Confidence Intervals

Gov 2001: Quantitative Social Science Methods I

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Today's Reading

Required

- **Aronow & Miller**, §3.3.1: Confidence intervals (pp. 124–130)
- **Blackwell**, Ch. 4: Hypothesis tests (preview)

Goal: Learn to quantify uncertainty about our estimates.

The Problem

Situation: We've computed an estimate $\hat{\theta}$ from our sample.

Question: How close is $\hat{\theta}$ to the true θ?

We know:

- $\hat{\theta}$ is random—it varies from sample to sample
- The CLT tells us $\hat{\theta}$ is approximately normal for large n
- We can compute the standard error of $\hat{\theta}$

Idea: Use the sampling distribution to build an interval around $\hat{\theta}$.

From CLT to Confidence Interval

Setup: Estimating μ with $\bar{Y}$.

By CLT: For large n,

$$Z = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \approx N(0, 1)$$

Key fact: For standard normal, $\Pr(-1.96 < Z < 1.96) \approx 0.95$

Therefore:

$$\Pr\left(-1.96 < \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} < 1.96\right) \approx 0.95$$

Rearranging:

$$\Pr\left(\bar{Y} - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{Y} + 1.96\frac{\sigma}{\sqrt{n}}\right) \approx 0.95$$

The 95% Confidence Interval

95% Confidence Interval for μ

$$\text{CI}: \quad \bar{Y} \pm 1.96 \times \frac{\sigma}{\sqrt{n}}$$

or equivalently:

$$\left[\bar{Y} - 1.96 \times \text{SE}, \quad \bar{Y} + 1.96 \times \text{SE}\right]$$

Where $\text{SE} = \sigma/\sqrt{n}$ is the **standard error**.

General formula for $(1-\alpha)\%$ **CI**:

$$\bar{Y} \pm z_{\alpha/2} \times \text{SE}$$

Common values: $z_{0.025} = 1.96$ (95%), $z_{0.005} = 2.58$ (99%)

What Does "95% Confident" Mean?

Common misconception: "There's a 95% probability that μ is in this interval."

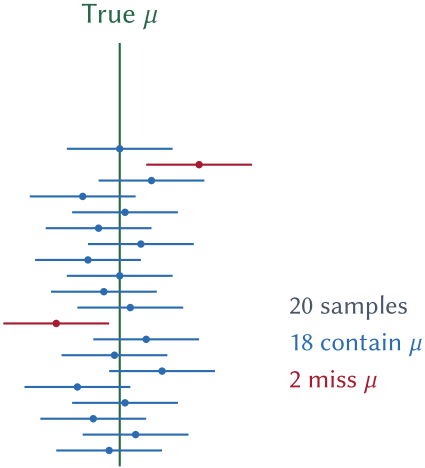
Correct interpretation: If we repeated this procedure many times (new samples, new CIs), 95% of the intervals would contain μ.

The key distinction:

- μ is **fixed** (unknown, but not random)
- The **interval** is random (depends on the sample)
- Either μ is in the interval or it isn't—no probability about it

The probability statement is about the *procedure*, not the *parameter*.

Visualizing Confidence Intervals



Each horizontal line is a 95% CI from a different sample.

The Standard Error Problem

Issue: The CI formula uses σ, but σ is unknown!

$$\bar{Y} \pm 1.96 \times \frac{\sigma}{\sqrt{n}}$$

Solution: Estimate σ with $\hat{\sigma} = s$:

$$s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2}$$

The estimated standard error:

$$\widehat{\text{SE}} = \frac{s}{\sqrt{n}}$$

This is what software reports as "Std. Error" or "SE."

The t-Distribution

New problem: When we use $\hat{\sigma}$ instead of σ:

$$T = \frac{\bar{Y} - \mu}{s/\sqrt{n}} \sim t_{n-1}$$

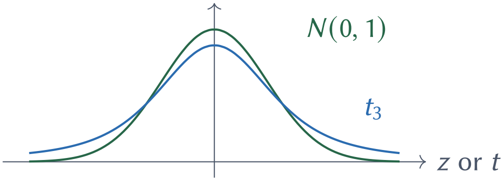
This follows a **t-distribution** with $n - 1$ degrees of freedom, not standard normal.

Key properties of t_{n-1}:

- Symmetric, bell-shaped (like normal)
- Heavier tails than normal (more extreme values possible)
- As $n \to \infty$, $t_{n-1} \to N(0, 1)$

For large n: t and z are nearly identical. Use $z = 1.96$.
For small n: Use t critical values (wider intervals).

t-Distribution vs. Normal



The t-distribution has heavier tails:

- Extreme values more likely
- Accounts for uncertainty in estimating σ
- Critical values larger: $t_{0.025,10} = 2.23$ vs. $z_{0.025} = 1.96$

Practical CI Construction

95% Confidence Interval (Practical)

$$\bar{Y} \pm t_{0.025,n-1} \times \frac{s}{\sqrt{n}}$$

In practice (for $n \geq 30$):

$$\bar{Y} \pm 2 \times \widehat{\text{SE}}$$

The "2" is a convenient approximation to 1.96.

Reporting: "The estimated approval rating is 45% (95% CI: 42% to 48%)."

Coverage in Finite Samples

Important caveat: The 95% coverage is an *asymptotic* property.

A&M simulations show:

- At $n = 10$ for uniform data: actual coverage $\approx$ 92%, not 95%
- Bernoulli data: coverage can be even worse for small n
- Skewed distributions need larger n for accurate coverage

"$n \geq 30$ is enough for CLT" is folklore, not theorem.

Implications:

- Small samples: reported CIs may be overconfident
- For critical decisions: consider bootstrap CIs (Week 7)

Example: Poll Margin of Error

Setup: Poll of $n = 1{,}000$ voters. $\hat{p} = 0.48$ support candidate A.

Standard error: $\widehat{\mathsf{SE}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \sqrt{\frac{0.48\times0.52}{1000}} = 0.0158$

95% CI:

$$0.48 \pm 1.96 \times 0.0158 = 0.48 \pm 0.031 = [0.449, 0.511]$$

Interpretation: We're 95% confident the true support is between 44.9% and 51.1%.

The "margin of error" of $\pm 3.1\%$ is $1.96 \times \mathsf{SE}$.

Factors Affecting CI Width

The CI width is:

$$\text{Width} = 2 \times z_{\alpha/2} \times \frac{\sigma}{\sqrt{n}}$$

Wider CIs come from:

- Higher confidence level ($\uparrow z_{\alpha/2}$)
- More variability in population ($\uparrow \sigma$)
- Smaller sample size ($\downarrow n$)

To cut width in half: Need 4× the sample size!
Precision is expensive.

Common Mistakes with CIs

Wrong: "There's a 95% probability μ is in this interval."
Right: "If we repeated sampling, 95% of intervals would contain μ."

Wrong: "95% of the data falls in this interval."
Right: "This is an interval for the *mean*, not for individual observations."

Wrong: "A wider CI means my estimate is worse."
Right: "A wider CI honestly reflects more uncertainty. Narrow CIs can be false precision."

CIs for Any Asymptotically Normal Estimator

General Principle (A&M Theorem 3.4.2)

If $\hat{\theta}$ is asymptotically normal (CLT applies), then:

$$\text{95\% CI}: \quad \hat{\theta} \pm 1.96 \times \text{SE}(\hat{\theta})$$

This covers almost everything you'll estimate:

- Difference of means: $(\bar{Y}_1 - \bar{Y}_2) \pm 1.96 \times \text{SE}$
- Regression coefficient: $\hat{\beta} \pm 1.96 \times \text{SE}(\hat{\beta})$
- Treatment effect: $\hat{\tau} \pm 1.96 \times \text{SE}(\hat{\tau})$

One formula, many applications. Just need the SE.

Connection to Hypothesis Testing

Key relationship:
A 95% CI contains all values of θ_0 that would *not* be rejected by a two-sided test at $\alpha = 0.05$.

Equivalently:

- If θ_0 is outside the CI $\Rightarrow$ reject $H_0 : \theta = \theta_0$
- If θ_0 is inside the CI $\Rightarrow$ fail to reject H_0

CIs are more informative than tests: They tell you the range of plausible values, not just whether one value is rejected.

Summary

1. **Confidence intervals** quantify uncertainty about estimates
2. **Construction**: $\hat{\theta} \pm z_{\alpha/2} \times \text{SE}$
3. **Interpretation**: 95% of intervals (across repeated samples) contain θ
4. **The interval is random**, the parameter is fixed
5. Use t-**distribution** for small samples (when estimating σ)
6. **Precision costs**: Halving width requires 4× sample size

Next week: Hypothesis testing—a different framework for inference.

Looking Ahead

Week 7: Hypothesis Testing

- Null and alternative hypotheses
- Test statistics and p-values
- Type I and Type II errors
- Power
- The bootstrap (when CLT doesn't apply)

Reading:

- A&M §3.3.2–3.3.3 (hypothesis testing)
- A&M §3.4.3 (bootstrap)
- Blackwell Ch. 4

Then: Midterm exam covering Weeks 1–7!