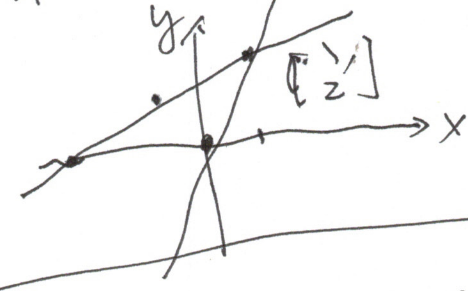
Lecture: 1

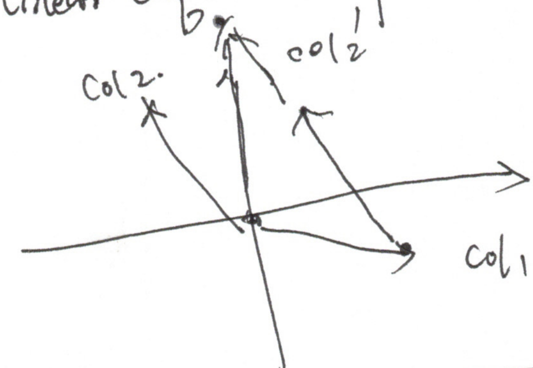
1. $\begin{bmatrix} 2, & -1 \\ -1, & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ → $\boxed{\text{Row picture}}$

$$A \quad x = b$$



2. $x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ → $\boxed{\text{Column picture}}$

Linear combination of columns



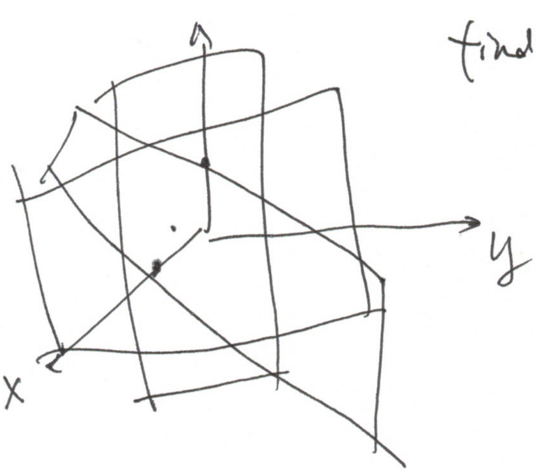
$x=1 \quad y=2.$

3x3 example.

$$\begin{aligned} 2x - y \quad &= 0 \\ -x + 2y - z &= -1 \\ -3y + 4z &= 4 \end{aligned}$$

$A \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}$ $\quad b = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$

$\boxed{\text{Row Picture:}}$

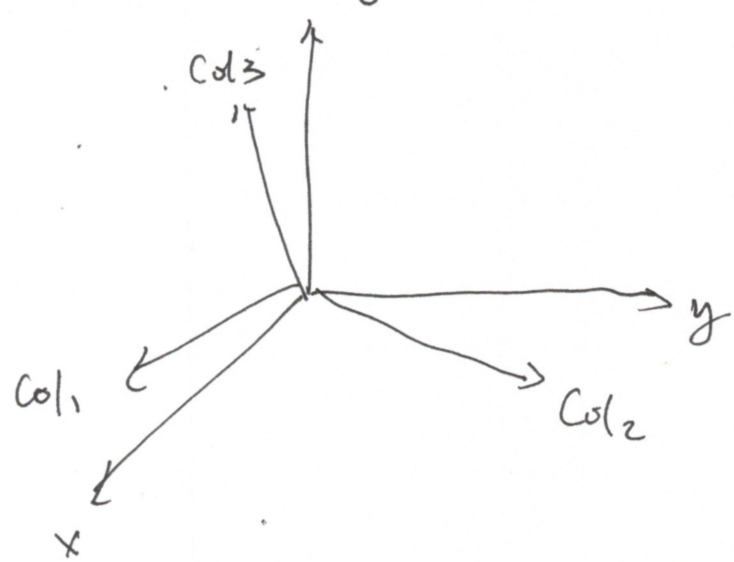


find the point where 3 plane meet.

Column Picture

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix}+y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}0\\-1\\4\end{bmatrix}$$

$b=\begin{bmatrix}1\\1\\-3\end{bmatrix}$ $x=1$ $y=1$ $z=0$



$x=0$
$y=0$
$z=1$

Can I solve $Ax=b$? for every b.
or do the linear combination of all columns fill the R^3 space.

Yes,

Intuitively, if col1/col2/col3 are on the same plane. It is singular/non-invertible. (not independent)

$Ax=b$ is a combination of columns of A

Lecture 2: elemination expressed as matrix

elimination
$$\begin{bmatrix}1&2&1\\3&8&1\\0&4&1\end{bmatrix} \qquad \begin{bmatrix}2\\12\\2\end{bmatrix}$$
$Ax \qquad = b$

pivot
$$\left[\begin{array}{ccc|c}\boxed{1}&2&1&2\\3&8&1&12\\0&4&1&2\end{array}\right] \xrightarrow{(2,1)} \left[\begin{array}{ccc|c}\boxed{1}&2&1&2\\0&\boxed{2}&-2&6\\0&4&1&2\end{array}\right] \xrightarrow{(3,2)} \left[\begin{array}{ccc|c}\boxed{1}&2&1&2\\0&\boxed{2}&-2&6\\0&0&\boxed{5}&-10\end{array}\right]$$
$A \quad b \qquad\qquad U \quad C$

elemanation from A to U. (upper triangle)

determinant = multple by diagles.

---

back substitution

$x + 2y + z = 2 \qquad x = 2.$

$2y - 2z = 6 \qquad y = 1$

$5z = -10 \qquad z = -2$

Matrices.

$$\underset{E32}{\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}} \underset{E21}{\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}} \underset{A}{\begin{bmatrix}1&2&1\\3&8&1\\0&4&1\end{bmatrix}} \qquad [a, b, c]\begin{bmatrix}A\end{bmatrix} =$$

$E_{32}(E_{21}A) = U$ it is associative

Permutation matrix

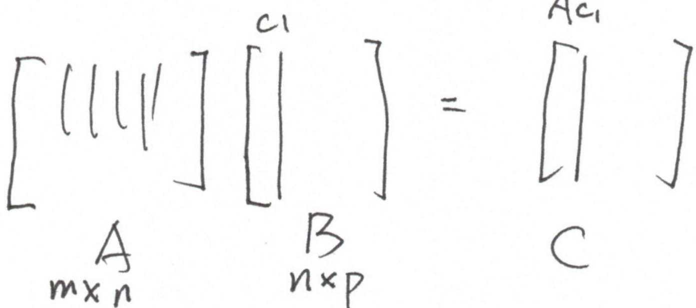
Inverse: $\underset{E^{-1}}{\begin{bmatrix}1&0&0\\3&1&0\\0&0&1\end{bmatrix}} \underset{E}{\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}} = I$

Lecture 3

Matrix multiplication $\quad AB = C$

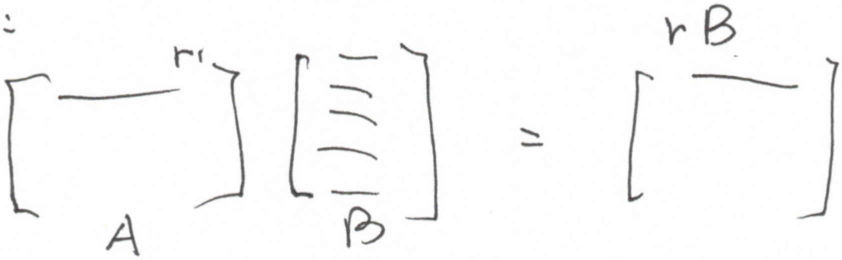
1. element wise : $c_{ij} = \sum_{k=0}^{n} a_{ik} b_{kj}$

2. coloum wise :



each ~~row~~ coloum of C ~~B~~ is a linear combination of ~~A~~ ~~A's Colounms~~ Coloumns of A

3. row wise :

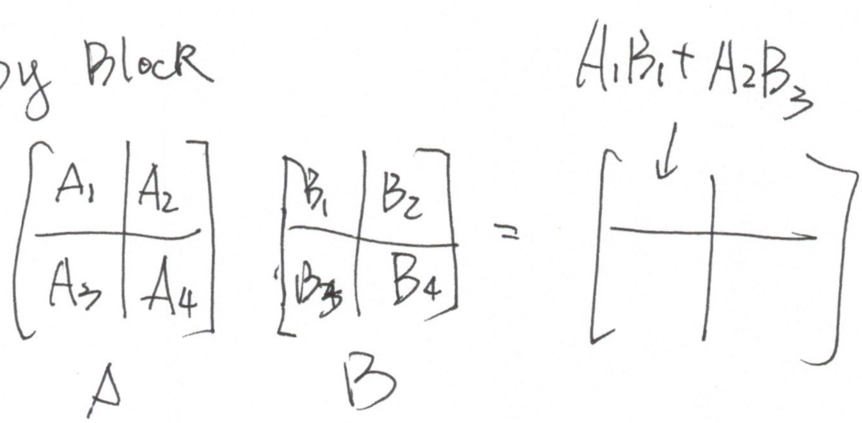


each row of C is a linear combination of rows of B

4. Colum of A * Row of B $\qquad$? why

AB = Sum of (Colum of A)(Row of B)

5. by Block



Inverse matrix:

$$A^{-1}A = I = AA^{-1}$$ for square matrix.

Inverted or non-singular.

(hard to prove left inverse equal to right inverse)

Singular matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$ = any AB → result column space on the same line.

We can find $\underline{Ax = 0}$, so it is singular.

proof:
because if A^{-1} exist → $A^{-1}Ax = A^{-1}0 = 0$
→ $x = 0$

Solving A^{-1}

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$A \times$ Column j of A^{-1} = Column j of I.

Gauss-Jordan (solve 2 at once)

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Augmented Matrix

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ \boxed{2} & 7 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \boxed{3} & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

1. $ABB^{-1}A^{-1} = I$

2. $AA^{-1} = I$

$(A^{-1})^T A^T = I \rightarrow (A^{-1})^T = (A^T)^{-1}$

$$\underset{E}{\begin{bmatrix}1 & 0\\ -4 & 1\end{bmatrix}} \underset{A}{\begin{bmatrix}2 & 1\\ 8 & 7\end{bmatrix}} = \underset{U}{\begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}}$$

$$A = L_{(ower)} U_{(pper)}$$

$$\begin{bmatrix}2 & 1\\ 8 & 7\end{bmatrix} = \begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix} \begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\ 4 & 1\end{bmatrix} \begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix} \begin{bmatrix}1 & \frac{1}{2}\\ 0 & 1\end{bmatrix}$$

$E_{32}E_{31}E_{21}A = U$

$A = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$ ← this is better because less compute dependency.

$A = LU$ if no row exchanges, the multipliers go directly into L. (~~ID~~ ie, E_{31}^{-1} have no dependency of E_{21}^{-1})

How many operation it takes to eliminate? $O(n^3)$

Permutations: $\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix} \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1\end{bmatrix} \begin{bmatrix} & & 1\\ & 1 & \\ 1 & & \end{bmatrix}$ Permutation group

$\begin{bmatrix}1 & & \\ & & 1\\ & 1 & \end{bmatrix} \begin{bmatrix} & 1 & \\ & & 1\\ 1 & & \end{bmatrix} \begin{bmatrix} & & 1\\ 1 & & \\ & 1 & \end{bmatrix}$

$P^{-1} = P^T$

Permutation in $A = LU$ → allow row exchange

$$PA = LU$$

P = identity matrix with reordered rows......

$n!$ counts the possible reordering.

P is invertible, and $\underline{P^T = P^{-1}}$? $P^TP = I$

Traspose = $(A^T)_{ij} = A_{ji}$

Symmetric matrices : $A^T = A$

R^TR is always symmetic :

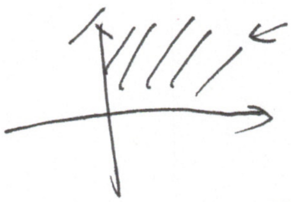
$$\cancel{R_{ij}} \cdot (R^TR)^T = \cancel{R^TR} \quad R^TR^{TT} = R^TR$$

Lecture 5. Vector spaces (chapter 3)

·1. R^2 = all 2 dim real number vectors

= the plane.

R^n = all vectors with n Real components.



not vector space

Example of R^2 subspace: 1. lines through origin

2. origin /zero vector

3. R^2

$A=\begin{bmatrix}1&3\\2&3\\4&1\end{bmatrix}$ coloumn in R^3

$\boxed{\text{all their linear combination form a subspace}}$.

called coloumn space. $C(A)$.

Lecture 6.: 2 subspaces = P and L

$P \cup L$ ~~并~~ is not a subspace. (usually)

$P \cap L$ is a subspace for any two subspaces P and L

becaus any v, w, v+w in both subspaces.

v+w and cv are in the subspace:

all linear combination form a subspace.

column space of A is ~~a~~ subspace of R^4 $C(A)$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix}$$ = all linear combination of columns.

Does $Ax = b$ have a solution for every b. No

$Ax = b$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$ ~~how~~ which b have answers?

we can solve $Ax = b$ when b is in A's column space $C(A)$ in R^4

Null (space) of A = all solution $x \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in R^3 of $Ax = 0$

$N(A)$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$N(A)$ contains $\cancel{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}, c\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

if v, w is in Nullspace, then any linear combination is also in Nullspace

i.e $Av = 0$ $Aw = 0$

$A(cv + dw) = 0$

~~the~~

$Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ is not a subspace → as zero ~~is~~ not in the space

is a plane not through origin.

2 ways of construct subspace

1. column combinations
2. solve equations

Lecture 7.

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

elimination ~~is~~ won't chang null space

Rank of A = # of pivots = 2.

echelon $\rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$

2 pivot columns

free columns

1. elimination find free
2. give 1, 0 to free varibles
3. find special solutions, combine them

$r = 2$.

$n - r = 2 \Rightarrow$ free variables. ~~of~~ for $m \times n$ matrix A.

rref(A) R reduced row echelon form

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R = \text{rref}(A) = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$ ← r pivot rows

r pivot colum

n − r free colums.

Magic!!! ...

$RN = 0 \rightarrow N = \begin{bmatrix} -F \\ I \end{bmatrix} \rightarrow N$ is nullspace

Lecture 8. solve $Ax=b$

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \rightarrow \begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-3b_1 \end{array} \quad \begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{array}$$

Soluability Condition on the right hand size.

$Ax=b$ solvable when b is in $C(A)$

---

if combination of rows of A give zero row
then the same combination of entries of b must give 0.

To find complete sol'n to $Ax=b$.

① X particular : 1. set free variable to zero
2. solve $Ax=b$ for piviot

+

② X nullspace

$Ax_p = b$

$X = X_p + X_n$

$Ax_n = 0$

$A(X_p + X_n) = b$

$$X = \begin{bmatrix} -2 \\ 3/2 \\ 0 \end{bmatrix} + \underbrace{c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\text{nullspace}}$$

a plane though X particular.

m by n matrix A of rank r $(r \leq m,\ r \leq n)$

1. Full rank column means $\boxed{r = n}$, no free variable

$N(A) = \{\text{zero vector}\}$

Solution to $Ax = b$: $x = x_p$ unique solution if it exists

(0 or 1 solution)

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2. Full Row Rank means $r = m$

Can solve $Ax = b$ for every b

Left with $n - r$ free variables

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & \boxed{F} \\ 0 & 1 & \end{bmatrix}$$

3. $r = n = m$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$ $N(A) = \{\text{zero}\}$. $Ax = b$ exists for any b.

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$r = m = n$

$R = I$

1 solution

$n<m$

$r = \cancel{m<n}$

$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

((0 or 1 solution

$r = m < n$

$R = \begin{bmatrix} I & F \end{bmatrix}$

((~~1 or~~ infinite many solution)

$r < m, r < n$

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

((0 or ∞ solutions)

Lecture 9: Independence spanning & basis

1. suppose A is $m \times n$ with $m<n$,
Then there are infinite solutions to $Ax=0$
(because there is free variables.

2. $\boxed{\text{Independence}}$ Vectors $x_1, x_2 \cdots x_n$ are independent
if no combinations give zero vector (except zero comb)

Repeat : when $v_1 \cdots v_n$ are columns of A
They are independent if $N(A)$ is {zero vector} $\boxed{\text{rank} = n}$
They are dependent $Ac=0$ for some non zero c.
$\boxed{\text{rank} < n}$

Vectors $v_1 \cdots v_n$ $\boxed{\text{span}}$ ~~a~~ space means:
The space consists of all combs of those vector.

$\boxed{\text{Basis}}$ for a space is a sequence of vectors
$v_1, v_2 \cdots v_d$ with two properties
1. They are independent
2. They span the space

Example $\cancel{R^3}$

space is R^3

one basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ $\begin{bmatrix}1\\1\\2\end{bmatrix}\begin{bmatrix}2\\2\\5\end{bmatrix}$

n vectors give basis if $n \times n$ matrix use n as column vectors if $n \times n$ matrix is invertible

given a space

every basis for the space has the same number of vectors

Dimension of the space.

rank(A) = # pivot column = dimension of C(A)

matrix space

$\dim C(A) = r$

$\dim N(A)$ = number of free varible $= n - r$

Null space vector really tells the vectors make columns dependent.

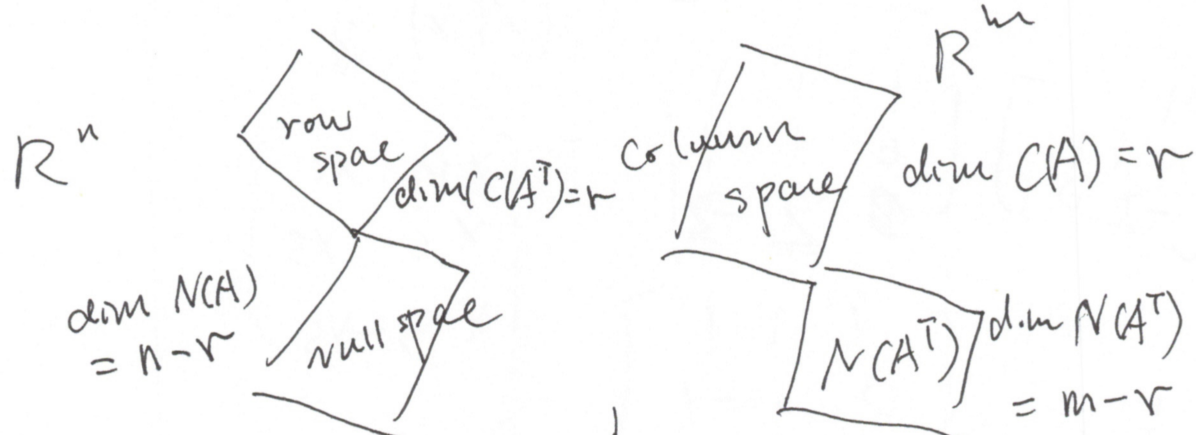
Lecture 10. 4 subspaces A in $m \times n$

column space. $C(A)$ in R^m

Null space. $N(A)$ in R^n

Row space $C(A^T)$ in R^n

left Null space $N(A^T)$ in R^m



	$C(A)$	~~$C(A^T)$~~	$N(A)$
Basis:	pivot columns		special solutions
dim :	r	~~r~~	$n-r$

	$C(A^T)$	$N(A^T)$
Basis :	first r rows of R (R is the reduced form)	the Rows in E produce 0 rows in R.
dim :	r	$m-r$

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 \end{bmatrix} \rightarrow$$

$$R \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rref} \begin{bmatrix} A_{m\times n} & I_{m\times m} \end{bmatrix} \rightarrow \begin{bmatrix} R_{m\times n} & E_{m\times m} \end{bmatrix}$$

$$EA = R$$

new vector space, M

all 3×3 matrices !! $\quad A + B,\ CA$

Subspace of M: all upper triangles

symmetric matrices

diagonal matrices $D \to \dim = 3$

Lature 14. Orthorgnal vectors and subspaces

Orthogral vectors : $x^T y = 0$

Pythagoras : $\|x\|^2 + \|y\|^2 = \|x+y\|^2$

$$x^T x + y^T y = (x+y)^T (x+y)$$

$$= x^T x + y^T y + x^T y + y^T x$$

$$2x^T y = 0$$

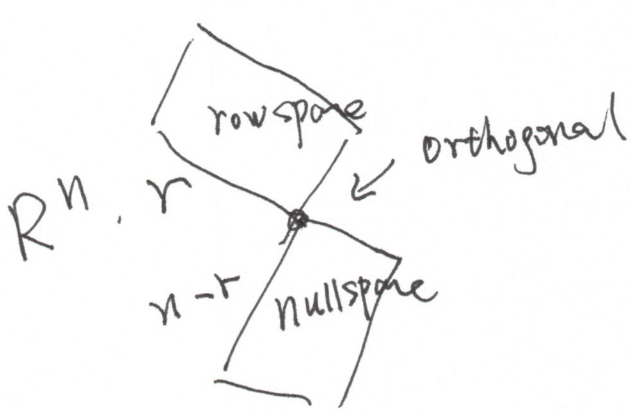
Subspace ~~S~~ S is Othorgnal to subspace T :

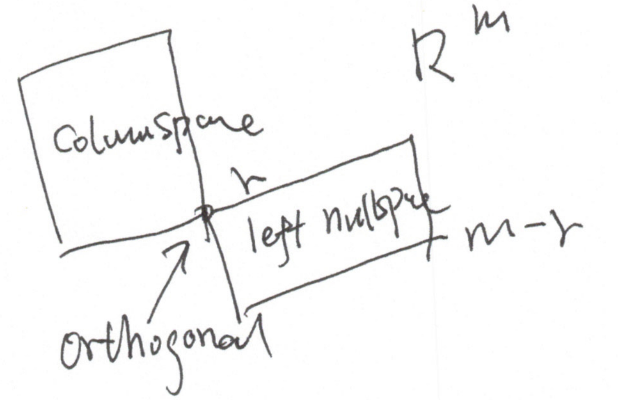
means: <u>every vector</u> in S in orthognal to <u>evey vector</u> in T

---

row space is orthogaralto nullspace.

why? ~~x~~ $Ax = 0$ $\quad \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$





nullspace and Row space are orthogonal, their dim add up to R^n! $\to$ $\boxed{\text{Orthogaral Complements}}$

Nullspace contains ALL vectors $\perp$ row space.

Convrg: "solve" $Ax=b$ when ~~there~~ is no solution. (i.e noise)

$(m>n)$

A^TA symmetric:

square: $n \times m \quad m \times n$

$n \times n$

$$A^TA\hat{x} = A^Tb$$

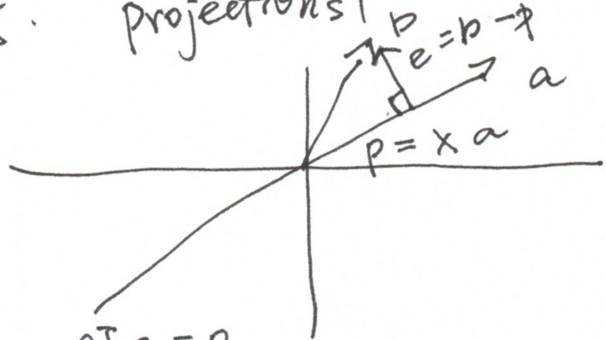
$N(A^TA) = N(A)$. rank of A^TA = rank of A	A^TA is invertible if A has independent columns

---

Any solution of $Ax=b$ cand ~~have~~ be presented as $x_r^* + x_n$ where x_r is in Rowspace and X_n is in Nullspace.

why? because $C(A^T)$ and $N(A)$ are ortho components of R^n → so any x canbe presented as a $x_r + x_n$ ~~where~~

Lecture 15. Projections



$a^T e = 0$

$a^T(b - xa) = 0$

$x\, a^T a = a^T b$

$\boxed{x = \frac{a^T b}{a^T a}}$ $\boxed{p = ax}$ $\boxed{p = a\frac{a^T b}{a^T a}}$

Projection = Pb $\boxed{P = \frac{aa^T}{a^T a}}$

projection matrix

$C(P)$ = line through a.

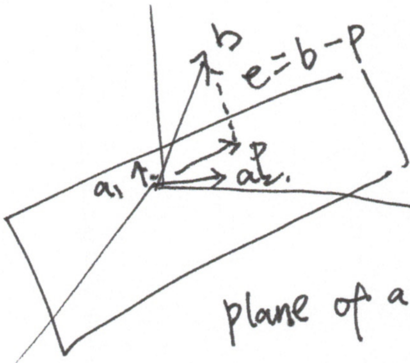
rank$(P) = 1$

* $\boxed{P^T = P. \quad P^2 = P}$ (projection twice)

Why project? because $Ax = b$ may have no solution.

solve $A\hat{x} = p$ → instead.

project b onto column space.



is prep to plane

$p = \hat{x}_1 a_1 + \hat{x}_2 a_2$

$= A\hat{x}$

$p = A\hat{x}$

plane of $a_1 a_2$ = colspace of

$A = [a_1 \; a_2]$

$p = A\hat{x}$ Find $\hat{x}$

key: $b - A\hat{x}$ is prep to plane

$a_1^T (b - A\hat{x}) = 0 \quad a_2^T (b - A\hat{x}) = 0$

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$e \; (b - A\hat{x})$ in $N(A^T)$

$e \perp C(A)$

$A^T(b - A\hat{x}) = 0.$

$$\boxed{A^T A\hat{x} = A^T b}$$

$\hat{x} = (A^T A)^{-1} A^T b$

$p = \underbrace{A(A^T A)^{-1} A^T}_{} b$

matrix $P = A(A^T A)^{-1} A^T \rightarrow$ id $\dfrac{aa^T}{a^T a}$

A is not inversible.

$I - P$ is also projection
it is projecting to e

~~b(I-P)~~

$(I-P)b \perp Pb$

$(I-P)$ space otho

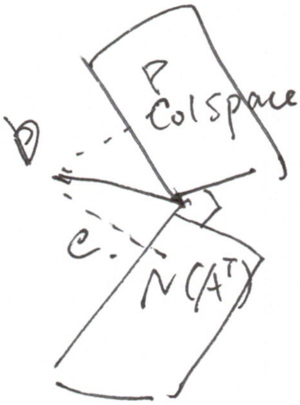
$P^T = P$

$P^2 = P$: $P^2 = A(A^T A)^{-1} \cancel{A^T A (A^T A)^{-1}} A^T$

$= A(A^T A)^{-1} A^T$

Lecture 16. projection matrix and least square error

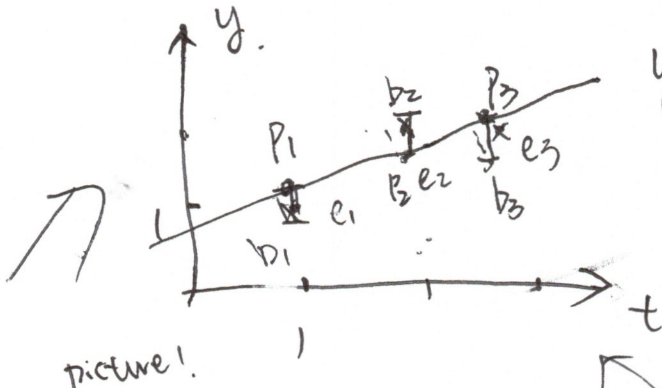
$$\boxed{P = A(A^TA)^{-1}A^T}$$



Pb

$p + e = b$.

$(I-P)b$. ☆ $\boxed{I-P}$ is a projection onto $\perp$ space, $N(A^T)$



picture 1.

$y = C + Dt$ $\quad Ax = b$

$C + D = 1$
$C + 2D = 2$
$C + 3D = 2$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

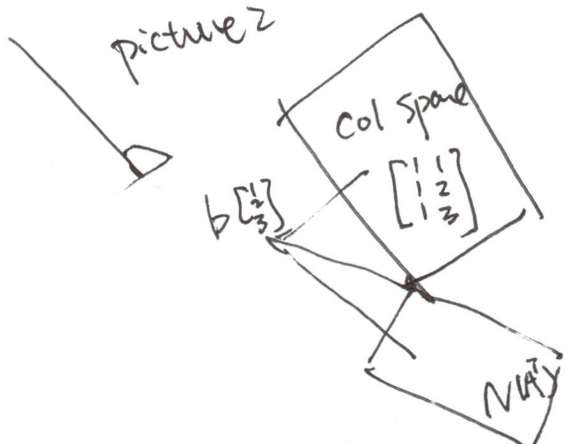
Minimize $||Ax - b||^2 = ||e||^2$.

$= e_1^2 + e_2^2 + e_3^2$

$= (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$

equal to find a projection onto the column span

picture 2



Find $\hat{x} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, P

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$3C + 6D = 5$
$6C + 14D = 11$
$D = \frac{1}{2}$
$C = \frac{2}{3}$

$A^TA\hat{x} = A^Tb$.

If A has independent colums. then A^TA is invertible.

---

prof. suppose $A^TAx=0$, x must be 0.

1) $x^TA^TAx=0$

$\Rightarrow ||Ax||=0$.

$\Rightarrow Ax=0$.

because A has independent colums, so x is zero.

Columns are definitely independent, if they are perpendicular unit vector. like $\begin{bmatrix}1\\0\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\\0\end{bmatrix}$ $\begin{bmatrix}0\\0\\1\end{bmatrix}$

orthonormal vectors.

Lecture 17.

Orthogonal basis.

Orthogonal matrix Q

Gram-Schmidt $A \to Q$

Orthonormal vectors.

$$q_i^T q_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

$$Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix} \quad Q^T Q = I = \begin{bmatrix} - q_1 - \\ \vdots \\ - q_n - \end{bmatrix} \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} = I$$

(Q doesn't have to be square!!!).

Orthogonal matrix $\to$ matrix is <u>square</u> and orthonormal

If Q is square. $Q^T Q = I$. ~~the~~ tell us. $Q^T = Q^{-1}$

i.e permutation matrix. $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = I$

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \qquad Q = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$$

Project onto Q' colum space.

$P = Q(Q^T Q)^{-1} Q^T = \underline{QQ^T}$ (symmetric. $(QQ^T)^2$ square = itself). {I is Q is square}

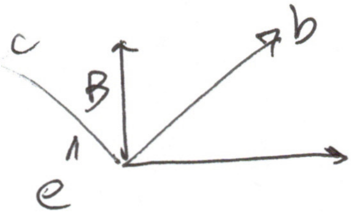
$A^T A x = A^T b$. (for least squre error)

$Q^T Q \hat{x} = Q^T b$.

$\boxed{\hat{x} = Q^T b}$. $\to$ $\boxed{\hat{x}_i = q_i^T b}$

Gram - Schmidt.

independent.
vectors a, b, c → orthogonal → orthonormal



A, B, C $q_1 = \frac{A}{\|A\|}$ $q_2 = \frac{B}{\|B\|}$ $q_3 = \frac{C}{\|C\|}$.

$a = A$. $B = b - P_b = b - \cancel{A}\, a \frac{a a^T}{a^T a} \cancel{b}$

$$C = c - P_c = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B.$$

ratio of A

C's projection on A

C's projection on B

$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$A \perp B$

$\cancel{q_1 =}$ $Q = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

same column space!!!

$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$

$$A = QR.$$

$\begin{bmatrix} a_1 & a_2 \end{bmatrix} =$

$\begin{bmatrix} q_1 & q_2 \end{bmatrix} \begin{bmatrix} a_1^T q_1 & * \\ 0 & * \end{bmatrix}$

$a_1 q_2$

$\boxed{q_2 \perp a_1}$ now because of Gram Schmidt does

$\boxed{R}$ is upper triangular

Lecture 18..

Determinants $\det A = |A|$

property of det

① $\det I = 1$.

② Exchange rows = reverse the sign of det

$\det P = 1$ even / -1 odd number of exchanges.

---

① $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ 3a $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

② $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$ 3b $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$.

linear combinations

FOR EACH ROW

$\det(A+B) \neq \det A + \det B$

④ 2 equal rows $\to$ det = 0

Exchange those rows $\to$ same matrix. $\Rightarrow$ det 0.

⑤ Substract $l \times$ row i from row k.

DET dosen't change

$\to \begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix}$ (3.a/3.b)

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l\begin{vmatrix} a & b \\ a & b \end{vmatrix}$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} - 0$ ④

⑥ Row of zeros $\rightarrow \det A = 0$

$$\begin{vmatrix} \cancel{a} & \cancel{b} \\ \cancel{c} & \cancel{d} \end{vmatrix} = \begin{vmatrix} 0\cdot a & 0\cdot b \\ c & d \end{vmatrix} = 0 \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0.$$

---

⑦ $$\det U = \begin{vmatrix} d_1 & & * \\ & \ddots & \\ 0 & & d_n \end{vmatrix} = d_1 \cdot d_2 \cdots d_n.$$

$\Downarrow$ prove by eliminations then

rule 5

we get $\det U' = \begin{vmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{vmatrix}$

rule 3a

$\Downarrow$ $d_n \cdots d_2 d_1 \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix}$

rule 1

$\Downarrow$ $d_n \cdots d_2 d_1 * 1$

---

⑧ $\det A = 0$ $\rightarrow$ row of zeros when done LR elimination

when A is singular.

$\det A \neq 0$ $\rightarrow$ LR elimination

when A is invertible. $\searrow$ $D \searrow d_1 d_2 \cdots d_n$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ 0 & d - \frac{c}{a} b \end{bmatrix} \rightarrow \det A = ad - bc$$

⑨ ☆ $\det AB = (\det A)(\det B)$ // ~~det~~ A^{-1}

$\det A^{-1} = 1/\det A$] use above property.

~~⑩~~ for dignal matrix, trival. to prove.

for other matrix, need go through elimination.

$\det A^2 = (\det A)^2$

$\det 2A = 2^n \det A$. --- (like volume)

⑩ $\det A^T = \det A$. → all previous rows rule apply to columns.

prove this $|A^T| = |A|$

$|U^T L^T| = |LU|$

$|U^T||L^T| = |L||U|$

↑↑ all triangular, so the det is only on dignals!

※ rule (2) implies permutations are either odd or even!

Lecture 19. Determinant formula and cofactors.

① $\det I = 1$

② SIGN REVERSE. Exchage two rows

③ LINEAR ON EACH Row sperately.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \underset{③}{=} \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \cancel{\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix}} \underset{②}{+} \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \cancel{\begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}}$$

2×2

$$= ad - bc.$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}$$

3×3

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}$$
$$- a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31}$$
$$+ a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Big formula: $n \times n$ det

$$\det A = \sum_{n!\text{ terms}} \pm\, a_{1\alpha} a_{2\beta} a_{3\gamma} \cdots a_{n\omega}$$

$(\alpha\ \beta\ \gamma \cdots \omega) \Leftarrow$ Permutation of $(1, 2 \cdots n)$

Example

$$\begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 0 \to \text{singular}$$

$(4,3,2,1) \to +1$ $(3,2,1,4) \to -1$

Cofactors! 3x3 in parens.

$$\det = a_{11}(a_{22}a_{33} - a_{23}a_{32}) \Leftarrow \text{det of } 2\times 2.$$
$$+ a_{12}(-a_{21}a_{33} + a_{23}a_{31})$$
$$+ a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix}$$

$$\begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

$$\boxed{\begin{array}{l} \text{Cofactor of } a_{ij} = C_{ij} \\ \pm \det\left(\begin{array}{l} n-1 \text{ matrix} \\ \text{with row } i,\ \text{col } j \text{ erased} \end{array}\right) \end{array}}$$

$i+j$	$i+j$
$+$ even	$-$ odd

Cofactor formula: (along row 1)

$$\boxed{\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots a_{1n}C_{1n}}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad + b(-c)$$

Ex.

$$A_4 = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$|A_1| = 1 \quad |A_2| = 0 \quad |A_3| = -1 \quad \cancel{|A_4|} =$

$$|A_4| = 1 \cdot |A_3| - 1 \cdot |A_2| = 1$$

$$|A_n| = |A_{n-1}| - |A_{n-2}|.$$

Lecture 20.

1. Formular for A^{-1}

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad \boxed{A^{-1} = \frac{1}{\det A} C^T}$$

($C \to$ cofactor matrix)

Check $A C^T = (\det A) I$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ C_{12} & & \vdots \\ \vdots & & \vdots \\ C_{1n} & & C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \ddots \ \det A \end{bmatrix}$$

exacty cofactor formaular to calculate det(A)

the none diagnal value is zero, is because it is equivalent to take det of a matrix with two equal rows.

$Ax = b$

$x = A^{-1}b = \frac{1}{\det A} C^T b.$

CRAMER'S RULE

$x_1 = \frac{\det B_1}{\det A}$ ······ $(C_{11}b_1 + b_{21} b_2 \cdots)$.

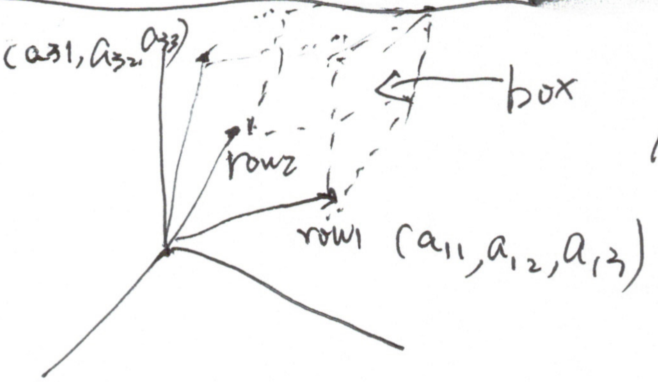
$x_2 = \frac{\det B_2}{\det A}$

$$\boxed{x_j = \frac{\det B_j}{\det A}}$$

A with column 1 replaced by b. $= B_1 = \begin{bmatrix} b & \begin{matrix} n-1 \\ \text{columns} \\ \text{of } A \end{matrix} \end{bmatrix}$

$B_j = \begin{bmatrix} A \text{ with column } j \\ \text{replaced by } b \end{bmatrix}$

$\boxed{|\det A| = \text{volume of box}}$ 3×3



$A = I \to 1$

$A = Q \to 1$

to prove it: just check 3 properties!!

$Q^TQ = I \to \det(Q^T)\det(Q) = \det I$

$\det(Q)^2 = 1$

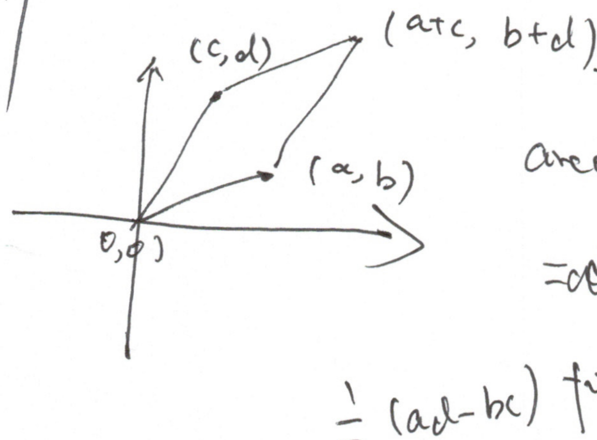
$|\det A|$ = volume box → property 1 ✓ ($\det I = 1$)

property 2 ✓ (sign. We don't mind.)

property 3.a ✓ factor a row, ~~ti~~ increase the row length by n

property 3.b

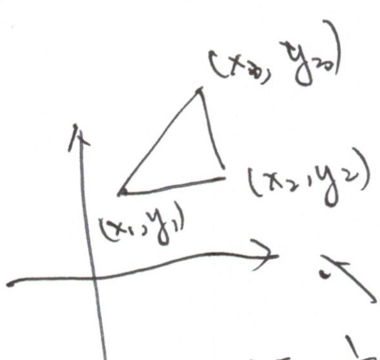
$$\begin{vmatrix} a+a', & b+b' \\ c, & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$



area $= \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= ad - bc$

$\frac{1}{2}(ad-bc)$ for triangle



area $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.