More Discrete Random Variables and Their Distributions

Previously on CSCI 3022

Def: a discrete random variable X is a function that maps the elements of the sample space Ω to a finite number of values $a_1, a_2, \ldots, a_n$ or an infinite number of values $a_1, a_2, \ldots$

Def: A discrete random variable $X \sim Ber(p)$, where $0 \leq p \leq 1$, if its probability mass function is given by

$$p_X(1) = P(X = 1) = p \quad \textbf{and} \quad p_X(0) = P(X = 0) = 1 - p$$

Def: A discrete random variable $X \sim Bin(n, p)$, where $n = 1, 2, \ldots$ and $0 \leq p \leq 1$, if its probability mass function is given by

$$p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \textbf{for } k = 0, 1, \ldots, n$$

The Gist: A Bernouilli random variable models a coin flip. A Binomial random variable represents the number of n independent coin flips that come up Heads.

Binomial-Like Distributions

There are several discrete distributions that are similar in spirit to the Binomial distribution. We'll look at three of them today:

- the Geometric Distribution
- the Negative Binomial Distribution
- the Poisson Distribution

Election Day Exit Polls

Example: You are doing an exit poll outside of a voting station on Election Day. As people exit, you ask them questions about their political affiliation, who they voted for, etc. In particular, you're interested in how registered Independents voted. Being well-prepared, you know that about 20% of registered voters are registered as Independents.

Goal: Suppose you hang out until you interview 100 people. Let X be a random variable describing the number of actual Independents that you encounter.

Distribution: Binomial Dist, $Bin(n=100, p=0.2)$

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Goal: Suppose you're talking to a lot of registered Republicans and Democrats, but haven't met an Independents yet. Let X be a random variable describing the number of people you have interviewed up to and including your first registered Independent voter.

Distribution: GEOMETRIC DISTRIBUTION

Election Day Exit Polls

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Goal: You're really interested in talking to a lot of Independents. Let X be the random variable describing the number of people you have to talk to in order to interview exactly 100 registered Independents.

Distribution: NEGATIVE BINOMIAL DIST.

Election Day Exit Polls

Example: You are doing an exit poll outside of a voting station on Election Day. As people exit, you ask them questions about their political affiliation, who they voted for, etc. In particular, you're interested in how registered Independents voted. Being well-prepared, you know that about 20% of registered voters are registered as Independents.

Goal: You're concerned about being overwhelmed during the peak voting time, so you keep an eye on the number of people arriving in line at the voting station, and plan to call a colleague for help if it gets too busy. Let X be a random variable describing the number of voters that arrive at the station over a 15 minute period.

Distribution: Poisson Distribution.

The Geometric Distribution

Simple Example: Suppose you flip the same biased coin repeatedly. How many times do you flip the coin before you see your first Heads?

P(HEADS) = p

1 FLIP: p

2 Flips: $(1-p)p$

3 Flips: $(1-p)^2 p$

$$p_X(k) = (1-p)^{k-1} p$$

The Geometric Distribution

Def: A discrete random variable X has a geometric distribution with parameter p, where $0 \leq p \leq 1$, if it's probability mass function is given by

$$p_X(k) = P(X = k) = (1-p)^{k-1}p \quad \textbf{for } k = 1, 2, 3, \ldots$$

We say that $X \sim Geo(p)$.

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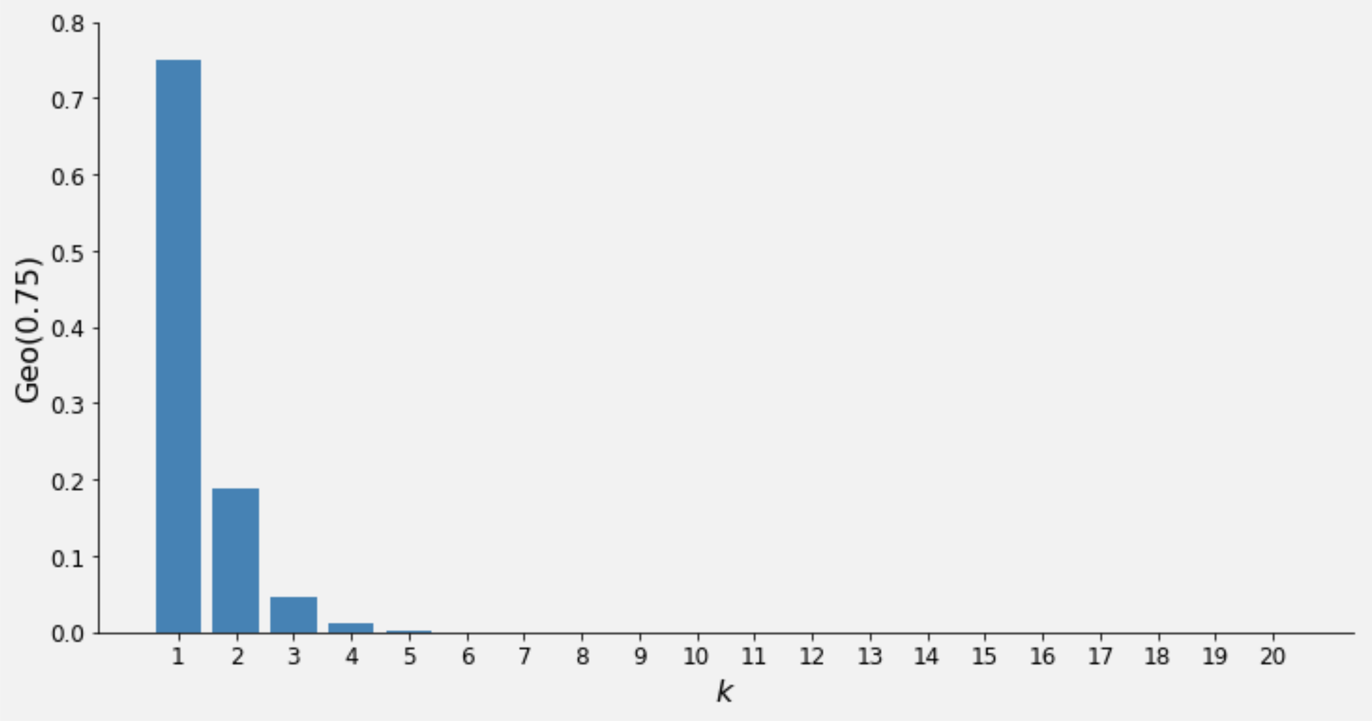

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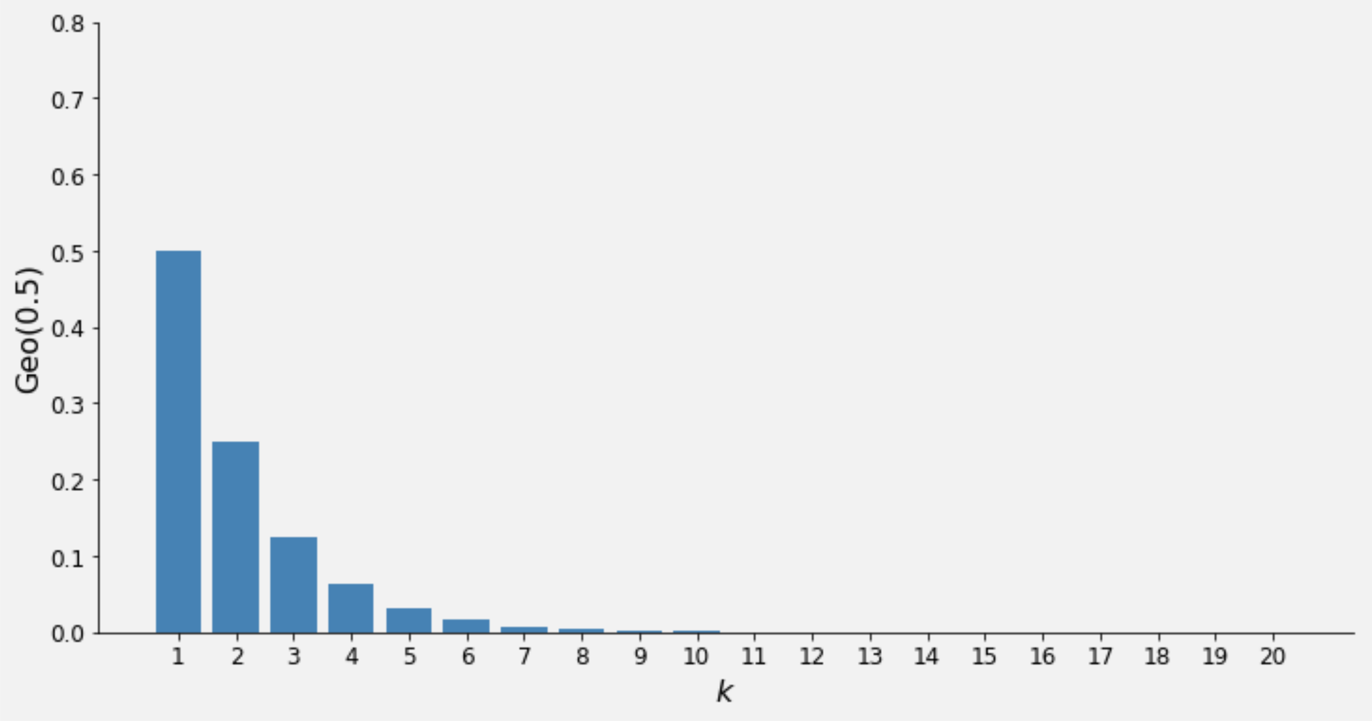

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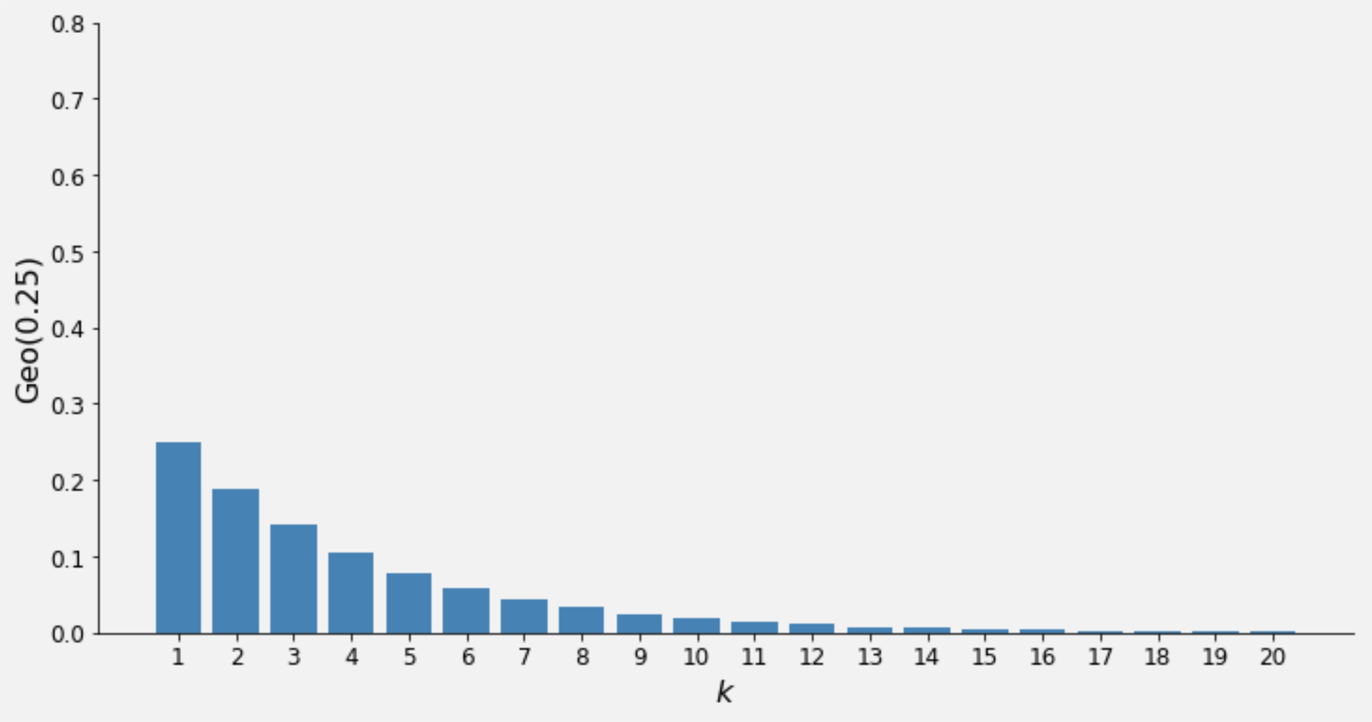

The Geometric Distribution

Question: What assumptions did we implicitly make in deriving the Geometric Distribution?

- Each trial is a Bernoulli RV with probability of success p
- Each trial is independent

The Geometric Distribution

Question: What assumptions did we implicitly make in deriving the Geometric Distribution?

- EACH TRIAL IS INDEPENDENT
- probability p for EACH flip IS the SAME.

The Negative Binomial Distribution

Simple Example: Suppose you flip the same biased coin repeatedly. How many times do you flip the coin before you see three Heads?

$X \in \{3, 4, 5 \ldots\}$

$$\boxed{\binom{8}{3} p^3 (1-p)^5}$$

$P_X(k) = $ [PROB OF 2 HEADS IN $k-1$] [PROB OF H ON LAST FLIP]

$= $ BINOMIAL W/ $n = k-1$ AND # 2 $\cdot p$

$$\binom{k-1}{2} p^2 (1-p)^{k-3} \cdot p = \binom{k-1}{2} p^3 (1-p)^{k-3}$$

$$= \binom{k-1}{r-1} p^r (1-p)^{k-r} \quad \text{FOR } k = 3, 4, 5 \ldots$$

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Def: A discrete random variable X has a negative binomial distribution with parameters r and p, where $r > 1$ and $0 \leq p \leq 1$, if it's probability mass function is given by

$$p_X(k) = P(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r} \quad \text{for } k = r, r+1, \ldots$$

We say that $X \sim NB(r, p)$.

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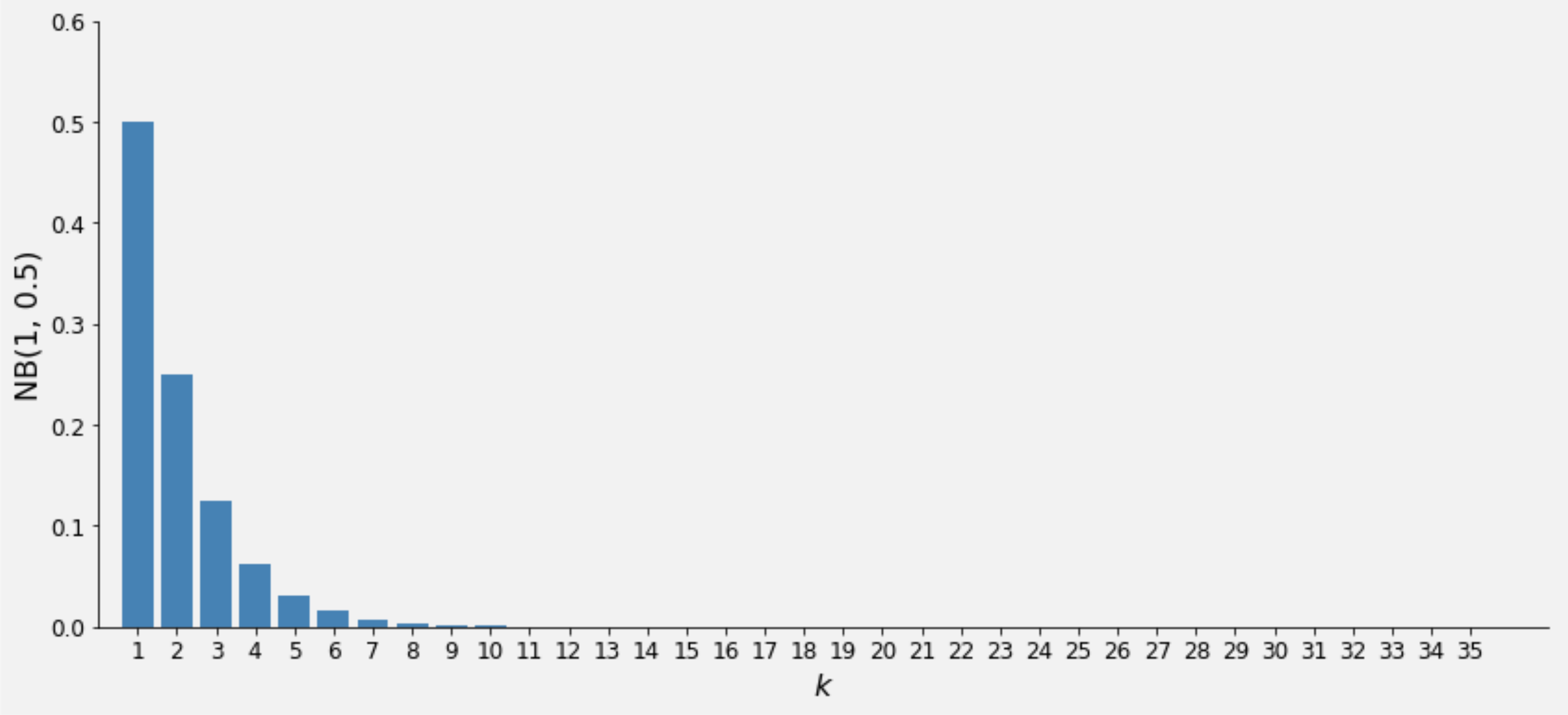

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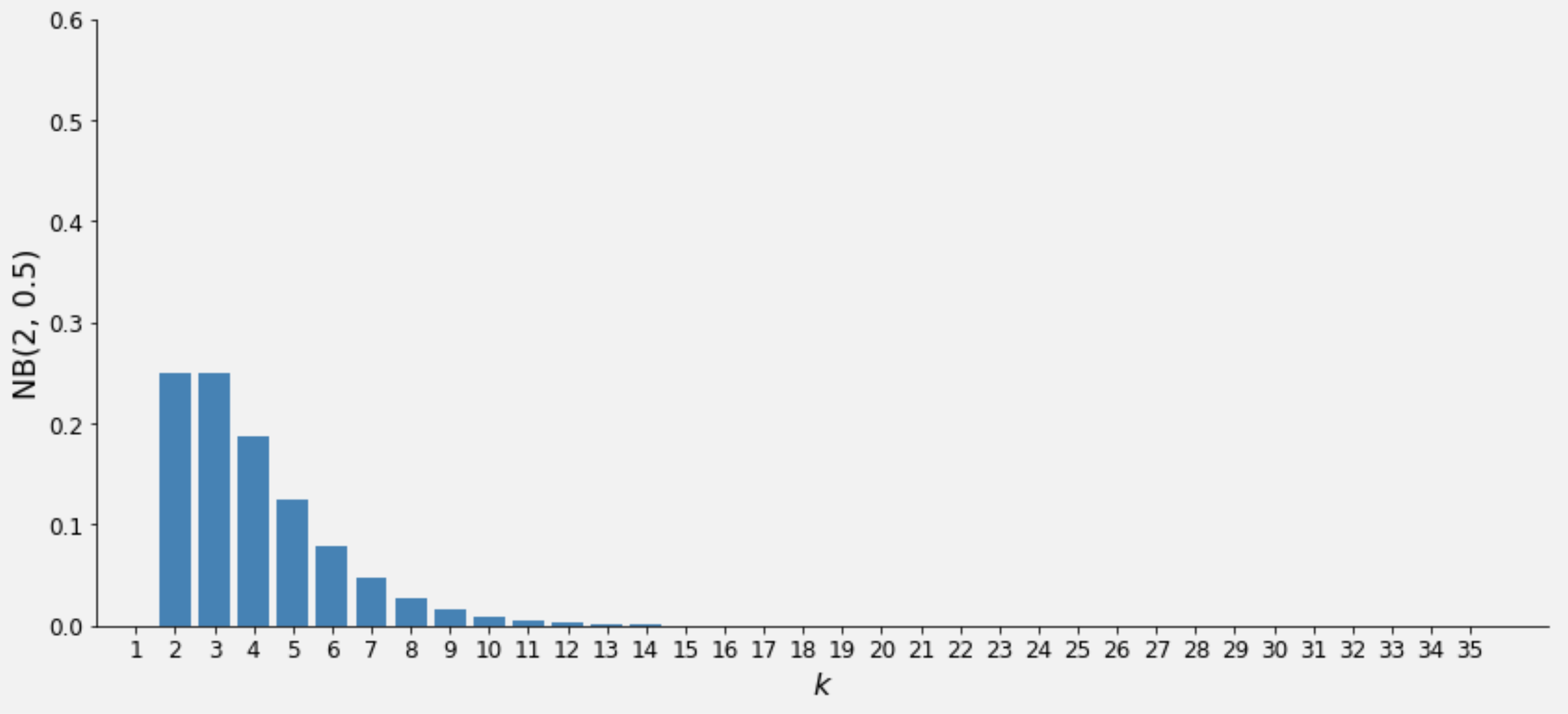

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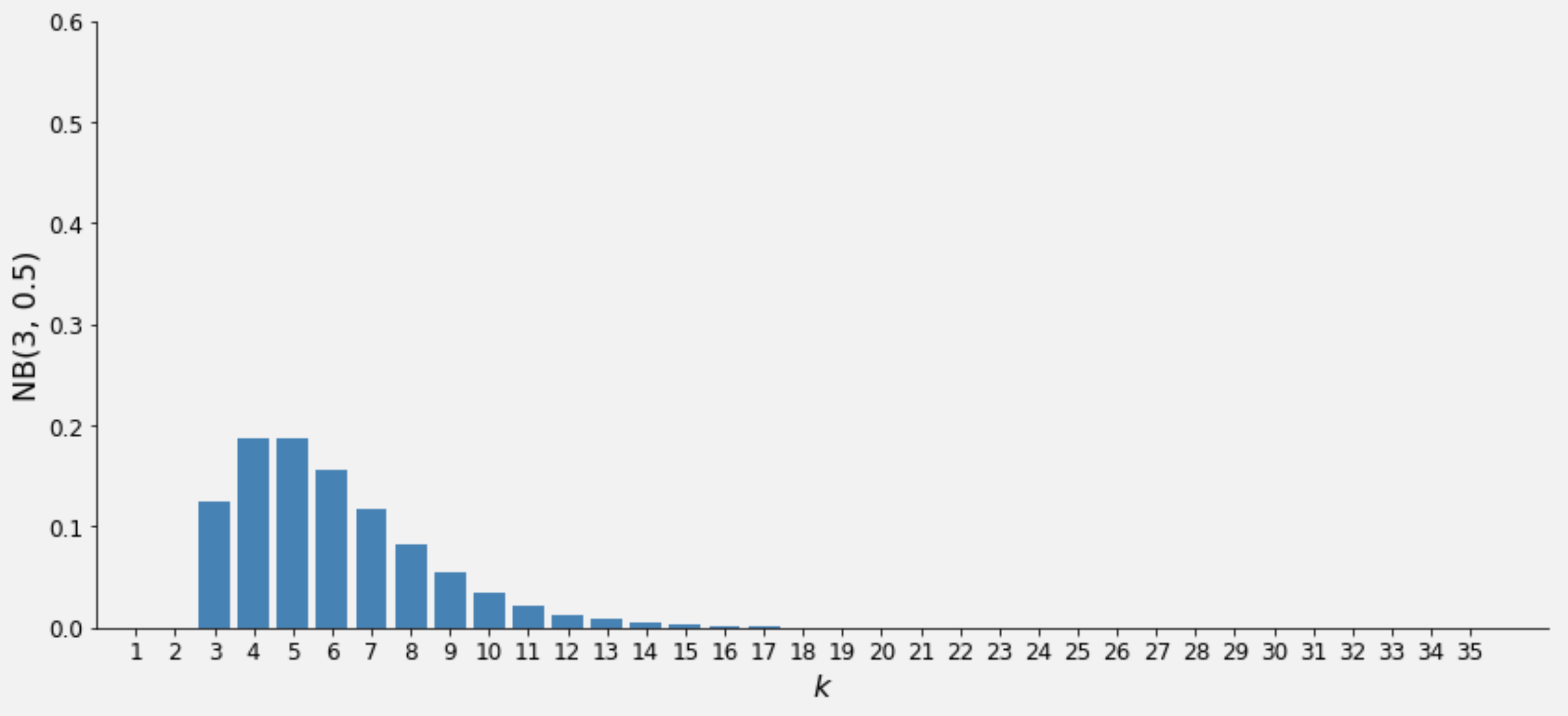

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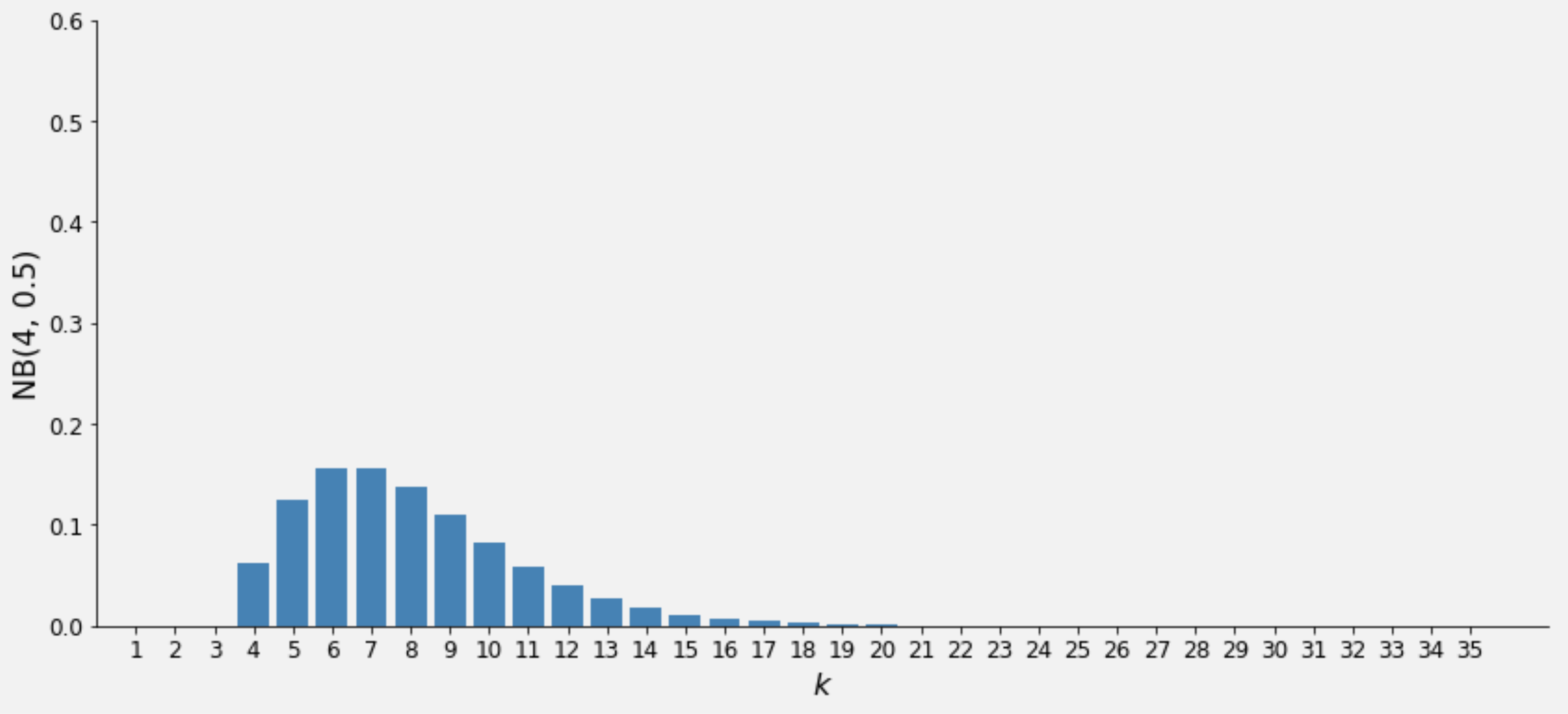

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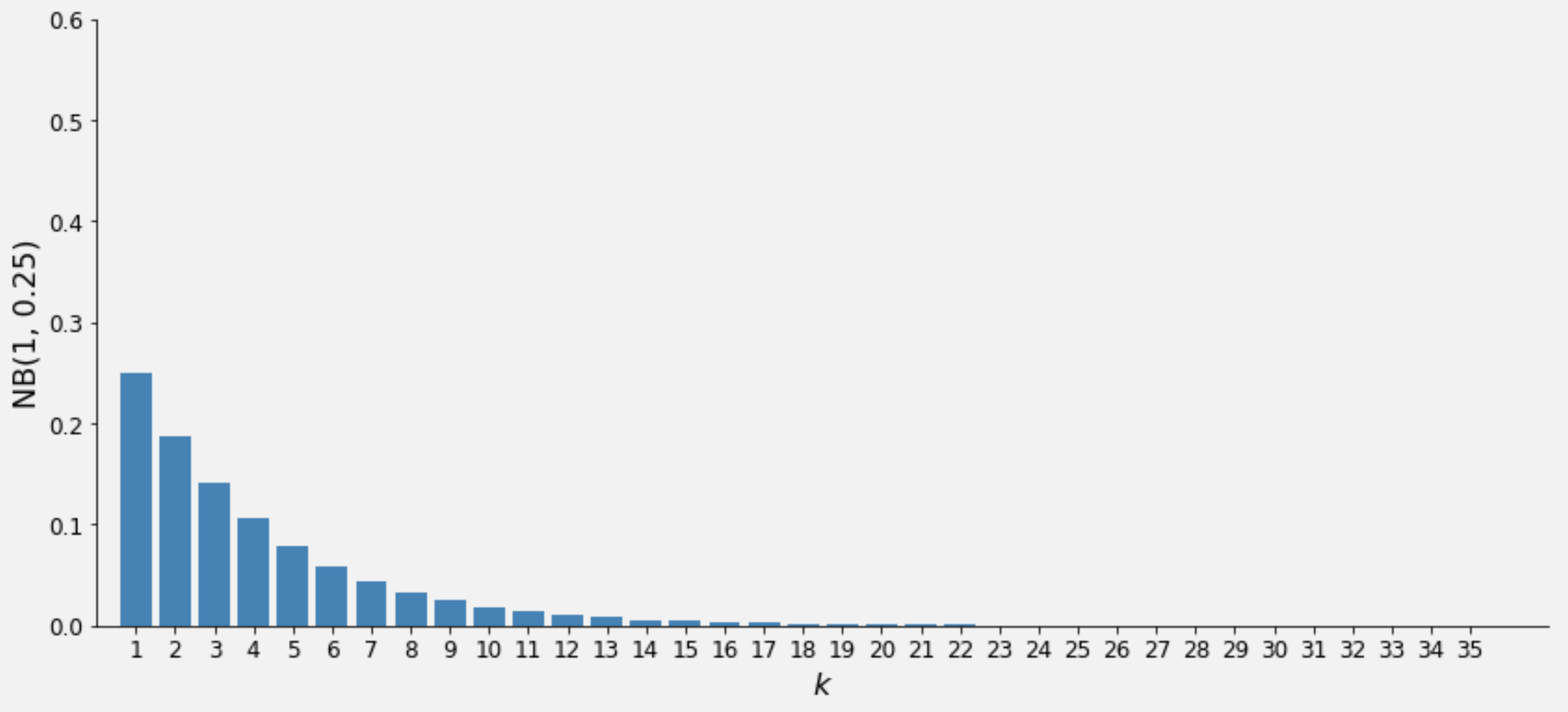

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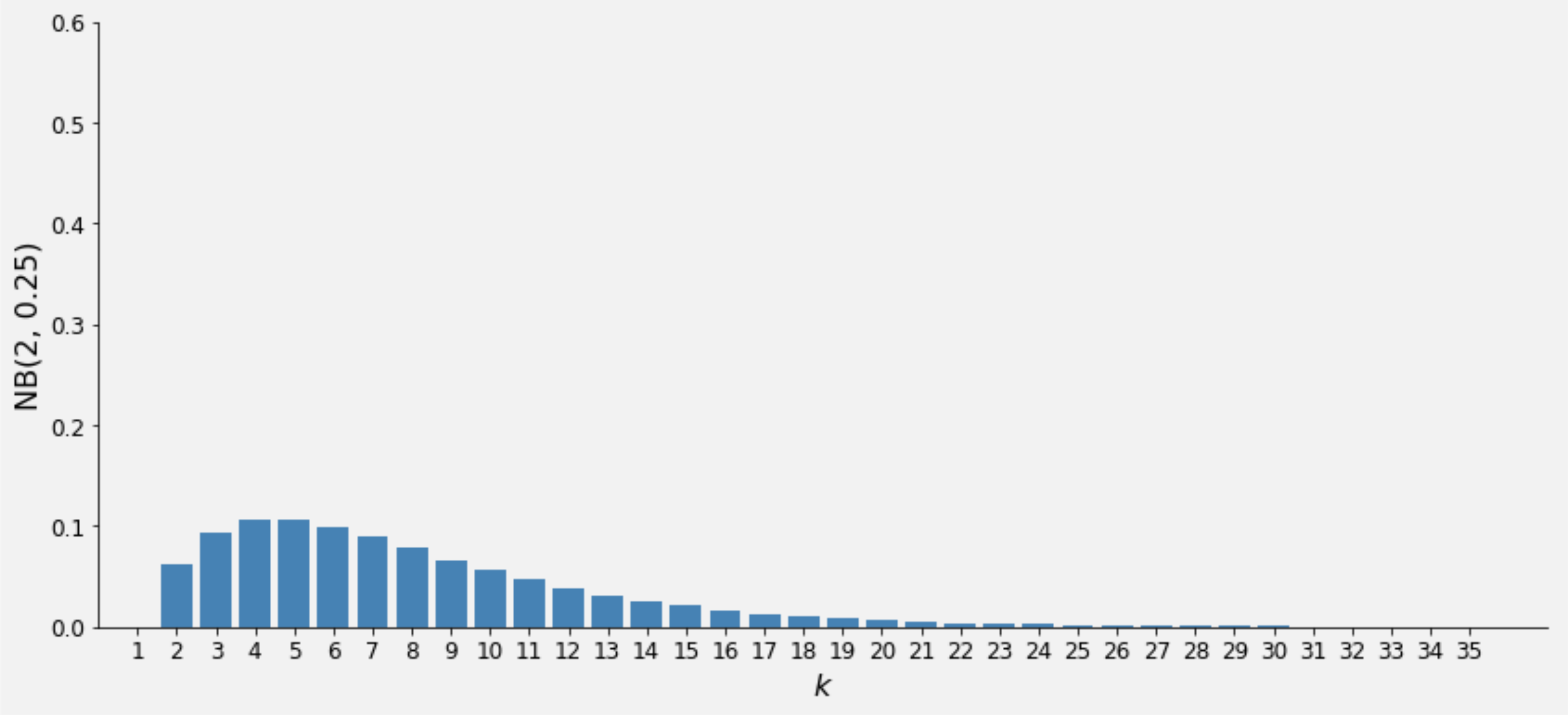

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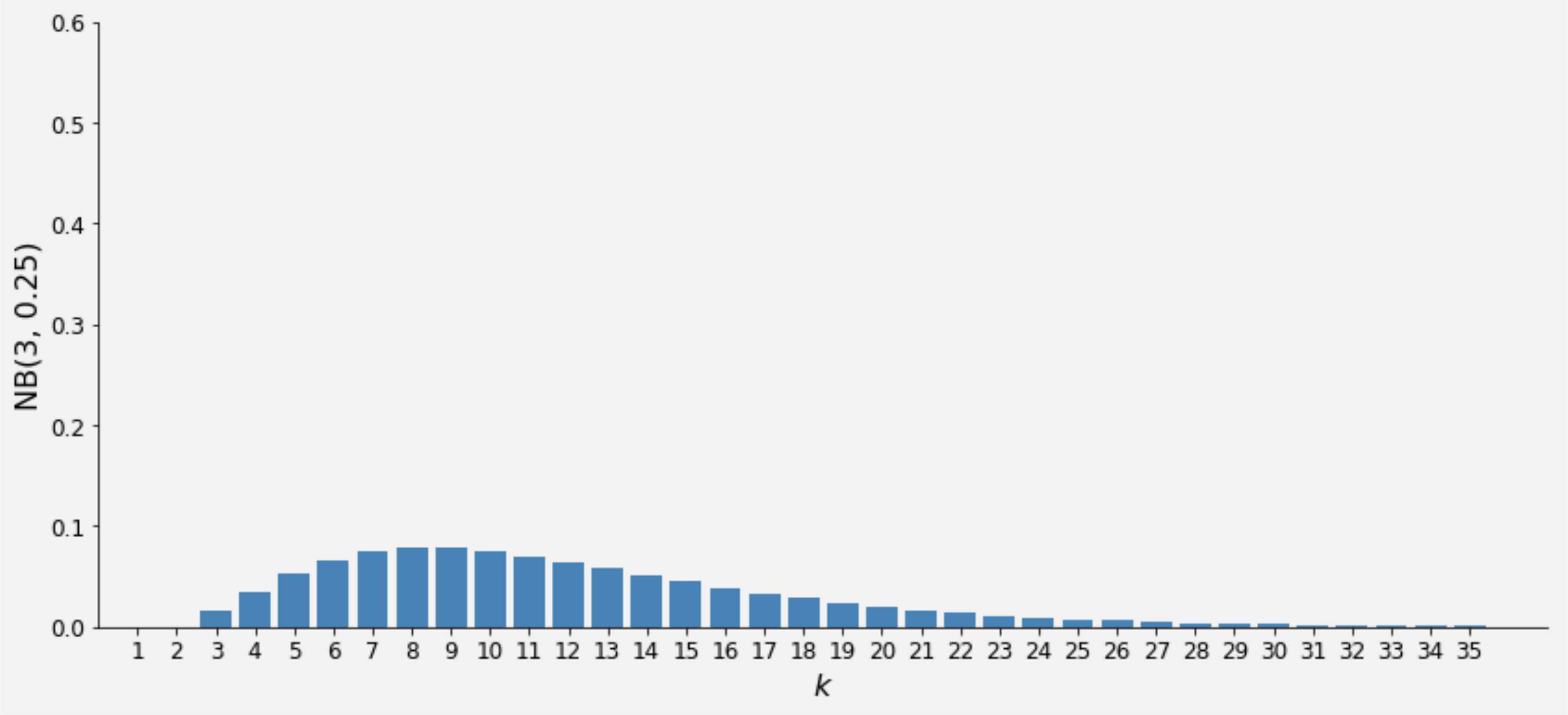

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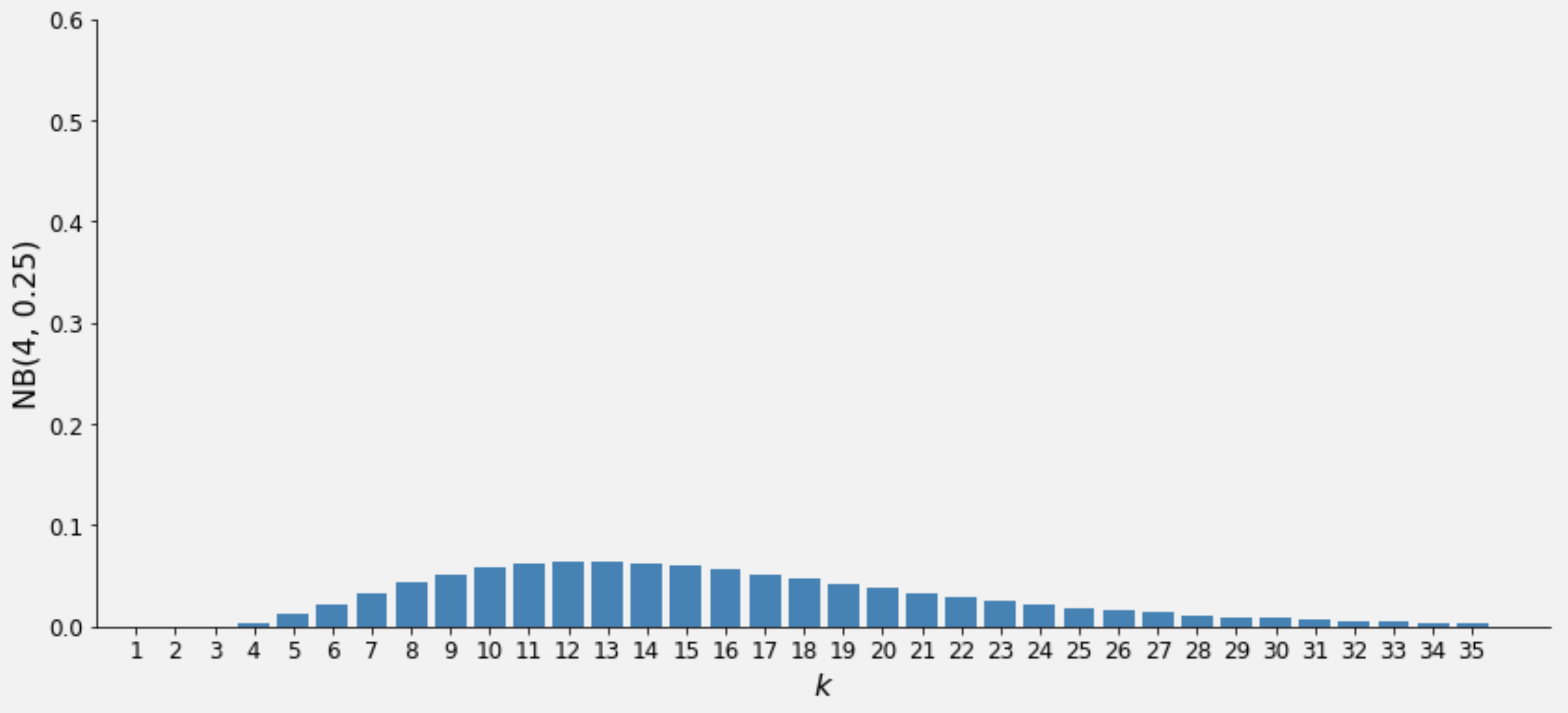

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The Poisson Distribution

Simple Example: A call center typically receives calls at a rate of 10 per minute. What is the probability that they receive: 1 call in a minute? 50 calls in a minute? 100 calls in a minute?

$\mu = np$

RATE

n TIME SLICES

p IS PROB OF A CALL IN A SLICE

$$\binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$\lim_{n\to\infty} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$\mu = np \Rightarrow p = \frac{\mu}{n}$$

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$$\lim_{n\to\infty} \frac{n!}{k!(n-k)!} \left(\frac{\mu}{n}\right)^k \left(1-\frac{\mu}{n}\right)^{n-k}$$

$$= \frac{1}{k!} \mu^k \lim_{n\to\infty} \underbrace{\frac{n!}{(n-k)!} \frac{1}{n^k}}_{\to 1} \underbrace{\left(1-\frac{\mu}{n}\right)^n}_{\to e^{-\mu}} \underbrace{\left(1-\frac{\mu}{n}\right)^{-k}}_{\to 1}$$

$$f(k) = \frac{\mu^k e^{-\mu}}{k!}$$

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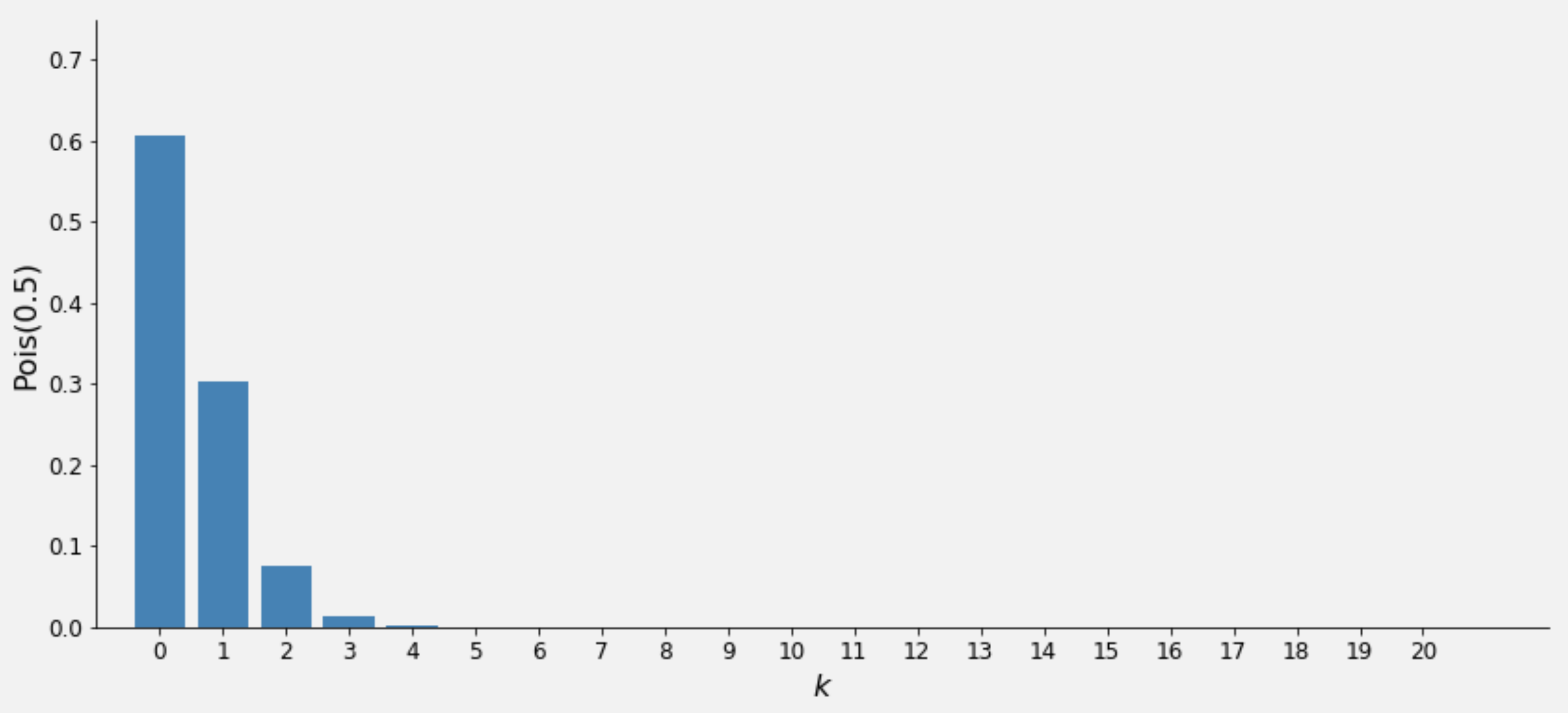
The Poisson Distribution

Def: A discrete random variable X has a Poisson distribution with parameters μ, where $\mu > 0$ if it's probability mass function is given by

$$p_X(k) = P(X = k) = \frac{\mu^k}{k!} e^{-\mu} \quad \textbf{for } k = 0, 1, 2, \ldots$$

We say that $X \sim Pois(\mu)$.



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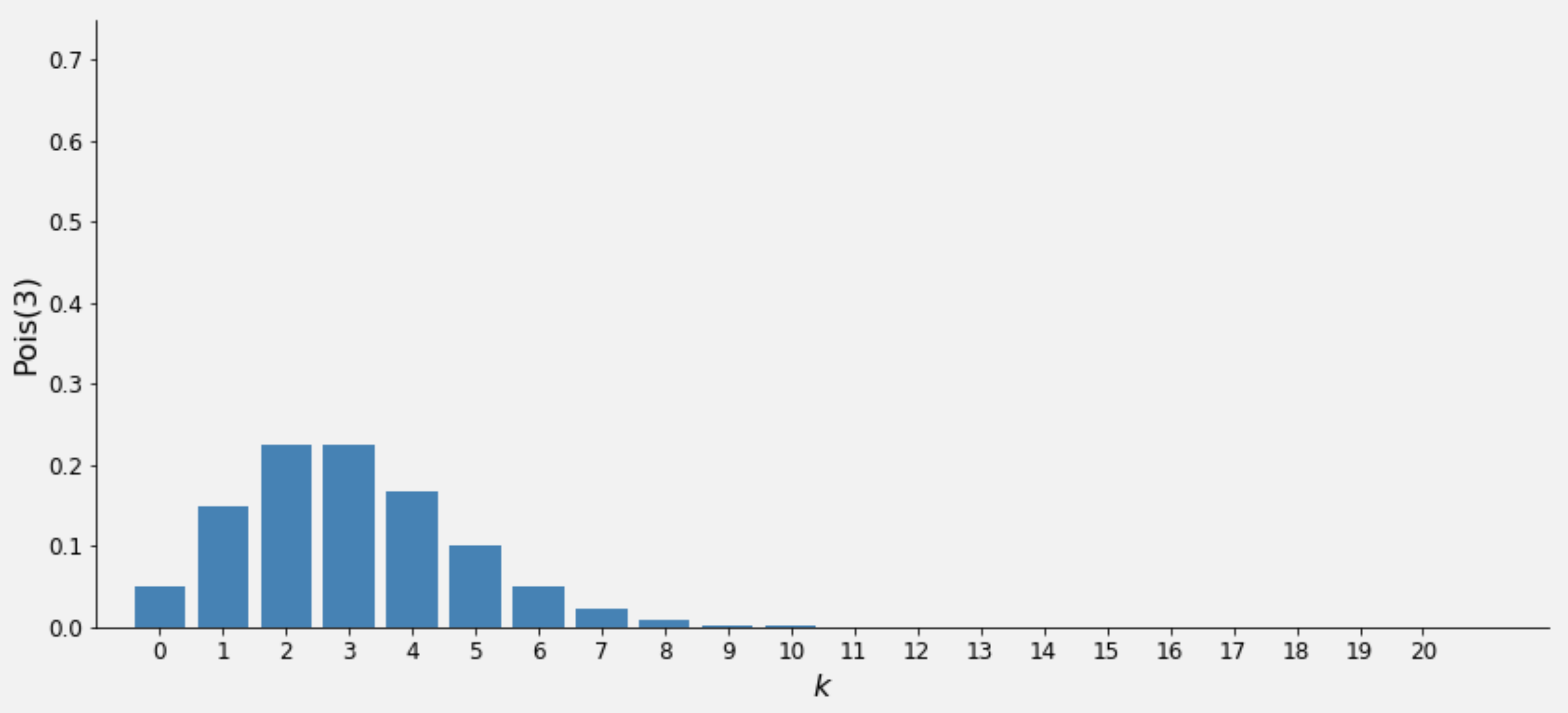

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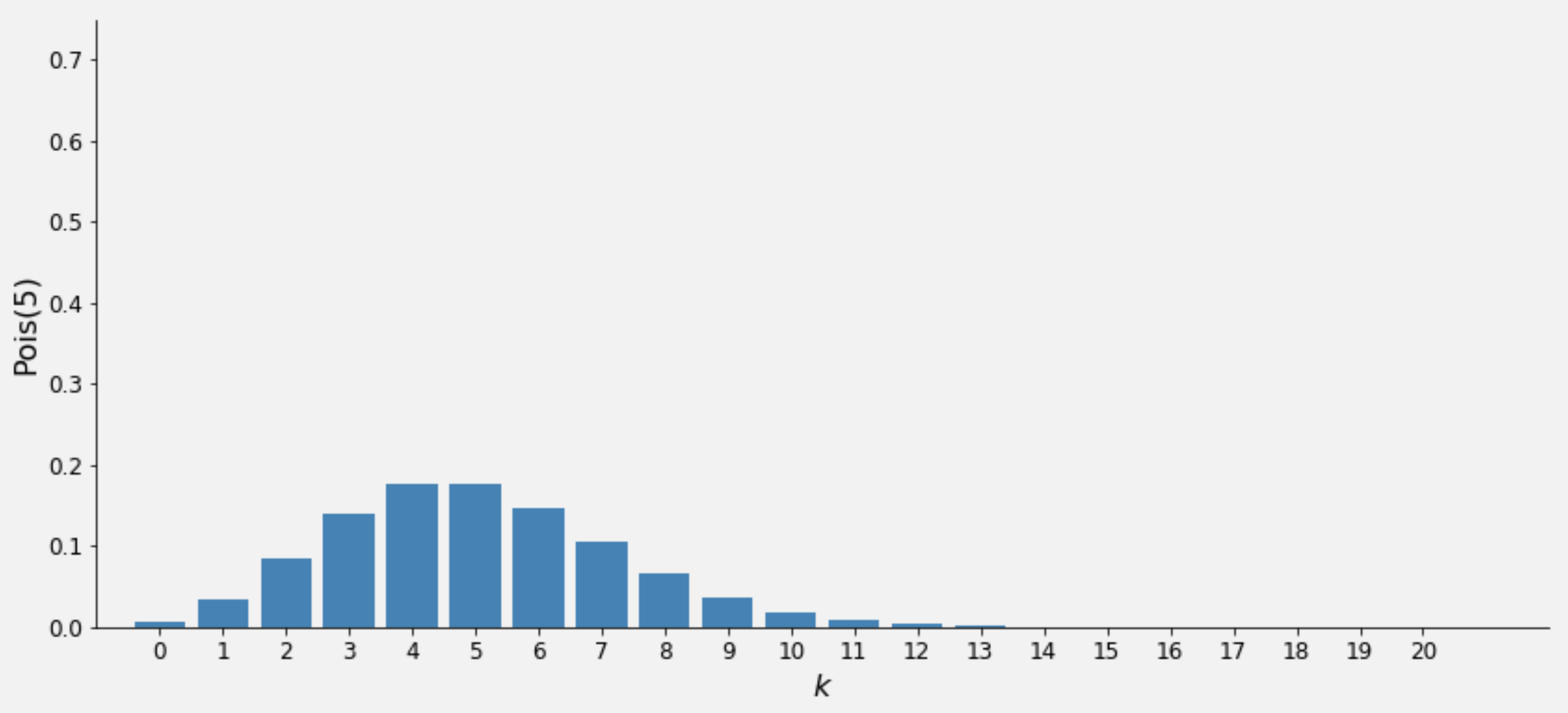

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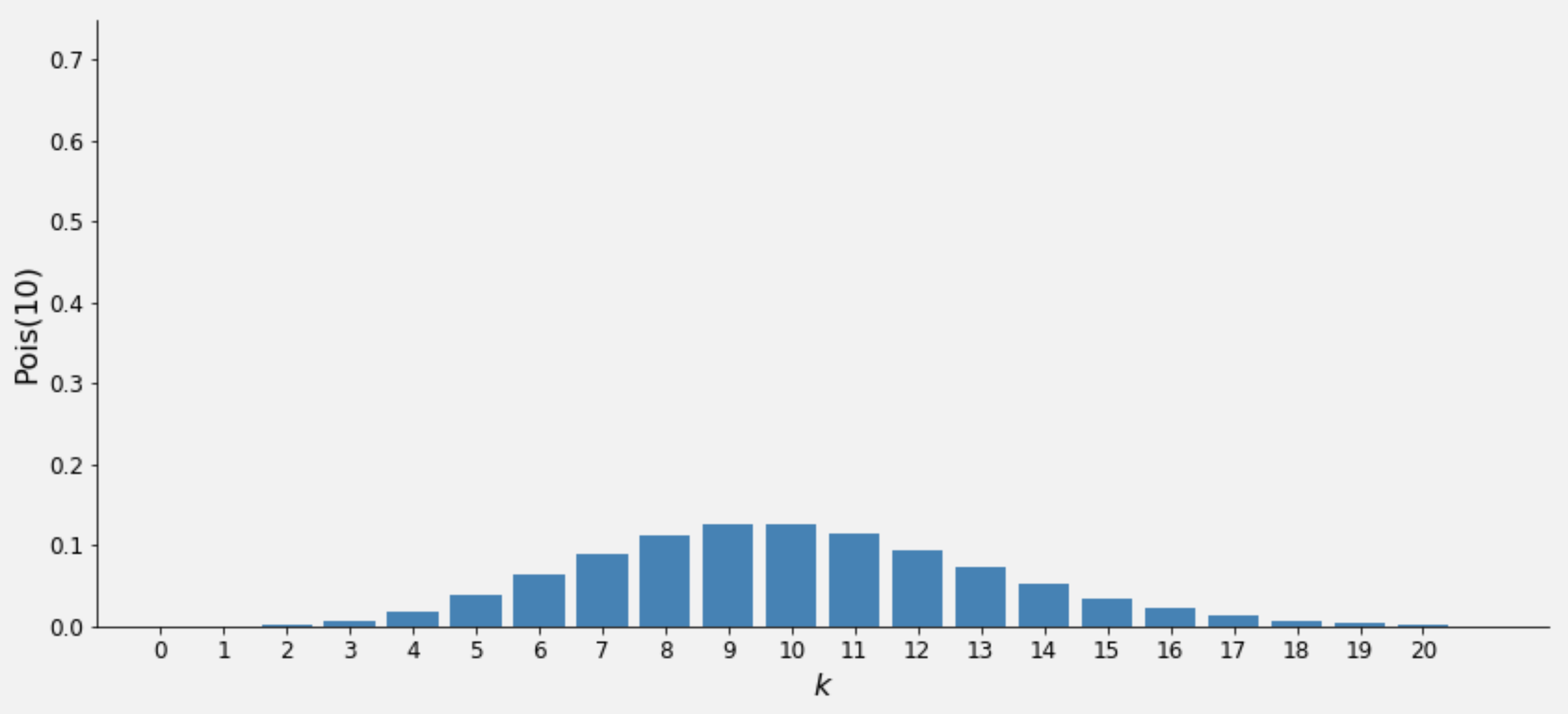


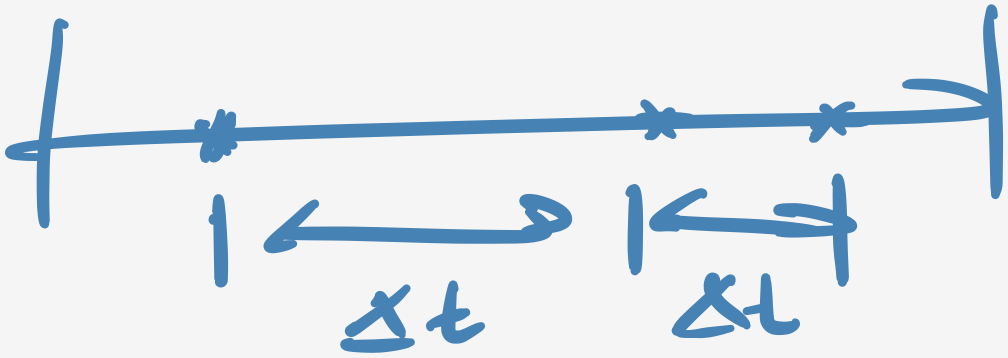
The Poisson Distribution

Question: What assumptions did we implicitly make in deriving the Poisson Distribution?

- The probability of observing an single event over a small interval is proportional to the size of the interval.
- Each event / arrival is independent

The Poisson Distribution with a Twist

Question: Suppose arrivals are described well by a Poisson random variable. What is the probability that the time between two arrivals is between 5 and 10 minutes?



Practice, Practice, Practice

Exercise: You and a friend want to go to a concert, but unfortunately only one ticket is still available. The man who sells the tickets decides to toss a coin until heads appears. In each toss heads appears with probability p , where $0 < p < 1$, independent of each of the previous tosses. If the number of tosses needed is odd, your friend is allowed to buy the ticket; otherwise you can buy it. Would you agree to this arrangement?

$$\underbrace{p, \ (1-p)p}, \ \underbrace{(1-p)^2p, \ (1-p)^3p}, \ \ldots$$

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OK! Let's Go to Work!

Get in groups, get out laptop, and open the Lecture 8 In-Class Notebook

Let's:

- Practice identifying applications for the distributions we've learned
- Confirm our theoretical distributions with simulation
- Look more at the Challenger Disaster