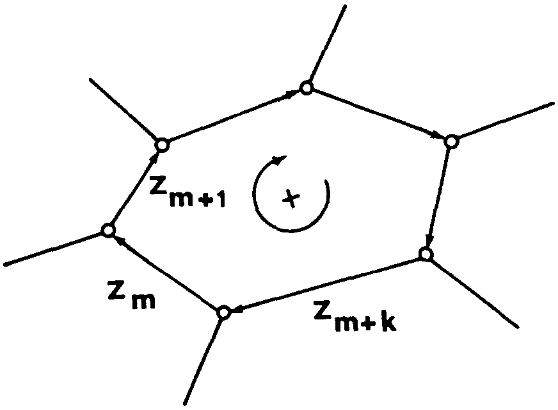

Fig. 1. Closed loop.

It is known [1] that such linkage with N degrees of freedom (that is N known input angles for the purpose of the analysis), ℓ links connected rigidly with the frame of reference and n links, will leave n-N-ℓ angular rotations to be determined by way of (n-N-ℓ)/2 complex equations, such as eq. (2). Without loss in generality, assume that N=1 and ℓ=1, as it usually happens. This lends to (n-2)/2 complex equations which will represent the equations of closure of (n-2)/2 closed loops in the displaced position. This system can be expressed in the form:

$$\alpha_{31} e^{i\vartheta_3} Z_3 + \alpha_{41} e^{i\vartheta_4} Z_4 + \ldots + \alpha_{n1} e^{i\vartheta_n} Z_n = \alpha_{11} Z_1 - $$
$$ - \alpha_{21} e^{i\vartheta_2} Z_2$$

$$\alpha_{32} e^{i\vartheta_3} Z_3 + {}_{42} e^{i\vartheta_4} Z_4 + \ldots + {}_{n2} e^{i\vartheta_n} Z_n = {}_{12} Z - $$
$$ - \alpha_{22} e^{i\vartheta_2} Z_2$$

$$\alpha_{3r} e^{i\vartheta_3} Z_3 + \alpha_{4r} e^{i\vartheta_4} Z_4 + \ldots + \alpha_{nr} e^{i\vartheta_n} Z_n = \qquad (3)$$
$$ = \alpha_{ir} Z_1 - {}_{2r} e^{i\vartheta_2} - Z_2$$

where the parameters α_{ij} take the values 0, 1, -1 as follows, where positive route is arbitrary.

$$\alpha_{ij} = \begin{cases} 0 & \text{when the link i does not belong to the loop j} \\ 1 & \text{when link i of loop j has positive direction} \\ -1 & \text{when link i of loop j has negative direction} \end{cases}$$

Each equation represents a closed loop and the equations must be linearly independent, which has an apparent geometric interpretation: the closure of a new loop must not be already achieved by the closure of previous loops.

It is further supposed that link Z_1 is fixed on the reference frame and link Z_2 takes the input, therefore the input, thus known, angle is ϑ_2. Differentiation in respect to ϑ_2 yields:

$$\alpha_{31}\vartheta'_3 Z_3 + \alpha_{41}\vartheta'_4 Z_4 + \ldots + \alpha_{n1}\vartheta'_n Z_n = -\alpha_{21} Z_2$$
$$\alpha_{32}\vartheta'_3 Z_3 + \alpha_{42}\vartheta'_4 Z_4 + \ldots + \alpha_{n2}\vartheta'_n Z_n = -\alpha_{22} Z_2$$
$$\alpha_{33}\vartheta'_3 Z_3 + \alpha_{43}\vartheta'_4 Z_4 + \ldots + \alpha_{n3}\vartheta'_n Z_n = -\alpha_{23} Z_2$$

--- (4)

$$\alpha_{3r}\vartheta'_3 Z_3 + \alpha_{4r}\vartheta'_4 Z_4 + \ldots + \alpha_{nr}\vartheta'_n Z_n = -\alpha_{2r} Z_2$$

The system of equations (4) can be written in the form

$$\bar{A}\, Z\, \vartheta' \overset{R}{=} \bar{B}\, Z_2 \qquad (5)$$

where the symbol R means that from the resulting complex equation one keeps only the real parts, and

$$A = \begin{bmatrix} \alpha_{31} & \alpha_{41} & \cdots\cdots & \alpha_{n1} \\ \alpha_{32} & \alpha_{42} & & \alpha_{n2} \\ \hline \alpha_{3r} & \alpha_{4r} & & \alpha_{nr} \end{bmatrix}$$

$$B = \begin{bmatrix} \alpha_{21} & \alpha_{22} & \cdots\cdots & \alpha_{2r} \end{bmatrix}^T$$

Here, A is a symbolic matrix describing the interconnections of the links and B is a symbolic vector describing the interconnections of the input link 2 with the others. Moreover

$$\bar{A} = \begin{bmatrix} A \\ \hline iA \end{bmatrix}, \quad B = \begin{bmatrix} B \\ \hline iB \end{bmatrix}$$

Furthermore

$$Z = \begin{bmatrix} Z_3 & 0 & 0 & 0 & \ldots & 0 \\ 0 & Z_4 & 0 & 0 & \ldots & 0 \\ 0 & 0 & Z_5 & 0 & \ldots & 0 \\ \hline 0 & 0 & 0 & 0 & \ldots & Z_n \end{bmatrix}$$

$$\vartheta' = \begin{bmatrix} \vartheta'_3 & \vartheta'_4 & \vartheta'_5 & \vartheta'_6 & & \vartheta'_n \end{bmatrix}^T$$

The system (5) has n-2 algebraic, linear equations in n-2 unknowns ϑ'_3, ϑ'_4, ..., ϑ'_n, where again differentiations is meant in respect to the input angle ϑ_2. The output angles ϑ_3, ϑ_4, ..., ϑ_n can now be computed with appropriate numerical methods in a digital computer and the functions $\vartheta_3(\vartheta_2)$, $\vartheta_4(\vartheta_2)$, ..., $\vartheta_n(\vartheta_2)$ can be constructed.

Angular velocities are obtained directly from eq. (5) in the form

$$\dot{\vartheta}_j = \frac{d\vartheta j}{dt} = \frac{d\vartheta j}{d\vartheta_2} \cdot \frac{d\vartheta_2}{dt} = \vartheta_j'\omega_2 \qquad (6)$$

It should be noticed that in every integration step the matrix $\mathcal{Z}$ changes because the vector $(\mathcal{Z}_j)_2$ representing the link j in the new position 2 has the value $(\mathcal{Z}_j)_1\, e^{i(\vartheta_j)_{1,2}}$. In every new position the linkage should satisfy the initial equations of closure, eq. 1. The left hand side of the equation instead of 0 yields a residual vector. This feature allows for a continous check of the accuracy of the algorithm and the stability of the numerical integration method. This is a very usefull companion for methods which present difficulties in estimating the intergation error.

THE FOUR-BAR LINKAGE

For the analysis of four-bar linkages, closed form algebraic solutions are available. This allows for a direct evaluation of the computation error of the devised algorithm. With the notation adopted in Fig. 2, it is evident that the eq. 5 can be used with

$$A = \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 & 1 \\ i & i \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} -1 & -i \end{bmatrix}^T$$

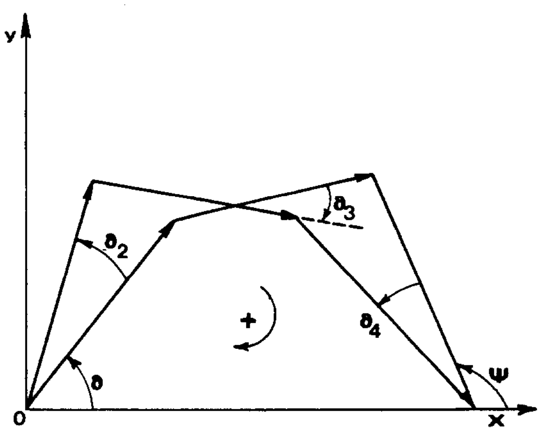

Fig. 2. 4-bar linkage, displaced position

Equations 5 take the form

$$\begin{bmatrix} 1 & 1 \\ i & i \end{bmatrix} \cdot \begin{bmatrix} \mathcal{Z}_3 & 0 \\ 0 & \mathcal{Z}_4 \end{bmatrix} \vartheta' \stackrel{R}{=} \begin{bmatrix} -1 & -i \end{bmatrix}^T Z_2 \qquad (7)$$

Eq. (7) are further decomposed in the form of a linear system of algebraic equations (real):

$$\begin{bmatrix} Z_{3X} & Z_{4X} \\ -Z_{3\psi} & -Z_{4\psi} \end{bmatrix} \begin{bmatrix} \vartheta_3' \\ \vartheta_4' \end{bmatrix} = \begin{bmatrix} -Z_{2X} \\ Z_{2\psi} \end{bmatrix} \qquad (8)$$

Although higher order numerical integration methods could be used, it was found adequate to employ the simple Euler method. To this end, one starts with a linkage position $\mathcal{Z}_1$, $\mathcal{Z}_2$, ..., $\mathcal{Z}_n$ and calculates the derivatives ϑ_3', ϑ_4' using eq. (8). For a sufficiently small angular displacement input $\Delta\vartheta_2$, the output angular displacements $\Delta\vartheta_3$ and $\Delta\vartheta_4$ are

$$\Delta\vartheta_3 = \vartheta_3'\,\Delta\vartheta_2 \qquad (9)$$

$$\Delta\vartheta_4 = \vartheta_4'\,\Delta\vartheta_2 \qquad (10)$$

The new position of the linkage is determined as

$$(\mathcal{Z}_1)_2 = (\mathcal{Z}_1)_1$$

$$(\mathcal{Z}_2)_2 = e^{i\Delta\vartheta_2}\,(\mathcal{Z}_2)_1$$

$$(\mathcal{Z}_3)_2 = e^{i\Delta\vartheta_3}\,(\mathcal{Z}_3)_1$$

$$(\mathcal{Z}_4)_2 = e^{i\Delta\vartheta_4}\,(\mathcal{Z}_4)_1$$

This will be the starting position for the next step. For comparison, the exact solution is

$$\tan\frac{\psi}{2} = \frac{A \pm \sqrt{A^2+B^2-C^2}}{B+C} \qquad (11)$$

where

$$A = \sin\varphi$$

$$B = -\frac{\alpha_1}{\alpha_2} + \cos\varphi$$

$$C = \frac{\alpha_1^2+\alpha_2^2+\alpha_4^2-\alpha_3^2}{2\alpha_2\alpha_4} - \frac{\alpha_1}{\alpha_4}\cos\varphi$$

The crack and rocker linkage of fig 3 was employed, drawn to scale. The ratio

$$\varepsilon = \left[(\psi-\psi_0) - \vartheta_4 \right] / \vartheta_4$$

giving the relative output error of the link 4xas plotted vs the output ϑ_4 with the input angle step $\Delta\vartheta_2$ used as a parameter, fig.4,5.

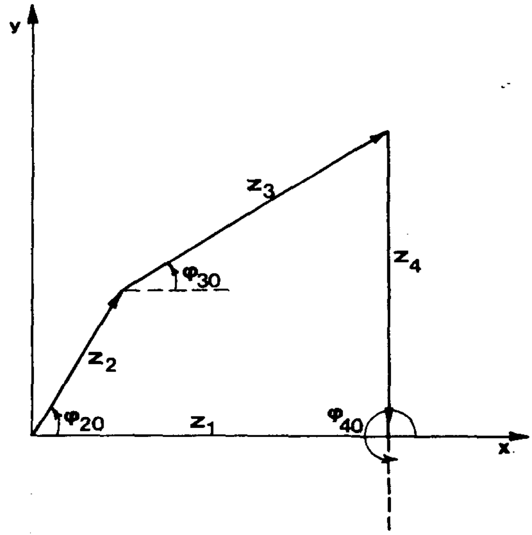

Fig. 3. Test linkage, 4-bar.

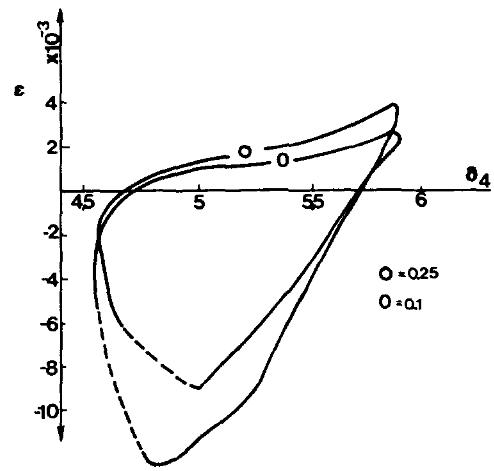

Fig. 5. Error, integration step 0.1, 0.25 rad.

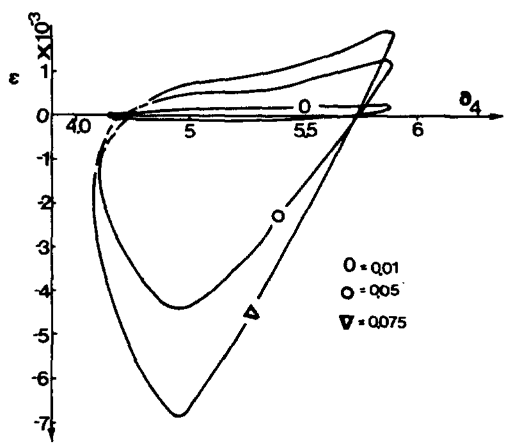

Fig. 4. Error, integration step .01 to .075 rad.

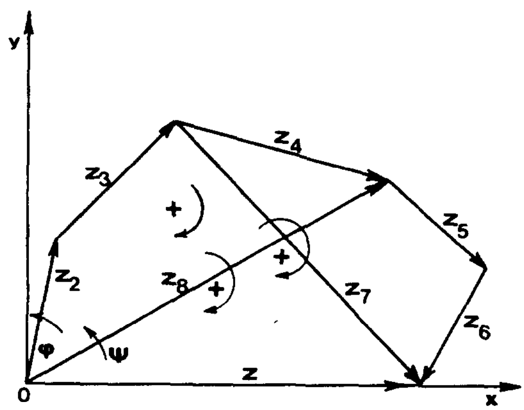

Fig. 6. Test linkage, 8-bar.

It is evident that large integration steps can be used without appreciable error. Even for a step of 0.25 rad, the maximum error is about 13%. Since, even for plotting purposes the step should be below 0.01 rad, the corresponding maximum error is below 0,02%.This explains why higher order integration methods are not necessary for this case.

EIGHT-BAR LINKAGE

The same algorithm was used to analyze the 8-bar linkage shown on scale in Fig. 6. With the notation shown and with input on link 2, the symbolic representation of the linkage is

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -1 \\ i & i & i & i & 0 & 0 \\ i & 0 & 0 & 0 & i & 0 \\ i & i & 0 & 0 & 0 & -i \end{bmatrix} \qquad (12)$$

$$B = \begin{bmatrix} 1 & 1 & 1 & -i & -i & -i \end{bmatrix}$$

In this case there is no explicit expression for the performance of the linkage to be used for the computation of the error. An experimental procedure was used mainly to search for chaotic programming errors because the numerical accuracy of the algorithm was far beyond the manufacturing accuracy which could be obtained for the experimental procedure. The output in link 8 was plotted in fig. 7.

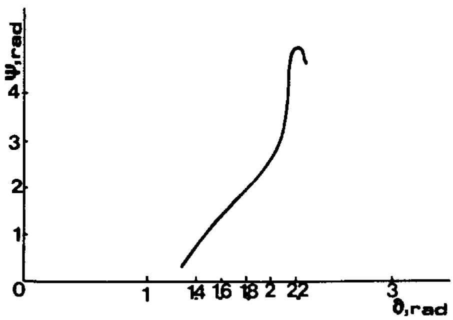

Fig. 7. 8-bar linkage function.

CONCLUSION

A general purpose algorithm was devised for kinematic analysis of planar linkages. The algorithm needs only the initial position of the mechanism and a symbolic representation of the link interconnections. In this work, only linkages with links and pins were considered. Extension to other elements, such as sliding elements and gears is straighforward.

Application to a four-bar linkage showed that excellent numerical accuracy can be obtained with rather large intergration steps which makes the method economical to run. Further application to an eight-bar linkage demostrated the applicability of the method to complex problems.

Since the method is sequential it does not suffer from the double-solution problems encountered in analytical methods.

LITTERATURE

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