

# **Discrete Distributions**

Bernoulli, Binomial, and Poisson

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Gov 2001 · Harvard University

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# Where We Are

From abstract machinery to named distributions

**Last week:** The general framework

- Random variables, PMFs, PDFs, CDFs
- Expected value, variance, Jensen's inequality
- Independence and indicator variables

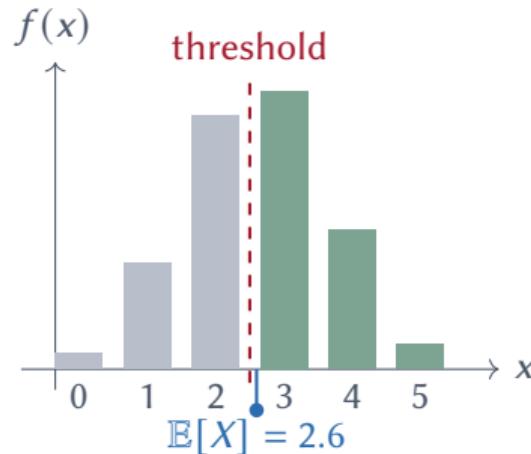
**This week:** Named distributions — the **vocabulary** of statistics

- **Today:** Discrete distributions (Bernoulli, Binomial, Poisson)
- **Wednesday:** Continuous distributions (Uniform, Normal, Exponential)

Reading: Aronow & Miller §1.2–1.3, Blackwell 2.2–2.3

## A Puzzle from Thursday

How can you win 54% of the time but expect only 2.6 votes?



$$P(X \geq 3) = 0.54$$

Win more than half

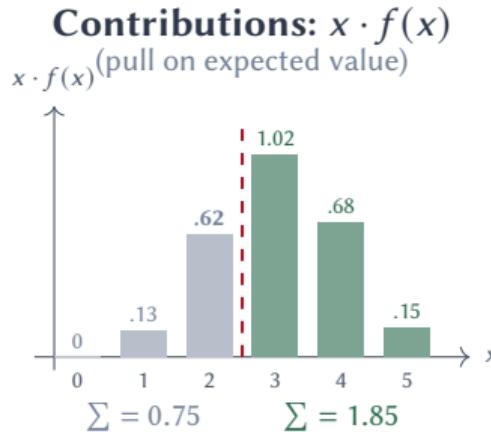
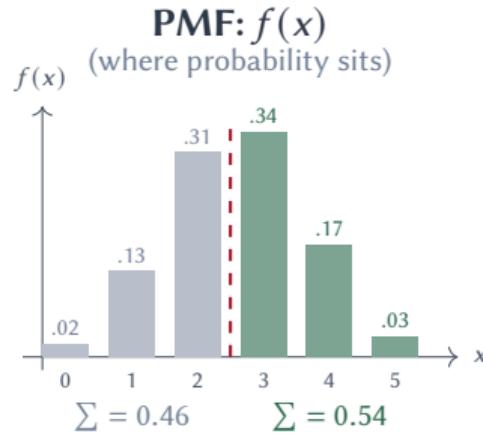
$$E[X] = 2.60$$

Below threshold

**How can both be true?**

## PMF vs. Contributions to $E[X]$

The  $x = 2$  bar looks small in the PMF, but contributes a lot to  $E[X]$



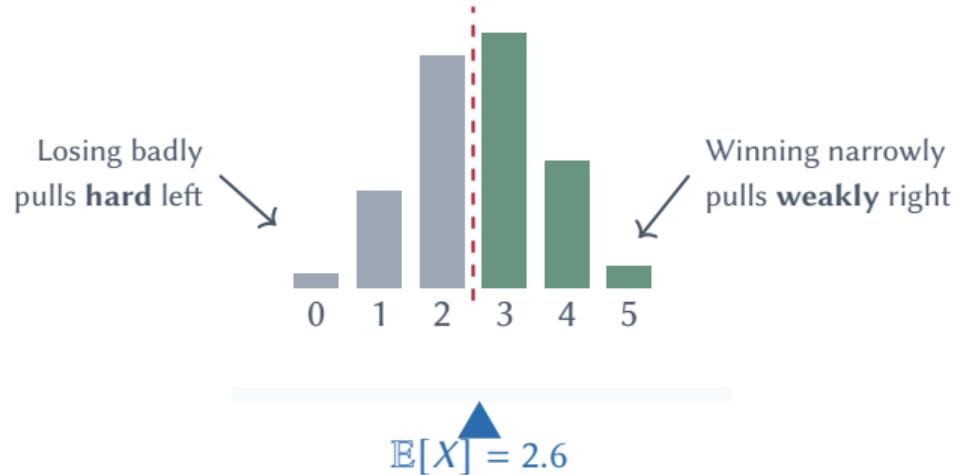
**Left:** PMF shows probability mass. More mass on the right (54%), so the map is *usually* struck down.

**Right:** Contributions to  $E[X]$ . The  $x = 2$  bar is now substantial (0.62). That's a lot of "pull" just below threshold.

$$E[X] = 0.75 + 1.85 = 2.60 \quad (\text{sum of all contributions})$$

## Expected Value is a Center of Mass

$E[X]$  is pulled by magnitude, not just frequency



**Resolution:** More than half the probability mass is above 3, but the *center of mass* is at 2.6. Losing badly (0, 1) pulls harder than winning narrowly (3). The asymmetry pulls  $\mathbb{E}[X]$  below threshold.

# Today: Famous Discrete Distributions

Building our toolkit

The redistricting puzzle showed us why random variables matter. Now we build a toolkit of **named distributions** — probability models that appear constantly in political science.

Distribution	What it models
Bernoulli	A single yes/no outcome
Binomial	Number of successes in $n$ trials
Poisson	Counts of rare events

For each, we'll ask: *Where did it come from? Where do we see it? What are its properties?*

# Why Named Distributions?

These aren't arbitrary — they model real phenomena

Three reasons to learn these:

1. **They appear everywhere:** Elections, counts, durations, measurements
2. **They're tractable:** We can derive  $E[X]$ ,  $\text{Var}[X]$ , and more
3. **They connect:** Understanding one helps you understand others

Today's cast:

- **Bernoulli** — yes/no, success/failure (the foundation)
- **Binomial** — counting successes in  $n$  trials
- **Poisson** — counting rare events

# The Big Picture

Why distributions matter for inference

## What we're building toward:

1. We **describe populations** with distributions (this week)
2. We **estimate parameters** ( $p, \lambda, \mu$ ) from sample data (coming soon)
3. We **quantify uncertainty** about those estimates (requires understanding distributions)

**Example:** A poll samples 100 voters. 58 support candidate A.

- The *population* has some true support rate  $p$  (unknown)
- Our *estimate* is  $\hat{p} = 0.58$  (from sample)
- How good is this estimate? *That depends on the Binomial distribution.*

Distributions aren't just mathematical objects — they're the foundation for learning from data.

## Part I

# The Bernoulli Distribution

The Simplest Random Variable

## Jacob Bernoulli (1655–1705)

The mathematician who formalized probability

**Jacob Bernoulli** was a Swiss mathematician from a family that produced eight prominent mathematicians across three generations. His posthumous work *Ars Conjectandi* (1713) was foundational to probability theory.

### Key contributions:

- Proved the first version of the **Law of Large Numbers**
- Showed that with enough trials, observed frequencies converge to true probabilities
- Established probability as a rigorous mathematical field

The Bernoulli distribution — a single binary trial — is named in his honor. Simple as it is, it's the building block for much of what follows.

# Where You See Bernoulli in Social Science

Any yes/no outcome is a Bernoulli trial

## Political science:

- Did this citizen vote? ( $p \approx 0.6$  in US presidential elections)
- Did the incumbent win this district?
- Does this respondent approve of the president?

## Economics:

- Did this worker find a job this month?
- Did this firm enter the market?

## Sociology:

- Did this couple divorce within 5 years?
- Did this student graduate?

Whenever your outcome is binary (0 or 1), you're in Bernoulli territory.

# The Bernoulli Distribution

The simplest random variable: just 0 or 1

**The model:**  $X \sim \text{Bernoulli}(p)$  means:

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

**PMF:**  $f(x) = p^x(1 - p)^{1-x}$  for  $x \in \{0, 1\}$

**Support:**  $\{0, 1\}$  — only two possible values

The Bernoulli is a single trial. What if we have many trials? That's where the Binomial comes in.

## A Quick Definition: Support

What values can a random variable take?

**Definition:** The **support** of  $X$  is the set of values where the PMF (or PDF) is positive:

$$\text{Supp}[X] = \{x : f(x) > 0\}$$

**Examples:**

- Bernoulli( $p$ ): Support =  $\{0, 1\}$
- Binomial( $n, p$ ): Support =  $\{0, 1, 2, \dots, n\}$
- Poisson( $\lambda$ ): Support =  $\{0, 1, 2, \dots\}$  (unbounded)

**Why it matters:** When computing expectations or conditioning on  $X$ , you sum/integrate over the support. Knowing what values are possible is fundamental.

**For Bernoulli,  $\mathbb{E}[X] = p$  and  $\text{Var}[X] = p(1 - p)$**

For  $X \sim \text{Bernoulli}(p)$ :

**Expected value** (this is key — memorize it):

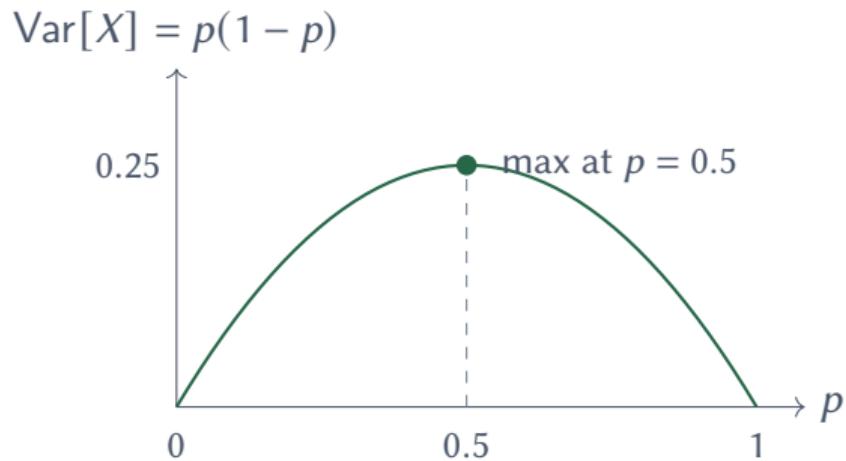
$$\mathbb{E}[X] = 0 \cdot (1 - p) + 1 \cdot p = p$$

**Variance:**

$$\begin{aligned}\text{Var}[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= p - p^2 = p(1 - p)\end{aligned}$$

**Key insight:**  $\mathbb{E}[X] = \mathbb{P}(X = 1)$ . For a 0/1 variable, the mean *is* the probability.  
This is why we can estimate probabilities by computing sample means.

## Bernoulli Variance is Maximized at $p = 0.5$



When outcomes are most uncertain ( $p = 0.5$ ), variance is highest.

Certainty ( $p = 0$  or  $p = 1$ ) means zero variance — no spread if there's only one possible outcome.

## Part II

# The Binomial Distribution

Counting Successes

## Pascal, Fermat, and the Gamblers

Probability theory was invented to solve gambling problems

In 1654, a French gambler named **Chevalier de Méré** had a problem: how should you fairly divide the pot when a dice game is interrupted mid-play?

He asked **Blaise Pascal**, who wrote to **Pierre de Fermat**. Their letters that summer are often called *the birth of probability theory*.

**The “problem of points”:** Each player has some probability of eventually winning the remaining rounds. Pascal and Fermat realized you need to count the *ways* each could win — the binomial coefficient:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Weirdly, gamblers drove these early innovations. And they still do: modern finance and speculation run on probability theory that traces back to these 1654 letters about dice.

The math has far exceeded gambling — but gambling remains where it started, and where a lot of the money still is.

# Where You See Binomial in Social Science

Counting successes in a fixed number of trials

## Political science:

- Out of 10 voters polled, how many support candidate A?
- Out of 50 precincts, how many have wait times over 1 hour?
- Out of 9 Supreme Court justices, how many vote to strike down?

## Economics:

- Out of 100 loan applicants, how many default?
- Out of 20 firms, how many survive their first year?

## Public health:

- Out of 500 vaccinated people, how many get infected?
- Out of 30 patients, how many respond to treatment?

Whenever you're counting successes in  $n$  independent trials, you're in Binomial territory.

## From Bernoulli to Binomial

Sum of independent trials

**Setup:** Run  $n$  independent Bernoulli trials, each with success probability  $p$ .

**Question:** What's the distribution of the **total number of successes**?

If  $X_1, X_2, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Bernoulli}(p)$ , then:

$$Y = \sum_{i=1}^n X_i \sim \text{Binomial}(n, p)$$

You saw this logic last week with the redistricting court — same idea, but now  $p$  is the same for all trials.

## The Binomial PMF Counts Ways to Achieve $k$ Successes

Three ingredients: successes, failures, arrangements

For  $Y \sim \text{Binomial}(n, p)$ :

$$f(k) = \mathbb{P}(Y = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

for  $k \in \{0, 1, 2, \dots, n\}$

**Decomposing the formula:**

- $p^k$ : probability that the  $k$  successes happen
- $(1 - p)^{n-k}$ : probability that the  $n - k$  failures happen
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ : number of ways to choose *which*  $k$  trials are successes

You saw this logic last week with the redistricting court — same idea, but now  $p$  is the same for all trials.

$\mathbb{E}[Y] = np$  Because It's a Sum of Bernoullis

Linearity of expectation does the heavy lifting

For  $Y \sim \text{Binomial}(n, p)$ :

**Expected value:** Since  $Y = \sum_{i=1}^n X_i$  where  $X_i \sim \text{Bernoulli}(p)$ :

$$\mathbb{E}[Y] = \sum_{i=1}^n \mathbb{E}[X_i] = np$$

**Variance:** Since the  $X_i$  are **independent**:

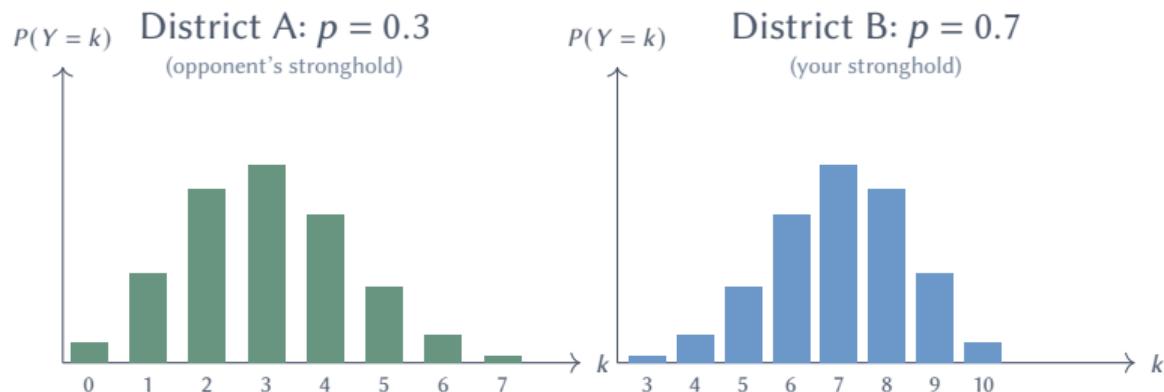
$$\text{Var}[Y] = \sum_{i=1}^n \text{Var}[X_i] = np(1 - p)$$

Linearity of expectation **always** works. Additivity of variance requires **independence**.

# How Likely Is Your Candidate's Support Level?

Visualizing the Binomial: same sample size, different support rates

Poll 10 voters in two different districts:



Same sample size  $n = 10$ , different  $p$ . The distribution shifts with the true support rate.

## Key Property: Binomials Add

If they share the same  $p$

**Theorem:** If  $X \sim \text{Binomial}(n, p)$  and  $Y \sim \text{Binomial}(m, p)$  are **independent**, then:

$$X + Y \sim \text{Binomial}(n + m, p)$$

**Intuition:**  $X$  is successes in  $n$  trials,  $Y$  is successes in  $m$  trials. Together, you have  $n + m$  trials.

**Example:** Two precincts, same underlying turnout rate  $p = 0.6$ .

- Precinct A: 100 voters,  $X \sim \text{Bin}(100, 0.6)$
- Precinct B: 150 voters,  $Y \sim \text{Bin}(150, 0.6)$
- Combined:  $X + Y \sim \text{Bin}(250, 0.6)$

This only works when  $p$  is the same across both. Different  $p$ 's? No nice closed form.

## Your Turn: Binomial Practice

Try these before we move on

A poll samples **200 voters**. Suppose the true approval rate is  $p = 0.45$ .

Let  $X$  = number who approve. Then  $X \sim \text{Binomial}(200, 0.45)$ .

### Questions:

1. What is  $\mathbb{E}[X]$ ?
2. What is  $\text{Var}[X]$ ? What is the standard deviation?
3. What is  $\mathbb{P}(X = 90)$ ? (Just set up the formula)

### Answers:

1.  $\mathbb{E}[X] = np = 200 \times 0.45 = 90$
2.  $\text{Var}[X] = np(1 - p) = 200 \times 0.45 \times 0.55 = 49.5$ , so  $\sigma = \sqrt{49.5} \approx 7.04$
3.  $\mathbb{P}(X = 90) = \binom{200}{90} (0.45)^{90} (0.55)^{110}$  (use software to compute)

## Part III

# The Poisson Distribution

Counting Rare Events

## Siméon Denis Poisson (1781–1840)

From artillery to actuarial science

**Poisson** was a French mathematician who studied under Laplace and Lagrange. He introduced his distribution in 1837 in a work on criminal justice — studying the number of wrongful convictions in court cases.

But the distribution became famous through a morbid application: **Ladislaus Bortkiewicz** (1898) used it to model *deaths by horse kicks in the Prussian army*. He showed that these rare, random events followed the Poisson beautifully.

**Why it works:** When you have many opportunities for something rare to happen (many soldiers, many days), and each event is independent, the Poisson emerges naturally.

The “law of rare events” — Poisson appears whenever you’re counting unlikely things in large populations.

# Where You See Poisson in Social Science

Counting rare events in large populations or time windows

## Political science:

- Supreme Court vacancies per presidential term (average  $\approx 1.5$ )
- Coups in a region per decade
- Mass protests in a country per year
- Terrorist attacks per month

## Economics:

- Number of bankruptcies in a sector per quarter
- Patent applications per firm per year

## Criminology / Public health:

- Homicides per city per year
- Disease outbreaks per region per decade

Whenever you're counting events that happen rarely and independently, you're in Poisson territory.

# The Poisson Distribution

Counting events that occur at a constant rate

**The model:**  $X \sim \text{Poisson}(\lambda)$  counts events occurring at rate  $\lambda$ :

$$f(k) = \mathbb{P}(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{for } k \in \{0, 1, 2, \dots\}$$

**Support:**  $\{0, 1, 2, 3, \dots\}$  – can be any non-negative integer (no upper bound!)

**The parameter**  $\lambda$  is the *average rate* of events.

This formula tells us  $\mathbb{P}(X = k)$  given  $\lambda$ . Later, we'll flip it: given observed counts, which  $\lambda$  makes the data most probable? That's **maximum likelihood** thinking.

## Poisson Has $\mathbb{E}[X] = \text{Var}[X] = \lambda$

For  $X \sim \text{Poisson}(\lambda)$ :

**Expected value:**  $\mathbb{E}[X] = \lambda$

**Variance:**  $\text{Var}[X] = \lambda$

**The mean equals the variance.** This is the defining characteristic.

**Why this matters:** If you see count data where variance  $\approx$  mean, Poisson might be a good model. If variance  $\gg$  mean (“overdispersion”), you need something else.

## Poisson Emerges When $n$ Is Large and $p$ Is Small

The “law of rare events”

### The Poisson approximation:

If  $n$  is large and  $p$  is small (with  $np = \lambda$  moderate):

$$\text{Binomial}(n, p) \approx \text{Poisson}(\lambda = np)$$

**Example:** Rare disease in a large population

- $n = 1,000,000$  people
- $p = 0.00001$  (1 in 100,000 chance)
- $\lambda = np = 10$

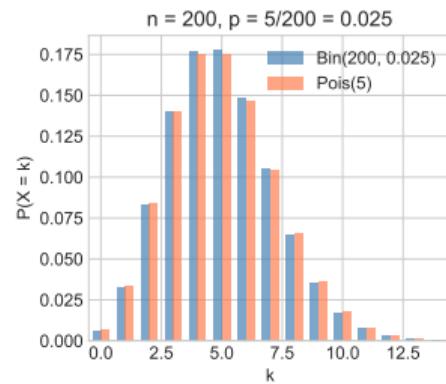
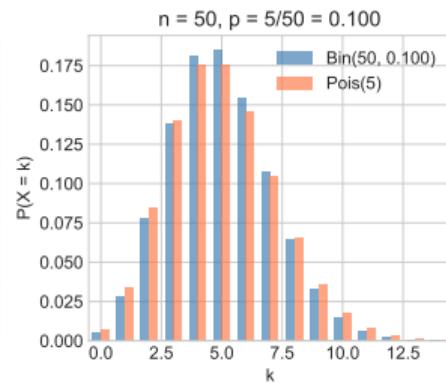
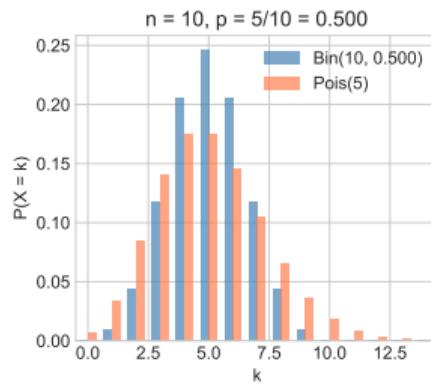
Number of cases  $\approx$  Poisson(10), much easier to work with than Binomial(1M, 0.00001).

This is the “law of rare events” — Poisson emerges naturally when counting unlikely things in large populations.

# As $n$ Grows and $p$ Shrinks, Binomial $\rightarrow$ Poisson

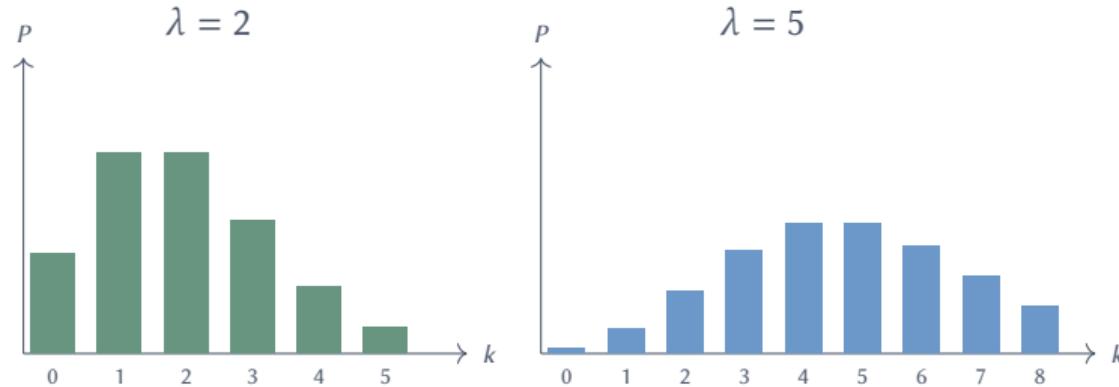
Visualization: Kaixiao Liu

Binomial  $\rightarrow$  Poisson: As  $n \rightarrow \infty$ ,  $p \rightarrow 0$ ,  $np = \lambda$  fixed



Rule of thumb: Use Poisson approximation when  $n \geq 20$  and  $p \leq 0.05$ .

## Visualizing the Poisson



As  $\lambda$  increases, the distribution shifts right and becomes more symmetric.  
For large  $\lambda$ , Poisson looks increasingly normal (another CLT application).

## Key Property: Poissons Add

**Theorem:** If  $X \sim \text{Poisson}(\lambda_1)$  and  $Y \sim \text{Poisson}(\lambda_2)$  are **independent**, then:

$$X + Y \sim \text{Poisson}(\lambda_1 + \lambda_2)$$

**Intuition:** If events arrive at rate  $\lambda_1$  from one source and  $\lambda_2$  from another, the combined rate is  $\lambda_1 + \lambda_2$ .

**Example:** Counting legislative hearings

- House committees:  $X \sim \text{Poisson}(12)$  per month
- Senate committees:  $Y \sim \text{Poisson}(8)$  per month
- Total:  $X + Y \sim \text{Poisson}(20)$

Unlike Binomial, Poisson addition works even when the rates differ.

## How These Distributions Connect



- **Bernoulli** → **Binomial**: Sum of independent 0/1 trials
- **Binomial** → **Poisson**: Many trials, small probability (law of rare events)

Wednesday: Continuous distributions (Uniform, Normal, Exponential) and the Poisson–Exponential connection.

## Summary: Three Discrete Distributions

Distribution	Support	$\mathbb{E}[X]$	$\text{Var}[X]$	Use case
Bernoulli( $p$ )	$\{0, 1\}$	$p$	$p(1 - p)$	Binary outcomes
Binomial( $n, p$ )	$\{0, \dots, n\}$	$np$	$np(1 - p)$	Count successes
Poisson( $\lambda$ )	$\{0, 1, 2, \dots\}$	$\lambda$	$\lambda$	Rare event counts

### Sum properties:

- $\text{Binomial}(n, p) + \text{Binomial}(m, p) = \text{Binomial}(n + m, p)$  (same  $p$ !)
- $\text{Poisson}(\lambda_1) + \text{Poisson}(\lambda_2) = \text{Poisson}(\lambda_1 + \lambda_2)$

These distributions will reappear constantly. Know their moments by heart.

# Looking Ahead

## Wednesday: Continuous distributions

- Uniform — and the “universality of uniform”
- Normal — the star of the show
- Exponential — waiting times and memorylessness
- The Poisson–Exponential connection

## Reading:

- Aronow & Miller, §1.2 continued (pp. 25–50)
- Blackwell, Chapter 2.3