Mathematical Probability

Birthday Survey

- Go to Moodle and do the quiz titled Part 3 on the calendar
- This counts as your participation grade for today
- Quiz password: 3022bday
- The Quiz closes at 4:10, so do it now please
- Quiz has one question: What is your birthday? Please enter in format specified.
- If you don't feel comfortable telling us your birthday you can lie. Won't change results.
- If you *still* haven't signed up for Moodle, enrollment key is csci3022_F17_001

Last Time

- **Sample Space** Ω : Set of all possible outcomes of an experiment
- **Event**: A set of one or more outcomes
- **Probability Function** P: Assigns value in [0,1] to each outcome or event
- Probability functions satisfy $P(\Omega) = 1$ and $P(A \cup B) = P(A) + P(B)$ if A and B are disjoint
- If events A and B are not disjoint then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- If results of two trials don't affect each other we say they are *independent*

Warm-Up Problem

Suppose you draw a single card from a standard 52-card deck.

Question: What is the probability that the card is the $A\diamond$

Intuition $\Rightarrow P(A\diamond) = \frac{1}{52}$

comp: $P(A \cap \diamond) = P(A)P(\diamond)$

$= \frac{1}{13} \cdot \frac{1}{4} = \frac{1}{52}$

Question: What is the probability that the card an A or $\diamond$

$$P(A \cup \diamond) = P(A) + P(\diamond) - P(A \cap \diamond)$$

$$= \frac{1}{13} + \frac{1}{4} - \frac{1}{52} = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

A Rigorous Way to Compute Probabilities

Suppose we know $P(\omega)$ for each outcome ω in Ω

We can compute the probability of an event A (which is a set of one or more outcomes) as the sum of the probabilities of the outcomes in A

$$P(A) = \sum_{\omega \in A} P(\omega)$$

Question: Suppose we flip a biased coin with probability function $P(\{H, T\}) = \{p, 1-p\}$ three times. What is the probability that we get two or more Tails?

$A = \{TTH, THT, HTT, TTT\}$

$P(A) = \sum_{\omega \in A} P(\omega) = (1-p)^2 p + (1-p)^2 p + (1-p)^2 p + (1-p)^3$

$= 3p(1-p)^2 + (1-p)^3$

To Infinity and Beyond!

Suppose you flip a biased coin until a Heads comes up. Show that the probability that you eventually flip a Heads is 1.

Question: What is the sample space for this experiment?

$$\Omega = \{H, TH, TTH, TTTH, \ldots\}$$

$$P(\Omega) = p + (1-p)p + (1-p)^2 p + \cdots = p\left[1 + (1-p) + (1-p)^2 + \cdots\right]$$

$$= p \sum_{k=0}^{\infty} (1-p)^k = p\left[\frac{1}{1-(1-p)}\right] = p\left[\frac{1}{p}\right] = 1$$

GEOMETRIC SERIES

$$\sum_{k=0}^{\infty} a r^k = \frac{a}{1-r} \quad \text{FOR } |r| < 1$$

Conditional Probability

Question: If you stop a random person on the street and ask them what month they were born, what is the probability they were born in a long month month (i.e. with 31 days in it)?

Question: What is the probability that they were born in a month with an r in the name?

Conditional Probability

Question: If you stop a random person on the street and ask them what month they were born, what is the probability they were born in a long month month (i.e. with 31 days in it)?

Let L = {Jan, Mar, May, Jul, Aug, Oct, Dec} be the event the person's birth month has 31 days

$P(L) = 7/12$

Question: What is the probability that they were born in a month with an r in the name?

Conditional Probability

Question: If you stop a random person on the street and ask them what month they were born, what is the probability they were born in a long month month (i.e. with 31 days in it)?

Let L = {Jan, Mar, May, Jul, Aug, Oct, Dec} be the event the person's birth month has 31 days

$$P(L) = \frac{7}{12}$$

Question: What is the probability that they were born in a month with an r in the name?

Let R = {Jan, Feb, Mar, Apr, Sep, Oct, Nov, Dec} be the event the person's birth month has an r

$P(R) = 8/12$

Conditional Probability

Question: Suppose the man tells you that he was born in a long month. Now what is the probability that he was born in a month with an r in it?

L = {Jan, Mar, May, Jul, Aug, Oct, Dec} and R = {Jan, Feb, Mar, Apr, Sep, Oct, Nov, Dec}

$$= \frac{\#\text{ Long months w/ r}}{\#\text{ Long months}} = \frac{4}{7}$$

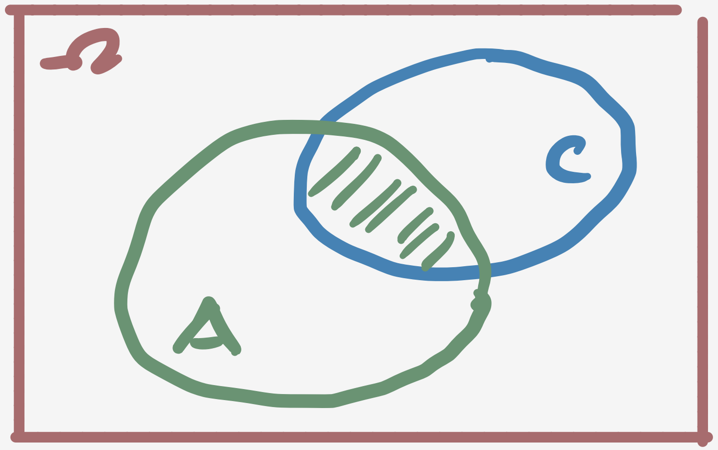
$$= \frac{P(L \cap R)}{P(R)}$$

Conditional Probability

Def: The conditional probability of A given C is defined by

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)}$$

provided that $P(C) > 0$



Example: A bit string (1s and 0s) of length 4 is generated at random so that each of the 16 possible bit strings is equally likely. What is the probability that it contains at least two consecutive 1's given that the first bit is a 1?

LET A = "At least 2 cons. 1s" & C = "1st BIT IS 1"

$C = \{1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111\}$

$A \cap C = \{1011, 1100, 1101, 1110, 1111\}$ $\quad P(A \mid C) = \frac{5/16}{8/16} = \frac{5}{8}$

Conditional Probability

Follow-Up: A bit string (1s and 0s) of length 4 is generated at random so that each of the 16 possible bit strings is equally likely. What is the probability that it **does not** contain two consecutive 1's given that the first bit is a 1?

FROM BEFORE: $A^c =$ "DOES not contain 11"

$$C = \{1000,\ 1001,\ 1010,\ 1011,\ 1100,\ 1101,\ 1110,\ 1111\}$$

$$A^c \cap C = \{1000, 1001, 1010\}$$

$$P(A^c | C) = \frac{3/16}{8/16} = \frac{3}{8}$$

Conditional Probability

Follow-Up: A bit string (1s and 0s) of length 4 is generated at random so that each of the 16 possible bit strings is equally likely. What is the probability that it does not contain two consecutive 1's given that the first bit is a 1?

$P(A|C) = \frac{5}{8}$ $\qquad$ $P(A^c|C) = \frac{3}{8}$

$P(A|C) + P(A^c|C) = 1$

Take-Away: The conditional probability $P(\,\cdot\mid C)$ is a valid probability function!

The Product Rule of Probability

The definition of conditional probability can be manipulated into other useful formulas

Def: The following is called the product rule or multiplication rule of probability:

$$P(A \cap C) = P(A \mid C)\ P(C)$$

The product rule is useful when the conditional probability is easy to compute, but the probability of intersections of events are not.

Example: You draw two cards from a standard deck. What is the probability that they are both black?

LET B_i = "i^{th} CARD IS BLACK". WANT $P(B_1 \cap B_2)$

$= P(B_2 \mid B_1)\, P(B_1) = \frac{12}{51} \cdot \frac{13}{52} \approx 0.0588$

Independent Events – Intuition

Previous Example: You draw two cards from a standard deck. What is the probability that they are both black?

Are the events of drawing the first black card and the second black card **independent**?

NO WAY! RESULT OF FIRST DRAW AFFECTS RESULT OF SECOND DRAW

Independent Events - Mathiness

Def: An event A is said to be independent of event B if $P(A \mid B) = P(A)$

This definition, combined with the product rule or the definition of conditional probability gives us many equivalent tests for independence of two events:

1. $P(A \mid B) = P(A)$
2. $P(B \mid A) = P(B)$
3. $P(A \cap B) = P(A)P(B)$

CARD CHECK

$$P(B_2 \mid B_1) = \frac{12}{51} \neq \frac{13}{52} = P(B_2)$$

Subtleties of Independence

Def: Events $A_1, A_2, \ldots, A_m$ are independent if

$$P(A_1 \cap A_2 \cap \cdots \cap A_m) = P(A_1)P(A_2)\cdots P(A_m)$$

Question: Is independence of 3 events A, B, and C the same as: A and B are independent; B and C are independent; and A and C are independent?

Example: Suppose you flip a fair coin twice. Let A be the event "Heads on flip 1", B be the event "Heads on flip 2", and C be the event "the two flips match".

$\Omega = \{HH, HT, TH, TT\}$ $\quad P(A) = P(B) = P(C) = \frac{1}{2}$

$P(A | B) = P(B | C) = P(A | C) = 1/2$

$P(A \cap B \cap C) = P(\{HH\}) = \frac{1}{4} \neq \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

EVENT C COMPLETELY DETERMINED BY A & B!

Law of Total Probability

Example: Suppose I have two bags of marbles. The first bag contains 6 white marbles and 4 black marbles. The second bag contains 3 white marbles and 7 black marbles. Now suppose I put the two bags in a box. If I close my eyes, grab a bag from the box, and then grab a marble from the bag, what is the probability that it is black?

P(CHOOSE BAG 1) × P(BLACK | CHOSE BAG 1)

+ P(CHOOSE BAG 2) × P(BLACK | CHOSE BAG 2)

$$= \frac{1}{2} \cdot \frac{4}{10} + \frac{1}{2} \cdot \frac{7}{10} = \frac{4+7}{20} = \frac{11}{20}$$

Law of Total Probability

Example: Same scenario as before, but now suppose that the first bag is much larger than the second bag, so that when I reach into the box I'm twice as likely to grab the first bag as the second. What is the probability of grabbing a black marble?

$$= \frac{2}{3} \cdot \frac{4}{10} + \frac{1}{3} \cdot \frac{7}{10} = \frac{8+7}{30} = \frac{15}{30} = \frac{1}{2}$$

CAN WE GENERALIZE THIS RULE?

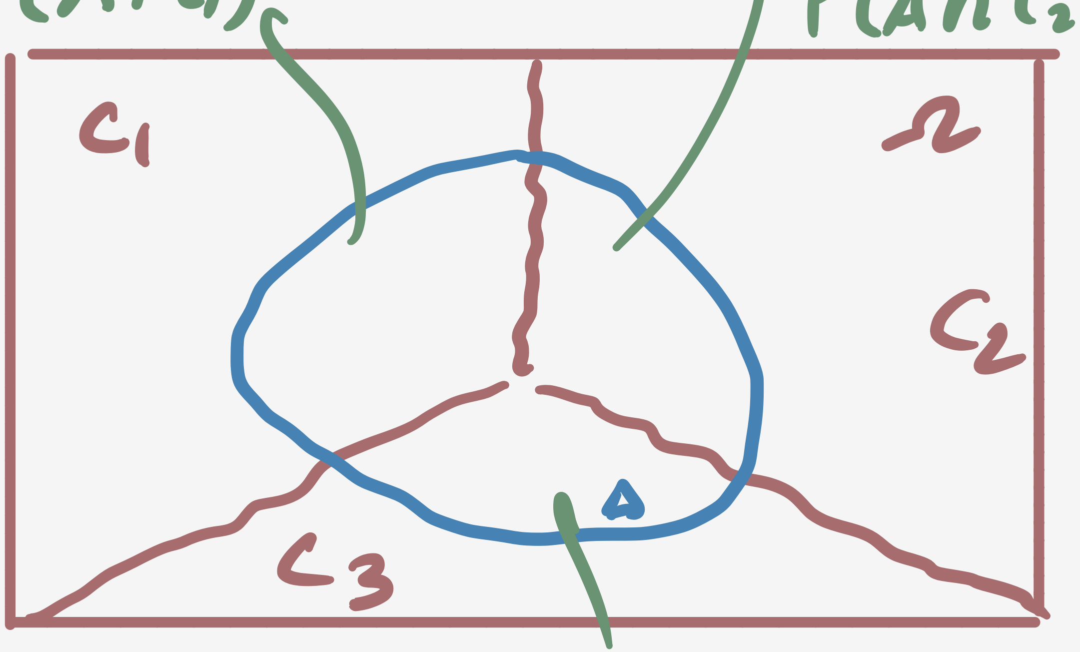
Law of Total Probability

Def: Suppose $C_1, C_2, \ldots, C_m$ are disjoint events such that $C_1 \cup C_2 \cup \cdots \cup C_m = \Omega$. The probability of an arbitrary event A can be expressed as:

$$P(A) = P(A \mid C_1)P(C_1) + P(A \mid C_2)P(C_2) + \cdots + P(A \mid C_m)P(C_m)$$

$P(A \cap C_1)$ $\quad$ $P(A \cap C_2)$

LET $m = 3$



$P(A \cap C_3) = P(A \mid C_3)P(C_3)$

The Birthday Paradox

Suppose there are two random people in a room. What is the probability that they have different birthday?

LET B_2 = "THE TWO PEOPLE HAVE DIFF BDAYS"

$$P(B_2) = 1 - P(B_2^c) = 1 - \frac{1}{365}$$

SUPPOSE 3RD PERSON ENTERS

LET A_3 BE EVENT THAT NEW PERSON HAS DIFF BDAY:

$$P(B_3) = P(B_2 \cap A_3) = P(A_3 \mid B_2) P(B_2) = \left(1 - \frac{2}{365}\right)\left(1 - \frac{1}{365}\right)$$

The Birthday Paradox

Suppose there are two random people in a room. What is the probability that they have different birthday?

$$P(B_n) = P(B_{n-1} \cap A_n) = P(A_n \mid B_{n-1}) P(B_{n-1})$$

$$= \left(1 - \frac{n-1}{365}\right) P(B_{n-1}) = \left(1 - \frac{n-1}{365}\right) P(A_{n-1} \mid B_{n-2}) P(B_{n-2})$$

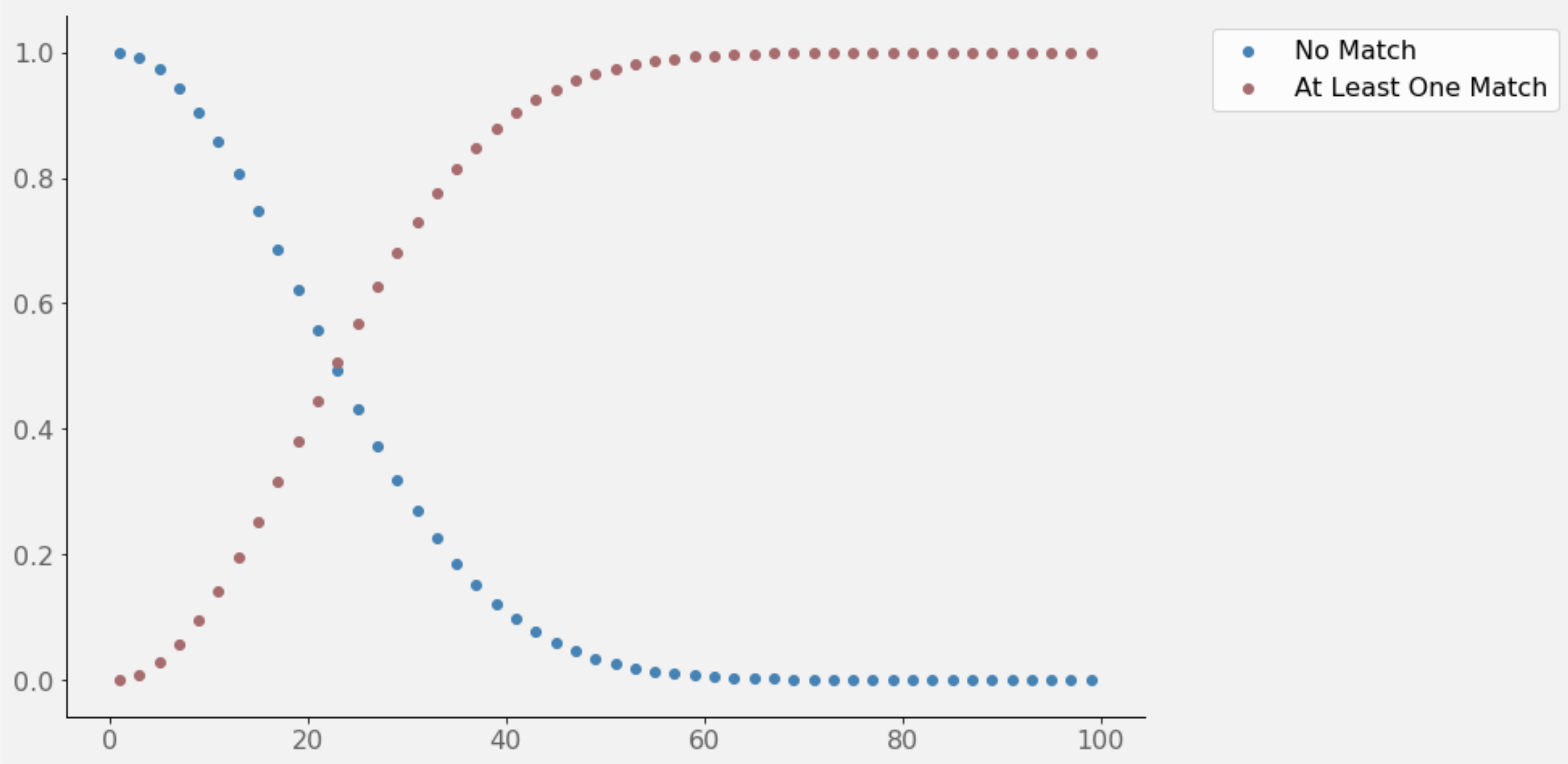
$$= \left(1 - \frac{n-1}{365}\right)\left(1 - \frac{n-2}{365}\right) P(A_{n-2} \mid B_{n-3}) P(B_{n-3})$$

$$\vdots$$

$$= \left(1 - \frac{n-1}{365}\right)\left(1 - \frac{n-2}{365}\right) \cdots \left(1 - \frac{2}{365}\right)\left(1 - \frac{1}{365}\right)$$

The Birthday Paradox

- If you have 23 people in the room, probability of a shared birthday is 0.5973
- If you have 58 people in the room, probability of a shared birthday is 0.9917
- If you have 120 people in the room, probability of a shared birthday is 0.9999999998



OK! Let's Go to Work!

Get in groups, get out laptop, and open the Lecture 5 In-Class Notebook

Let's:

- Get some more practice with unions, intersections, etc. of outcomes
- Get some more practice with our new probability rules
- See how we can simulate the more complicated things like conditional probabilities