Why is Hume's Principle not good enough for Frege?

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1 Introduction

Unfair as it is, the most well-known property of Frege's foundation of arithmetic seems to be "inconsistent". His attempt was to interpret arithmetic in second-order logic with the following "Basic Law V":

(1) $$\nabla F = \nabla G \leftrightarrow \forall x(F(x) = G(x))$$

where ∇F denotes the *extension* of a concept (equivalent to its graph). This axiom expresses that two concepts have the same extension iff they assign the same evaluations for all the entities. Russell discovered that this axiom, together with the comprehension principle (which asserts that every formula corresponds to some predicate) in second-order logic, leads to inconsistency

(2) $$\exists X(\nabla X \in X \leftrightarrow \nabla X \notin X).$$

This paradox is regarded as the deathblow for Frege's system. However, in recent years it has been realized that there are various ways of salvaging Frege's system from such inconsistency. Frege's arithmetic can be shown consistent if any of the following weakening is applied:

(i) First-order Frege arithmetic, whose only nonlogical vocabulary is the extension operator ∇, and whose axioms are all first-order instances of Basic Law V is shown to be consistent [2].

(ii) If the concept in the comprehension principle (see Section 2) is restricted to a certain form, the system can become equi-interpretable with various subsystems of second-order arithmetic [11].

(iii) If the Basic Law V is substituted for the so-called Hume's Principle, which expresses that two concepts have the same cardinality if there exists a bijection between them, then the resulted system is equi-interpretable with full second-order arithmetic.

The third direction retains the most of Frege's original system, and has drawn most philosophical attention. Most notably, it inspires the so-called *neo-logicism*, which aims to resurrect Frege's logicism program on the base of Hume's Principle. Neo-logicists hold the position that since arithmetic can be deduced from second-order logic and Hume's Principles alone, and it is possible to build “a form of epistemic foundationalism in which logic is intended to play a foundational role in resolving specific epistemic challenges, such as our knowledge of arithmetic and analysis.” [4]

Interestingly, Frege himself had the chance of being a neo-logicist. If we take a close examination of Frege’s original program of interpreting arithmetic, it was divided in two steps:

(i) Reduce arithmetic to second-order logic plus Hume’s principle, as Frege wrote in §87 of [7]:

> I hope I may claim in the present work to have made it probable that the laws of arithmetic are analytic judgements and consequently a priori. Arithmetic thus becomes simply a development of logic, and every proposition of arithmetic a law of logic, albeit a derivative one.

(ii) Further reduce Hume’s Principle to more basic laws, as in §0 pf [6]:

> In my Grundlagen der Arithmetik, I sought to make it plausible that arithmetic is a branch of logic and need not borrow any ground of proof whatever from either experience or intuition. In the present book, this shall be confirmed, by the derivation of the simplest laws of Numbers by logical means alone.

Given such a clear separation of the two steps, it is reasonable to imagine that Frege had the option of claiming that he has reduced arithmetic to a few very basic principles: Although the second step of the program was not completed, he could claim second-order logic plus Hume’s Principle to be the foundation of arithmetic, as is now suggested by the neo-logicists. Richard Heck in [9] explicitly asked this question as follows (after showing Frege’s original proofs of the Dedekind-Peano axioms from Hume’s Principle).

> All of this having been said, the question arises why, upon receiving Russell’s famous letter, Frege did not simply drop Axiom V, install Hume’s Principle as an axiom, and claim himself to have established logicism anyway.
>
> What I do know is that the questions lately raised need answers: for, until we have such answers, we shall not understand the significance that Axiom V had for Frege, since we shall not understand why he could not abandon it in favor of Hume’s Principle. That is to say, we shall not understand how he conceived the logicist project.

This is the question I aim to investigate.

This question can have a short answer: Russell’s paradox put something very fundamental in Frege’s belief into question, which can hardly be circumvented by a technical solution or partial claim. Frege’s bewilderment about how to rethink about his logicism program is best reflected in his letter to Russell [5]:

> Your discovery of the contradiction has surprised me beyond words and, I should almost like to say, left me thunderstruck, because it has rocked the ground on which I meant to build arithmetic...It is all the more serious as the collapse of my law V seems to undermine not only

> the foundations of my arithmetic but the only possible foundations of arithmetic as such... The question is, how do we apprehend logical objects? And I have found no other answer to it than this, We apprehend them as extensions of concepts, or more generally, as value-ranges of functions. I have always been aware that there were difficulties with this, and your discovery of the contradiction has added to them; but what other way is there?

The fact that the underlying notions themselves can give rise to false knowledge baffled Frege. The notions of concepts, functions and their course-of-values are regarded as self-evident to Frege, about which he holds a rationalist's view. He regarded them as "the only possible foundation of arithmetic as such". Frege's commitment to the existence of the third realm, and more importantly our epistemological competence of *grasping* the realm, is fundamental to his program. Russell's paradox shows that the very foundation of this program is shaken because the properties of the logical objects, if they exist at all, can be even more puzzling than arithmetic itself. This very fact is disturbing in a much deeper sense for Frege, beyond the failure of deriving arithmetic *per se*.

Along these lines, I aim to understand the reason why Frege regarded Hume's Principle as not fundamental enough, while Basic Law V is, had it been correct. To do this, we need to first understand Frege's plan of using Hume's Principle to deduce the Dedekind-Peano Axioms. Then, I look at Frege's own objection to relying on Hume's Principle as a basic law, to which he raise the famous Julius Caesar Problem. By examining Frege's view of this problem, I aim to clarify Frege's implicit commitment on both ontological and epistemological levels.

On the other hand, the fact that Frege choosed to accept Law V instead, putting him under the colored-glasses of ours, shows his naivety about our competence in "grasping"the logical objects. I compare our contemporary skeptic attitude towards finding any analytic foundation for arithmetic with Frege's optimism, and conclude that this shows the valuable progress we have achieved in our understanding of the notion of analyticity through logic and analytic philosophy.

2 Frege's Theorem

Frege's theorem, as termed by George Boolos, states that the Dedekind-Peano Axioms can be interpreted in second-order logic with Hume's Principle as the only non-logical axiom. The proof of the theorem has been extracted from Frege's original works and nicely presented in [9, 13]. In this section, I outline the basic constructions in the proof to facilitate later discussions.

We assume a standard system of second-order logic. The only non-logical axiom that is needed in Frege's arithmetic is the Hume's Principle, which we now formally state.

Definition 2.1 (Cardinality Operator)**.** *Let P be any predicate. The cardinality of P is denoted as $\#P$.*

Definition 2.2 (Hume's Principle)**.** *The cardinality operator is interpreted by the Hume's principle:*

(3) $$\#P = \#Q \leftrightarrow P \simeq Q$$

where

$$P \simeq Q = \exists R(\forall x(Px \to \exists! y(Qy \wedge Rxy) \wedge \forall x(Qx \to \exists! y(Py \wedge Ryx)))) \tag{4}$$

. This condition shows that there exists a bijection between P and Q.

Naturally, cardinal numbers can be defined using existentially quantification:

Definition 2.3 (Cardinal numbers)**.**

$$Nx \leftrightarrow \exists P(x = \#P) \tag{5}$$

A main rule in Frege's system is the substitution rule, which asserts that any atomic formula can be substituted by a complex formula. It leads to the following so-called comprehension principle, which is important in the definition of numbers:

Definition 2.4 (Comprehension Principle)**.**

$$\exists P \forall x(Px \leftrightarrow \varphi(x)) \tag{6}$$

where $\varphi(x)$ does not contain free occurrence of P or x. (Here x can be a vector of n variables to cover the case of n-ary concepts.)

It is easy to see how this principle is derived:

$$\begin{array}{lll} (7) & \forall x(Fx \leftrightarrow Fx) & \text{(Axiom)} \\ (8) & \exists G \forall x(Gx \leftrightarrow Fx) & \text{(Existential Quantification)} \\ (9) & \exists G \forall x(Gx \leftrightarrow \varphi(x)) & \text{(Substitution)} \end{array}$$

Note that the step 8 suggested that a strong existential commitment (first suggested by Boolos in [1]). We will return back to this point in Section 4.

The principle says that for any formula $\varphi(x)$, there is a corresponding concept P. This allows us to introduce the λ expressions as a familiar notation.

Definition 2.5 (λ-conversion)**.** *For any formula $\varphi(x)$, $\lambda\varphi(x)$ is a predicate that satisfies:*

$$\forall y \ (\ (\lambda x.\varphi(x))y \leftrightarrow \varphi[y/x]) \tag{10}$$

where $\varphi[y/x]$ denotes the usual substitution of x by y.

Now we are ready to define zero and precedence relation that are needed for interpreting Peano arithmetic.

Definition 2.6 (Definition of 0)**.** *0 is defined to be the cardinal of the empty concept*

$$0 = \#(\lambda x.x \neq x) \tag{11}$$

Definition 2.7 (Precedence Relation)**.**

$$Pre(x, y) \leftrightarrow \exists P \exists w(Pw \wedge y = \#P \wedge x = \#[\lambda z.Pz \wedge z \neq w]) \tag{12}$$

Intuitively, a number x precedes y, if y is the cardinal of a some concept P, and x is the cardinal of the new concept which has exactly one fewer element than P, by requiring that the new concept is given by all the element in P with the exception of some w.

To define the natural number predicate $\mathbb{N}$, Frege further needs to define the closure of the predecessor relation. Frege defines this through using the idea that if two... Intuitively, suppose R is a binary relation, then the

Definition 2.8 (Ancestral Relation)**.** *Let R be a relation. Then define the ancestral relation to be:*

$$R^*(x,y) \leftrightarrow \forall F[\forall z(Rxz \to Fz) \wedge (\forall x, y(Rxy \to (Fx \to Fy)) \to Fy)] \tag{13}$$

and the so-called weak ancestral relation as:

$$R^+(x,y) \leftrightarrow R^*(x,y) \vee x = y \tag{14}$$

Then the predicate N can be defined by applying the relations and the newly defined constant 0.

Definition 2.9 (The number predicate $\mathbb{N}$)**.** *Frege gives the following definition for the natural number predicate $\mathbb{N}$:*

$$\mathbb{N}x \leftrightarrow Pre^+(0,x)$$

Now, the Dedekind-Peano axioms are expressible in Frege's arithmetic, and the Frege theorem states that they can be proved as valid propositions in the system.

Theorem 2.10. *The following Dedekind-Peano axioms can be proved in second-order logic from Hume's Principle.*

(i) $\mathbb{N}0$

(ii) $\neg\exists x(\mathbb{N}x \wedge Pre(x,0))$

(iii) $\forall x, y, z(x \neq y \to \neg(Pre(x,z) \wedge Pre(y,z)))$

(iv) $\forall x(\mathbb{N}x \to \exists y(\mathbb{N} \wedge Pre(x,y)))$

(v) $P0 \wedge \forall x, y(Pre(x,y) \to (Px \to Py)) \to \forall nPn$

In Frege's original proof, the extension operator, as defined by Basic Law V, was used. But as Heck 1993 shows, it was not used in an essential way in Frege's proof, and an "extension-free" proof of theorem can be extracted. Moreover, it is arguable from Frege's various remarks, that he himself was cognizant about the limited formal necessity of the extensions (for instance, in §107 of [7] that "I too attach no great importance to the introduction of extensions of concepts.").

3 The Julius Caesar Problem

We now return to the key question: given that it could have been a option for Frege to fall back to the Hume's Principle (HP for short) and claim that the logicism program can be realized at least to some extent, why did he deserted the whole program instead?

In fact, if we compare the two axioms side by side

$$\nabla F = \nabla G \leftrightarrow \forall x(F(x) = G(x)) \tag{BV}$$

$$\#F = \#G \leftrightarrow F \text{ and } G \text{ are equinumerous} \tag{HP}$$

It is easily seen that the two axioms are structurally very similar: They are both "contextual" definitions that use biconditionals to introduce new operators on the left-hand sides. In other words, if Basic Law V is replaced by HP, technically the obtained system retains a similar "shape". Thus Frege's commitment to BV but not HP is unlikely a result of technical considerations.

It is important that we first look at Frege's own consideration about the insufficiency of HP. After reducing arithmetic to HP, he raised the main objection to using HP in a fundamental way. This is the well-known Julius Caesar objection. Frege writes in §56 of [6]:

> But we can never – to take an extreme example - decide by means of our definitions whether the number Julius Caesar belongs to a concept, or whether that well-known conqueror of Gaul is a number or not. Further more, we cannot prove with the help of our attempted definitions that if the number a belongs to the concept F and the number b belongs to the same concept, then necessarily a=b. The expression 'the number that belongs to the concept F' could not therefore be justified and it would thus be quite impossible to prove a numerical equality, since we would be unable to apprehend a definite number at all.

First, the objection would make no sense if Frege subscribed to any structuralist's view of numbers. (For instance, a structuralist can regard any progression sequence as defining numbers, and Julius Casar may asl well be thought as a unit in some counting system.) Instead, Frege is certain that Julius Caesar is *not* a number, and he believes we all have strong intuition about this. That is, there must exist an ontological realm of numbers that is different from "ordinary" objects, and Hume's Principle does not convey this intuition properly. This is the *ontological concern* of Frege.

Second, Frege argues that the definition of numbers should be explicit and informative in such a way that, when a number is not given in the form of some $\#F$, there has to be a way of recognizing that number. This can be called the *epistemological concern.* As he further claims in §62 [7]:

> If we are to use the symbol a to signify an object, we must have a criterion for deciding in all cases whether b is the same as a, even if it is not always in our power to apply this criterion.

In other words, Hume's Principle does not provide us a criteria of recognizing numbers that are presented to us not in for form of "the cardinality of F".

From these two concerns (they are studied in detail in [10, 8]) we can extract an implicit strong requirement that Frege has set for himself: *The logical foundation of arithmetic should be at the same time ontologically and epistemologically informative.* It is unacceptable even if arithmetic has only been *technically* reduced logic.

Frege has never justified such a belief for himself, but it is not hard for us to understand this view of his. To Frege, after all, the whole mathematics should be analytic. Thus, it is not a matter of finding the just-right system that have all these desirable properties, which would appear rather difficult; instead, Frege thought that he only needed to "pick" the right axioms in the pool of principles that are all "equally analytic", such that the picked ones are more epistemologically convincing and suggest justifiable ontological commitments. This implicit belief of Frege reflects his strong commitment to the existence of the "third realm", as well as our epistemological capability of knowing the realm. He always regards the notions of logical truth and concepts "self-evident", without ever feeling the need of defining the term. As a reflection of this, Frege even pointed out questions about the way that Basic Law V defines extensions, although he regarded this as no more than a side-remark (§10, [6])

> ... This by no means fixes completely the denotation of a name like $\varepsilon\Phi(\varepsilon)$. We have only a means of always recognizing a course-of-values if it is designated by a name like '$\varepsilon\Phi(\varepsilon)$', by which it is already recognizable as a course-of-values. But, we can neither decide, so far, whether an object is a course-of-values that is not given us as such.

Again, the interesting problem for Frege, as opposed to any ontological concerns about logical objects and their properties, is the search for fundamental and interesting facts that are epistemologically useful establishing the foundation of mathematics. This was the central goal of his logicisim. The fact that Russell's paradox shows how our (most importantly, his) intuition about the third realm, which appeared so natural and reliable, can lead to contradictions, is thus astonishing. This leaves Frege with no option of finding other foundations. Hence the question: "What other way is there?"

When Heck asks the question "why didn't Frege just replace Basic Law V by Axiom V", it is important to note that this is already assuming a contemporary point of view. Indeed, the clear distinction of the "logic" part (Basic Law I-IV) and the nonlogical part (Basic Law V or Hume's Principle) in Frege's system is not readily available for Frege yet. To quote his own words (Preface in [6]):

> A dispute can arise, so far as I can see, only with regard to my Basic Law concerning courses-of-values (V)... I hold that it is a law of pure logic.

For Frege, all the axioms were self-evident principles about calculations of concepts, which should be intuitively true and analytic. That is, in a sense, Basic Law V should be *equi-consistent* with the other four axioms. Thus, the difficulty with the Basic Law V does not just affect this single axiom; there is no reason not to suspect that the other parts of the system would subject to new paradoxes. Such doubt about all the logical principles themselves is serious. In fact, as we now understand, indeed much of the power in Frege's system is due to the purely logic part as well. (We have mentioned that by restricting to weaker logics, the system can also be consistent.)

Thus, there was no point for Frege to go on with the logicist program, regardless of whether temporary technical patches can be found, if there is no systematic method of assuring the reliability of the underlying logic principles. To even talk about this consideration, we are already looking beyond Frege's time, and it is unreasonable to request him to accomplish that. We have to bare in mind that Frege was standing at the beginning of decades of research in logic. The fact that Frege stopped at the skeptical view of his own logicism is thus understandable.

The combination of the aspects above, namely, the fact that Frege had optimistic ontological and epistemological requirements for his foundation of arithmetic, and that the paradox shakes the whole foundation instead of just a single technical system, serves as an explanation of why Frege did not accept Hume's Principle as his axiom. The fact that Frege has successfully established so much of the reduction from arithmetic to logic, is much more attributed to his genius, whose success, as accidental as it is, can not be accepted by Frege himself. We have to assume a contemporary view to fully appreciate the technical achievement of Frege, and has no reason to blame him for neglecting his own work.

4 Hume's Principle: Too Weak or Too Strong?

I now proceed to a wider context to compare Frege's own rejection of Hume's Principle and the contemporary arguments against neo-logicist's approach. I find this comparison a good example of how our mindset has changed with the development of analytic philosophy.

Neo-logicism aims to develop a notion of analyticity such that arithmetic can be regarded as properly based on logic, based on the basic fact that Frege's theorem already reduces arithmetic to second-order logic and Hume's Principle. As was argued in the previous section, neo-logicism can only take its position after the advancement we have made in analytic philosophy and logic. It is widely admitted among neo-logicists [12] that Hume's Principle is not analytic in a standard sense, and the neo-logicists' position has been clarified to open up the possibility of defining a new notion of analyticity. It should be able to clarify the intuition that axioms like Hume's Principle is already "analytic enough".

Frege's rejection of using Hume's Principle is that it is too *weak*. Here weakness means that, although Frege subscribes to the analyticity of HP (to him, mathematics is analytic), he does not admit that it has the strength to serve as the fundamental axiom. As discussed in the previous section, Frege's criteria for choosing axioms is mainly their epistemological utility. Thus Hume's Principle, suffering from the Julius Caesar problem, is not a suitable candidate for laying the foundation of arithmetic. Note that Frege has this point view because of his optimism about the analyticity of mathematics. Such optimism can still be seen in the Appendix of [6]:

> The prime problem of arithmetic is the question, In what way are we to conceive logical objects, in particular, numbers? By what means are we justified in recognizing numbers as objects? Even if this problem is not solved to the degree I thought it was when I wrote this volume, still I do not doubt that the way to the solution has been found.

In sharp contrast, the contemporary argument against neo-logicism is that Hume's Principle may be too strong.

First, on the technical level, it has been shown that Frege's arithmetic, i.e., second-order logic plus Hume's principle, is equi-interpretable with the full second-order arithmetic. Thus, a notable concern expressed by Boolos in [3], is that consequently all subsystems of second-order arithmetic have to be regarded as analytic, and we have to develop notions of sub-analyticity notions to separate the strength of interesting subsystems of second-order arithmetic. To quote Boolos [3]:

> Learning that HP is analytic will not help us in the slightest with the problem of assessing the strength of various theorems, fragments and subtheories of analysis, all of which would, I suppose, have to count as analytic. The first part of my worry about content is that HP, when embedded into axiomatic second-order logic, yields an incredibly powerful mathematical theory.

On the conceptual level, the fact that Hume's principle together with second-order logic implies the existence of infinitely many numbers seem to be too strong an ontological commitment. (This is implied by Definition 2.6 and 2.7, since the predecessor can be applied to 0 to generate the infinite sequence of numbers.) It is now widely accepted that the key difficulty of giving an analytic foundation to mathematics is that analytic principles should never have existential commitments, and hence incapable of laying the foundation for facts in mathematics such as "there exists more than two prime numbers". Since Hume's Principle is capable of producing existential claims, it is too strong to be analytic, at least in the sense usually conceived.

The neo-logicists' reaction to this challenge is that, it is not a matter of whether Hume's Principle is analytic in the traditional sense of analyticity, because quite evidently it isn't (again, because of the existential claims that it implies). However, the fact that Hume's Principle is intuitively evident suggests that the very notion of analyticity may be open to revision such that we can treat the valid non-analytic principles in a more special manner. One direction of this is to evaluate, still somewhat related to Frege's original idea, the epistemological obligation that is needed in adpoting a principle. To quote Wright [12]:

> Hume's Principle may be laid down without significant epistemological obligation: that it may simply be stipulated as an explanation of the meaning of statements of numerical identity, and that—beyond the issue of the satisfaction of the truth-conditions it thereby lays down for such statements—no competent demand arises for an independent assurance that there are objects whose conditions of identity are as it stipulates.

The question about whether Hume's Principle is suitable for foundational use, and what kind of notion of analyticity is suitable for further development of foundations of mathematics, is still quite open-ended. It requires significant effort from the neo-logicists to formulate and justify a new notion of analyticity, which may open up renewed interest in the early works of analytic philosophy. Yet, the message that is revealed in the comparison of the present and Frege's own concerns about Hume's Principle shows that analytic philosophy, turning out-of-fashion to some degree, has achieved a great deal in clarifying our understanding about the fundamental notion of analyticity. In Frege's time, he could have such confidence in taking a rationalist's view towards the self-evident facts, accepting the existence and our epistemological competence in grasping the concepts. In our "post-analytic" era now, the general atmosphere is the skepticism against our ability of grasping the fundamental logical objects, if they exist at all.

5 Conclusion

In the preceding sections I investigated the question of why Frege did not accept Hume's Principle as a basic axiom as foundations of arithmetic. I observe that there are two main reasons:

(i) Frege posed the requirement that his foundation for arithmetic must be ontologically and epistemologically informative. This is not satisfied by Hume's Principle, as suggested by the Julius Caesar problem.

(ii) Russell's paradox shakes Frege's fundamental belief about the self-evident principles of logical objects, and does not suggest an easy fix of the system as we now realize is possible.

I further compared reasons for Frege's objection of using Hume's Principle, and the contemporary debates on neo-logicism. I believe it shows the significant progress we have gained in understanding analytic notions through the development of logic and analytic philosophy. From the case of the Hume's Principle, we do get a view of the progress of our understanding of the very concept of analyticity. The change from Frege's optimism to the skeptical attitude we now hold towards foundational questions, and the need of redefining analyticity, shows our progress in understanding the foundations of mathematics.

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