8.1 Multiway Cut Problem and Minimum-Cut-Based Algorithm

Approximating Multi-Terminal Cuts
via Disjoint Isolating Regions

Graduate Institute of Telecommunications and Communication, National Chung Cheng University ,Chen Hong

Multiway Cut and the Breakdown of Planarity

- Elias Dahlhaus et al., 1992

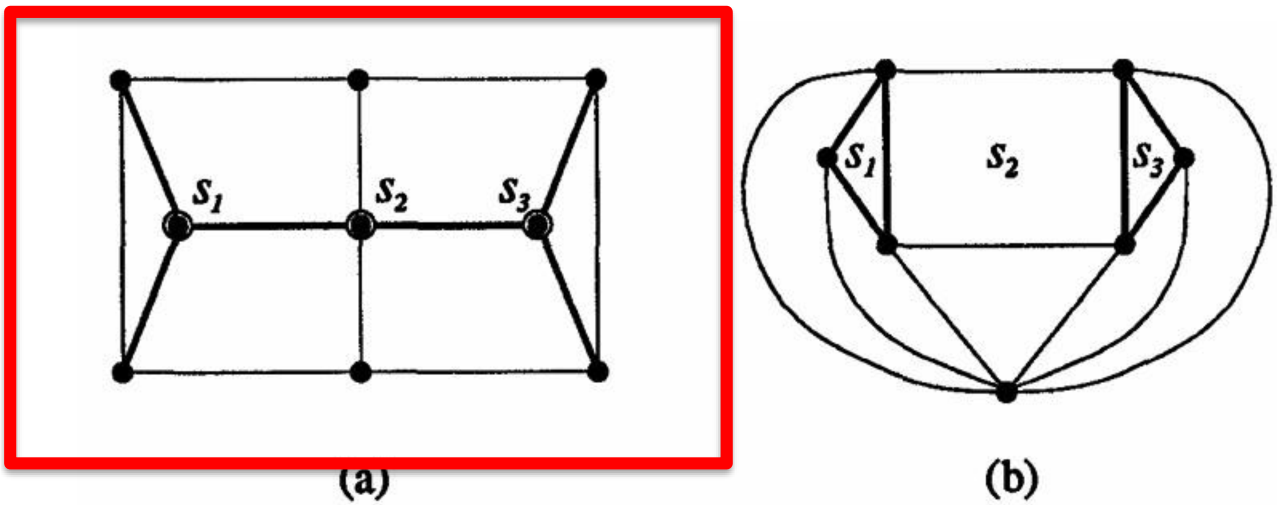
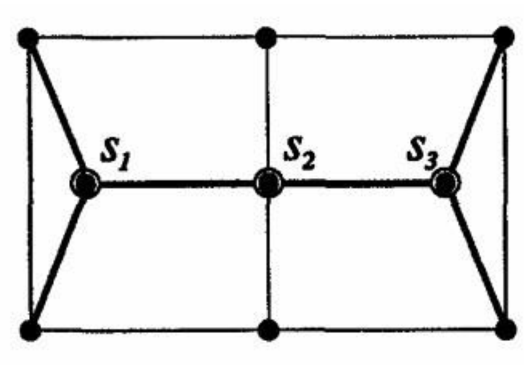


FIGURE 1. A planar 3-way cut (a) and its dual (b).

"Before introducing the algorithm, let's clearly define what the Multiway Cut Problem is."

Multi-way Cut Problem



In this graph, every edge can be **cut**, and removing an edge changes the **connectivity** of the graph.

Each edge has a **weight (or cost)** representing the expense of cutting it.

Our goal is to remove a set of edges so that the three terminals

s_1, s_2, s_3, are **no longer connected** to each other — that is, there is **no path** between any pair of terminals.

Multi-way Cut Problem

- Given an undirected graph $G = (V, E)$ with nonnegative edge costs

$$c_e \geq 0 \;\; \forall e \in E,$$

- and a set of k designated terminals

$$S = \{ s_1, s_2, \ldots, s_k\} \subseteq V.$$

Multi-way Cut Problem

Goal :

- Find a subset of edges $F \subseteq E$ such that, after removing F from G,
- every pair of distinct terminals $s_i, s_j \in S$ lies in **different connected components** of $G(V, E \setminus F)$, and the total cost of the removed edges is minimized.

- Minimize

$$c(F) = \sum_{e \in F} c_e$$

- subject to s_i and s_j are disconnected in

$$G(V, E \setminus F), \forall i \neq j.$$

Key Concept

- **Input:** A weighted undirected graph $G(V, E)$ and a set of terminals $S = \{s_1, s_2, s_3\}$
- **Output:** A minimum-cost edge set $E' \subseteq E$ such that, after removing E', all terminals become disconnected from one another.
- **Difficulty:** When the number of terminals $k \geq 3$, this problem becomes **NP-Hard**.

- Therefore, we typically rely on **approximation algorithms**, such as the *Isolating Cut Heuristic*, to obtain solutions that are *near-optimal* rather than exactly optimal.

Example Intuition

- \- Cutting an edge changes the graph’s connectivity.
- \- Objective: separate all terminals with minimum total cost.
- \- Common applications: network security, clustering, distributed systems isolation.

Motivation

- - In a simple min s-t cut, we can find the exact minimum using max-flow.
- - When number of terminals $k > 2 \rightarrow$ Multiway Cut Problem.
- - Dahlhaus et al. (1992):
- • NP-hard for general graphs.
- • Polynomial-time solvable only when planar and k fixed.

Intuitive Example — Distributed Computing

- Each vertex = an object or process.
- Each edge = communication between objects.
- c_e = communication cost.
- Terminals s_i must be placed on machine i.
- Objective: minimize inter-machine communication.

Isolating Cuts

- For each terminal s_i, define its region C_i as the set of vertices connected to s_i after removing F.
- $F_i = \delta(C_i)$
- Each F_i is an isolating cut separating s_i from the other terminals $\{s_1, \dots, s_k\}$.
- A single edge e may appear in multiple F_i's if it connects two regions C_i, C_j.

Algorithm Idea

- For each $i \in \{1, \ldots, k\}$:

  1. Add a virtual sink t.

  2. Connect all other terminals s_j , $j \neq i$ to t with infinite-cost edges.

  3. Compute the minimum $s_i - t$ cut — this gives the smallest F_i.

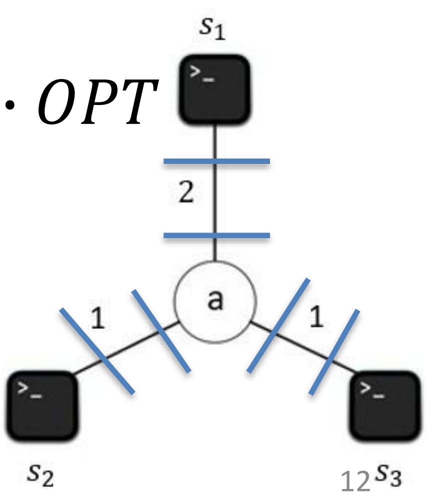
  Output $F = \bigcup_{i=1}^{k} F_i$ as the final multiway cut.
- Then:

$$c(F) \leq 2\left(1 - \frac{1}{k}\right) \cdot OPT$$

Theorem 8.1 — 2-Approximation

- Let $F*$ be the optimal multiway cut. For each s_i, let F_i* be its isolating cut in $F*$.
- Because Fi is a minimum isolating cut for si, we know that $c(F) \leq c(F*)$. Hence the cost of the solution of the algorithm is at most $\sum_{i=1}^{k} c(F_i) \leq \sum_{i=1}^{k} c(F_i*)$.
- Observed : each edge can belong to at most two F_i*'s:

$$\sum_{i=1}^{k} c(F) \leq \sum_{i=1}^{k} c(F_i*) \leq 2 \cdot c(F*) = 2 \cdot OPT$$



Improved Version — (2 – 2/k)-Approximation

- If we discard the most expensive among the k isolating cuts and keep only the cheapest $(k-1)$ cuts: $F = \bigcup_{i=1}^{k} F_i$
- Then:

$$c(F) \;\leq\; 2\left(1 - \frac{1}{k}\right) \cdot OPT$$

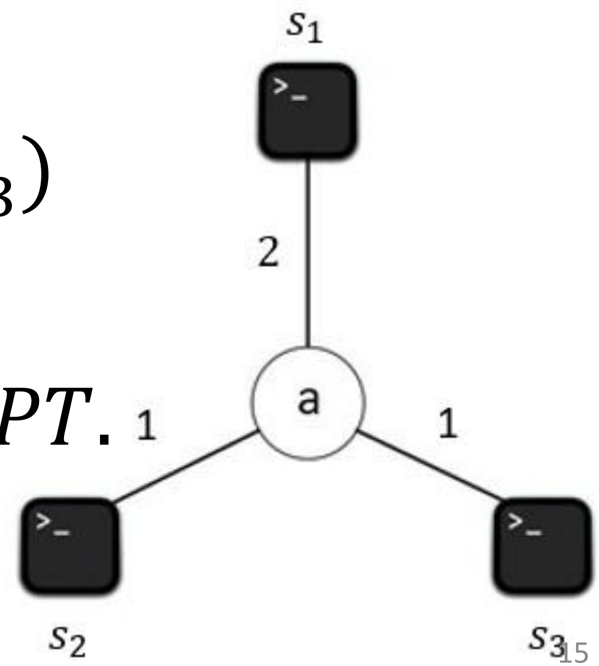
Main text begins

Planar Case (Normal Situation)

- Algorithm:
- $F_1 = \{(s_1, \text{a})\}$ or $\{(\text{a}, s_2), (\text{a}, s_3)\}$ cost=2
- $F_2 = \{(\text{a}, s_2)\}$ cost=1
- $F_3 = \{(\text{a}, s_3)\}$ cost=1
- That is , $c(F_1) + c(F_2) + c(F_3)$

$= 2 + 1 + 1 = 4.$

- Union=4, take $k - 1$=2 → OPT.



Well done, But The Multiway Cut problem is NP-hard.


不可能 絕對不可能

“The Complexity of Multiterminal Cuts” (Dahlhaus et al., 1992)

- established a complete complexity classification of the Multiway Cut problem:
- For **two terminals** ($k = 2$), it reduces to the *minimum s–t cut*, solvable in polynomial time via max-flow.
- For **three or more terminals** ($k \geq 3$), the problem becomes **NP-hard** in general graphs.
- However, it is **polynomial-time solvable** when the graph is **planar** and the number of terminals k is fixed.

"The Complexity of Multiterminal Cuts" (Dahlhaus et al., 1992)

No.	Result Type	Summary
(1)	Exact algorithm for planar graphs (k = 3)	In planar graphs, the three-terminal Multiway Cut can be solved exactly in $O(n^3 \log n)$ time, using specialized flow-network and dual-graph techniques.
(2)	Exact algorithm for planar graphs (fixed k)	For any fixed number of terminals k, the Multiway Cut in planar graphs remains polynomial-time solvable, but the runtime grows exponentially with k (e.g., $O\left(n^{O(k)}\right)$).
(3)	NP-hardness results	When k is part of the input (not fixed), the problem remains NP-hard even in planar graphs with unit edge weights. In general (non-planar) graphs, it is NP-hard even for k = 3.
(4)	Approximation algorithm	The first polynomial-time approximation algorithm achieves a ratio of 2 - 2/k. Later studies proved this bound is essentially tight unless P = NP.

“The Complexity of Multiterminal Cuts” (Dahlhaus et al., 1992)

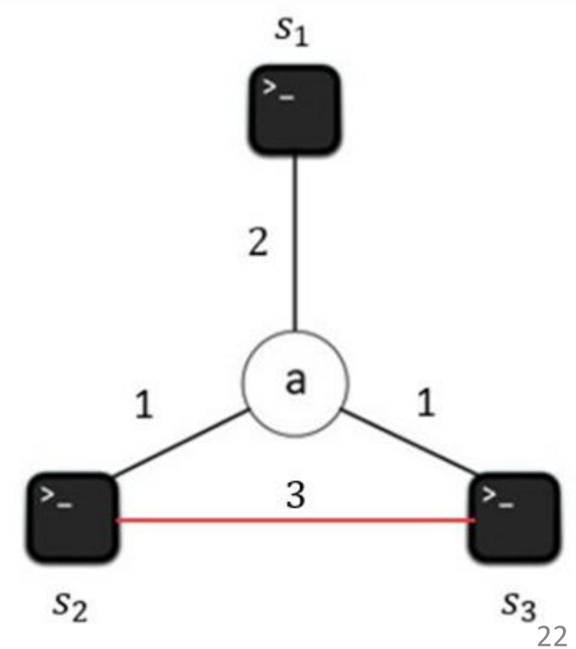
- “The results clarify the boundary between the tractable and intractable cases of the Multiway Cut problem, and give a simple, near-optimal approximation algorithm.”

"The Complexity of Multiterminal Cuts" (Dahlhaus et al., 1992)

- The study clearly distinguishes the boundary between tractable and intractable cases of the Multiway Cut problem:
  planar graphs with fixed k are solvable in polynomial time, while the problem becomes NP-hard for non-planar graphs or unbounded k.
  Moreover, they provide a simple polynomial-time algorithm achieving a near-optimal approximation ratio of $2 - \frac{2}{k}$.

Breaks topological separability between regions.

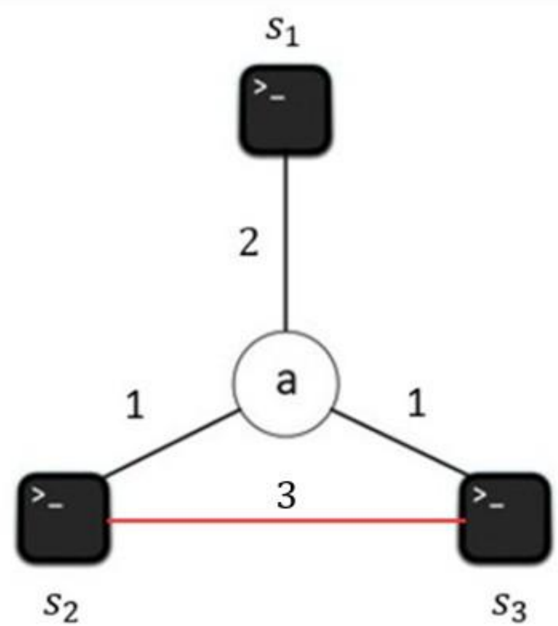
- Add edge $(s_2\ , s_3) = 3$
- Now terminals s_2 and s_3 are directly connected → **breaks topological separability** between regions.



 # "destroying the topology" refers to breaking the **disjointness** (i.e., the separability of terminal regions), rather than making the graph **geometrically non-planar**.

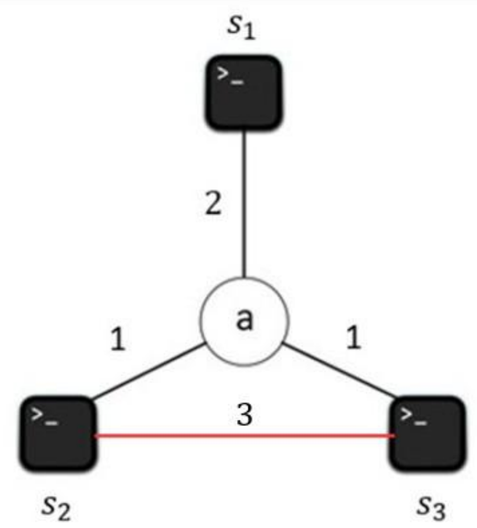
Breaks topological separability between regions.

- The direct link between terminals destroys the disjoint structure of isolating cuts — the regions now overlap topologically even though the graph is still planar.



Start by looking at the **isolating cuts** for the three terminals:

- Isolating cuts:
  s_1: $\{(s_1, a)\} \to \text{cost} = 2$ (if select this way)
  s_2: $\{(a, s_2)\} \to \text{cost} = 1$
  s_3: $\{(a, s_3)\} \to \text{cost} = 1$



- Without (s_2,s_3): total = 4
  Add $(s_2,s_3)=3 \to$ overlaps occur
  $F' = \{(s_1, a), (a, s_2), (a, s_3), (s_2, s_3)\} \to \text{cost} = 7$

(The total cost = 7 here does **not** come from the theoretical step of "dropping the largest cut.")

$OPT = \{(a, s_2), (a, s_3), (s_2, s_3)\} \to \text{cost} = 5$

Overlapping isolating regions $\Rightarrow$ higher cost

Why It Fails

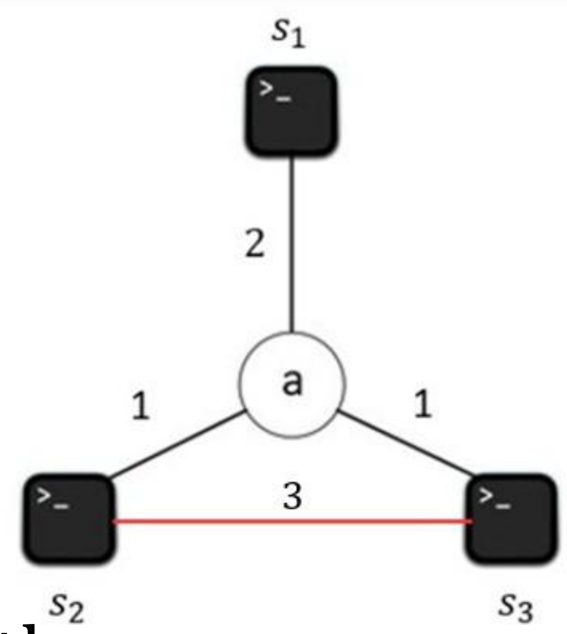
- **Direct edge between terminals breaks topological separability.**
(Terminals are no longer isolated by disjoint regions.)

- **Isolating regions overlap → edges are double-counted.**
(Cuts share common edges, increasing total cost.)

- **Each edge can appear in ≤ 2 isolating cuts → total ≤ 2 × OPT.**
(By double counting argument, total cost $\leq 2 \times OPT$.)

- **Removing the most expensive cut → approximation ratio =**$2 - \frac{k}{2}$.

Isolating Cut Heuristic → 2 – 2/k Approximation



Possible isolating cuts for s_1:

$F_1 = \{(s_1, a)\}$ → cost = 2

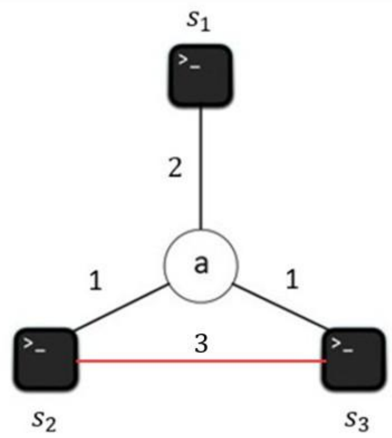
$F_1' = \{(a, s_2), (a, s_3)\}$ → cost = 2

(alternative choice)

Although both have the same cost, they affect the overall ratio differently:

- F_1 isolates one region → minimal overlap.
- F_1' connects two terminals → larger overlap in union.

Isolating Cut Heuristic → 2 – 2/k Approximation



Other isolating cuts:

$F_2 = \{(a, s_2), (s_2, s_3)\}$ $\rightarrow$ cost $= 4$

$F_3 = \{(a, s_3), (s_2, s_3)\}$ $\rightarrow$ cost $= 4$

Sum of isolating cuts $= 2 + 4 + 4 = 10$

Union of all edges $= \{(s_1, a), (a, s_2), (a, s_3), (s_2, s_3)\}$

$\rightarrow$ cost $= 7$

True $OPT = \{(a, s_2), (a, s_3), (s_2, s_3)\}$ $\rightarrow$ cost $= 5$ (OPT)

$\Rightarrow$ Equal cost $\neq$ equal ratio impact.

ALG depends on overlap structure, not just edge weights.

2-Approximation Algorithm

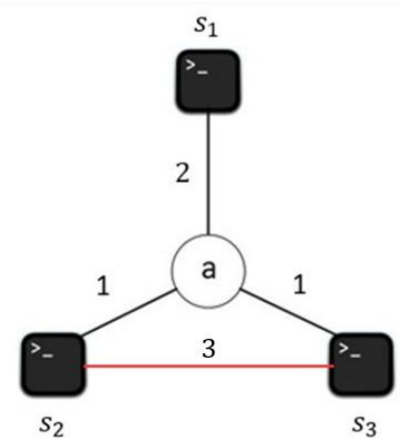
Isolating Cut Heuristic → 2 – 2/k Approximation

Possible isolating cuts for s_1:

$F_1 = \{(s_1, a)\}$ → cost = 2

$F_1' = \{(a, s_2), (a, s_3)\}$ → cost = 2

(alternative choice)



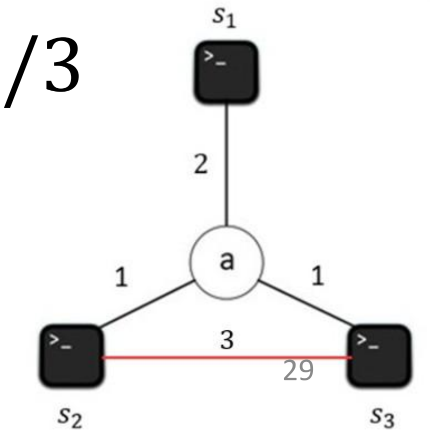
Other isolating cuts:

$F_2 = \{(a, s_2), (s_2, s_3)\}$ → cost = 4

$F_3 = \{(a, s_3), (s_2, s_3)\}$ → cost = 4

Isolating Cut Heuristic → 2 – 2/k Approximation

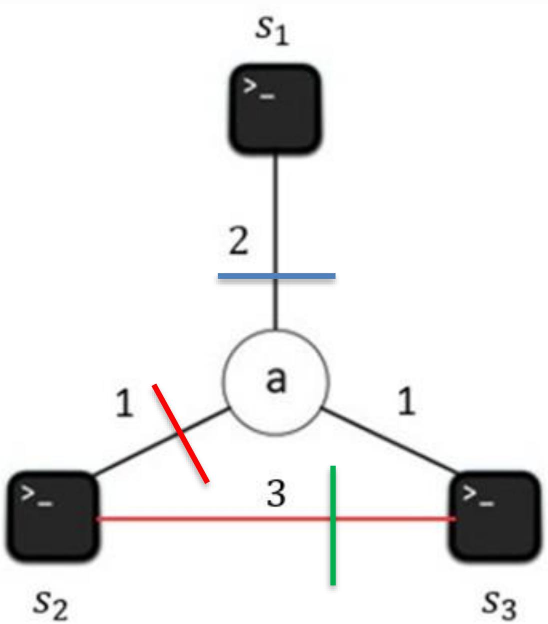
- Case A (choose F_1 and F_2; i.e., s_1 takes F_1)
  Union $= \{(s_1, a), (a, s_2), (s_2, s_3)\} \rightarrow ALG = 6$
  With $OPT = 5$,
  ratio $= 6/5 = 1.20 \leq (2 - 2/3) = 4/3$
- Case B (choose $F_1{}'$ and F_2; i.e., s_1 takes $F_1{}'$)
  Union $= \{(a, s_2), (a, s_3), (s_2, s_3)\} \rightarrow ALG = 5$
  With $OPT = 5$, ratio $= 5/5 = 1.00 \leq 4/3$

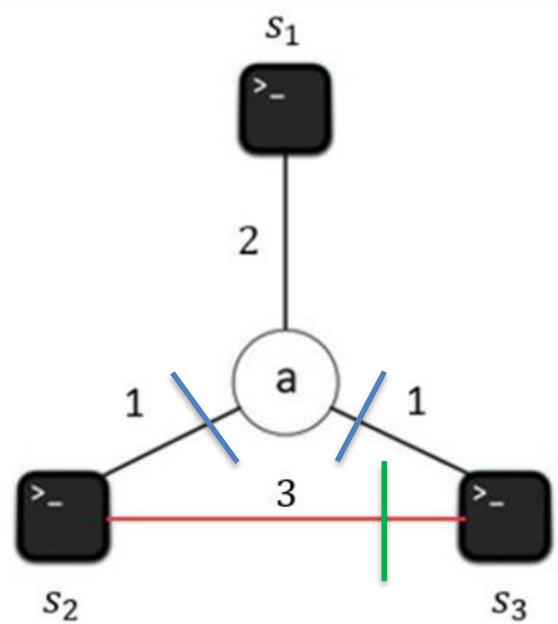


2-2/k -Approximation Algorithm

Compare Case A and Case B

Case A (choose F_1 and F_2; i.e., s_1 takes F_1)





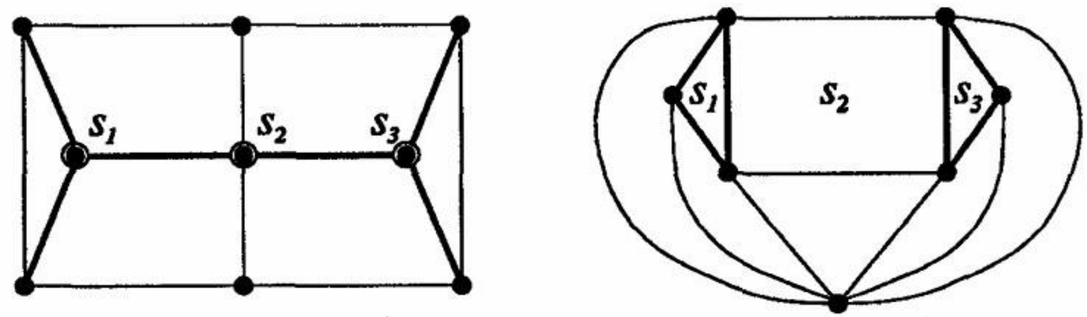
Case B (choose $F_1{}'$ and F_2; i.e., s_1 takes $F_1{}'$)

Topological Meaning of Multiway Cuts — Separation, not Geometry

- Dahlhaus et al. (1992) used many planar and dual constructions,but their use of topology was not about geometry—it was about separability.
- In other words, they cared about whether there exists a set of edges that can separate each terminal into different connected components, such that those separating cuts are disjoint in topology.
- This "topological separability" defines when a graph is tractable:

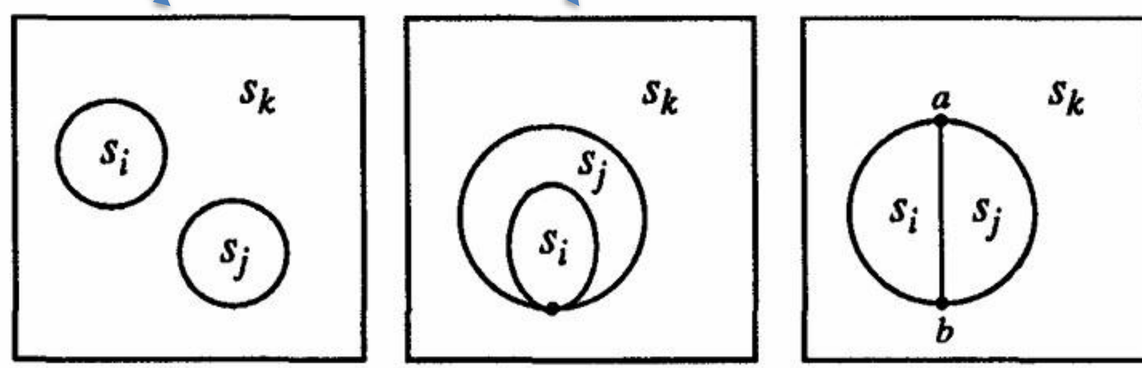
  When the cuts are separable (planar, disjoint regions) → solvable in polynomial time.

  When cuts overlap (non-separable regions) → NP-hard.
- So, topology here means the structure of separability, not whether lines cross in a geometric drawing.



(a) (b)

FIGURE 1. A planar 3-way cut (a) and its dual (b).



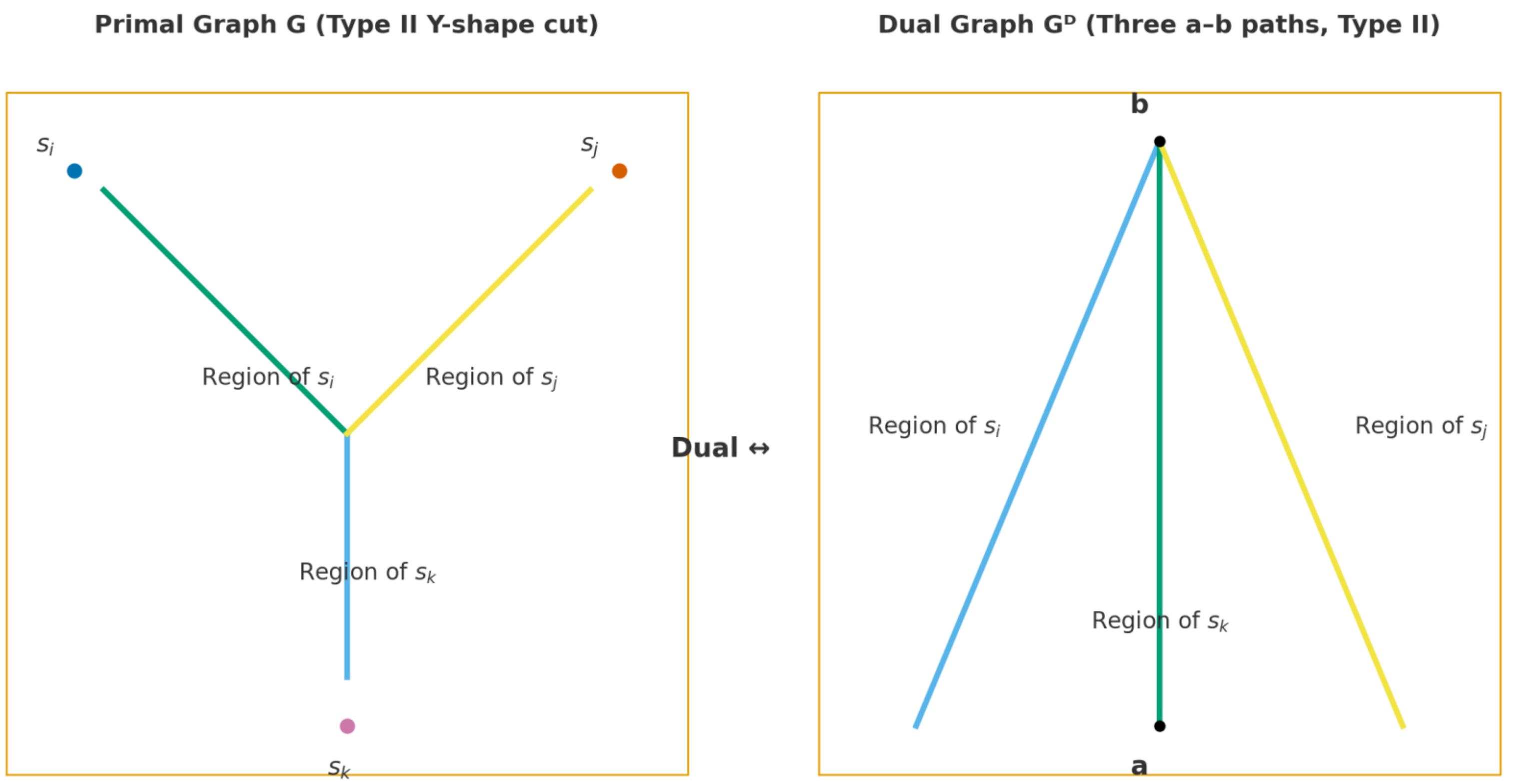
(a) (b) (c)

FIGURE 2. Types of 3-way cuts: Type I (a) and (b), Type II (c).

Classification of 3-Way Cut Topologies

Type	Dual Graph (C^D) Shape	Geometric Intuition	Meaning in Primal Graph
Type I	Two non-overlapping cycles (possibly tangent at one point)	Like two separate loops dividing the plane into three disjoint regions	Each cycle corresponds to a terminal’s isolating cut; regions are topologically disjoint
Type II	Three boundary paths connecting the same pair of vertices (a–b)	Like a 'three-petal flower' or Y-shaped structure	Terminal regions are pairwise adjacent, sharing boundaries → non-disjoint separability

Dahlhaus et al. (1992) *Type II 3-way cut*



Our example belong to Type II

Levels of Topological Breakdown in Multiway Cuts

情況	拓撲型態	幾何外觀	被破壞的性質	結果
直接加入 (s_2,s_3)	Edge cycle (Type II)	形成三角形	Edge disjointness 破壞	強破壞 → NP-hard
共用節點 a 但無 (s_2,s_3)	Node overlap (Type I)	區域在 a 接觸	Region disjointness 破壞	弱破壞 → 仍 NP-hard
完全分離	Disjoint regions (Type I)	各區域互不接觸	Topological DAG 保留	可 polytime 解

Key Insight

- When terminals are directly connected, the graph loses its original **DAG-like hierarchical separability**.
  As a result, **isolating cuts begin to overlap**, leading to **duplicate edge counting** and increased total cost.
  Among all tractable cases, **planarity** is the last structural property that preserves **topological separability**.

Summary

- **Planar graphs** possess a DAG-like topological separability,
  so the solution produced by the **Isolating Cut Heuristic** is close to the optimal (≈ OPT).
- In **non-planar graphs**, direct connections between terminals cause overlaps between cuts, resulting in higher costs and approximation errors.
- Thus, the loss of **planarity and separability** marks the **structural boundary between polynomial-time solvable and NP-hard cases**.

- **CS 598CSC: Approximation Algorithms**
  Instructor: Chandra Chekuri
  Scribe: Charles Blatti
  Lecture date: February 11, 2009
- **Title:**
  “Multiway Cut and k-Cut Problems”

- **# 備註 : 這個版本是用Chekuri, C.的版本，以下的演算法比較是用該版本的數學表達，而非David P. Williamson and David B. Shmoys.(8.1章節教材的)**

1.1 Isolating Cut Heuristic

Input:

- An undirected weighted graph $G = (V, E)$
- Edge weights $w : E \to \mathbb{R}^+$
- A set of terminals $S = \{s_1, s_2, ..., s_k\} \subseteq V$

Goal:

Find a set of edges $A \subseteq E$ whose removal separates all terminals, minimizing the total weight $w(A)$.

Algorithm:

1. **For each terminal $s_i \in S$:**
   (a) Connect all other terminals $S \setminus \{s_i\}$ to a new vertex t using edges of **infinite weight**.
   (b) Compute the **minimum $s_i - t$ cut** in this modified graph.
   (c) Let E_i denote the set of edges in that minimum isolating cut.
2. **Sort** the isolating cuts $E_1, E_2, ..., E_k$ by their total weight $w(E_i)$, so that $w(E_1) \leq w(E_2) \leq ... \leq w(E_k)$.
3. **Select the (k – 1) lightest cuts:**

$$A = E_1 \cup E_2 \cup ... \cup E_{k-1}.$$

4. **Output:**
   The edge set A as the **multiway cut**.

1.2 Greedy Splitting Algorithm

Input:

- An undirected weighted graph $G = (V, E)$
- Edge weights $w : E \to \mathbb{R}^+$
- A set of terminals $S = \{s_1, s_2, ..., s_k\} \subseteq V$

Goal:

Find a minimum-weight set of edges $A \subseteq E$
whose removal separates all terminals into distinct connected components.

Algorithm:

1. **Initialization:**
   Start with the entire graph as one partition:
   $$P_0 = \{V\}.$$
2. **Iteratively split the graph:**
   For each iteration $i = 1, 2, ..., k-1$:
   1. Find the **cheapest cut** in the current graph that divides
      one of the existing components into two smaller components,
      such that after the split,
      each component contains at least one terminal.
   2. Add the edges of this cut to the solution set.
   3. Update the partition:
      $$P_i = P_{i-1} \text{ with one component split into two.}$$
3. **Stop** when there are exactly k components,
   each containing one distinct terminal.
4. **Output:**
   The union of all edges removed during the process as the **multiway cut**.

Aspect / 比較面向	1.1 Isolating Cut Heuristic (ICH)	1.2 Greedy Splitting Algorithm (GSA)
Algorithm Type / 類型	Parallel independent heuristic（平行獨立啟發式）	Iterative greedy algorithm（逐步貪婪演算法）
Basic Idea / 核心概念	For each terminal, find the minimum isolating cut independently, then take the union of the k–1 lightest cuts.	Start from the full graph and repeatedly find the cheapest cut to split one component, until there are k components.
Execution Order / 執行順序	All cuts are computed **independently and simultaneously.**	Cuts are found **one by one**; each new cut depends on the previous partition.
Graph Update / 是否更新圖結構	❌ No — Each isolating cut is computed on the original graph.	✅ Yes — Each iteration refines the current partition of the graph.
Greedy Nature / 是否為貪婪法	Weakly greedy (選最小的 k–1 cuts, but not iterative).	Strictly greedy (每步都根據當前狀態選最便宜的 cut).
Computation Structure / 計算架構	Parallelizable — all isolating cuts can be solved concurrently using max-flow.	Sequential — each iteration's result affects the next decision.
Mathematical Analysis / 分析重點	Uses the property $2w(E^*) = \sum_i w(\delta(V_i))$; discards one heaviest cut to get $2 - 2/k$.	Uses induction on partition refinement; each split adds at most $w(\delta(V_h))$, summing to $2 - 2/k$.
Approximation Ratio / 近似比	$2 - \frac{2}{k}$	$2 - \frac{2}{k}$
Advantages / 優點	Simple, parallel, easy to implement; intuitive flow-based structure.	True greedy behavior; theoretical proof generalizes to k-Cut and Gomory-Hu Tree methods.
Disadvantages / 缺點	Lacks dynamic adjustment; may double-count overlapping cuts.	Requires multiple min-cut computations; sequentially dependent.
Interpretation / 直觀比喻	"Cut everyone apart first, then keep the cheapest $k - 1$."	"Split one region at a time, always choosing the cheapest next split."

備註(我自己要深入進去看的)

- Min-Cut / Max-Flow 定理

Min-Cut / Max-Flow 定理

🧩 一、故事直覺：水流與水壩的比喻

想像你有一張網路圖（Graph）：

```nginx
s —▶—▶—▶— t
```

- 每條邊代表一條「管線」或「水道」。
- 每條邊有容量 (capacity) $c(u, v)$，
  表示最多能通過的水量。
- 你的目標：從 **source s** 把水盡可能多地送到 **sink t**。

🧠 在這張圖中有兩種問題：

類別	**問題名稱**	**問什麼？**
流量問題	**Max-Flow**	在不違反容量限制的前提下，最多能流多少水？
切割問題	**Min-Cut**	若要完全阻止水流，最少要「切掉」多少管線？

結果：

> 這兩個問題雖然看起來相反，
> 但它們的**最佳解數值永遠相等**。
> 這就是 **最大流最小割定理（Max-Flow Min-Cut Theorem）**。

Min-Cut / Max-Flow 定理

🧩 二、形式定義（數學版）

◆ **給定一張有向圖 $G=(V,E)$，**

每條邊有容量 $c(u,v) \geq 0$。

指定 source s 與 sink t。

◆ **目標：找到最大流量 f**

$$\max \sum_{(s,v)\in E} f(s,v)$$

subject to

$$\begin{cases} f(u,v) \leq c(u,v) & \text{容量限制} \\ f(u,v) = -f(v,u) & \text{反向流} \\ \sum_{(u,v)} f(u,v) = 0, & \forall u \in V \setminus \{s,t\} \ \text{（流守恆）} \end{cases}$$

Min-Cut / Max-Flow 定理

◆ **最小割問題（Min-Cut）**

選一個節點子集 $S \subseteq V$，使 $s \in S, t \notin S$。

割集 (cut) 定義為所有從 S 指向 V–S 的邊：

$$C(S) = \{(u, v) \in E \mid u \in S, v \notin S\}$$

其「容量」是

$$\text{cap}(S) = \sum_{(u,v) \in C(S)} c(u, v)$$

目標：

$$\min_S \text{cap}(S)$$

Min-Cut / Max-Flow 定理

🧩 三、定理內容

> **Max-Flow Min-Cut Theorem (Ford & Fulkerson, 1956):**
> 在任意網路 G 中，最大可行流的流量
> 等於最小 s–t cut 的總容量。

$$\boxed{\max_{f} |f| = \min_{S} \operatorname{cap}(S)}$$

這代表：

- 找到最大流量 $\Longrightarrow$ 自動知道最小割容量。
- 找到最小割 $\Longrightarrow$ 自動知道最大流量。

  它們是 **線性規劃的強對偶關係**。

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🧩 四、幾何直覺（平面圖的對偶）

在平面圖中：

原圖	對偶圖
一條被切掉的邊	對偶圖上一段邊界曲線
最小割	對偶圖上最短路徑
最大流	對偶圖上最短距離（電位差）

所以：「在原圖找最小 cut」＝「在對偶圖找最短路徑」。
這是 Multiway Cut 延伸時「planarity 與 duality」概念的基礎。

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🧩 五、演算法流程簡述

1. **初始化流量 f=0。**
2. **尋找可增廣路徑 (augmenting path)：**
   從 s → t 找出沿途還有剩餘容量的路徑。
3. **沿路徑增加流量。**
4. 重複步驟 2–3，直到找不到增廣路徑。
5. **結果：**
   - 流量值 = 最小割容量。
   - S 集合 = 從 s 可達節點；
     T 集合 = 其他節點。
   - $E(S,T)$ 即最小割邊集。

這就是 Edmonds–Karp 或 Dinic 演算法的核心思想。

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- **Max-Flow Min-Cut** 定理說明：「阻止水流的最小代價」 = 「能流出的最大水量」。
- 幾何上是「最短路徑 vs. 最小割」的對偶；演算法上是「增廣流 vs. 切斷路」的對偶；理論上是「Primal–Dual 線性規劃」的強對偶關係。

(By ChatGPT)

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