

# MATH JUNKYARD

BO MA

## LINEAR ALGEBRA ESSENCE

### Vector

#### Geometric View

An arrow inside in a coordinate system with the tail sitting at the origin.

- The **span** of  $n$  vectors is the set of all their **linear combinations**.
- When dealing with a collection of vectors, it is useful to think of them all as points. For example, the span can be thought of as a space filled by points.
- The **basis** of a vector space is a set of linearly independent vectors that span the full space.

#### Linear Transformations and Matrices

- Linear Transformation:** All lines remain lines (parallel and evenly spaced grid lines). Origin remains fixed.
- Think of matrix multiplication as linear transformation.  $[x, y]^T$  is the coordinate in a vector space with basis  $\hat{i} = [1, 0]^T$  and  $\hat{j} = [0, 1]^T$ .  $[a, c]^T$  and  $[b, d]^T$  are where the  $\hat{i}$  and  $\hat{j}$  landed (respect to the original coordinate system) after the transformation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- A **matrix defines a certain transformation in space**.
- Determinant:** How much are the areas in the original space scaled by a linear transformation (or matrix multiplication)? Negative determinant means the orientation of the space has been inverted. 0 determinant means squish 2D space into line or point and 3D space into plane, line, or point.
- Linear system of equations:**  $A\vec{x} = \vec{b}$ ,  $A$  is the matrix that transforms  $\vec{x}$  to  $\vec{b}$ .
  - If  $\det(A) \neq 0$ ,  $A^{-1}$  exists.
  - If  $\det(A) = 0$ , input is squished to a lower dimensional output; therefore, no  $A^{-1}$  exists. The solution could still exist as long as  $\vec{b}$  lives in the transformed low dimensional space.
- Rank:** Number of dimensions in the output of a transformation. The matrix is full rank if the number of dimensions in the input space is the same as the output space, e.g.,

$$\begin{bmatrix} 2 & 0 \\ -1 & 1 \\ 2 & 1 \end{bmatrix}$$

- Column Space of  $A$ :** Set of all possible outputs of  $A\vec{v}$ , i.e., span of columns.

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- Null Space of  $A$ :** Set of all possible solutions of  $A\vec{v} = 0$ . If 2D squishes to 1D, there is a full line of vectors get squished to the origin. If 3D squishes to 2D/1D, there is a full line/plane of vectors get squished to the origin.

#### Dot Product and Duality

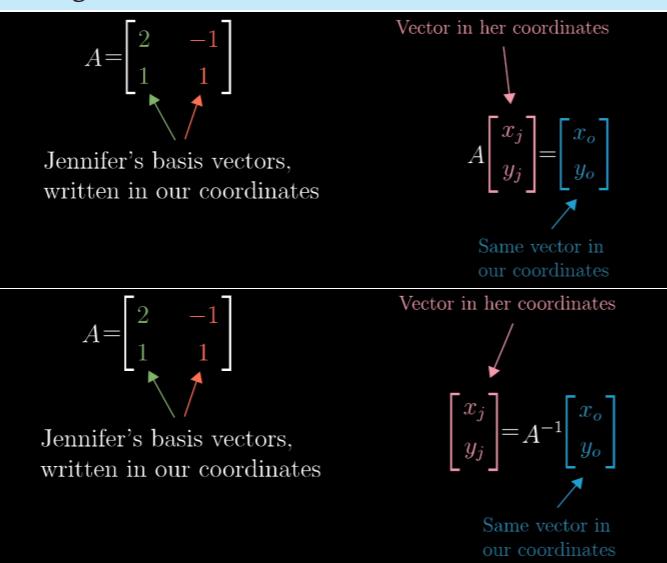
- $\vec{v} \cdot \vec{w} = (\text{length of projected } \vec{w})(\text{length of } \vec{v})$ .
- Why on earth the dot product has anything to do with projection?

- Anytime you have a 2D-to-1D linear transform whose output is the number line. This is always a unique vector correspond to the transformation, in the sense that applying the transformation is the same as taking a dot product with that vector.
- Duality:** The dual of a vector is the linear transformation it encodes. The dual of linear transformation from some space to 1D is a certain vector in that space.

#### Cross Product

$\vec{v} \times \vec{w} = \vec{p}$ , where  $\vec{p}$  is perpendicular to the parallelogram with length equals to  $\det([\vec{v} \ \vec{w}])$

#### Change of Basis



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### Eigenvectors and eigenvalues

To understand what a matrix transform is doing, you could read off the columns of the matrix as the landing spots for basis vectors. Often, to understand what the matrix transformation really does less dependent on the coordinate system is to find the eigenvectors and eigenvalues.

- $A\vec{v} = \lambda\vec{v}$ , the goal is to find  $\lambda$  and  $\vec{v}$  for a given transformation  $A$ .  $\vec{v}$  is a eigenvector that stays on its span after the transformation and scaled by  $\lambda$ .
- $(A - \lambda I)\vec{v} = \vec{0}$ , we want to find a nonzero solution for  $\vec{v}$ . In other words,  $(A - \lambda I)$  needs to squish  $\vec{v}$  to  $\vec{0}$   $\Rightarrow \det(A - \lambda I) = 0$

### Eigenbasis

$$\begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Diagonal matrix has all basis as eigenvectors with diagonal values being eigenvalues. If we have enough eigenvectors for us to transform our coordinate system to the eigenspace, matrix operations would be much easier there.

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## SYSTEMS OF LINEAR EQUATIONS

### Gaussian Elimination

$$\begin{array}{l} X + Y - Z = -2 \\ 2X - Y + Z = 5 \\ -X + 2Y + 2Z = 1 \end{array} \rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & 2 \end{array} \right] \quad \begin{matrix} \downarrow \\ Z = 2 \\ Y - Z = -3 \\ X + Y - Z = 2 \end{matrix}$$

Reduced row echelon form

$$A = \begin{pmatrix} 1 & 1 & -1 & -2 \\ 2 & -1 & 1 & 5 \\ -1 & 2 & 2 & 1 \end{pmatrix} \xrightarrow{\text{rref}(A)} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\xrightarrow{\text{pivot columns}}$

Computing inverses

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 6 \\ 3 & 6 & 7 & 8 \\ 4 & 8 & 9 & 10 \end{pmatrix} \xrightarrow{\text{rref}(A)} \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$AA^{-1} = I$$

$$A^{-1} = \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{\text{pivot columns}}$$

### LU Decomposition

LU decomposition is essentially the result of Gaussian elimination.

LU decomposition

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 6 & -6 & 2 \\ 3 & -4 & 4 \end{pmatrix} \quad M_1^{-1} M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_3 M_2 M_1 A = U$$

$$U = \begin{pmatrix} 3 & 2 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -2 \end{pmatrix} \quad A = M_1^{-1} M_2^{-1} M_3^{-1} U$$

$$M_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad M_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

## VECTOR SPACES

### Definition:

- Compose of set of vectors and scalars.
- Closed under vector addition and scalar multiplication.

Vector spaces associated with matrices:

- Null space
- Column space
- Row space
- Left null space

### Linear Independence

The set of vectors  $\{u_1, u_2, \dots, u_n\}$  are linearly independent if  $c_1 u_1 + c_2 u_2 + \dots + c_n u_n = 0$  has the only solution  $c_1 = c_2 = \dots = c_n = 0$ .

### Span, Basis and Dimension

#### Span, basis and dimension

$\{v_1, v_2, \dots, v_n\}$  span a vector space consisting of all linear combinations of  $v_1, v_2, \dots, v_n$ .

Example:  $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

span a vector space of  $3 \times 1$  matrices with 0 in 3rd row.

This is a vector subspace of all  $3 \times 1$  matrices.

The dimension of a vector space is the # of basis vectors.

Here the dimension is 2.

Basis of a vector space is a set of minimum # of vectors that span the space.

Eg. basis:  $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

or  $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

or  $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

orthonormal basis

### Gram-Schmidt Process

Construct orthonormal basis.

### Fundamental Subspaces of a Matrix

#### Null space

$$A = \begin{pmatrix} -3 & 6 & -1 & 2 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$$

$$\text{rref}(A) = \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Null(A) is a vector space of all column vectors  $X$  s.t.

$AX=0$ . Here Null(A) is a subspace of all  $5 \times 1$  matrices.

$x_1, x_3$  basic variables

$x_2, x_4, x_5$  free variables

$x_3 = -2x_4 + 2x_5$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{pmatrix}$$

$$= x_2 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

Basis for Null(A)

Application of the null space

$AX=b$  Fewer equations than unknowns. # rows < # columns

Let  $u$  be a general vector in Null(A)

Let  $v$  be any vector that solves  $AX=b$ .

$X=u+v$  general solution to  $AX=b$

$AX=A(u+v)=Au+Av=b$

① Null(A)

$X_1=0$

$X_2=-\frac{1}{2}X_3$

basis  $\begin{pmatrix} 0 \\ -\frac{1}{2}X_3 \\ X_3 \end{pmatrix}$

② Find  $v$  "particular solution"

$ZX_1+2X_2+X_3=0$

$2X_2-X_3=1$

$X_1=\frac{1}{4}X_3$  set

$X_2+\frac{1}{2}X_3=-\frac{1}{2}X_3=0$

$X_3=0$

$\begin{pmatrix} 2 & 2 & 1 & 0 \\ 0 & 2 & -1 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 & 0 & \frac{1}{4} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ 0 \end{pmatrix}$

Column space

$(a \ b)(x_1) = (ax_1 + bx_2)$

$= x_1(a) + x_2(b)$

Find a basis for Col(A) and its dimension

$A = \begin{pmatrix} -3 & 6 & -1 & 2 & -7 \\ 1 & -2 & 0 & -1 & 3 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$

$\text{rref}(A) = \begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

$\uparrow \text{columns}$

$\text{Col}(A) = \text{Span} \left\{ \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 6 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix} \right\}$

$\dim(\text{Col}(A)) = \# \text{ of pivot columns} = 3$

Row space, left null space and rank

$A: m \times n$

$\text{rref}(A)$

$\dim(\text{Null}(A)) = \# \text{ of non-pivot columns}$

$\dim(\text{Row}(A)) = \# \text{ of pivot columns}$

$\text{Row}(A) = \text{Col}(A^T): \quad (n \times 1)$

$\text{Leftnull}(A) = \text{Null}(A^T): \quad (m \times 2)$

$\text{Ax}=0$

Vectors in row space are orthogonal to vectors in left null space

$\dim(\text{Row}(A)) = \dim(\text{Col}(A)) = \# \text{ of pivot columns} = \text{rank}(A)$

## ORTHOGONAL PROJECTIONS

### Orthogonal projections

$V$  n-dimensional vector space

$W$  p-dimensional subspace of  $V$

$\{s_1, s_2, \dots, s_p\}$  orthonormal basis for  $W$

Let  $v$  be a vector in  $V$ .

The orthogonal projection of  $v$  onto  $W$

$$v_{\text{proj}_W} = (v^T s_1) s_1 + (v^T s_2) s_2 + \dots + (v^T s_p) s_p$$

$v_{\text{proj}_W}$  is the vector in  $W$

that is closest  $v$ .

Extend the basis for  $W$  to a basis for  $V$ :

Basis for  $V$ :

$$\{s_1, s_2, \dots, s_p, t_1, t_2, \dots, t_{n-p}\}$$

Write

$$v = a_1 s_1 + a_2 s_2 + \dots + a_p s_p + b_1 t_1 + b_2 t_2 + \dots + b_{n-p} t_{n-p}$$

$v_{\text{proj}_W}$  is the vector in  $W$

that is closest  $v$ .

### Least Squares

#### The least-squares problem

$$\begin{array}{ccc} y & \uparrow & \\ & \nearrow & \\ & & x \end{array}$$

Data:  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$

$x$ 's exact

$y$ 's noisy data

$$\text{fit } y = \beta_0 + \beta_1 x$$

Solve  $Ax = b$  overdetermined

Can not solve!

Solve  $Ax = b$

$$\xrightarrow{\text{proj}_{\text{Col}(A)}}$$

Solution of the least-squares problem

$Ax = b$  overdetermined

write  $b = b_{\text{proj}_{\text{Col}(A)}} + (b - b_{\text{proj}_{\text{Col}(A)}})$

$$A^T A x = A^T b$$

Normal eqs.

$$A^T A x = A^T b$$

$b_{\text{proj}_{\text{Col}(A)}} = b - (b - b_{\text{proj}_{\text{Col}(A)}})$

Data

$$(1, 1), (2, 3), (3, 2)$$

$\beta_0 = 1, \beta_1 = \frac{1}{2}$

$$y = 1 + \frac{1}{2}x$$

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## DETERMINANTS

### Three-by-Three Determinants

Two-by-two & three-by-three determinants

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\det A = ad - bc$$

$$Ax = b \Rightarrow x = A^{-1}b$$

$$Ax = 0 \Rightarrow x = 0 \text{ if } \det A \neq 0.$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

### Laplace expansion

Laplace expansion

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} d & e & f \\ g & h & i \end{vmatrix} + b \begin{vmatrix} e & f \\ h & i \end{vmatrix} + c \begin{vmatrix} d & f \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(gh - di) + c(dh - eg)$$

$$= a \begin{vmatrix} 1 & 0 & -1 \\ 3 & 0 & 5 \\ 2 & 4 & 3 \end{vmatrix} - b \begin{vmatrix} 2 & 0 & -1 \\ 3 & 0 & 5 \\ 2 & 4 & 3 \end{vmatrix} + c \begin{vmatrix} 1 & 0 & -1 \\ 3 & 5 & 0 \\ 2 & 4 & 3 \end{vmatrix}$$

$$= 10 \begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} = 80$$

### Leibniz Formula

Leibniz formula

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh + bdi - bgi + cdh - cei$$

3x3:  $3 \cdot 2 \cdot 1 = 3!$  terms

$n \times n$ :  $n!$  terms

term column sign #flips type

aei	2,3,1	+	0 even
afh	2,3,2	-	1 odd
bgi	2,3,1	+	2 even
bdi	2,1,3	-	1 odd
cdh	3,1,2	+	2 even
cei	3,2,1	-	1 odd

even permutation  $\oplus$   
odd permutations  $\ominus$

### Determinant Properties

#### Properties of a determinant

property 1:  $\det I = 1$

property 2:  $\det$  changes sign under row interchange

property 3:  $\det$  is a linear function of the 1st row.

- linear set of all rows
- $\det = 0$  if equal rows
- $\det = 0$  if 2 rows of 0's
- $\det = 0 \Rightarrow$  not invertible
- $\det D, \det L, \det U$  product of diagonal elements
- $\det (AB) = \det A \det B$
- $\det (A') = 1/\det A$
- $\det A' = \det A$

## EIGENVALUES AND EIGENVECTORS

### The eigenvalue problem

$$A \text{ } nxn : Ax = \lambda x$$

$$Ax = \lambda Ix \quad \begin{matrix} \uparrow \\ \text{eigenvalue} \\ \downarrow \\ \text{eigenvector} \end{matrix}$$

$$Ax - \lambda Ix = 0$$

$$(A - \lambda I)x = 0$$

$$\det(A - \lambda I) = 0$$

"characteristic equation of  $A$ "  $\xrightarrow{\text{n-th order polynomial}}$   $c_n \lambda^n + c_{n-1} \lambda^{n-1} + \dots + c_0 = 0$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

$$= 0$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$(a-d)/2 \text{ real } \lambda$$

$$\det(A - \lambda I) = 0 \text{ complex } \lambda's$$

$$\text{I real } \lambda \text{ (degenerate)}$$

Matrix Diagonalization

### Matrix diagonalization

$$2 \times 2 : \lambda_1, x_1 = \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}$$

$$A \lambda_2, x_2 = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$$

$$Ax_1 = \lambda_1 x_1$$

$$Ax_2 = \lambda_2 x_2$$

$$= \begin{pmatrix} \lambda_1 x_{11} & \lambda_2 x_{12} \\ \lambda_1 x_{21} & \lambda_2 x_{22} \end{pmatrix}$$

$$= \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$AS = S\Lambda$$

$$A = S\Lambda S^{-1}$$

$$\Lambda = S^{-1}AS$$

### Powers of a matrix

$$A = S\Lambda S^{-1}$$

$$A^2 = (S\Lambda S^{-1})(S\Lambda S^{-1})$$

$$= S\Lambda^2 S^{-1}$$

$$\vdots$$

$$A^p = S\Lambda^p S^{-1}$$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^p = \begin{pmatrix} \lambda_1^p & 0 \\ 0 & \lambda_2^p \end{pmatrix}$$

$$\det \text{ doesn't change when mult. a row by a number and add it to another row!}$$

$$\begin{vmatrix} a & b & c \\ a' & b' & c \\ c & d & d \end{vmatrix} = \begin{vmatrix} a & b \\ a' & b' \\ c & d \end{vmatrix}$$

$$\uparrow$$

$$\begin{vmatrix} a & b \\ a' & b' \\ c & d \end{vmatrix}$$