8.1 Multiway Cut Problem and Minimum-Cut-Based Algorithm

Approximating Multi-Terminal Cuts via Disjoint Isolating Regions

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Intuitive Example — Distributed Computing

- Each vertex $=$ an object or process.
- Each edge $=$ communication between objects.
- $c_e =$ communication cost.
- Terminals s_i must be placed on machine i.
- Objective: minimize inter-machine communication.

Isolating Cuts

- For each terminal s_i, define its region C_i as the set of vertices connected to s_i after removing F.
- $F_i = \delta(C_i)$
- Each F_i is an isolating cut separating s_i from the other terminals $\{s_1, \ldots, s_k\}$.
- A single edge e may appear in multiple F_i's if it connects two regions C_i, C_j.

Algorithm Idea

- For each $i \in \{1, \dots, k\}$:

  1. Add a virtual sink t.

  2. Connect all other terminals $s_j, j \neq i$ to t with infinite-cost edges.

  3. Compute the minimum $s_i - t$ cut — this gives the smallest F_i.

  Output $F = \bigcup_{i=1}^{k} F_i$ as the final multiway cut.

- Then:

$$c(F) \leq 2(1 - 1/k) \cdot OPT$$

Theorem 8.1 — 2-Approximation

- Let $F*$ be the optimal multiway cut. For each s_i, let F_i* be its isolating cut in $F*$.
- Because each edge can belong to at most two F_i*'s:
- $\Sigma_{i=1}^{k} c(F_i*) \leq 2 \cdot c(F*) = 2 \cdot OPT$

- Since F_i is the minimum isolating cut for s_i:
- $c(F_i) \leq c(F_i*) \Rightarrow c(F) \leq 2 \cdot OPT$

Improved Version — (2 − 2/k)-Approximation

- If we discard the most expensive among the k isolating cuts and keep only the cheapest (k−1) cuts:
- $F = \cup_{i=1}^{k-1} F_i$
- Then:

$$c(F) \;\leq\; 2(1 - \frac{1}{k}) \cdot OPT$$

Main text begins

Multiway Cut and the Breakdown of Planarity

- Elias Dahlhaus et al., 1992

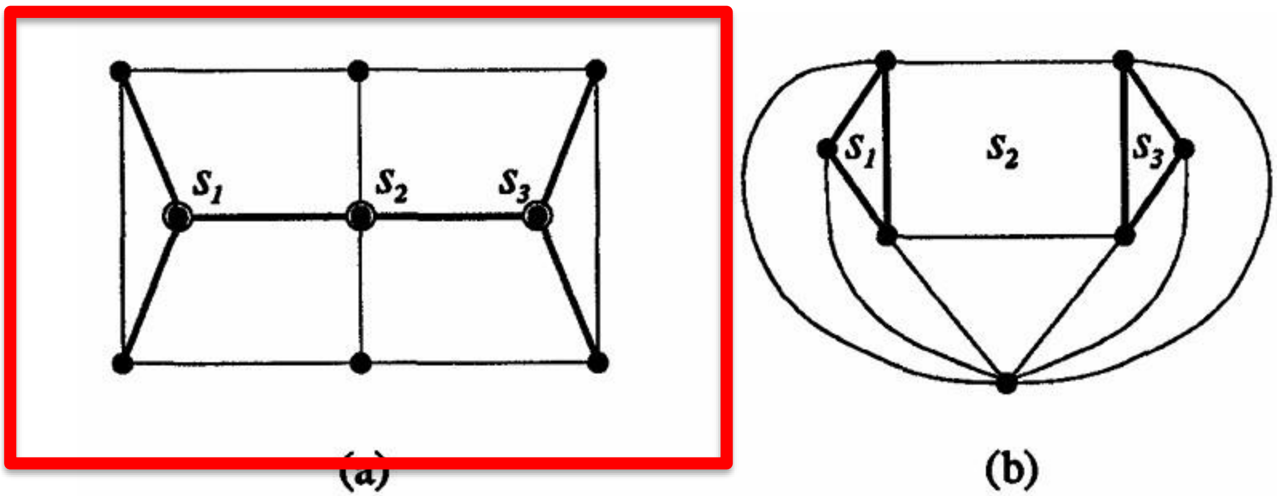
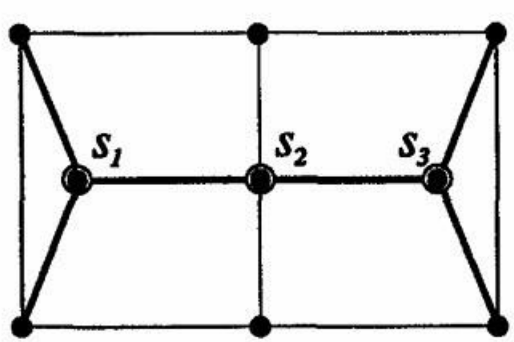


FIGURE 1. A planar 3-way cut (a) and its dual (b).

Multi-way Cut Problem



In this graph, every edge can be **cut**, and removing an edge changes the **connectivity** of the graph.

Each edge has a **weight (or cost)** representing the expense of cutting it.

Our goal is to remove a set of edges so that the three terminals

s_1, s_2, s_3, are **no longer connected** to each other — that is, there is **no path** between any pair of terminals.

Multi-way Cut Problem

- Given an undirected graph $G = (V, E)$ with nonnegative edge costs

$$c_e \geq 0 \;\; \forall e \in E,$$

- and a set of k designated terminals

$$S = \{ s_1, s_2, \dots, s_k \} \subseteq V.$$

Multi-way Cut Problem

Goal :

- Find a subset of edges $F \subseteq E$ such that, after removing F from G,
- every pair of distinct terminals $s_i, s_j \in S$ lies in **different connected components** of $G(V, E \setminus F)$, and the total cost of the removed edges is minimized.

Mathematical Formulation

- Minimize

$$c(F) = \sum_{e \in F} c_e$$

- subject to s_i and s_j are disconnected in

$$G(V, E \setminus F), \forall i \neq j.$$

Motivation

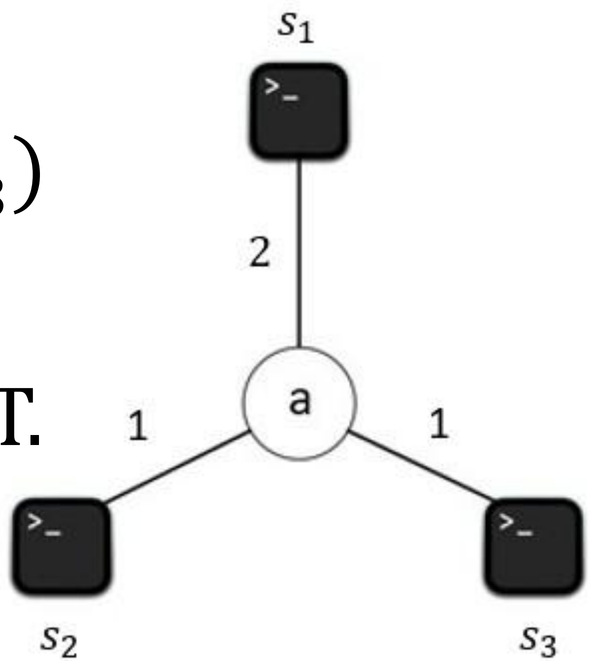
- - In a simple min s-t cut, we can find the exact minimum using max-flow.
- - When number of terminals $k > 2 \rightarrow$ Multiway Cut Problem.
- - Dahlhaus et al. (1992):
- • NP-hard for general graphs.
- • Polynomial-time solvable only when planar and k fixed.

Planar Case (Normal Situation)

- Algorithm:
- $F_1 = \{(s_1, \mathrm{a})\}$ or $\{(\mathrm{a}, s_2), (\mathrm{a}, s_3)\}$ cost=2
- $F_2 = \{(\mathrm{a}, s_2)\}$ cost=1
- $F_3 = \{(\mathrm{a}, s_3)\}$ cost=1
- That is , $c(F_1) + c(F_2) + c(F_3)$

$= 2 + 1 + 1 = 4.$

- Union=4, take k−1=2 → OPT.



Well done, But The Multiway Cut problem is NP-hard.


不可能 絕對不可能

“The Complexity of Multiterminal Cuts” (Dahlhaus et al., 1992)

- established a complete complexity classification of the Multiway Cut problem:
- For **two terminals** ($k = 2$), it reduces to the *minimum s–t cut*, solvable in polynomial time via max-flow.
- For **three or more terminals** ($k \geq 3$), the problem becomes **NP-hard** in general graphs.
- However, it is **polynomial-time solvable** when the graph is **planar** and the number of terminals k is fixed.

“The Complexity of Multiterminal Cuts” (Dahlhaus et al., 1992)

No.	Result Type	Summary
(1)	Exact algorithm for planar graphs (k = 3)	In planar graphs, the three-terminal Multiway Cut can be solved exactly in $O(n^3 \log n)$ time, using specialized flow-network and dual-graph techniques.
(2)	Exact algorithm for planar graphs (fixed k)	For any fixed number of terminals k, the Multiway Cut in planar graphs remains polynomial-time solvable, but the runtime grows exponentially with k (e.g., $O\left(n^{O(k)}\right)$).
(3)	NP-hardness results	When k is part of the input (not fixed), the problem remains NP-hard even in planar graphs with unit edge weights. In general (non-planar) graphs, it is NP-hard even for k = 3.
(4)	Approximation algorithm	The first polynomial-time approximation algorithm achieves a ratio of 2 - 2/k. Later studies proved this bound is essentially tight unless P = NP.

“The Complexity of Multiterminal Cuts” (Dahlhaus et al., 1992)

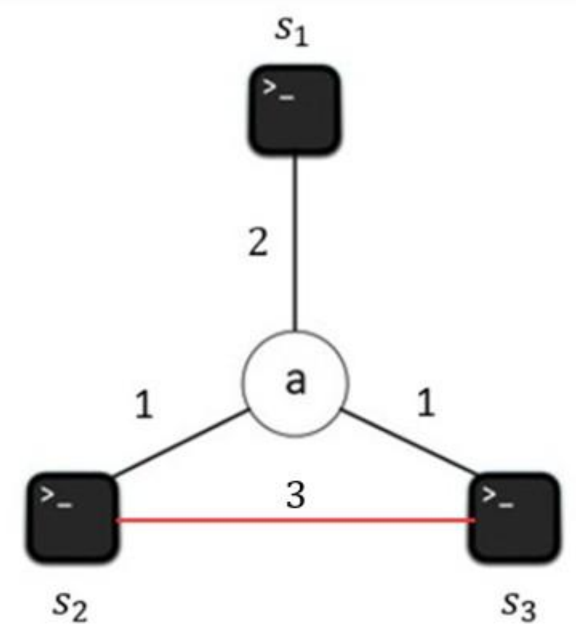
- “The results clarify the boundary between the tractable and intractable cases of the Multiway Cut problem, and give a simple, near-optimal approximation algorithm.”

“The Complexity of Multiterminal Cuts” (Dahlhaus et al., 1992)

- The study clearly distinguishes the boundary between tractable and intractable cases of the Multiway Cut problem:
  planar graphs with fixed k are solvable in polynomial time, while the problem becomes NP-hard for non-planar graphs or unbounded k.
  Moreover, they provide a simple polynomial-time algorithm achieving a near-optimal approximation ratio of $2 - \frac{2}{k}$.

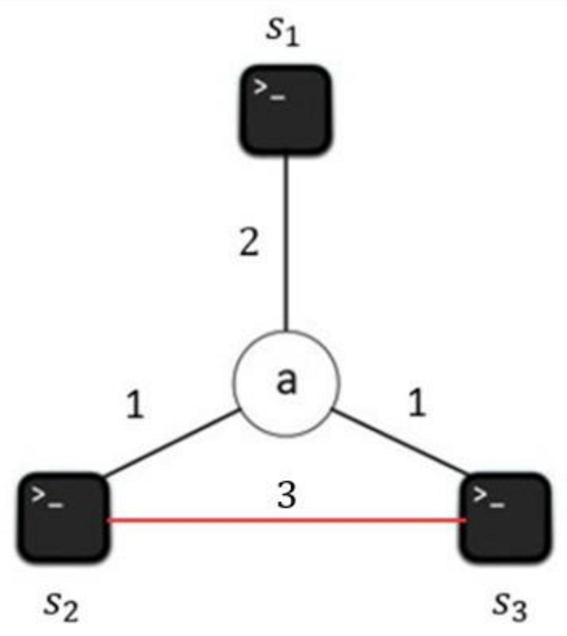
Breaks topological separability between regions.

- Add edge $(s_2\ , s_3) = 3$
- Now terminals s_2 and s_3 are directly connected → **breaks topological separability** between regions.

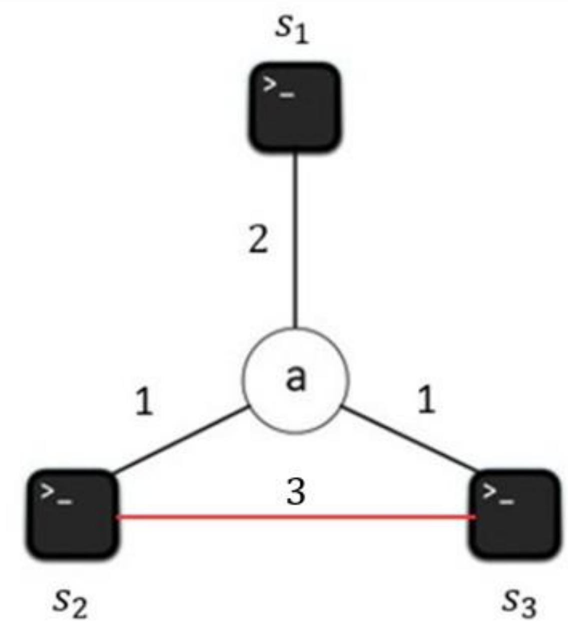


Breaks topological separability between regions.

- The direct link between terminals destroys the disjoint structure of isolating cuts — the regions now overlap topologically even though the graph is still planar.



Start by looking at the **isolating cuts** for the three terminals:



- Isolating cuts:
  s_1: $\{(s_1, a)\} \rightarrow \text{cost} = 2$
  s_2: $\{(a, s_2)\} \rightarrow \text{cost} = 1$
  s_3: $\{(a, s_3)\} \rightarrow \text{cost} = 1$

- Without (s_2,s_3): total $= 4$
  Add (s_2,s_3)=3 → overlaps occur
  $F' \;=\; \{(s_1, a), (a, s_2), (a, s_3), (s_2, s_3)\} \rightarrow \text{cost} = 7$
  $OPT \;=\; \{(a, s_2), (a, s_3), (s_2, s_3)\} \rightarrow \text{cost} = 5$
  Overlapping isolating regions ⇒ higher cost

Why It Fails

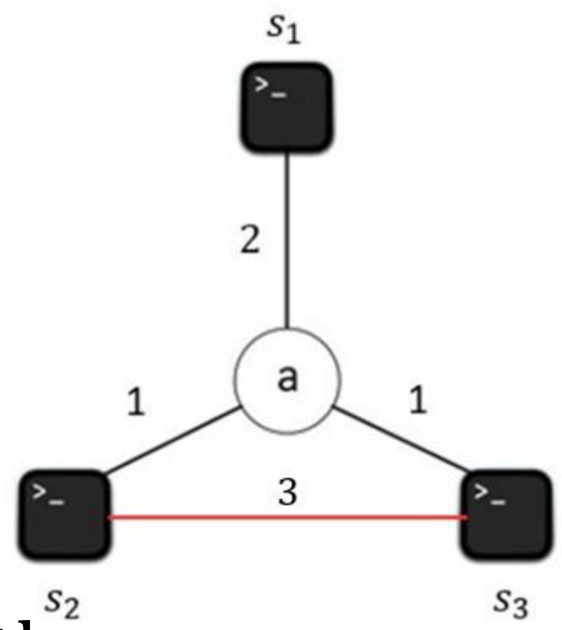
- **Direct edge between terminals breaks topological separability.**
  (Terminals are no longer isolated by disjoint regions.)

- **Isolating regions overlap → edges are double-counted.**
  (Cuts share common edges, increasing total cost.)

- **Each edge can appear in ≤ 2 isolating cuts → total $\leq 2 \times$ OPT.**
  (By double counting argument, total cost $\leq 2 \times$ OPT.)

- **Removing the most expensive cut → approximation ratio $= 2 - 2/k$.**

Isolating Cut Heuristic → 2 – 2/k Approximation



Possible isolating cuts for s_1:

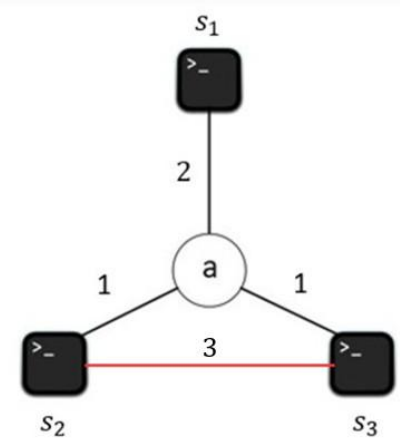
$F_1 = \{(s_1, a)\}$ $\rightarrow$ cost = 2

$F_1' = \{(a, s_2), (a, s_3)\}$ $\rightarrow$ cost = 2 (alternative choice)

Although both have the same cost, they affect the overall ratio differently:

- F_1 isolates one region → minimal overlap.
- F_1' connects two terminals → larger overlap in union.

Isolating Cut Heuristic → 2 – 2/k Approximation



Other isolating cuts:

$F_2 \;=\; \{(a, s_2), (s_2, s_3)\} \qquad \rightarrow \text{cost} = 4$

$F_3 \;=\; \{(a, s_3), (s_2, s_3)\} \qquad \rightarrow \text{cost} = 4$

Sum of isolating cuts $= 2 + 4 + 4 = 10$

Union of all edges $= \{(s_1, a), (a, s_2), (a, s_3), (s_2, s_3)\}$

$\rightarrow$ cost $= 7$

True OPT $= \{(a, s_2), (a, s_3), (s_2, s_3)\}$ $\rightarrow$ cost $= 5$ (OPT)

$\Rightarrow$ Equal cost $\neq$ equal ratio impact.

ALG depends on overlap structure, not just edge weights.

2-Approximation Algorithm

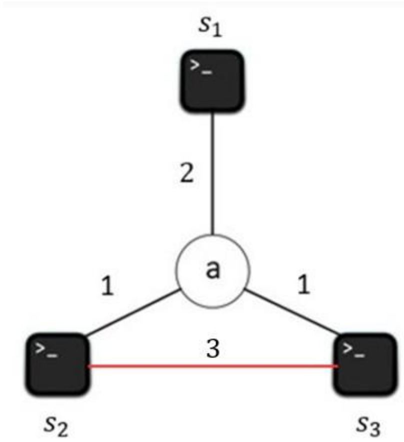
Isolating Cut Heuristic → 2 – 2/k Approximation

Possible isolating cuts for s_1:

$F_1 = \{(s_1, a)\}$ → cost = 2

$F_1{}' = \{(a, s_2), (a, s_3)\}$ → cost = 2

(alternative choice)



Other isolating cuts:

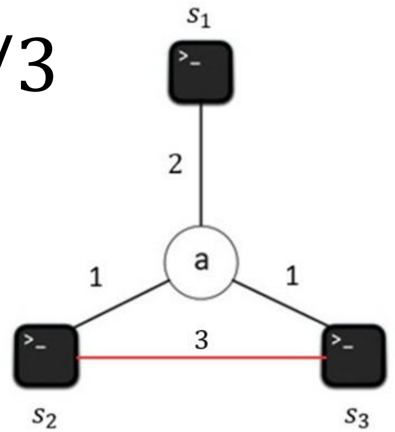
$F_2 = \{(a, s_2), (s_2, s_3)\}$ → cost = 4

$F_3 = \{(a, s_3), (s_2, s_3)\}$ → cost = 4

Isolating Cut Heuristic → 2 – 2/k Approximation

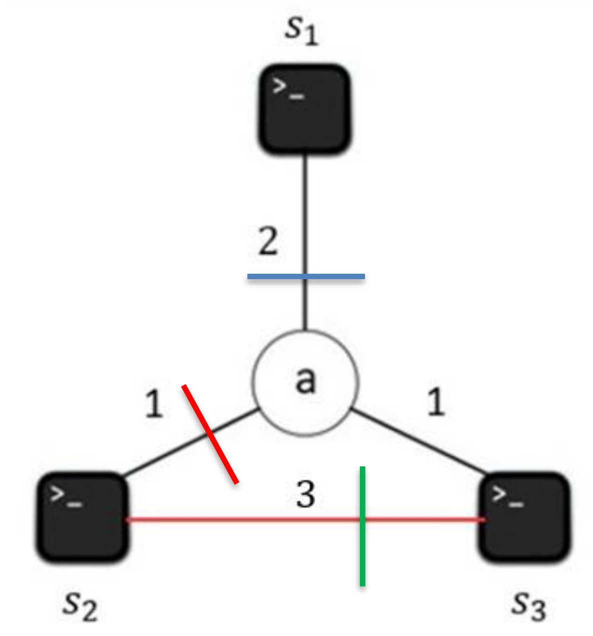
- Case A (choose F_1 and F_2; i.e., s_1 takes F_1)
  Union $= \{(s_1, a), (a, s_2), (s_2, s_3)\} \rightarrow ALG = 6$
  With $OPT = 5$,
  ratio $= 6/5 = 1.20 \leq (2 - 2/3) = 4/3$
- Case B (choose $F_1{}'$ and F_2; i.e., s_1 takes $F_1{}'$)
  Union $= \{(a, s_2), (a, s_3), (s_2, s_3)\} \rightarrow ALG = 5$
  With $OPT = 5$, ratio $= 5/5 = 1.00 \leq 4/3$

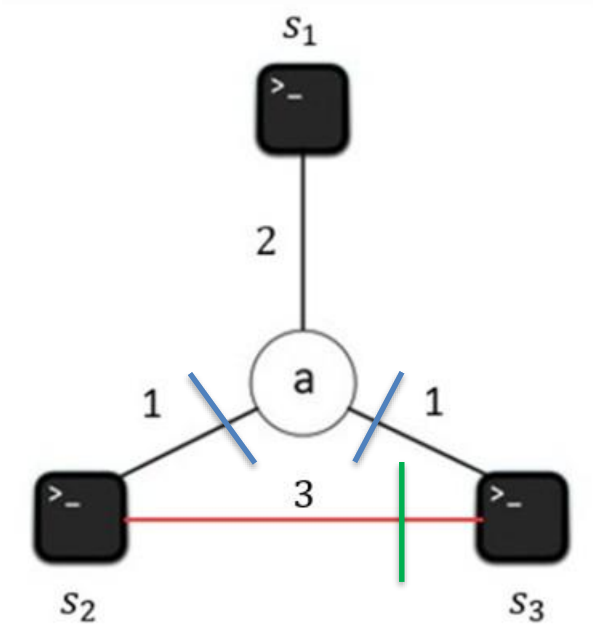
2-2/k -Approximation Algorithm



Compare Case A and Case B

Case A (choose F_1 and F_2; i.e., s_1 takes F_1)





Case B (choose $F_1{}'$ and F_2; i.e., s_1 takes $F_1{}'$)

Topological Meaning of Multiway Cuts — Separation, not Geometry

- Dahlhaus et al. (1992) used many planar and dual constructions,but their use of topology was not about geometry —it was about separability.
- In other words, they cared about whether there exists a set of edges that can separate each terminal into different connected components, such that those separating cuts are disjoint in topology.
- This “topological separability” defines when a graph is tractable:

  When the cuts are separable (planar, disjoint regions) → solvable in polynomial time.

  When cuts overlap (non-separable regions) → NP-hard.
- So, topology here means the structure of separability, not whether lines cross in a geometric drawing.

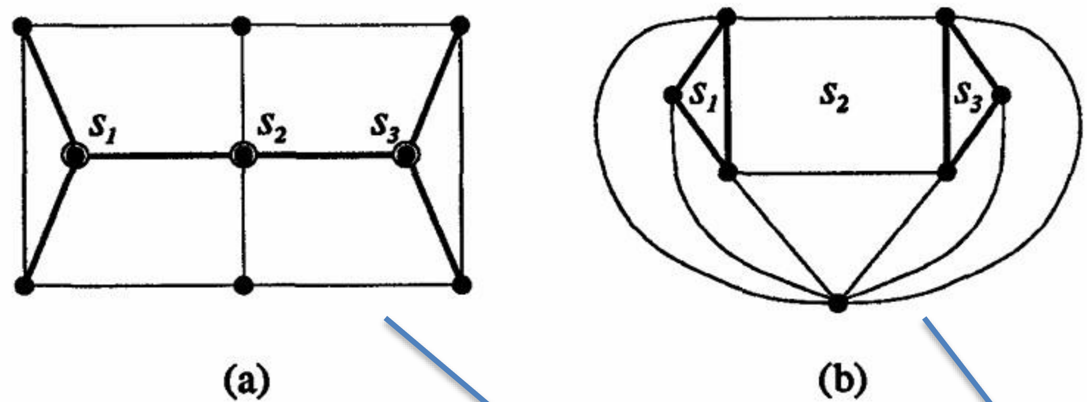


FIGURE 1. A planar 3-way cut (a) and its dual (b).

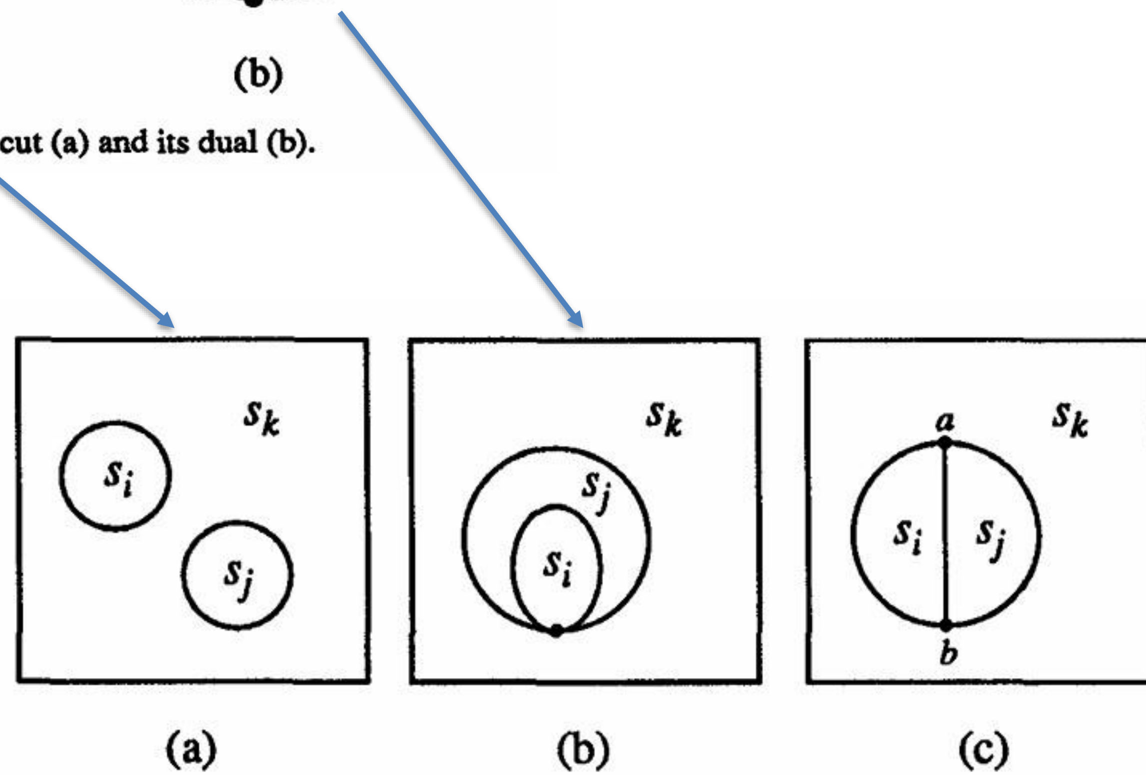
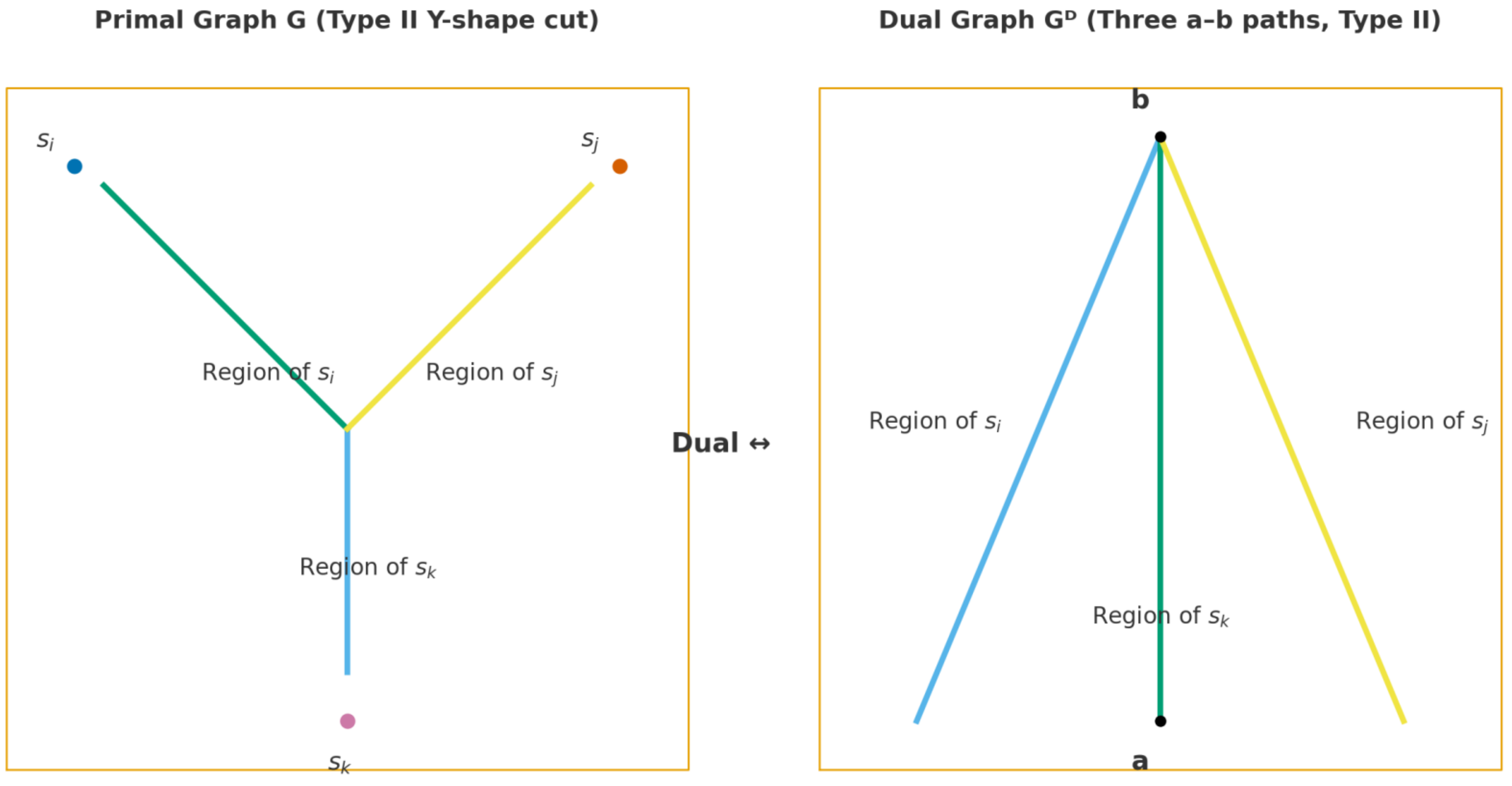


FIGURE 2. Types of 3-way cuts: Type I (a) and (b), Type II (c).

Classification of 3-Way Cut Topologies

Type	Dual Graph (C^D) Shape	Geometric Intuition	Meaning in Primal Graph
Type I	Two non-overlapping cycles (possibly tangent at one point)	Like two separate loops dividing the plane into three disjoint regions	Each cycle corresponds to a terminal's isolating cut; regions are topologically disjoint
Type II	Three boundary paths connecting the same pair of vertices (a–b)	Like a 'three-petal flower' or Y-shaped structure	Terminal regions are pairwise adjacent, sharing boundaries → non-disjoint separability

Dahlhaus et al. (1992) *Type II 3-way cut*



Our example belong to Type II

Key Insight

- When terminals are directly connected, the graph loses its original **DAG-like hierarchical separability**.
  As a result, **isolating cuts begin to overlap**, leading to **duplicate edge counting** and increased total cost.
  Among all tractable cases, **planarity** is the last structural property that preserves **topological separability**.

Summary

- **Planar graphs** possess a DAG-like topological separability,
  so the solution produced by the **Isolating Cut Heuristic** is close to the optimal (≈ OPT).
- In **non-planar graphs**, direct connections between terminals cause overlaps between cuts, resulting in higher costs and approximation errors.
- Thus, the loss of **planarity and separability** marks the **structural boundary between polynomial-time solvable and NP-hard cases**.

Reference

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