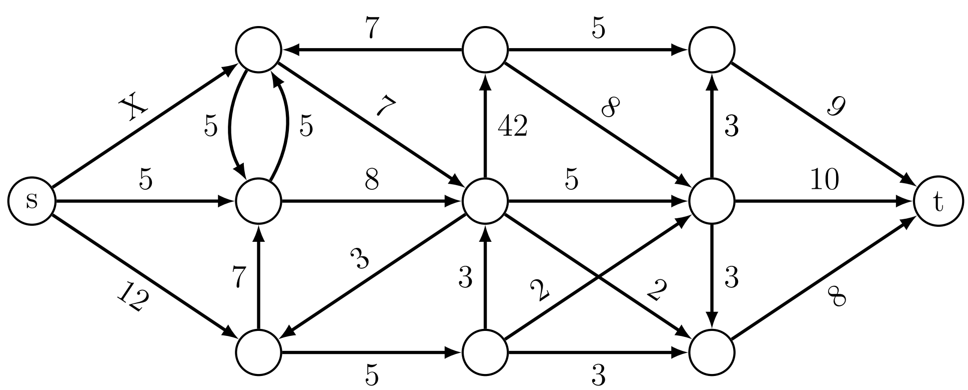
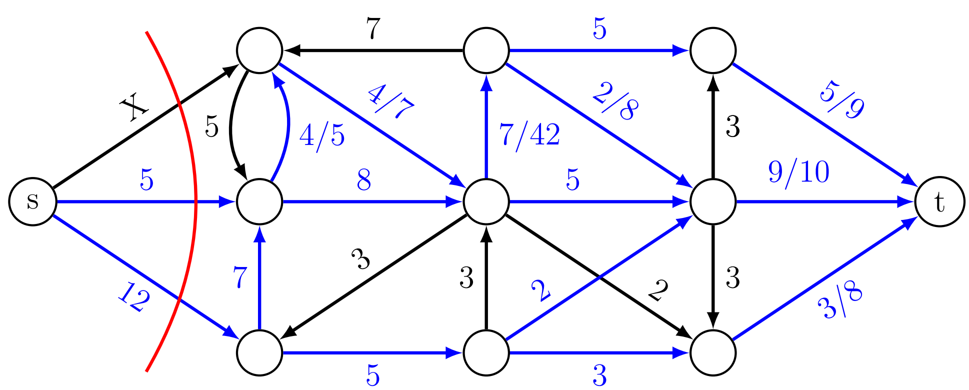
Grant Baker (07/23)
Sam Cuthbertson (06/16)
Connor Hudson (05/07)

2. *(30 pts total) Hagrid, the half-giant gamekeeper at Hogwarts, has installed a set small canals that convey water from a spring in the Forbidden Forest s to his house t. (He couldn't install just one big canal for some vague reason having to do with the Forbidden Forest getting mad about it.) Now, he's considering adding a new canal connecting the spring to his directed distribution network G. However, he's not sure how much additional water he will be able to push through G after adding the proposed canal. Hagrid needs your help to figure it out. The diagram below shows G', the network G plus the proposed canal X; edge labels indicate edge capacities.*

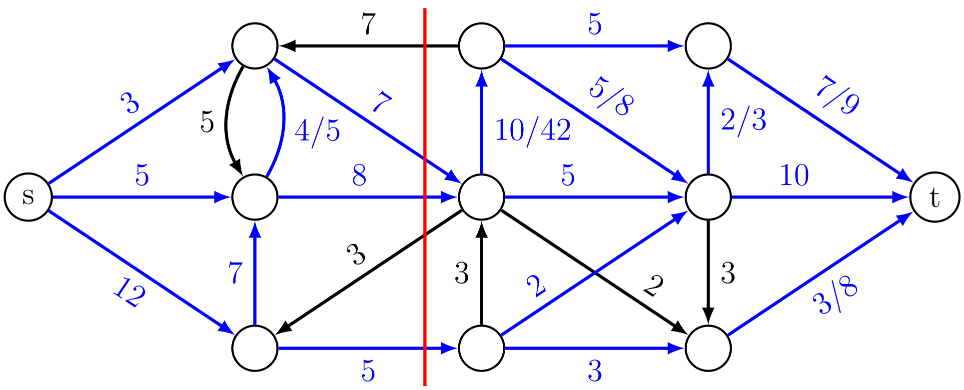


   (a) *Make a diagram showing the minimum cut corresponding to the maximum flow for G (where $X = 0$). Explain in terms of saturated and avoided edges why this is the minimum cut. Give the weight of this cut.*



   The red line above shows the minimum cut, where all edges under the cut are saturated. Notably, any other potential places for a cut would include either avoided forward edges, or include non-saturated forward edges.

(b) *If Hagrid adds the canal X, what is the smallest capacity that would maximize the increase in the water flow across the network? Explain.*



X can be at most 3, as that is what saturates the next ”layer” of forward pipes, as shown above. The red line above is a possible minimum cut while X is 3, and the only minimum cut if X is greater than 3.

(c) *Describe how Hagrid could use a min-cut/max-flow algorithm to decide what capacity X should be used for an arbitrary graph $G = (V, E)$ and arbitrary proposed edge $(u, v) \notin E$ with capacity X.*

Using a min-cut/max-flow algorithm, Hagrid could find the min-cut and then add an edge with infinite capacity. Then, Hagrid could find the min-cut on the augmented graph, find the capacity of that min-cut, and reduce the capacity of the added edge until there are two possible min-cuts.

3. *(30 pts total) After a brilliant prank goes awry, your wizard friends Fred and George Weasley have bitter argument. You intervene to keep the peace and they agree to stay away from each other, for the time being. In particular, they have agreed that when navigating the halls of Hogwarts, each will not walk on any section of stone that the other wizard has stepped on that day. The wizards have no problem with their paths crossing at an intersection. The problem, however, is that they both still need to get to the Great Hall each day to eat. Fortunately, both the Gryffindor house entrance s and Great Hall t are at intersections. You have a map of Hogwarts' hallways and their intersections, on which the Gryffindor entrance and the Great Hall are marked.*

   Your task is to determine whether there exists a pair of edge disjoint paths that would allow your friends to both get from s to t. Explain how to represent the problem as a graph G for which a straight-forward application of a max-flow/min-cut algorithm will yield the answer, and the paths.

   This problem can be turned into a max-flow/min-cut problem by creating a graph from the map of Hogwarts. This graph has each intersection as the nodes of the graph, and each hallway as an undirected edge with weight 1. If we mark the Gryffindor house entrance as the source s and the Great Hall as the sink t, then we can apply the max-flow/min-cut algorithm to see if we can push a max flow of 2 through the graph.

   If the max flow is 2 or greater, then that means since each edge has weight 1, there are at least two paths from s to t which do not share edges. Each of these paths will be pushing through 1 unit of flow, so if each Weasley brother takes one of these paths, they will keep the peace.

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4. *(10 pts extra credit) Preparing for a big end-of-semester party at Hogwarts, you crack open the Gryffindor cellar and count n bottles of fine drink. Dumbledore has previously warned you that exactly k of these bottles have been poisoned (he wouldn't go into detail as to how exactly this came to be), and consuming poisoned drink will result in an unpleasant death. The party starts in one hour, and you do not want to poison any of your guests.*

   Luckily, a family of ℓ docile rats occupies a corner of the cellar, and they have graciously volunteered to be test subjects for identifying the poisoned bottles. Let $\ell = o(n)$ and $k = 1$, and assume it takes just under one hour for poisoned drink to kill a rat. (Hence, you only get one shot at solving this problem.)

   Describe a scheme by which you can feed the drink to rats and identify with complete certainty the poisoned bottle, prove that the scheme is correct, and give a tight bound on the number of rats ℓ necessary to solve the problem.

   Dumbledore's hint: 1010101111[1]

   Since $k = 1$, there is an elegant solution that involves using a binary labeling system. We label the bottles from 1 to n in binary, and then assign each valiant rat a digit. For example, if there were 16 bottles of fine drink, we would label them like 0000, 0001, 0010, 0011, ..., 1111 so that each bottle has a unique binary representation. Then, we have four rats: one drinking a drop from each of the bottles with 1 in the leftmost digit, one drinking a drop from each of the bottles with a 1 in the second digit, and so on. After an hour, we see which rats have died and recreate the number: if a digit's rat died, then it is 1. Otherwise, it is 0. From there we have constructed a unique binary number that tells us exactly which bottle has been poisoned.

   We require exactly $\ell = \lceil \lg n \rceil$ rats for a solution to this problem, since to represent a number n in binary, we need $\lceil \lg n \rceil$ digits.

[1]Fortunately for Dumbledore, a king of the ancient times encountered a similar problem, and posted it on Reddit: `https://www.reddit.com/r/riddles/comments/16b5il/`