Power and Bootstrap

Gov 2001: Quantitative Social Science Methods I

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Today's Reading

Required

- **Aronow & Miller**, §3.3.3: Power (pp. 138–142)
- **Aronow & Miller**, §3.4.3: Bootstrap (pp. 145–150)
- **Blackwell**, Ch. 4 (finish)

Last probability lecture before the midterm!

Two Types of Errors

When we make a decision, we might be wrong:

	H_0 True	H_0 False
Reject H_0	Type I Error	Correct!
Fail to Reject	Correct!	Type II Error

- **Type I Error**: False positive. Convicting an innocent person.
- **Type II Error**: False negative. Letting a guilty person go free.

Type I Error Rate = α

Type I Error

$$\alpha = \Pr(\text{Reject } H_0 \mid H_0 \text{ true})$$

This is our significance level!
When we set $\alpha = 0.05$, we're accepting a 5% chance of Type I error.

Why 5%?

- Tradition (thanks, Fisher)
- Balances false positives against power
- Other fields use different conventions (particle physics: 5σ)

Type II Error and Power

Type II Error

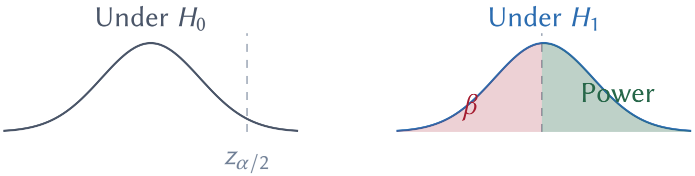
$$\beta = \Pr(\text{Fail to reject } H_0 \mid H_0 \text{ false})$$

Power

$$\text{Power} = 1 - \beta = \Pr(\text{Reject } H_0 \mid H_0 \text{ false})$$

Power = probability of detecting a real effect when one exists.
Higher power is better. We want to find effects that are really there.

Visualizing Power



Left: Distribution under H_0. **Right**: Distribution under H_1.
Power = green area. β = red area.

What Affects Power?

Power increases when:

1. **Effect size is larger**: Easier to detect big effects
2. **Sample size is larger**: More precise estimates, smaller SE
3. **Variance is smaller**: Less noise, clearer signal
4. α **is larger**: More willing to reject $\Rightarrow$ more rejections

The tradeoff: Increasing α increases power but also Type I error.
We typically fix $\alpha = 0.05$ and increase n to get power.

Power Calculation Example

Setup: Testing $H_0 : \mu = 0$ vs. $H_1 : \mu \neq 0$
True effect: $\mu = 0.5$, Standard deviation: $\sigma = 2$, Sample size: $n = 64$

Standard error: $\text{SE} = \sigma/\sqrt{n} = 2/8 = 0.25$

Under H_0: Reject if $|\bar{Y}| > 1.96 \times 0.25 = 0.49$

Under H_1 (true $\mu = 0.5$):

$$\begin{aligned}
\text{Power} &= \Pr(|\bar{Y}| > 0.49 \mid \mu = 0.5) \\
&\approx \Pr(\bar{Y} > 0.49) \quad \text{(ignoring left tail)} \\
&= \Pr\left(Z > \frac{0.49 - 0.5}{0.25}\right) = \Pr(Z > -0.04) \approx 0.52
\end{aligned}$$

Only 52% power—we'd miss this effect half the time!

Power and Sample Size Planning

Before running a study: Calculate required sample size for adequate power.

Convention: Target power = 0.80 (80%)

Formula (for two-sided test of mean):

$$n = \left(\frac{(z_{\alpha/2} + z_{\beta}) \cdot \sigma}{\mu_1 - \mu_0} \right)^2$$

where z_β is the z-value for desired power (e.g., $z_{0.20} = 0.84$ for 80% power).

Example: $\sigma = 2$, $\mu_1 - \mu_0 = 0.5$, 80% power:

$$n = \left(\frac{(1.96 + 0.84) \times 2}{0.5} \right)^2 = (11.2)^2 \approx 126$$

Power in Political Science Research

Many studies are underpowered:

- Median power in social science: $\sim$35% (Button et al., 2013)
- Small effects + limited samples = low power

Political science examples:

- GOTV effects ($\sim$2–3 pp) need $n \approx 5{,}000+$ for 80% power
- Survey experiments with many conditions: power drops rapidly
- Cross-national studies: 30 countries $\Rightarrow$ low power for small effects

Best practice: Power analysis before collecting data.

When CLT Doesn't Apply

The CLT requires:

- I.I.D. observations
- Finite variance
- "Large enough" n

What if:

- Sample size is small?
- Distribution is highly skewed?
- We want inference for a complicated estimator (median, ratio, etc.)?

Solution: The Bootstrap

The Bootstrap Idea

The problem: We want to know the sampling distribution of $\hat{\theta}$, but we only have one sample.

The insight: Treat the sample as a "stand-in" for the population.

The procedure:

1. Resample *with replacement* from your data
2. Compute $\hat{\theta}$ on the resample
3. Repeat many times (e.g., 10,000)
4. Use the distribution of resampled $\hat{\theta}$s as the sampling distribution

Bootstrap Procedure

Original sample: $Y_1, Y_2, \ldots, Y_n$

For $b = 1, 2, \ldots, B$:

1. Draw a sample of size n **with replacement** from $(Y_1, \ldots, Y_n)$
2. Call this $Y_1^{*b}, Y_2^{*b}, \ldots, Y_n^{*b}$
3. Compute $\hat{\theta}^{*b}$ on this bootstrap sample

Result: $\hat{\theta}^{*1}, \hat{\theta}^{*2}, \ldots, \hat{\theta}^{*B}$

Use this distribution to:

- Estimate SE: $\widehat{\mathrm{SE}} = (\hat{\theta}^{*1}, \ldots, \hat{\theta}^{*B})$
- Construct CI: Use percentiles (e.g., 2.5th and 97.5th)

Bootstrap Example: Median Income

Data: 50 income observations. Median = \$52,000.

Problem: No simple formula for SE of the median!

Bootstrap:

1. Resample 50 incomes with replacement
2. Compute median of resample
3. Repeat 10,000 times

Result: 10,000 bootstrap medians

- Bootstrap SE: \$3,200
- 95% CI: [\$46,000, \$58,500] (2.5th and 97.5th percentiles)

Bootstrap Confidence Intervals

Two common methods:

1. Percentile method (simplest):

$$\text{CI} = \left[\hat{\theta}^*_{(\alpha/2)}, \hat{\theta}^*_{(1-\alpha/2)}\right]$$

Use the $\alpha/2$ and $(1-\alpha/2)$ quantiles of bootstrap distribution.

2. Normal approximation:

$$\text{CI} = \hat{\theta} \pm z_{\alpha/2} \times \widehat{\text{SE}}_{boot}$$

Use bootstrap SE with normal critical values.

The percentile method is more robust to skewness.

Why Does Bootstrap Work?

Key insight: The relationship between

$$\text{Sample} \leftrightarrow \text{Population}$$

is similar to the relationship between

$$\text{Bootstrap sample} \leftrightarrow \text{Original sample}$$

For large n:

- The sample distribution approximates the population distribution
- Resampling from the sample mimics resampling from the population
- The bootstrap distribution approximates the true sampling distribution

This is the "plug-in principle" applied to distributions.

When Bootstrap Works (and Doesn't)

Bootstrap works well for:

- Means, medians, quantiles
- Regression coefficients
- Most "smooth" functions of the data

Bootstrap can fail for:

- Extremes (max, min)
- Very small samples
- Non-I.I.D. data (need modified versions)
- Parameters on the boundary (e.g., variance = 0)

Rule of thumb: If the estimator is consistent and asymptotically normal, bootstrap usually works.

Bootstrap in R

Simple implementation:

```
# Original statistic
theta_hat <- median(data)
# Bootstrap
B <- 10000
theta_boot <- numeric(B)
for (b in 1:B) {
    boot_sample <- sample(data, replace = TRUE)
    theta_boot[b] <- median(boot_sample)
}
# SE and CI
se_boot <- sd(theta_boot)
ci_boot <- quantile(theta_boot, c(0.025, 0.975))
```

Or use the boot package for more features.

Summary: Errors and Power

Concept	Definition	Typical Value
Type I Error (α)	$\Pr(\text{reject } H_0 \mid H_0 \text{ true})$	0.05
Type II Error (β)	$\Pr(\text{fail to reject} \mid H_0 \text{ false})$	0.20
Power	$1 - \beta$	0.80

Power depends on: Effect size, sample size, variance, α

Key Takeaways

1. **Type I error** = false positive; controlled by α
2. **Type II error** = false negative; related to power
3. **Power** = probability of detecting a real effect
4. **Plan sample size** to achieve adequate power (usually 80%)
5. **Bootstrap** provides inference when CLT is questionable
6. **Bootstrap CI**: Resample, compute statistic, use percentiles

Midterm Preview

Midterm Exam: Covers Weeks 1–7

Topics:

- Probability: axioms, conditional probability, Bayes' Rule
- Random variables: PMF, PDF, CDF, expectation, variance
- Joint distributions, conditional expectation, CEF
- Sampling distributions, LLN, CLT
- Estimation: bias, variance, MSE, consistency
- Confidence intervals and hypothesis testing

After spring break: We start regression!

Looking Ahead

Spring Break: March 15–23

Week 8: What Is Regression?

- The Best Linear Predictor (BLP)
- OLS as sample BLP
- Connection to CEF

Reading:

- Blackwell Ch. 5
- A&M §2.2.4
- Angrist & Pischke Ch. 3.1

The second half of the course: applying what we've learned to regression.