Math Junkyard
Bo Ma

Linear Algebra Essence

Vector

Geometric View	Coordinate View
An arrow inside in a coordinate system with the tail siting at the origin.	The coordinate of a vector gives instructions how to get from the tail of the vector at the origin to its tip.

- The **span** of n vectors is the set of all their **linear combinations**.
- When dealing with a collection of vectors, it is useful to think of them all as points. For example, the span can be think of as a space filled by points.
- The **basis** of a vector space is a set of linearly independent vectors that span the full space.

Linear Transformations and Matrices

- **Linear Transformation:** All lines remain lines (parallel and evenly spaces grid lines). Origin remains fixed.
- Think of matrix multiplication as linear transformation. $[x, y]^T$ is the coordinate in a vector space with basis $\hat{i} = [1, 0]^T$ and $\hat{j} = [0, 1]^T$. $[a, c]^T$ and $[b, d]^T$ are where the $\hat{i}$ and $\hat{j}$ landed (respect to the original coordinate system) after the transformation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- **A matrix defines a certain transformation in space.**
- **Determinant**: How much are the areas in the original space scaled by a linear transformation (or matrix multiplication)? Negative determinant means the orientation of the space has been inverted. 0 determinant means squish 2D space into line or point and 3D space into plane, line, or point.
- **Linear system of equations**: $A\vec{x} = \vec{b}$, A is the matrix that transform $\vec{x}$ to $\vec{b}$.
  - If $\det(A) \neq 0$, A^{-1} exists.
  - If $\det(A) = 0$, input is squishes to a lower dimensional output; therefore, no A^{-1} exists. The solution could still exist as long as $\vec{b}$ lives in the transformed low dimensional space.
- **Rank**: Number of dimensions in the output of a transformation. The matrix is full rank if the number of dimensions in the input space is the same as the output space, e.g.,

$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}$$

- **Column Space** of A: Set of all possible outputs of $A\vec{v}$, i.e., span of columns.

Linear Algebra Essence

- **Null Space** of A: Set of all possible solutions of $A\vec{v} = 0$. If 2D squishes to 1D, there is a full line of vectors get squished to the origin. If 3D squishes to 2D/1D, there is a full line/plane of vectors get squished to the origin.

Dot Product and Duality

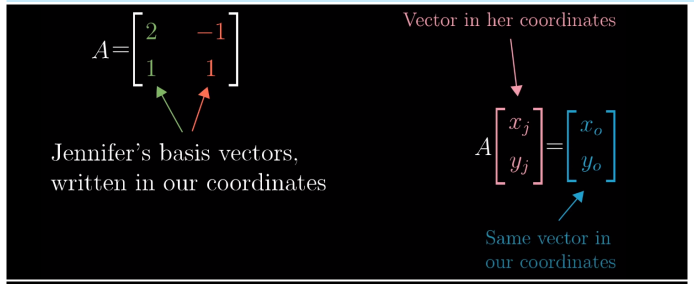
- $\vec{v} \cdot \vec{w} =$ (length of projected $\vec{w}$)(length of $\vec{v}$).
- Why on earth the dot product has anything to do with projection?
  - Anytime you have a 2D-to-1D linear transform whose output is the number line. This is always a unique vector correspond to the transformation, in the sense that applying the transformation is the same as taking a dot product with that vector.
  - **Duality**: The dual of a vector is the linear transformation it encodes. The dual of linear transformation from some space to 1D is a certain vector in that space.

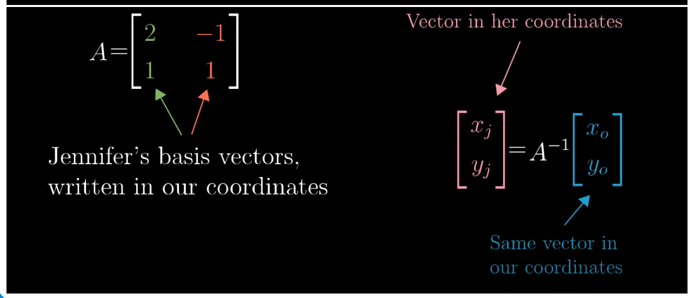
Cross Product

$\vec{v} \times \vec{w} = \vec{p}$, where $\vec{p}$ is perpendicular to the parallelogram

with length equals to det($\begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$)

Change of Basis





Linear Algebra Essence

Eigenvectors and eigenvalues

To understand what a matrix transform it doing, you could read off the columns of the matrix as the landing spots for basis vectors. Often, to understand what the matrix transformation really does less dependent on the coordinate system is to find the eigenvectors and eigenvalues.

1. $A\vec{v} = \lambda\vec{v}$, the goal is to find λ and $\vec{v}$ for a given transformation A. $\vec{v}$ is a eigenvector that stays on its span after the transformation and scaled by λ.
2. $(A - \lambda I)\vec{v} = \vec{0}$, we want to find a nonzero solution for $\vec{v}$. In other words, $(A - \lambda I)$ needs to squish $\vec{v}$ to $\vec{0}$ => $\det(A - \lambda I) = 0$

Eigenbasis

$$\begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

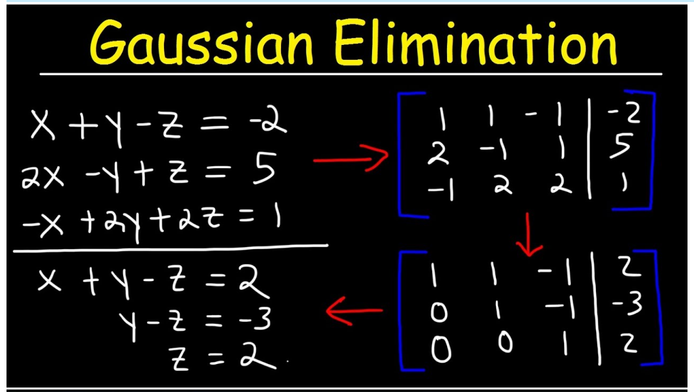
Diagonal matrix has all basis as eigenvectors with diagonal values being eigenvalues. If we have enough eigenvectors for us to transform our coordinate system to the eigenspace, matrix operations would be much easier there.

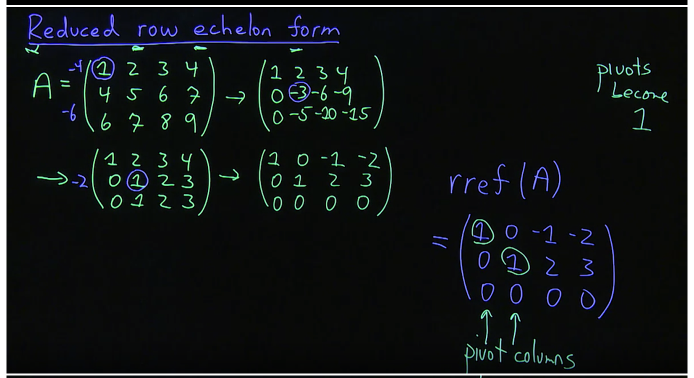
Math Junkyard

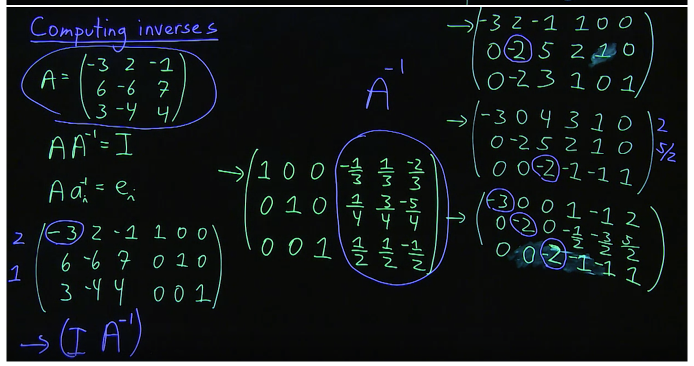
Bo Ma

Systems of Linear Equations

Gaussian Elimination

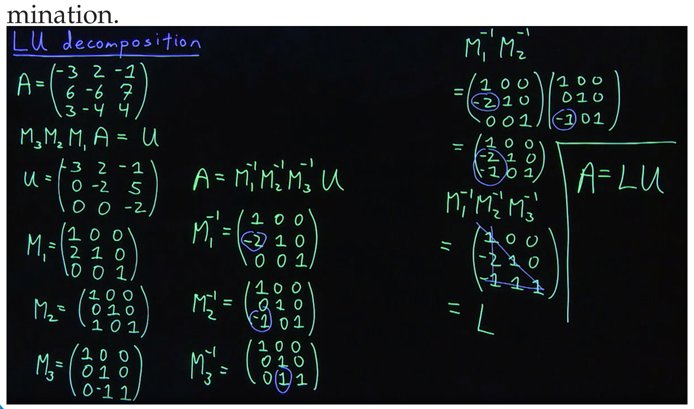






LU Decomposition

LU decomposition is essentially the result of Gaussian elimination.



Vector Spaces

Definition:

- Compose of set of vectors and scalars.
- Closed under vector addition and scalar multiplication.

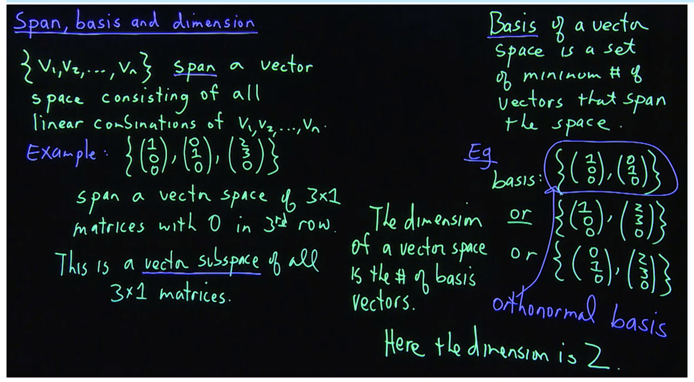
Vector spaces associated with matrices:

- Null space
- Column space
- Row space
- Left null space

Linear Independence

The set of vectors $\{u_1, u_2, \ldots, u_n\}$ are linearly independent if $c_1u_1 + c_2u_2 + \cdots + c_nu_n = 0$ has the only solution $c_1 = c_2 = \cdots = c_n = 0$.

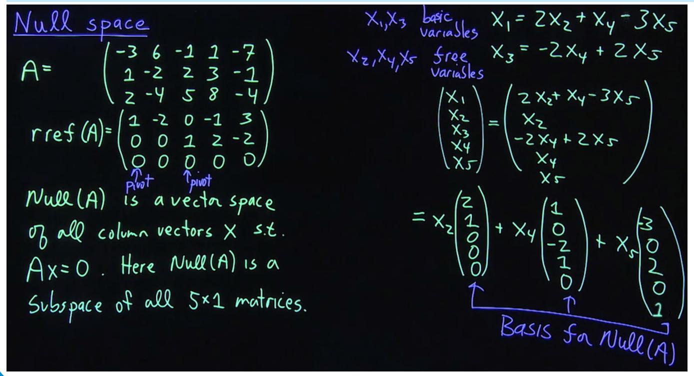
Span, Basis and Dimension

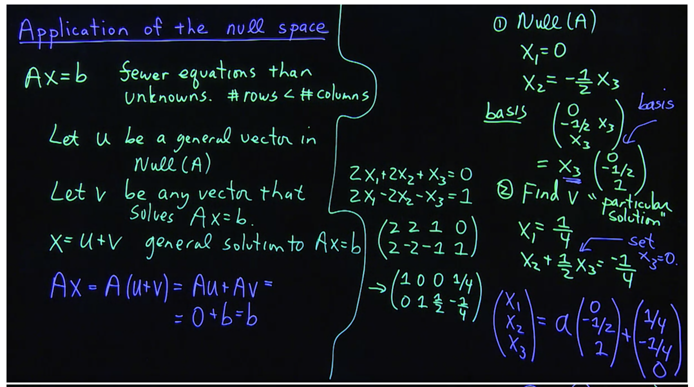


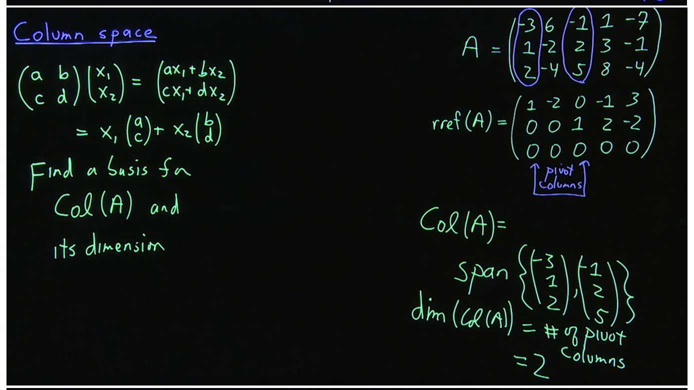
Gram-Schmidt Process

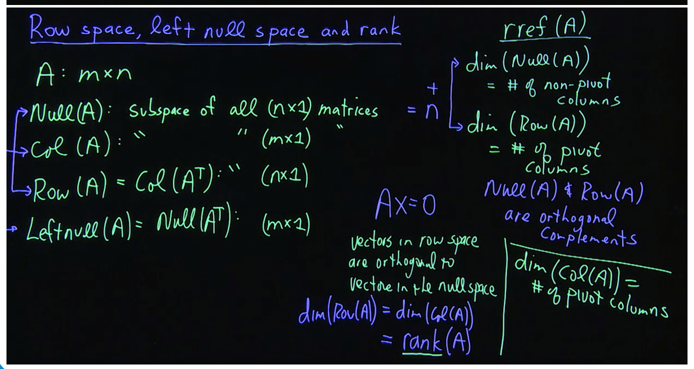
Construct orthonormal basis.

Fundamental Subspaces of a Matrix

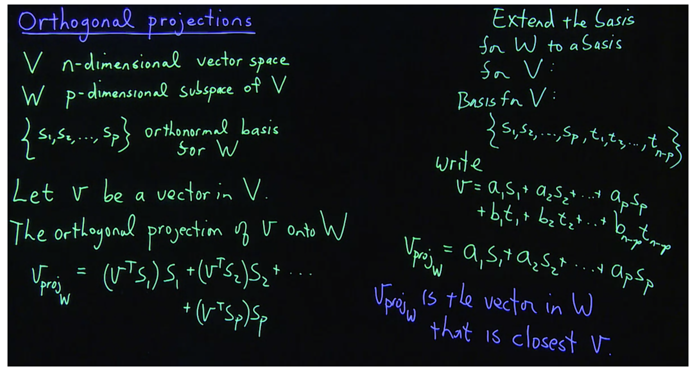




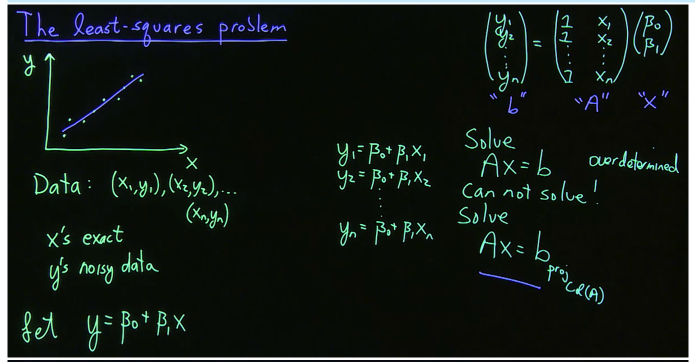


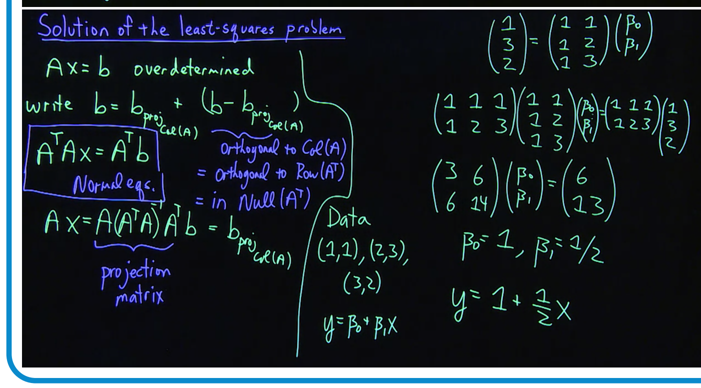


Orthogonal Projections



Least Squares



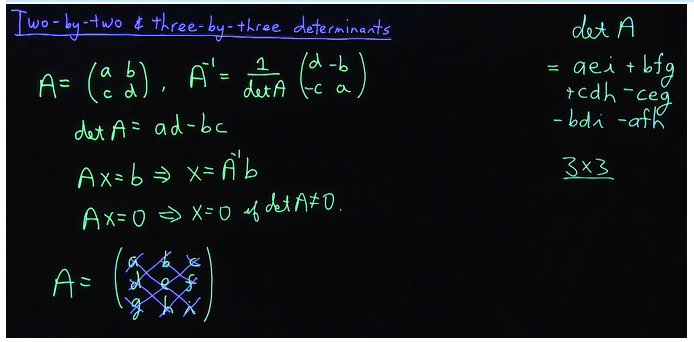


Math Junkyard

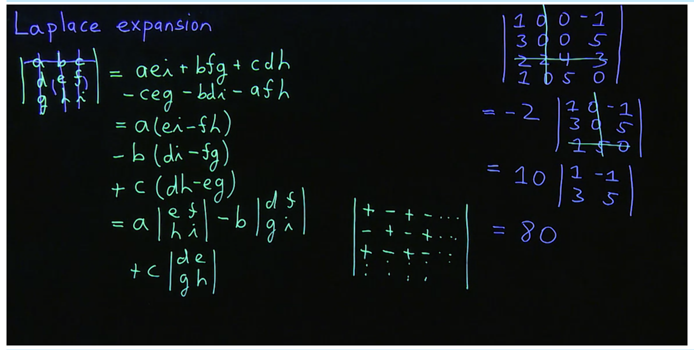
Bo Ma

Determinants

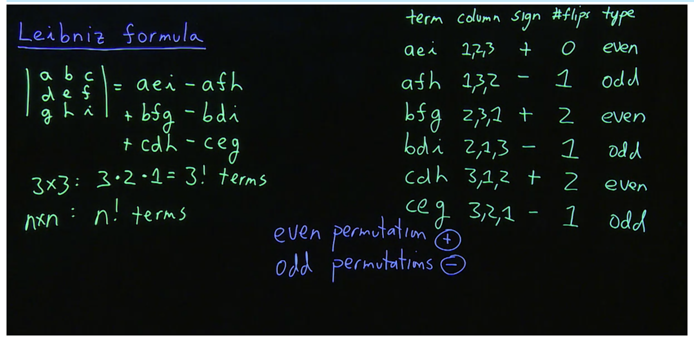
Three-by-Three Determinants



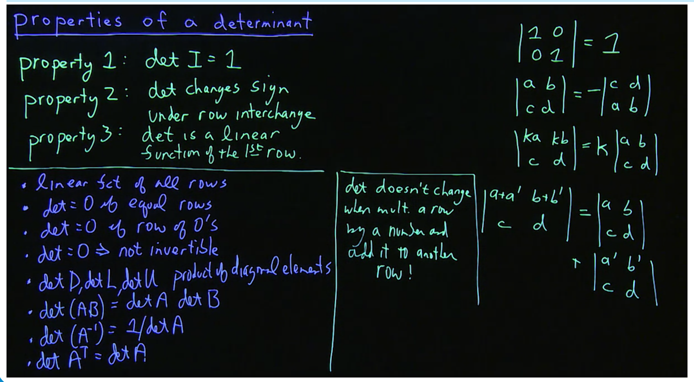
Laplace expansion



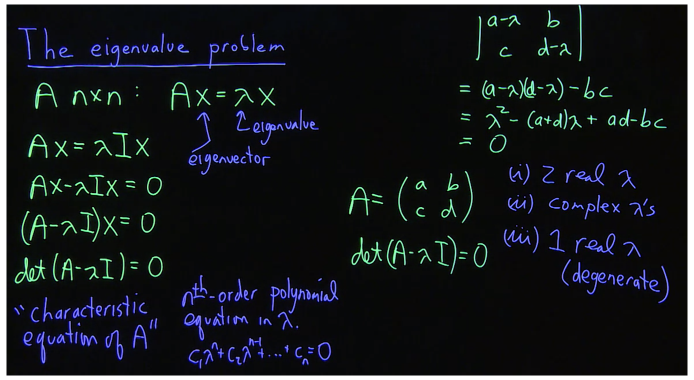
Leibniz Formula



Determinant Properties



Eigenvalues and Eigenvectors



Matrix Diagonalization