Expectation of
Discrete and Continuous Random
Variables

Previously on CSCI 3022

Def: a probability mass function is the map between the discrete random variable's values and the probabilities of those values

$$f(a) = P(X = a)$$

Def: A random variable X is **continuous** if for some function $f : \mathbb{R} \to \mathbb{R}$ and for any numbers a and b with $a \leq b$

$$P(a \leq X \leq b) = \int_a^b f(x)\ dx$$

The function has to satisfy $f(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x)\ dx = 1$. We call f the probability density function of X.

Chuck-a-Luck

Recall: Chuck-a-luck involves placing a bet on a number and rolling 3 dice. If your number appears you win your bet times the number of times your number appears. If your number does not appear you lose your bet.

Question: How much money to you expect to win/lose on average if you play many games?

LET X BE BINOMIAL RV w/ $n=3$ & $p=\frac{1}{6}$

$$3\cdot\binom{3}{3}\left(\tfrac{1}{6}\right)^3+2\binom{3}{2}\left(\tfrac{1}{6}\right)^2\left(\tfrac{5}{6}\right)+1\binom{3}{1}\left(\tfrac{1}{6}\right)\left(\tfrac{5}{6}\right)^2-1\binom{3}{0}\left(\tfrac{5}{6}\right)^3$$

$$=\frac{3}{216}+\frac{30}{216}+\frac{75}{216}-\frac{125}{216}=\boxed{-\frac{17}{216}}\approx -0.079$$

$$+3\,P(\text{3 FIVES})+2\,P(\text{2 FIVES})+1\,P(\text{1 FIVE})-1\,P(\text{0 FIVES})$$

Expectation of a Discrete RV

Def: The expectation, expected value, or mean of a discrete random variable X taking the values $a_1, a_2, \ldots$ and with probability mass function p is the number

$$E[X] = \sum_i a_i P(X = a_i) = \sum_i a_i p(a_i)$$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} \frac{1}{n} x_i$$

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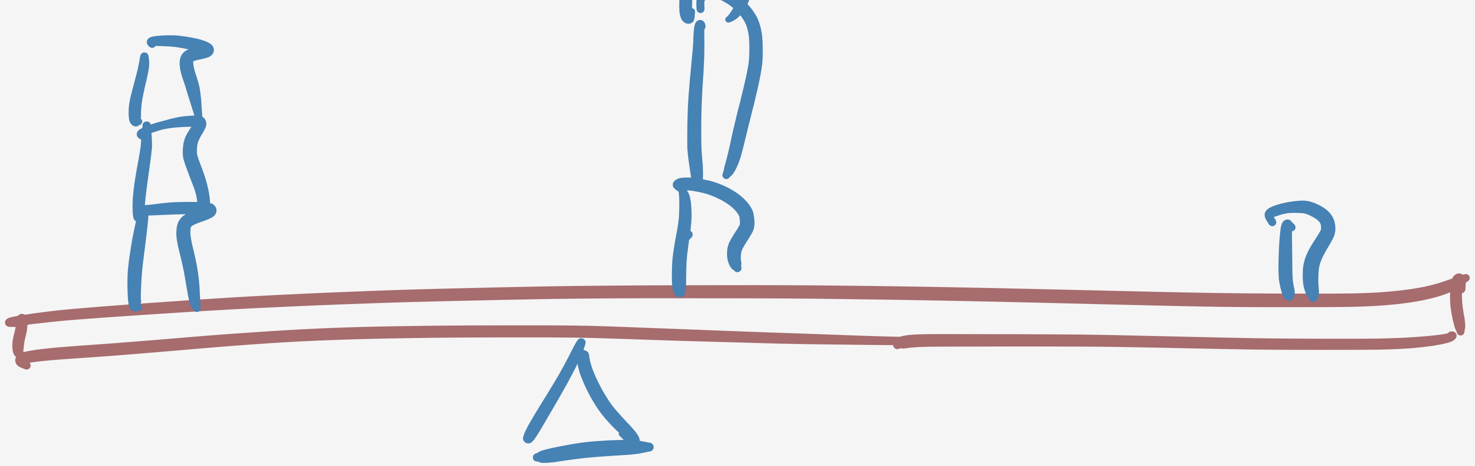
Intuition: Think of masses of weight $p(a_i)$ placed at the points a_i



Expectation of a Discrete RV

Example: Let X be a Bernoulli random variable with parameter p. What is $E[X]$?

$$X = \{0, 1\} \qquad p(0) = (1-p) \qquad p(1) = p$$

$$E[X] = \cancel{0 \cdot (1-p)} + 1 \cdot p = p$$

$\sum_i a_i P(X=a_i)$

Expectation of a Discrete RV

$p = 1/6$

Example: Suppose you and a friend are avoiding studying by each rolling a fair die. You decide that the first time that you roll the same number you'll go back to work. What is the expected number of times you'll roll the dice before getting a match?

LET X BE THE RV DESCRIBING # OF ROLLS

$$E[X] = 1\cdot p + 2\cdot p(1-p) + 3\cdot p(1-p)^2 + \cdots$$

$$E[X] = \sum_{k=1}^{\infty} k\, p(1-p)^{k-1} = \sum_{m=0}^{\infty} (m+1)\, p(1-p)^{m}$$

$$= \sum_{m=0}^{\infty} p(1-p)^{m} + \sum_{m=0}^{\infty} m\, p(1-p)^{m}$$

Expectation of a Discrete RV

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$$E[X] = \sum_{n=0}^{\infty} p(1-p)^n + \sum_{n=1}^{\infty} n p(1-p)^n$$

$$= p\sum_{n=0}^{\infty}(1-p)^n + (1-p)\underbrace{\sum_{n=1}^{\infty} n p(1-p)^{n-1}}_{E[X]}$$

$$E[X] = p \cdot \frac{1}{1-(1-p)} + (1-p)E[X]$$

$$E[X] + (p-1)E[X] = 1 \Rightarrow \boxed{E[X] = \frac{1}{p}}$$

From Discrete to Continuous

Example: Let X be a continuous random variable whose density function is nonzero on [0,1].

DISCRETE

$$E[X] = \sum_i a_i P(X=a_i)$$

CONTINUOUS

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

Expectation of a Continuous RV

Def: The expectation, expected value, or mean of a continuous random variable X with probability density function f is the number

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Expectation of a Continuous RV

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Intuition: Think of a big (one-dimensional) rock balancing on a fulcrum

Expectation of a Continuous RV

4 YEARS

Example: The lifetime (in years) of a certain brand of battery is Exponentially distributed with parameter $\lambda = 0.25$. How long, on average, will this battery last?

$$X \sim \mathrm{Exp}(\lambda)$$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x(\lambda e^{-\lambda x})\,dx$$

$= \ldots$ MIRACLE $\ldots$ $E[X] = \dfrac{1}{\lambda}$

Expectation of a Continuous RV

Example: The lifetime (in years) of a certain brand of battery is Exponentially distributed with parameter $\lambda = 0.25$. How long, on average, will this battery last?

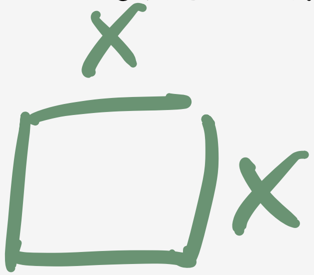
Follow-Up: Suppose you have observed that, on average, 300 cars cross a particular bridge every day. How much time do you expect to wait between two cars crossing the bridge?

$T \sim Exp(\lambda = 300)$ $\quad E[T] = \frac{1}{\lambda} = \frac{1}{300}$

Expectation of Functions of RVs

Often times we want to compute the expectation of a function of a random variable instead of the random variable itself. For instance, we might want to compute $E[X^2]$ instead of $E[X]$

Example: Suppose an architect is designing a community and wants to maximize the diversity in the size of his square buildings that are of both width and depth X , but X is uniformly distributed between 0 and 10 meters. What is the distribution of the area X^2 of a building?



$X \sim U[0,10]$ $\quad 0 \le x \le 10$

$Y = X^2 =$ AREA OF HOUSE

$0 \le y \le 100$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y}) = \frac{\sqrt{y}}{10}$$

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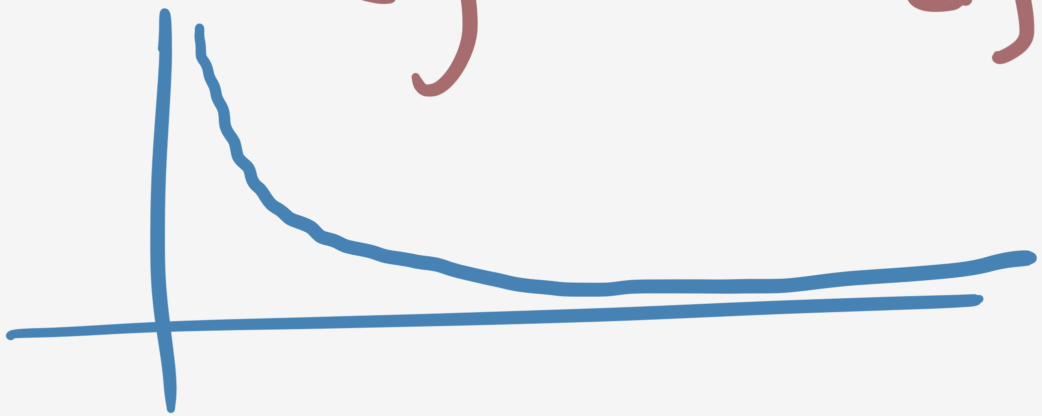
$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left(\frac{\sqrt{y}}{10}\right) = \frac{1}{20\sqrt{y}}$$



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$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^{10} \frac{x^2}{10}\, dx = \frac{1}{10} \cdot \frac{x^3}{3}\Big|_0^{10}$$

$$= \frac{100}{3} = 33\frac{1}{3} \text{ METERS}^2$$

Expectation of Functions of RVs

Often times we want to compute the expectation of a function of a random variable instead of the random variable itself. For instance, we might want to compute $E[X^2]$ instead of $E[X]$

Change-of-Variables Formula: Let X be a random variable and let $g : \mathbb{R} \to \mathbb{R}$ be a function.

If X is discrete, take the values $a_1, a_2, \ldots,$ then

$$E[g(x)] = \sum_i g(a_i)P(X = a_i)$$

If X is continuous, with probability density function , then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

Linearity of Expectation

Super-Useful Fact: Expectation is a linear function.

$$E[aX + b] = aE[X] + b$$

OK! Let's Go (Back) to Work!

Get in groups, get out laptop, and open the Lecture 10 In-Class Notebook

Let's:

- Look at the (un)profitability of playing Casino Roulette
- Get some practice computing expected values of continuous random variables