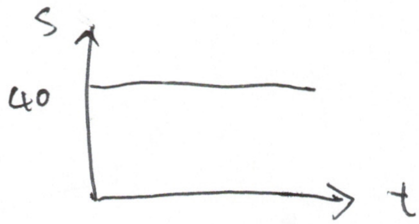
Hight of calculus : Big picture. Lecture1

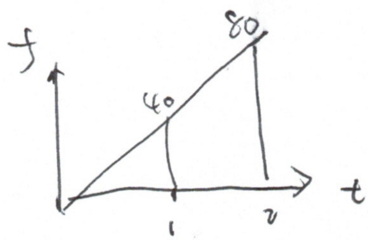
1. Distance speed

Height Slope

$f(t)$ $y(x)$ $s = \frac{df}{dt}$ or $\frac{dy}{dx}$

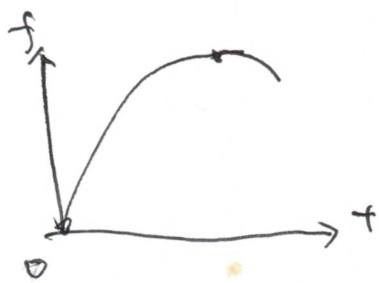


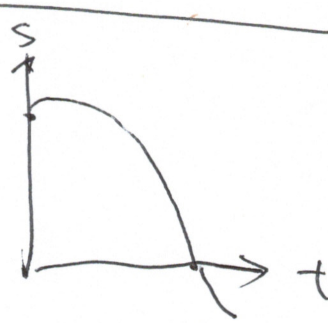
speed
slope $= \frac{\text{up}}{\text{across}} = \frac{\Delta f}{\Delta t} = s$



distance
height $= f = st$

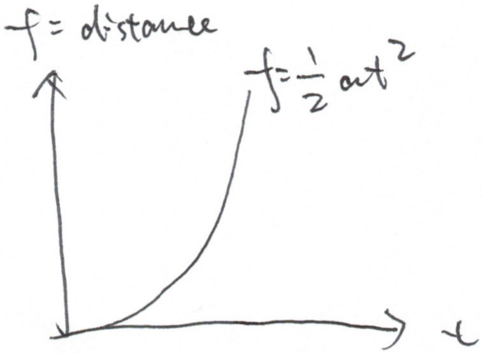
$y = sx$

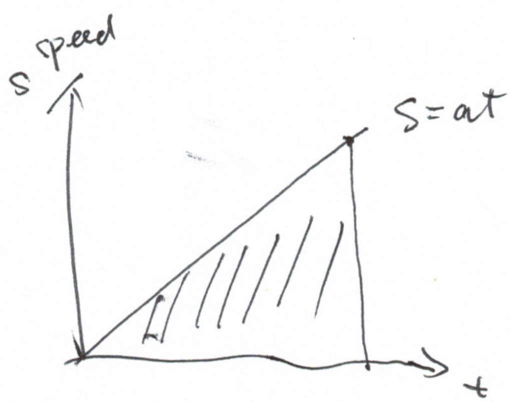




Differential calculus: $f_1 \to f_2$

Integral calculus: $f_2 \to f_1$





$f = \frac{1}{2}at^2$ $\frac{df}{dt} = at$

Big picture: Derivatives. Lecture 2.

(1) → (2)

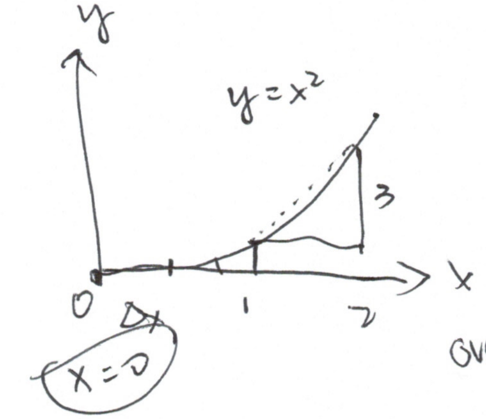
Distance → speed $\frac{df}{dt}$

Height → slope. $\frac{dy}{dx}$

$y = x^n \quad \frac{dy}{dx} = nx^{n-1}$

$y = \sin x \quad \frac{dy}{dx} = \cos x$

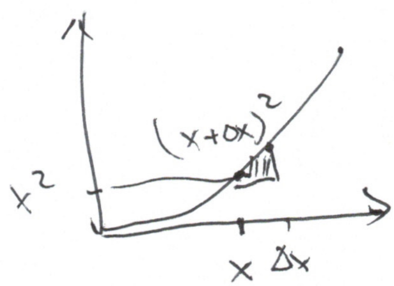
$y = e^x \quad \frac{dy}{dx} = e^x$



slope average $= \frac{3}{1}$ (calculus is look at istant)

average over the 1st piece $= \frac{(\Delta x)^2}{\Delta x} = \Delta x$

instance slope $= 0$

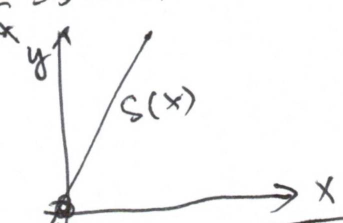


$$\frac{\Delta y}{\Delta x} = \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \frac{2x\Delta x + \Delta x^2}{\Delta x}$$

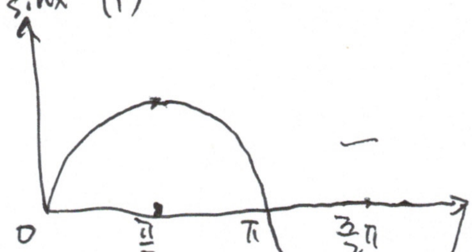
$$= 2x + \Delta x$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 2x$$

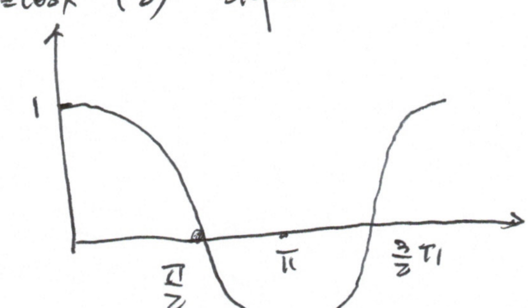
$\frac{dy}{dx} = s = 2x$



$y = \sin x$ (1)

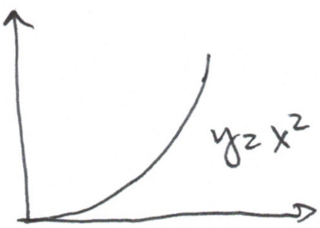


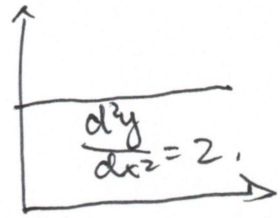
$y = \cos x$ (2) slope



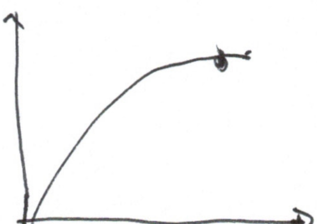
Lecture 3. Max Min and $\frac{d^2y}{dx^2}$

(1)	(2)	(3)
Distance $f(t)$	speed $s = \frac{df}{dt}$	acceleration $\frac{d^2y}{dx^2}$
Height	slope	Bending



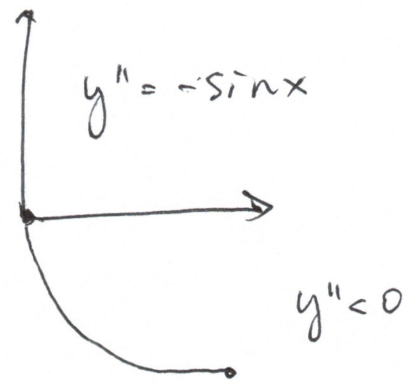


Bending up (convex)

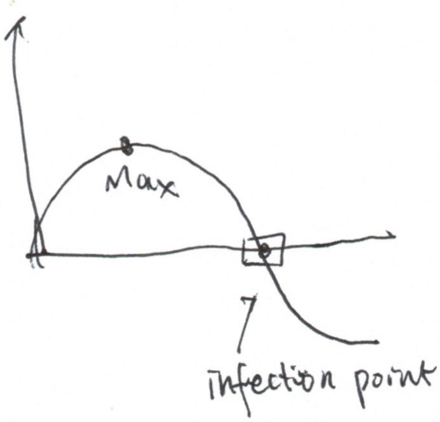


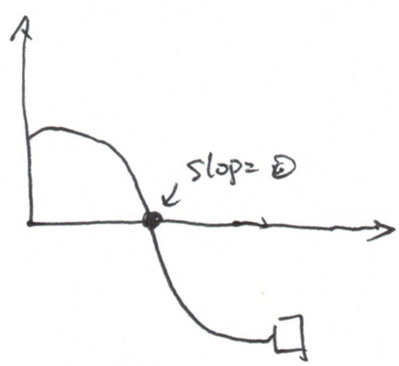
$y = \sin x$

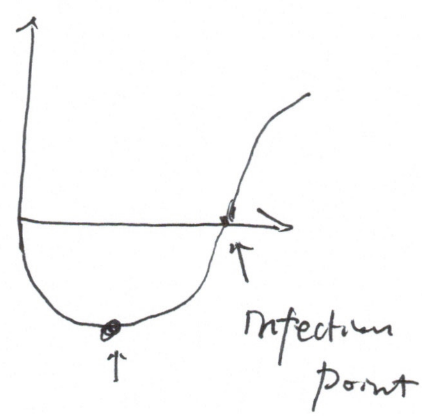
$y' = \cos x$



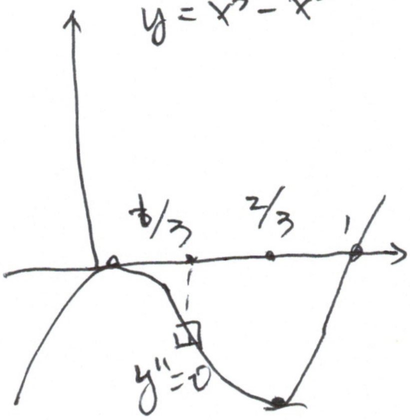
Bending down
concave







$y = x^3 - x^2$ $\quad y' = 3x^2 - 2x$ $\quad y'' = 6x - 2$



Max/Min

$y' = 3x^2 - 2x = 0$

$3x^2 = 2x$

$x = 0$

$x = 2/3$

Max

$x = 0 \quad y'' = -2$ (bending down)

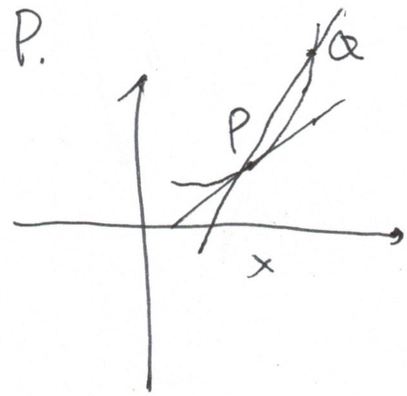
Min

$x = \frac{2}{3} \quad y'' = 2$ (bending up)

inflection point: $y'' = 0 \rightarrow x = \frac{1}{3}$

Unit 1. Lecture 1: Rate of Change

Def. $f'(x_0)$ the derivative of f at x_0. is the slop of the tangent line to $y=f(x)$ at P.



$y-y_0=m(x-x_0)$

point $y_0=f(x_0)$

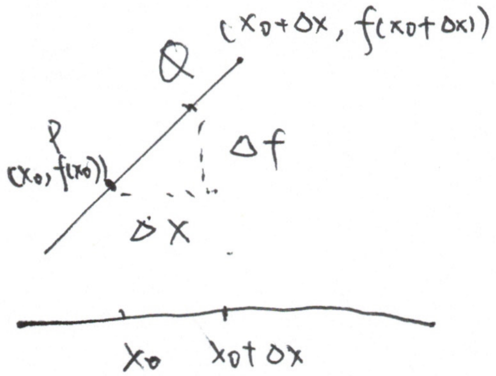
slope $m=f'(x_0)$

Intuition:

Tangent Line =

Limit of secant lines PQ

As $Q\to P$ (P is fixed)



$$m=\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}$$

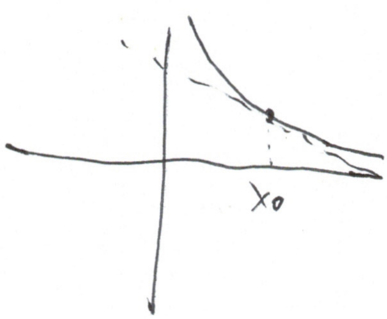
$\Vert$

slope of tangent.

$$\boxed{f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}}$$

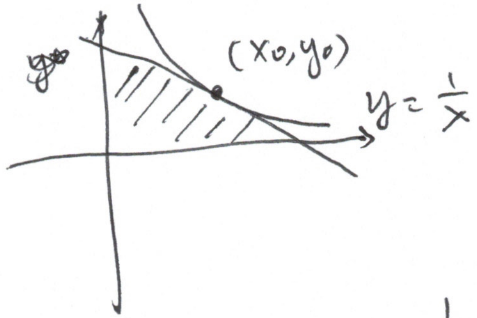
Example $f(x)=\frac{1}{x}$

$f'(x_0)=-\frac{1}{x^2}$



$$\frac{\Delta f}{\Delta x}=\frac{\frac{1}{x_0+\Delta x}-\frac{1}{x_0}}{\Delta x}=\frac{1}{\Delta x}\left(\frac{-\Delta x}{(x_0+\Delta x)\,x_0}\right)$$

$$=-\frac{1}{x_0^2}$$



find the area size of trangle shape by x/y/ tangent line of $y = \frac{1}{x}$.

$$y - y_0 = -\frac{1}{x_0^2}(x - x_0)$$

Find x-intersect : $0 - \frac{1}{x_0} = -\frac{1}{x_0^2}(x - x_0)$

$$x = 2x^0$$

x - intersect : $y = 2y_0$ (by symetric)

$$= 2\frac{1}{x_0}$$

Area $= \frac{1}{2}(2x_0)(2y^0) = 2x_0y_0 = 2.$

More notations

$y = f(x) \quad \Delta y = \Delta f$

$$\underset{\text{newtons}}{f'} = \underbrace{\frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}y = \frac{d}{dx}f}_{\text{Lebniz}}$$

$f(x) = x^n, \quad n = 1, 2, 3$

$\frac{d}{dx}x^n \qquad \frac{\Delta f}{\Delta x} = \frac{(x+\Delta x)^n - x^n}{\Delta x}$

$(x+\Delta x)^n = (x+\Delta x) \cdots (x+\Delta x)$

$= x^n + nx^{n-1}\Delta x + O(\Delta x^2)$

$= \frac{1}{\Delta x}(x^n + nx^{n-1}\Delta x + O(\Delta x)^2 - x^n)$

$= \frac{1}{\Delta x}(nx^{n-1}\Delta x + O(\Delta x)^2)$

$= nx^{n-1} + O(\Delta x)$

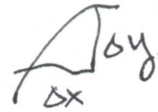
polonomio.

Lecture 2: Limit.

Recap: Derivitive = slope of tangent line.

What is Derivitive: Rate of change

$y = f(x)$



$$\frac{\Delta y}{\Delta x} \to \frac{dy}{dx}$$

average   instantanous

Rate of change

Examples

1. q = charg. $\frac{dq}{dt}$ = current

2. s = distance $\frac{ds}{dt}$ = speed

$h = 80 - 5t^2$. $\frac{\Delta h}{\Delta t} = \frac{0-80}{5} = -20$.

$t = 4$

$\frac{dh}{dt} = 0 - 10t$. $t = 4$. $h' = -40$/sec

3 T = tempertune

$\frac{dT}{dx}$ = temperature gradient.

4. Sensitivity of measurement.

Limits + Continuity

1. Easy limit

$$\lim_{x \to 4} \frac{x+3}{x^2+1} = \frac{7}{17}.$$

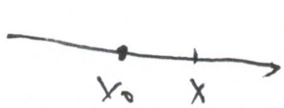
2. Derivatives are always harder:

$$\lim_{x \to x_0} \frac{f(x+\Delta x) - f(x)}{x - x_0}$$. always need eliminating

②.

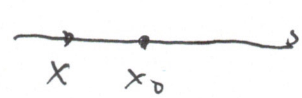
$\lim_{x \to x_0^+} f(x)$ = right hand limit.

$\begin{bmatrix} x \to x_0 \\ x > x_0 \end{bmatrix}$



$\lim_{x \to x_0^-} f(x)$ = left hand limit.

$\begin{bmatrix} x \to x_0 \\ x < x_0 \end{bmatrix}$



Example:

$$f(x)=\begin{cases} x+1, & x>0 \\ -x+2, & x<0 \end{cases}$$

$\lim_{x\to 0^+} = \lim_{x\to 0} x+1 = 1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} -x+2 = 2.$

Define f is $\boxed{\text{continous}}$ at x_0 means

at x_0 means

$\lim_{x\to x_0} f(x) = f(x_0)$ at x_0

$x \neq x_0$

"easy" limit

means,

1. $\lim_{x\to 0} f(x)$ exists, both from L. R

2 $f(x)$ is defined

3. they are the same.

Jump discontinuaty.

lim from left and right exists but are not equal. (above example)

Removable discontinuaty.

lim ~~are~~ from left rig are equal

Example: $g(x) = \frac{\sin x}{x}$: $g(0)$??

$$\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$$

$$h(x) = \frac{1-\cos x}{x}$$

$$\boxed{\lim_{x\to 0} \frac{1-\cos x}{x} = 1}$$

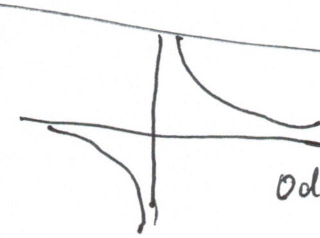
Infinite discontinuaty

$\lim_{x\to 0^+} \frac{1}{x} = \infty$

$\lim_{x\to 0^-} \frac{1}{x} = -\infty$

$\lim_{x\to 0} \frac{1}{x^2} = -\infty$

$y = \frac{1}{x}$



odd

$y' = \frac{1}{x^2}$

other (Ugly) Discontinuity

$y = \sin\frac{1}{x}$ as $x \to 0$

no $\lim_{x \to 0} y$

Theorem (Diff = CTS)

if f is differentiable at x_0, then f is continuous a x_0.

Proof

$$\lim_{x \to x_0} f(x) - f(x_0) \stackrel{?}{=} 0$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} (x - x_0) = f'(x_0)\, 0 = 0$$

$x \neq x_0$

Lecture 3: Derivatives

1. specil $f'(x)$
2. General $(u+v)' = u' + v'$

   $(cu)' = cu'$

A: $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$

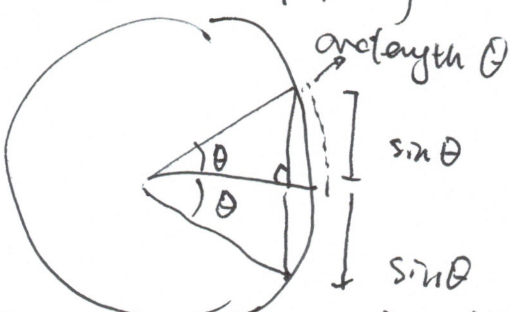
B: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

$$\frac{d}{dx}\sin x = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \frac{\sin x \cos\Delta x + \cos x \sin\Delta x - \sin x}{\Delta x} = \underbrace{\frac{\sin x \cos x - \sin x}{\Delta x}}_{0} + \cos x \underbrace{\boxed{\frac{\sin\Delta x}{\Delta x}}}_{1}$$

$= \cos x$

$$\frac{d}{dx}\cos x = -\sin x$$

B. geomertry proof

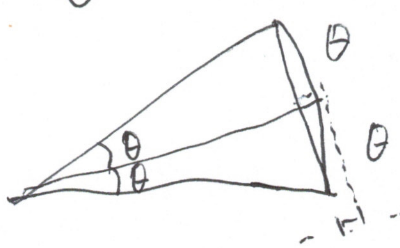


$$\frac{2\sin\theta}{2\theta \text{ (arclength)}} = \frac{\sin\theta}{\theta} \to 1$$

$\theta \to 0$

intuition: Short curves are nearly streight.

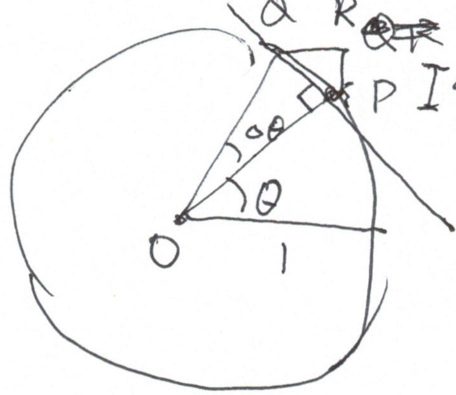
A. $\frac{1-\cos\theta}{\theta}$ as $\theta \to 0$



Geom Prof of

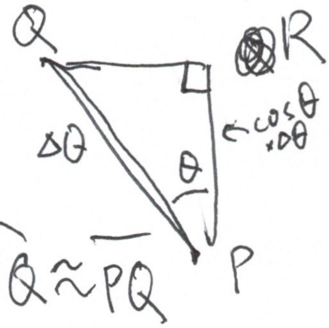
$$\frac{d}{dy}\sin\theta = \cos\theta$$

(all θ)



$\Delta y = \cancel{PQ}\ PR$

$$\Delta y = \cancel{PQ}\,\cancel{PQ}\ PR \approx \Delta\theta \cos\theta$$



$\widehat{PQ} \approx \overline{PQ}$

$PQ \approx \Delta\theta$

$PQ \underset{\approx}{\perp} OP$. $\cancel{PQ}$ PR is vertical.

$\angle QPR \approx \theta$

$$\frac{\Delta y}{\Delta\theta} \approx \cos\theta$$

$$\lim_{\Delta\theta \to 0} \frac{\Delta y}{\Delta\theta} = \cos\theta$$

General rules:

$(uv)' = v'u + \cancel{v'u}\ u'v$ ← change one at a time

$$(u/v)' = \frac{u'v - uv'}{v^2}.$$

Lecture 4.

product Rule:

$$\boxed{(uv)' = u'v + uv'}$$

$$\frac{d}{dx}(x^n \sin x) = n x^{n-1} \sin x + x^n \cos x$$

Prof $\Delta(uv)$

$= u(x+\Delta x)v(x+\Delta x) - v(x)u(x)$

$= (u(x+\Delta x) - u(x))v(x+\Delta x) + u(x)($

$\quad v(x+\Delta x) - v(x))$

$= \Delta u\, v(x+\Delta x) + \Delta v \cdot u(x)$

$$\lim_{\Delta x \to 0} \frac{\Delta(ux)}{\Delta x} = \frac{\Delta u}{\Delta x} v(x+\Delta x) + u\frac{\Delta v}{\Delta x}$$

$$= \frac{du}{dx} v(x) + \frac{dv}{dx} u(x)$$

Quotient Rule

$$\boxed{\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}}$$

$\downarrow$ use product rule

$$\left(\frac{u}{v}\right)' = \left(u\frac{1}{v}\right)' = \frac{u'}{v} - \frac{uv'}{v^2}$$

$$= \frac{u'v - uv'}{v^2}$$

$$\Delta\left(\frac{u}{v}\right) = \frac{u+\Delta u}{v+\Delta v} - \frac{u}{v}$$

$$= \frac{uv + (\Delta u)v - uv - u\Delta v}{(v+\Delta v)v}$$

$$= \frac{(\Delta u)v - u\Delta v}{(v+\Delta v)v}$$

$$\frac{\Delta\left(\frac{u}{v}\right)}{\Delta x} = \frac{\frac{\Delta u}{\Delta x}v - u\frac{\Delta v}{\Delta x}}{(v+\Delta v)v}$$

$\downarrow \Delta x \to 0$

$$\frac{d\left(\frac{u}{v}\right)}{d(x)} = \frac{\frac{du}{dx}\cdot v - u\frac{dv}{dx}}{v \cdot v}$$

$u=1$

$$\frac{d}{dx}\left(\frac{1}{v}\right) = \frac{-v'}{v^2} = -v^{-2}v'$$

$u=1$

$v=x^n$

$$\frac{d}{dx}\left(\frac{1}{x^n}\right) = \frac{-nx^{n-1}}{\dots} \quad -n-1$$

Composition rule.

i.e. $y = (\sin t)^{10}$

Pf. $\dfrac{\Delta y}{\Delta t} = \dfrac{\Delta y}{\Delta x} \cdot \dfrac{\Delta x}{\Delta t}$

$\Delta t \to 0 \downarrow$

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

chain rule = Differentiation of a composition is a product.

Ex. $\dfrac{d}{dt}(\sin t)^{10}$

$= 10x^9 \cos t$

$= 10 \sin t^9 \cos t$

Ex2 $\dfrac{d}{dt} \sin(10t)$

$= 10 \cos 10t$

Higher derivatives

$u = u(x) \quad u' \quad u'' = (u')' \quad u''' \quad u^{(4)}$

$u = \sin(x) \quad u' = \cos(x), \quad u'' = -\sin(x), \quad u''' = -\cos(x) \quad u^{(4)} = \sin(x)$

operator $\quad (D = \dfrac{d}{dx}).$

$u' = \dfrac{du}{dx} = \dfrac{d}{dx} u = Du$

$u'' = \dfrac{d}{dx} \dfrac{du}{dx} = \dfrac{d}{dx} \dfrac{d}{dx} u$

$= \left(\dfrac{d}{dx}\right)^2 u = \dfrac{d^2}{(dx)^2} u = \dfrac{d^2 u}{dx^2} = D^2 u.$

Ex. $D^n x^n$.

$D x^n = n x^{n-1}$

$D^2 x^n = n(n-1) x^{n-2}.$

$D^3 x^n = n(n-1)(n-2) x^{n-3}$

$\vdots$

Lecture 5. Implicit differentiation.

Example 1. $\frac{d}{dx} x^a = a x^{a-1}$

$a = m/n$ $\quad m, n$ is Integers

$y = x^{m/n}$ (1)

$y^n = x^m$ (2) implicit.

$$\frac{d}{dx} y^n = \frac{d}{dx} x^m$$

$$\left(\frac{d}{dy} y^n\right) \frac{dy}{dx} = m x^{m-1}$$

$$n y^{n-1} \frac{dy}{dx} = m x^{m-1}$$

$$\frac{dy}{dx} = \frac{m x^{m-1}}{n y^{n-1}} = \frac{m}{n} \frac{x^{m-1}}{(x^{m/n})^{n-1}} = a\, x^{m-1-(n-1)\frac{m}{n}} = a x^{a-1}$$

Example 2. $x^2 + y^2 = 1$

(explicit)

$y = +\sqrt{1-x^2}$

$y' = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x)$

$= -\frac{x}{\sqrt{1-x^2}}$

implicit.

$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \cancel{0}\, -\frac{x}{y} = -\frac{x}{\sqrt{1-x^2}}$

Examp 4. $y^4 + xy^2 + 2 = 0$

$y^2 = \frac{-x \pm \sqrt{x^2 - 4(-2)}}{2}$

$y = \pm\sqrt{\frac{-x \pm \sqrt{x^2+8}}{2}}$ $\quad (*)$

implicit

$4y^3 y' + y^2 + 2yxy' = 0$

$(4y^3 + 2xy) y' = -y^2$

$$\boxed{y' = -\frac{y^2}{4y^3 + 2xy}}$$

$x = 1$ $y = 1$ is a solution to ↑

$y'(1) = -\frac{1}{4+2} = -\frac{1}{6}$

if $x = 2$

we are stuck using $(*)$ to find y

then find y'

Inverse functions

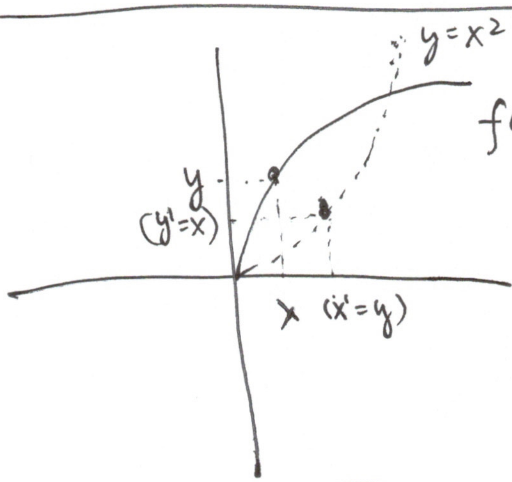
Ex. $y=\sqrt{x}$, $x>0$, $y^2=x$

$g(y)=x$ $g(y)=y^2$

In general

$y=f(x)$ $g^{-1}(y)=x$

$g(f(x))=x$. $g=f^{-1}$ $f=g^{-1}$



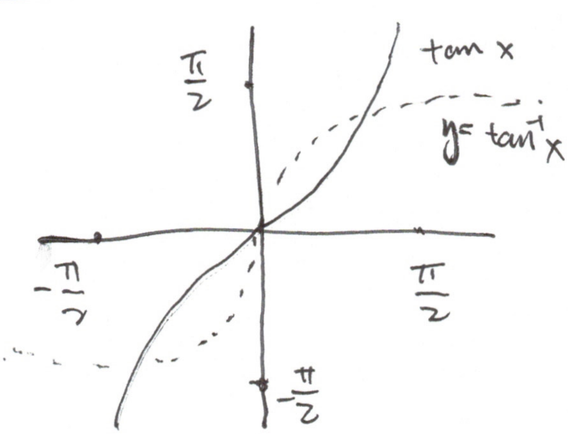
f and f^{-1} switch x and y. (reflect) thru diagnal

$g(y)=x$

$g(x)=?$

implicit differentiation allow us to find derivative of any inverse funtion, provided we know derivative of the function

Ex: $y=\tan^{-1}x$ = $\tan y=x$



$$\frac{d}{dy}\tan y=\frac{d}{dy}\frac{\sin y}{\cos y}=\dots=\frac{1}{\cos^2 y}$$

$$\frac{d}{dx}(\tan y=x)$$

$$\frac{d}{dy}\frac{1}{\cos^2 y}y'=1$$

$$y'=\cos^2 y$$

$y'=\cos^2(\tan^{-1}x)$ ← too complicated

$\tan y=x$ $\sqrt{1+x^2}$, x, y, 1

$\downarrow$

$\cos y=\frac{1}{\sqrt{1+x^2}}$

$y'=\frac{1}{1+x^2}$

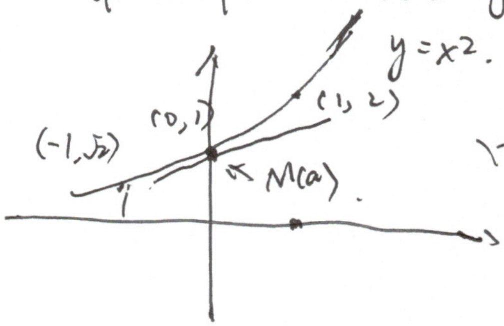
Lecture 6. Exponents and Logarthms.

$$a^{m+n} = a^m \cdot a^n$$

$$(a^m)^n = a^{mn}$$

$$a^{m/n} = \sqrt[n]{a^m}.$$

a^x is defined for all numbers by contunuity. ie. $a^{\sqrt{2}}$.



$\boxed{\text{Goal: } \frac{d}{dx} a^x = ?}$

$$\lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{a^x}(a^{\Delta x} - 1)}{\Delta x} a^x$$

$$\boxed{\frac{d}{dx} a^x = a^x \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}}$$

$$M(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}$$

$$\frac{d}{dx} a^x = M(a)\, a^x$$

plug in 0

$$\left.\frac{d}{dx} a^x\right|_{x=0} = M(a).$$

$M(a)$ is the slope for a^x at $x=0$

What is $M(a)$?

Define base $\underline{\underline{e}}$ as the unique number

So $M(e) = 1$

$$\boxed{\frac{d}{dx} e^x = e^x}$$

$M(e)=1$

$$\left.\frac{d}{dx} e^x\right|_{x=0} = 1$$

slope 1 at $x=0$

Why e exists?

$f(x) = 2^x$. $f'(0) = M(2)$.

Stretch by k.

$f(kx) = 2^{kx} = (2^k)^x = b^x$

resaling →

$b = 2^k$.

$$\frac{d}{dx} b^x = \frac{d}{dx} f(kx) = k f'(kx).$$

$$\left.\frac{d}{dx} b^x \right|_{x=0} = k f'(0) = k M(2)$$

$b = e$ when $k = 1/M(2)$

Nature Log

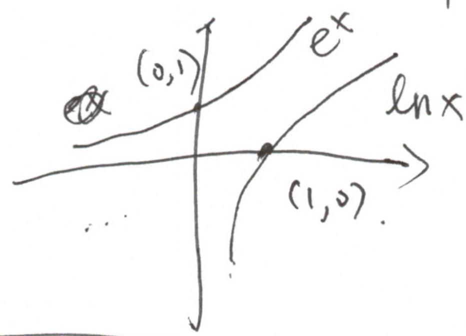
$W = \ln x$

$y = e^x \iff ly\cancel{x}g = x$

defines "ln"

$\ln(x_1 x_2) = \ln(x_1) + \ln(x_2)$

$\ln 1 = 0$; $\ln e = 1$.



$x > 0$.

to find $\boxed{\frac{d}{dx} \ln x = \frac{1}{x}}$

$w = \ln x \Rightarrow e^w = x$

$(e^{\ln x} = x)$

$$\frac{d}{dx} e^w = \frac{d}{dx} x = 1$$

$$\left(\frac{d}{dw} e^w\right) \frac{dw}{dx} = 1$$

$$\frac{dw}{dx} = \frac{1}{e^w} = \frac{1}{x}$$

To differentiate any exp: two methods.

Method 1. $\frac{d}{dx} a^x$

use base e. $a^x = (e^{\ln a})^x = e^{x \ln a}$.

$\frac{d}{dx} 2^x = \ln 2 \, 2^x$

$\frac{d}{dx} 10^x = \ln 10 \, 10^x$

$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a}$

$= (\ln a) e^{x \ln a}$

$$\boxed{\frac{d}{dx} a^x = (\ln a)\, a^x}$$

$M(a) = \ln a$.

Method 2. $\boxed{\text{Logarithmic differentiation}}$

$\frac{d}{dx} u$?? . $\frac{d}{dx} \ln u$.

$= \left(\frac{d \ln u}{du}\right)\left(\frac{du}{dx}\right) = \frac{1}{u} \frac{du}{dx}$

$$\boxed{(\ln u)' = u'/u.}$$

$\frac{d}{dx} a^x$ | $u = a^x$

$\ln u = x \ln a$.

$(\ln u)' = \ln a$

$u'/u = \ln a$

$u' = \ln a \, u$

$\frac{d}{dx} a^x = (\ln a)\, a^x$

Ex 2. Moving exponents.:

$v = x^x \quad \ln v = x \ln x$

$(\ln v)' = \ln x + x \cdot \frac{1}{x}$

$= \ln x + 1$

$v'/v = 1 + \ln x$

$v' = x^x (1 + \ln x)$

$\frac{d}{dx} x^x = x^x (1 + \ln x)$

Ex 3. $\lim_{n \to \infty} (1 + \frac{1}{n})^n \qquad \Delta x = \frac{1}{n} \to 0$

$\ln ((1 + \frac{1}{n})^n) = n \ln (1 + \frac{1}{n}) = \frac{1}{\Delta x} (\ln (1 + \Delta x) - \ln 1)$

$\downarrow \; \Delta x \to 0$

$\frac{d}{dx} \ln x \Big|_{x=1} = \frac{1}{x} = 1$

$\lim_{n \to \infty} (1 + \frac{1}{n})^n = e^{\left[\lim_{n \to \infty} \ln (1 + \frac{1}{n})^n\right]} = e^1 = e$

$\quad \| $

$\quad e$

$\approx (1 + \frac{1}{100})^{100}$ numerical approx.

Lecture 7: Exponents Cont'

$$a_k = \left(1+\frac{1}{k}\right)^k$$

$$\lim_{k\to\infty} a_k = e.$$

$$e = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k$$

$$\ln a_k = 1 \quad (k\to\infty)$$

$$e^{\ln a_k} \to e^1 = e$$

$$\parallel$$

$$a_k \qquad \left(e^{\ln a} = a\right)$$

$$\frac{d}{dx} x^r = r x^{r-1} \quad \text{all } r$$

Method 1. (base e)

$$x^r = \left(e^{\ln x}\right)^r = e^{r\ln x}$$

$$\frac{d}{dx} x^r = \left(e^{r\ln x}\right)' = e^{r\ln x}(r\ln x)'$$

$$= x^r \frac{r}{x}$$

$$= r x^{r-1}$$

Method 2.

$$u = x^r \qquad \ln u = r\ln x$$

$$\frac{u'}{u} = (\ln u)' = \frac{r}{x}$$

$$u' = u\frac{r}{x} = r x^{r-1}$$

Natural log is natural.!

Example: economics.

REVIEW OF UNIT 1.

General.

$(u+v)'$, $(cu)'$, $(uv)'$, $(u/v)'$

$\frac{d}{dx}f(u) = f'(u)\,u'(x)$ $[u = u(x)]$

implicit differentiation:

(inverse, log diff.)

specific diff

x^r, $\sin x$, $\cos x$, $\tan x$, $\sec x$

e^x, $\ln x$.

$\tan^{-1}x$, $\sin^{-1}x$

Definition of derivitives.: $f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$

Unit 2. Application of differentiation.

1. Linear Approximation.

$$\boxed{f(x) \approx f(x) + f'(x)(x-x_0)}$$

$x \approx x_0$.

curve $y=f(x)$

$\approx$ $y = f(x_0) + f'(x_0)(x-x_0)$

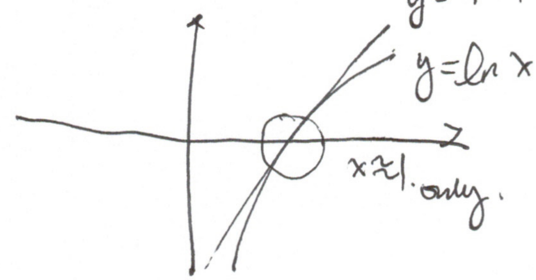
tangent line.

Example $f(x) = \ln x$, $f'(x) = \frac{1}{x}$

$x_0 = 1$, $f(1) = \ln 1 = 0$, $f'(x) = 1$,

$f(x) \approx 0 + 1 \cdot (x-1)$.

$\ln x \approx x - 1$



$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$

$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} \approx f'(x_0)$

$$\boxed{\frac{\Delta f}{\Delta x} \approx f'(x_0)}$$

Δx is small

$\Leftrightarrow$ $\Delta f \approx f'(x_0) \Delta x$

$\Leftrightarrow$ $f(x) - f(x_0) \approx f'(x_0)(x - x_0)$

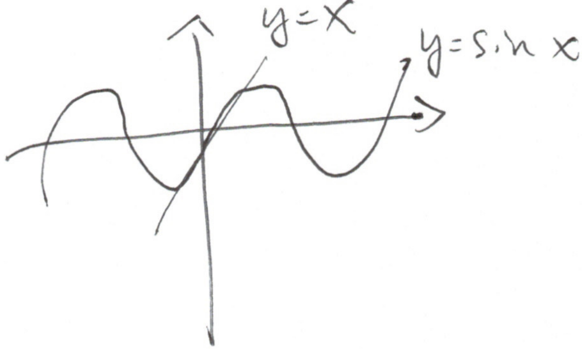
$x_0 = 0$. $f(x) = f(0) + f'(0)\, x$

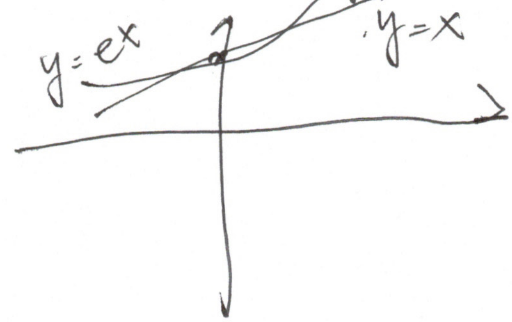
$\sin x \approx x$

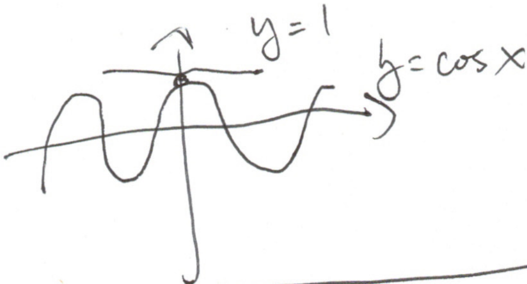
$\cos x \approx 1$

$e^x \approx 1 + x$

f'	$f(0)$	$f'(0)$
$\cos x$	0	1
$-\sin x$	1	0
e^x	1	1







$\ln(1+x) \approx x$

$(1+x)^r \approx 1+rx$

$x \approx 0$

	f'	$f(0)$	$f'(0)$
			1
	$1/(1+x)$	0	
	$r(1+x)^{r-1}$	1	r

Ex 2. $\ln 1.1$ (Hard) $\quad \ln 1.1 \approx \frac{1}{10}$ (easy)

$\ln(1+x) \approx x. \quad x = \frac{1}{10}$

Ex 3 Find linear approximation of $\frac{e^{-3x}}{\sqrt{1+x}}$ near 0

$= e^{-3x}(1+x)^{-1/2}$

$e^{-3x}(1+x)^{-\frac{1}{2}} \approx (1-3x)\left(1-\frac{1}{2}x\right)$

$= 1-3x-\frac{1}{2}x + \frac{3}{2}x^2$ ← drop quadratic term. (that's why it's called linear approximation)

$\approx 1-\frac{7}{2}x$

Ex 4.

$T' = \frac{T}{\sqrt{1-v^2/c^2}}$ $\qquad T' = T(1-u)^{-\frac{1}{2}}$

$= T\left(1+\frac{1}{2}\frac{v^2}{c^2}\right)$

$T' = T\left(1+\frac{1}{2}\frac{v^2}{c^2}\right).$

o return

Quadratic approximation.

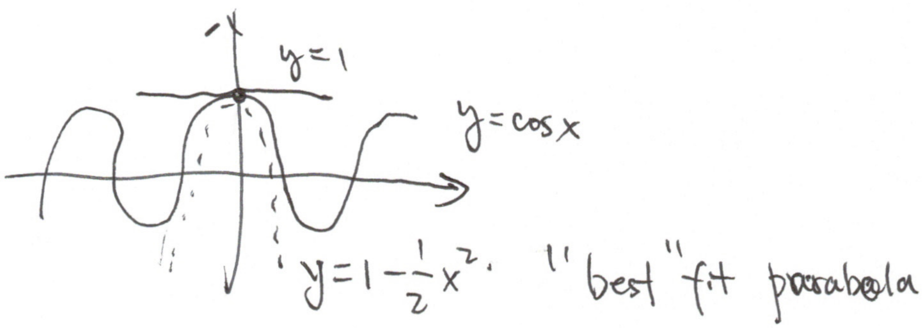
$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2.$$

$\ln(1+x) \approx x$ linear; quadratic: $\ln(1+x) \approx x - x^2/2$.

$$\ln(1.1) = \ln(1+\tfrac{1}{10}) \approx \frac{1}{10} - \frac{1}{2}\left(\frac{1}{10}\right)^2 = 0.95$$

	$f''(\ )$	$f''(0)$
$\sin x \approx x$	$-\sin x$	0
$\cos x \approx 1 - \frac{1}{2}x^2$	$-\cos x$	-1
$e^x \approx 1 + x + \frac{1}{2}x^2$	e^x	1

Geometric significance. (of quadratic appox.)



Lecture 10.

Appox. Con'?

$$T' = T(1 - v^2/c^2)^{1/2} \approx T(1 + \frac{1}{2}\frac{v^2}{c^2})$$

$$\Delta T = T - T'$$

$$\frac{\Delta T}{T} = \frac{1}{2}\frac{v^2}{c^2}.$$

Error fraction is proportional to v^2/c^2. with factor $1/2$.

Quadratic Appox.

use when Linear are not enough.

$$\boxed{f(x) = f(0) + f'(0)(x - x_0) + \frac{f''(0)}{2}(x - x_0)^2 \quad x \approx 0}$$

why $\frac{1}{2}$ $f''(0)$?

$$f(x) = a + bx + cx^2.$$

$$f'(x) = b + 2cx$$

$$f''(x) = 2c.$$

$$f(0) = a$$

$$f'(0) = b$$

$$\frac{1}{2}f''(0) = c$$

why $\frac{1}{2}''(0)$

x near 0.

$$\sin x \approx 1$$

$$\cos x \approx 1 - \frac{1}{2}x^2$$

$$e^x \approx 1 + x + \frac{1}{2}x^2$$

$$\ln(1+x) \approx x - \frac{1}{2}x^2$$

$$(1+x)^r \approx 1 + rx + \frac{r(r-1)}{2}x^2.$$

$a_k = \left(1+\frac{1}{k}\right)^k$

$\ln a_k = k \ln\left(1+\frac{1}{k}\right)$

$\approx k\left(\frac{1}{k}\right) = 1$ (as $k \to \infty$)

$\uparrow$

$\ln(1+x) \approx x$ $\quad (x = 1/k \approx 0)$

rate of convergence.

$(\ln a_k) - 1 \to 0$

use quadratic approx

Find quadratic approx in x near 0

$$e^{-3x}(1+x)^{\frac{1}{2}} \approx \left(1 + -3x + \frac{(-3x)^2}{2}\right)\left(1 - \frac{1}{2}x + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^2\right)$$

$x \approx 0$

$$\approx 1 - 3x - \frac{1}{2}x + \frac{3}{2}x^2 + \frac{9}{2}x^2 + \frac{3}{8}x^2 + (\text{drop } x^3, x^4 \text{ etc})$$

$$= 1 - \frac{7}{2}x + \frac{51}{8}x^2.$$

Curve sketching.

GOAL: Draw graph of f using f', f'' positive/negative

Warning:
don't abandon your precalculus skill/common sense.

i) $f' > 0 \Rightarrow f$ is increasing.

$f' < 0 \Rightarrow f$ is decreasing.

$f'' > 0 \Rightarrow f'$ is increasing.

$\boxed{f \text{ concave u}}$

$f'' < 0 \Rightarrow f'$ is decreasing

$\uparrow$

$\boxed{f \text{ concave down}}$

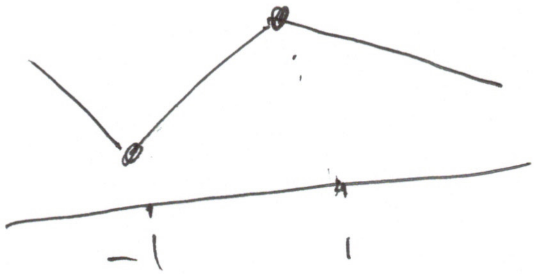
Ex1. $f(x) = 3x - x^3$

$f'(x) = 3(1-x)(1+x)$.

$-1 < x < 1$. $f'(x) > 0$

f increasing.

$x < -1 \,|\, x > 1$ $f'(x) < 0$. f decreasing schematic



turning point.

Definition If $f'(x_0)=0$.

x_0 is a $\boxed{\text{critical point}}$

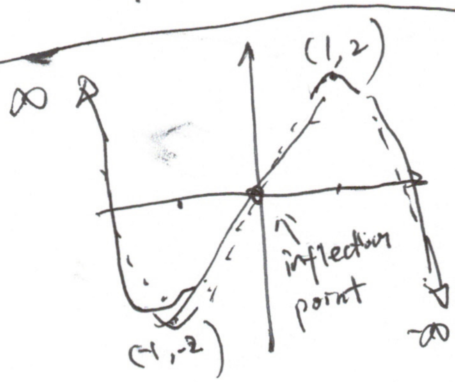
$y_0=f(x_0)$ is a $\boxed{\text{critical value}}$

Critical point/values

$f'(x)=0 \to x=\pm 1$

$f(1)=2$, $f(-1)=-\frac{1}{2}$

$f(0)=0.$



f is odd.

$f(0)=0.$

<u>Ends.</u> $x\to\infty$

$x\to-\infty$

$f(x)=3x-x^3 \sim -x^3 \to -\infty$ $x\to\infty$

$\to\infty$ $x\to-\infty$

$f''(x)=-6x$

$f''(x)<0$ if >0 (concave down)

$f''(x)>0$ if $x<0$ (concave up)

Lecture 11.

$f(x) = \frac{x+1}{x+2}$. $f'(x) = \frac{1}{(x+2)^2} \neq 0$.

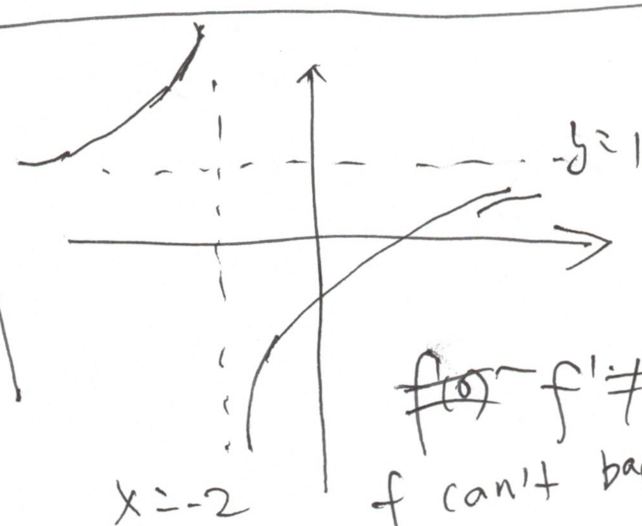
plot points : $x = -2$.

$f(-2^+) = -\infty$

$f((-2)^-) = +\infty$

Ends. $x \to \pm\infty$

$f(x) = \frac{x+1}{x+2} = \frac{1+1/x}{1+2/x} \to 1$.



~~f(0)~~ $f' \neq 0$.

f can't back track

$f'(x) = \frac{1}{(1+x)^2} > 0 \quad (x \neq -2)$

$f''(x) = -2\frac{1}{(1+x)^3} \quad (x \neq 2)$

$f''(x) < 0 \quad -2 < x < \infty$ concave down ← no wiggle

$f''(x) > 0 \quad -\infty < x < -2$ concave up.

General strategy.

1. Plot a) discontinuity, especially $\pm\infty$)
   b) endpoints
   c) easy points (optional)

2. a) solve $f'(x) = 0$
   b) plot critical point/value

3. Decide $f' \begin{smallmatrix} >0 \\ <0 \end{smallmatrix}$ on each interval between critical points/discont.
   (consistent with 1/2)

4. $f'' \begin{smallmatrix} >0 \\ <0 \end{smallmatrix}$ concave up/down.
   $f''(x) = 0 \Leftrightarrow x_0$ inflection p

5. combine everything

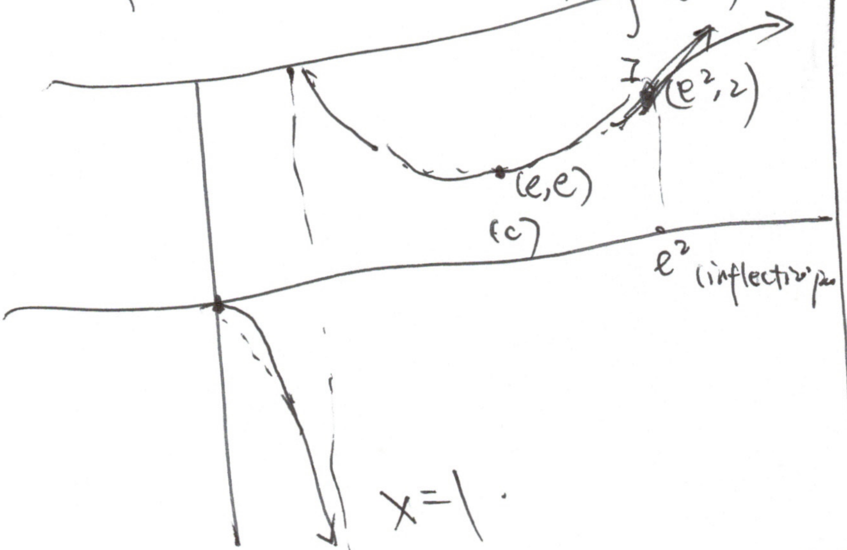
Ex $f(x)=\frac{x}{\ln x}$ $\quad x>0$

1 a) $f(1^+)=\frac{1}{\ln 1^+}=\frac{1}{0^+}=\infty$

$f(1^-)=-\infty$

1 b). ends. $f(0^+)=\frac{0^+}{-\infty}=0.$

$f(\infty)\Rightarrow\infty$



2). $f'(x)=\frac{\ln x-1}{(\ln x)^2}=\frac{\ln x-1}{(\ln x)^2}$

$f'(x)=0 \Rightarrow x=e$

$f(e)=\frac{e}{\ln e}=e$

Double check (3)

f is decreasing $\quad 0<x<1$

decreasing $\quad 1<x<e$

Increasing $\quad x>e.$

$f'(x)\ \frac{\ln x-1}{\ln x^2}.$

$\frac{-}{+}<0 \quad 0<x<1.$

$\frac{-}{+}<0 \quad 1<x<e$

$\frac{+}{+}>0. \quad x>e$

$f''(x)=-(\ln x)^{-2}\frac{1}{x}+2(\ln x)^{-3}\frac{1}{x}$

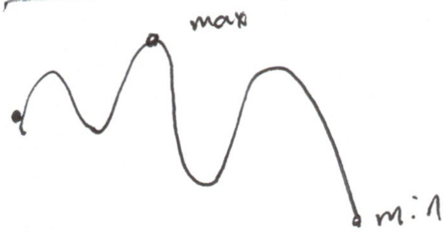
$=\frac{2-\ln x}{x(\ln x)^3}$

$\frac{+}{-}<0 \quad 0<x<1$

$f''\ \frac{+}{+}>0 \quad 1<x<e^2$

$f''\ \frac{-}{+} \quad e^2<x<+\infty$

Max / Min. problems.



Easy to find max/min with sketch.

goal : short cuts. without sketch

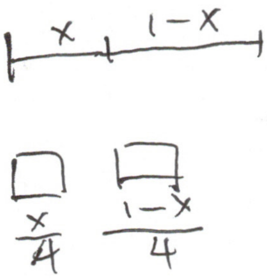
Key to find Max & Min.:

only look at $\boxed{\text{critical points}}$ and $\boxed{\text{End points}}$

and $\boxed{\text{discontinuity}}$.

Lecture 12. MAX/MiN.

Ex. 1. cut the wire two 2 piece, and form a two square. find the largest area it encloses.



$$A = \left(\frac{x}{4}\right)^2 + \left(\frac{1-x}{4}\right)^2$$

Find critical points

$A' = 0$.

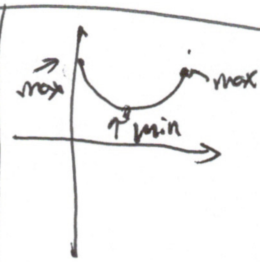
$$A' = \frac{x}{8} - \frac{(1-x)}{8}$$

$\Rightarrow$ $x = \frac{1}{2}$ is critical poin

$$A\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{8}\right)^2 + \left(\tfrac{1}{8}\right)^2 = \frac{1}{32}$$

Not done! Endpoints:

$$A(0^+) = \frac{1}{16}$$
$$A(1^-) = \frac{1}{16}$$



Least area enclosed $1/32$ when $x = \frac{1}{2}$. (equal squares)

Most when only on square

What is the minimum: $1/32$ (value)

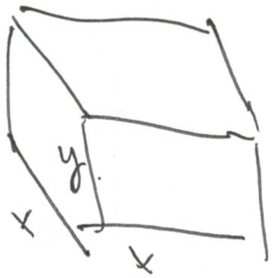
where is the minimum. $x = 1/2$. (point)

minimum point/value!

Alternatively. minimum point:

$$\left(\frac{1}{2}, \frac{1}{32}\right)$$

Ex2. Consider a box without a top. with least surface area for a fixed volume.



$V = x^2 y$ ↙ constraint.

$A = x^2 + 4xy$.

$y = \frac{V}{x^2}$.

$A = x^2 + 4x\frac{V}{x^2}$

$= x^2 + 4\frac{V}{x}$

critical points.

$A = x^2 + 4\frac{V}{x}$

$A' = 2x - 4\frac{V}{x^2}$

$2x = 4\frac{V}{x^2}$

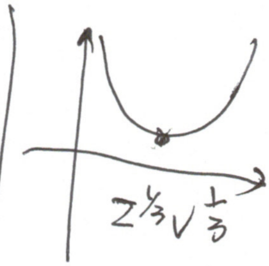
$x^3 = 2V$.

$x = 2^{1/3} V^{1/3}$

Ends $0 < x < \infty$

$A(0^+) = x^2 + \frac{4V}{x}\Big|_{x=0^+} = \infty$

$A(\infty) = x^2 + \frac{4V}{x}\Big|_{x=\infty} = \infty$



second derivative test (do not recommend).

$A' = 2x - 4\frac{V}{x} = 2 + 8\frac{V}{x^3}$ (positive, concave up)!

$x = 2^{\frac{1}{3}} V^{\frac{1}{3}}$

$y = \frac{V}{2^{\frac{2}{3}} V^{\frac{2}{3}}} = 2^{-\frac{2}{3}} V^{\frac{1}{3}}$

$A = (2^{\frac{1}{3}} V^{\frac{1}{3}})^2 + 4\frac{V}{2^{\frac{1}{3}} V^{\frac{1}{3}}}$

$= 3 \cdot 2^{\frac{2}{3}} V^{\frac{2}{3}}$.

More meaningful Answers:

dimensionless variable

$A / V^{2/3} = 3 \cdot 2^{1/3}$

$x/y = \frac{2}{1} = 2$.

optimal shape: $x/y = 2$.

Ex2 by implicit differentiation.

$V = x^2 y$. $A = x^2 + 4xy$.

goal min A w/ V constant.

$\frac{d}{dx}(V = x^2 y) \Rightarrow 0 = 2xy + x^2 y'$

$y' = -\frac{2xy}{x^2} \Rightarrow -\frac{2y}{x}$

$\frac{dA}{dx} = 2x + 4y + 4xy'$

$= 2x + 4y + 4x\left(-\frac{2y}{x}\right) = 0$

$2x + 4y - 8y = 0$

$2x = 4y$

$\boxed{\frac{x}{y} = 2}$.

Faster.

nicer.

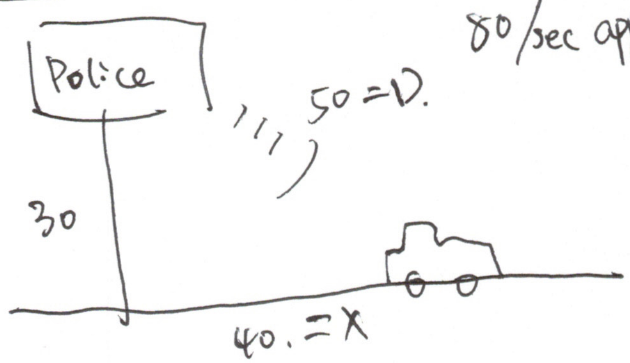
disadvantage: did not check the critical point max/min/neither.

Related rates.:

$\frac{dD}{dt}$.

80/sec approaching 95 ft/sec.

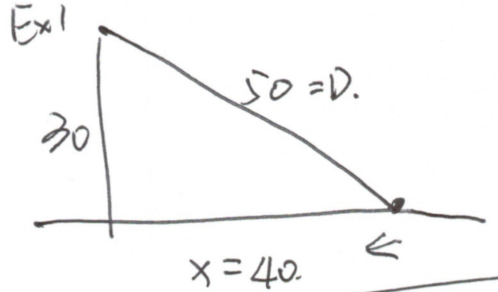
t time in second



$\frac{dx}{dt} > 95$ ft ?

Lecture 13: Related Rates

Ex1



Are you speeding:

$\frac{dD}{dt} = -80$ $\frac{dx}{dt} > 95$?

$x^2 + 30^2 = D^2$. $\frac{dD}{dt} = -80$

Easiest with implicit diff.

$$2x\frac{dx}{dt} = 2D\frac{dD}{dt}$$

$$2 \cdot 40 \cdot \frac{dx}{dt} = 2 \cdot 50 \cdot \frac{dD}{dt}\ -80$$

$$\boxed{\frac{dx}{dt} = -100 \text{ feet/sec}}$$

Ex2. A conical tank.

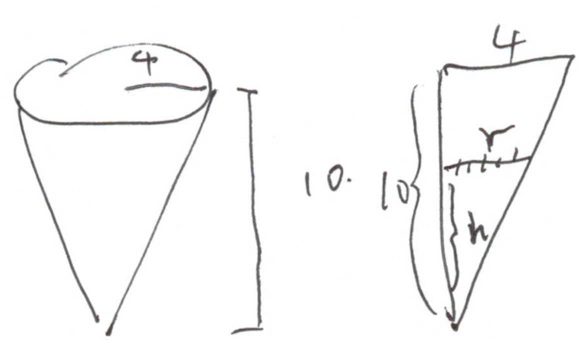
top radius 4 ft

depth 10 ft:

filled at 2 cu ft/min

how fast are the water rising when depth is 5 feet.

Diagram, variable



$\frac{r}{h} = \frac{4}{10}$.

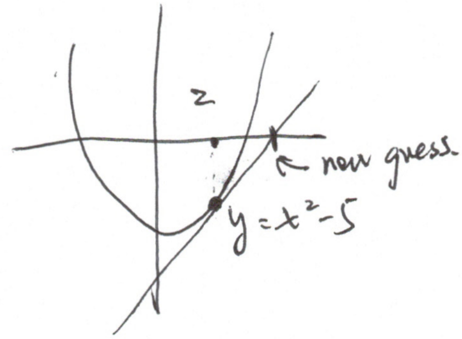
$V = \frac{1}{3}\pi r^2 h$ $\frac{dV}{dt} = 2$.

$\frac{dh}{dt}$? when $h = 5$

Newton's method

Example. solve $x^2=5$.

$f(x)=x^2-5$; solve $f(x)=0$



Start with inital guess $x_0=2$.

tang line

$y-y_0=m(x-x_0)$

x_1 is the x-intercept

$0-y_0=m(x_1-x_0)$.

$\frac{-y_0}{m}=x_1-x_0$.

$x_1=x_0-\frac{y_0}{m}$.

$$\boxed{x_1=x_0-\frac{f(x_0)}{f'(x_0)}}$$

newton's method

$$\boxed{x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}}$$

$x_0=2$. $f(x)=x^2-5$

$f'(x)=2x$

$x_1=x_0-\frac{x_0^2-5}{2x_0}$

$x_1=\frac{1}{2}x_0+\frac{5}{2}x_0$

$x_1=\frac{1}{2}\cdot 2+\frac{5}{4}=9/4$

$x_2=\frac{1}{2}\frac{9}{4}+\frac{5}{2}\cdot\frac{4}{9}=\frac{161}{72}$

$x_3=\frac{1}{2}\frac{161}{72}+\frac{5}{2}\cdot\frac{72}{161}$

n	$\sqrt{5}-x_n$
0	2×10^{-1}
1	10^{-2}
2	4×10^{-5}
3	4×10^{-10}

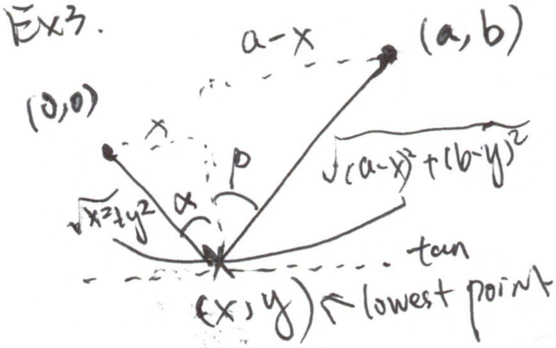
$r = \frac{2}{5}h$. $V = \frac{1}{3}\pi\left(\frac{2}{5}h\right)^2 h$

$$2 = \frac{dV}{dt} = \frac{\pi}{3}\left(\frac{2}{5}\right)^2 3h^2 \frac{dh}{dt}$$

$$2 = \frac{\pi}{3}\left(\frac{2}{5}\right)^2 3 \cdot 5^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{2\pi} \text{ feet/minute}$$

Ex3.



$$\sqrt{x^2+y^2} + \sqrt{(a-x)^2+(b-y)^2} = L \text{ (constant)}$$

: Find min y. | From diagram, the bottom point is $y' = 0$ (critical point).

Implicit diff on the constraint

$$\frac{x + yy'}{(x^2+y^2)^{1/2}} + \frac{(a-x) + (b-y)y'}{\sqrt{(a-x)^2+(b-y)^2}} = 0$$

$y' = 0$,

$$\frac{x}{(x^2+y^2)^{\frac{1}{2}}} = \frac{a-x}{\sqrt{(a-x)^2+(b-y)^2}}$$

$\sin\alpha = \sin\beta$

$\alpha = \beta$