Probability Foundations

The Language of Uncertainty

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Why Start Here?

Today: The language of probability

Probability is the vocabulary for describing **populations** and **uncertainty**.

Before we can estimate anything, we need language to describe *what we're trying to learn.*

This course takes a "population-first" approach: define what you want to know about the population before worrying about estimation.

Part I

Sample Spaces and Events

The building blocks

What Is Probability?

A **model** for describing uncertainty about outcomes.

Three ingredients:

1. A **sample space** Ω: all possible outcomes
2. An **event space** $\mathcal{S}$: subsets of outcomes we care about
3. A **probability measure** $\mathbb{P}$: assigns numbers to events

Together, $(\Omega, \mathcal{S}, \mathbb{P})$ is a **probability space**.

Probability Is a Model

Not a property of the world

Consider flipping a coin. If you knew *everything*—the exact force applied, the coin's initial orientation, air resistance, the surface it lands on—you could predict exactly whether it lands heads or tails. There's nothing inherently "random" about a coin flip.

So what is probability?

It's a **model of our uncertainty**, not a feature of physical reality. We use probability because we *don't* know everything—it describes what we believe given our ignorance.

This is a key feature of the **agnostic approach** we take in this course (following Aronow & Miller). It's worth noting upfront.

"All models are wrong, but some are useful." — George Box

Sample Space

All possible outcomes

The **sample space** Ω is the set of all possible outcomes of a random process.

Examples:

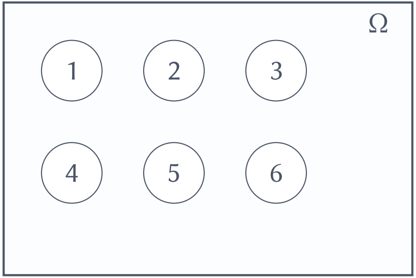
- Coin flip: $\Omega = \{\text{Heads}, \text{Tails}\}$
- Die roll: $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Temperature tomorrow: $\Omega = \mathbb{R}$ (or some interval)

The sample space can be finite, countably infinite, or uncountable.

Visualizing the Sample Space

The universe of possibilities

Die Roll: Sample Space



The **sample space** $\Omega = \{1, 2, 3, 4, 5, 6\}$ contains *every* possible outcome.

Think of Ω as the "universe" — nothing can happen outside of it.

Events

Questions we can ask

An **event** is a subset of the sample space: $A \subseteq \Omega$.

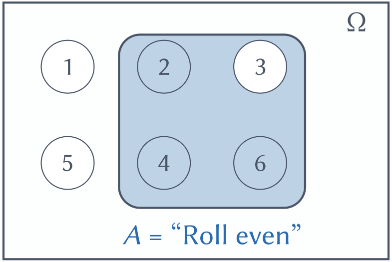
For a die roll ($\Omega = \{1, 2, 3, 4, 5, 6\}$):

- $A = \{6\}$: "Roll a six"
- $B = \{2, 4, 6\}$: "Roll an even number"
- $C = \{1, 2\}$: "Roll less than three"
- Ω: "Something happens" (the *certain* event)
- $\emptyset$: "Nothing happens" (the *impossible* event)

Events are the things we assign probabilities to.

Visualizing Events

Subsets of the sample space



The event $A = \{2, 4, 6\}$ is a **subset** of Ω: we write $A \subseteq \Omega$.

We assign probabilities to events: $\mathbb{P}(A) = \mathbb{P}(\text{"Roll even"}) = 3/6 = 1/2$

Sample Space vs. Events

$A \subseteq \Omega$

Sample Space Ω	**Event A**
All possible outcomes	*Some* possible outcomes
The "universe"	A subset of the universe
Fixed for a given experiment	Many different events possible
$\mathbb{P}(\Omega) = 1$ always	$0 \leq \mathbb{P}(A) \leq 1$

Die roll example:

- Sample space: $\Omega = \{1, 2, 3, 4, 5, 6\}$ — all six faces
- Event "roll even": $A = \{2, 4, 6\}$ — three of the six faces
- Event "roll a six": $B = \{6\}$ — just one face

Operations on Events

Events are sets, so we can combine them:

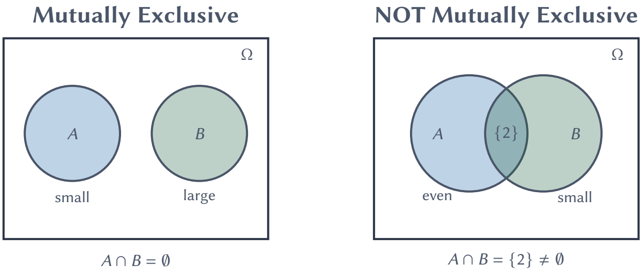
Operation	Notation	Meaning
Union	$A \cup B$	A or B (or both)
Intersection	$A \cap B$	A and B
Complement	A^c	not A
Difference	$A \setminus B$	A but not B

Example: Die roll, $A = \{2, 4, 6\}$ (even), $B = \{1, 2, 3\}$ (small)

- $A \cap B = \{2\}$ (even AND small)
- $A \cup B = \{1, 2, 3, 4, 6\}$ (even OR small)
- $A^c = \{1, 3, 5\}$ (odd)

Mutually Exclusive Events

Two events are **mutually exclusive** (or *disjoint*) if they cannot both occur: $A \cap B = \emptyset$



Example: Die roll — $A = \{1, 2, 3\}$ (small), $B = \{4, 5, 6\}$ (large) are mutually exclusive.

Why does this matter? It simplifies probability calculations: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

Event Spaces

Which subsets can we assign probabilities to?

Not just *any* collection of subsets will work. An **event space** (or σ-algebra) $\mathcal{S}$ on Ω must satisfy three properties:

1. **Contains the sample space**: $\Omega \in \mathcal{S}$
2. **Closed under complements**: If $A \in \mathcal{S}$, then $A^c \in \mathcal{S}$
3. **Closed under countable unions**: If $A_1, A_2, \ldots \in \mathcal{S}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$

Why does this matter?

- These properties ensure we can do set operations without "leaving" the event space
- From (1) and (2): $\emptyset = \Omega^c \in \mathcal{S}$ (the impossible event is always in $\mathcal{S}$)
- From (2) and (3): also closed under countable intersections (De Morgan's laws)

For PS1: You'll verify that specific collections of sets satisfy (or fail) these properties.

Event Space Examples

Building intuition

Let $\Omega = \{A, B, C\}$ (a finite sample space with three outcomes).

The trivial event space: $\mathcal{S}_1 = \{\emptyset, \Omega\}$

- Check: $\Omega \in \mathcal{S}_1$? ✓
- Check: $\Omega^c = \emptyset \in \mathcal{S}_1$? ✓ $\quad \emptyset^c = \Omega \in \mathcal{S}_1$? ✓
- Check: Unions? Only $\emptyset \cup \Omega = \Omega \in \mathcal{S}_1$ ✓

A larger event space: $\mathcal{S}_2 = \{\emptyset, \{A\}, \{B, C\}, \Omega\}$

- Check: $\{A\}^c = \{B, C\} \in \mathcal{S}_2$? ✓
- Check: $\{A\} \cup \{B, C\} = \Omega \in \mathcal{S}_2$? ✓

The power set $\mathcal{S}_3 = 2^\Omega$ (all $2^3 = 8$ subsets) is always a valid event space.

Note: $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \mathcal{S}_3$ — event spaces can be nested.

Kolmogorov Axioms

The rules probability must follow

A **probability measure** $\mathbb{P} : \mathcal{S} \to [0, 1]$ satisfies three axioms:

1. **Non-negativity**: $\mathbb{P}(A) \geq 0$ for all events A
   $\to$ Probabilities can't be negative
2. **Normalization**: $\mathbb{P}(\Omega) = 1$
   $\to$ Something must happen; probabilities sum to 1
3. **Countable additivity**: For mutually exclusive events $A_1, A_2, \ldots$:

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

   $\to$ If events can't overlap, just add their probabilities

Everything else we'll derive follows from these three axioms.

Consequences of the Axioms

From the three axioms, we can prove:

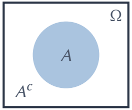
- **Complement rule**: $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$
- **Impossible event**: $\mathbb{P}(\emptyset) = 0$
- **Monotonicity**: If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$
- **Subtraction rule**: $\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$
- **Addition rule**: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

The addition rule corrects for double-counting the intersection.

Visualizing the Consequences

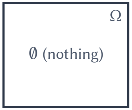
Quick reference

Complement



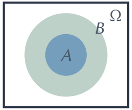
$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

Impossible



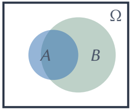
$\mathbb{P}(\emptyset) = 0$

Monotonicity



$A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

Subtraction

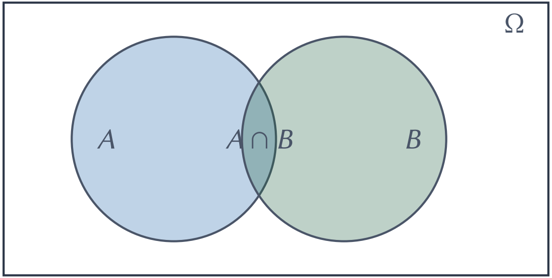


$\mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A \cap B)$

Each follows from the three axioms. Proofs are in the readings.

The Addition Rule

Visualizing inclusion-exclusion



$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

If we add $\mathbb{P}(A)$ and $\mathbb{P}(B)$, we count the intersection twice.

Conditional Probability

Updating beliefs with new information

Conditional Probability

The key definition

The **conditional probability** of A given B is:

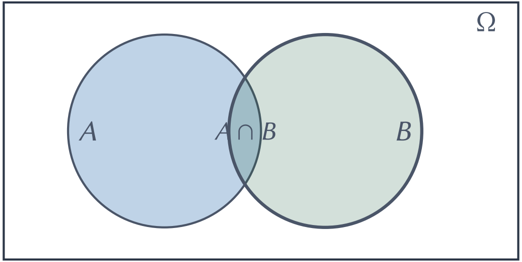
$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \qquad \text{provided } \mathbb{P}(B) > 0$$

Interpretation: The probability of A, *given that we know B occurred.*

We "zoom in" on the world where B happened and ask: how much of that world is A?

Conditional Probability

Visual intuition



$$\mathbb{P}(A \mid B) = \frac{\text{Probability of being in both } A \text{ and } B}{\text{Probability of being in } B}$$

Given that we're in B, what fraction is also in A?

Example: Two Dice

Roll two fair dice. What is $\mathbb{P}(\text{sum} = 8 \mid \text{first die} = 3)$?

Solution:

- Let $A = \{\text{sum} = 8\}$ and $B = \{\text{first die} = 3\}$
- $\mathbb{P}(B) = 6/36 = 1/6$ (six outcomes where first die is 3)
- $A \cap B = \{(3, 5)\}$ (only way to get sum 8 with first die 3)
- $\mathbb{P}(A \cap B) = 1/36$

$$\mathbb{P}(A \mid B) = \frac{1/36}{1/6} = \frac{1}{6}$$

Compare to $\mathbb{P}(\text{sum} = 8) = 5/36 \approx 0.14$. Knowing the first die changes things!

The Multiplicative Law

Rearranging the definition of conditional probability:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B) \cdot \mathbb{P}(B)$$

Or equivalently:

$$\mathbb{P}(A \cap B) = \mathbb{P}(B \mid A) \cdot \mathbb{P}(A)$$

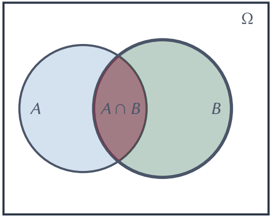
The chain rule (for three events):

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B \mid A) \cdot \mathbb{P}(C \mid A \cap B)$$

Visualizing the Multiplicative Law

Two ways to compute $\mathbb{P}(A \cap B)$

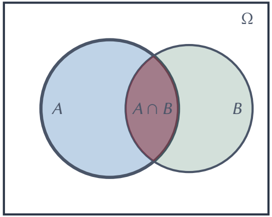
Start with B, then find A



$$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B) \cdot \mathbb{P}(B)$$

"What fraction of B is also A?"

Start with A, then find B



$$\mathbb{P}(A \cap B) = \mathbb{P}(B \mid A) \cdot \mathbb{P}(A)$$

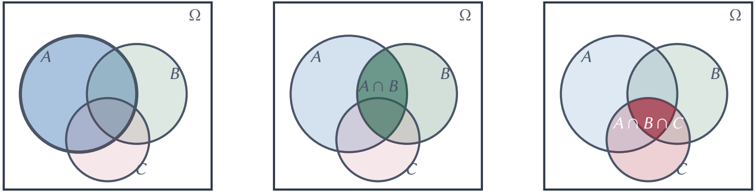
"What fraction of A is also B?"

Key insight: The intersection $A \cap B$ is the same region either way—we're just computing its probability through different "doors."

Visualizing the Chain Rule

$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B \mid A) \cdot \mathbb{P}(C \mid A \cap B)$

Step 1: Start with A | **Step 2: B given A** | **Step 3: C given $A \cap B$**



$\mathbb{P}(A)$ | $\mathbb{P}(B \mid A)$ | $\mathbb{P}(C \mid A \cap B)$

Intuition: Keep "zooming in." First restrict to A, then to $A \cap B$, then ask what fraction is also in C.

Each step narrows the universe; each conditional probability asks "what fraction of where we are is also in the next set?"

Part III

Independence

When knowing one thing tells you nothing about another

Independence of Events

Definition

Events A and B are **independent** if:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

Equivalent statement (when $\mathbb{P}(B) > 0$):

$$\mathbb{P}(A \mid B) = \mathbb{P}(A)$$

Knowing B occurred doesn't change the probability of A.

Independence means information is irrelevant. Learning B happened gives you no information about whether A happened.

Notation: $A \perp\!\!\!\perp B$ means "A is independent of B"

Independence vs. Mutual Exclusivity

These are NOT the same thing!

Mutually exclusive: $A \cap B = \emptyset$ (can't both happen)

Independent: $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ (knowing one doesn't affect the other)

In fact, they're almost opposites!

If A and B are mutually exclusive with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{0}{\mathbb{P}(B)} = 0 \neq \mathbb{P}(A)$$

So mutually exclusive events are **dependent** (strongly so!).

If I know B happened, I know A didn't happen.

Example: Coin Flips

Flip a fair coin twice. Let $A = \{\text{first flip is Heads}\}$ and $B = \{\text{second flip is Heads}\}$.

Are A and B independent?

Check:

- $\mathbb{P}(A) = 1/2, \quad \mathbb{P}(B) = 1/2$
- $\mathbb{P}(A \cap B) = \mathbb{P}(\{HH\}) = 1/4$
- $\mathbb{P}(A) \cdot \mathbb{P}(B) = (1/2)(1/2) = 1/4 \checkmark$

Yes, they are independent. The outcome of one flip doesn't affect the other.

Example: Drawing Cards

Draw two cards from a deck **without replacement**. Let:

- $A = \{\text{first card is an Ace}\}, \quad B = \{\text{second card is an Ace}\}$

Are A and B independent?

Check:

- $\mathbb{P}(A) = 4/52$
- $\mathbb{P}(B \mid A) = 3/51$ (if first was Ace, only 3 Aces left in 51 cards)
- $\mathbb{P}(B \mid A^c) = 4/51$ (if first wasn't Ace, still 4 Aces in 51 cards)

Since $\mathbb{P}(B \mid A) \neq \mathbb{P}(B \mid A^c)$, knowing A changes $\mathbb{P}(B)$.

No, they are **not** independent.

Part IV

Bayes' Rule

Reversing conditional probabilities

The Reverend Thomas Bayes (1701–1761)

A thought experiment with billiard balls

Thomas Bayes was an English Presbyterian minister and amateur mathematician. His famous essay was published posthumously in 1763 by his friend Richard Price.

Bayes' thought experiment: Imagine a billiard table. A ball is rolled and comes to rest somewhere—you don't see where. Then more balls are rolled, and you're told whether each lands to the left or right of the first ball.

The question: Given this evidence, what can you infer about where the first ball is?

Key insight: Each new observation lets us *update* our beliefs about the unknown quantity. We start with uncertainty (a "prior"), observe evidence, and arrive at refined beliefs (a "posterior").

This is the logic of Bayesian inference: prior $\times$ likelihood $\longrightarrow$ posterior.

A Controversial Idea

The frequentist-Bayesian debate

Bayes' approach was **controversial** for over 200 years. Why?

Frequentist objection: Probability should describe long-run frequencies of *repeatable* events. "There's a 70% probability the ball is in the left half" seemed unscientific—the ball is either there or it isn't!

Bayesian response: Probability describes our *uncertainty*, not physical randomness. It's sensible to have beliefs about fixed but unknown quantities.

Historical irony: Statisticians dismissed Bayesian methods as "subjective" well into the 20th century. Yet Bayesian reasoning proved essential in:

- Breaking the Nazi Enigma code (1940s)
- Finding lost submarines and aircraft (1960s–today)
- Modern machine learning and AI

Today, most statisticians use both approaches as tools for different problems.

Bayes' Rule in History

From codebreaking to search and rescue

Alan Turing (1940s): Used sequential Bayesian updating to crack the Nazi Enigma code at Bletchley Park. Each piece of intercepted message updated beliefs about machine settings. He called it "banburismus" after the town of Banbury, where the paper strips were made.

Search Theory (1960s–today):

- **USS *Scorpion* (1968)**: Navy used Bayesian search to find the lost submarine
- **Air France 447 (2009)**: After two years of failed searches, Bayesian methods found the wreckage
- **Steve Fossett (2007)**: Updated probability maps based on search patterns
- **MH370 (2014)**: Bayesian analysis of satellite data guided the search

The logic: Start with prior beliefs about location. Each failed search in an area *lowers* the probability there and *raises* it elsewhere.

Motivation: Strategic Thinking Under Uncertainty

Example: You're playing poker, and the person in front of you raises.

What's your best response?

- It depends on what you *learned* from that raise
- And what cards you're holding

This requires us to **update our beliefs** based on new information.

We need to calculate conditional probabilities—but often we know them "backwards."

Deriving Bayes' Rule

Step by step from definitions

Let A and B be two events. We want $\mathbb{P}(A \mid B)$.

Start with the definition of conditional probability:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \implies \mathbb{P}(A \cap B) = \mathbb{P}(A \mid B) \cdot \mathbb{P}(B)$$

Similarly:

$$\mathbb{P}(B \mid A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \implies \mathbb{P}(B \cap A) = \mathbb{P}(B \mid A) \cdot \mathbb{P}(A)$$

Deriving Bayes' Rule

The key insight

Since $\mathbb{P}(A \cap B) = \mathbb{P}(B \cap A)$:

$$\mathbb{P}(A \mid B) \cdot \mathbb{P}(B) = \mathbb{P}(B \mid A) \cdot \mathbb{P}(A)$$

Solve for $\mathbb{P}(A \mid B)$:

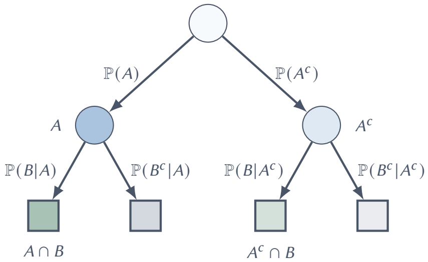
$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A) \cdot \mathbb{P}(A)}{\mathbb{P}(B)}$$

This is **Bayes' Rule** (naive form).

It lets us "flip" conditional probabilities: from $\mathbb{P}(B \mid A)$ to $\mathbb{P}(A \mid B)$.

Visualizing Bayes' Rule

The tree diagram: forward vs. backward



Forward (what we often know):
Nature "picks" A or A^c first,
then B happens (or not).

Backward (what we want):
We *observe* B. Given that,
was it A or A^c?

Bayes' Rule: Of all the ways to reach B, what fraction came through A?

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cap B) + \mathbb{P}(A^c \cap B)} = \frac{\text{green path through } A}{\text{all green paths}}$$

The Law of Total Probability

A consequence of the additivity axiom

Observation: We can decompose B using a **partition** of Ω: $B = (B \cap A) \cup (B \cap A^c)$

A partition is a collection of mutually exclusive, exhaustive "bins." Here $\{A, A^c\}$ partitions Ω.

These pieces are mutually exclusive, so by the **additivity axiom**:

$$\mathbb{P}(B) = \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A^c)$$

Apply the multiplicative law:

$$\mathbb{P}(B) = \mathbb{P}(B \mid A) \cdot \mathbb{P}(A) + \mathbb{P}(B \mid A^c) \cdot \mathbb{P}(A^c)$$

The unconditional probability is a weighted average of conditional probabilities.

This gives us what we need for Bayes' denominator.

Bayes' Rule: Full Form

Substituting the Law of Total Probability into Bayes' Rule:

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A) \cdot \mathbb{P}(A)}{\mathbb{P}(B \mid A) \cdot \mathbb{P}(A) + \mathbb{P}(B \mid A^c) \cdot \mathbb{P}(A^c)}$$

Terminology:

- $\mathbb{P}(A)$: **Prior** — belief before seeing B
- $\mathbb{P}(B \mid A)$: **Likelihood** — how likely is B if A is true?
- $\mathbb{P}(A \mid B)$: **Posterior** — updated belief after seeing B

"Ask Marilyn" and the Monty Hall Firestorm

September 9, 1990

Marilyn vos Savant—listed in the *Guinness Book of World Records* for highest recorded IQ—wrote a column in *Parade* magazine. A reader sent this puzzle:

> *"Suppose you're on a game show, and you're given the choice of three doors. Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says, 'Do you want to pick door No. 2?' Is it to your advantage to switch?"*

Marilyn's answer: **Yes, you should switch.** Switching gives you a $\frac{2}{3}$ chance of winning.

What happened next would reveal something uncomfortable about how experts respond to being corrected—especially by a woman.

The Backlash

When mathematicians get it wrong

Marilyn received approximately **10,000 letters**, nearly 1,000 from PhDs. Many were hostile:

> *"You blew it, and you blew it big! Since you seem to have difficulty grasping the basic principle at work here, I'll explain..." —PhD, Georgetown University*
>
> *"You are utterly incorrect... How many irate mathematicians are needed to get you to change your mind?" —PhD, U.S. Army Research Institute*
>
> *"You made a mistake, but look at the positive side. If all those PhDs were wrong, the country would be in very serious trouble." —PhD, Univ. of Florida*

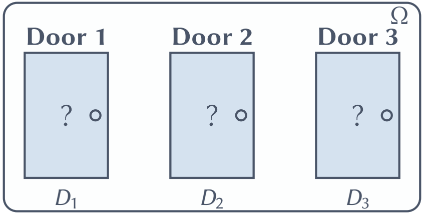
The twist: Marilyn was right. The PhDs were wrong.

Let's work through why, using Bayes' Rule.

The Monty Hall Problem

Step 1: The prior

Three doors. Behind one is \$1 million; behind the other two are goats.



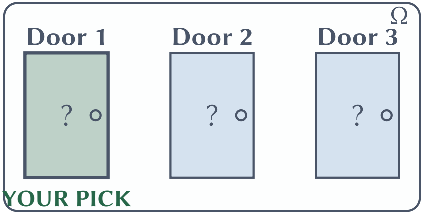
The events D_1, D_2, D_3 ("money behind door i") are **mutually exclusive** and **exhaustive**. With no information, each is equally likely:

$$\mathbb{P}(D_1) = \mathbb{P}(D_2) = \mathbb{P}(D_3) = \frac{1}{3} \qquad \text{(the } \textbf{prior}\text{)}$$

The Monty Hall Problem

Step 2: You choose

You pick Door 1. Since the doors are equally likely, it doesn't matter which you choose.



At this point, what's the probability you picked the winning door?

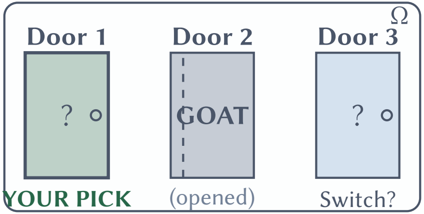
$$\mathbb{P}(D_1) = \frac{1}{3} \qquad \mathbb{P}(D_2 \cup D_3) = \frac{2}{3}$$

You have a 1/3 chance of being right. The money is *more likely* behind one of the other two doors.

The Monty Hall Problem

Step 3: Monty opens a door

Monty Hall, the host, *knows* where the money is. He opens Door 2: a goat.



Monty asks: *"Would you like to switch to Door 3?"*

The question: Has the probability changed? Is $\mathbb{P}(D_3 \mid O) = \frac{1}{3}$ still, or has Monty's action given us new information?

The Monty Hall Problem

The key insight

Key: Monty's choice is *not random*—it's constrained by his knowledge.

- Monty *knows* where the money is
- Monty will *never* open the door with the money
- Monty will *never* open the door you picked

This means his action **reveals information**. We need to update our beliefs.

Let O = "Monty opened door 2." We want to compute:

$$\mathbb{P}(D_3 \mid O) = ?$$

Is this still $\frac{1}{3}$, or has it changed? Let's use Bayes' Rule to find out.

The Monty Hall Problem

Setting up Bayes' Rule

We want $\mathbb{P}(D_3 \mid O)$ using Bayes' Rule:

$$\mathbb{P}(D_3 \mid O) = \frac{\mathbb{P}(O \mid D_3) \cdot \mathbb{P}(D_3)}{\mathbb{P}(O \mid D_1)\mathbb{P}(D_1) + \mathbb{P}(O \mid D_2)\mathbb{P}(D_2) + \mathbb{P}(O \mid D_3)\mathbb{P}(D_3)}$$

Priors: $\mathbb{P}(D_1) = \mathbb{P}(D_2) = \mathbb{P}(D_3) = \frac{1}{3}$

The Monty Hall Problem

The likelihoods

Key insight: Monty *knows* where the money is and will *never* open a door with money.

What is $\mathbb{P}(O \mid D_i)$? (Given the money is behind door i, what's the probability Monty opens door 2?)

1. $\mathbb{P}(O \mid D_1) = 0.5$
   Money behind door 1. Monty can choose door 2 or 3 randomly.
2. $\mathbb{P}(O \mid D_2) = 0$
   Money behind door 2. Monty would never open door 2!
3. $\mathbb{P}(O \mid D_3) = 1$
   Money behind door 3. Monty must open door 2 (can't open door 1 or 3).

The Monty Hall Problem

The calculation

$$\mathbb{P}(D_3 \mid O) = \frac{\mathbb{P}(O \mid D_3) \cdot \mathbb{P}(D_3)}{\mathbb{P}(O \mid D_1)\mathbb{P}(D_1) + \mathbb{P}(O \mid D_2)\mathbb{P}(D_2) + \mathbb{P}(O \mid D_3)\mathbb{P}(D_3)}$$

Substituting:

$$\mathbb{P}(D_3 \mid O) = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{6} + 0 + \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

The Monty Hall Problem

The answer

$$\mathbb{P}(D_3 \mid O) = \frac{2}{3} \qquad \mathbb{P}(D_1 \mid O) = \frac{1}{3}$$

Definitely switch to door 3!

Intuition:

- When you picked door 1, you had a $\frac{1}{3}$ chance of being right
- The other two doors collectively had $\frac{2}{3}$ probability
- Monty's action *concentrates* that $\frac{2}{3}$ onto door 3

Marilyn vos Savant was right. The angry PhDs were wrong. Bayes' Rule settles it.

Bayes and Strategic Behavior

When actions reveal information

In Monty Hall, the host's action (opening door 2) **reveals information** because it's constrained by what he knows. This is the foundation of **signaling theory**:

- **Michael Spence (1973)**: Education as a signal of ability. If degrees are *easier* for high-ability workers, employers can use education to update beliefs. (Nobel Prize, 2001)
- **Amotz Zahavi (1975)**: The peacock's tail is *costly*—only fit males can afford it. Costly signals are credible.
- **Diego Gambetta (2009)**: Criminals use tattoos and rituals as costly signals of commitment. (*Codes of the Underworld*)

In game theory, we solve for **Perfect Bayesian Equilibrium**: strategies are optimal given beliefs, and beliefs are updated via Bayes' Rule. David Lewis discovered signaling in his 1969 dissertation—before Spence.

Example: Medical Testing

Setup

A disease affects 1% of the population. A test has:

- 95% sensitivity: $\mathbb{P}(\text{positive} \mid \text{disease}) = 0.95$
- 90% specificity: $\mathbb{P}(\text{negative} \mid \text{no disease}) = 0.90$

Question: If you test positive, what's the probability you have the disease?

Given:

- $\mathbb{P}(D) = 0.01$, so $\mathbb{P}(D^c) = 0.99$
- $\mathbb{P}(+ \mid D) = 0.95$
- $\mathbb{P}(+ \mid D^c) = 0.10$ (false positive rate = $1 - 0.90$)

Example: Medical Testing

Applying Bayes' Rule

$$\mathbb{P}(D \mid +) = \frac{\mathbb{P}(+ \mid D) \cdot \mathbb{P}(D)}{\mathbb{P}(+ \mid D)\mathbb{P}(D) + \mathbb{P}(+ \mid D^c)\mathbb{P}(D^c)}$$

First, find $\mathbb{P}(+)$ using the Law of Total Probability:

$$\mathbb{P}(+) = (0.95)(0.01) + (0.10)(0.99) = 0.0095 + 0.099 = 0.1085$$

Then:

$$\mathbb{P}(D \mid +) = \frac{(0.95)(0.01)}{0.1085} = \frac{0.0095}{0.1085} \approx 0.088$$

Example: Medical Testing

The surprising result

Even with a positive test, there's only an **8.8%** chance you have the disease!

Why so low?

- The disease is rare (1% prevalence)
- Most positive tests are false positives from the 99% without disease
- The 10% false positive rate applied to 99% $\gg$ the 95% true positive rate applied to 1%

Base rates matter. This is called the "base rate fallacy" when people ignore priors.

Why Independence Matters

Independence dramatically simplifies calculations:

- $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$ (no need to find conditional)
- For n independent events: $\mathbb{P}(A_1 \cap \cdots \cap A_n) = \prod_{i=1}^{n} \mathbb{P}(A_i)$

In this course:

- The **i.i.d. assumption** (coming in a few weeks) assumes observations are independent
- Many of our results depend on independence
- When independence fails, we need different tools (clustering, time series)

Today's Key Ideas

1. **Sample spaces and events**: The vocabulary for describing outcomes
2. **Kolmogorov axioms**: Non-negativity, normalization, additivity
3. **Conditional probability**: $\mathbb{P}(A \mid B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$
4. **Independence**: $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$
5. **Law of Total Probability**: Follows from additivity axiom
6. **Bayes' Rule**: Derived from conditional probability; flips conditionals

This is the language. Next: the objects we'll actually work with.

Looking Ahead

Wednesday: Conditional probability and Bayes' Rule (continued)

- More examples of Bayes' Rule
- Law of Total Probability applications

Next week: Random variables, expectation, and variance

Week 3: Famous distributions (two full lectures)

We're building the vocabulary to describe populations precisely.

For Wednesday

Reading:

- Aronow & Miller, §1.1: Review today's material
- Blackwell, Chapter 2.1: Probability foundations

Think about:

- In the medical testing example, what would happen if prevalence were 10% instead of 1%?
- Can you think of real-world examples where base rate neglect causes problems?

Questions?