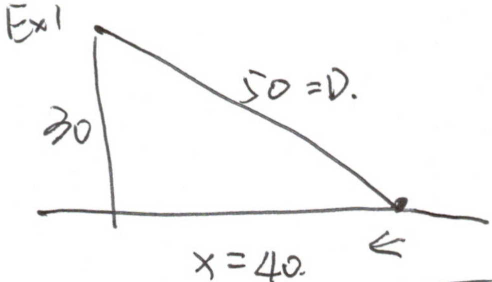
Lecture 13.: Related Rates

Ex1



Are you speeding:

$\frac{dD}{dt} = -80$ @ $\frac{dx}{dt} > 95$?

$x^2 + 30^2 = D^2$. $\frac{dD}{dt} = -80$

Easiest with implicit diff.

$2x\frac{dx}{dt} = 2D\frac{dD}{dt}$

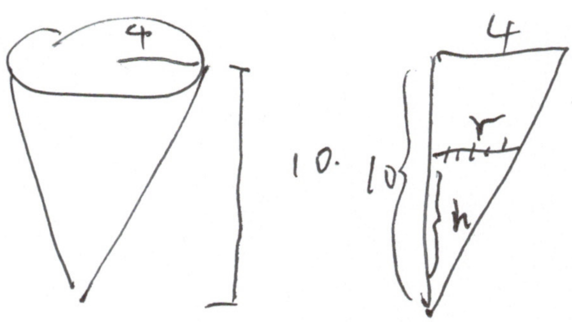
$2 \cdot 40 \cdot \frac{dx}{dt} = 2 \cdot 50 \cdot \frac{dD}{dt} \cdot 80$

$\boxed{\frac{dx}{dt} = -100 \text{ feet/sec}}$

Ex2. A conical tank.
top radius 4 ft
depth 10 ft:
filled at 2 cu ft/min

how fast are the water rising when depth is 5 feet.

Diagram, variable



$\frac{r}{h} = \frac{4}{10}$.

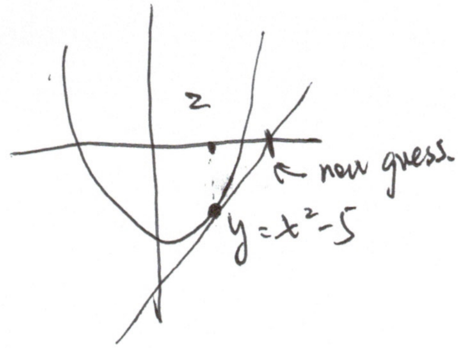
$V = \frac{1}{3}\pi r^2 h$ $\frac{dV}{dt} = 2$.

$\frac{dh}{dt}$? when $h = 5$

Newton's method

Example. solve $x^2=5$.

$f(x)=x^2-5$; solve $f(x)=0$



Start with initial guess $x_0=2$.

tang line

$y-y_0=m(x-x_0)$

x_1 is the x-intercept

$0-y_0=m(x_1-x_0)$.

$\frac{-y_0}{m}=x_1-x_0$.

$x_1=x_0-\frac{y_0}{m}$.

$$\boxed{x_1=x_0-\frac{f(x_0)}{f'(x_0)}}$$

Newton's method

$$\boxed{x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}}$$

$x_0=2$. $f(x)=x^2-5$

$f'(x)=2x$

$x_1=x_0-\frac{x_0^2-5}{2x_0}$

$x_1=\frac{1}{2}x_0+\frac{5}{2x_0}$

$x_1=\frac{1}{2}\cdot 2+\frac{5}{4}=9/4$

$x_2=\frac{1}{2}\frac{9}{4}+\frac{5}{2}\cdot\frac{4}{9}=\frac{161}{72}$

$x_3=\frac{1}{2}\frac{161}{72}+\frac{5}{2}\cdot\frac{72}{161}$

n	$\sqrt{5}-x_n$
0	2×10^{-1}
1	10^{-2}
2	4×10^{-5}
3	4×10^{-10}

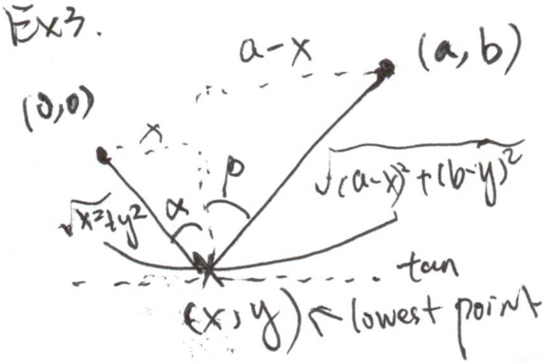
$r = \frac{2}{5}h$. $V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h$

$$2 = \frac{dV}{dt} = \frac{\pi}{3}\left(\frac{2}{5}\right)^2 3h^2\frac{dh}{dt}$$

$$2 = \frac{\pi}{3}\left(\frac{2}{5}\right)^2 3\cdot 5^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{2\pi} \text{ feet/minute}$$

Ex3.



$$\sqrt{x^2+y^2} + \sqrt{(a-x)^2+(b-y)^2} = L \text{ (constant)}$$

: Find min y. | From diagram, the bottom point is $y' = 0$ (critical point).

Implicit diff on the constraint

$$\frac{x + yy'}{(x^2+y^2)^{1/2}} - \frac{(a-x) + (b-y)y'}{\sqrt{(a-x)^2+(b-y)^2}} = 0$$

$y' = 0$ $\rightarrow$

$$\frac{x}{(x^2+y^2)^{\frac{1}{2}}} = \frac{a-x}{\sqrt{(a-x)^2+(b-y)^2}}$$

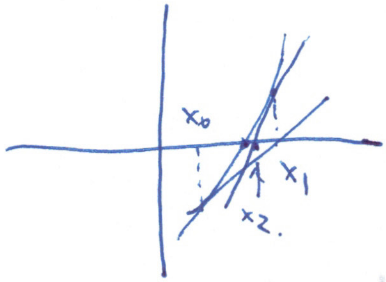
$\sin\alpha = \sin\beta$

$\Downarrow$

$\alpha = \beta$

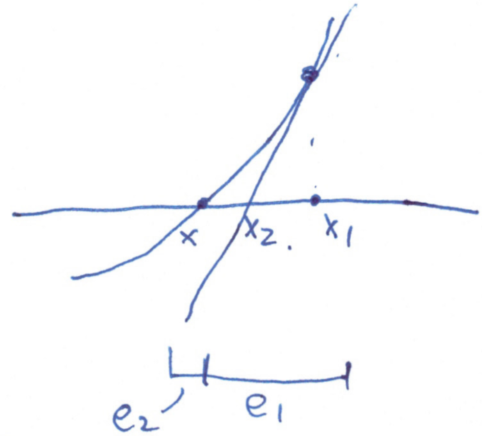
Lecture 14.

Newton's method.



$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$



ERROR ANALYSIS

$E_1 = |x - x_1|$

$E_2 = |x - x_2|$

$E_2 \sim E_1^2$?

E_0	E_1	E_2	E_3	E_4
10^{-1}	10^{-2}	10^{-4}	10^{-8}	10^{-16}

digits of accuracy doubles at each step.

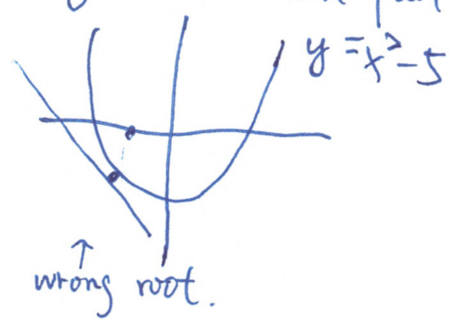
Newton's method works well if

$|f'|$ not too small

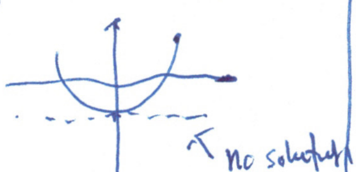
$|f''|$ not too big

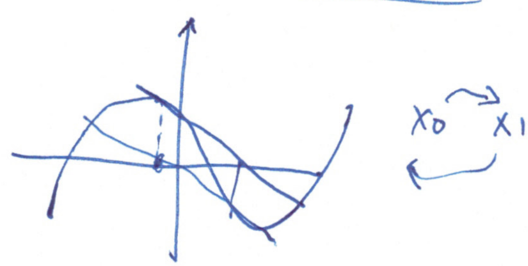
x_0 starts nearby the target x.

Ways method can fail



$f' = 0$ is a disaster for method as $\frac{f}{f'(0)}$ is undefined.





MEAN VALUE THEOREM

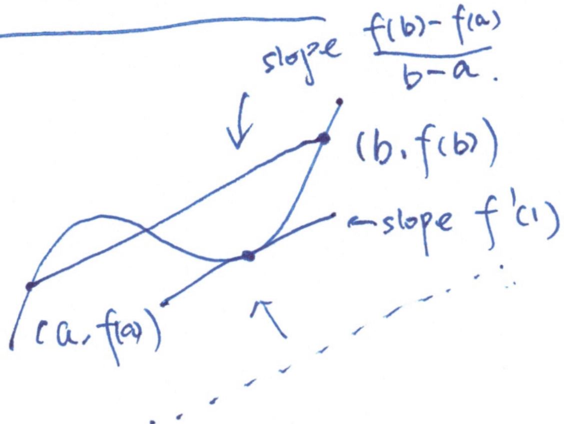
If you go boston to LA (3000 m) in 6 hrs. then at some time you are going at the average speed: $\frac{3000}{6} = 500$ miles/h

Theorem: MVT

$$\boxed{\frac{f(b)-f(a)}{b-a} = f'(c) \quad \text{for some } c, \text{ for } a < c < b}$$

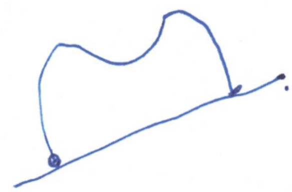
provided. f is differentiable in $a < x < b$ and continuous in $a \le x \le b$.

PROOF OF MVT



move up the parallel line until the line touches f

If it does not touch, then bring the parallel lines from ~~the to~~ above.



why hypothesis is needed.

not differentiable.

one bad point ruins the proof. We need $f'(x)$ exists at <u>all</u> x, $a<x<b$.

Application to graphing

1. If f' is positive (>0) then f is increasing
2. if f' is <0, then f is decreasing
3. if $f'=0$ the f is constant.

proof: REWITE (*)

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

$$f(b)-f(a) = f'(c)(b-a)$$

$$\boxed{f(b) = f(a) + f'(c)(b-a)} \quad (**)$$

$a<b$. 1. if $f'(c)>0$, then $f(b)>f(a)$

2. $f'(c)<0$ then $f(b)<f(a)$

3. $f'(c)=0$ then $f(b)=f(a)$

Compare with Linear approximation

$$\frac{\Delta f}{\Delta x} \approx f'(a) \quad b \text{ near } a. \ (b-a=\Delta x)$$

$$\frac{\Delta f}{\Delta x} = f'(c) \quad c \text{ is between } a \text{ and } b$$

$$(*) \quad \boxed{\min f' \le \frac{f(b)-f(a)}{b-a} \le \max f' \quad \text{on } a \le x \le b}$$

~~MTV~~ MVT

MVT
min ≤ avg speed ≤ max

Linear Appx
avg speed ≈ initial speed.

Ex. Inequalities.

1. $e^x > 1+x \quad (x>0)$

proof $f(x) = e^x - (1+x)$.

starts $f(0) = e^0 + (1+0) = 1$

and $f'(x) = e^x - 1 > 0$.

$\therefore f(x) > f(0)$.

$e^x - (1+x) > 0 \Rightarrow e^x > 1+x$

$$e^x > 1 + x + \frac{x^2}{2}.$$

$$g(x) = e^x - \left(1 + x + \frac{x^2}{2}\right) > 0.$$

$$g(0) = 1 - 1 = 0$$

$g'(x) = e^x - (1+x) > 0 \quad (x>0)$ ↑ step 1.

$\Rightarrow$ ~~g(x)~~ g is increasing

$\Rightarrow g(x) > g(0)$.

$$e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{3 \cdot 2} + \frac{x^4}{4 \cdot 3 \cdot 2} \cdots$$

Lecture 15. Integration.

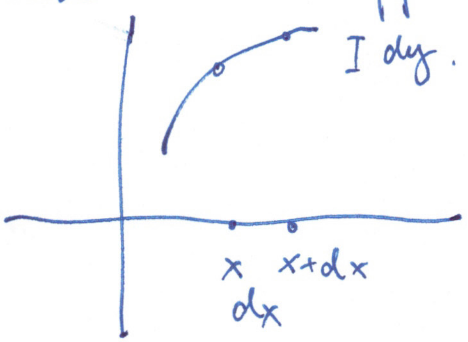
Differentials

$y = f(x)$.

Differential of y (or f): $\boxed{dy = f'(x)\,dx}$ $\Leftrightarrow$ $\dfrac{dy}{dx} = f'(x)$

Leibniz's interpretation of derivative as a ratio of "infinitesimals" 无穷小

Use in linear approximations.



dx replaces Δx

dy replaces Δy.

Ex. $(64.1)^{1/3} = ?$

$y = x^{1/3}$. $dy = \frac{1}{3} x^{-\frac{2}{3}} dx$

At $x = 64$

$y = 64^{\frac{1}{3}} = 4$.

$dy = \frac{1}{3}(64)^{-2/3} dx$

$= \frac{1}{3} \frac{1}{16} dx = \frac{1}{48} dx$

$x = 64$, $x + \cancel{dy}\, dx = 64.1$

$dx = \frac{1}{10}$

$(64.1)^{1/3} \approx y + dy = 4 + \frac{1}{48} dx = 4 + \frac{1}{480}$

$(64.1)^{1/3}$: $y + \Delta y$

$dy = \frac{1}{48} dx$

Compare to previous notation. (Review of linear approx.)

SAME!

$f(x) = f(a) + f'(a)(x-a)$

$a = 64$ $f(x) = \frac{1}{3}$

$f(a) = f(64) = 4$

$f'(a) = \frac{1}{3} a^{-2/3} = \frac{1}{48}$

$x^{1/3} \approx 4 + \frac{1}{48}(x-64)$

$64.1^{\frac{1}{3}} \approx 4 + \frac{1}{48}(0.1)$

$= 4 + \frac{1}{480}$

ANTIDERIVATIVES.

$G(x) = \int g(x)\,dx)$ integral sign

Antiderivative of g. = indefinite integral of g.

1. $\int \sin x\,dx = -\cos x + C$

C indefinite

$G(x) = -\cos x + C$ (C constant)

$G'(x) = \sin x$.

2. $\int x^a\,dx \overset{a \neq -1}{=} \frac{1}{a+1} x^{a+1} + C$

$d(x^{a+1}) = (a+1)x^a\,dx$ (for all a)

3. $\int \frac{dx}{x} = \ln|x| + C$

$x > 0$ ✓

Check $x < 0$ $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln -x$

$= \frac{1}{-x}\frac{d}{dx}(-x)$

$= \frac{1}{x}$

$y = \ln(-x)$

$x < 0$

$y' = \frac{1}{x}$

4. $\int \sec^2 x\,dx = \tan x + C$

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

$\int \frac{dx}{1+x^2} = \tan^{-1} x + C$

Uniqueness of antiderivative up to a constant.

Thm. If $F' = G'$ then

$F(x) = G(x) + C$

proof If $F' = G'$

then $(F - G)' = F' - G' = 0$,

hence. $F(x) - G(x) = C$ (from MVT)

$\Rightarrow F(x) = G(x) + C$

Ex.1. $\int x^3 (x^4+2)^5 dx$

method of substitution.

tailor made for differention notation

$u = x^4 + 2$, $du = 4x^3 dx$

$\int x^3 (x^4+2)^5 dx = \int \underbrace{(x^4+2)^5}_{u^5} \underbrace{x^3 dx}_{\frac{1}{4}du} = \int \frac{u^5}{4} du$

$= \frac{1}{24} u^6 + C = \frac{1}{24}(x^4+2)^6 + C$

Ex.2. $\int \frac{x\,dx}{\sqrt{1+x^2}}$

(sust: $u = 1 + x^2$

$du = 2x dx$

$\cdots \int u^{-1/2} du$

recommended method:

$\boxed{\text{advanced guessing.}}$

$\frac{d}{dx}(1+x^2)^{1/2} = \frac{1}{2}(1+x^2)^{-1/2} 2x$

$= \frac{x}{\sqrt{1+x^2}}$

Ex3. $\int e^{6x} dx = \frac{1}{6} e^{6x} + C$

gues: e^{6x}

$\frac{d}{dx} e^{6x} = 6e^{6x}$

$\int x e^{-x^2} dx = -\frac{1}{2} e^{-x^2} + C$

guess e^{-x^2}.

$\frac{d}{dx} e^{-x^2} = e^{-x^2}(-2x)$

$\int \sin x \cos x \, dx = \frac{1}{2}\sin^2 x + C_2$

guess $\frac{d}{dx}\sin^2 x = 2\sin x \cos x$

But also true $\frac{d}{dx}\cos^2 x = 2\cos x(-\sin x)$.

Another Answer.

$\int \sin x \cos x \, dx = -\frac{1}{2}\cos^2 x + C_1$

$-\frac{1}{2}\cos^2 x - \frac{1}{2}\sin^2 x = -\frac{1}{2}$ costant.

$\boxed{C_1 - C_2 = -\frac{1}{2}}$

$$\int \frac{dx}{x \ln x} \underset{\uparrow}{=} \int \frac{1}{\ln x} \frac{dx}{x} = \int \frac{1}{u}\, du = \ln|u| + C$$

$$= \ln|\ln x| + C$$

Subst: $u = \ln x$

$du = \frac{dx}{x}$

Lecture 18

Differential equations

Ex $\frac{dy}{dx} = f(x)$

$y = \int f(x)\,dx$ solved

FOR now only one method: Substitution, or advanced guessing.

Ex 2. $\underbrace{\left(\frac{d}{dx} + x\right)}y = 0$

annihilation operator in quantum mechanics.

$$\frac{dy}{dx} = -xy.$$

$$\frac{dy}{y} = -x\,dx$$

$$\int \frac{dy}{y} = -\int x\,dx$$

$$\ln y = -x^2/2 + C$$

$$e^{\ln y} = e^{(-x^2/2 + C)}$$

$$y = Ae^{-x^2/2} \quad (A = e^C)$$

SEPARATION of variables:

$$\frac{dy}{dx} = f(x)g(y)$$

$$\frac{dy}{g(y)} = f(x)\,dx$$

$$H(y) = \int \frac{dy}{g(y)};$$

$$F(x) = \int f(x)\,dx.$$

implicit $H(y) = F(x) + c$ (c is a constant).

$$y = H^{-1}(F(x) + c)$$

$y = ae^{-x^2/2}$ any a. (>0, <0, $=0$)

$$\frac{dy}{dx} = a\frac{d}{dx}e^{-x^2/2}$$

$$= a(-x)e^{-x^2/2}$$

$$= -xy. \checkmark$$

Remarks:

Could have written

$$\ln|y| = -x^2/2 + c \quad (y \neq 0)$$

$$|y| = Ae^{-x^2/2}.$$

$$y = Ae^{-x^2/2} \quad (A = \pm a)$$

miss $y=0$ because: $\frac{dy}{y}$ missing $y=0$

Remarks cont.

$$\ln y + c_1 = -\frac{x^2}{2} + c_2$$

$$\Rightarrow \ln y = -\frac{x^2}{2} + \underbrace{c_2 - c_1}_{= C}$$

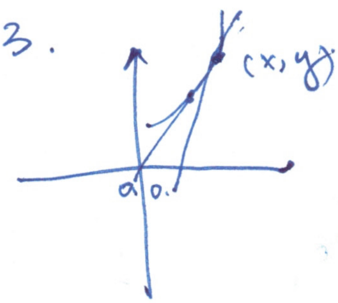
two constant can alwalys be combined.

Ex1 via separation of vars.

$$\frac{dy}{dx} = f(x) \Rightarrow dy = f(x)dx$$

$$y = \int dy = \int f(x)dx$$

Ex3.



slope of tang is = twice of ray from origin.

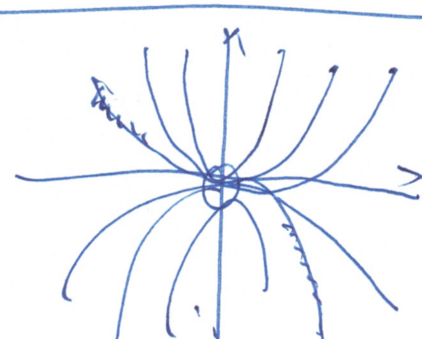
$$\frac{dy}{dx} = 2\frac{y}{x}$$

$$\int \frac{dy}{y} = \int 2\frac{dx}{x}$$

$$\ln y = 2\ln x + C \quad (y>0)$$

$$e^{\ln y} = e^{2\ln x + C}$$

$$y = Ax^2. \quad A = e^C.$$



$y = ax^2$. a can be any constant.

$$y = ax^2$$

$$\frac{dy}{dx} = 2ax = \frac{2ax^2}{x} = \frac{2y}{x}$$

warning: $\frac{dy}{dx} = \frac{2y}{x}$. undefined $x=0$

Example 4.

Find curves perpendicular to the parabolas.

$$\frac{dy}{dx} = \frac{-1}{\text{slope of tangent to parabola}} = \frac{1}{2(y/x)} = -\frac{x}{2y}$$

Separate variables:

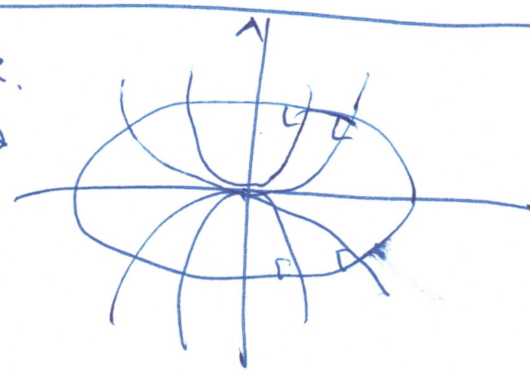
$$2y\,dy = -x\,dx$$

$$y^2 = -\frac{x^2}{2} + C.$$

solution: ellipse.

$$\frac{x^2}{2} + y^2 = C. = a^2.$$

ellipse



$y = +\sqrt{a^2 - x^2/2}$ - top halves

$y = -\sqrt{a^2 - x^2/2}.$ - bottom halves

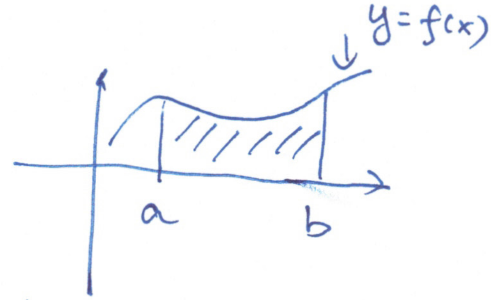
problem at $y=0$. vertical slope, solution stop!

information f' (or f'') tells us information about f.

Lecture 18. Intro to integration.

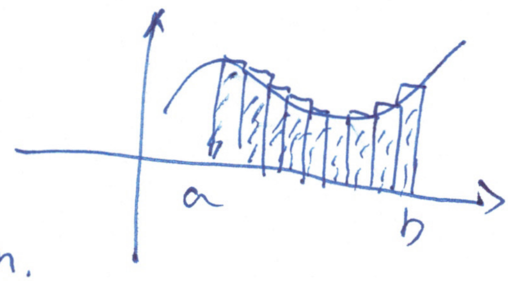
Definite integrals. $= \int_a^b f(x)\,dx$

Find area under a curve.
(Accumulative sum).

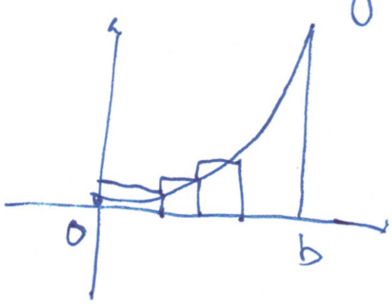


To compute the area.

1. divide into "rectangles"
2. add up the areas.
3. taking a limit as rectangles get thin.



Ex 1. $f(x) = x^2$; $a = 0$
(b arbitrary).



divide into n pieces

base length $= b/n$ (all equal intervals)

x	$f(x)$
b/n	b^2/n^2
$2b/n$	$(2b/n)^2$
$3b/n$	$(3b/n)^2$

total area (sum) of □'s

$$(b/n)(b/n)^2 + \frac{b}{n}\left(\frac{2b}{n}\right)^2 + \cdots \left(\frac{b}{n}\right)\left(\frac{nb}{n}\right)^2$$

$$= \left(\frac{b}{n}\right)^3 (1^2 + 2^2 + 3^2 + \cdots + n^2) = b^3\left(\frac{1^2+2^2+\cdots n^2}{n^3}\right)$$

$$\frac{1}{3}n^3 < 1^2 + 2^2 \cdots + (n-1)^2 + (n)^2 < \frac{1}{3}(n+1)^2(n+1)$$

top view

n — $(n-1)(n-1)$

n

side view

small cone and big cone

Vol $= \frac{1}{3}$ base · height.

Divide by n^3 $\frac{1}{3} < \frac{1^2+2^2 \cdots + n^2}{n^3} < \frac{1}{3}\left(1+\frac{1}{n}\right)^3$

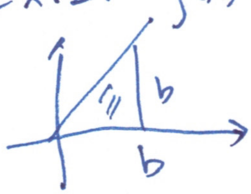
$\downarrow$ $n \to \infty$. $\frac{1}{3}$

Total area under x^2 =

$$\boxed{\int_0^b x^2\,dx = \frac{1}{3}b^3}$$

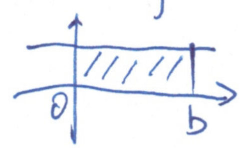
$$\frac{1}{n^3}\sum_{i=1}^{n} i^2 \underset{n\to\infty}{\longrightarrow} \frac{1}{3}$$

Ex. 2. $f(x) = x$



Area $= \frac{1}{2} b \cdot b = \frac{1}{2} b^2$.

Ex 3. $f(x) = 1$



Area $= b \cdot 1 = b$

Pattern

$f(x)$	$\int_0^b f(x)\,dx$
x^2	$b^3/3$
$x = x^1$	$b^2/2$
$1 = x^0$	$b^1/1$

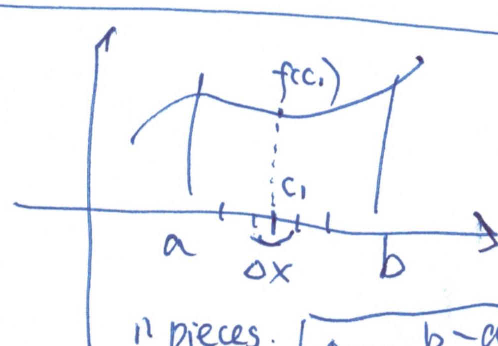
area under the curve between 0 & b.

guess: $f(x) = x^3$

$$\int_0^b x^3\,dx = b^4/4$$

Notation (Riemann Sums)

General procedure for definite integrals.



n pieces. $\boxed{\Delta x = \frac{b-a}{n}}$

Pick any hight of f

in each interval

$$\sum_{i=1}^{n} \underbrace{f(c_i)}_{\text{height}} \underbrace{\Delta x}_{\text{base}} \longrightarrow \int_a^b f(x)\,dx \quad (\Delta x \to 0)$$

Riemann Sum

Integrals as cumulative sums

t time in years

$f(t)$ \$/year. borrowing rate

$\Delta t = \frac{1}{365}$ yrs

In day 45 $\left(t = \frac{45}{365}\right)$

$f\left(\frac{45}{365}\right)\Delta t = f\left(\frac{45}{365}\right)\frac{1}{365}$

$$\sum_{i=1}^{365} f\left(\frac{i}{365}\right)\Delta t \Longrightarrow \int_0^1 f(t)\,dt$$

total borrowed

interest rate

P after tim T

you own. Pe^{rT}:

r: rate (as/yr)

$$\sum_{i=1}^{365} \left(f\left(\frac{i}{365}\right)\Delta t\right) e^{r\left(1-\frac{i}{365}\right)} \longrightarrow \int_0^1 e^{r(1-t)} f(t)\,dt$$

$T = 1 - t$

Derivative & Notation Cheat Sheet (Disambiguation Table)

Notation	Precise meaning
f	The function itself (mapping $f : \mathbb{R} \to \mathbb{R}$).
$f(x)$	Value of f at input x (a number).
f'	The derivative function of f (a new function).
$f'(x)$	Value of the derivative at x.
$f'(-x)$	Value of the derivative at $-x$ (this is *not* "the derivative of $f(-x)$").
$\dfrac{df}{dx}$	Derivative function (Leibniz notation); equivalent to f'.
$\left.\dfrac{df}{dx}\right\|_{x=a}$	Derivative evaluated at $x = a$; equivalent to $f'(a)$.
$\dfrac{d}{dx} f(x)$	Apply the derivative operator to the expression $f(x)$; equals $f'(x)$.
$\dfrac{d}{dx} f(-x)$	Derivative of the composite function $x \mapsto f(-x)$; equals $-f'(-x)$.
$\left.\dfrac{df}{dx}\right\|_{x=-x}$	Derivative evaluated at $x = -x$; equivalent to $f'(-x)$.

Key disambiguation: $f'(-x) \neq \dfrac{d}{dx} f(-x)$ in general. The left is "differentiate first, then evaluate at $-x$"; the right is "differentiate the composite $x \mapsto f(-x)$".

§. Fundeametal Therem. of calculus (FTC1)

If $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

$F = \int f(x)\,dx$

Notation

$F(b) - F(a)$

$F(x)\Big|_a^b$

$F(x)\Big|_{x=a}^{x=b}$

Ex1. $F(x) = x^3/3$

$F'(x) = x^2$

$\Rightarrow$ (FTC) $\int_a^b x^2\,dx = F(b) - F(a) = \frac{b^3}{3} - \frac{a^3}{3}$

$\int_0^b x^2\,dx = \frac{x^3}{3}\Big|_0^b = \frac{b^3}{3}$ $(a=0)$

Ex2. Area under one hump of $\sin x$

$$\int_0^\pi \sin x\,dx = (-\cos x)\Big|_0^\pi$$

$$= -\cos\pi - (-\cos 0)$$

$$= -(-1) - (-1) = 2$$

Ex3.

$$\int_0^1 x^{100}\,dx = \frac{x^{100}}{101}\Big|_0^1 = \frac{1}{101}$$

Intuitive intepretation

$x(t)$ position at time t

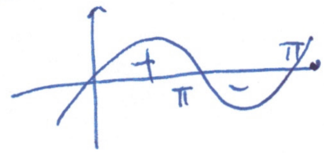
$x'(t)$ rate of change, speed $= \frac{dx}{dt} = v(t)$

$$\int_a^b \underbrace{v(t)\,dt}_{\text{speedometer}} = \underbrace{x(b) - x(a)}_{\text{distance travelled}}$$

$$\underbrace{\sum_{i=1}^{n} v(t_i)\,\underbrace{\Delta t}_{1\,\text{sec}}}_{\text{distance travelled in ith second}} \cong \int_a^b v(t)\,dt \underset{\text{exactly}}{=} x(b) - x(a)$$

(Riemann Sum)

Extend integration to the case $f<0$. (or $f>0$)



Ex. $\int_0^{2\pi} \sin x\,dx = (-\cos x)\Big|_0^{2\pi} = -\cos 2\pi - (-\cos 0)$

$= -1 - (-1)$

$= 0$

true geometric interp. of definite integral

is + area above x-axis

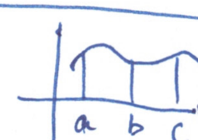
minus area below x-axis

Properties of integrals.

1. $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

2. $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$

3. ~~$a<b<c$~~ (because 5)



$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

4. $\int_a^a f(x)\,dx = 0 \quad (= F(a) - F(a))$

5. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$F(b) - F(a) = -(F(a) - F(b))$

6. (Estimation)

If $f(x) \le g(x)$. then

$\int_a^b f(x) \le \int_a^b g(x)\,dx \quad (a<b)$

Example of estimation.

$e^x \geq 1,\ x \geq 1$ (start)

$\int_0^b e^x dx > \int_0^b 1 dx$

$\int_0^b e^x dx = e^x \Big|_0^b = e^b - e^0 = e^b - 1$

$\int_0^b 1 dx = b$

$e^b - 1 \geq b \Leftrightarrow e^b \geq 1 + b \quad b \geq 0$

Repeat. $e^x \geq 1 + x,\ x \geq 0$

$\int_0^b e^x dx \geqslant \int_0^b (1+x) dx = (x + \frac{x^2}{2}) \Big|_0^b$

$e^b - 1 \qquad = b + b^2/2$

hence $e^b - 1 \geq b + b^2/2$ $(b \geq 0)$

$e^b \geq 1 + b + b^2/2$

Change of variables.

(= substitution)

$$\int_{u_1}^{u_2} g(u) du = \int_{x_1}^{x_2} g(u(x)) u'(x) dx$$

$u = u(x)$

$du = u'(x) dx$

$u_1 = u(x_1).\ u_2 = u(x_2)$

$\boxed{\text{Only works if } u' \text{ doesn't change sign}}$

单调

Example $\int_1^2 (x^3+2)^5 x^2 dx = \int_3^{10} u^5 \frac{1}{3} du = \frac{1}{18} u^6 \Big|_{u=3}^{u=10}$

$\frac{1}{3} du = x^2 dx \qquad u = x^3 + 2. \quad du = 3x^2 dx \qquad = \frac{1}{18}(10^6 - 3^6)$

Warning

$\int_{-1}^1 x^2 dx \neq \int_1^1 u \frac{1}{2\sqrt{u}} du = 0.$

$u = x^2,\ du = 2x dx$

$u = x^2 \begin{cases} x < 0 \\ x > 0. \end{cases}$

FTC1. if $F' = f$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

used to evaluate integrals

$$F(b) - F(a) = \int_a^b f(x)\,dx$$

use f to understand F

Info about F' => Info about F

Compare FTC with MVT

$\Delta F = F(b) - F(a)$, $\Delta x = b - a$

$$\Delta F = \int_a^b f(x)\,dx \quad (\text{FTC}_1)$$

$$\frac{\Delta F}{\Delta x} = \boxed{\frac{1}{b-a}\int_a^b f(x)\,dx}$$

Average (f)

Inequalities

$$(\min F')\Delta x \le \Delta F \overset{\text{FTC1}}{=} [\text{Average}(F')]\Delta x \le (\max F')\Delta x$$

$$(\min F')\Delta x \le \Delta F \underset{\text{MVT}}{=} \underbrace{F'(c)}_{\text{some } c,\ a<c<b}\Delta x \le (\max F')\Delta x$$

Exam2. $F'(x) = \dfrac{1}{1+x}$ $\quad F(0) = 1$

The MVT implies

$A < F(4) < B$

for which A, B ?

$$F(4) - F(0) = \frac{1}{1+c}4$$

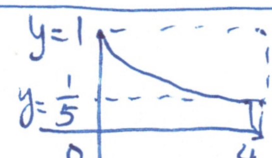
$$\frac{4}{5} < \frac{1}{1+c} < 4$$

$$\frac{4}{5} < F(4) - F(0) < 4$$

The FTC implies $\displaystyle F(4) - F(0) = \int_0^4 \frac{dx}{1+x}$

$$< \int_0^4 \frac{1}{1}\,dx = 4$$

$$> \int_0^4 \frac{dx}{5} = \frac{4}{5}$$



$y = \dfrac{1}{1+x}$

$$\text{low RS} < \int_0^4 \frac{dx}{1+x} < \text{upper RS}$$

FTC2. If f is continous and $G(x) = \int_a^x f(t)\,dt$; $(a \le t \le x)$

then $G'(x) = f(x)$

$G(x)$ solves the differential eq
$$\begin{cases} y' = f \\ y(a) = 0 \end{cases}$$

Example $\frac{d}{dx}\left[\int_1^x \frac{dt}{t^2}\right] = \frac{1}{x^2}$

$\int_1^x \frac{dt}{t^2} = G(x)$, $G'(x) = f(x)$

$\int_1^x f(t)\,dt$, $f(t) = \frac{1}{t^2}$

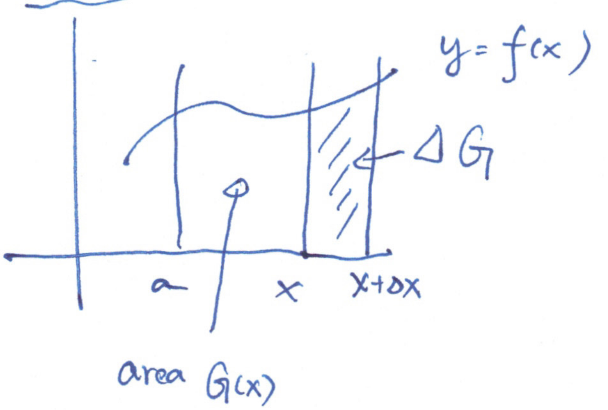
Check

$\int_1^x t^{-2}\,dt = -t^{-1}\Big|_1^x = -\frac{1}{x} - (-1)$

$\frac{d}{dx}\left(1 - \frac{1}{x}\right) = +\frac{1}{x^2}$

$1 - \frac{1}{x} = G(x)$

PROOF OF FTC2.



$\Delta G \approx \underbrace{\Delta x}_{\text{base}}\ \underbrace{f(x)}_{\text{ht.}}$

$\lim\limits_{\Delta x \to 0} \frac{\Delta G}{\Delta x} = f(x)$ (f cts)

QED

PROOF OF FTC1

(assume f is continous)

Start with $F' = f$

Define $G(x) = \int_a^x f(t)\,dt$

FTC2 $\Rightarrow G'(x) = f(x)$

$F'(x) = G'(x)$ $\underset{\text{use MVT}}{\boxed{\Rightarrow}}$ $F(x) = G(x) + C$

C = constant

Hence $F(b) - F(a)$

$$= (G(b) + C) - (G(a) + C)$$

$$= G(b) - G(a) = \int_a^b f(x)\,dx - 0$$

Q.E.D.

?

Ex. $L'(x) = \frac{1}{x}$; $L(1) = 0$

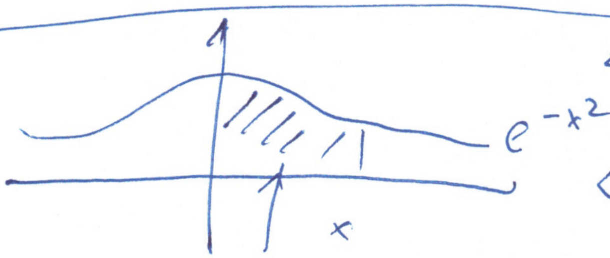
FTC2

$$L(x) = \int_1^x \frac{1}{t}\,dt$$

"New" functions.

$$y' = e^{-x^2}, \quad y(0) = 0$$

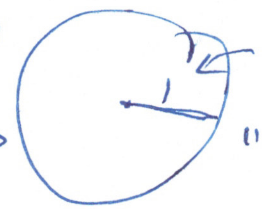
$$F(x) = \int_0^x e^{-t^2}\,dt$$



area. $F(x)$.

$F(x)$ cannot be expressed ... of loss, exp, ...

$\Longleftrightarrow$



area π

"New" number

π = not the root of an algebra equation with rational coeff