Variance of
Discrete and Continuous Random Variables

Administrivia

- **Homework 3** due October 13th. Good **Milestones**:
  - Problems 1, 2, and 3 done this week
  - Problems 4 and 5 done next week
- **Optional Coding Practice Boot Camp** Thursday @ 5pm in ECCR 265
  - Talk about how to go from **Problem Statement** to **Working Code**
  - Implement Black Jack and player / dealer strategies
  - Implement Connect-4
- **Midterm** coming up **in-class** on **October 18th**

Previously on CSCI 3022

Def: The expectation, expected value, or mean of a discrete random variable X taking the values $a_1, a_2, \ldots$ and with probability mass function p is the number

$$E[X] = \sum_i a_i P(X = a_i) = \sum_i a_i p(a_i)$$

Def: The expectation, expected value, or mean of a continuous random variable X with probability density function f is the number

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Change-of-Variables: Let X be a random variable and let $g : \mathbb{R} \to \mathbb{R}$ be a function. Then

$$E[g(X)] = \sum_i g(a_i) P(X = a_i) = \sum_i g(a_i) p(a_i)$$ and $$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

Easy Peasy Plinko

Let X be the random variable describing the winnings in each round of Plinko with n rows and a probability p of moving to the right off of each peg

Question: What distribution does X follow?

$$X \sim \text{Bin}(n, p)$$

Easy Peasy Plinko

Let X be the random variable describing the winnings in each round of Plinko with n rows and a probability p of moving to the right off of each peg

Question: What is the expected value of X?

LET Y_k FOR $k=1,2,\ldots,n$ BE BER(p) RVS

AND SET $X = Y_1 + Y_2 + \cdots + Y_n$

$$E[X] = E[Y_1 + Y_2 + \cdots + Y_n] = E[Y_1] + E[Y_2] + \cdots + E[Y_n]$$

$$= p + p + \cdots + p = np$$

RECALL: $E[rX+s] = rE[X]+s$

CHECK!

Easy Peasy Plinko

Let X be the random variable describing the winnings in each round of Plinko with n rows and a probability p of moving to the right off of each peg

Question: What is the expected value of X?

Easy Peasy Plinko

Let X be the random variable describing the winnings in each round of Plinko with n rows and a probability p of moving to the right off of each peg

Question: What is the variance of X?

Easy Peasy Plinko

Let X be the random variable describing the winnings in each round of Plinko with n rows and a probability p of moving to the right off of each peg

Question: What is the variance of X?

Better Question: What is variance?

Variance of a Random Variable

Recall: The sample variance of data $x_1, x_2, \ldots, x_n$ is given by

$$\bar{x} = \frac{1}{n} \sum_{k=1}^{n} x_k \qquad s^2 = \frac{1}{n-1} \sum_{k=1}^{n} (x_k - \bar{x})^2$$

s^2 IS (ALMOST) WEIGHTED AVERAGE OF SQUARED - DEVIATION FROM MEAN...

DO IT RIGHT: $\mathrm{Var}(X) = \sum_i (a_i - E[X])^2 P(X = a_i)$

OR $\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - E[X])^2 f(x)\, dx$

Variance of a Random Variable

Recall: The sample variance of data $x_1, x_2, \ldots, x_n$ is given by

Variance of a Random Variable

Def: The variance $\mathrm{Var}(X)$ of a random variable X is the number

$$\mathrm{Var}(X) = E[(X - E[X])^2]$$

Variance of a Random Variable

Def: The variance $\mathrm{Var}(X)$ of a random variable X is the number

$$\mathrm{Var}(X) = E[(X - E[X])^2]$$

Def: The standard deviation of X is the square-root of the variance $\sqrt{\mathrm{Var}(X)}$

To Compute $\mathrm{Var}(X)$ **:**

- First compute $\mu = E[X]$
- Then use change-of-variables formula:

$$\mathrm{Var}(X) = \sum_i (a_i - \mu)^2 P(X = a_i) \qquad\qquad \mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\, dx$$

Variance of a Random Variable

Def: The variance $\mathrm{Var}(X)$ of a random variable X is the number

$$\mathrm{Var}(X) = E[(X - E[X])^2]$$

A Better Way:

Useful Fact: Expectation is a linear function $E[rX + s] = rE[X] + s$

PROOF: $E[rX+s] = \int_{-\infty}^{\infty} (rx+s) f(x)\, dx$

$= r\int_{-\infty}^{\infty} x f(x)\, dx + s\int_{-\infty}^{\infty} f(x)\, dx$

$= rE[X] + s$ ✓

Variance of a Random Variable

Def: The variance $\text{Var}(X)$ of a random variable X is the number

$$\text{Var}(X) = E[(X - E[X])^2]$$

* JUST A NUMBER!!

A Better Way: $\text{Var}(X) = E[X^2] - (E[X])^2$

Proof: $\text{VAR}(X) = E[(X - E[X])^2] =$

$= E[X^2 - 2XE[X] + (E[X])^2]$

$= E[X^2] - 2E[XE[X]] + (E[X])^2]$

$= E[X^2] - 2(E[X])^2 + (E[X])^2]$

$= E[X^2] - (E[X])^2$ IMPLICATIONS OF THIS?!

Easy Peasy Plinko

Let X be the random variable describing the winnings in each round of Plinko with n rows and a probability p of moving to the right off of each peg

Question: What is the variance of $X \sim Bin(n, p)$?

AGAIN, USE SUM OF BER(p)!

Easy Peasy Plinko

Let X be the random variable describing the winnings in each round of Plinko with n rows and a probability p of moving to the right off of each peg

Question: What is the variance of $X \sim Bin(n,p)$?

First Step: What is the variance of $Y \sim Ber(p)$?

USE $E[Y^2] - (E[Y])^2$

$$E[Y^2] = \sum_i a_i^2 P(X=a_i) = 1^2 \cdot p + 0^2 \cdot (1-p) = p$$

$$VAR(Y) = p - p^2 = p(1-p)$$

FACT: IF $X \sim BER(p)$ THEN

$E[X] = p$ & $VAR(X) = p(1-p)$

Easy Peasy Plinko

Let X be the random variable describing the winnings in each round of Plinko with n rows and a probability p of moving to the right off of each peg

Question: What is the variance of $X \sim Bin(n, p)$?

SOOOO IMPORTANT!

Fact: If X and Y are independent, then $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

LET $Y_1, Y_2, \ldots, Y_n$ BE INDEPENDENT BER(p)

AND $X = Y_1 + Y_2 + \cdots + Y_n$

$$\mathrm{Var}(X) = \mathrm{Var}(Y_1 + Y_2 + \cdots + Y_n) = p(1-p) + \cdots + p(1-p)$$

$$= np(1-p)$$

CHECK!

Easy Peasy Plinko

Let X be the random variable describing the winnings in each round of Plinko with n rows and a probability p of moving to the right off of each peg

Question: What is the variance of $X \sim Bin(n,p)$?

Fact: If X and Y are **independent**, then $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

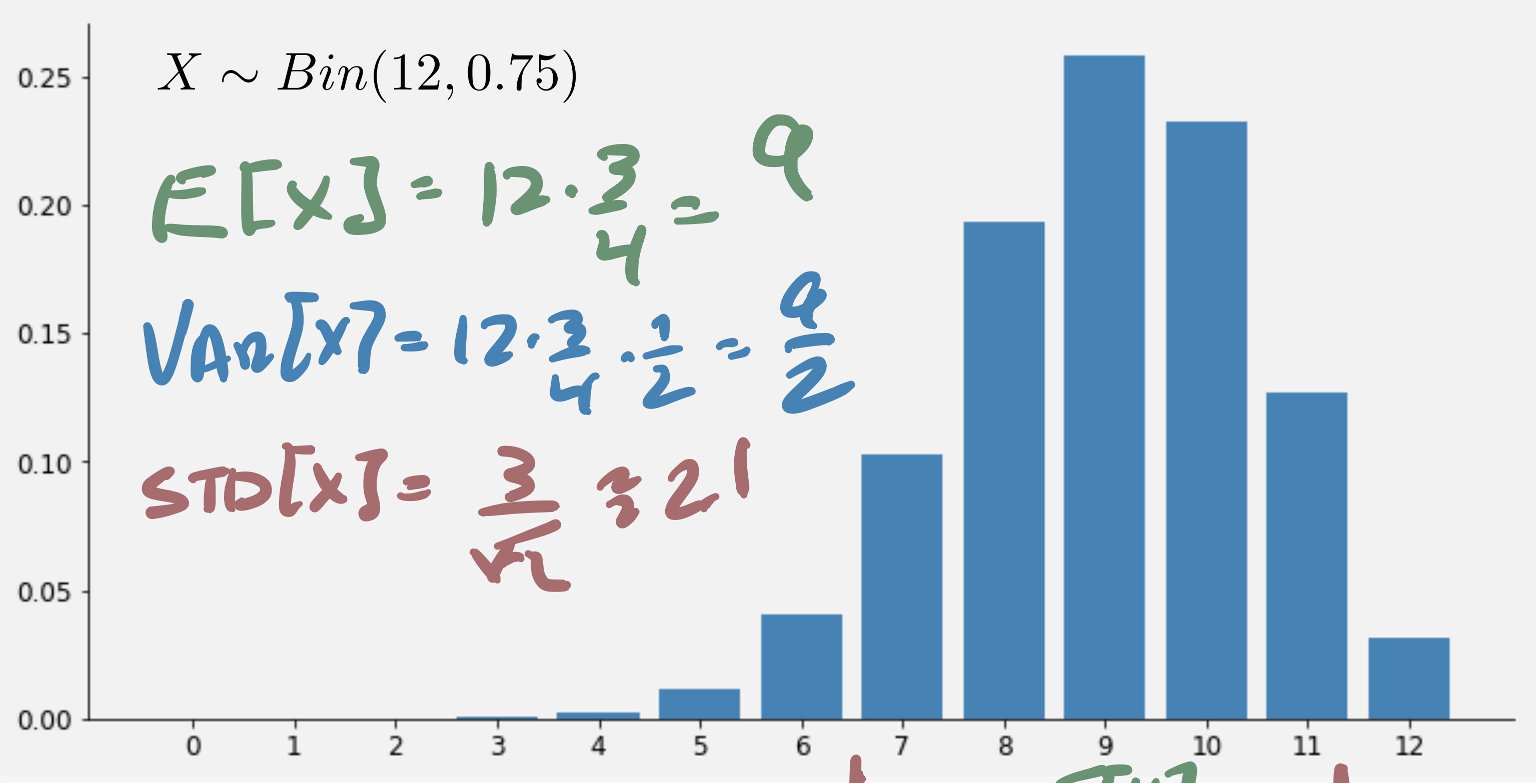
Thm: Let X be a Binomial random variable with parameters n and p. Then

$$E[X] = np \quad \text{and} \quad \mathrm{Var}(X) = np(1-p)$$

The Binomial Distribution

Thm: Let X be a Binomial random variable with parameters n and p. Then

$$E[X] = np \quad \text{and} \quad \mathrm{Var}(X) = np(1-p)$$



$E[X] = 12 \cdot \frac{3}{4} = 9$

$\mathrm{Var}[X] = 12 \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{9}{2}$

$\mathrm{STD}[X] = \frac{3}{\sqrt{2}} \approx 2.1$

$|—\ E[X]\ —|$

More Fun Facts about Variance

Recall: Expectation is linear: $E[rX + s] = rE[X] + s$

What about **Variance?**

WHAT HAPPENS IF WE SHIFT $X \to X+s$?

NOTHING! $\mathrm{VAR}(X+s) = \mathrm{VAR}(X)$

More Fun Facts about Variance

Recall: Expectation is linear: $E[rX+s] = rE[X]+s$

What about **Variance?**

WHAT HAPPENS IF WE SCALE $X \to rX$?

$$\begin{aligned} Var(rX) &= E[(rX)^2] - (E[rX])^2 \\ &= E[r^2X^2] - (rE[X])^2 \\ &= r^2E[X^2] - r^2(E[X])^2 \\ &= r^2(E[X^2] - (E[X])^2) \\ &= r^2 Var(X) \end{aligned}$$

More Fun Facts about Variance

Recall: Expectation is linear: $E[rX+s] = rE[X]+s$

Fact: Variance is not linear: $\mathrm{Var}(rX+s) = r^2\mathrm{Var}(X)$

Mean and Variance of a Uniform RV

Example: Let $X \sim U[\alpha, \beta]$. Find $E[X]$ and $\mathrm{Var}(X)$.

WHAT IS pdf OF X, f(x)?

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{IF } \alpha \le x \le \beta \\ 0 & \text{ELSE} \end{cases}$$

WHAT IS E[X]?

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta-\alpha}\,dx = \frac{x^2}{2\cdot(\beta-\alpha)}\Big|_{\alpha}^{\beta} = \frac{\alpha+\beta}{2}$$

Mean and Variance of a Uniform RV

Example: Let $X \sim U[\alpha, \beta]$. Find $E[X]$ and $\mathrm{Var}(X)$.

WHAT IS $E[X^2]$?

$$E[X^2] = \int_\alpha^\beta x^2 \frac{1}{\beta-\alpha}\,dx = \frac{1}{\beta-\alpha} \left.\frac{x^3}{3}\right|_\alpha^\beta = \frac{\beta^3-\alpha^3}{3(\beta-\alpha)}$$

$$= \frac{(\beta-\alpha)(\beta^2+\alpha\beta+\alpha^2)}{3(\beta-\alpha)} = \frac{\beta^2+\alpha\beta+\alpha^2}{3}$$

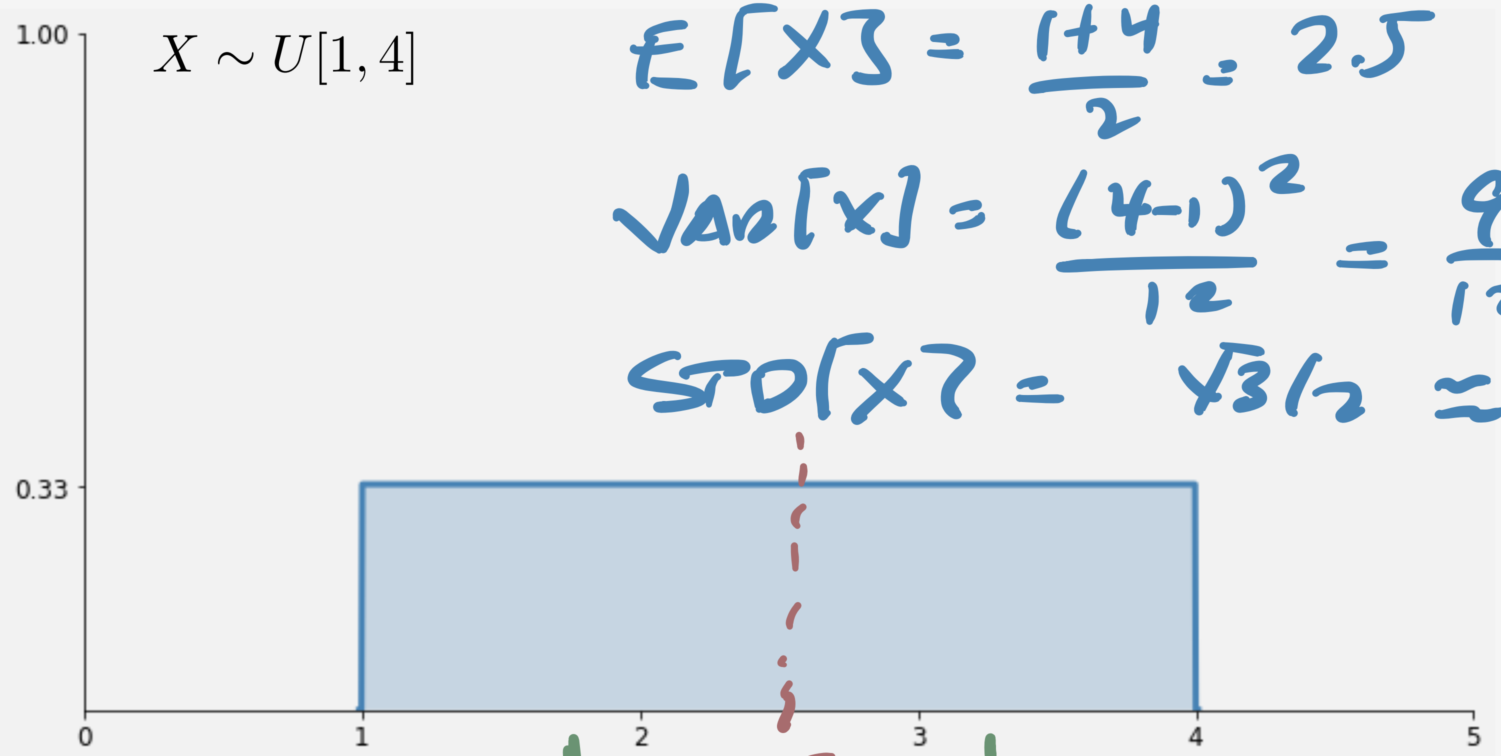
$$\mathrm{VAR}[X] = \frac{\beta^2+\alpha\beta+\alpha^2}{3} - \frac{(\alpha+\beta)^2}{4} = \frac{4(\beta^2+\alpha\beta+\alpha^2) - 3(\alpha+\beta)^2}{12}$$

$$= \frac{1}{12}\left[\cancel{4}\beta^2 + \cancel{4\alpha\beta} + \cancel{4}\alpha^2 - \cancel{3\alpha^2} - \cancel{6}\alpha\beta - \cancel{3}\beta^2\right] = \frac{1}{12}(\beta-\alpha)^2$$

Mean and Variance of a Uniform RV

Thm: Let X be a Uniform distribution defined on the interval $[\alpha, \beta]$

$$E[X] = \frac{\alpha + \beta}{2} \quad \text{and} \quad \text{Var}(X) = \frac{(\beta - \alpha)^2}{12}$$



$$E[X] = \frac{1+4}{2} = 2.5$$

$$VAR[X] = \frac{(4-1)^2}{12} = \frac{9}{12} = \frac{3}{4}$$

$$STD[X] = \sqrt{3}/2 \approx .85$$

1 — 2.5 — 1

OK! Let's Go (Back) to Work!

Get in groups, get out laptop, and open the Lecture 11 In-Class Notebook