ITMO UNIVERSITY

How to Win Coding Competitions: Secrets of Champions

Week 3: Sorting and Search Algorithms
Lecture 6: Mergesort

Maxim Buzdalov
Saint Petersburg 2016

Another Divide-and-Conquer algorithm

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- This algorithm needs extra scratch memory

procedure $\textsc{Mergesort}(A, W, \preceq, s, t)$
    if $s + 1 = t$ **then return end if**
    $m \leftarrow (s + t)/2$
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    $i \leftarrow s, j \leftarrow m, k \leftarrow s$
    while $i < m$ **or** $j < t$ **do**
        if $j = t$ **or** $(i < m$ **and** $A[i] \preceq A[j])$ **then**
            $W[t] \leftarrow A[i],\ i \leftarrow i + 1,\ t \leftarrow t + 1$
        else
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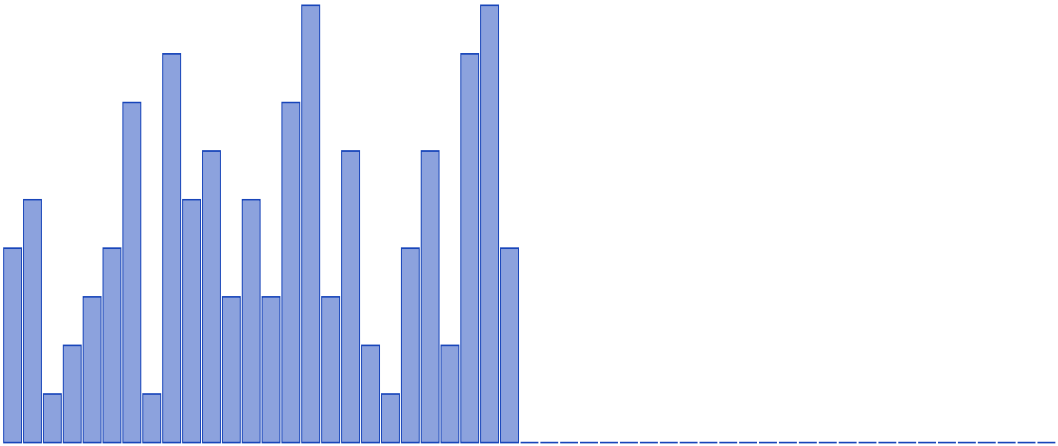
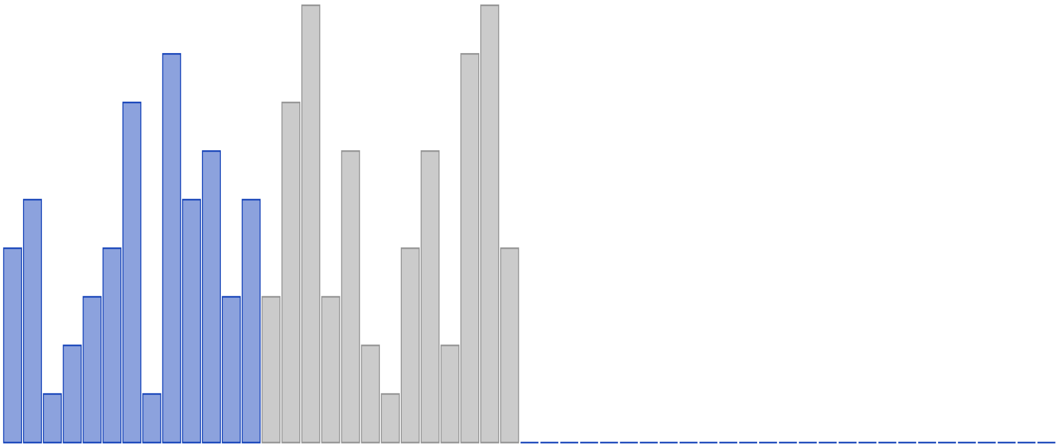
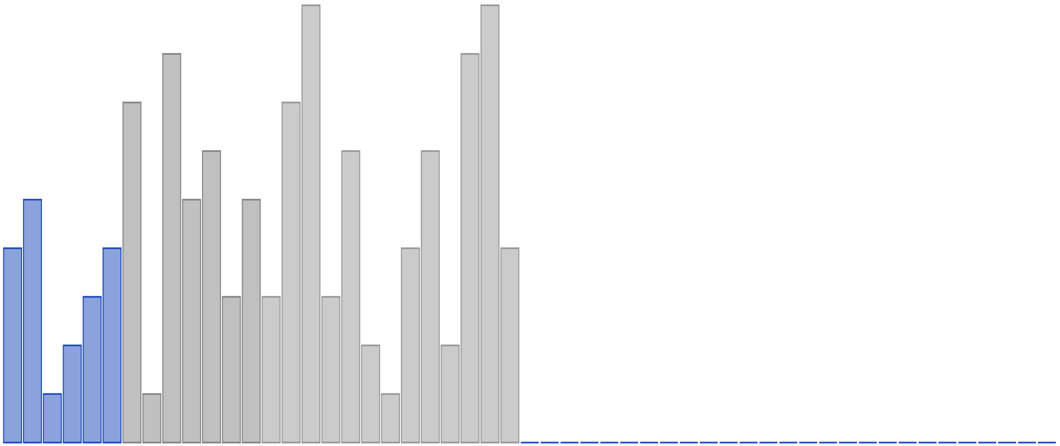
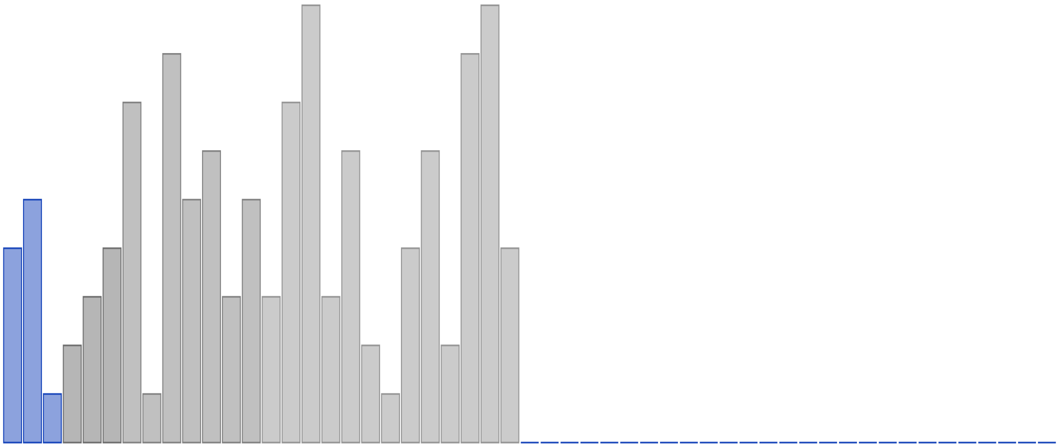
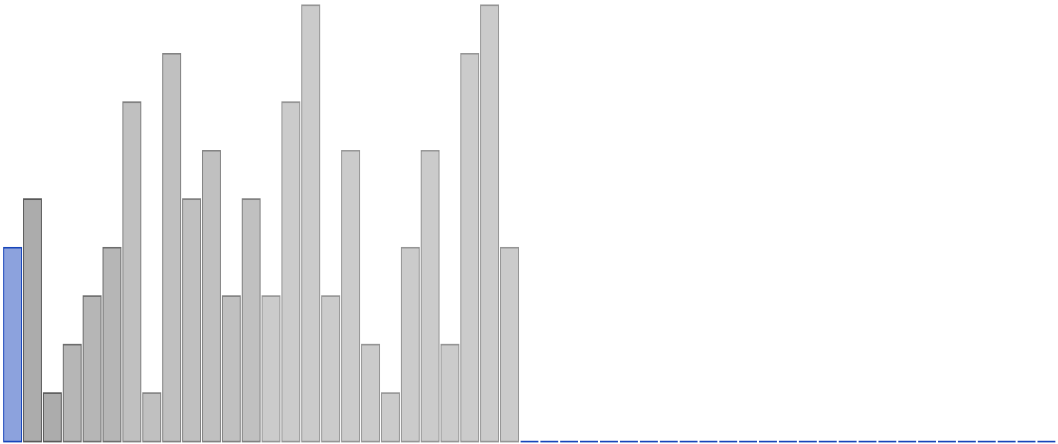
- Merge runs in $\Theta(t - s)$
  - Every scratch element is written exactly once

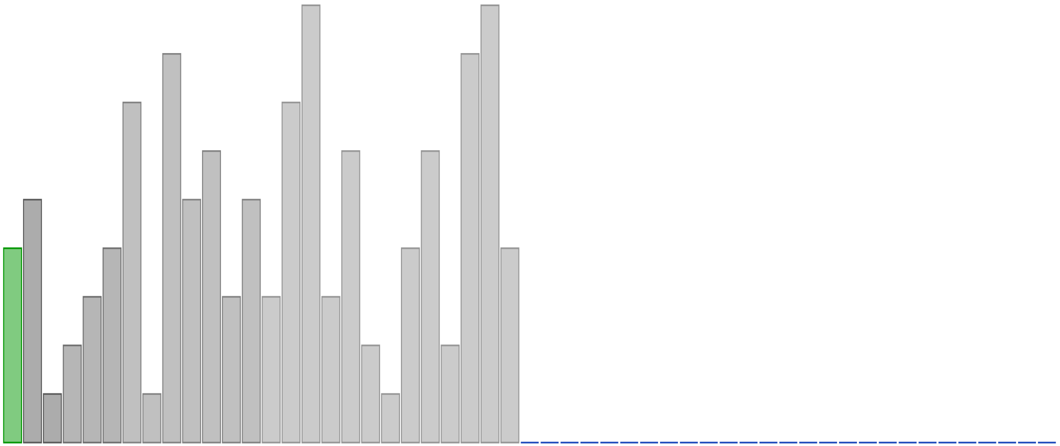


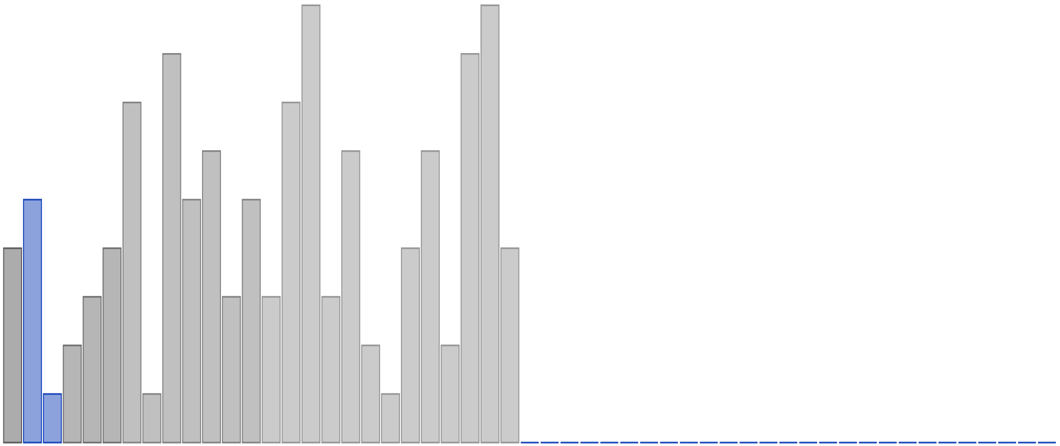


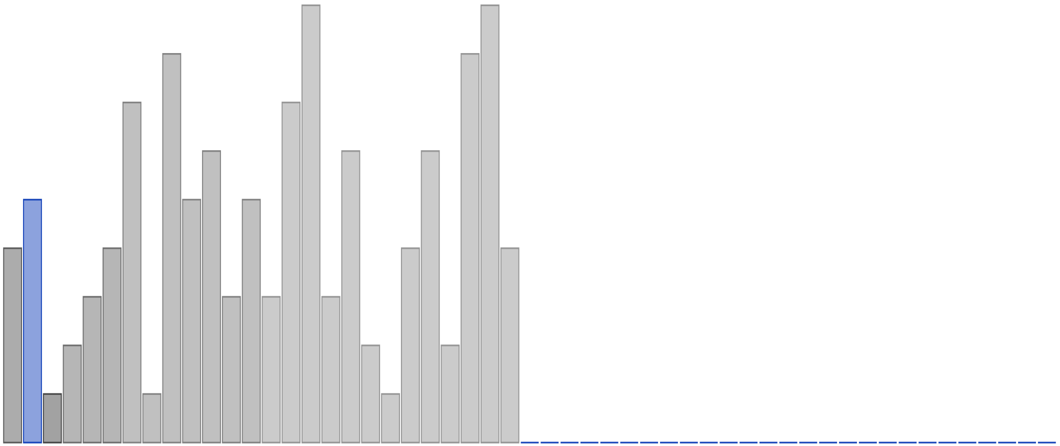


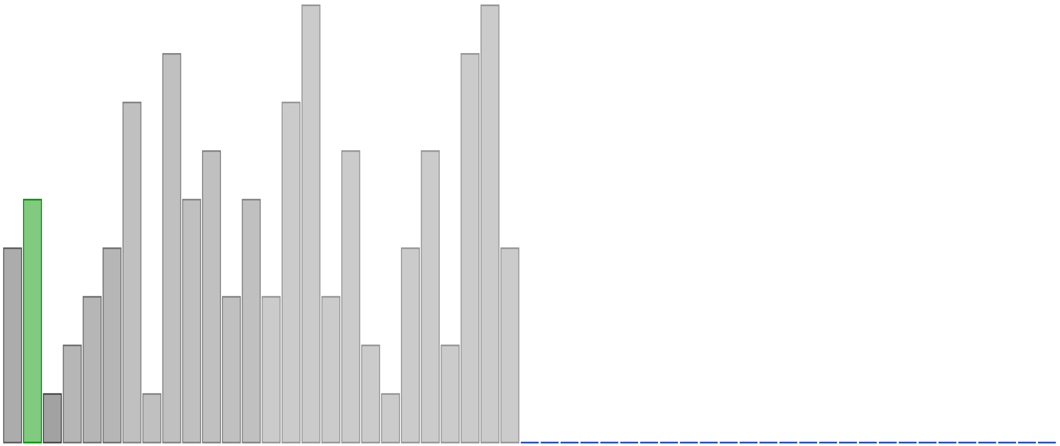


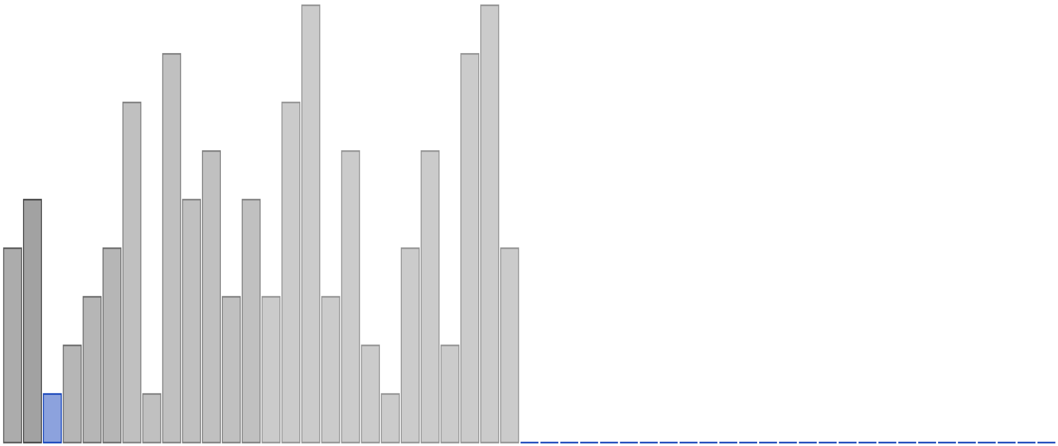


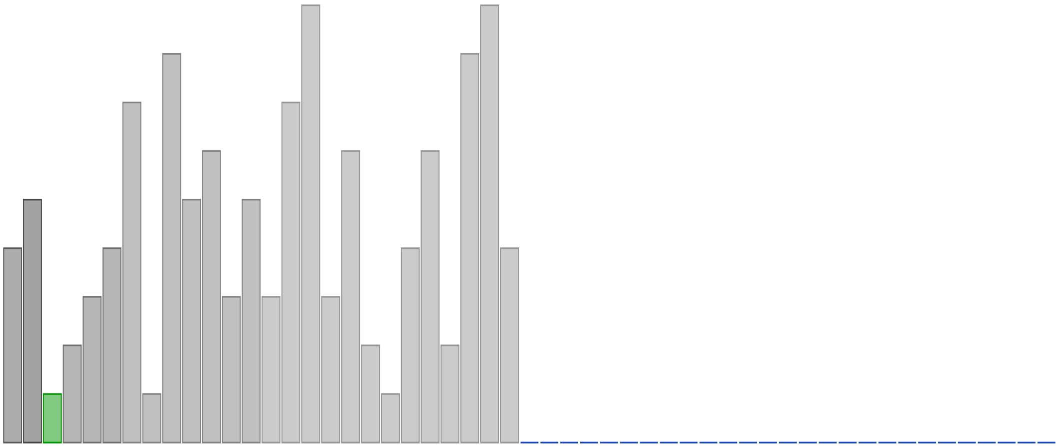


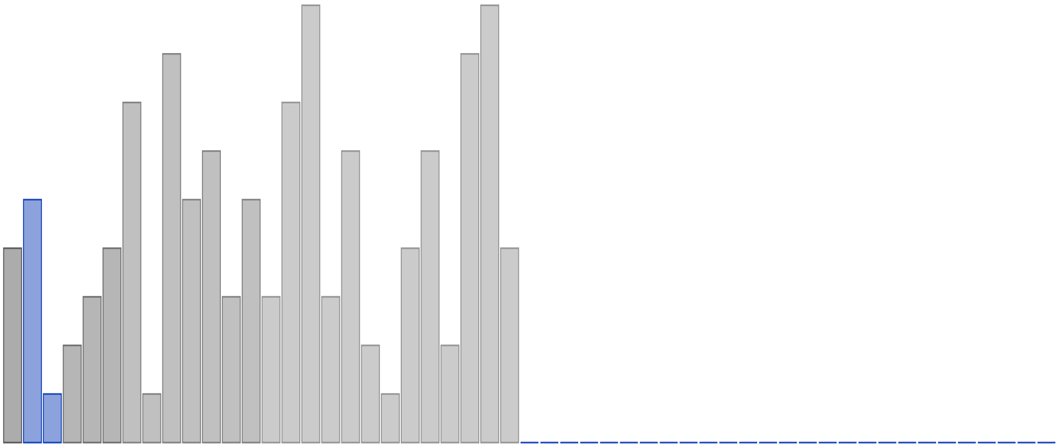


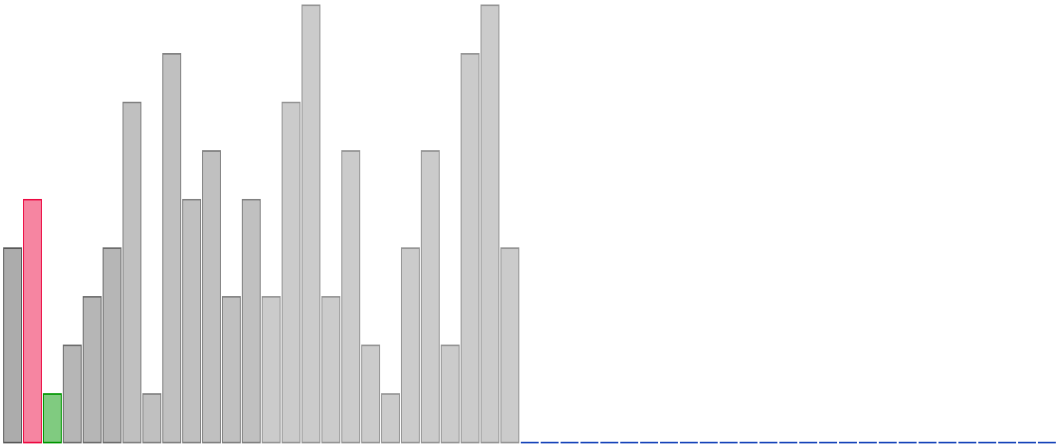


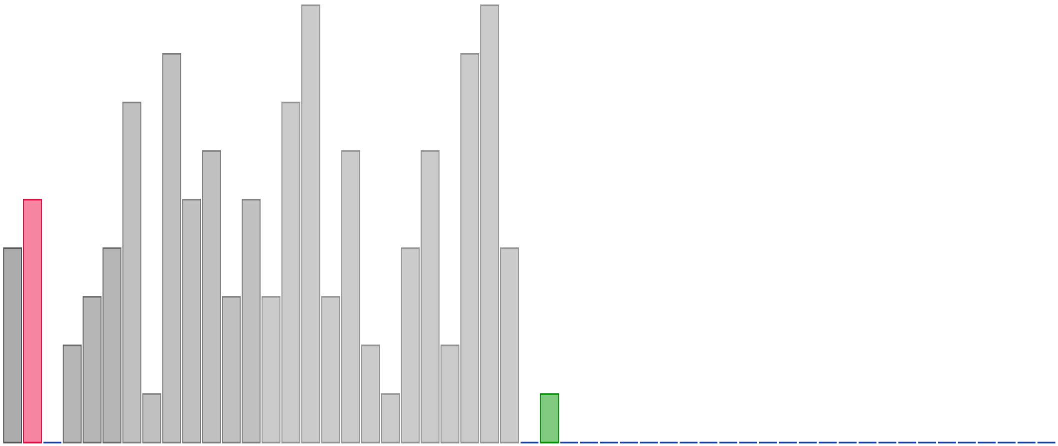


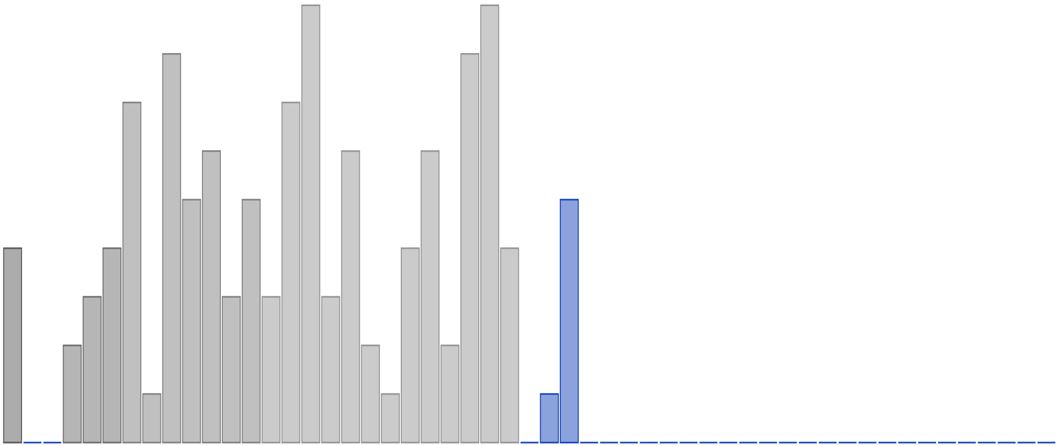


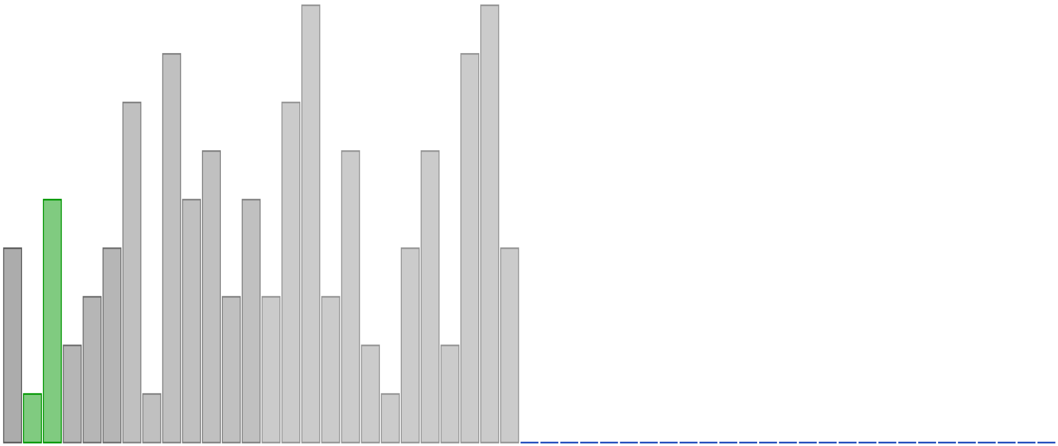


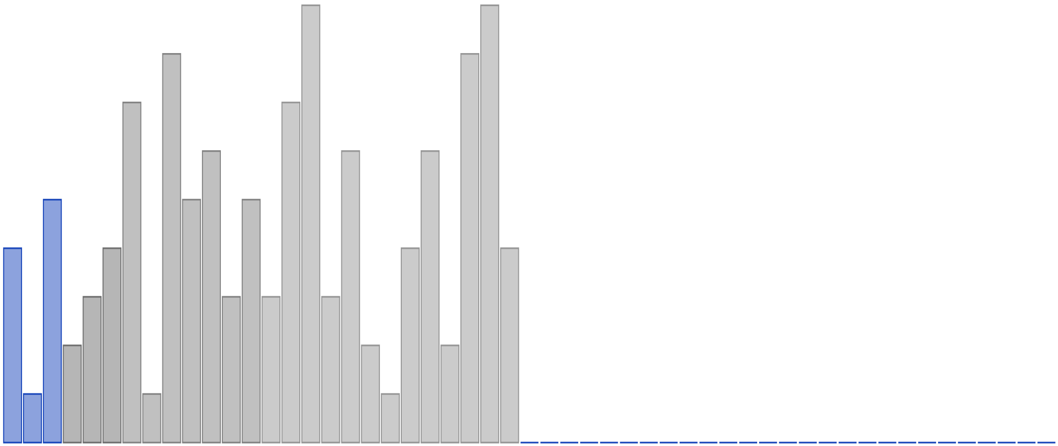


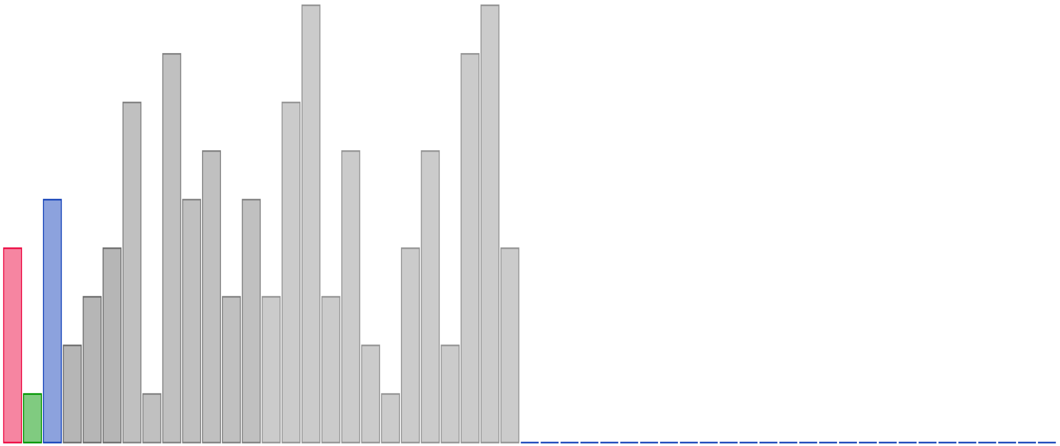


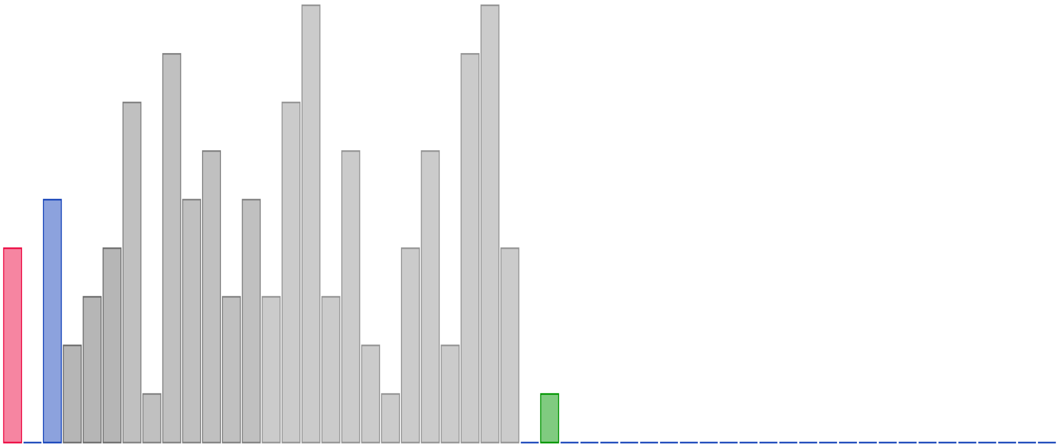


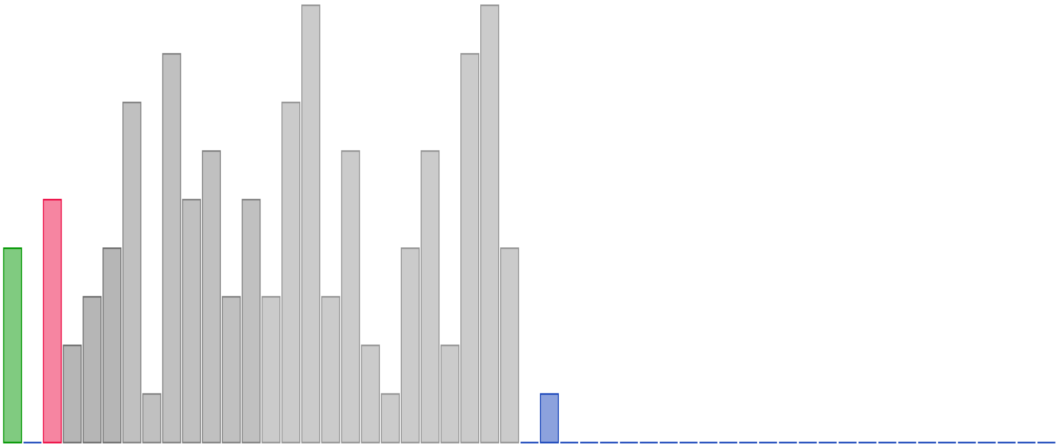


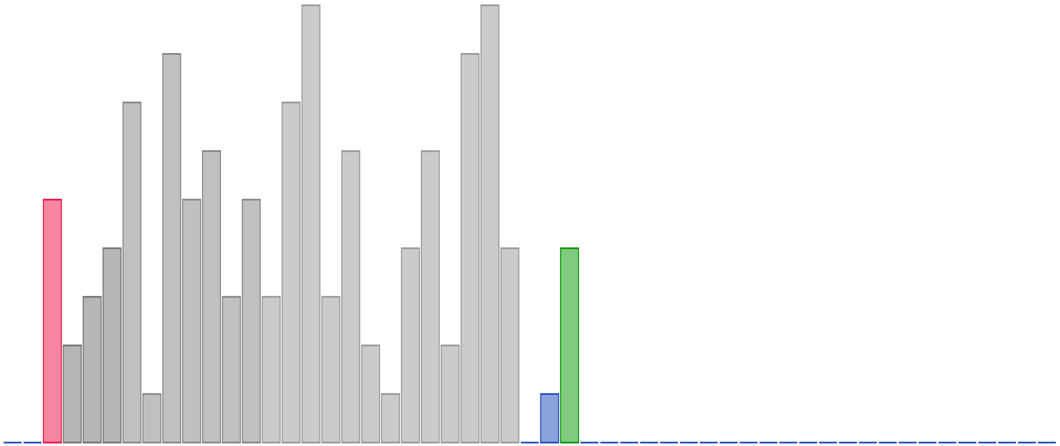


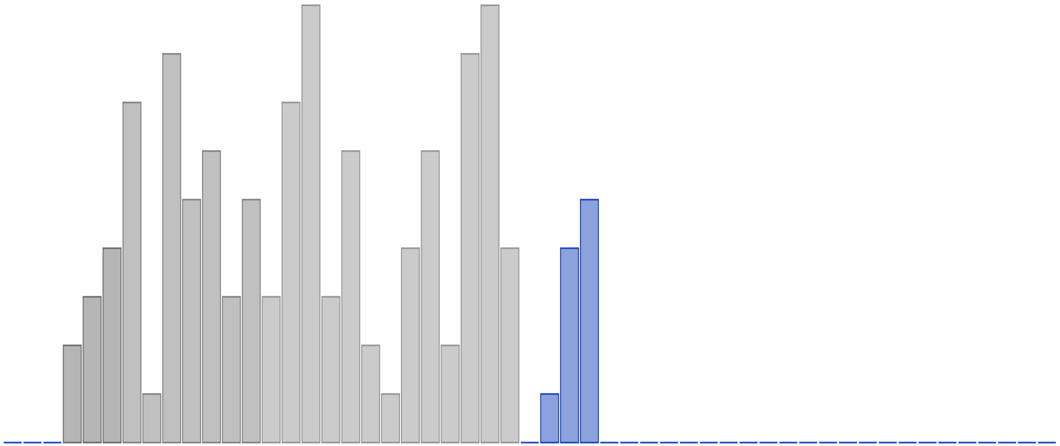


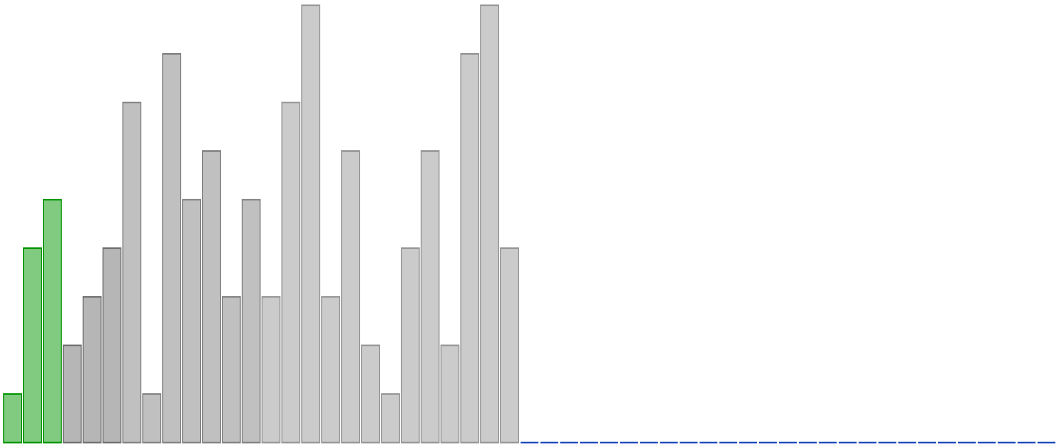


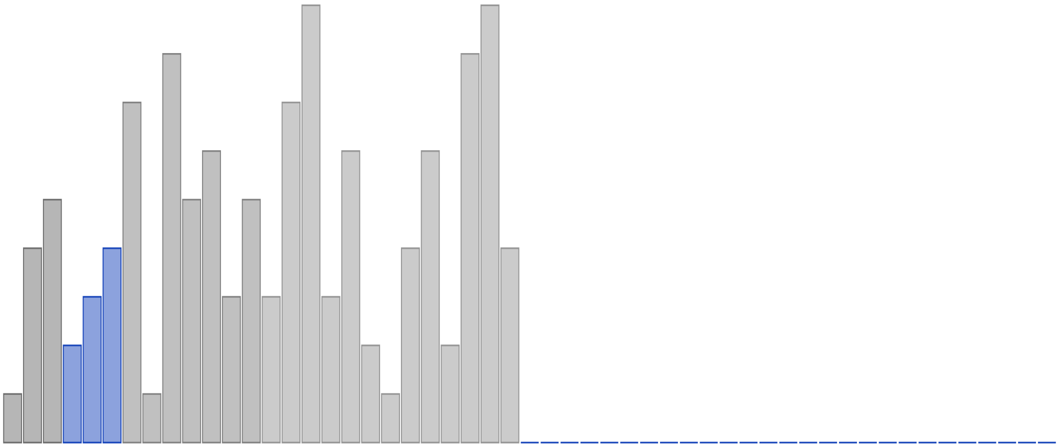


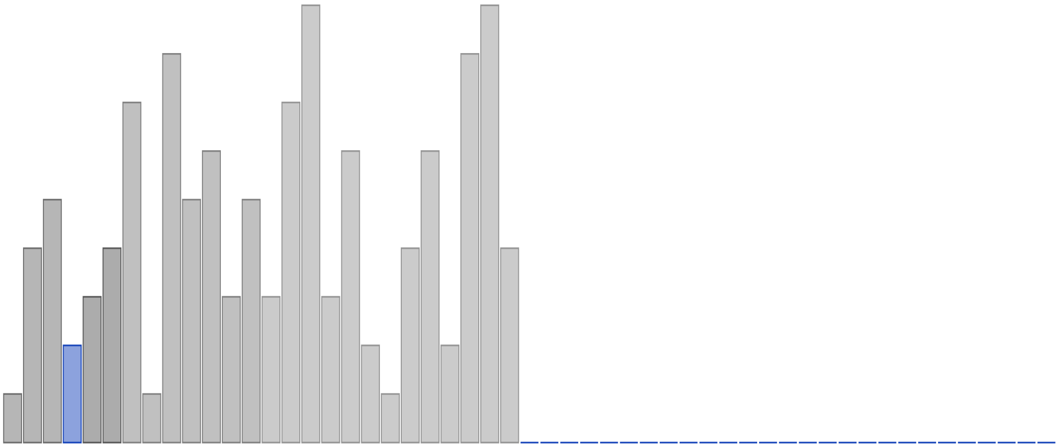


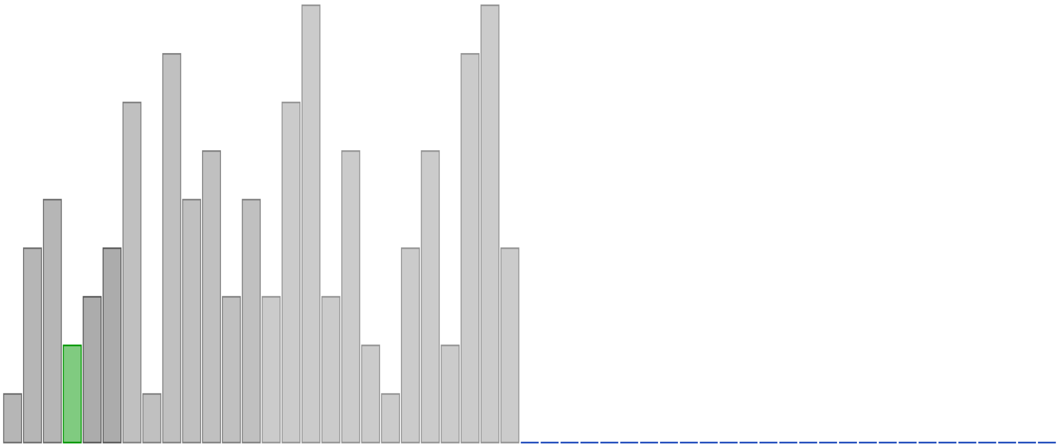


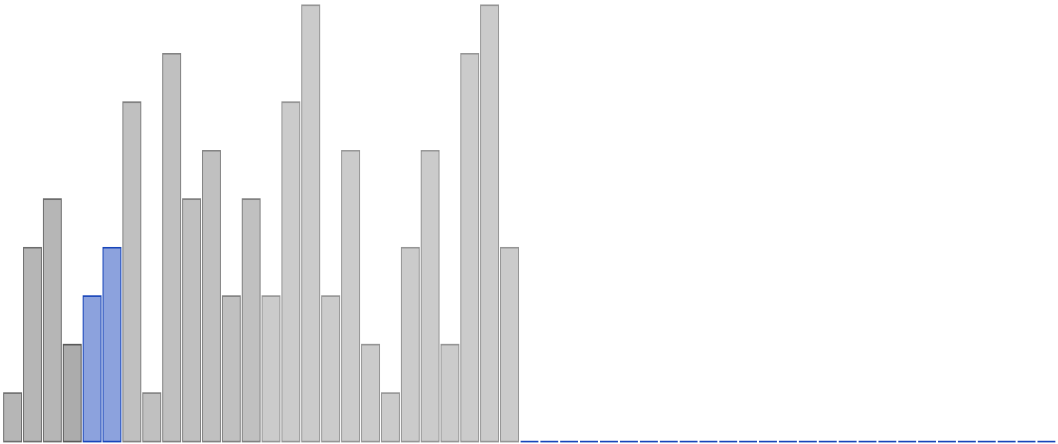


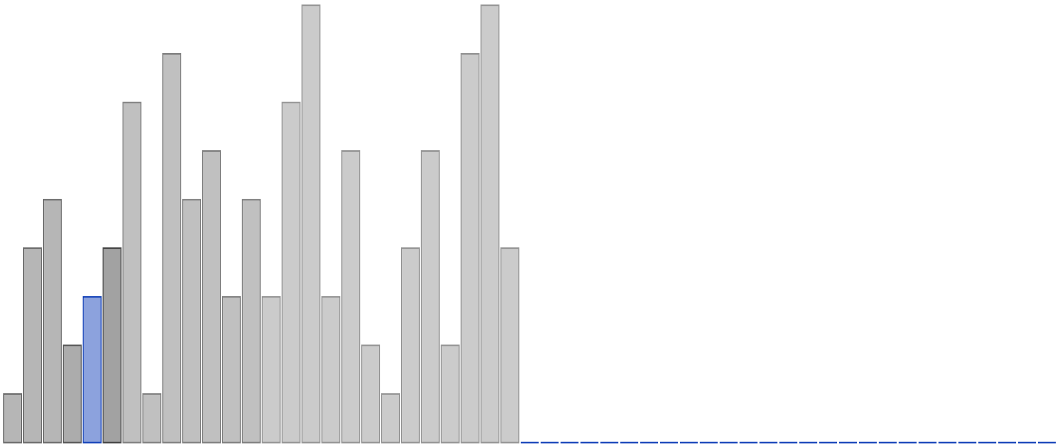


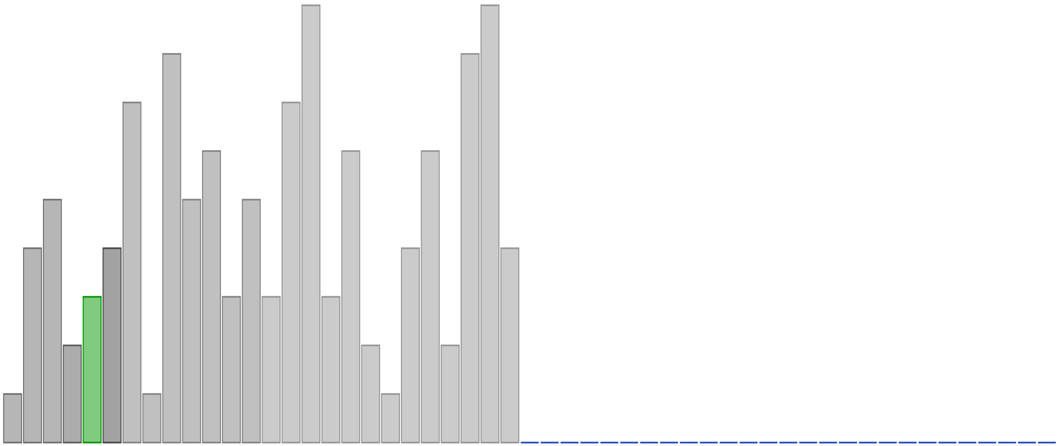


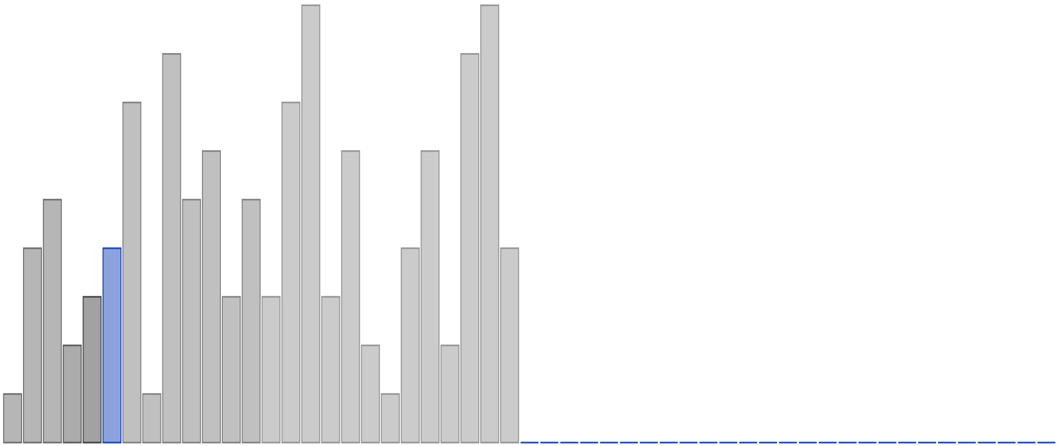


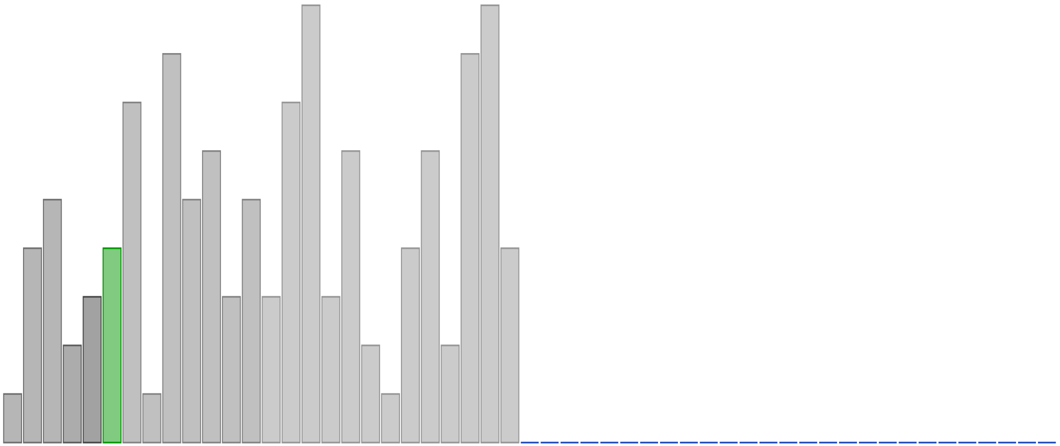


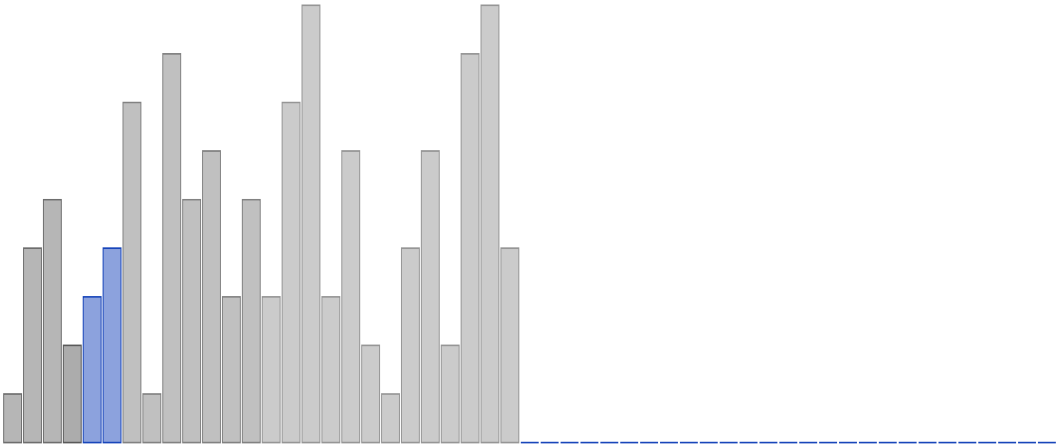


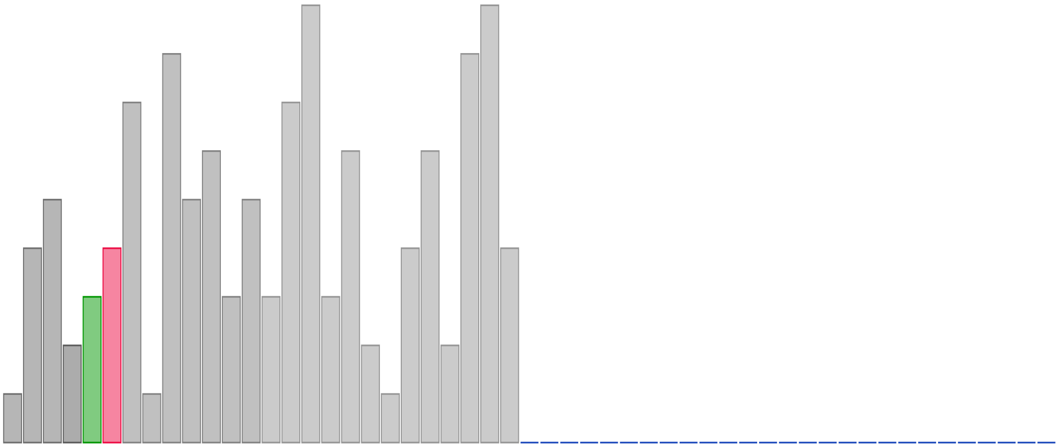


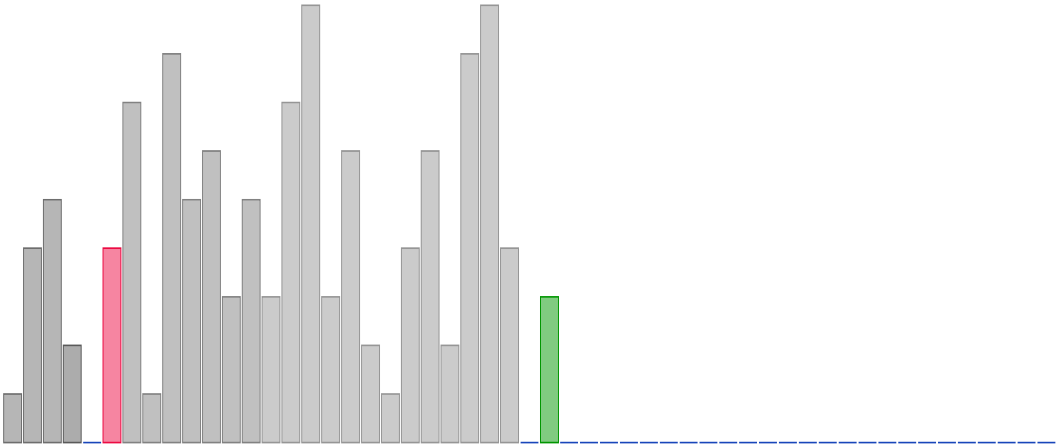


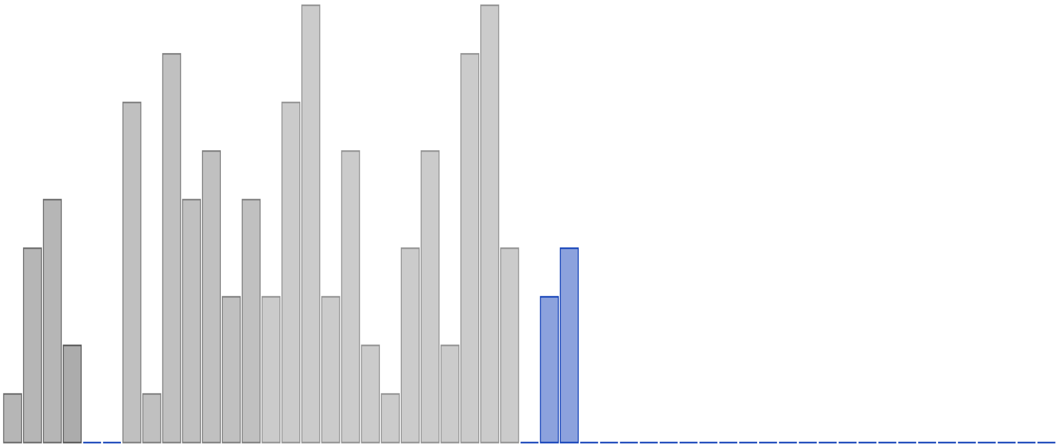


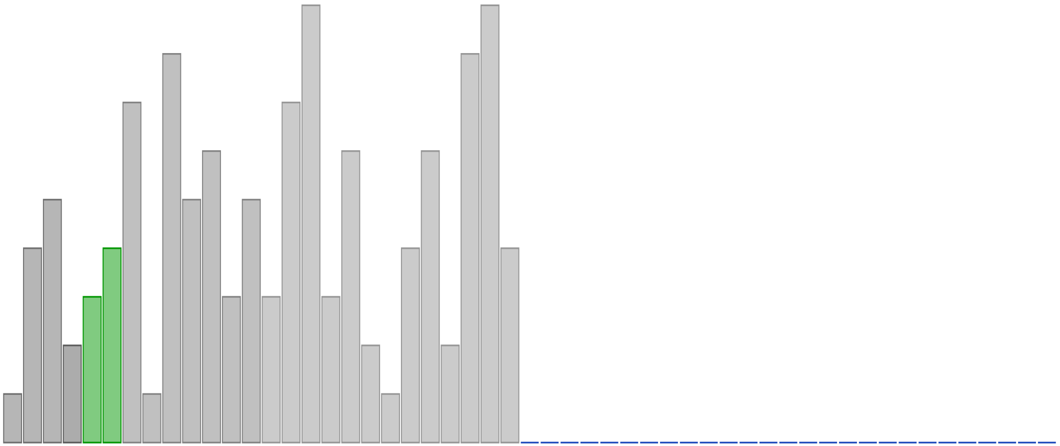


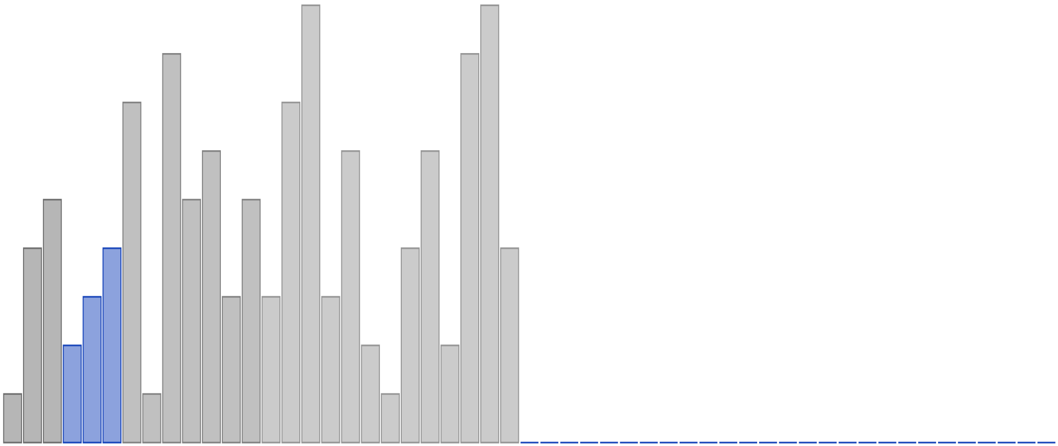


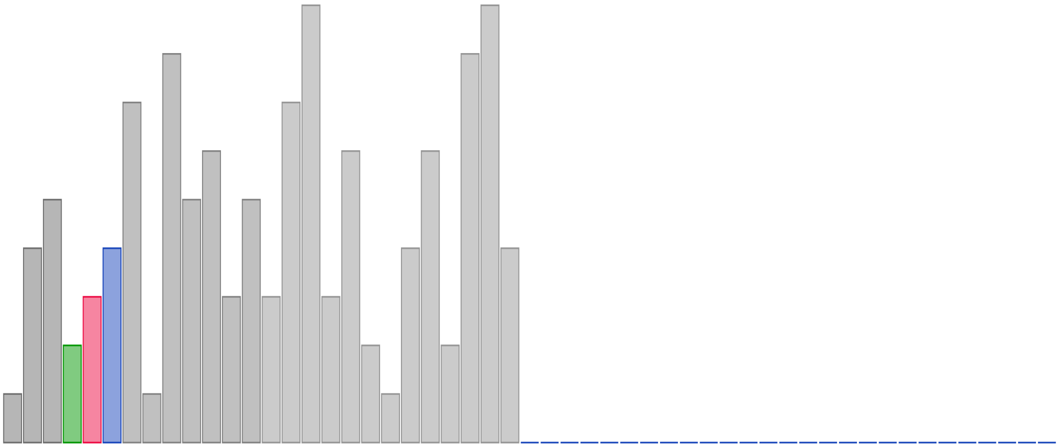


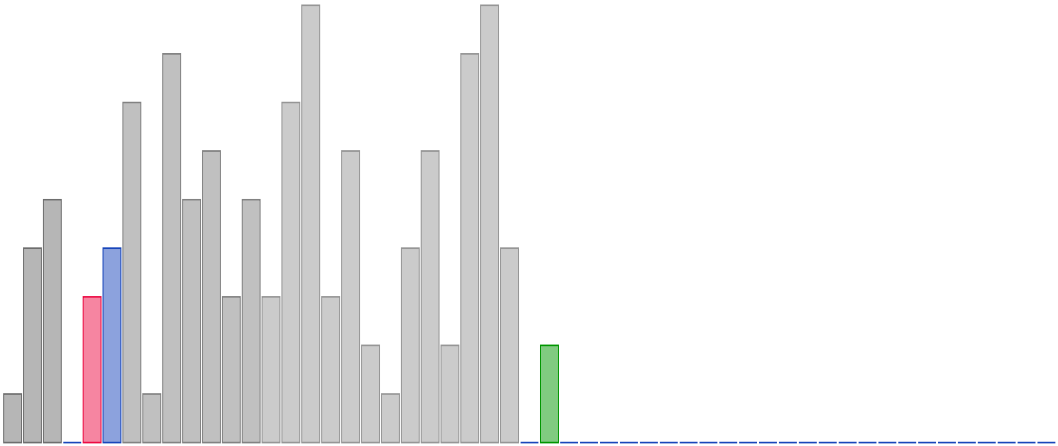


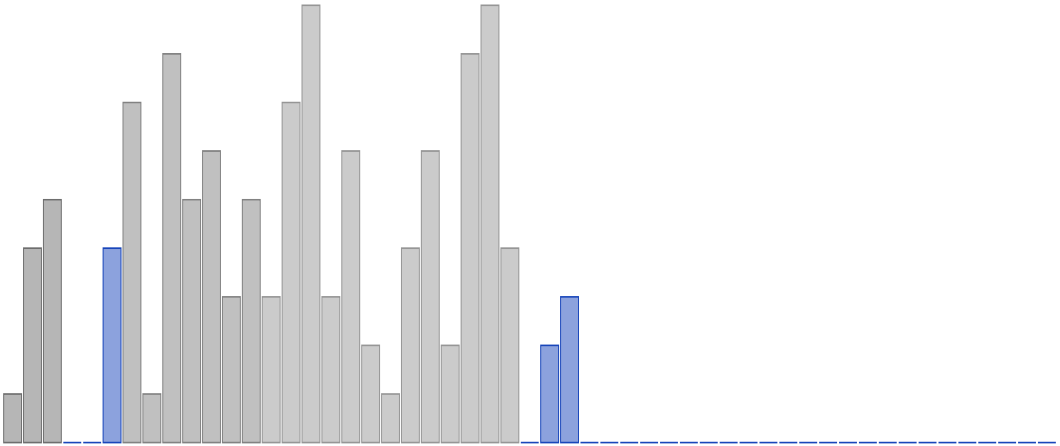


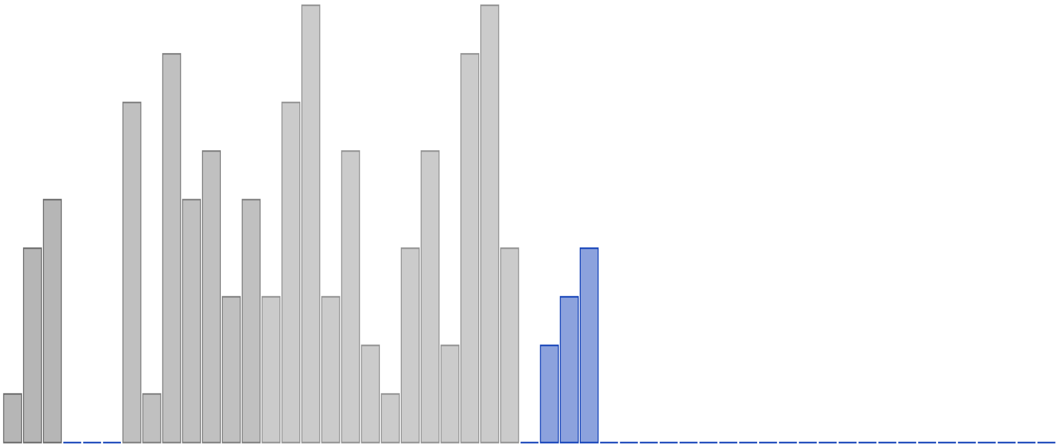


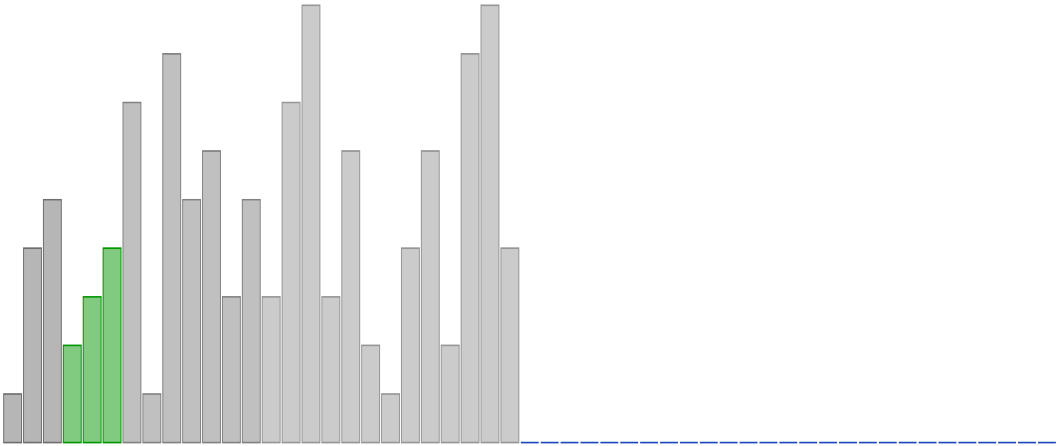


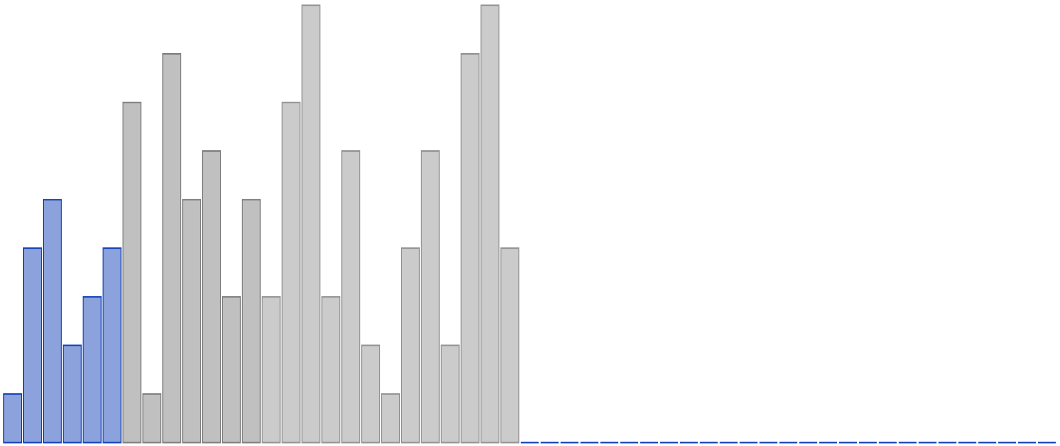


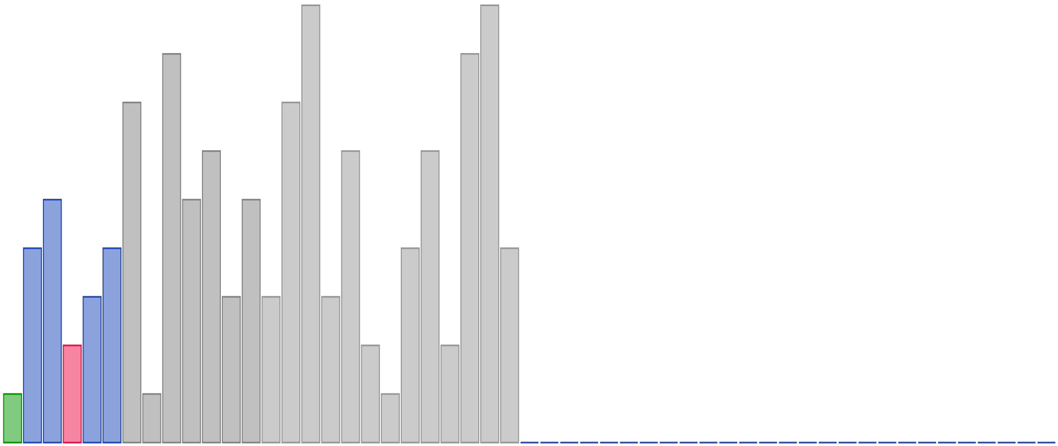


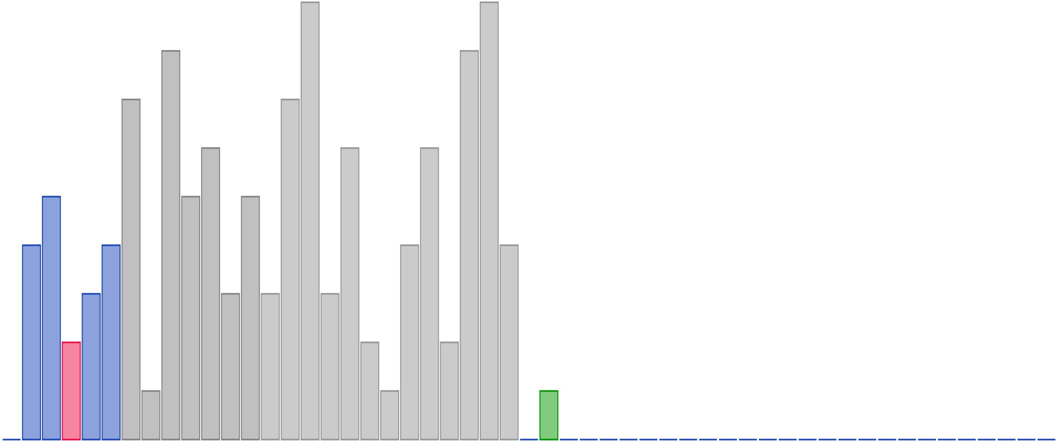


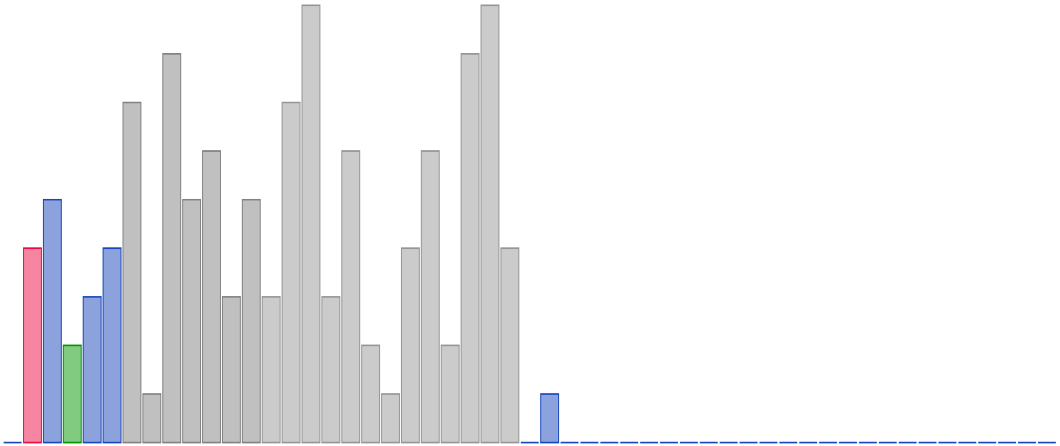


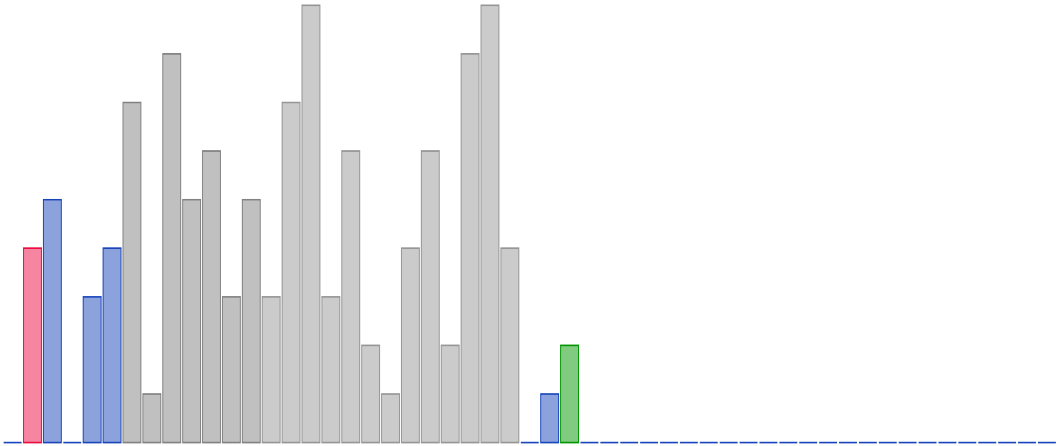


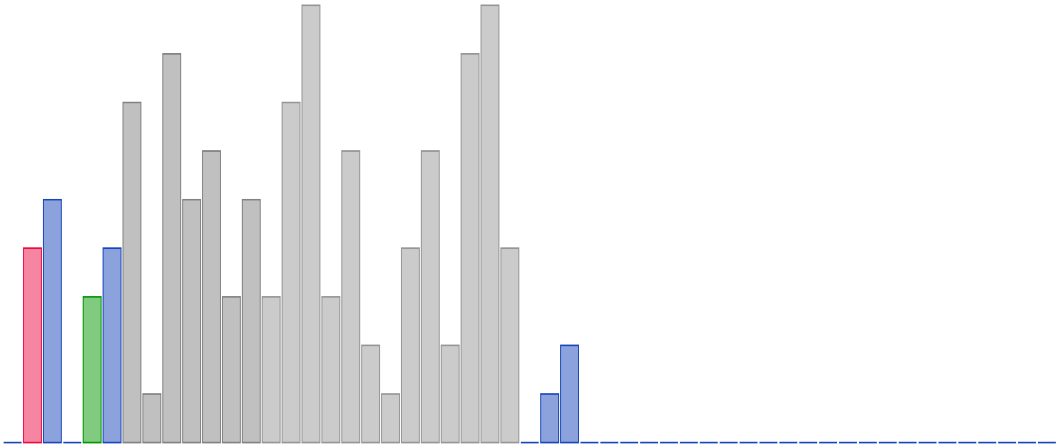


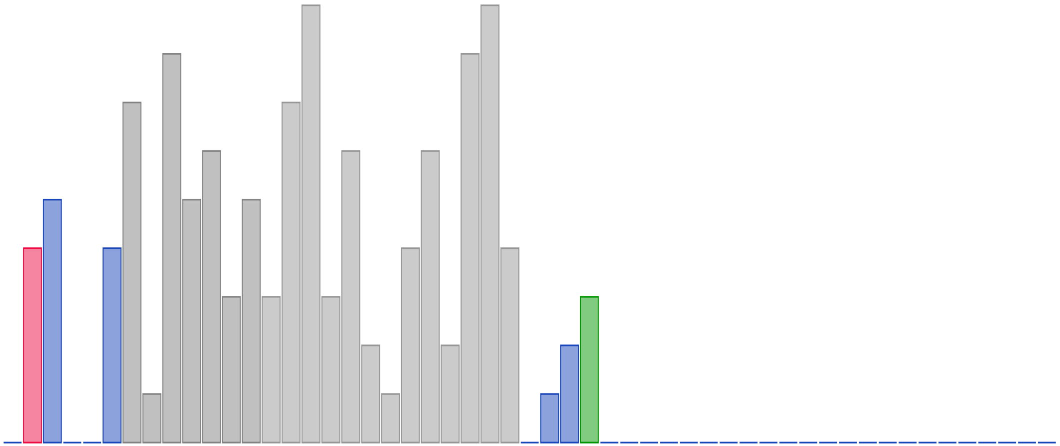


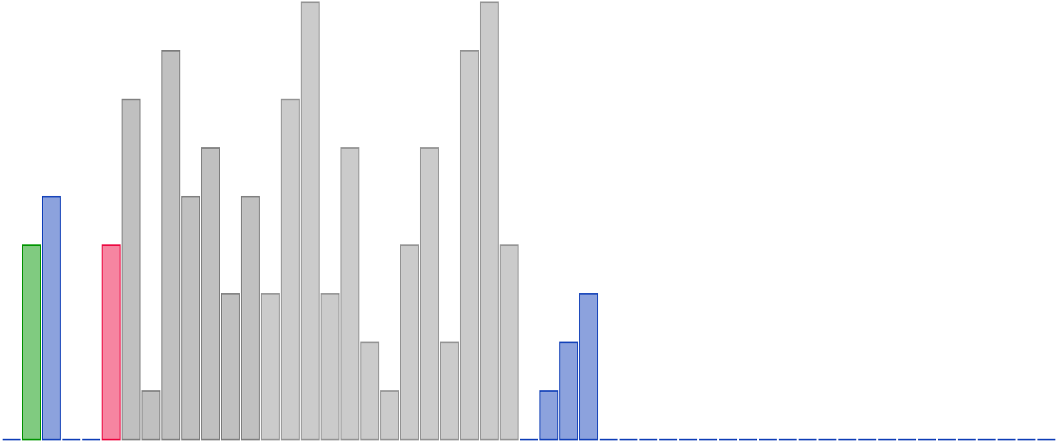


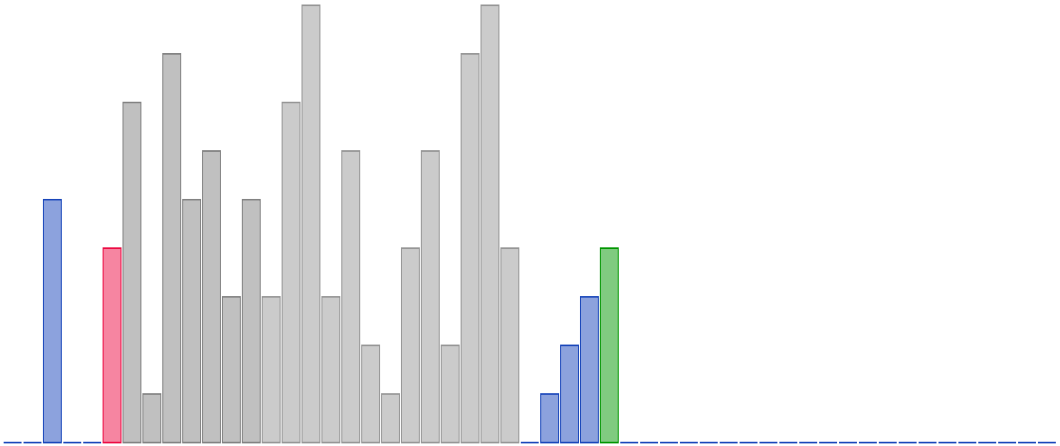


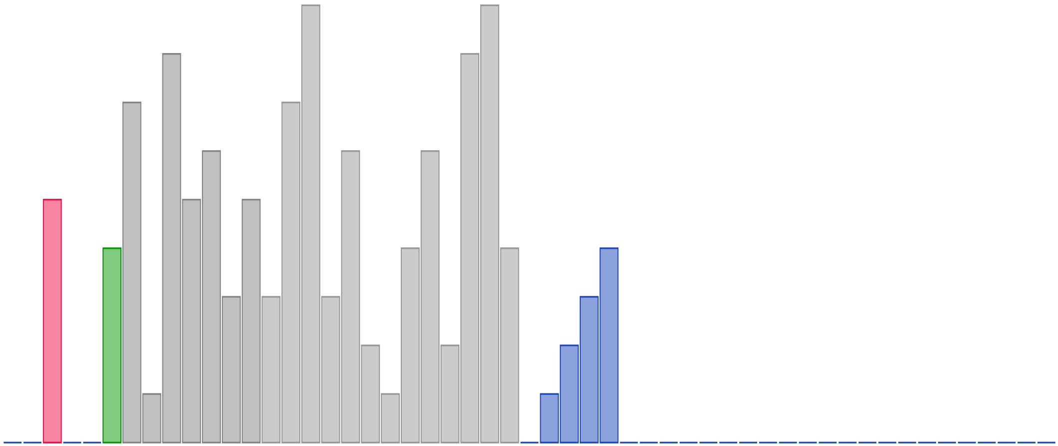


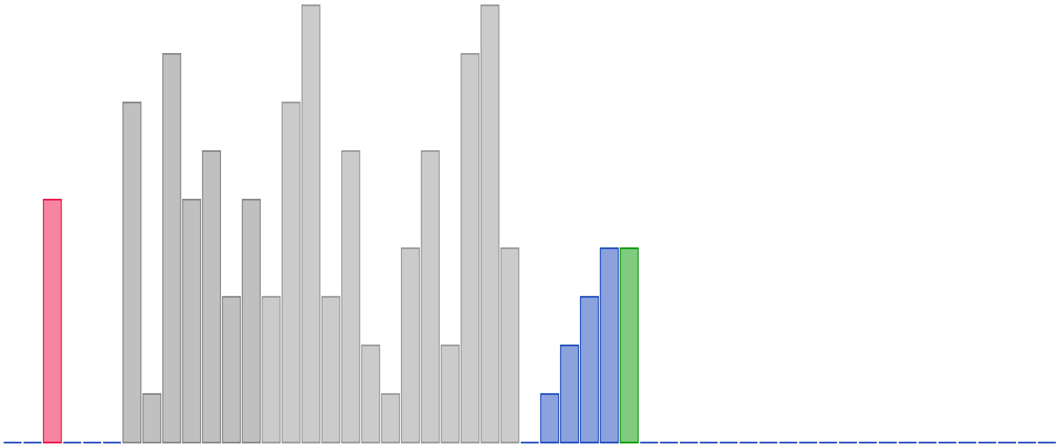


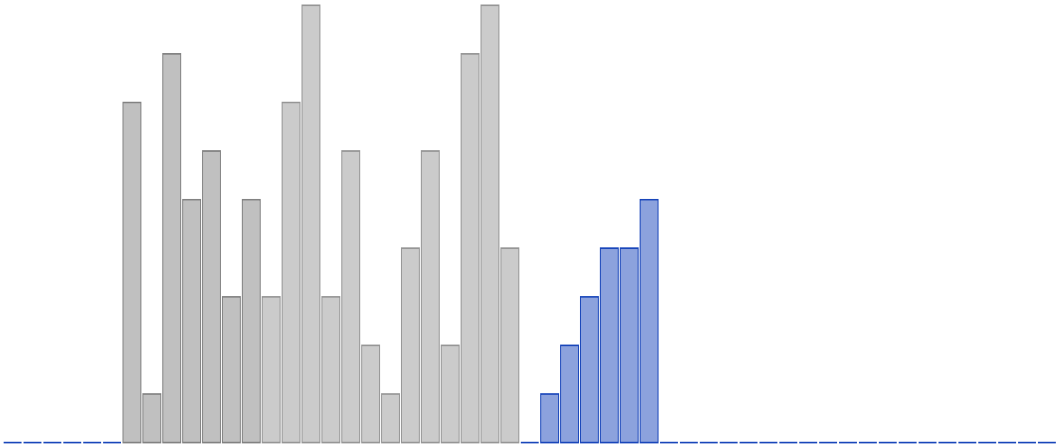


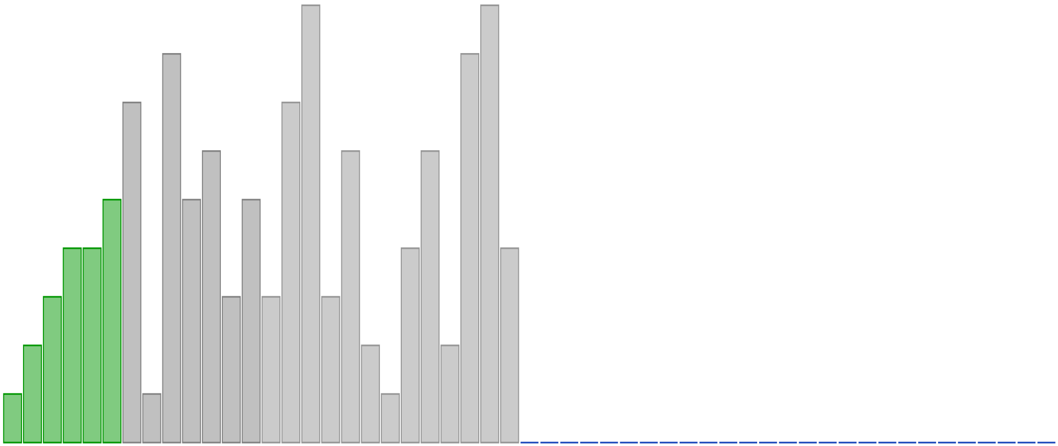


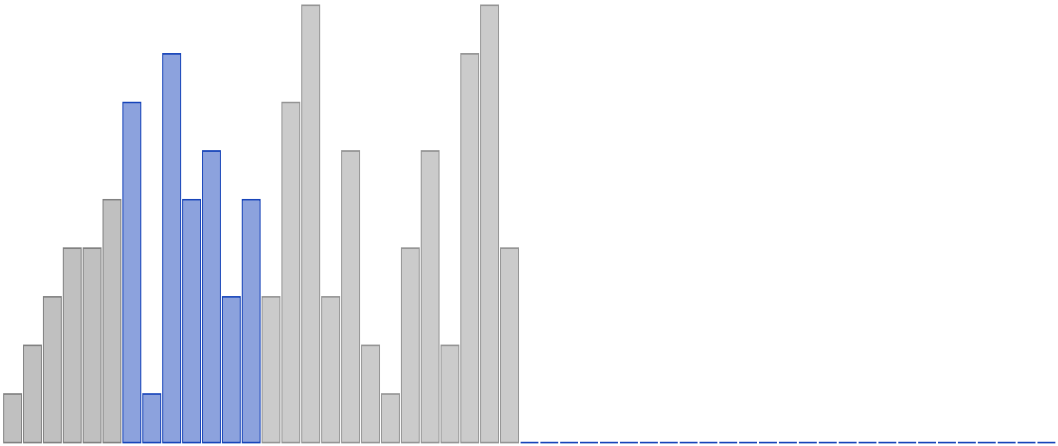


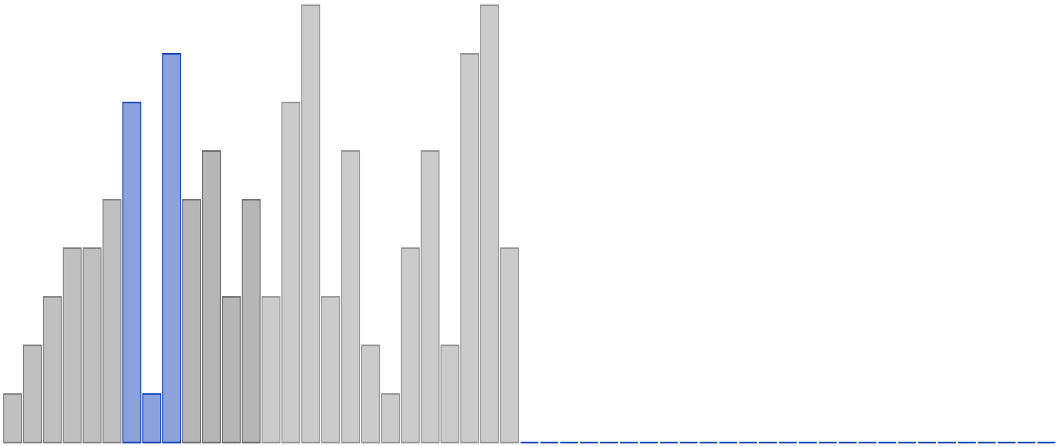


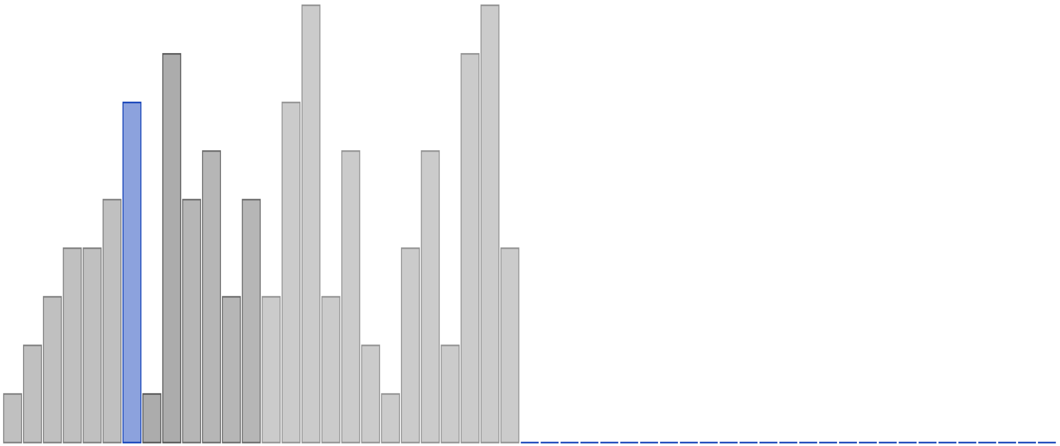


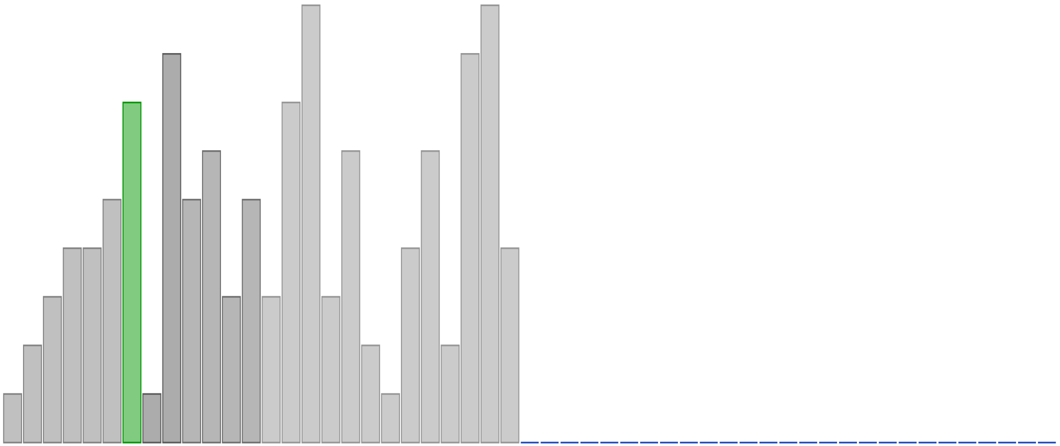


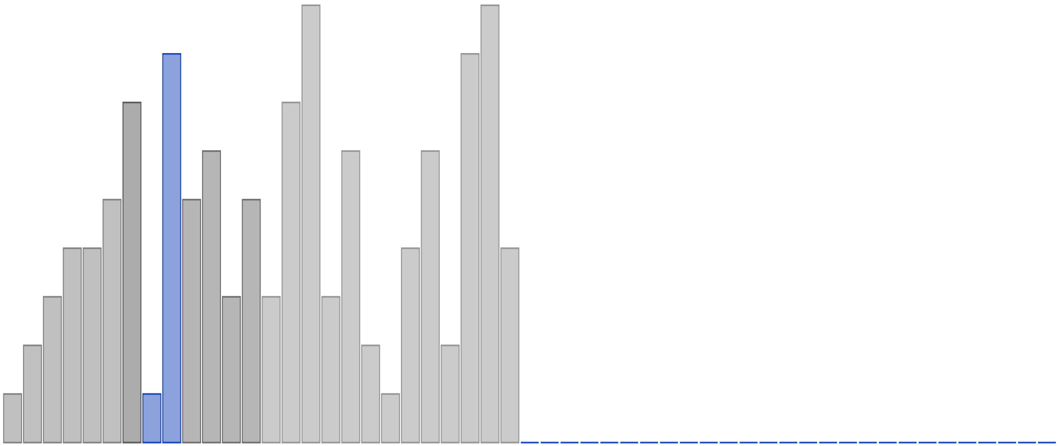


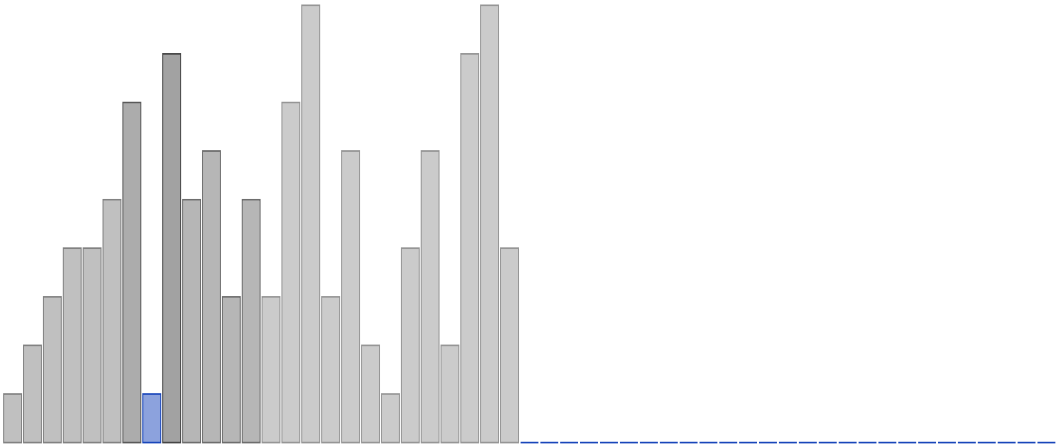


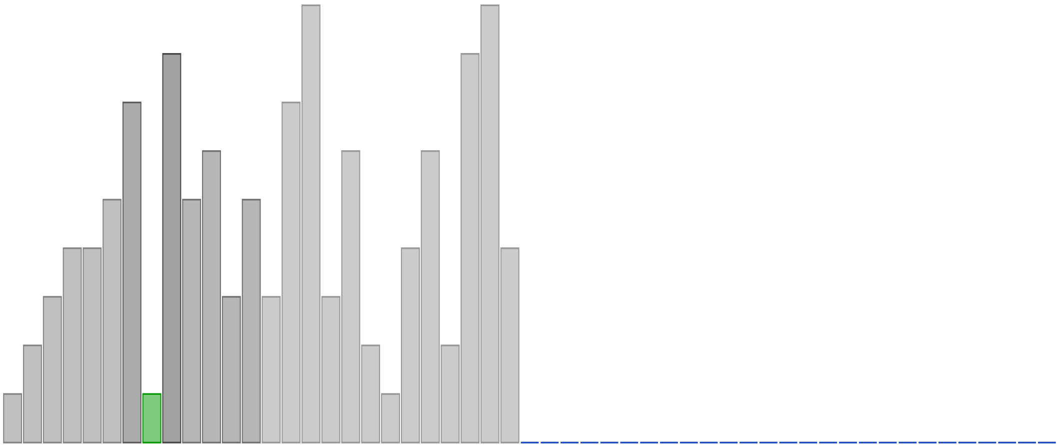


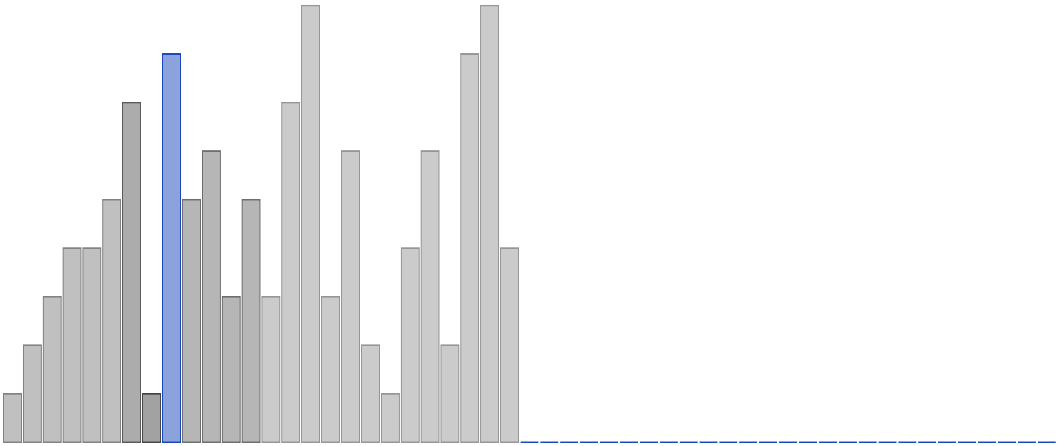


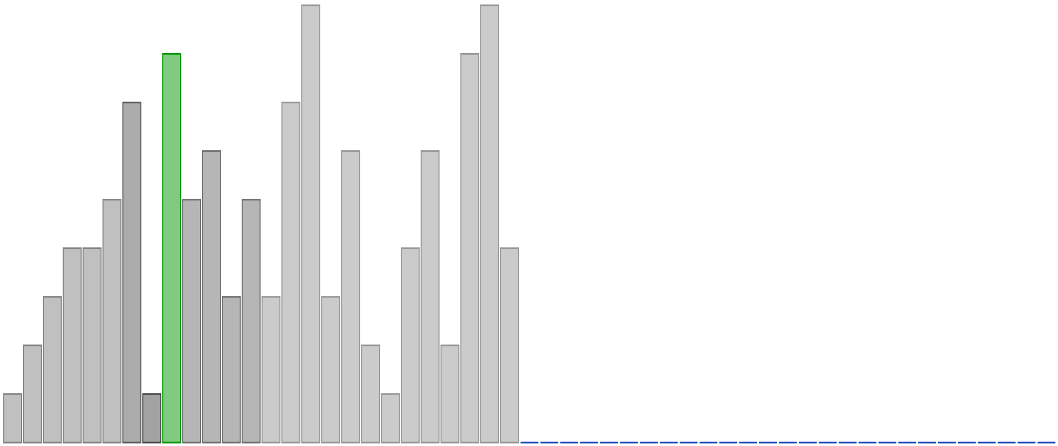


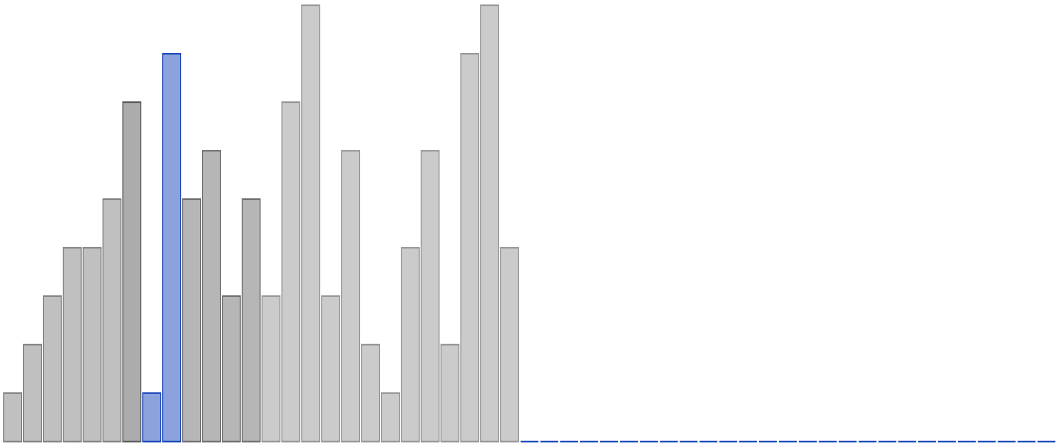


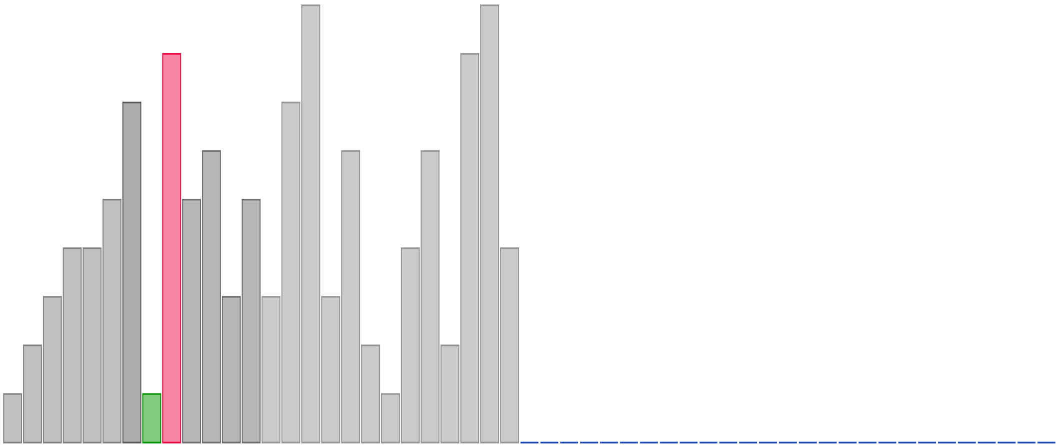




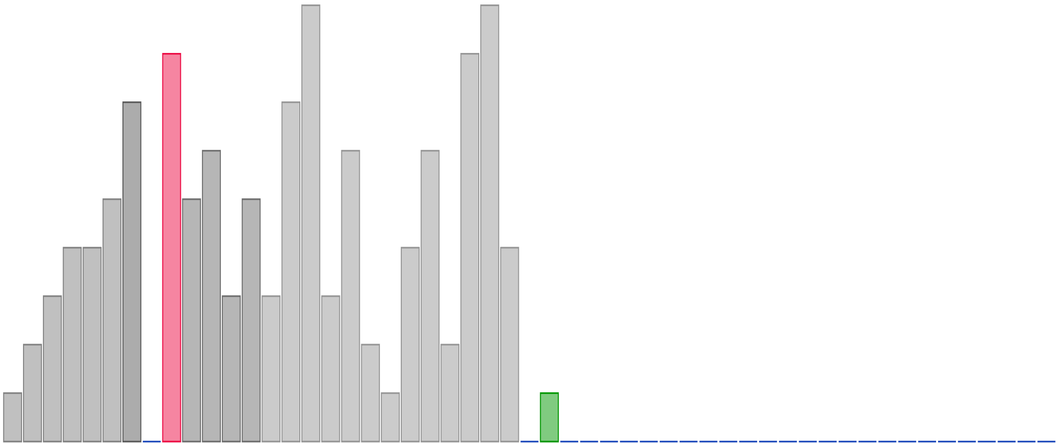


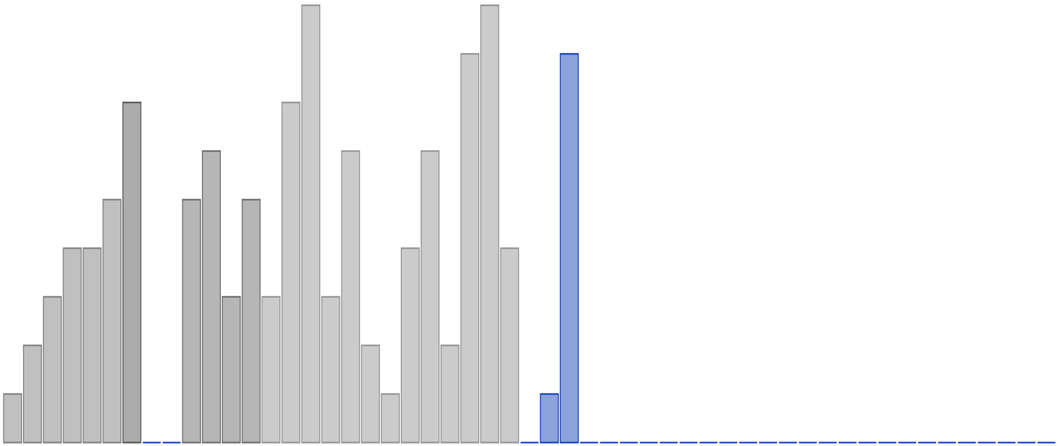


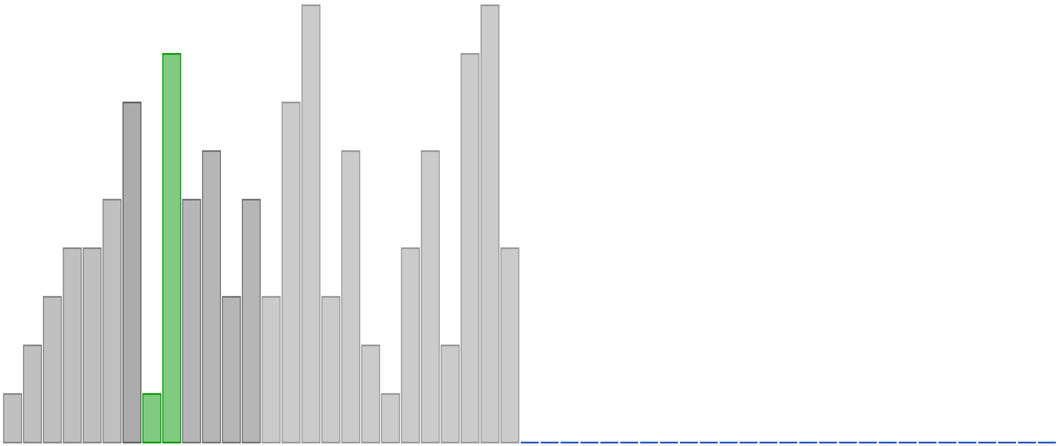


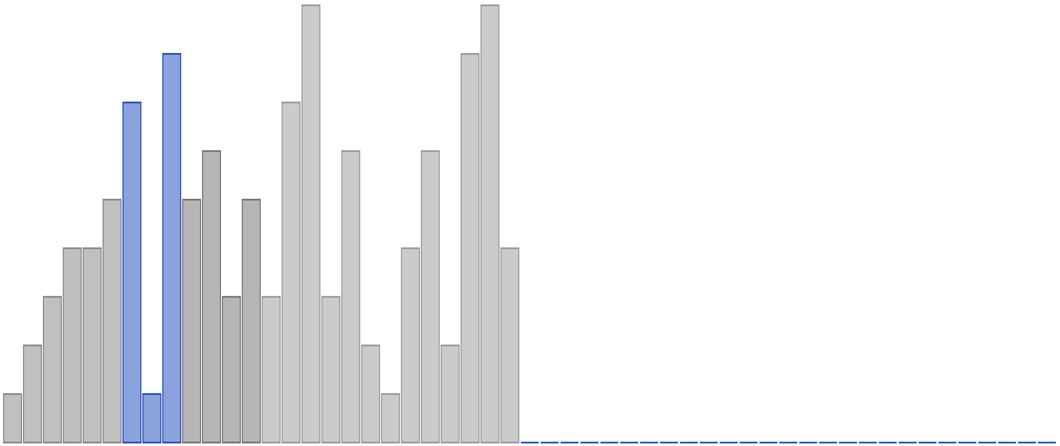


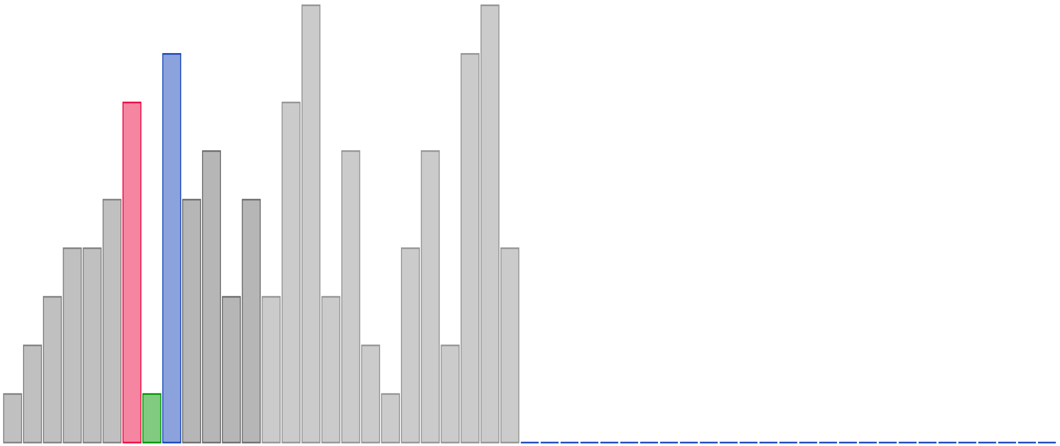
Mergesort: Example

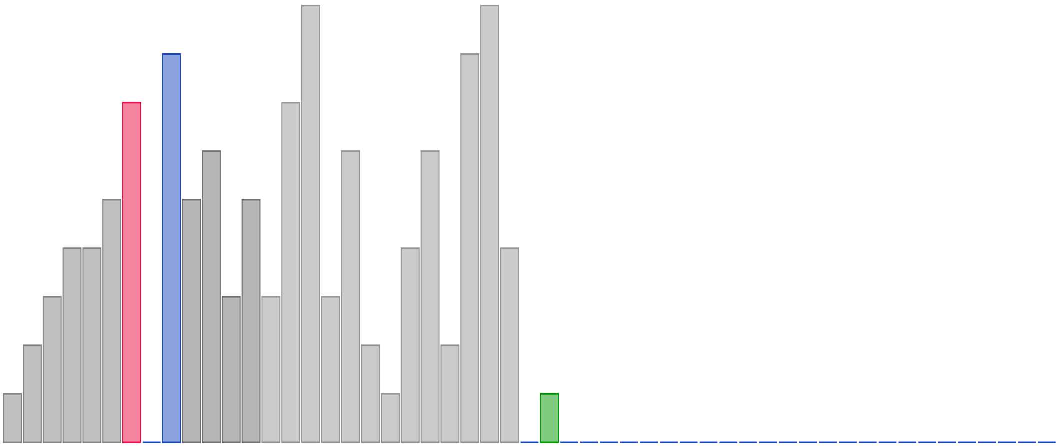


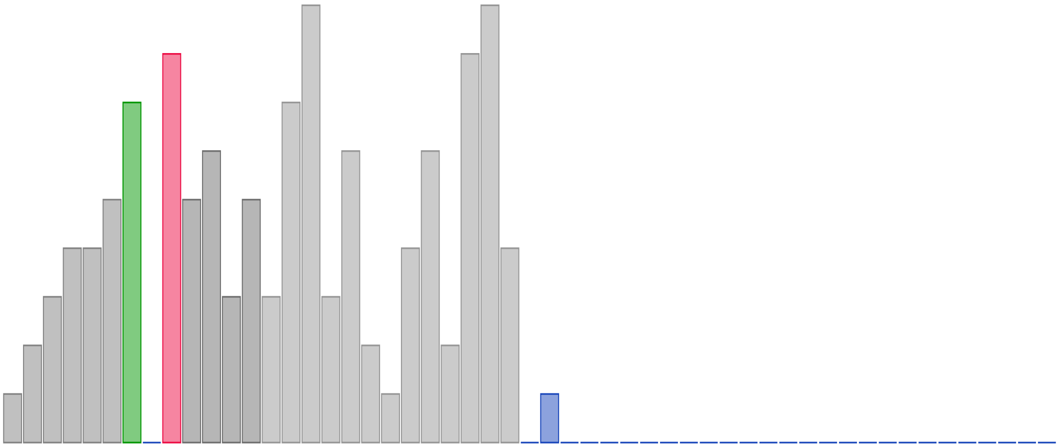


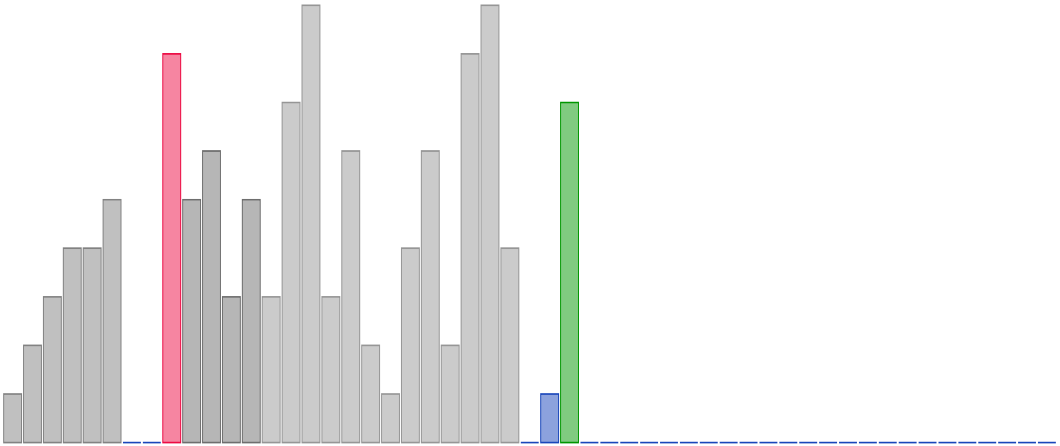


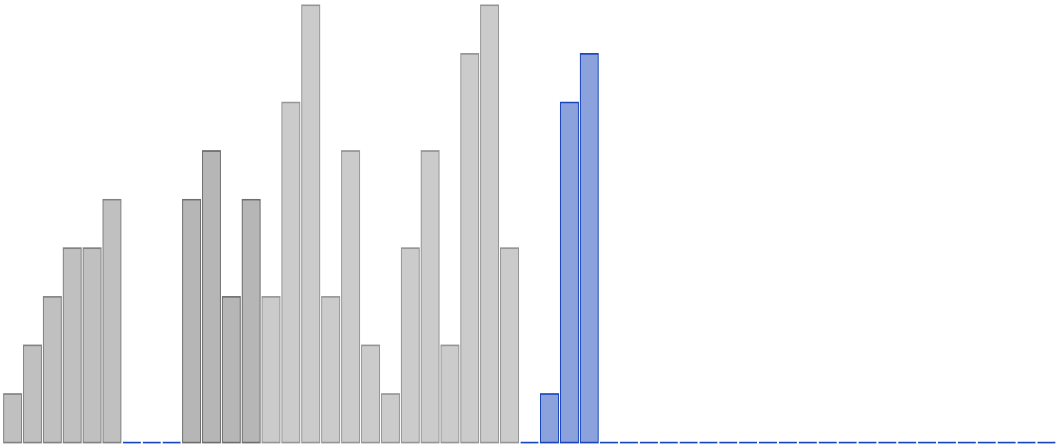


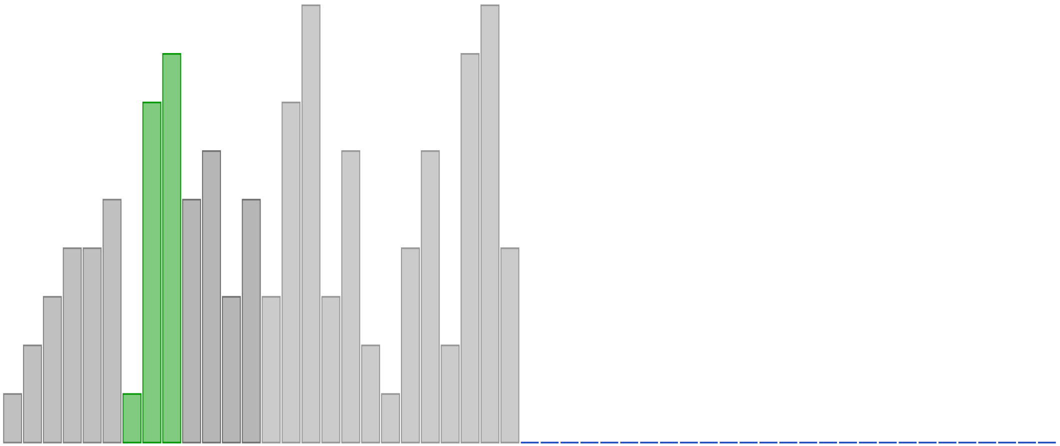


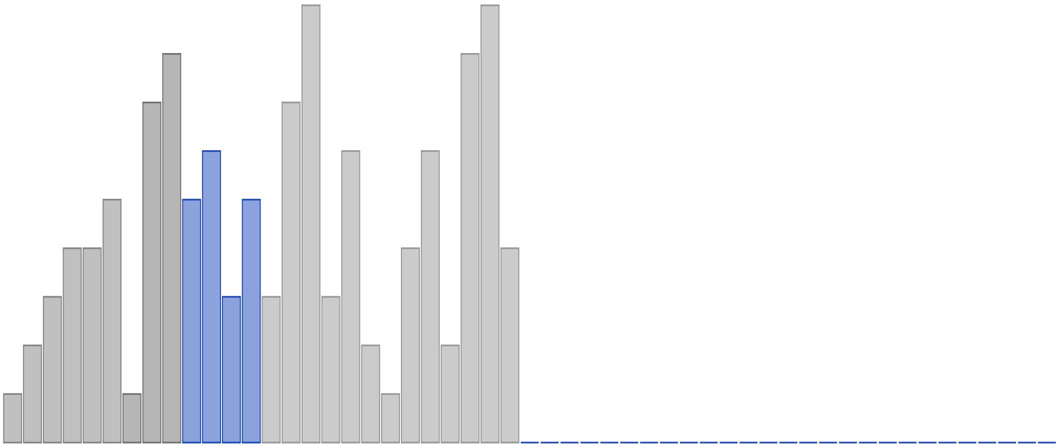


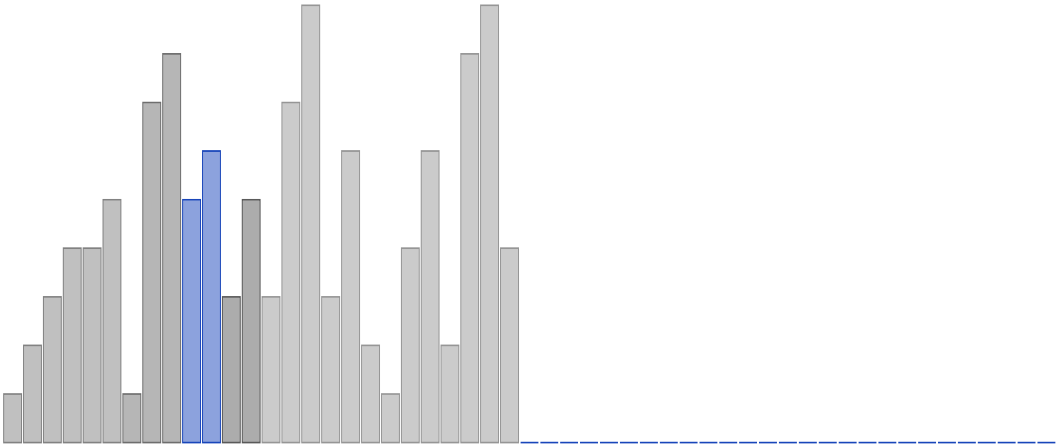


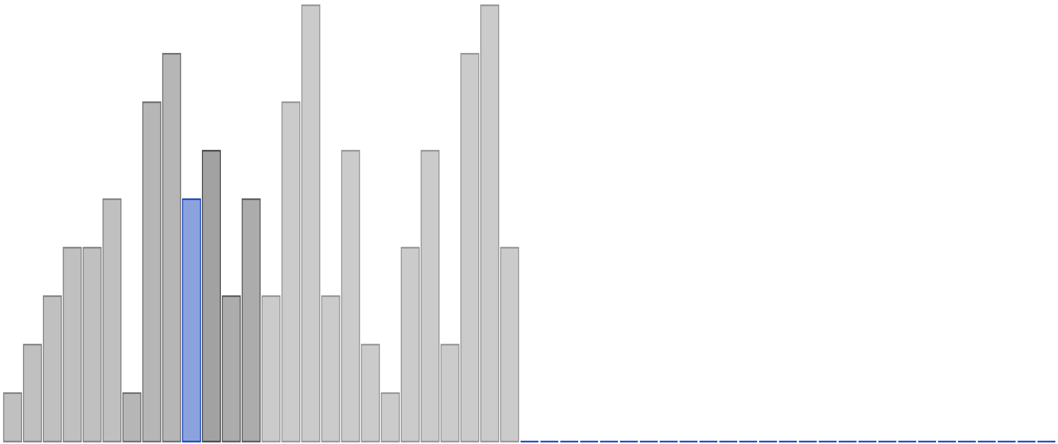


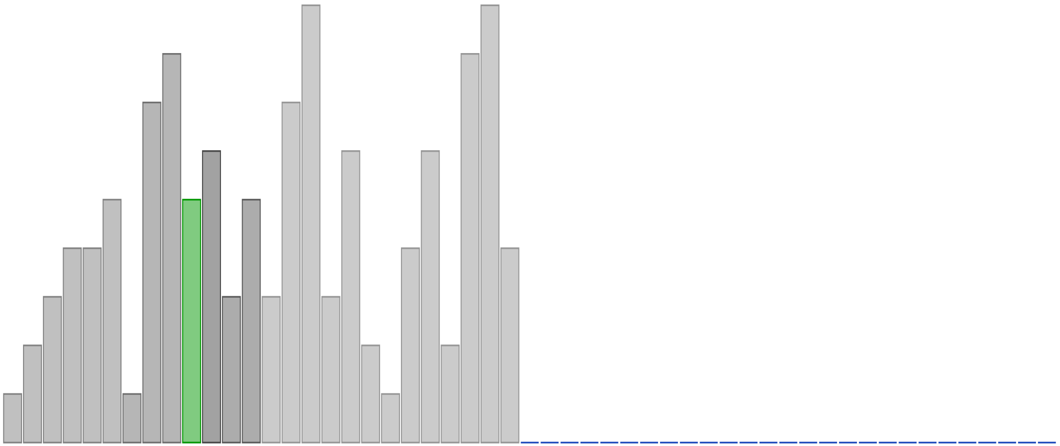


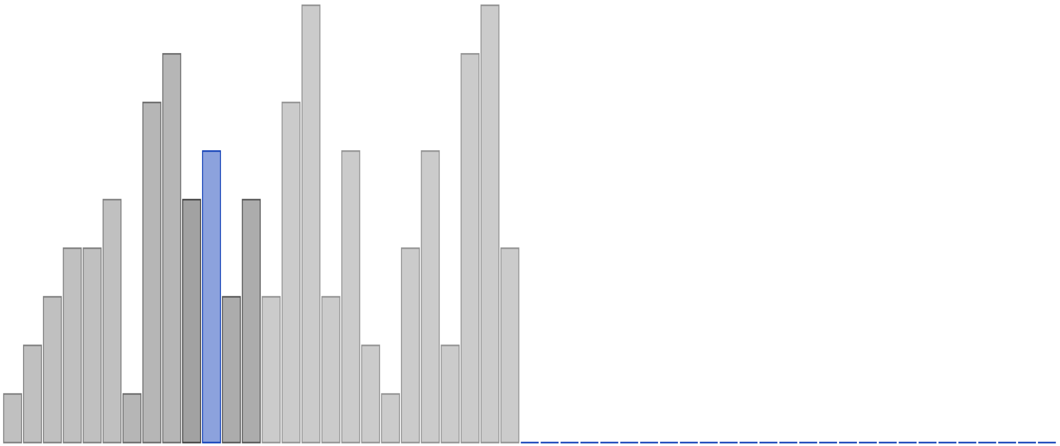


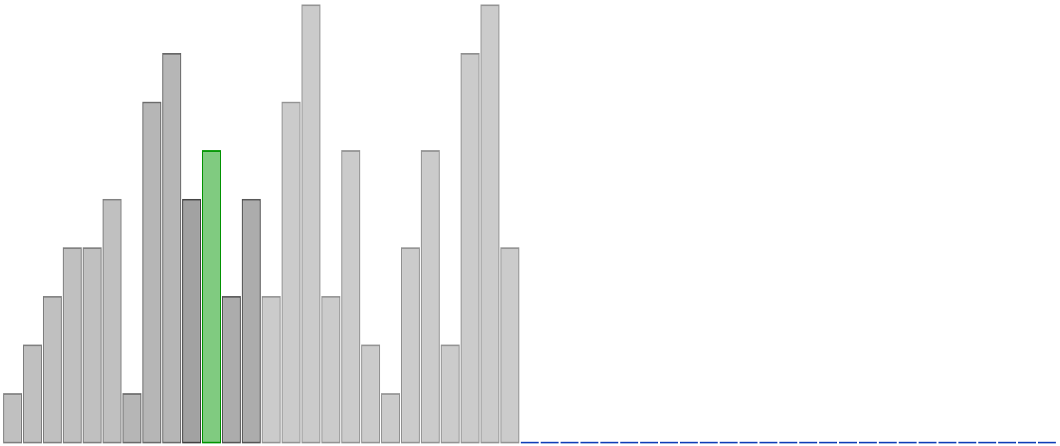


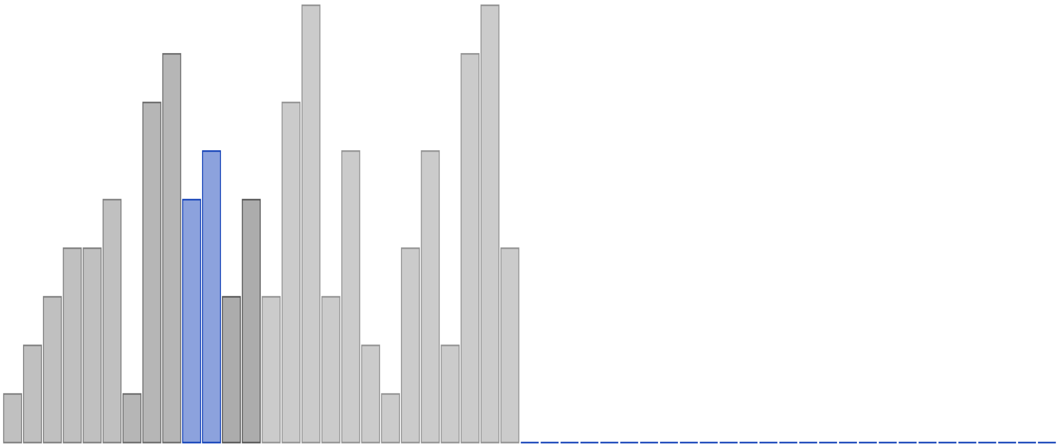


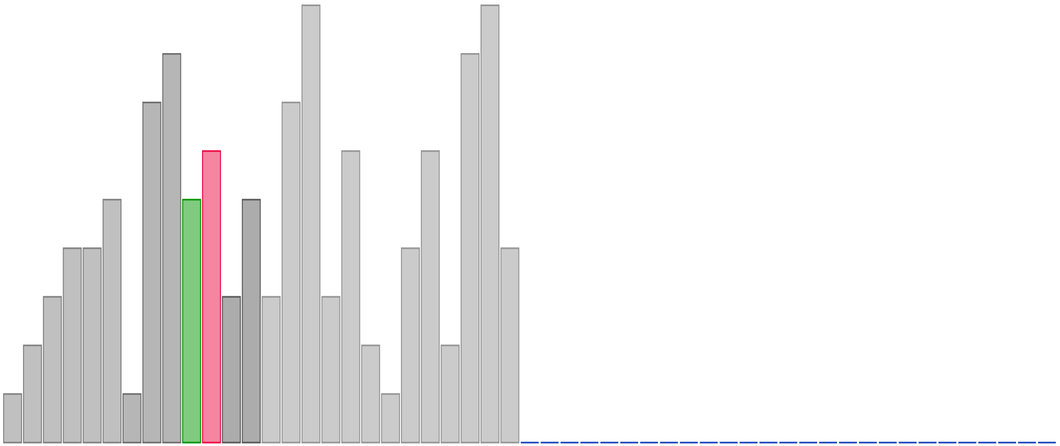


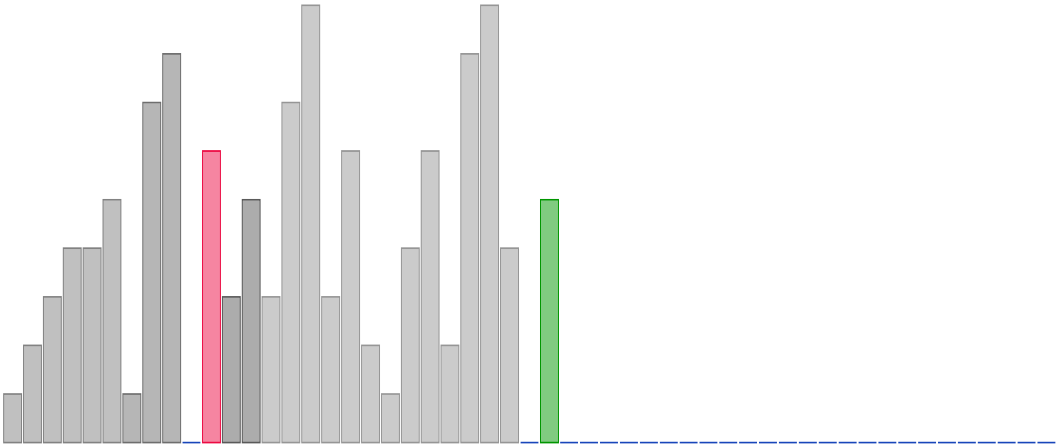


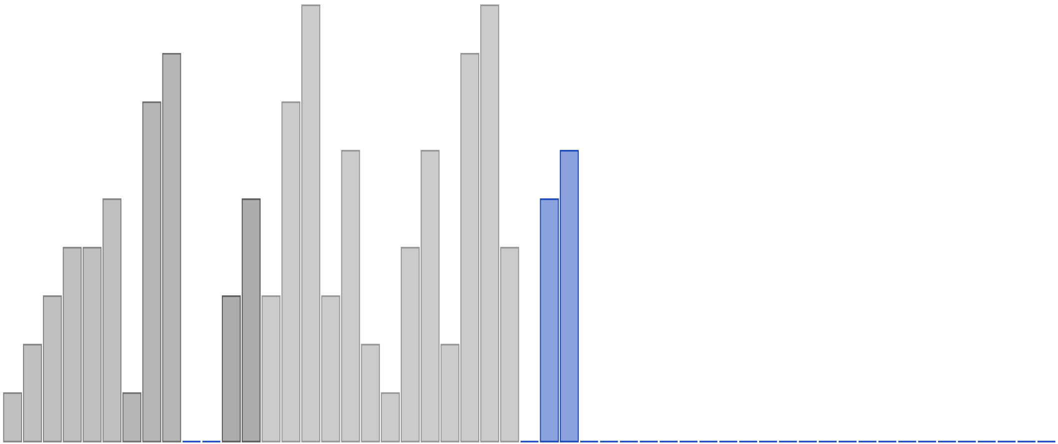


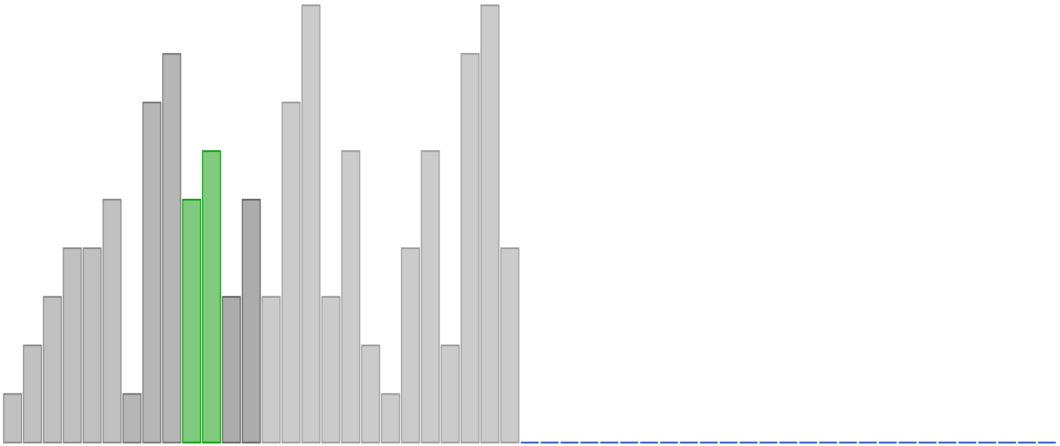


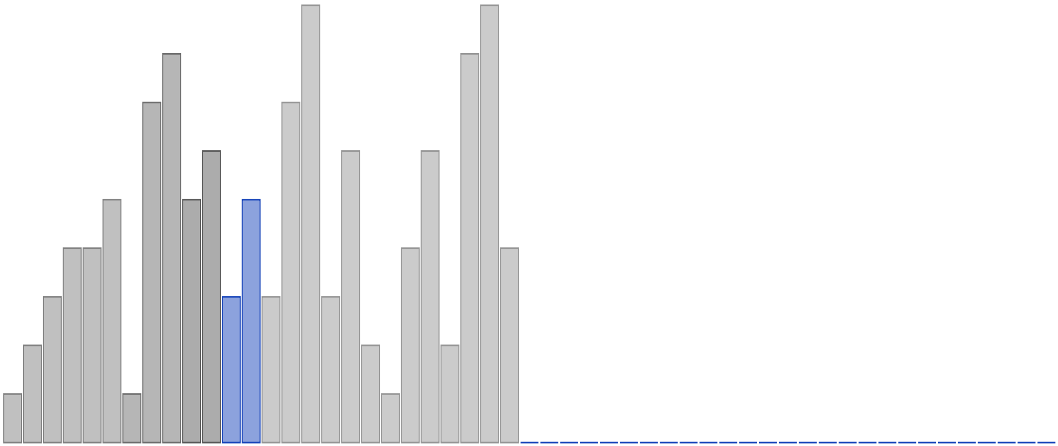


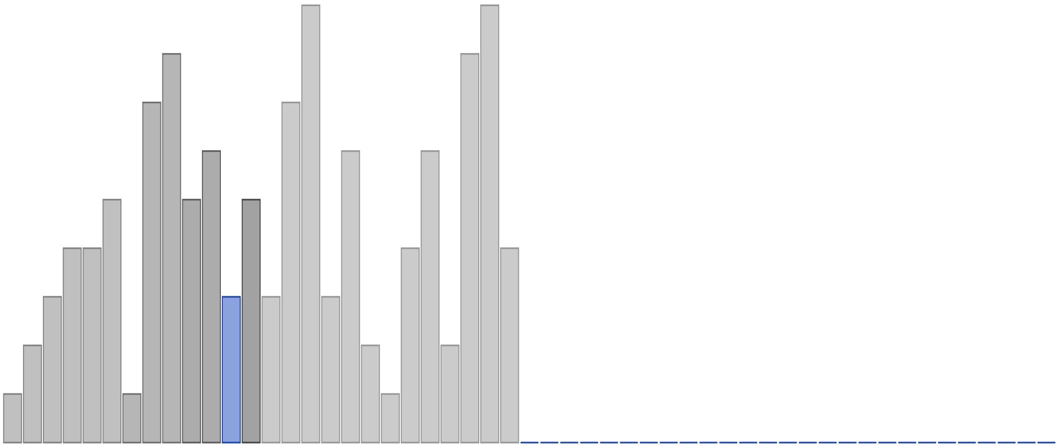


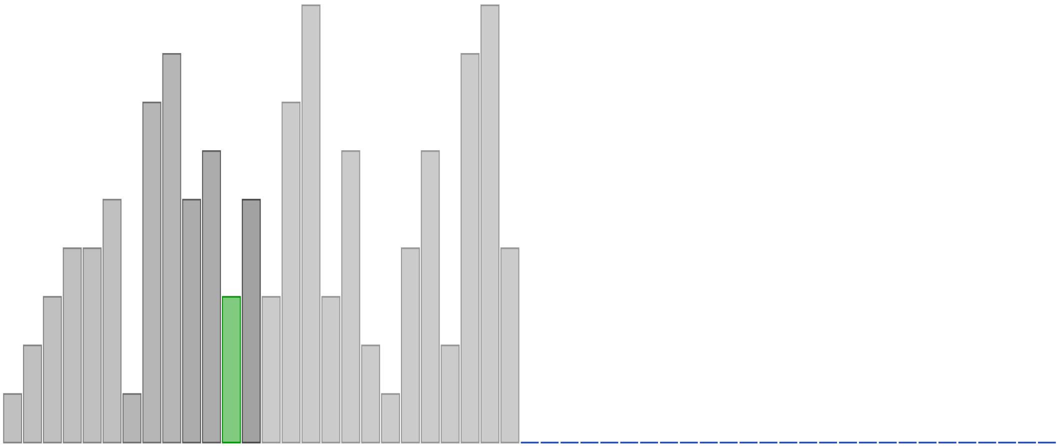


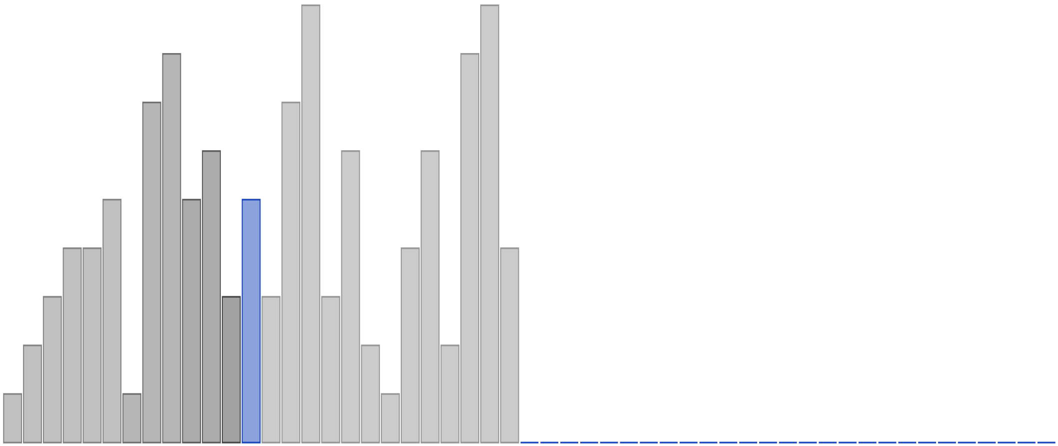


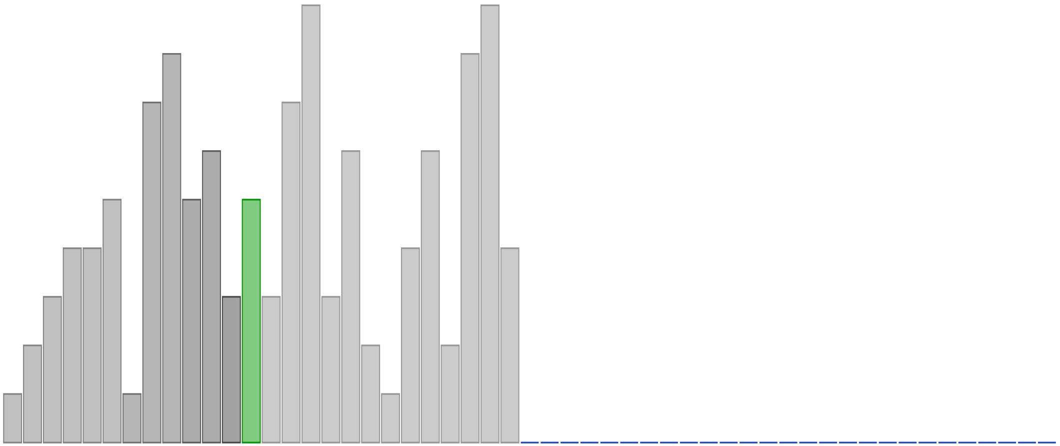


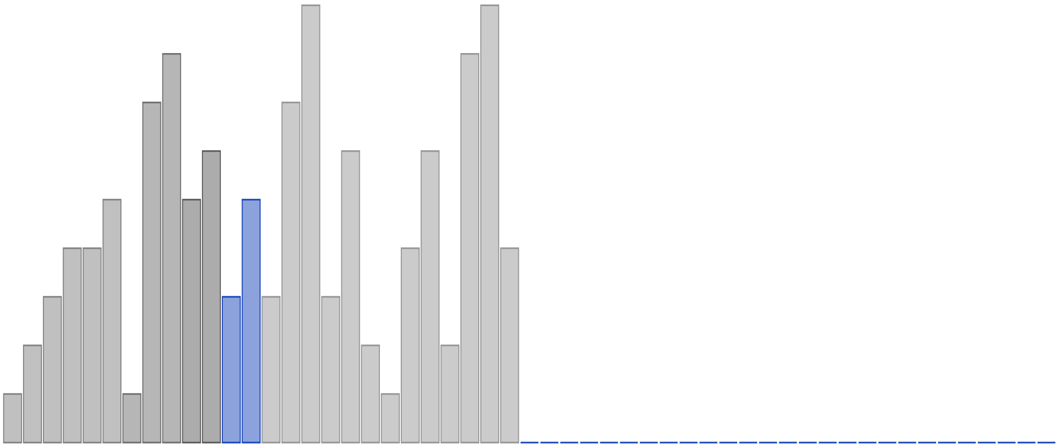


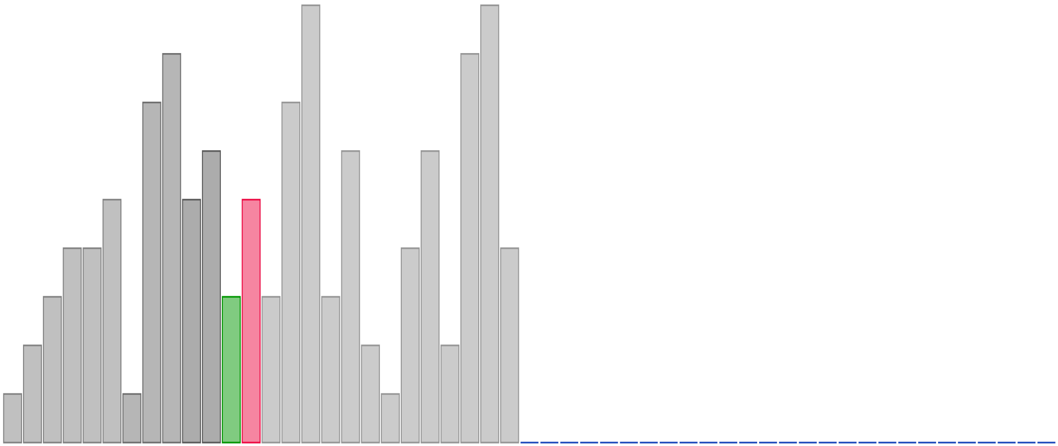


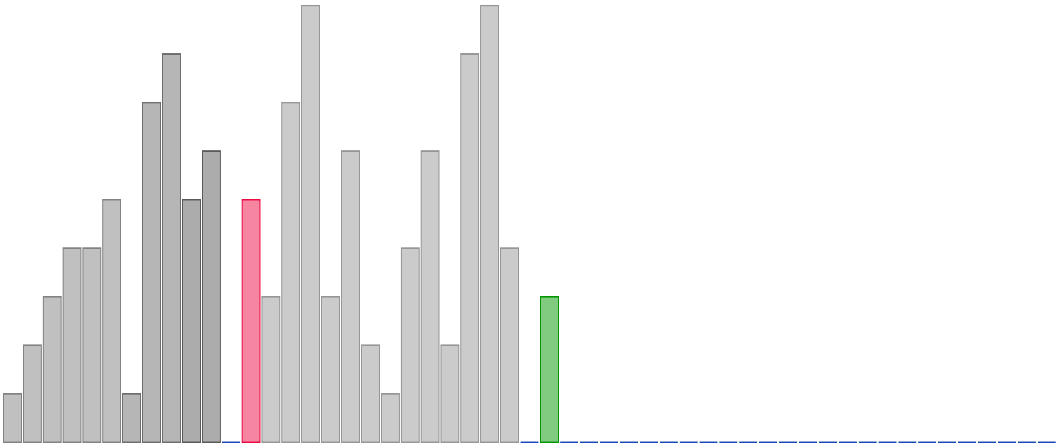


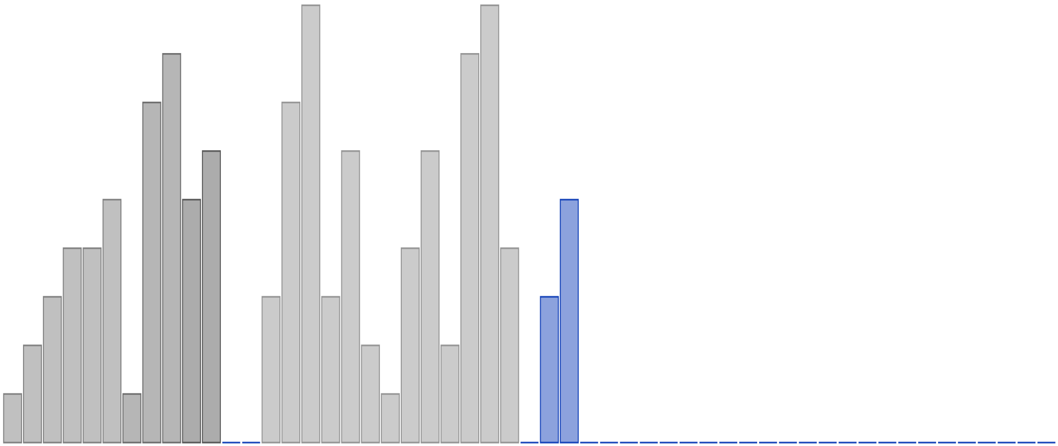


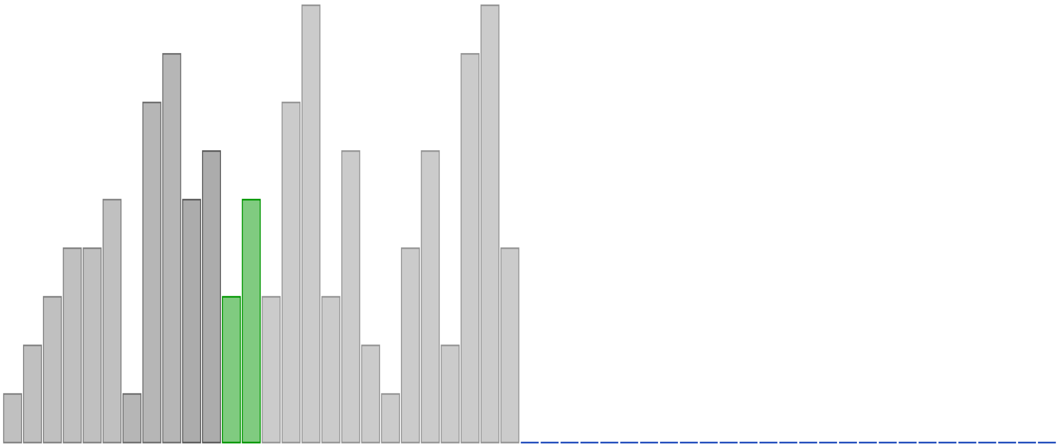


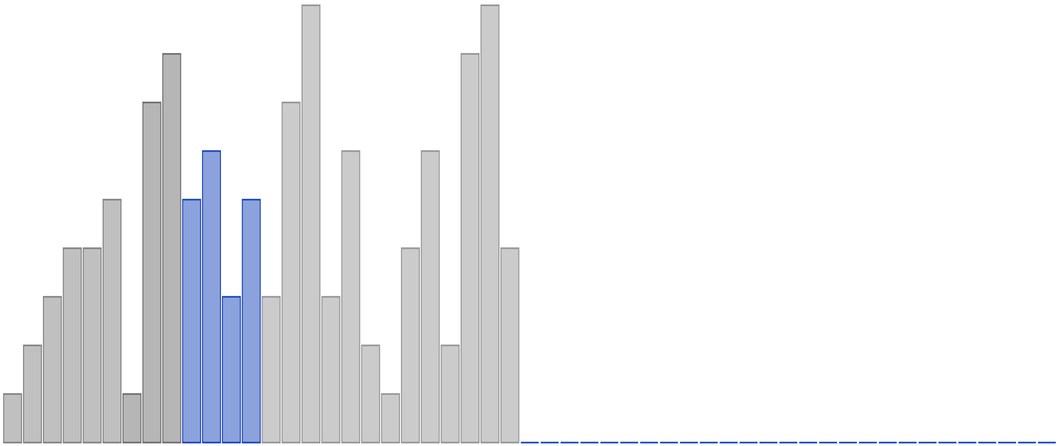


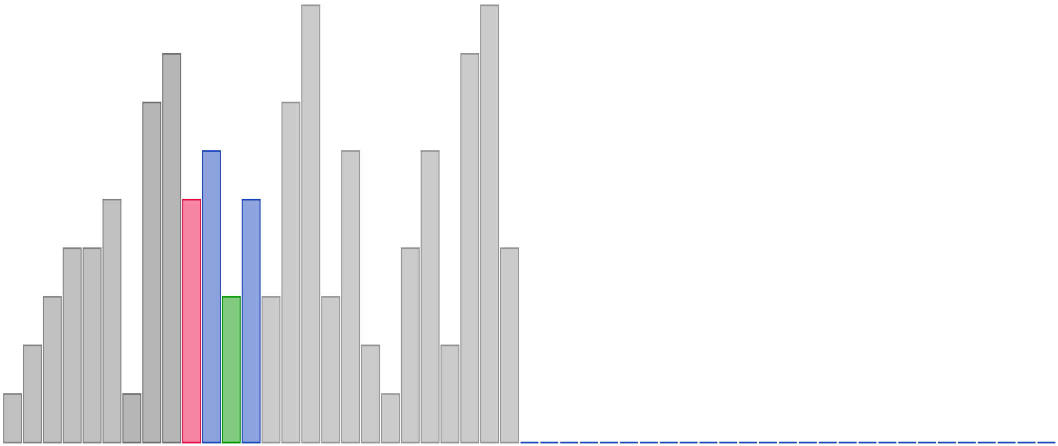


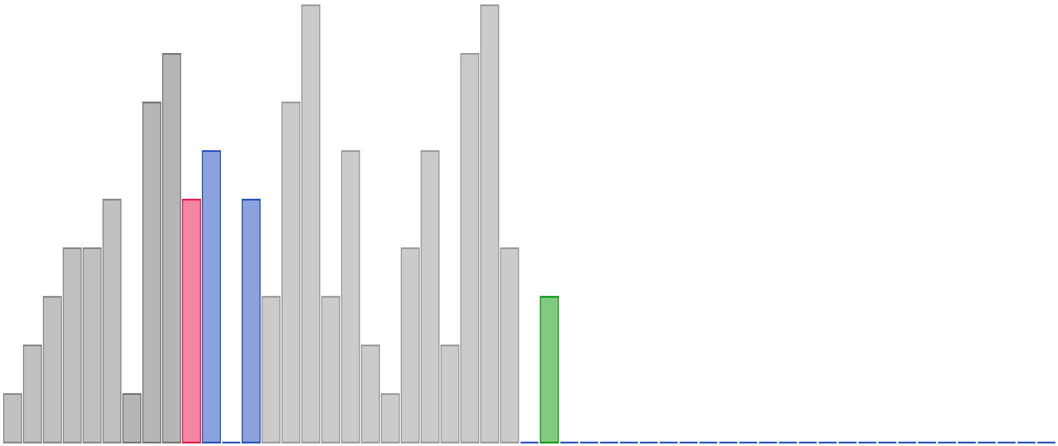


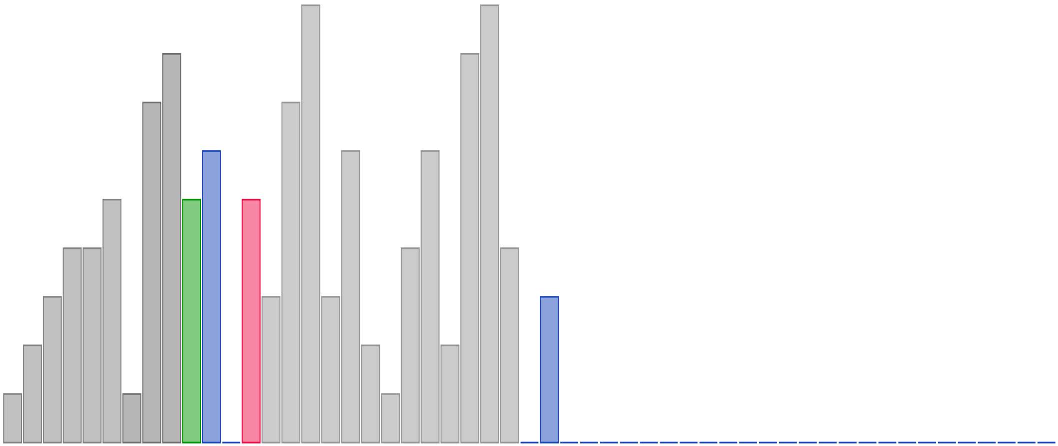


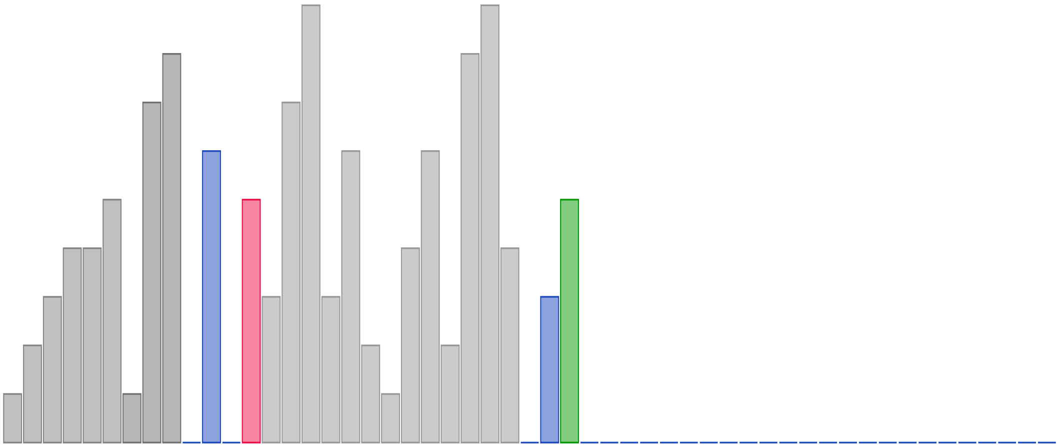


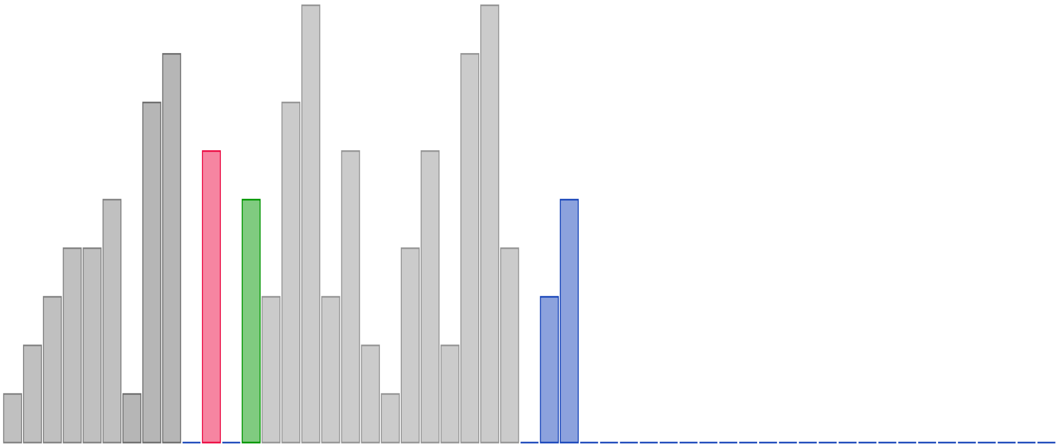


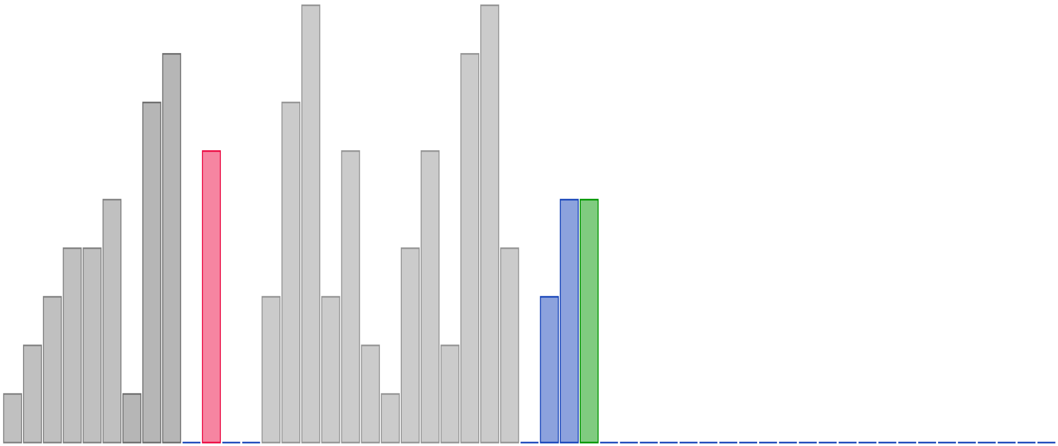


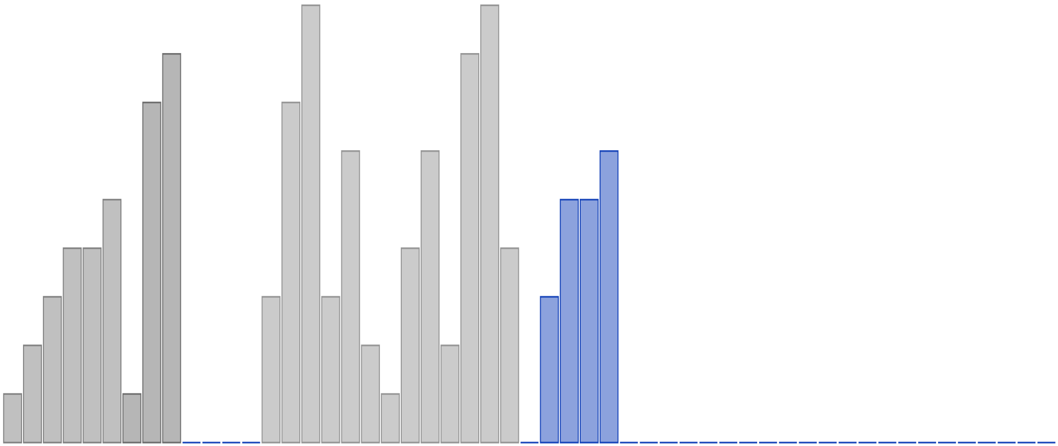


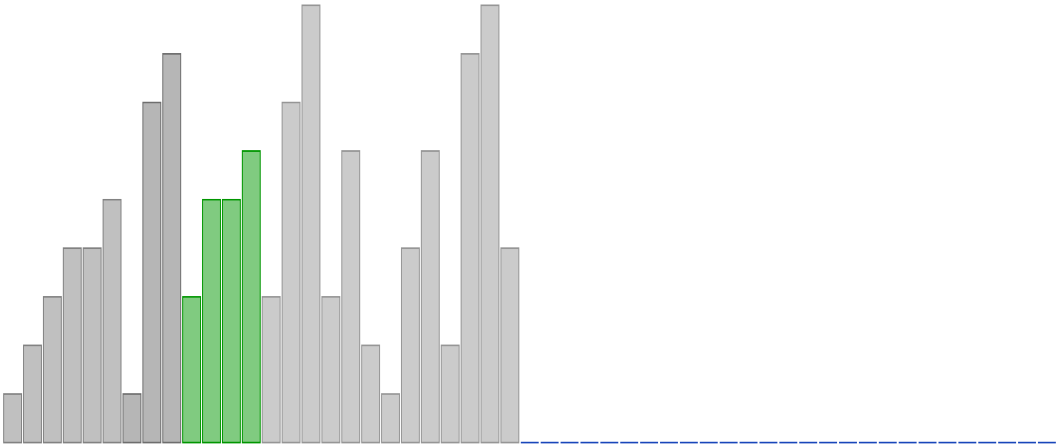


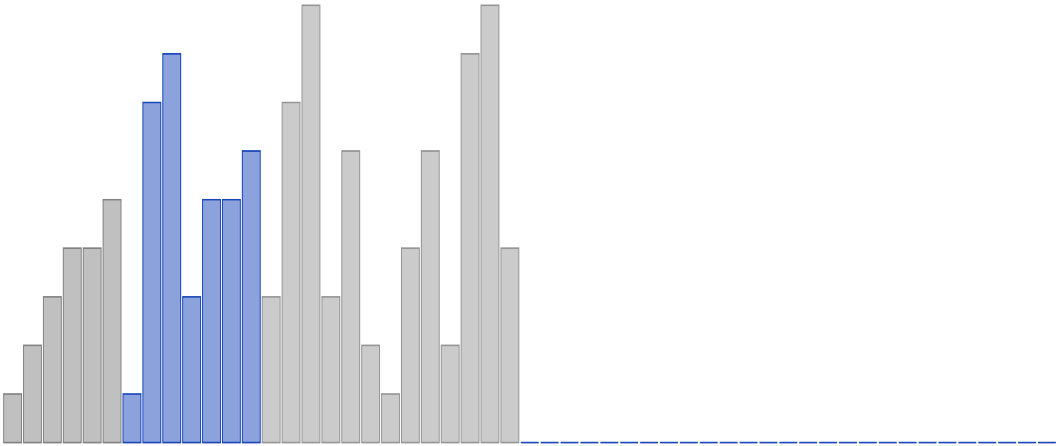


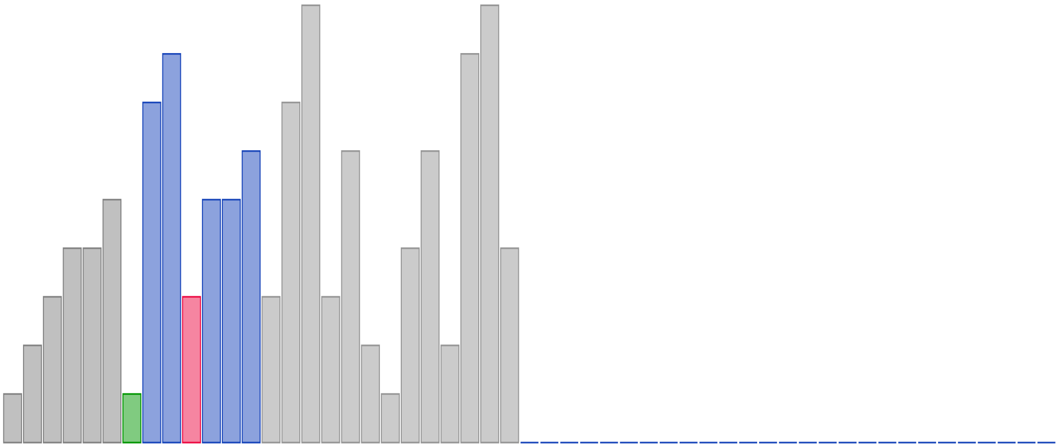


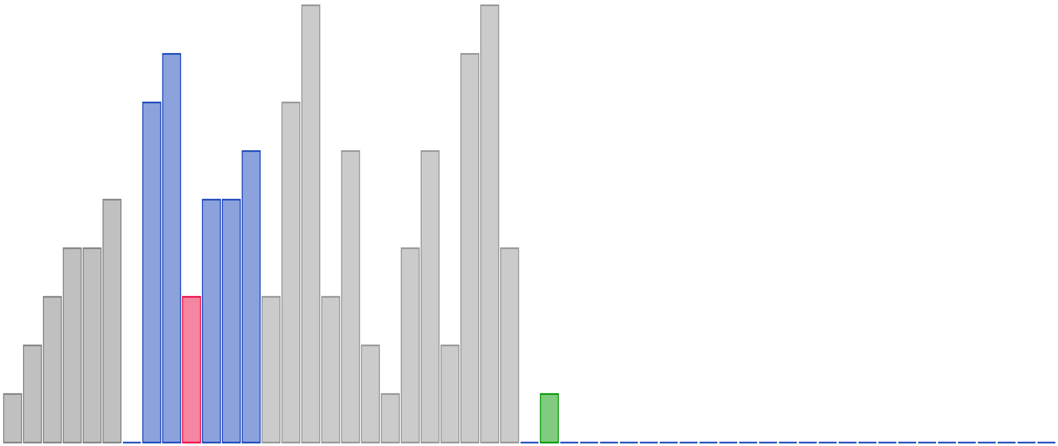


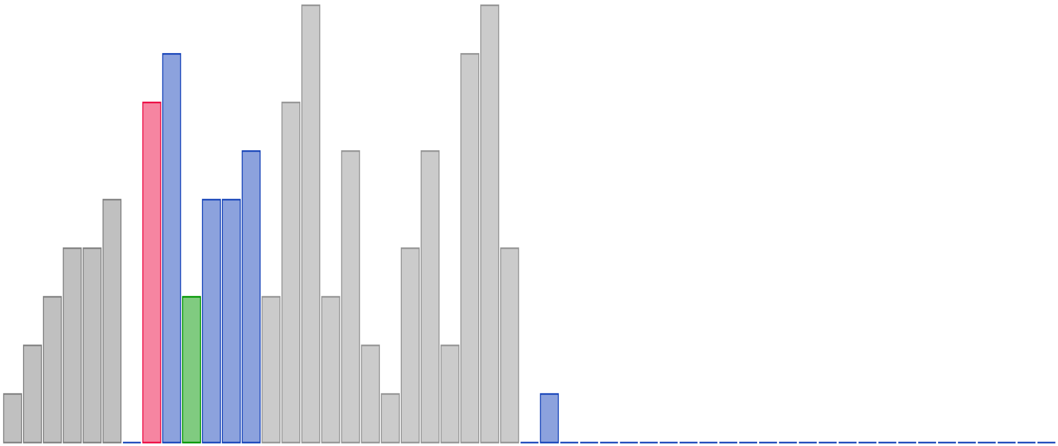


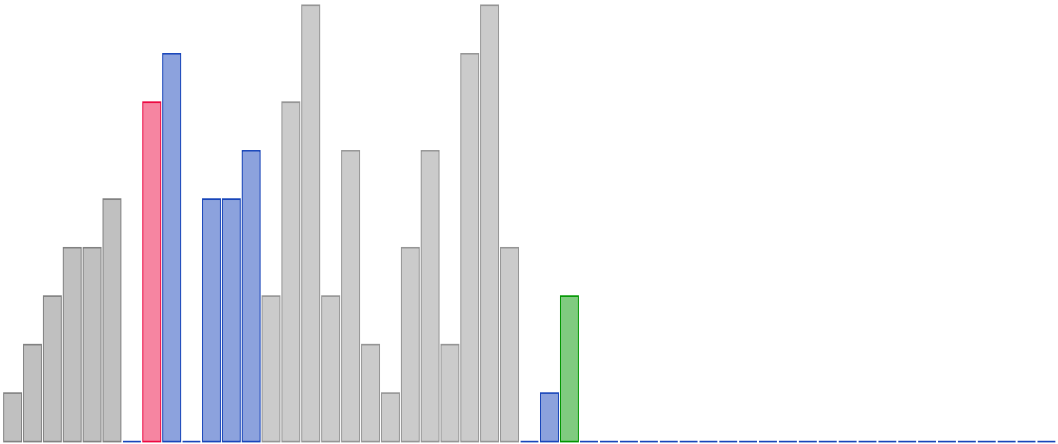


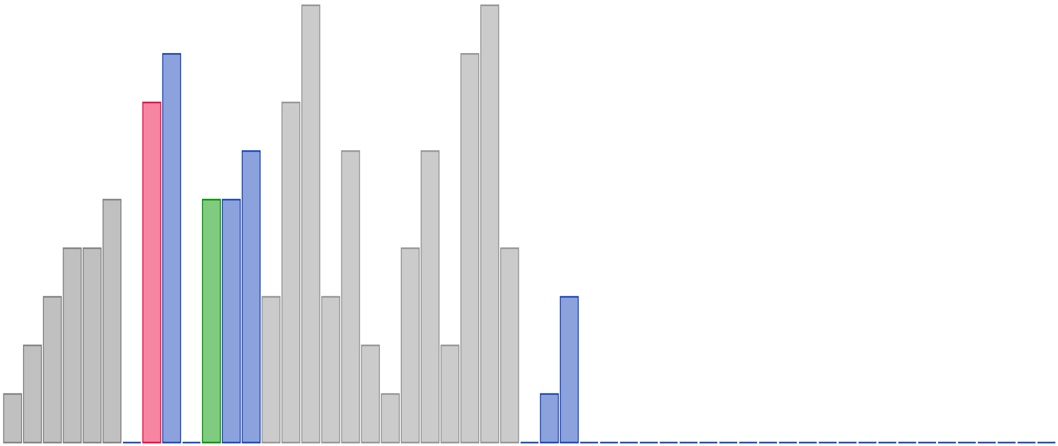


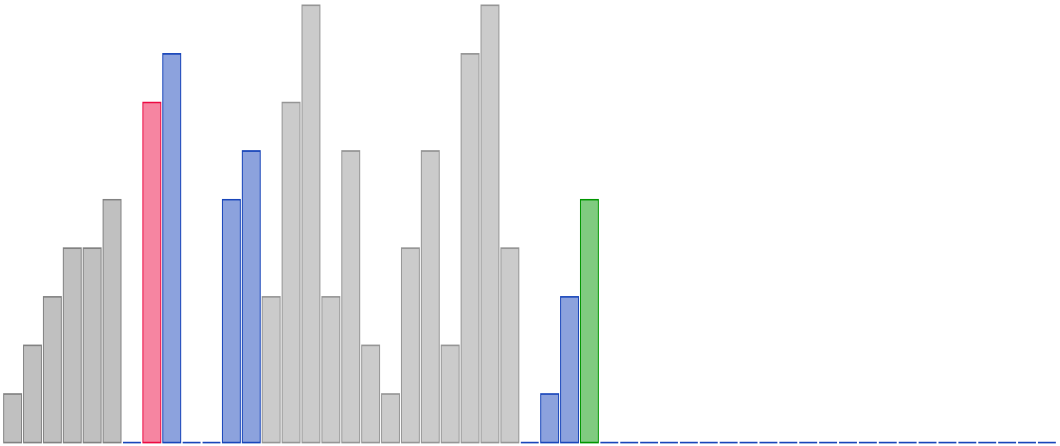


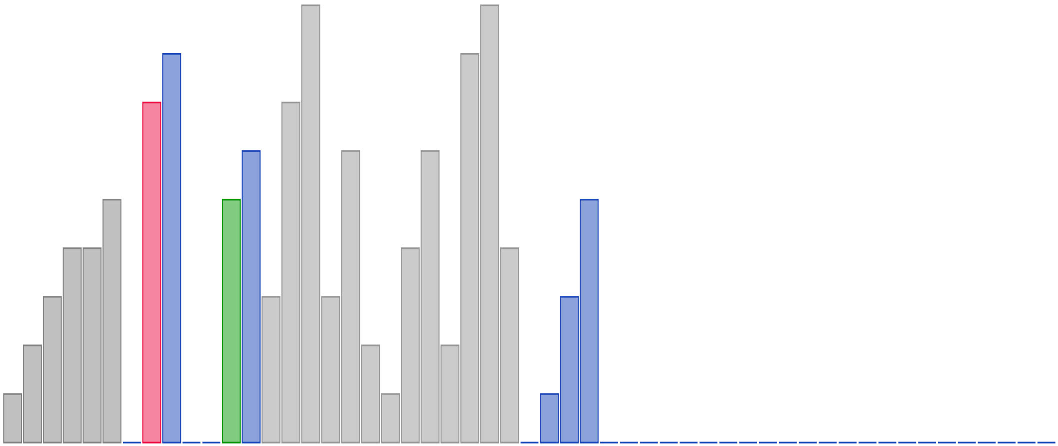


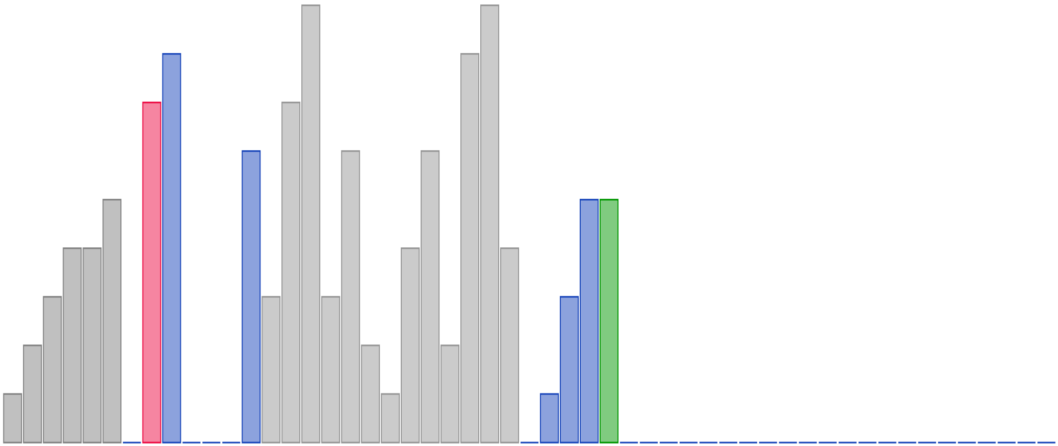


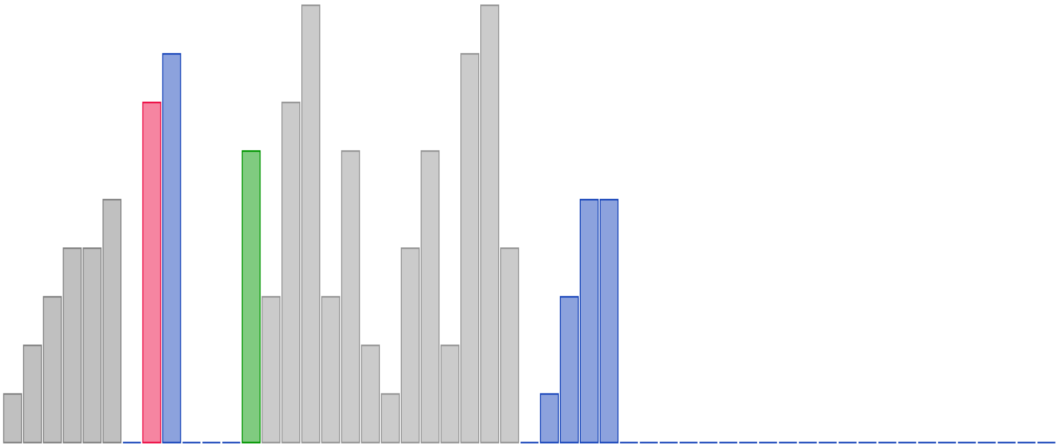


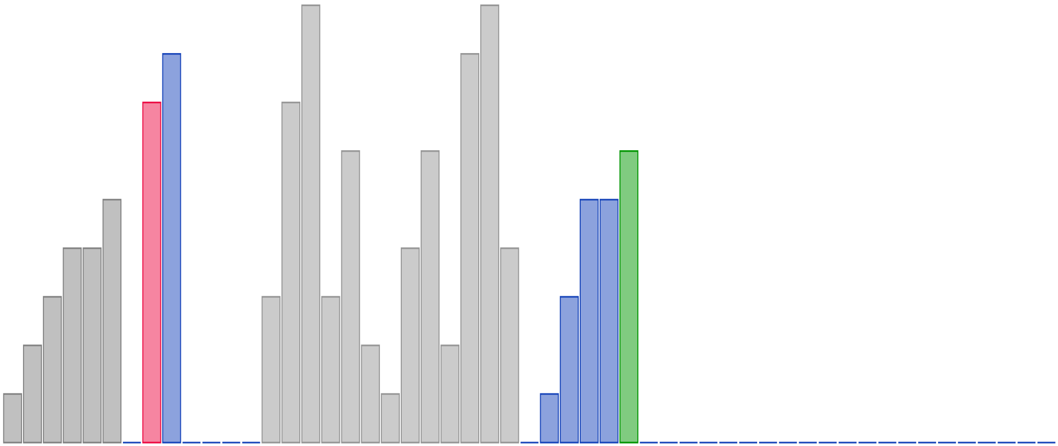


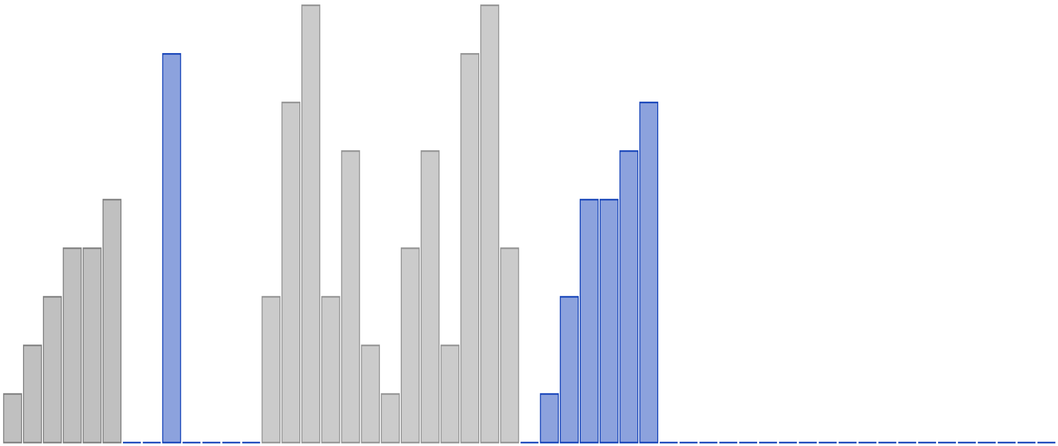


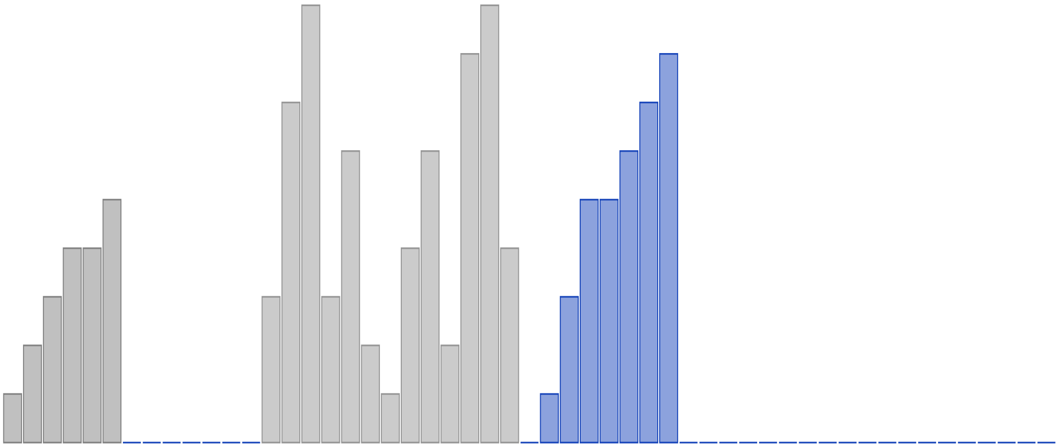


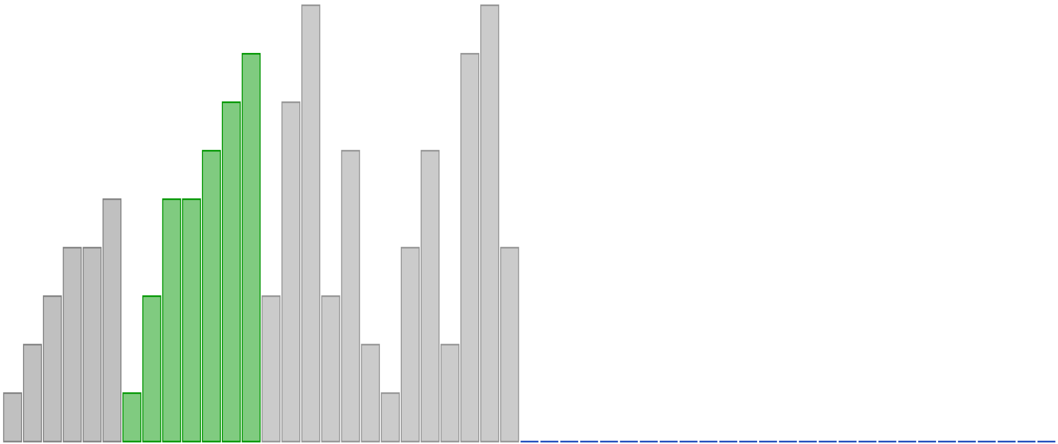


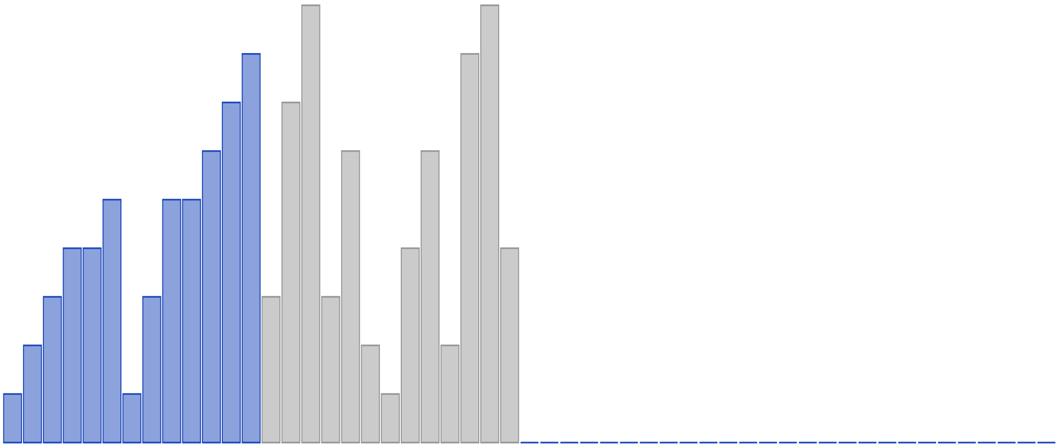


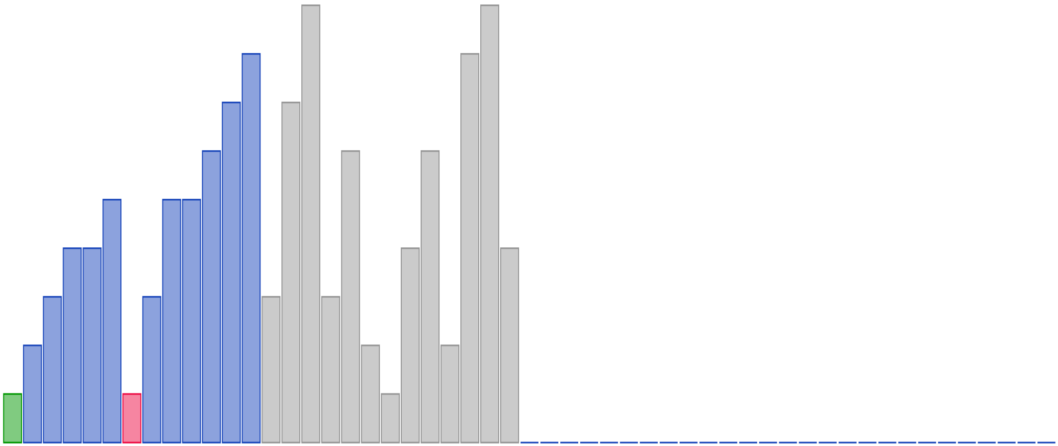


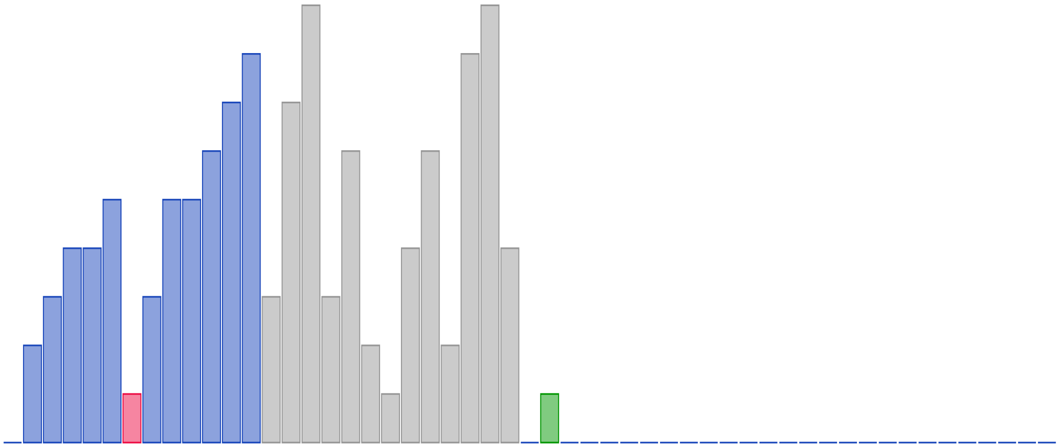


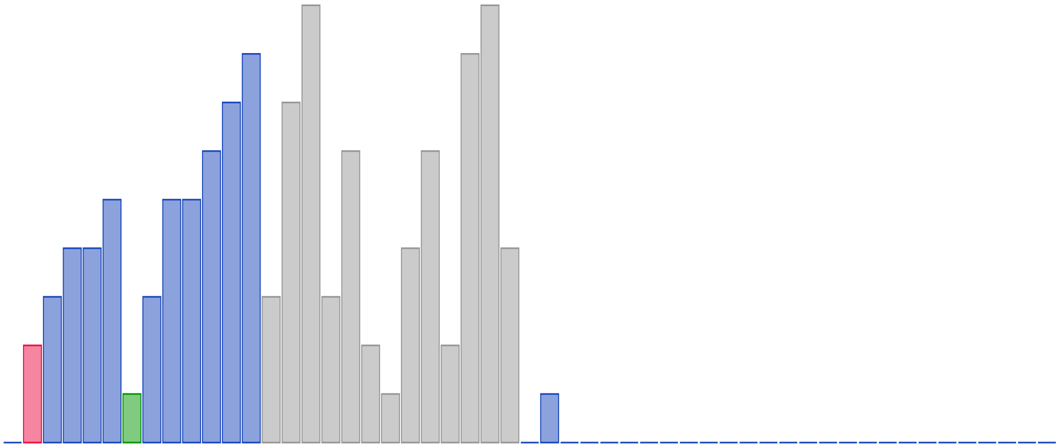


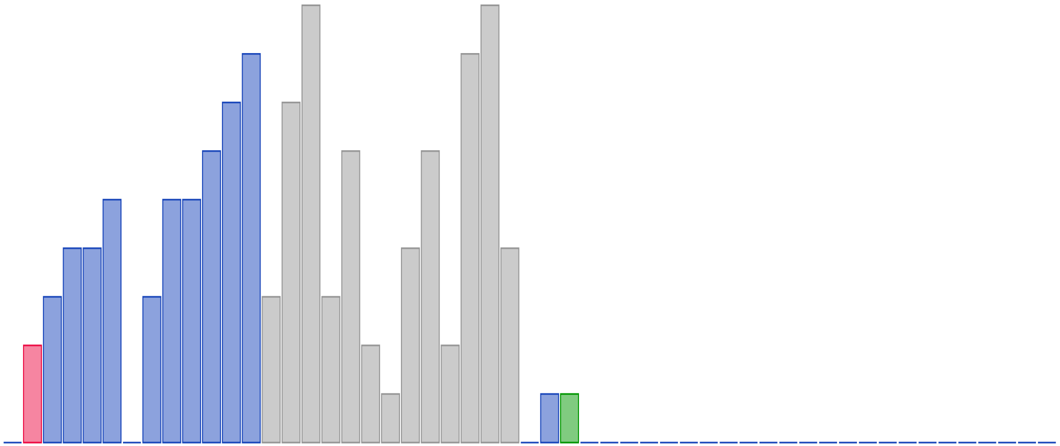


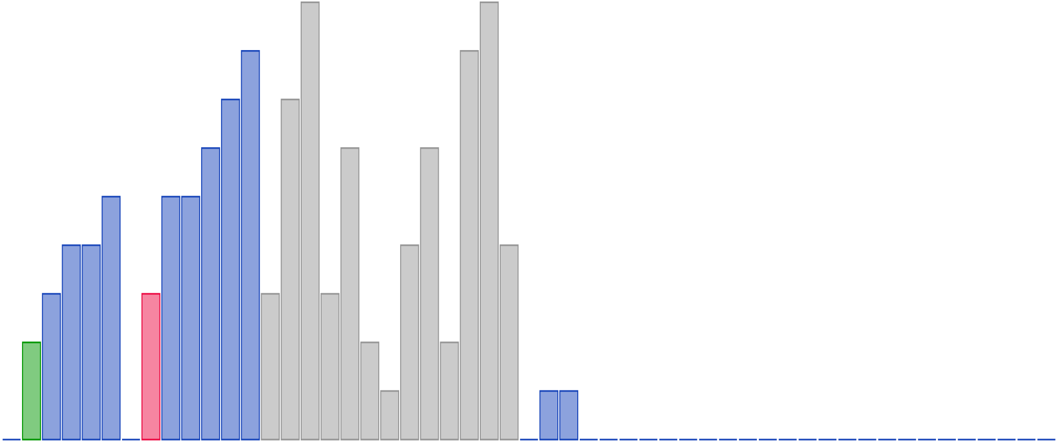


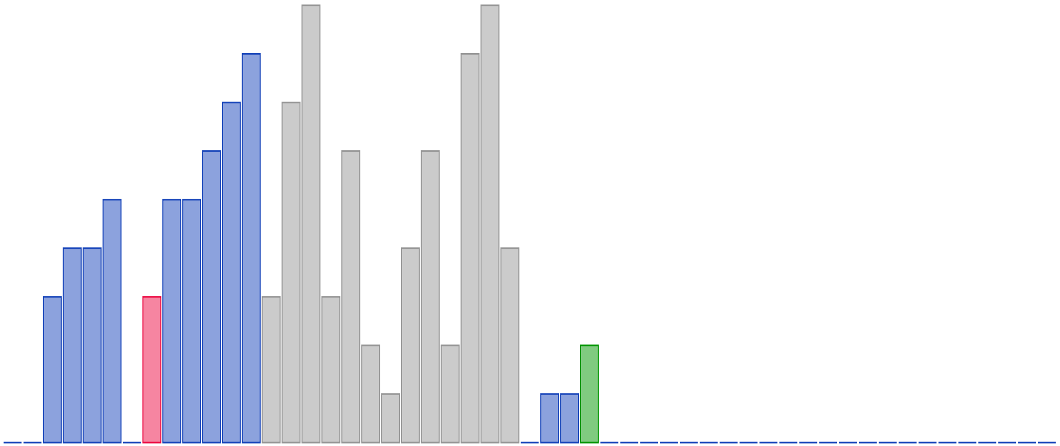


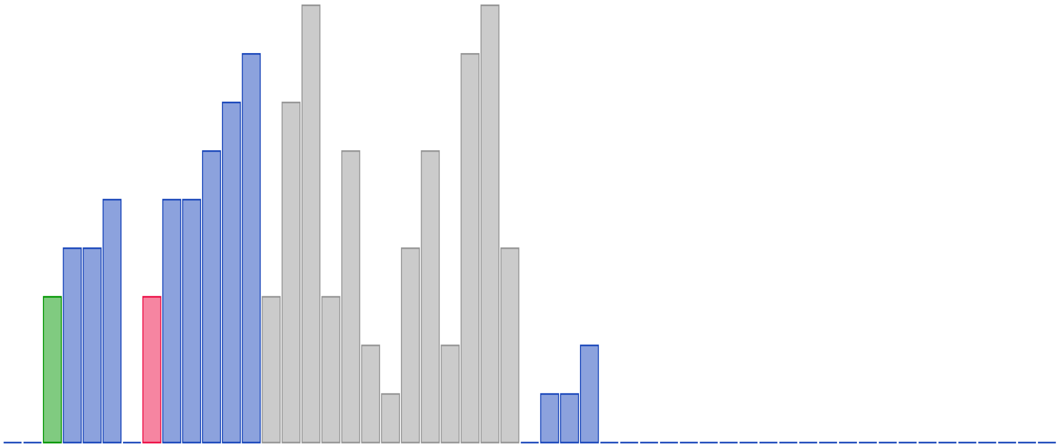


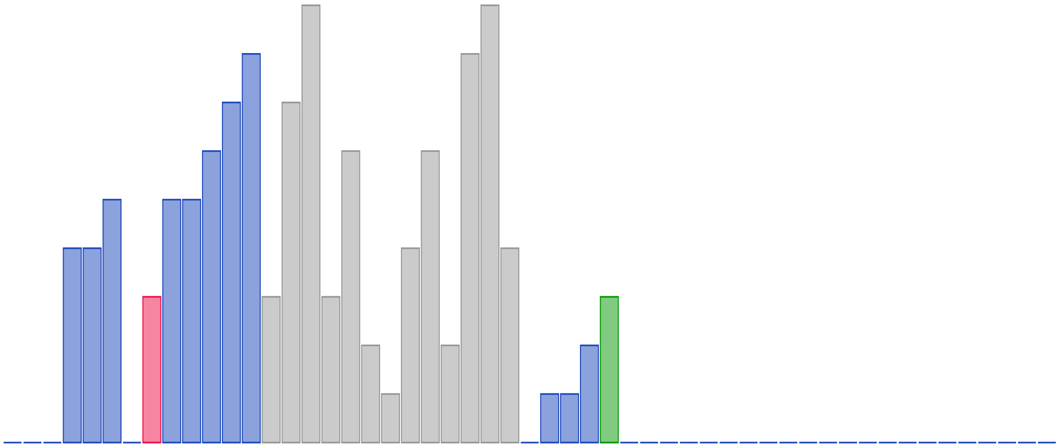


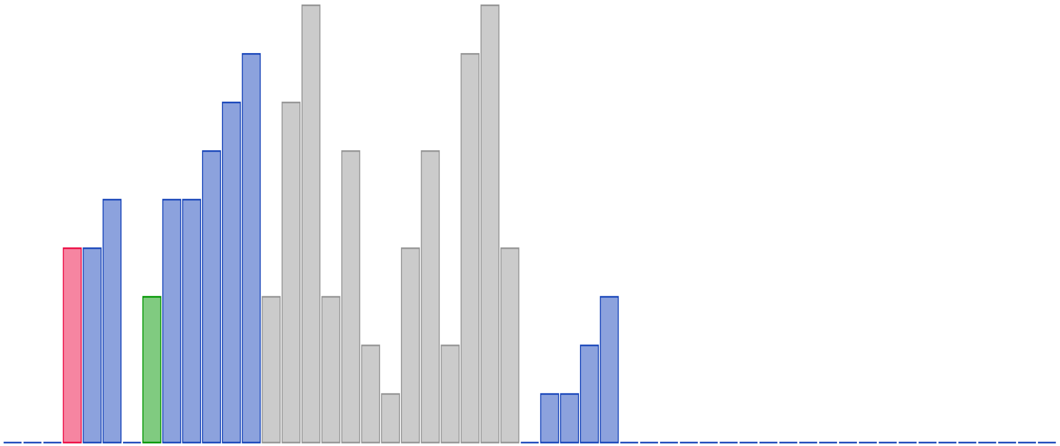


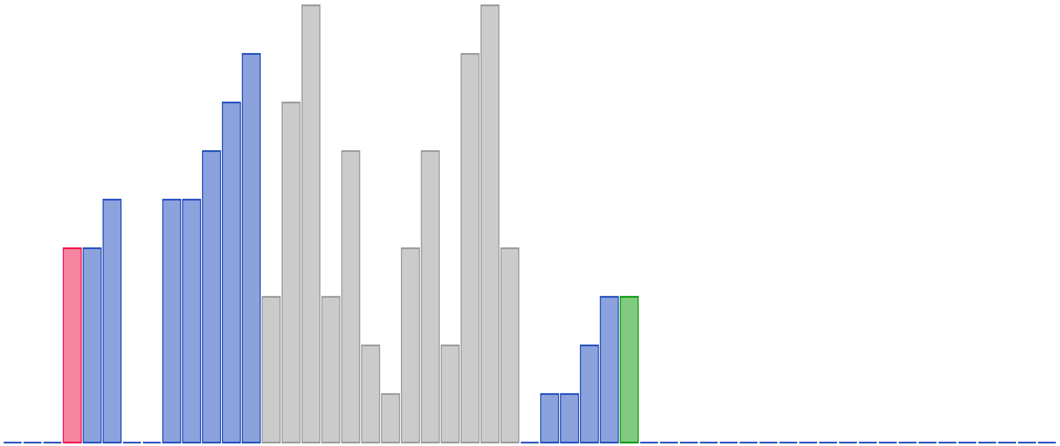


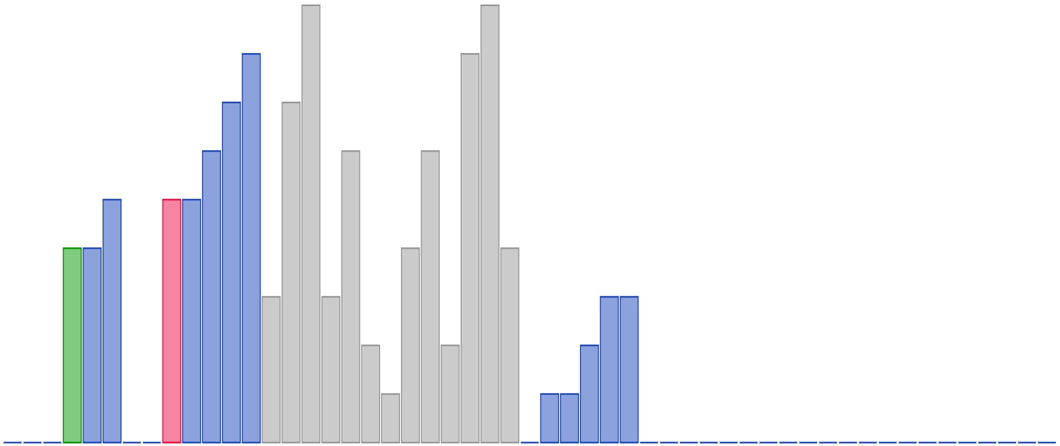


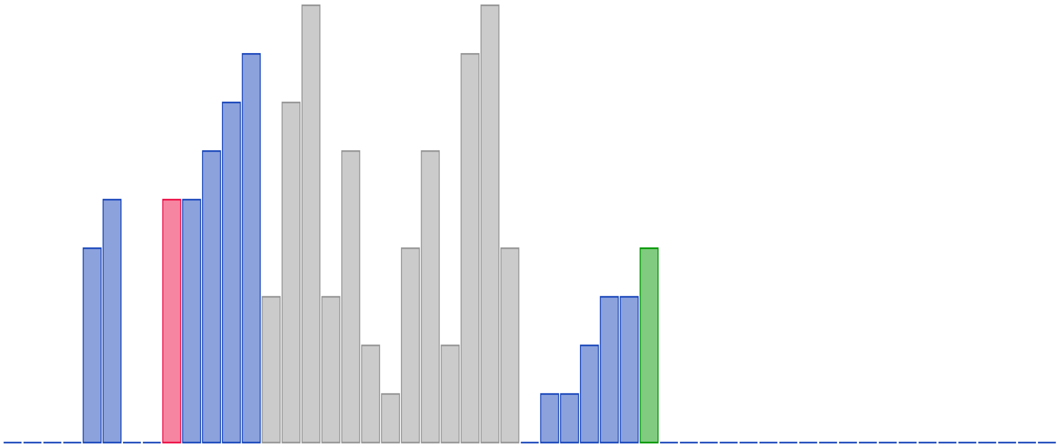


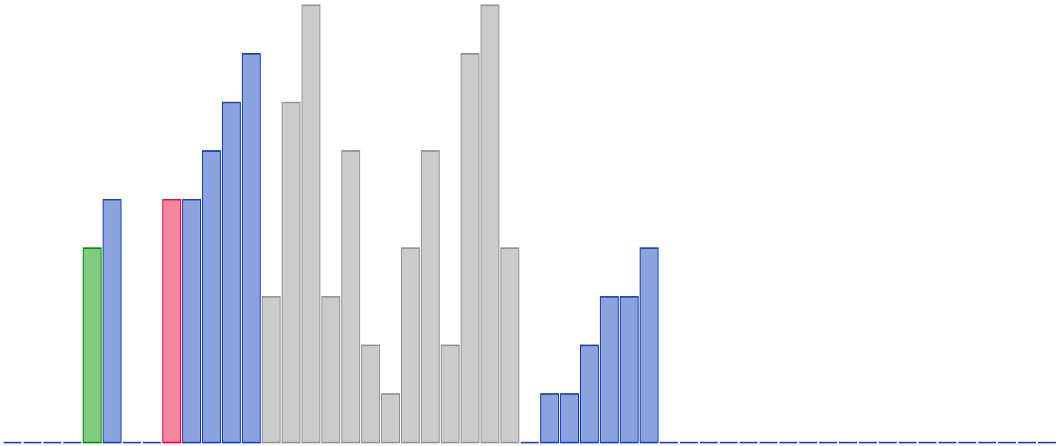


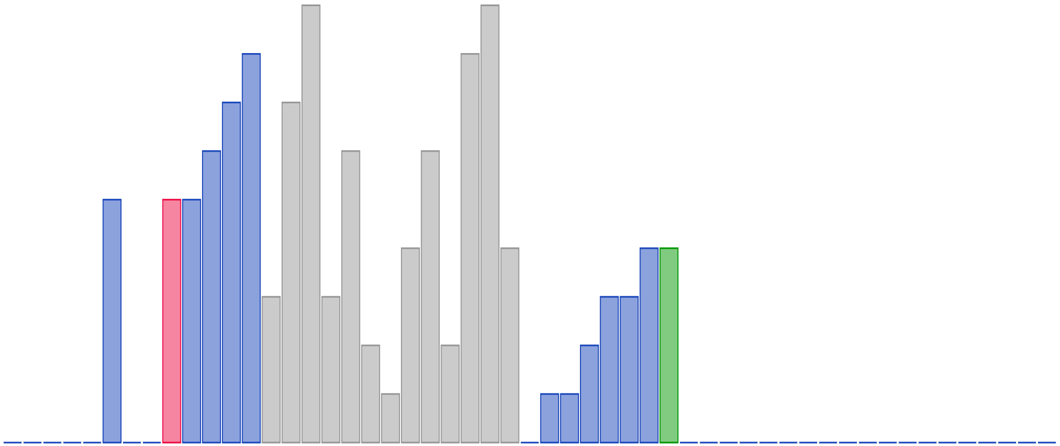


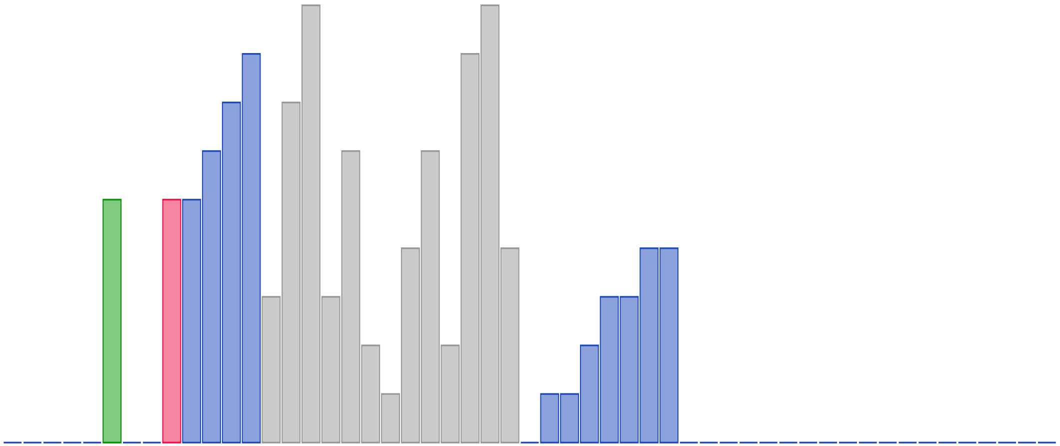


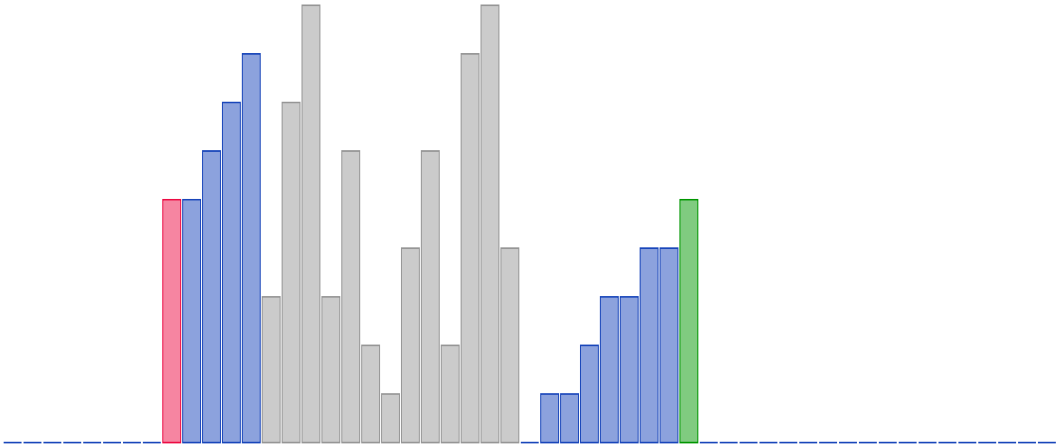


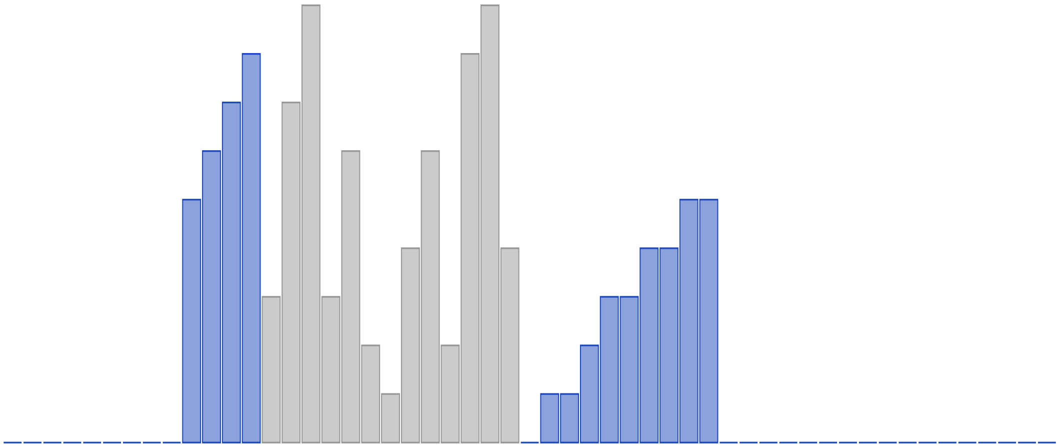


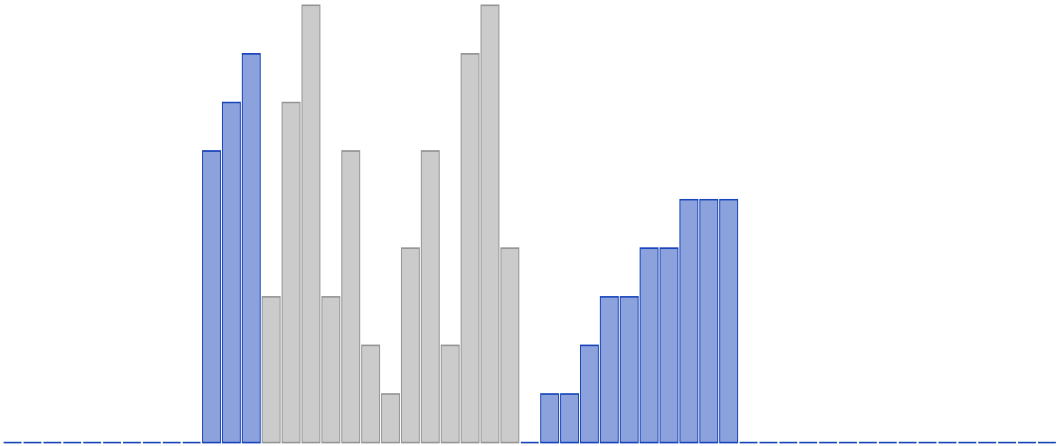


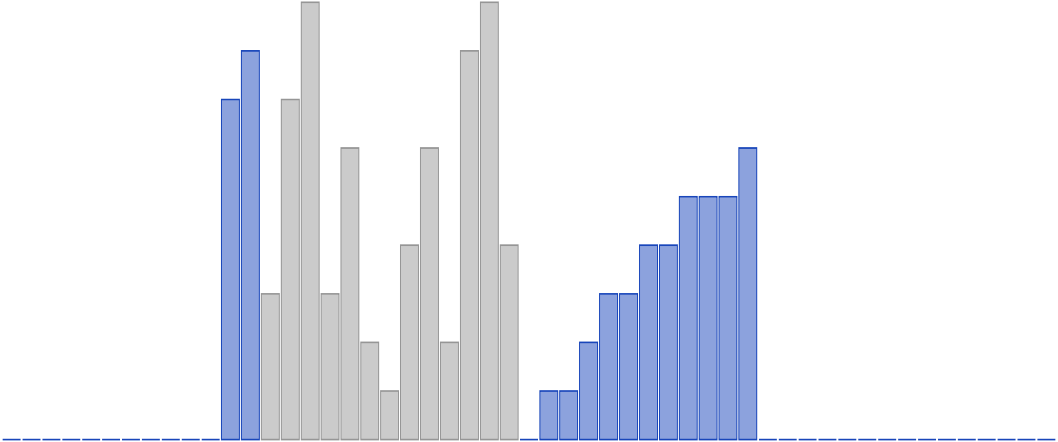


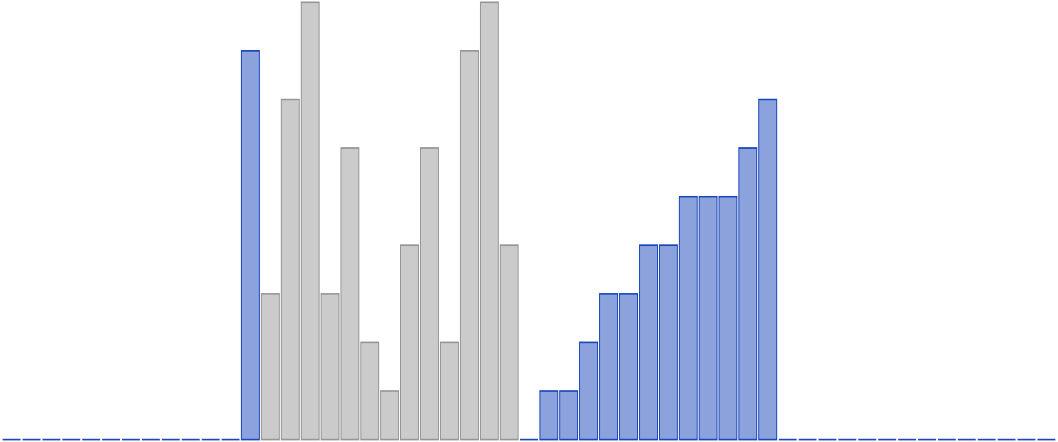


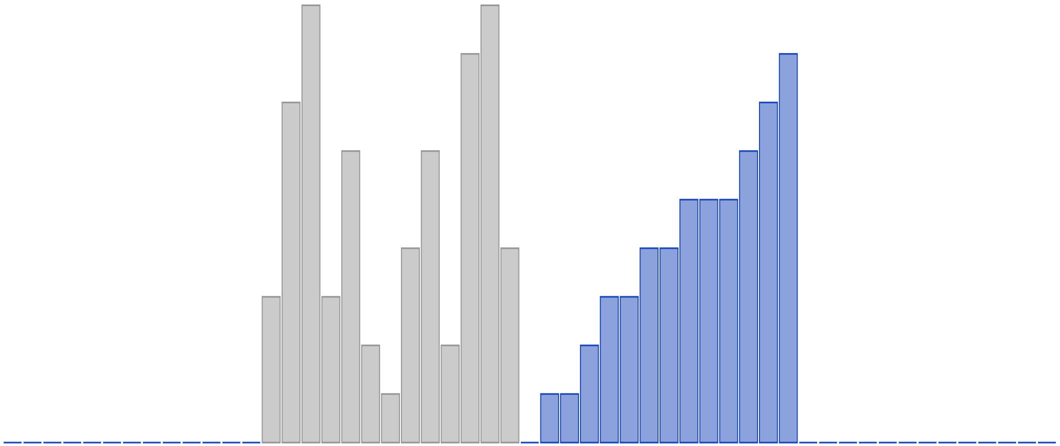


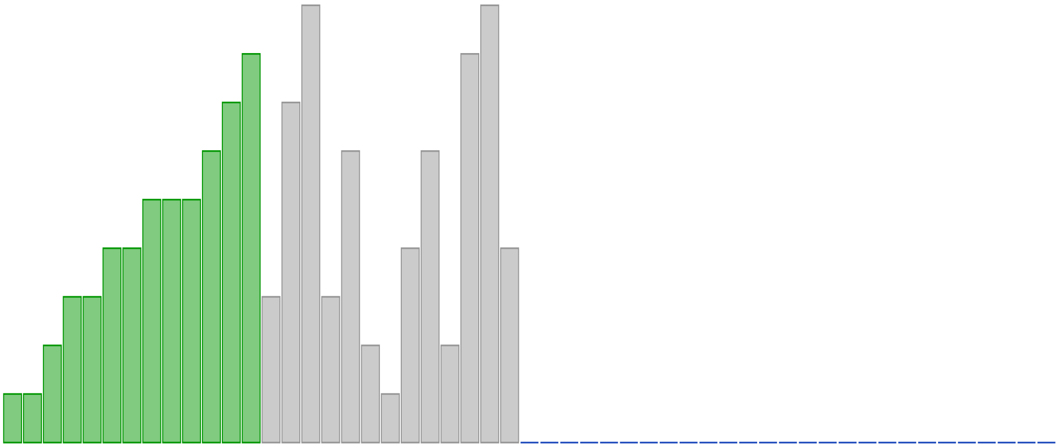


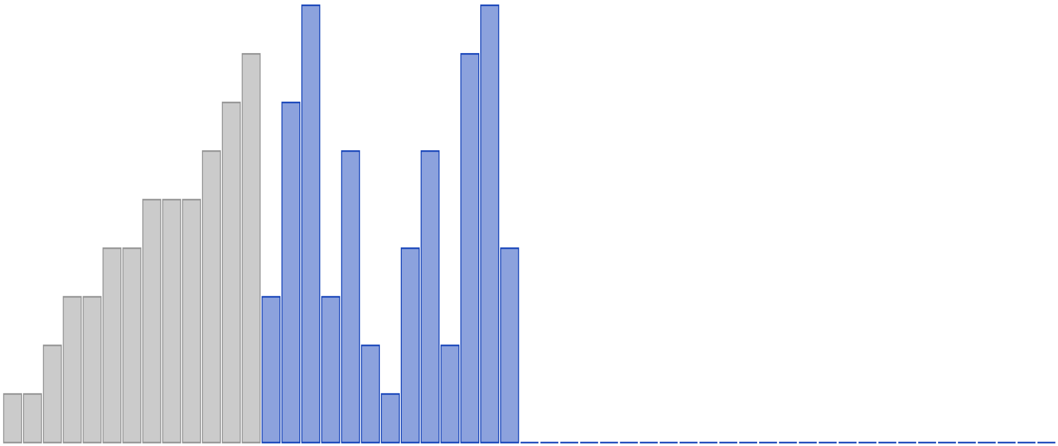


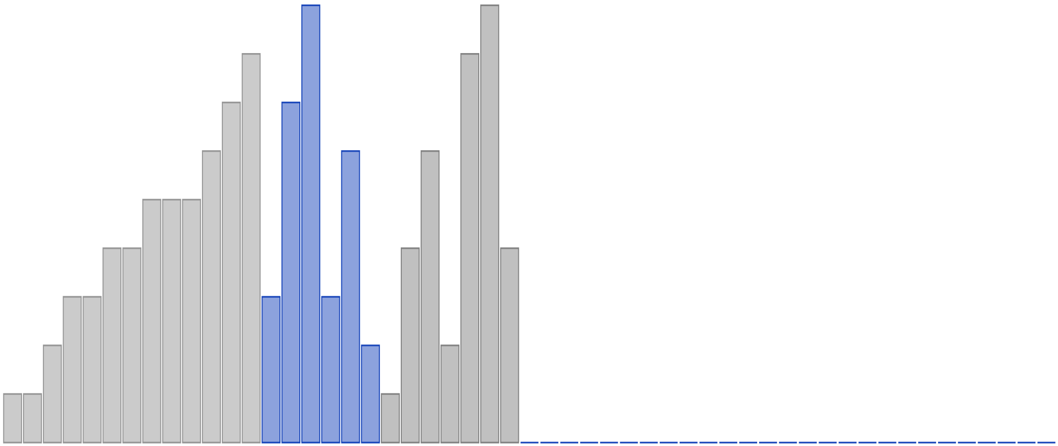


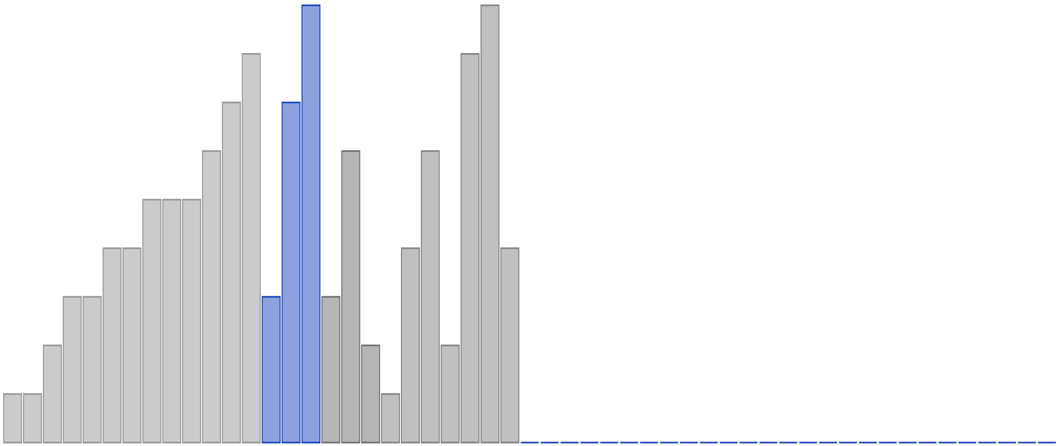


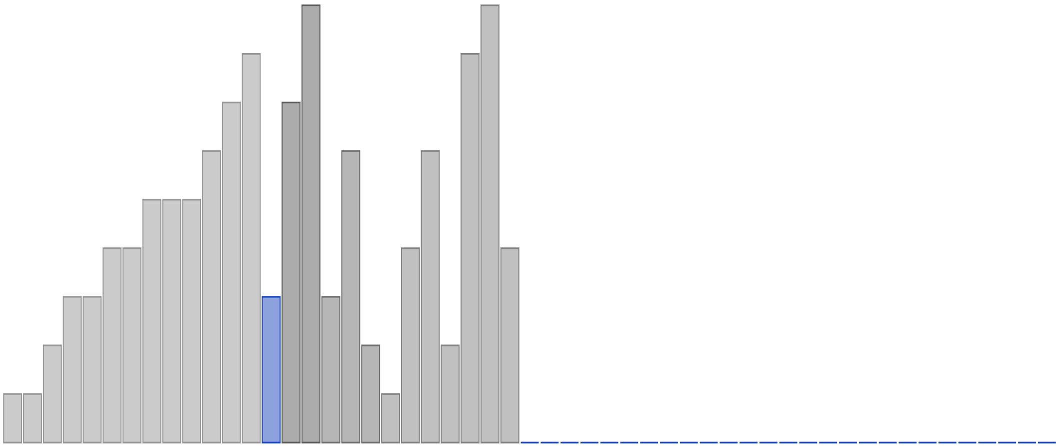


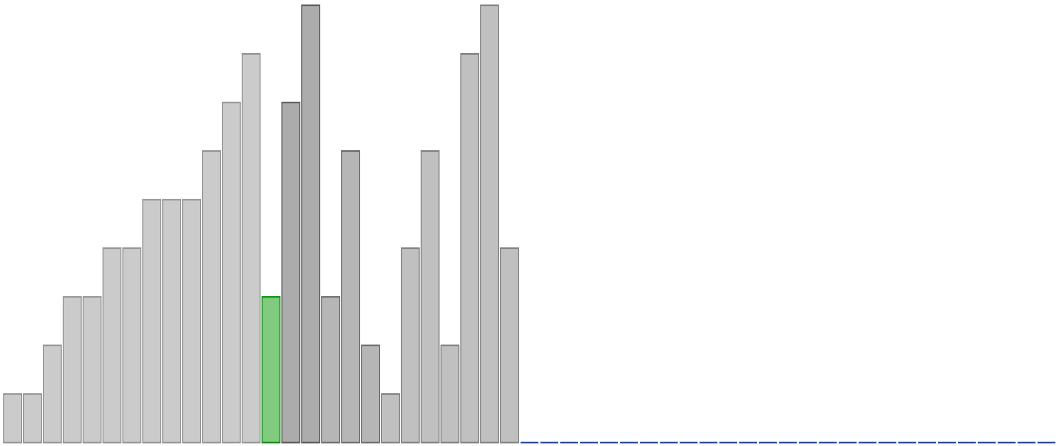


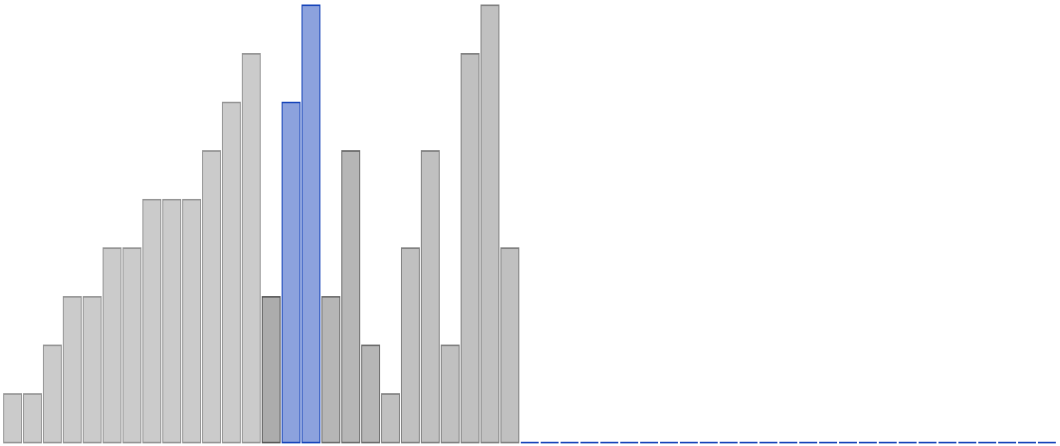


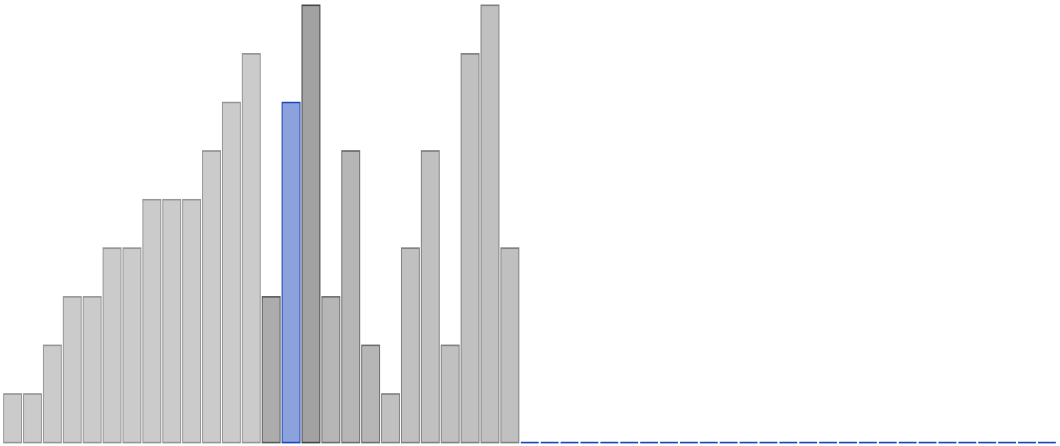


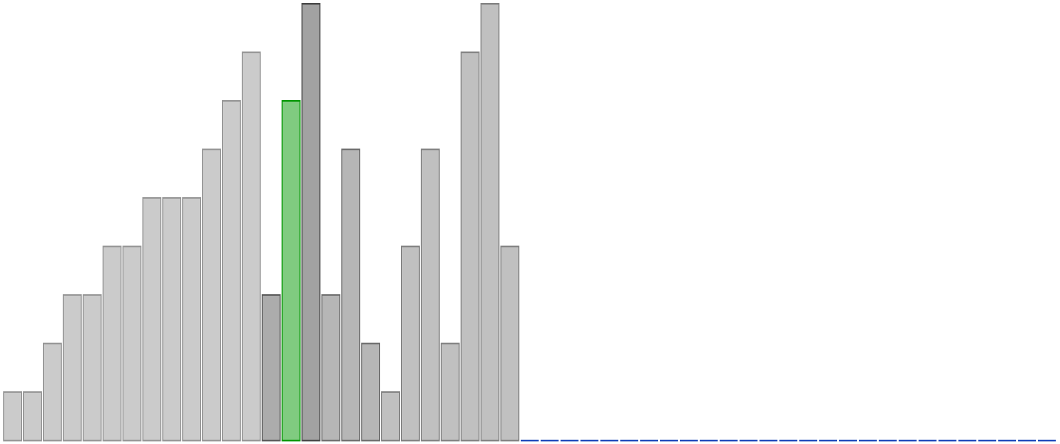


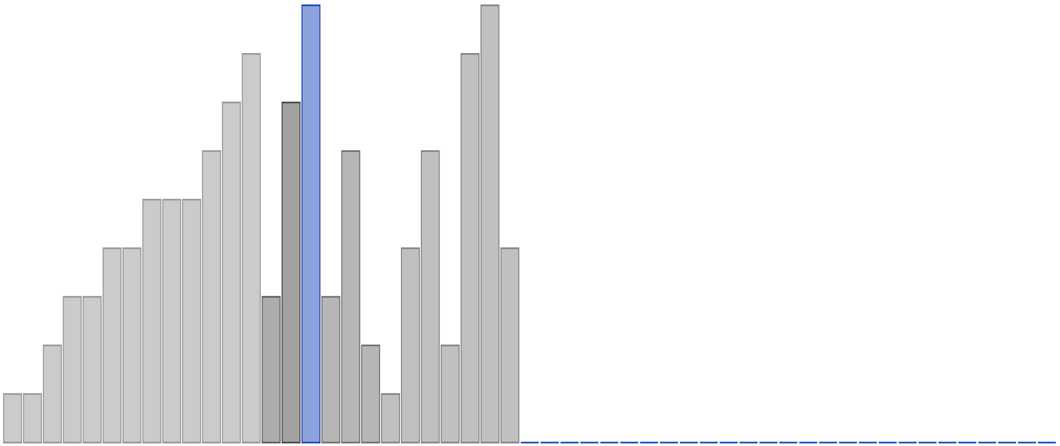


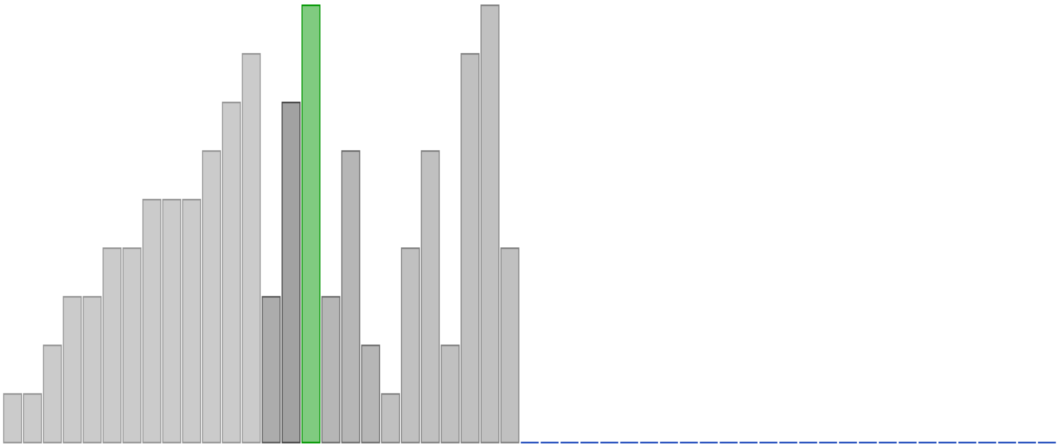


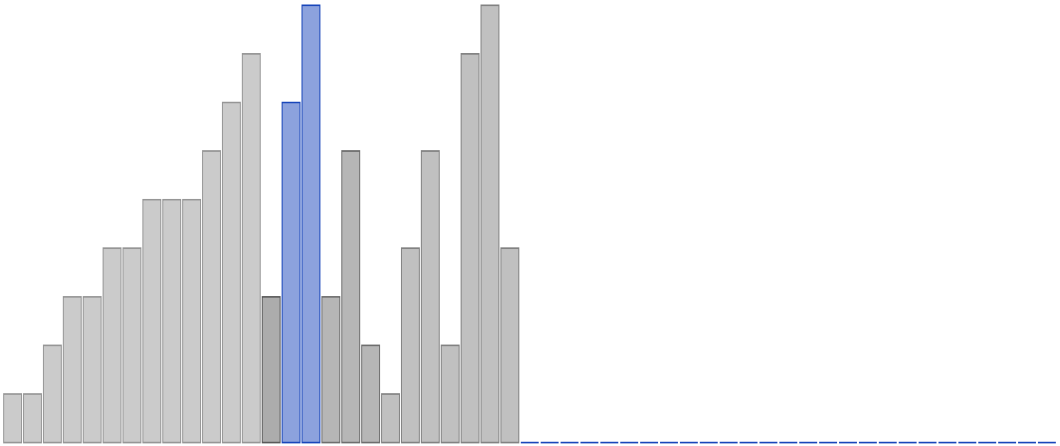


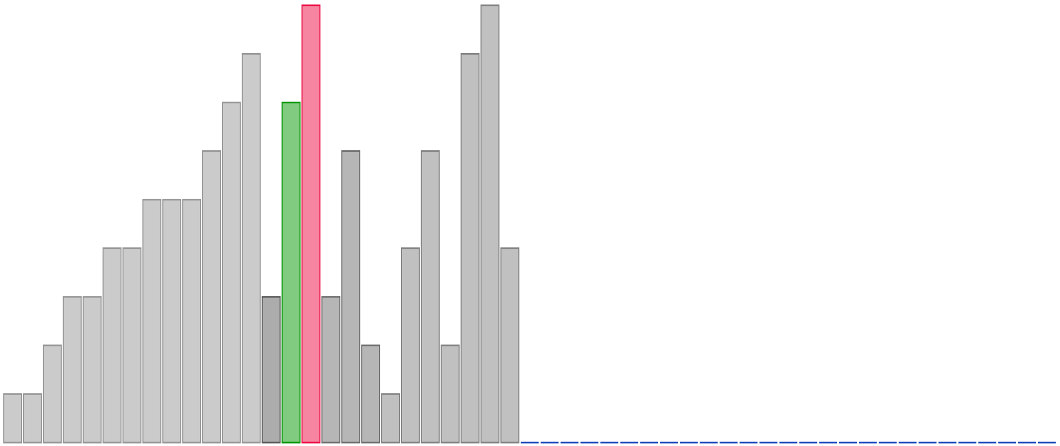


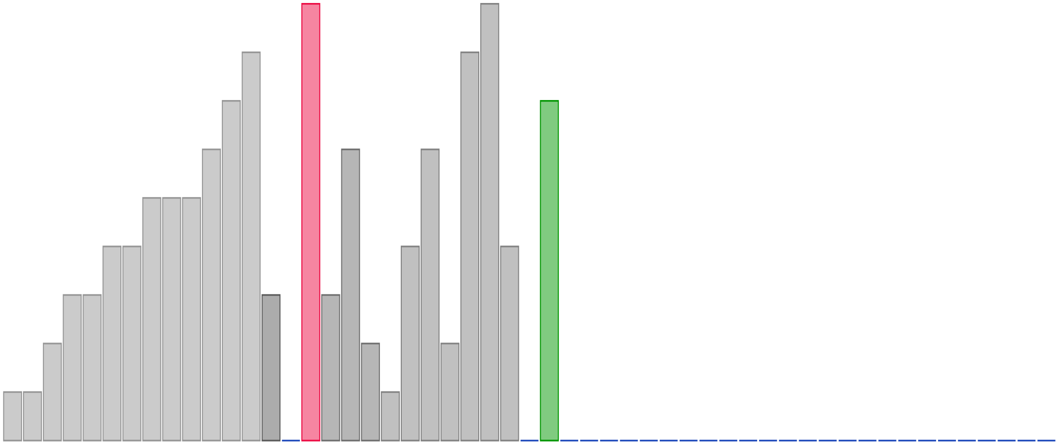


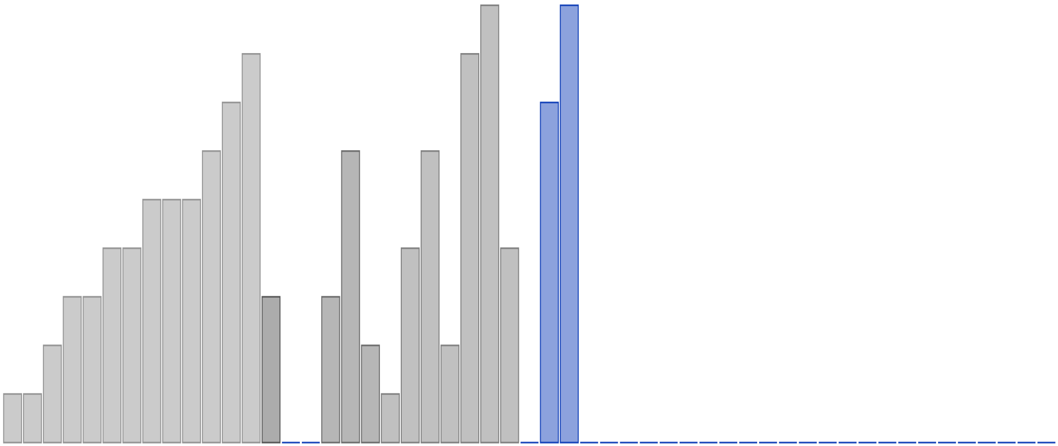


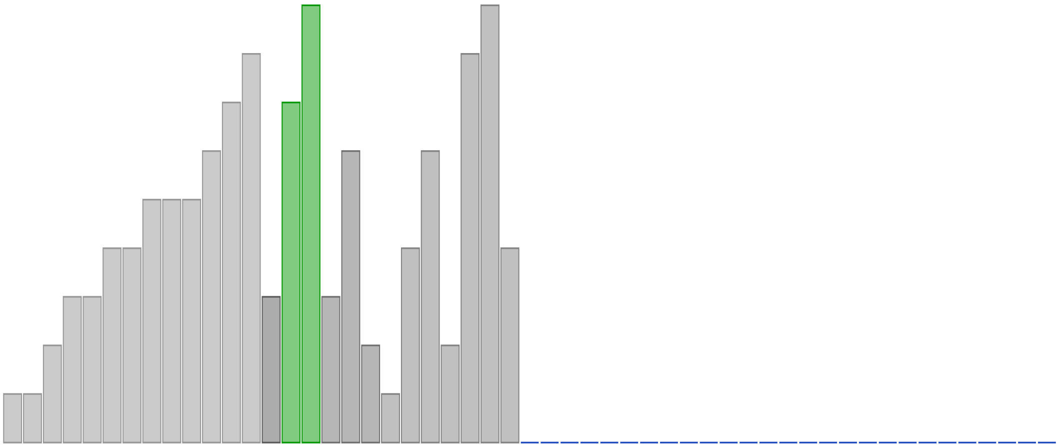


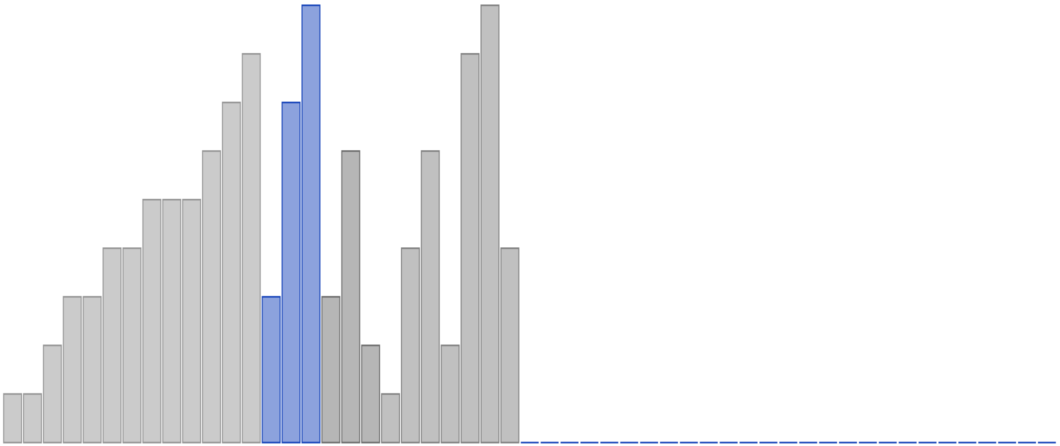


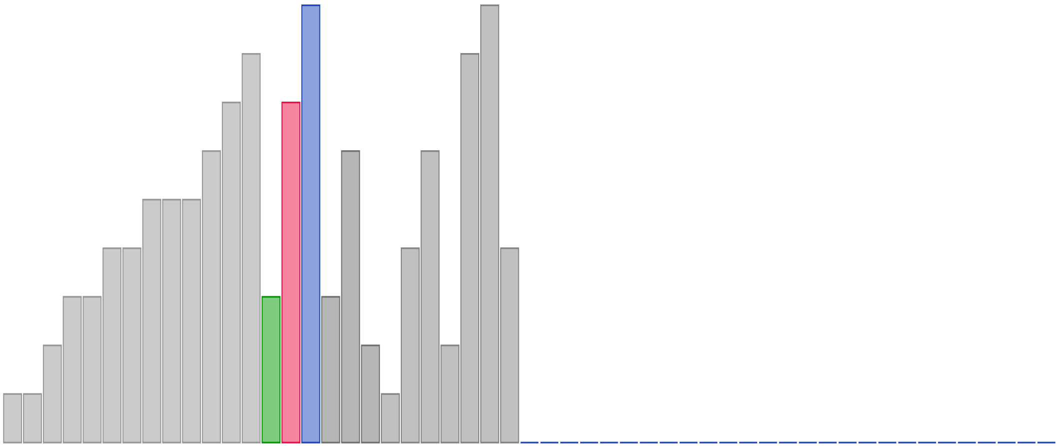


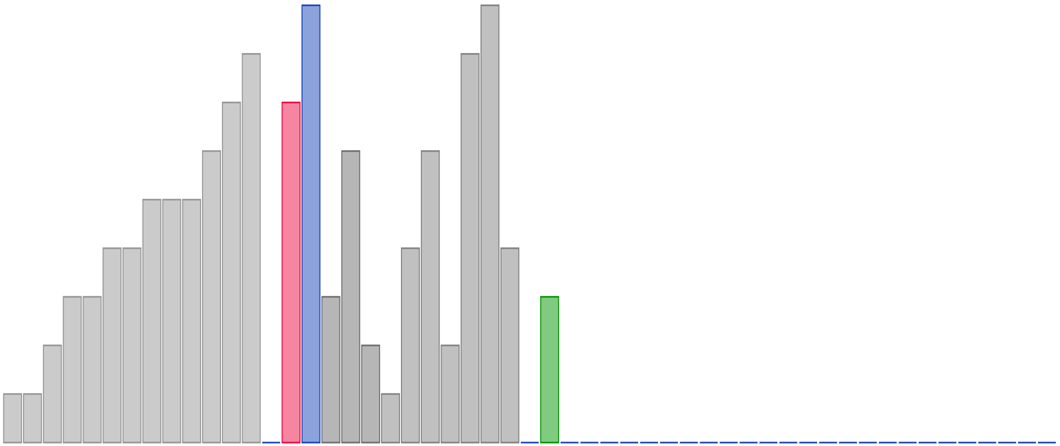


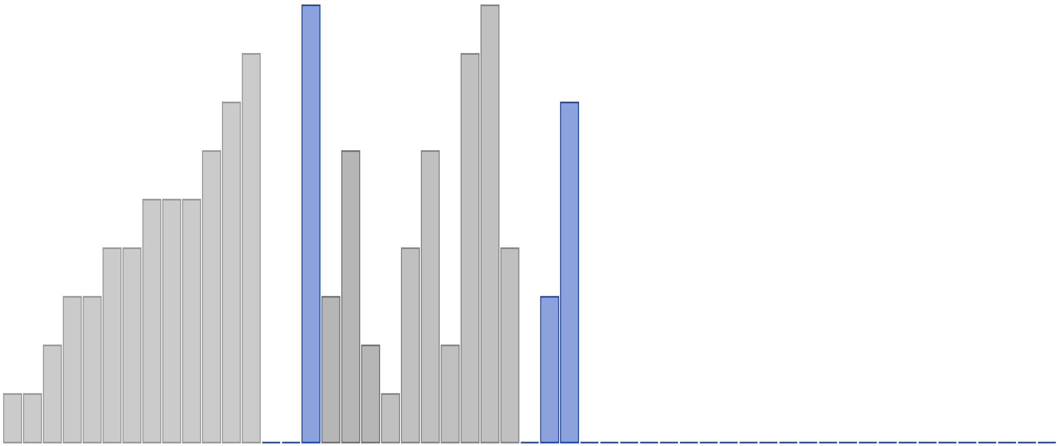


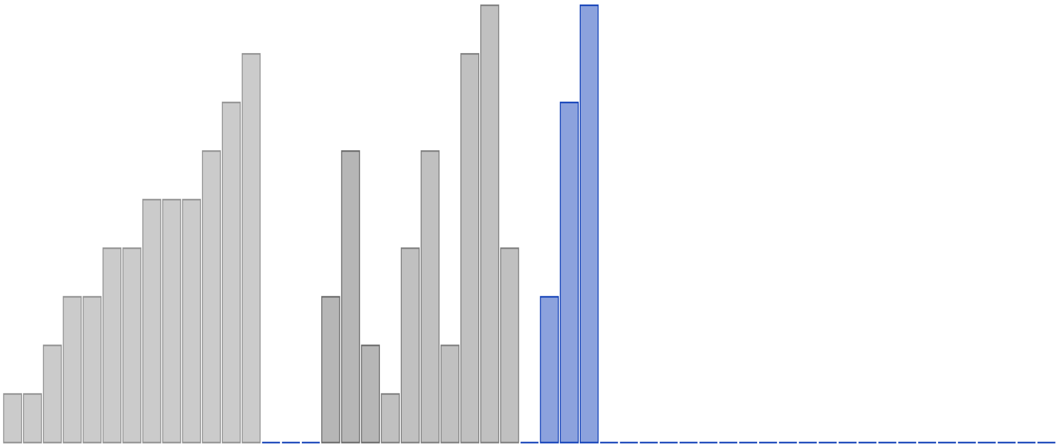


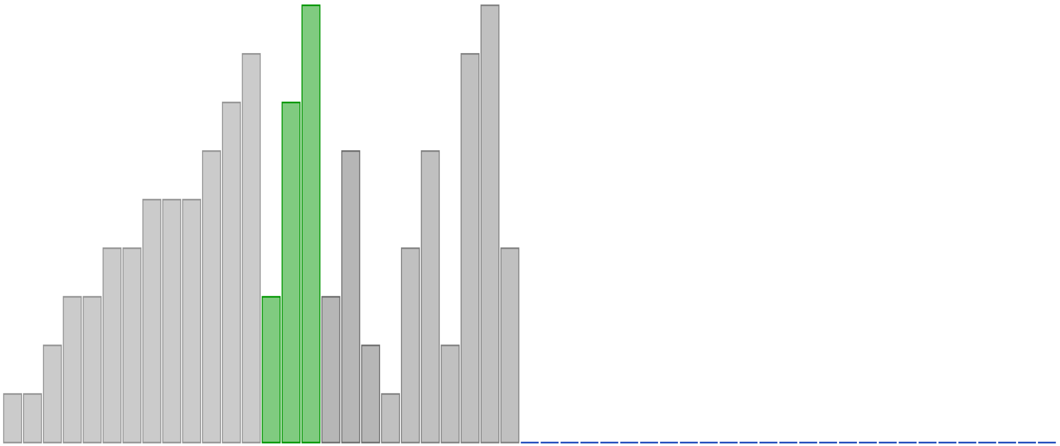


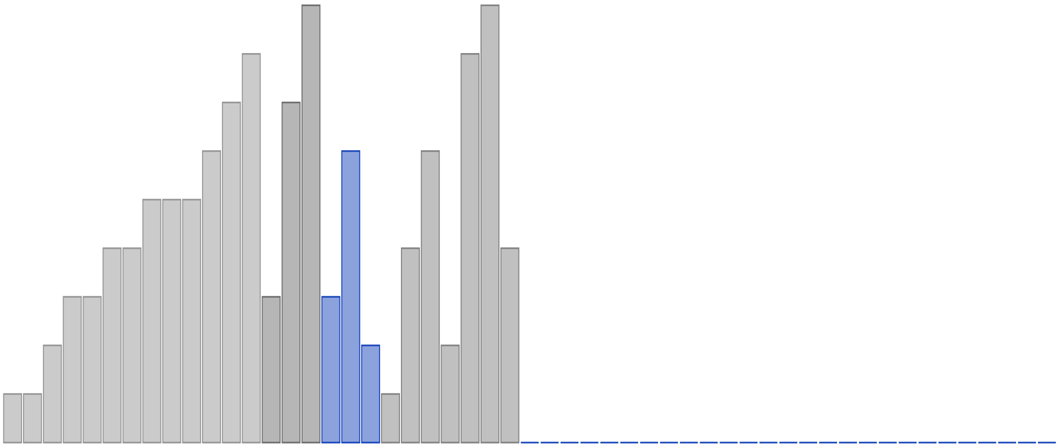


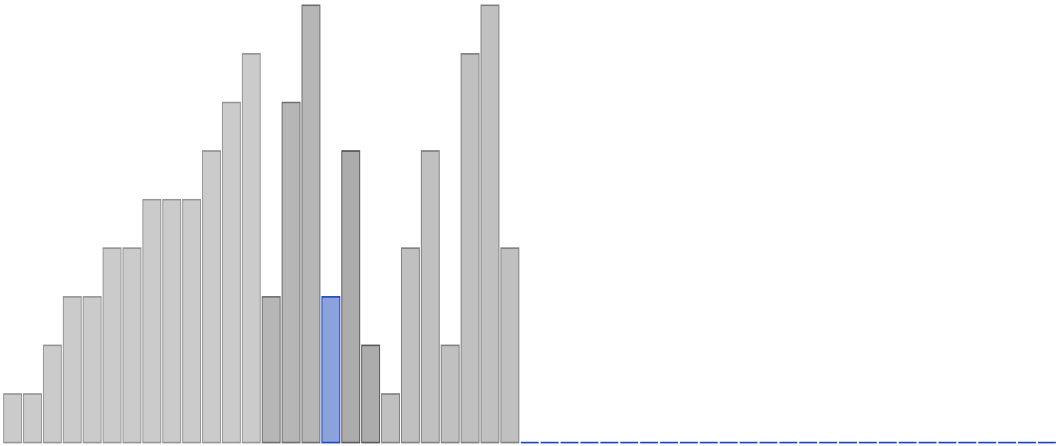


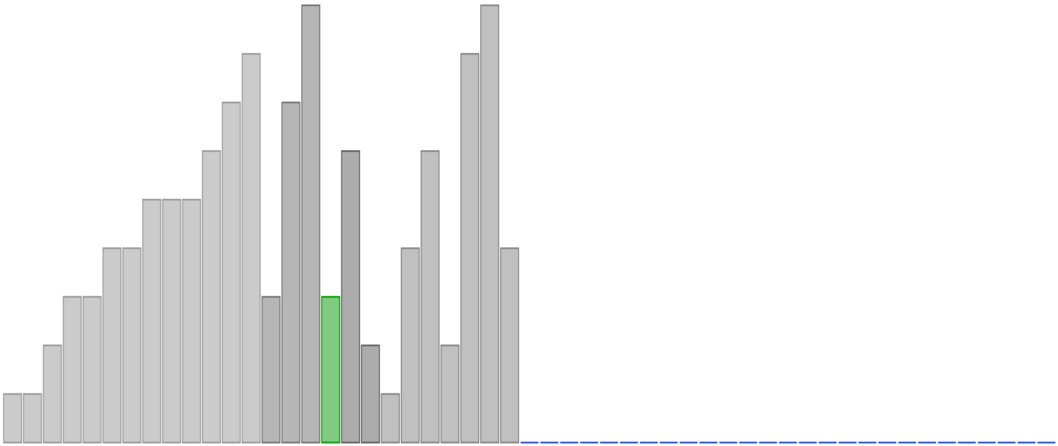


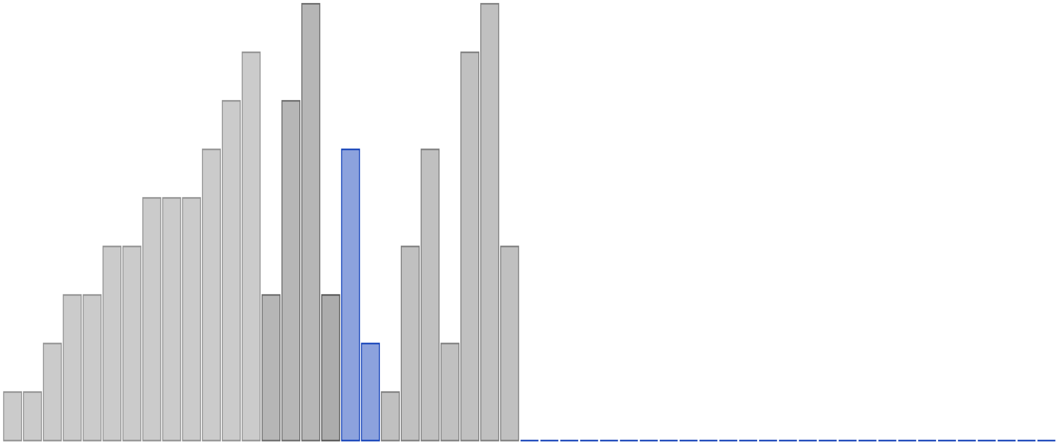


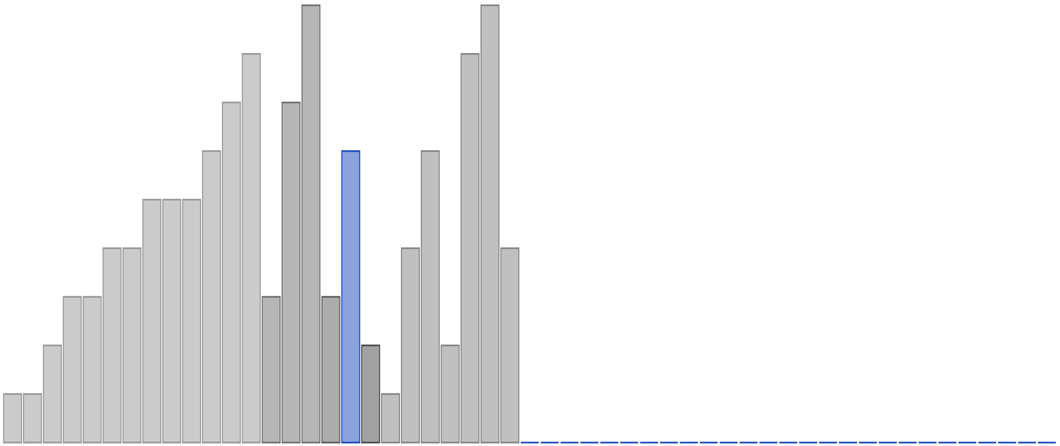


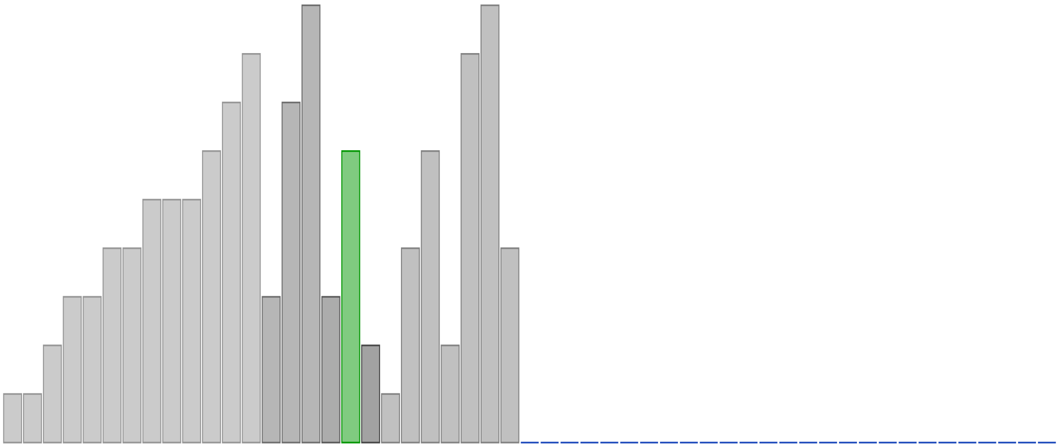


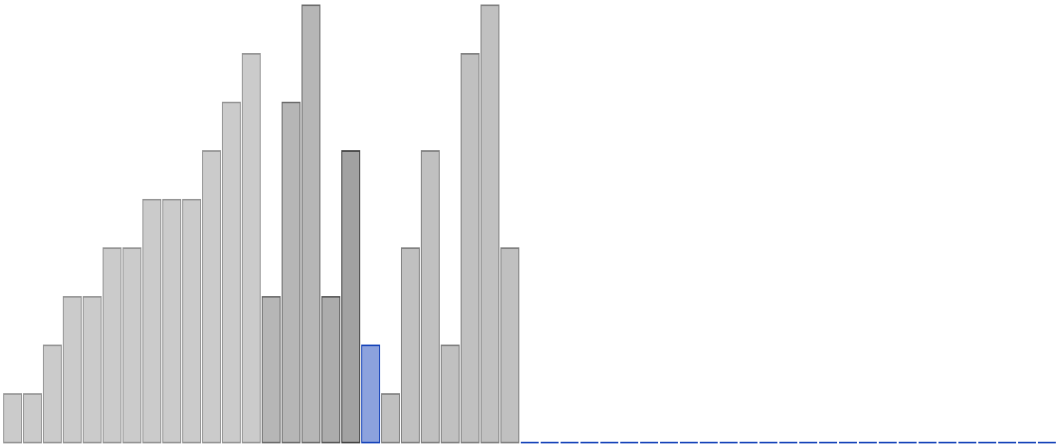


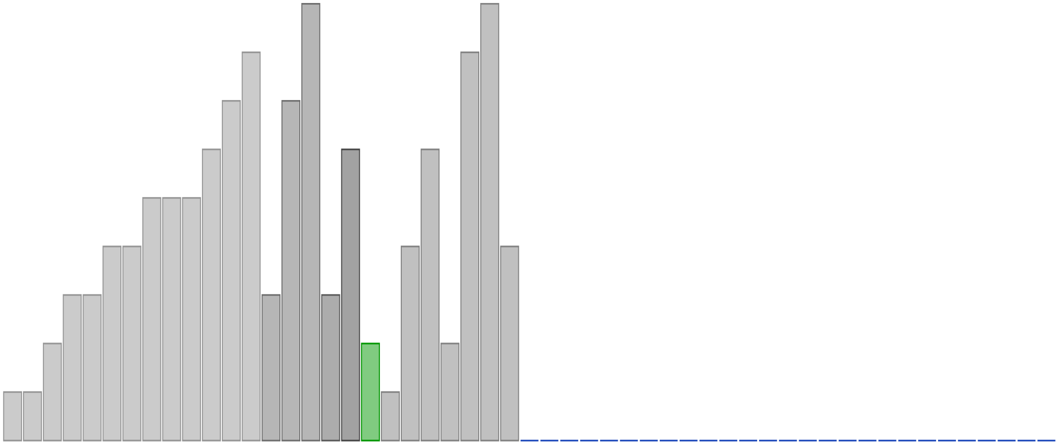


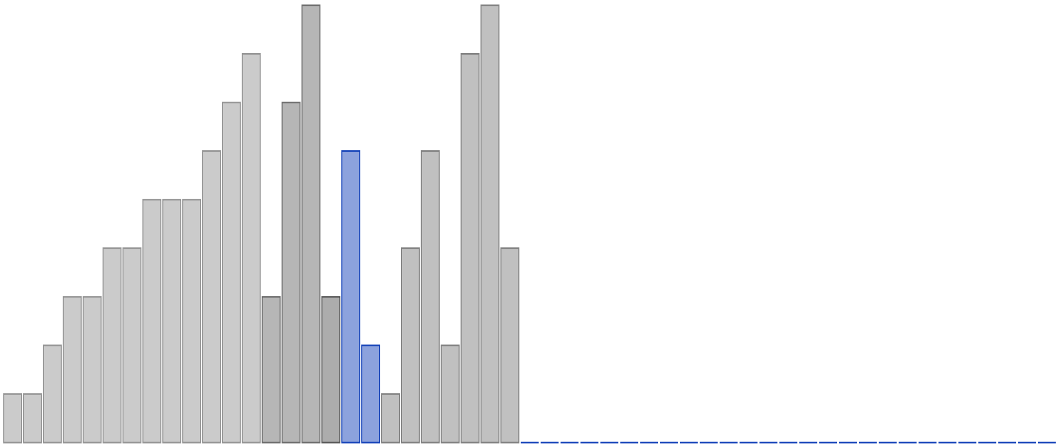


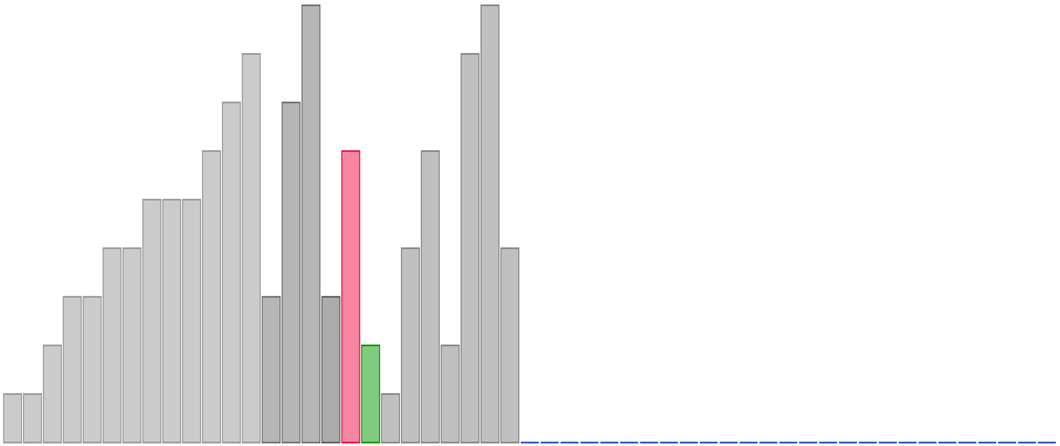


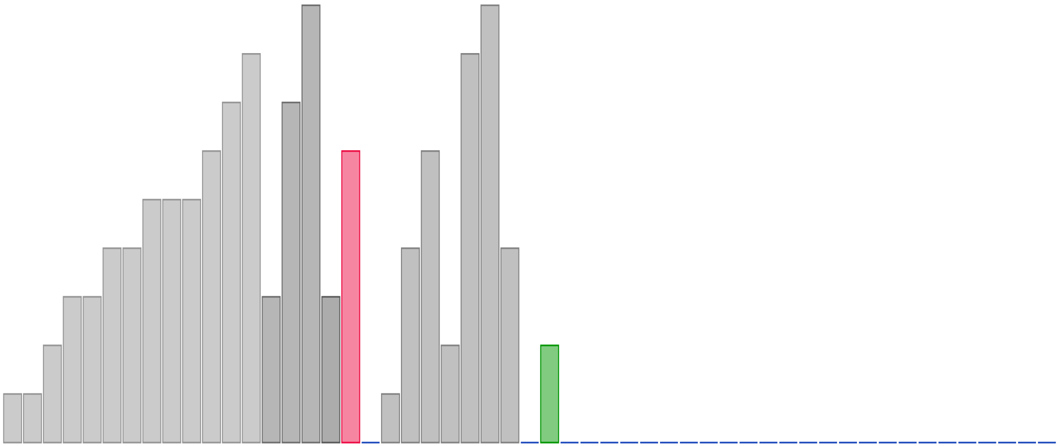


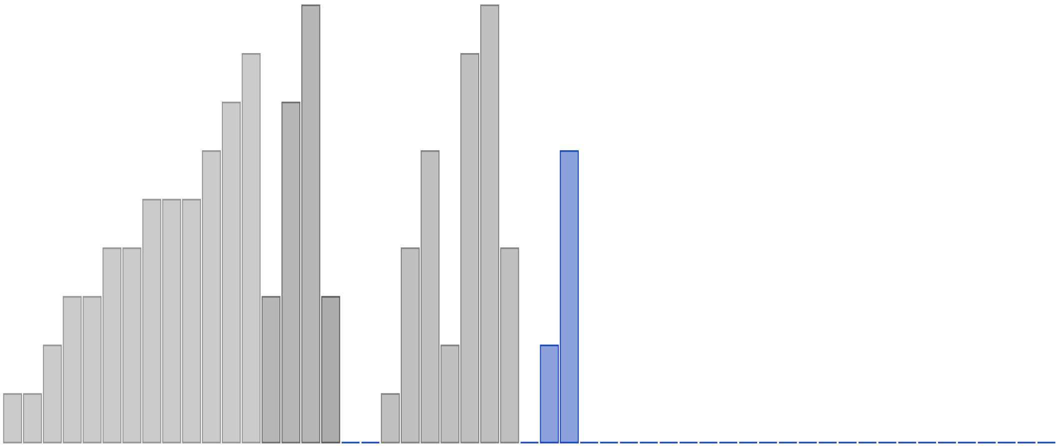


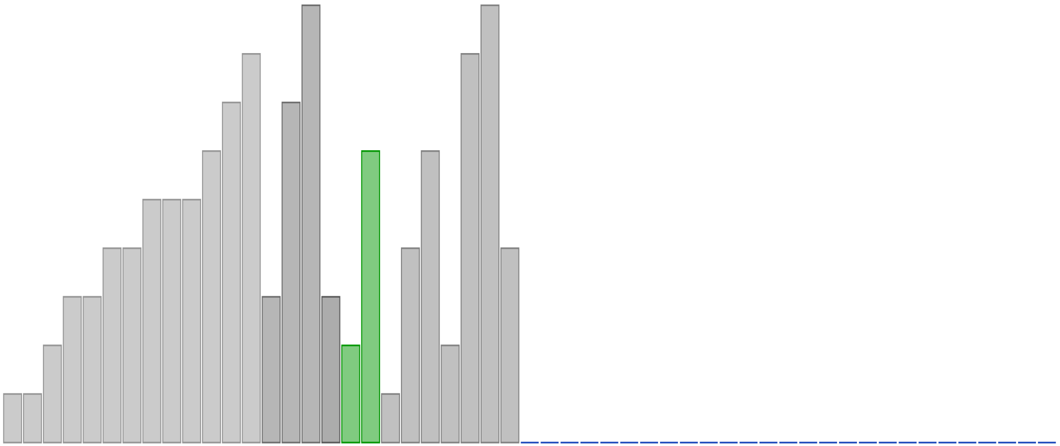


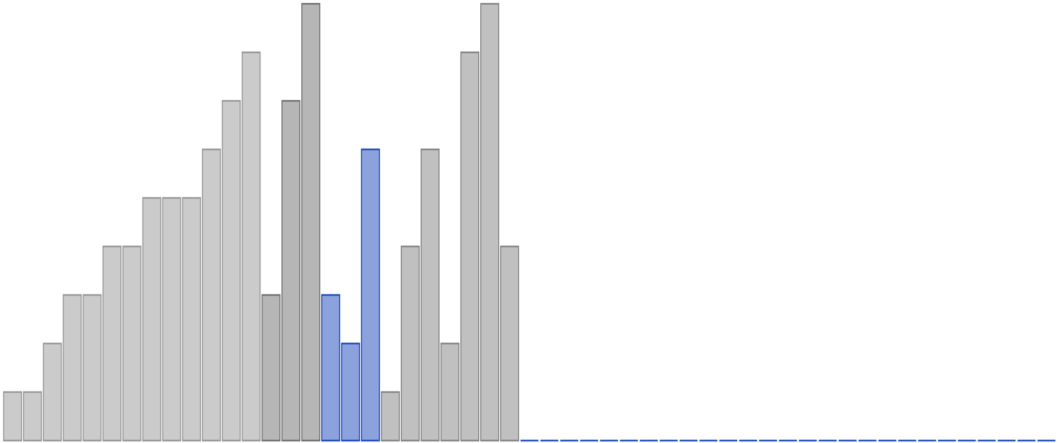


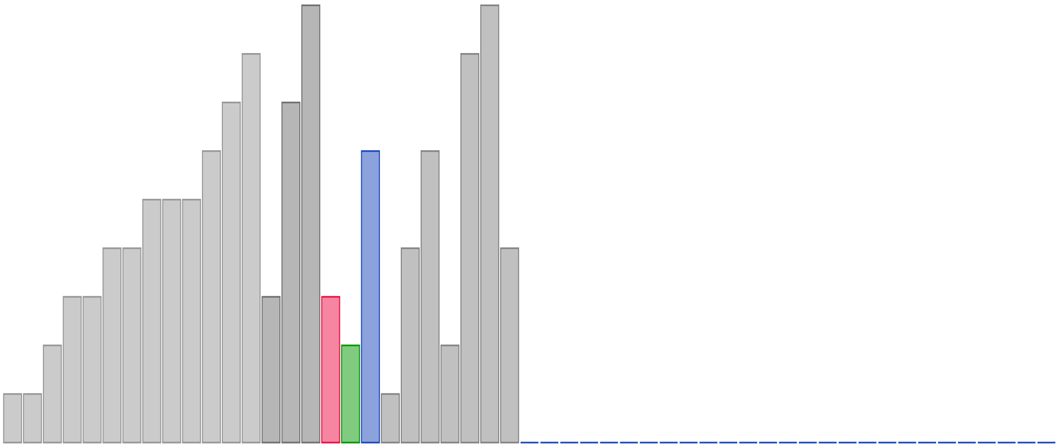


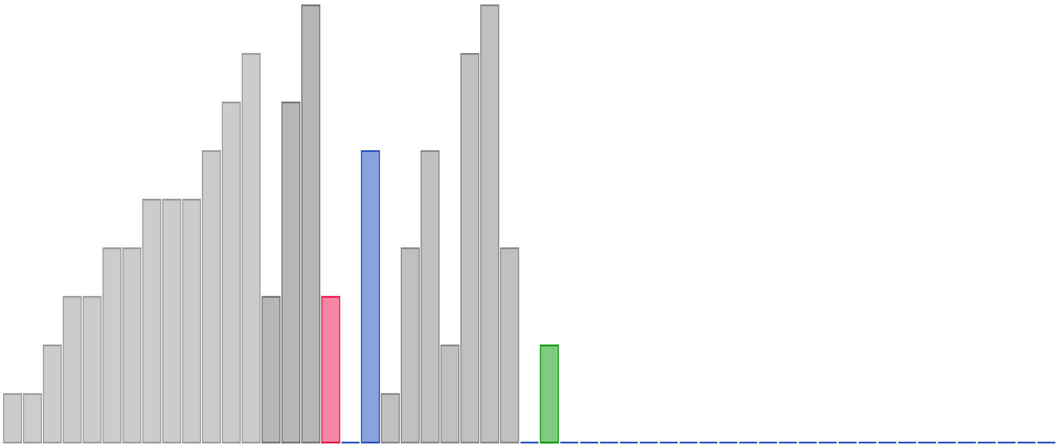


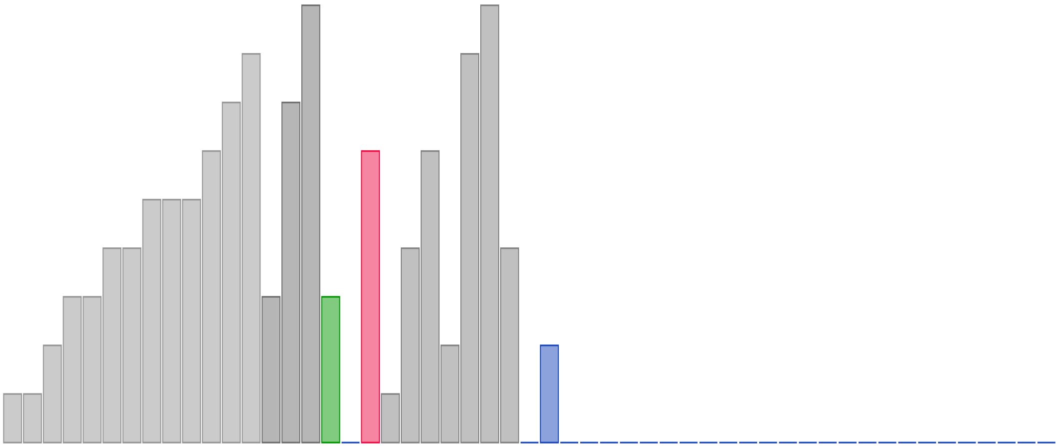


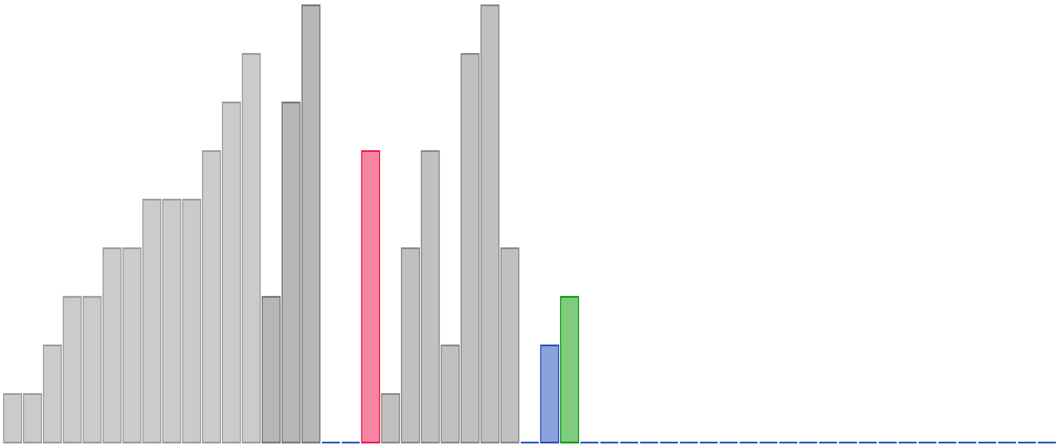


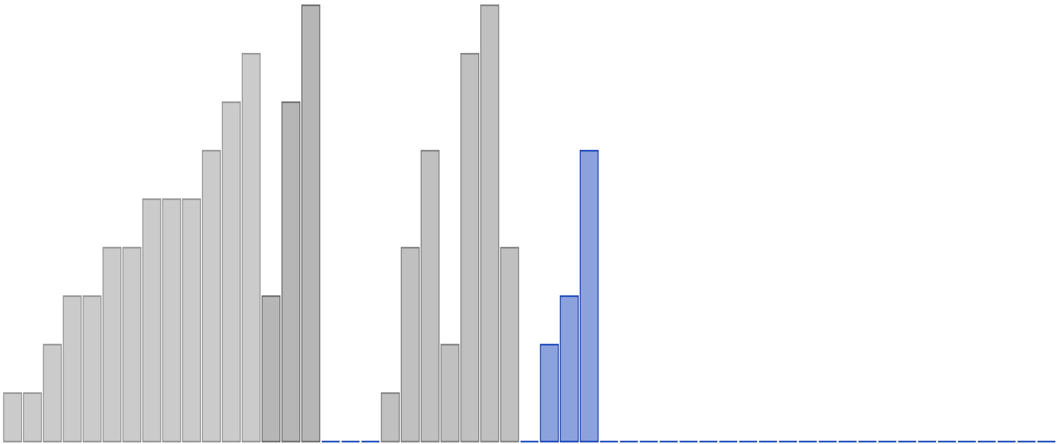


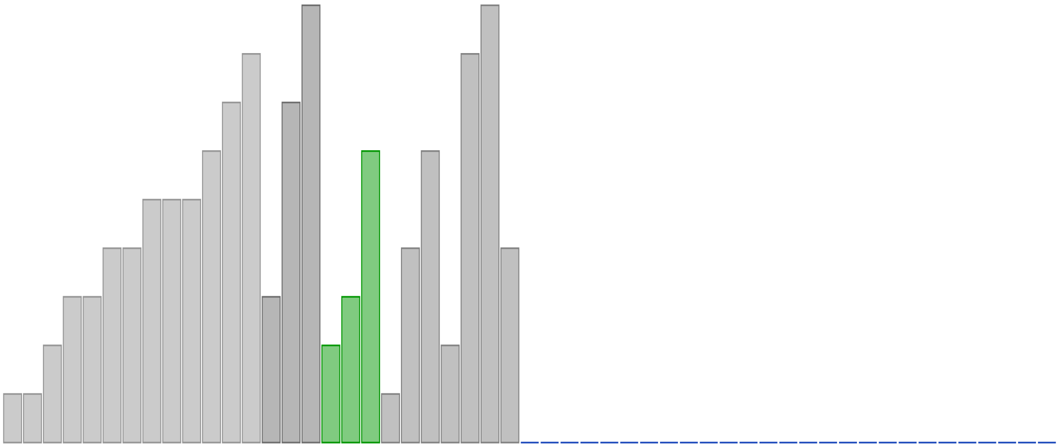


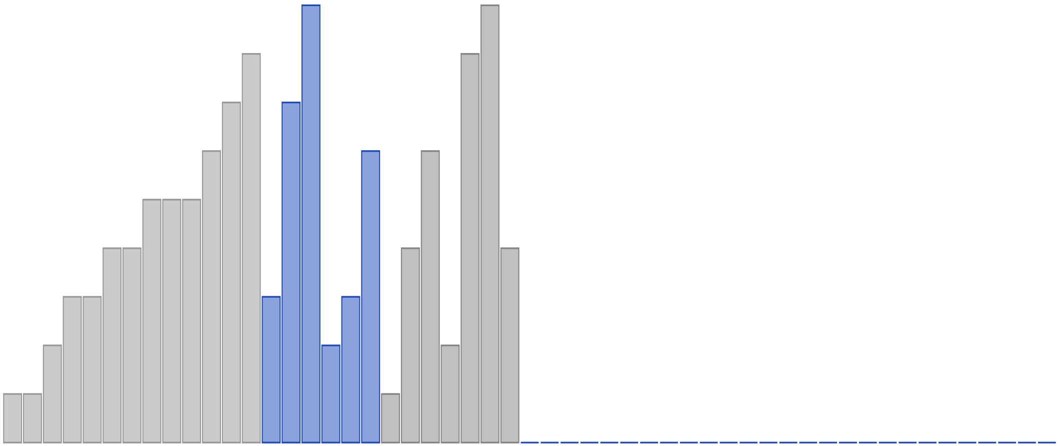


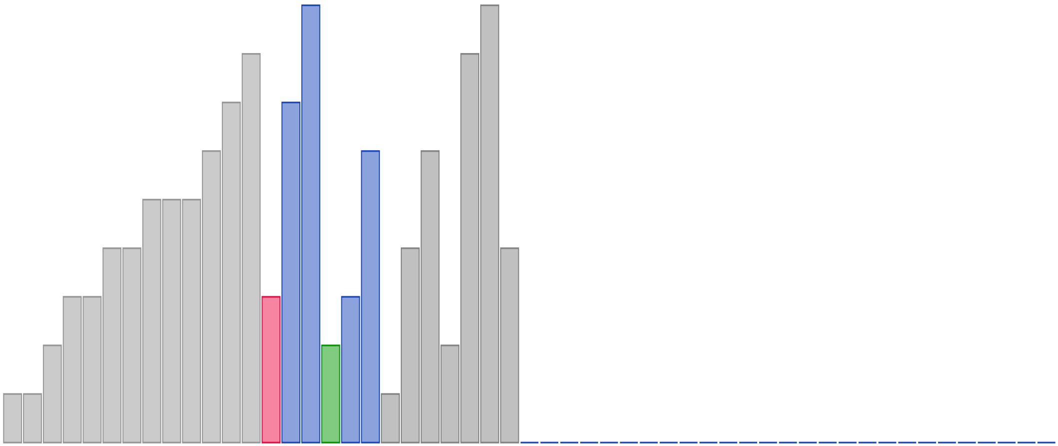


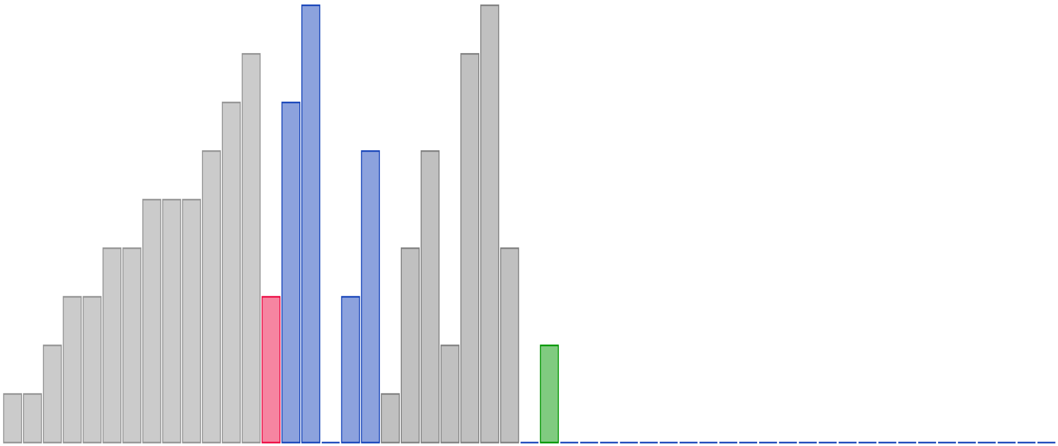


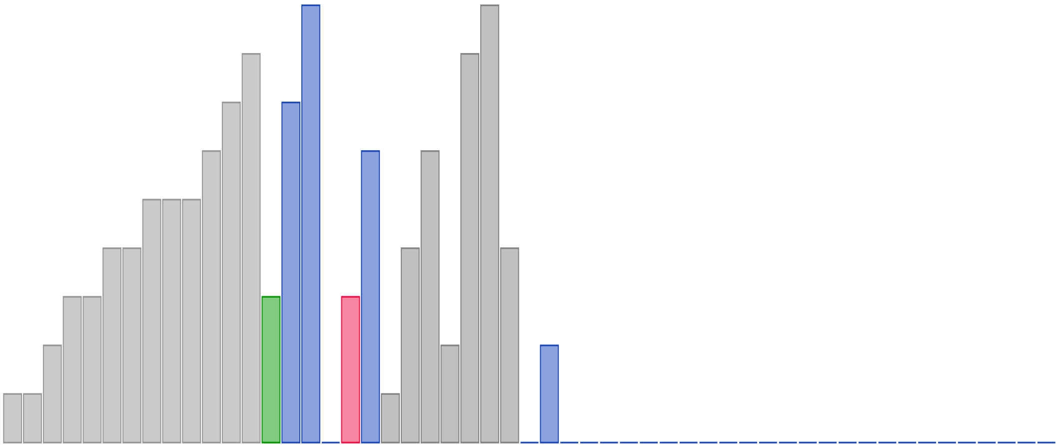


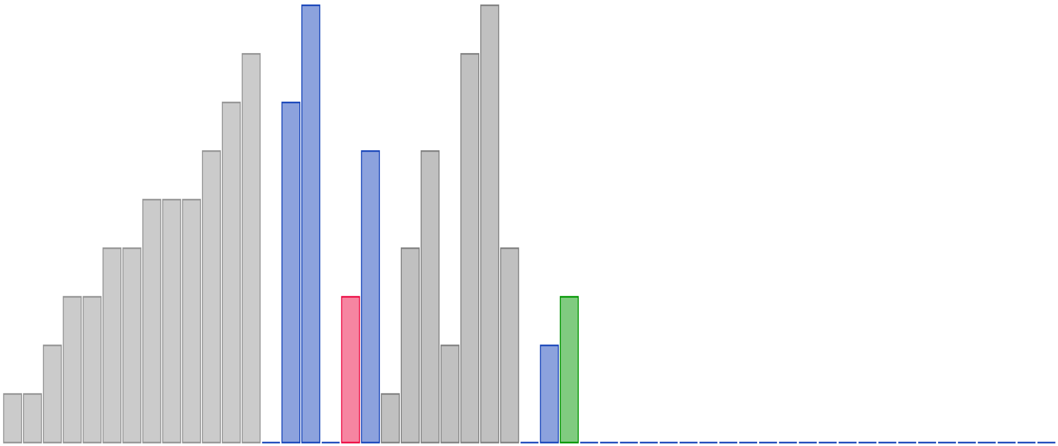


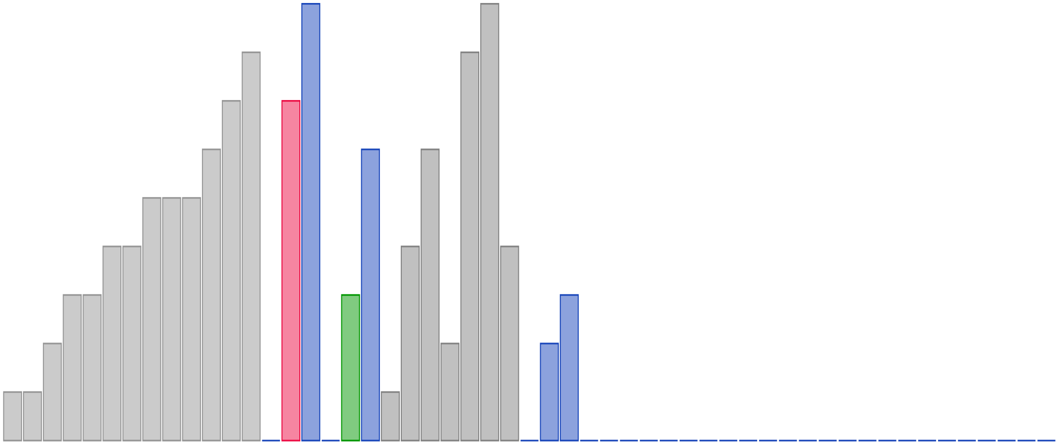


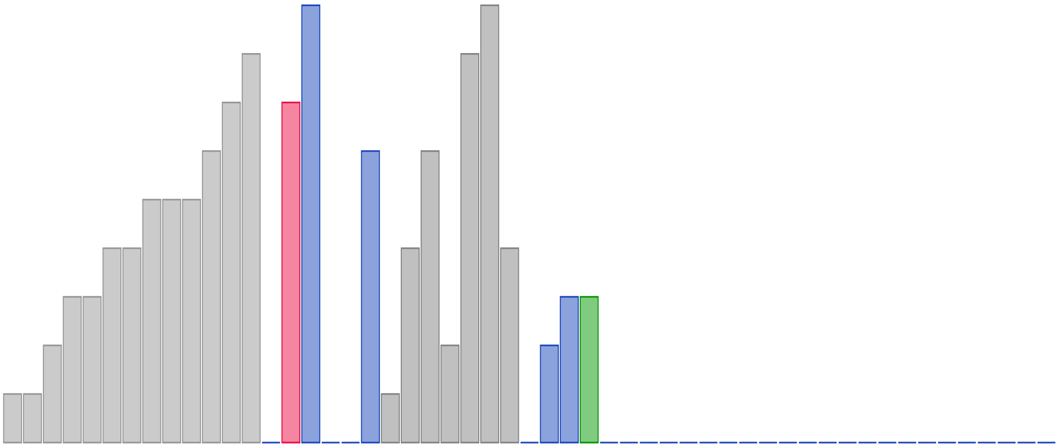


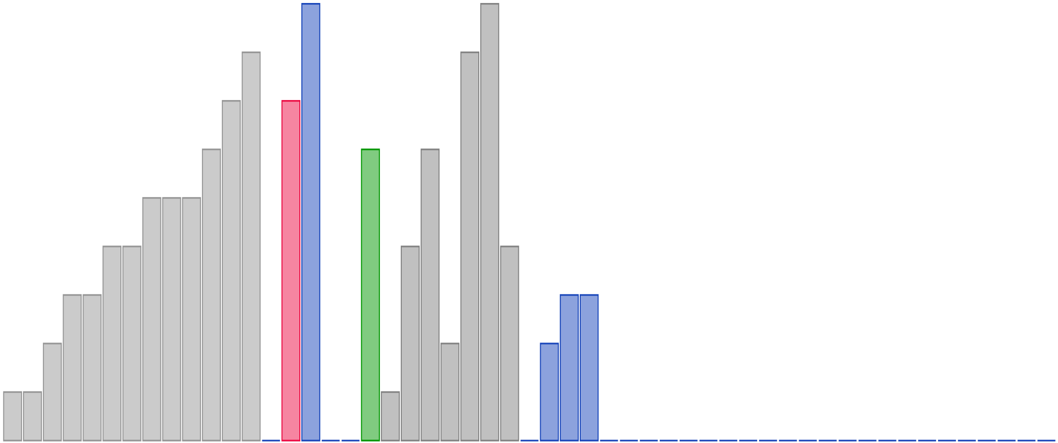


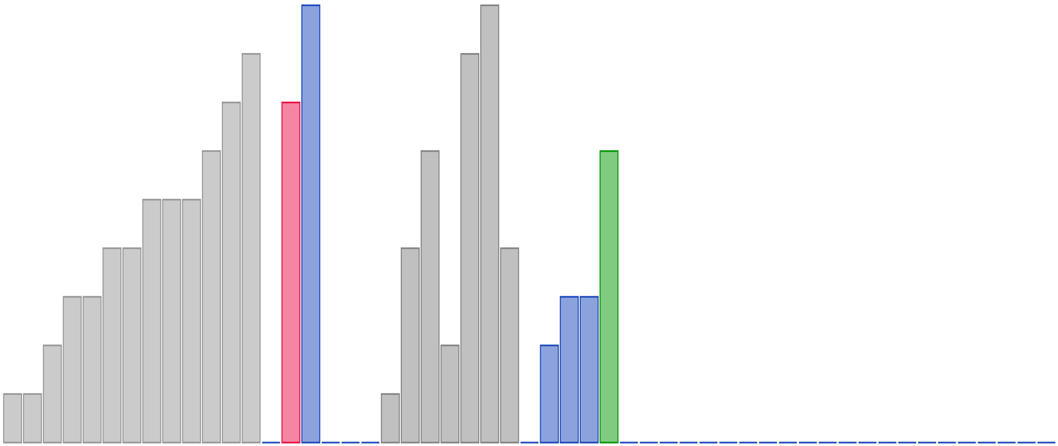


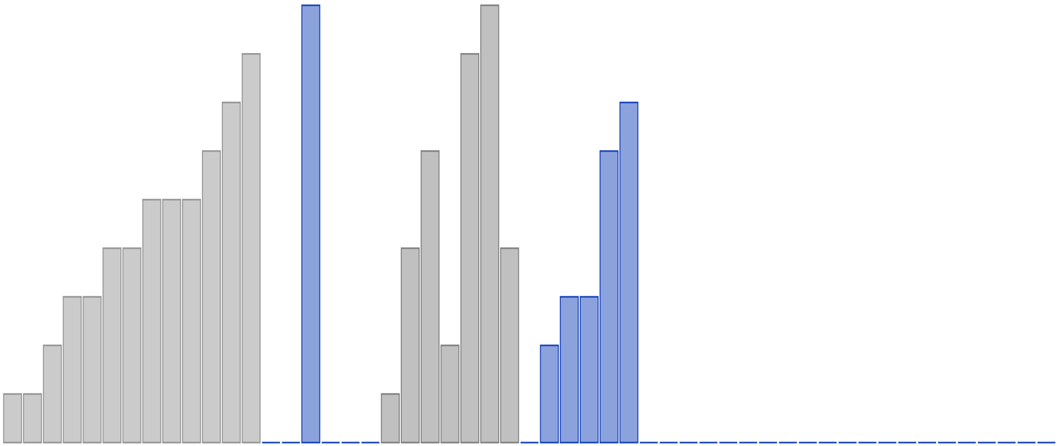


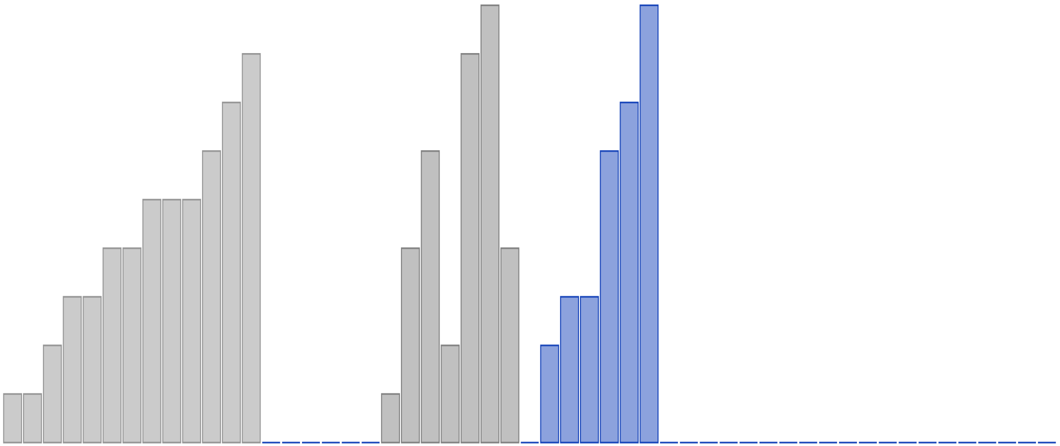


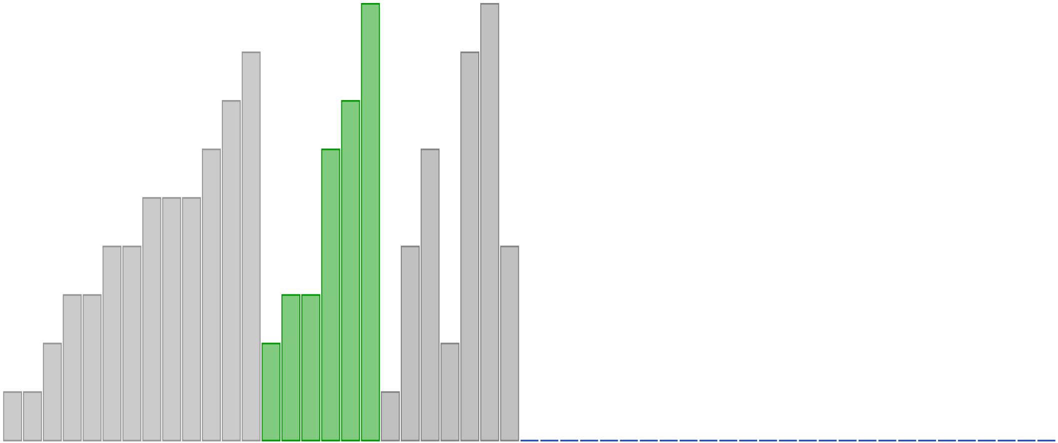


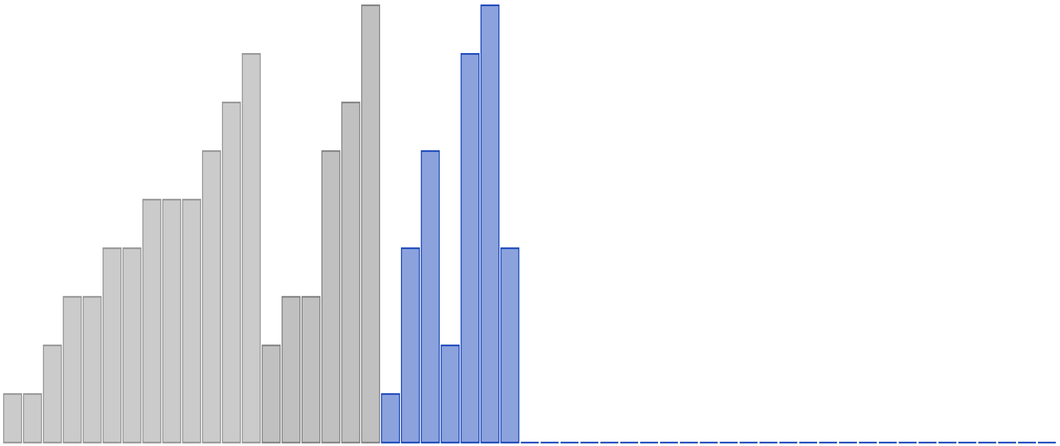


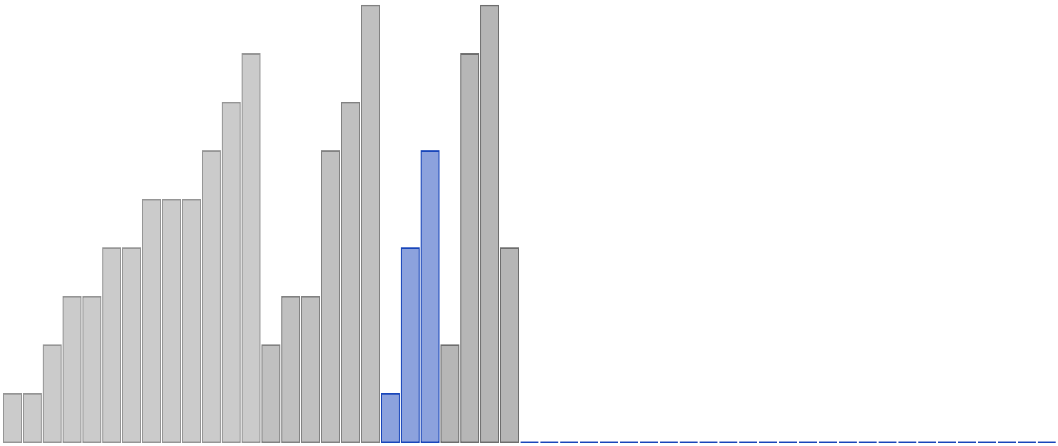


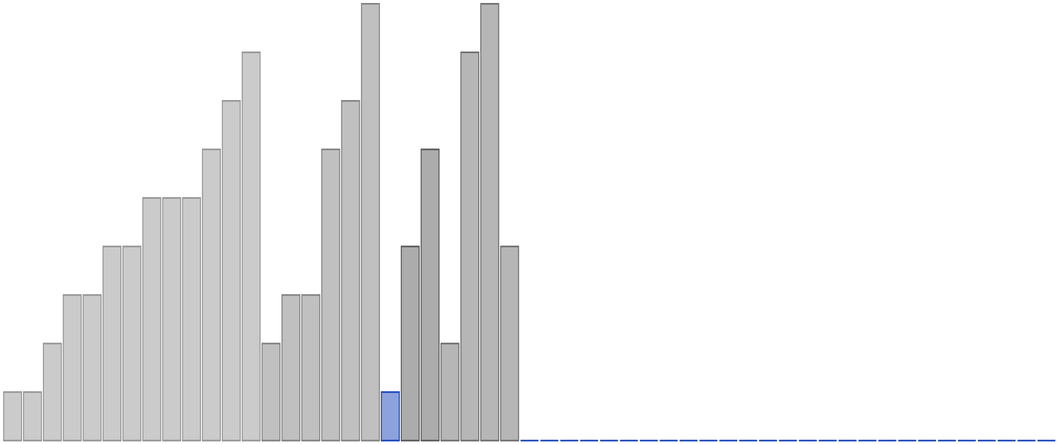


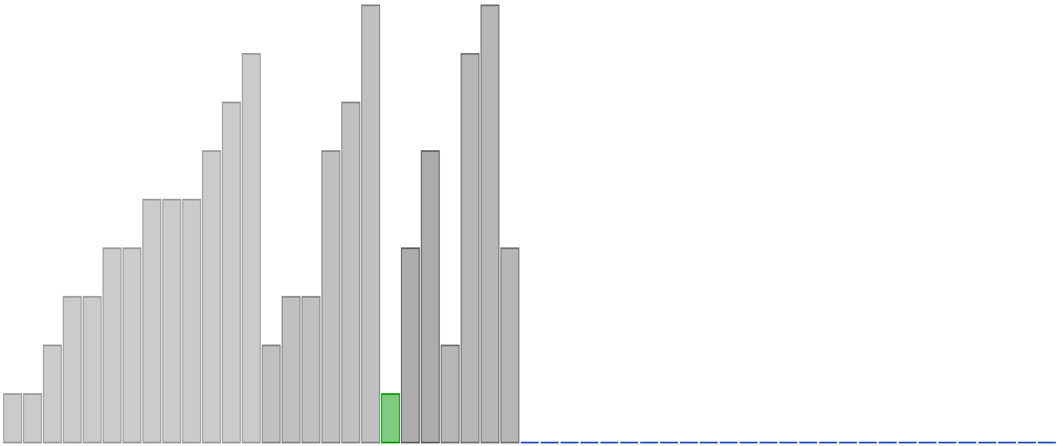


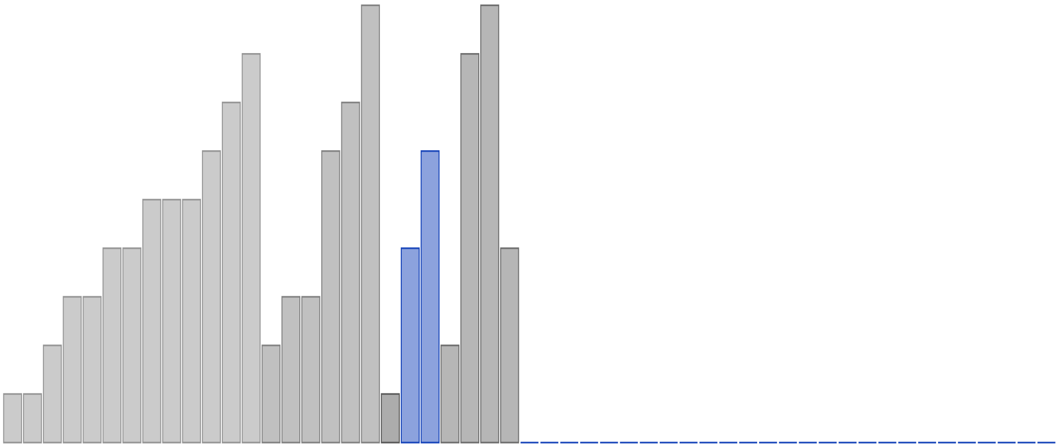


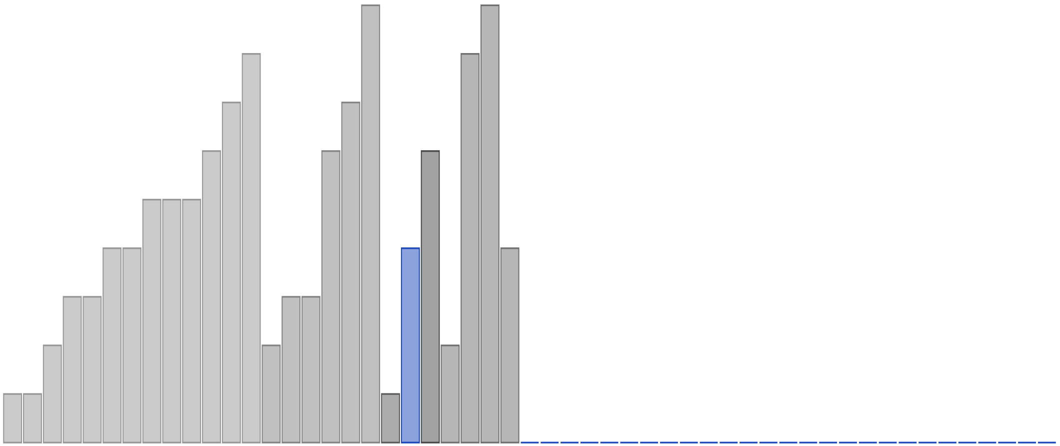


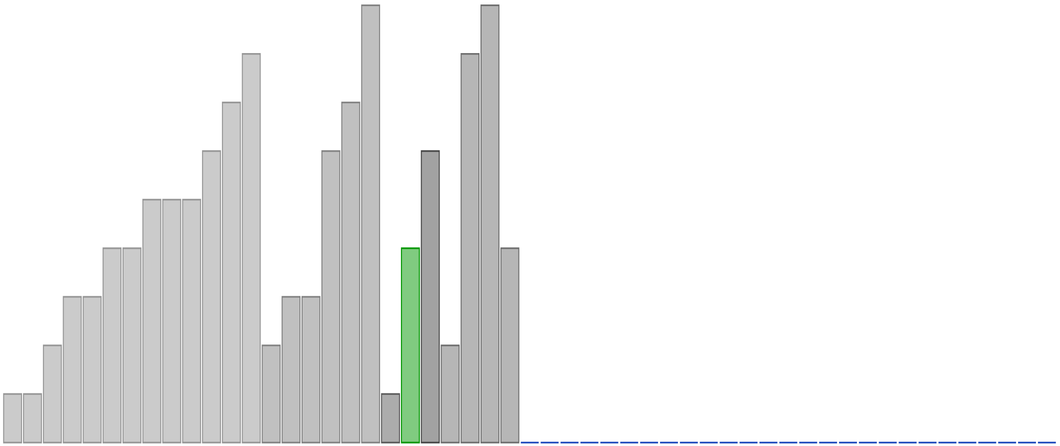


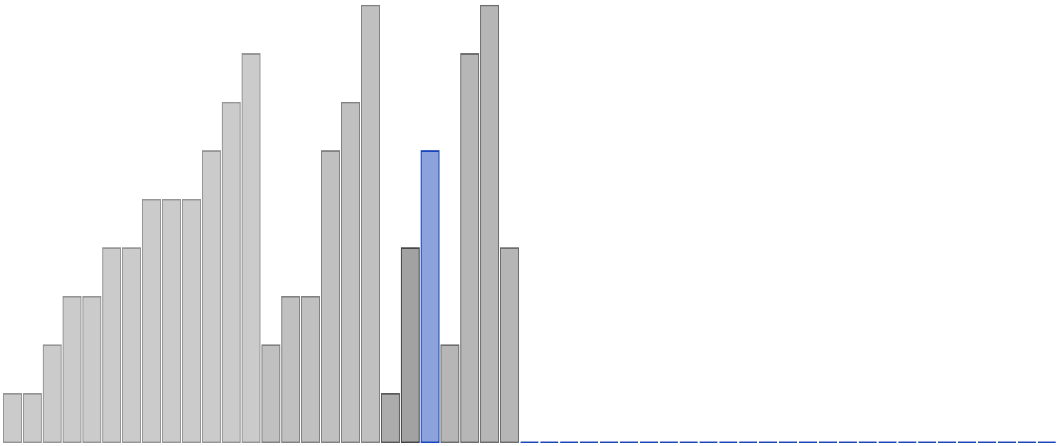


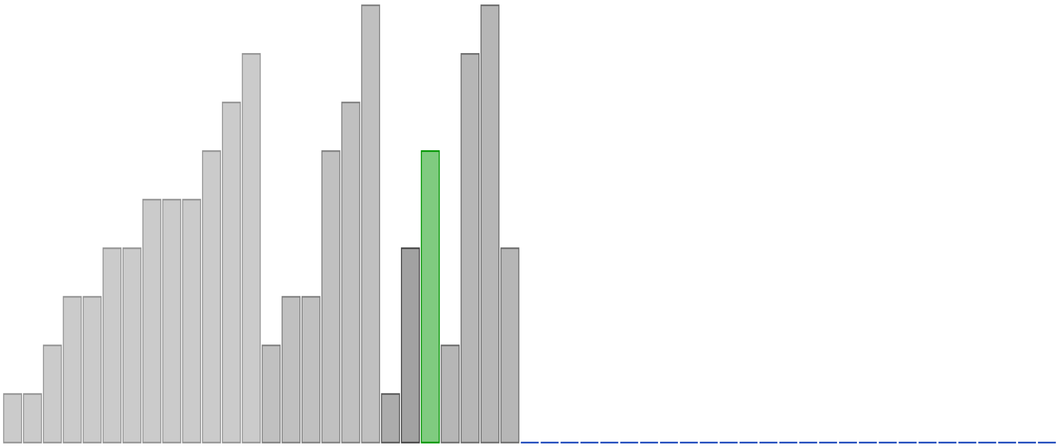


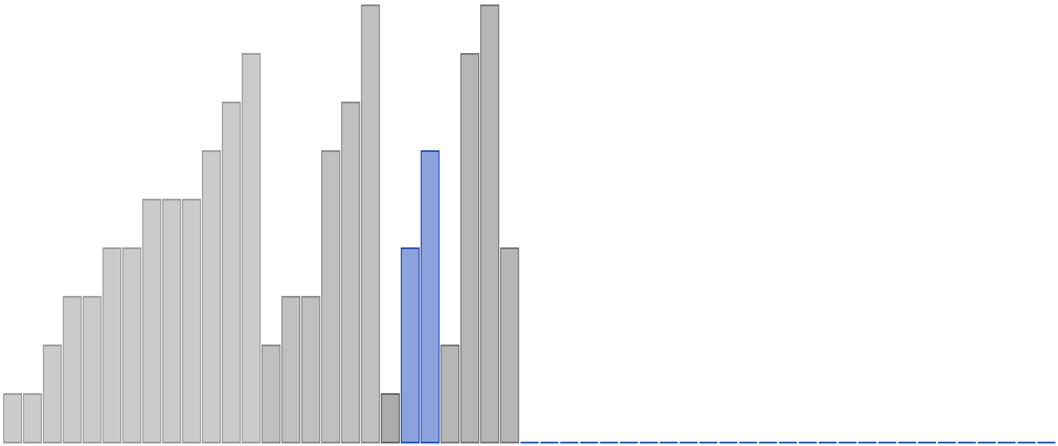


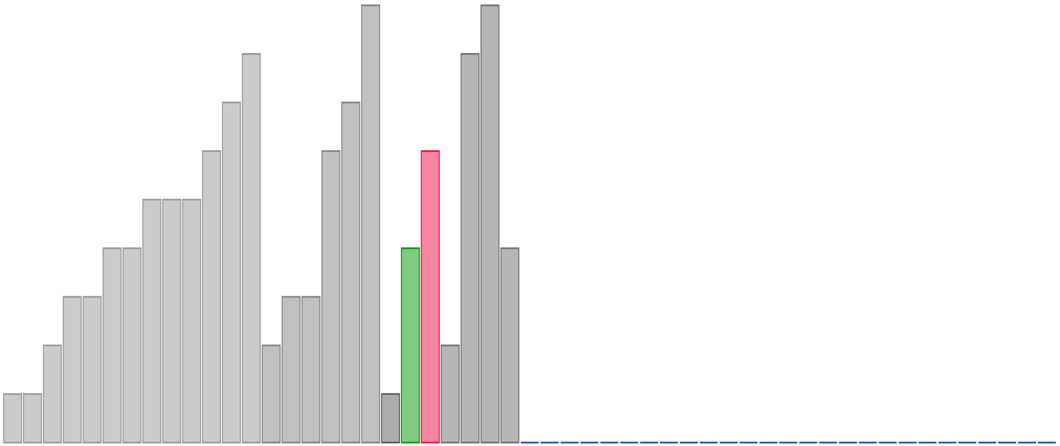


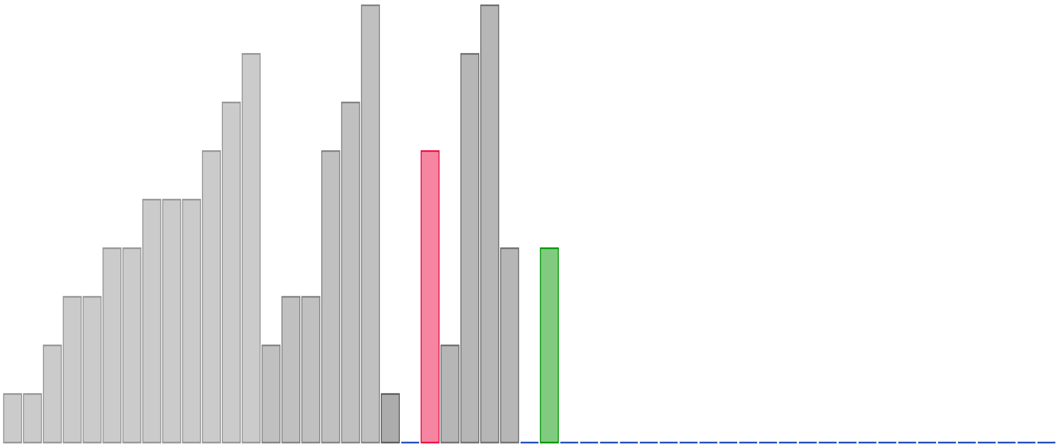


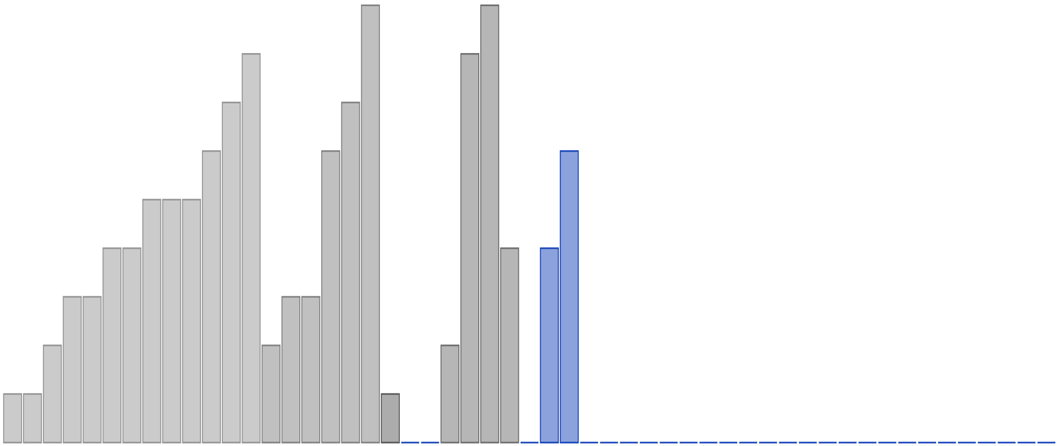


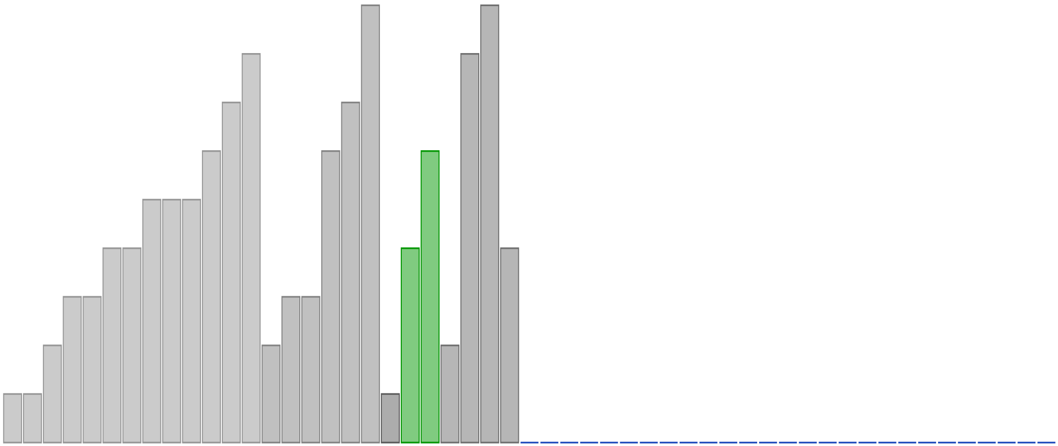


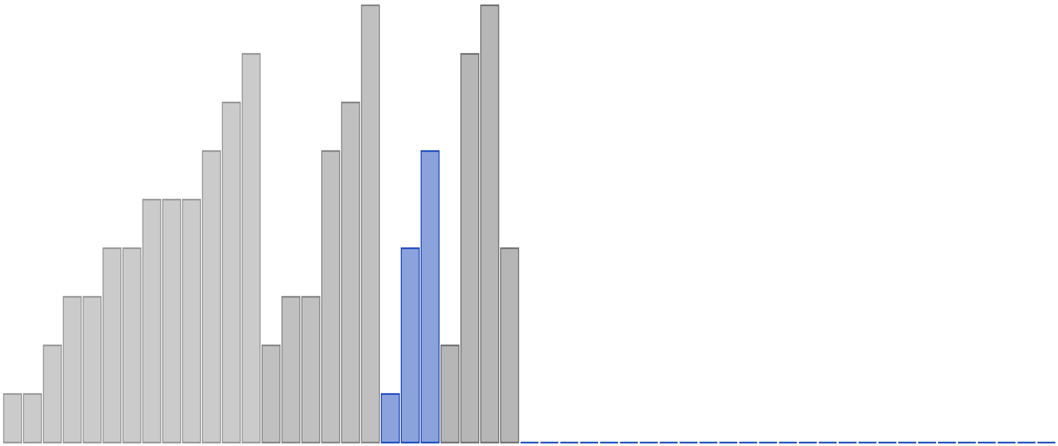


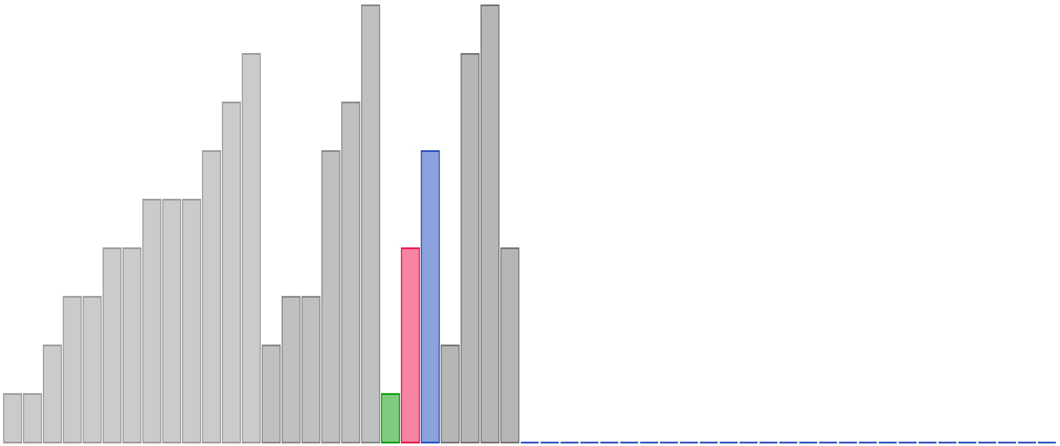


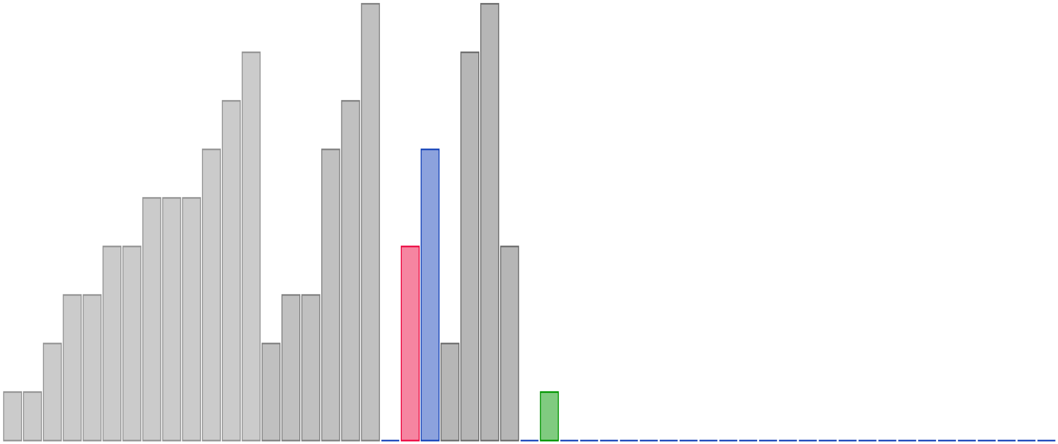


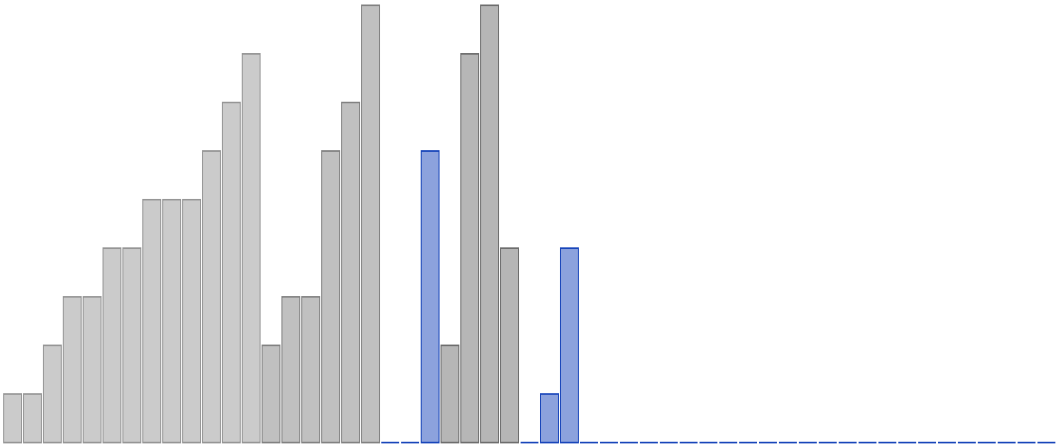


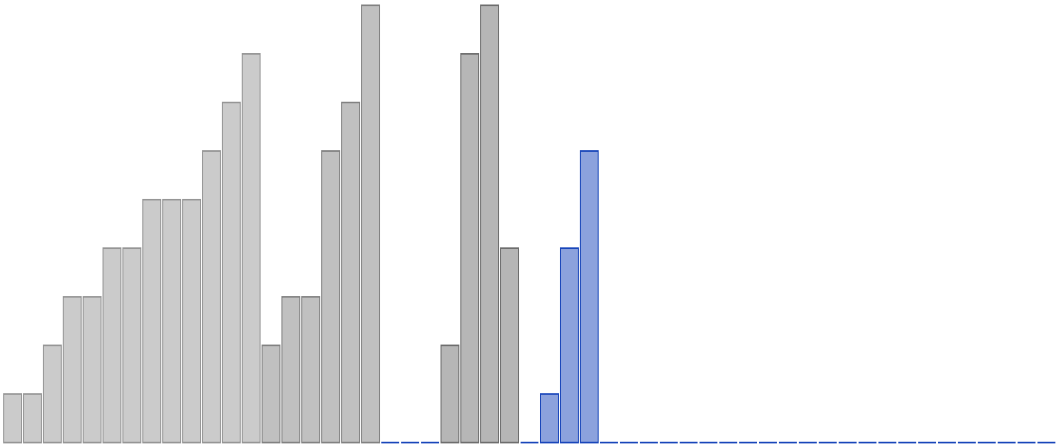


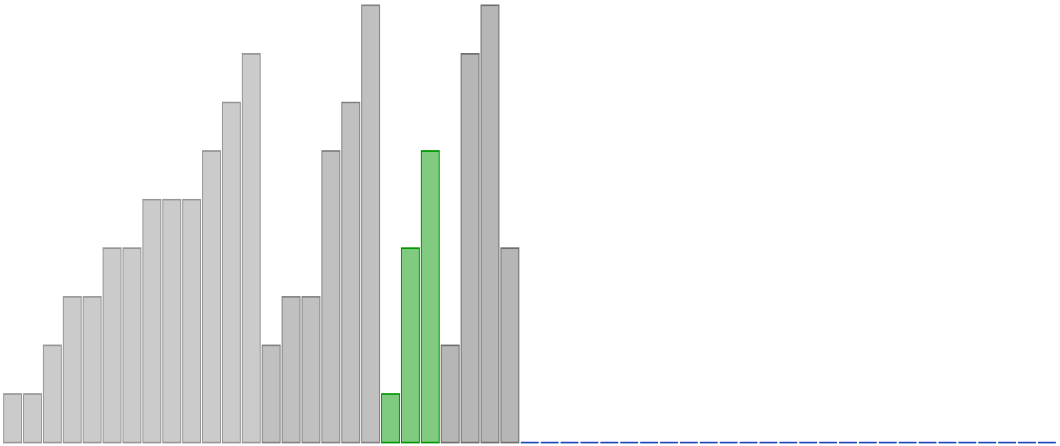


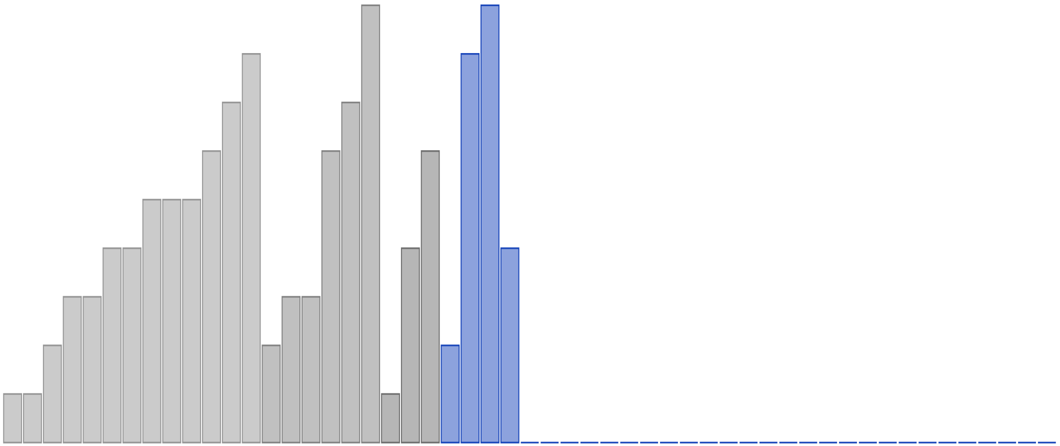


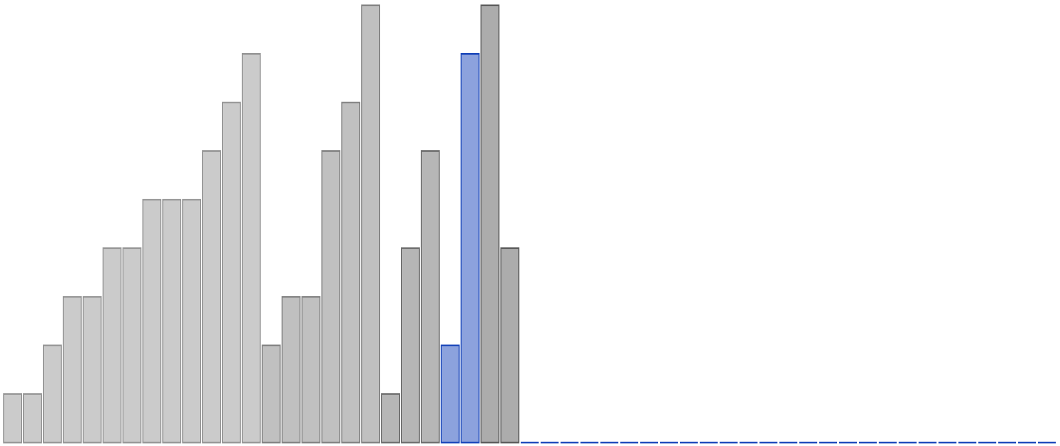




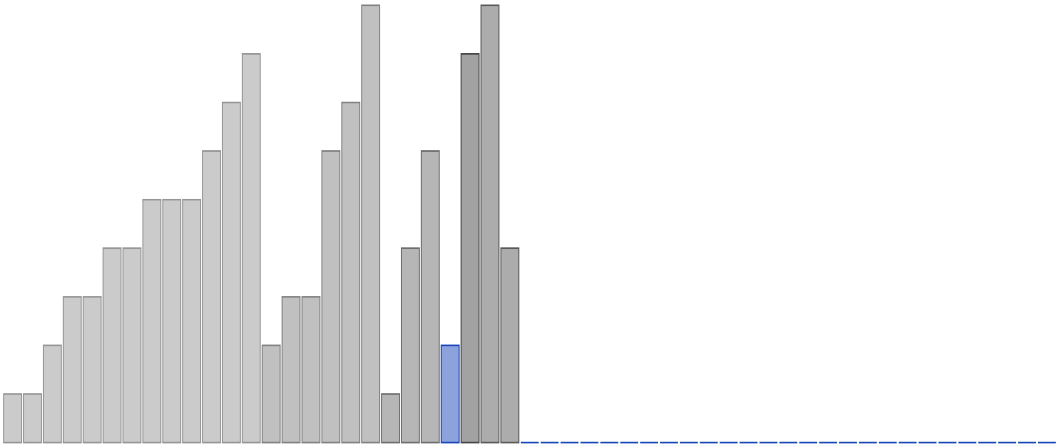


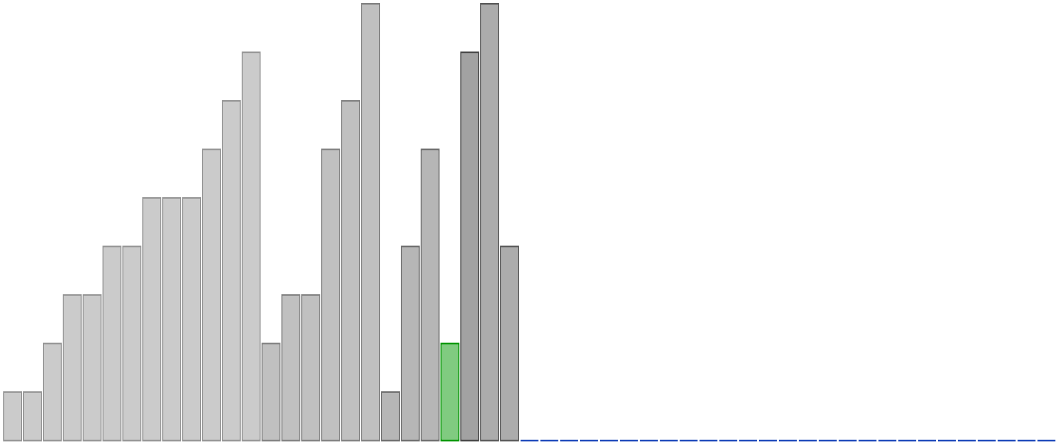


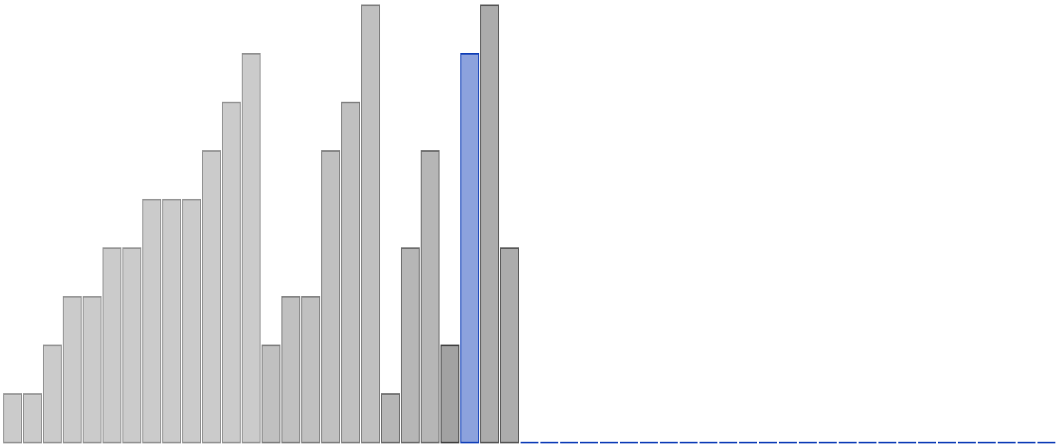


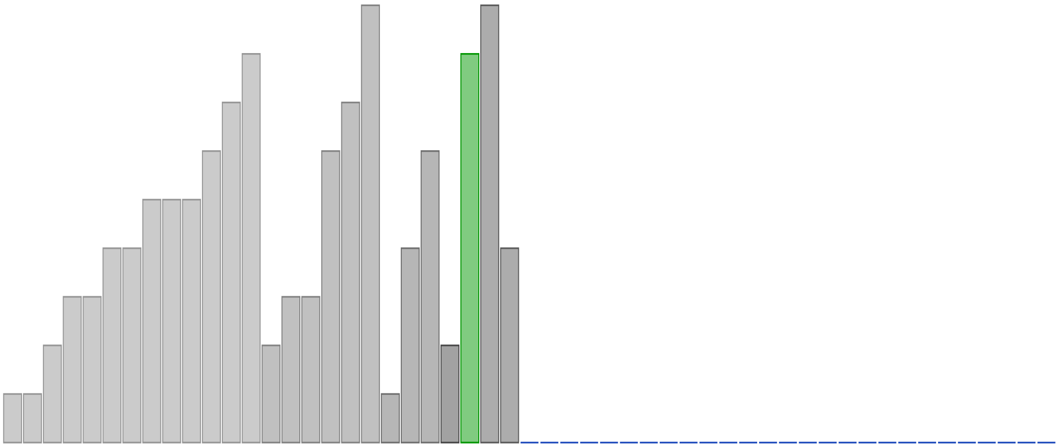


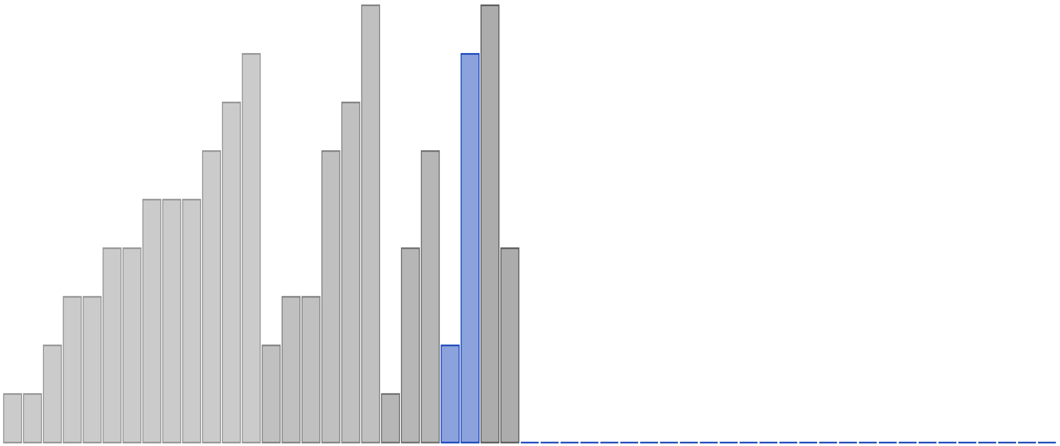
Mergesort: Example

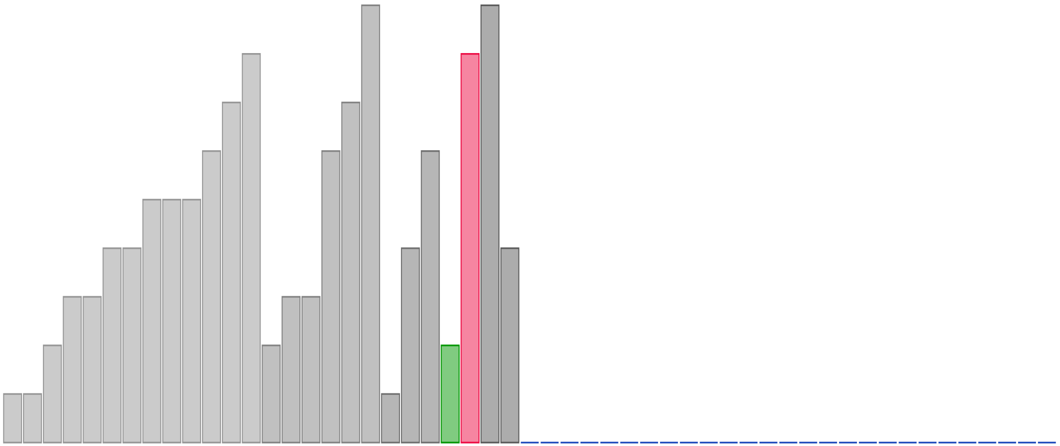


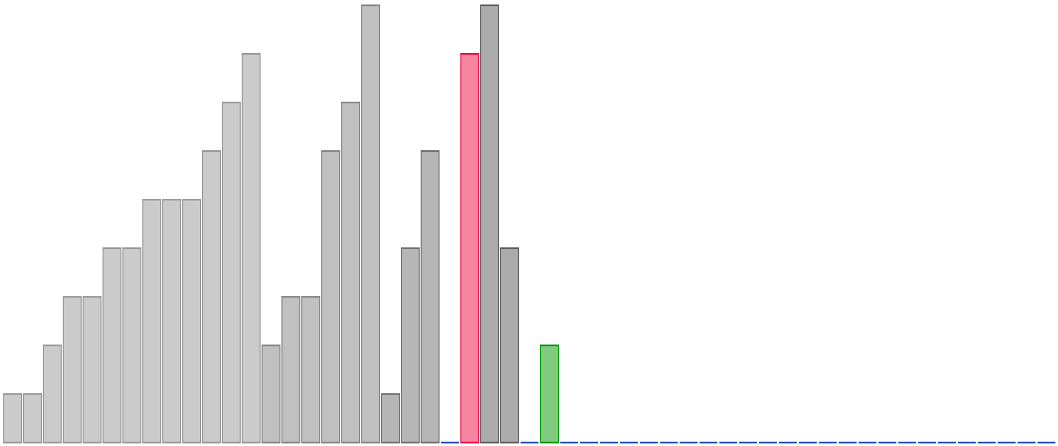


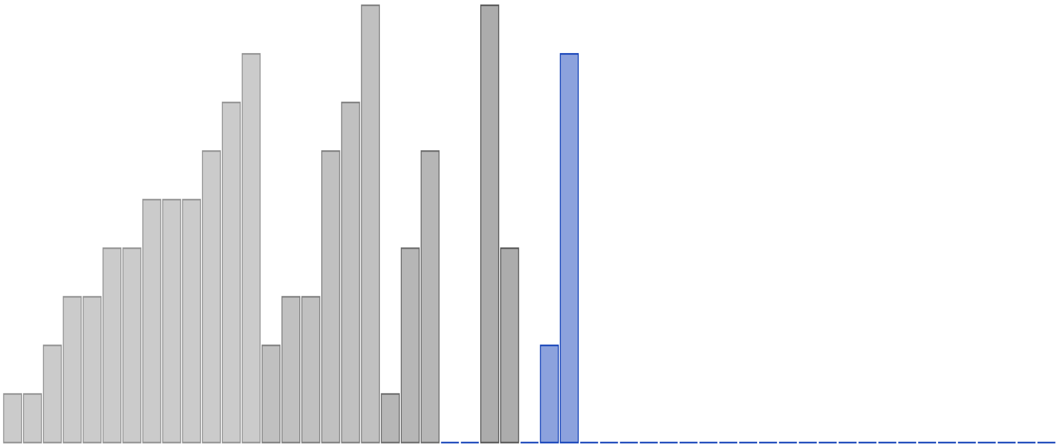


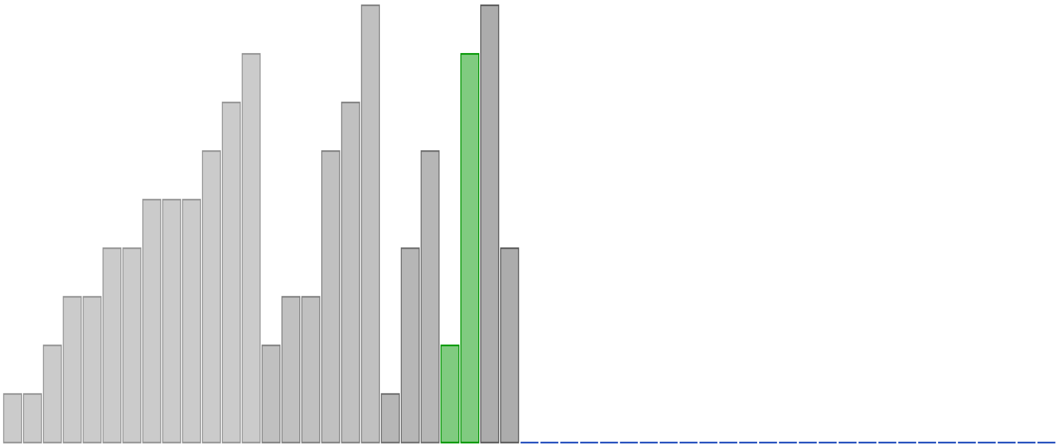


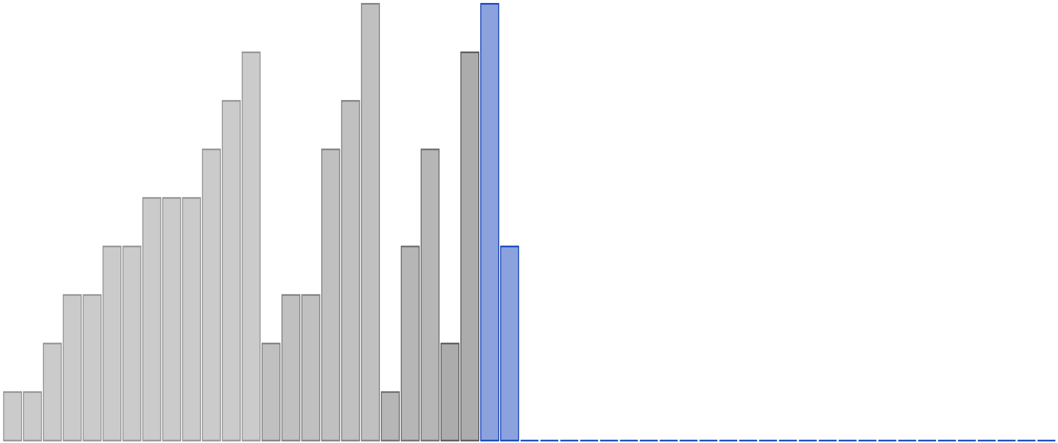


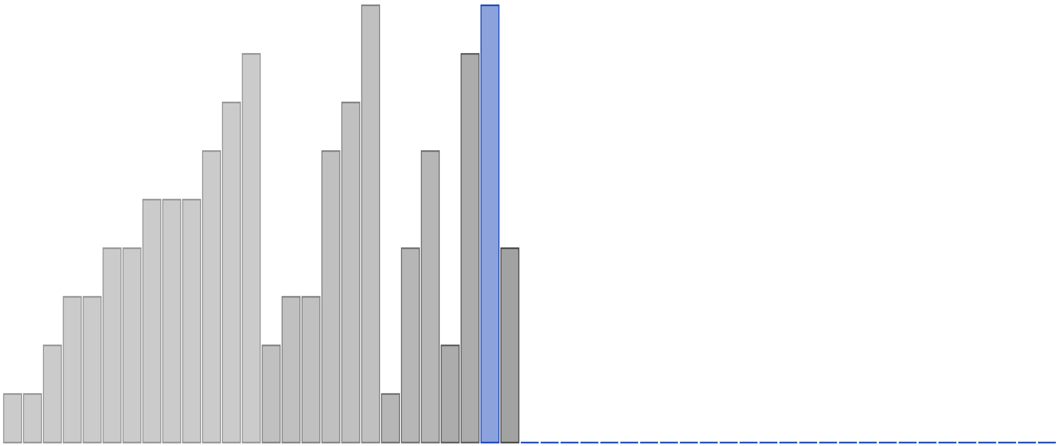


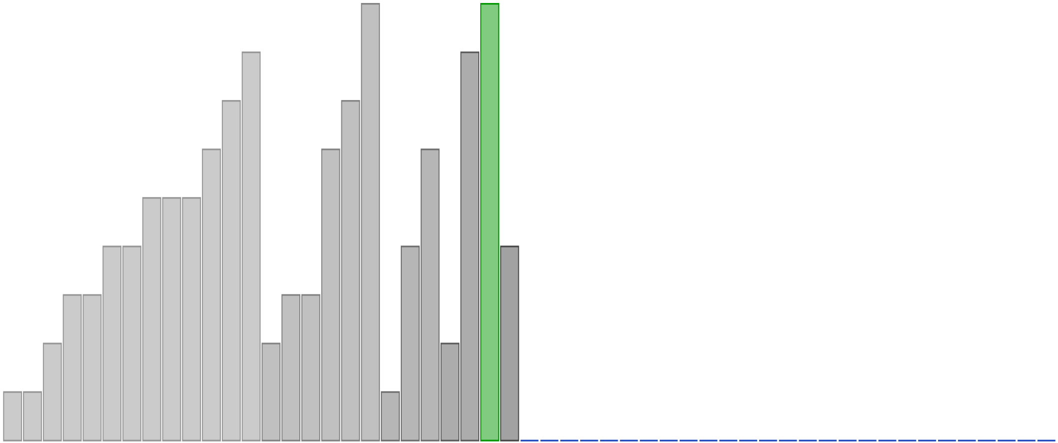


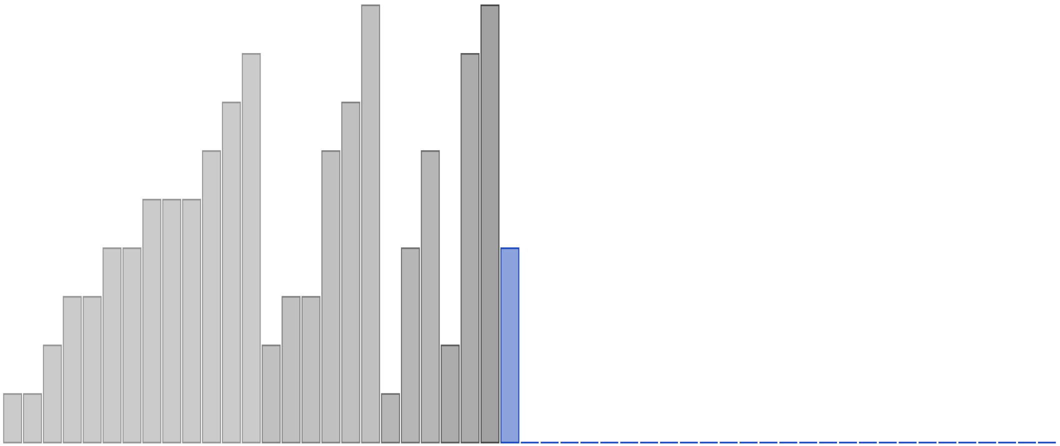


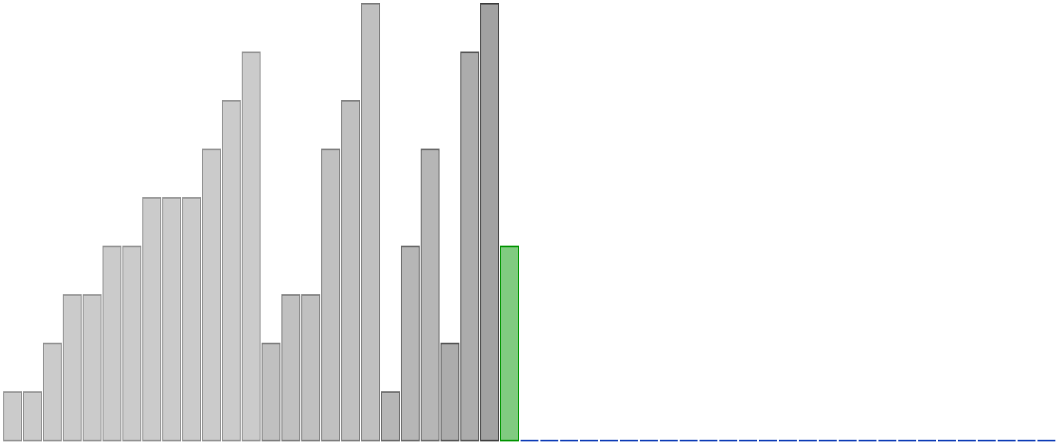


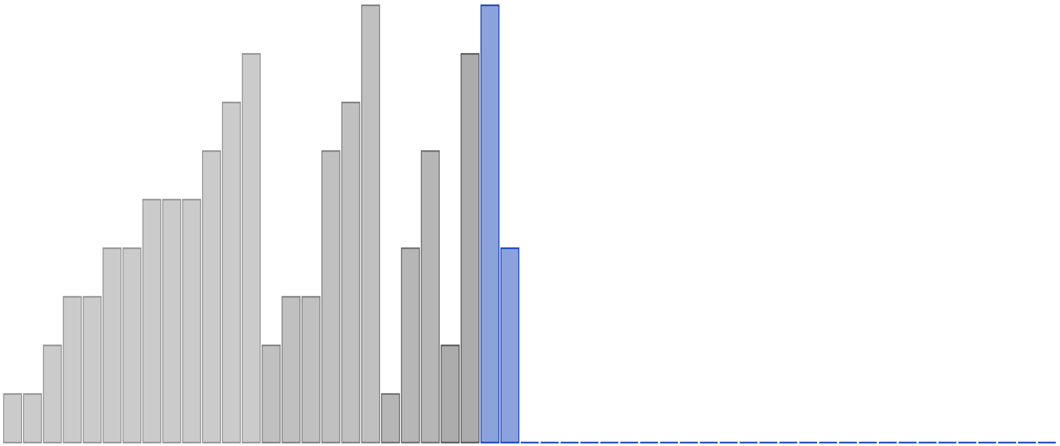


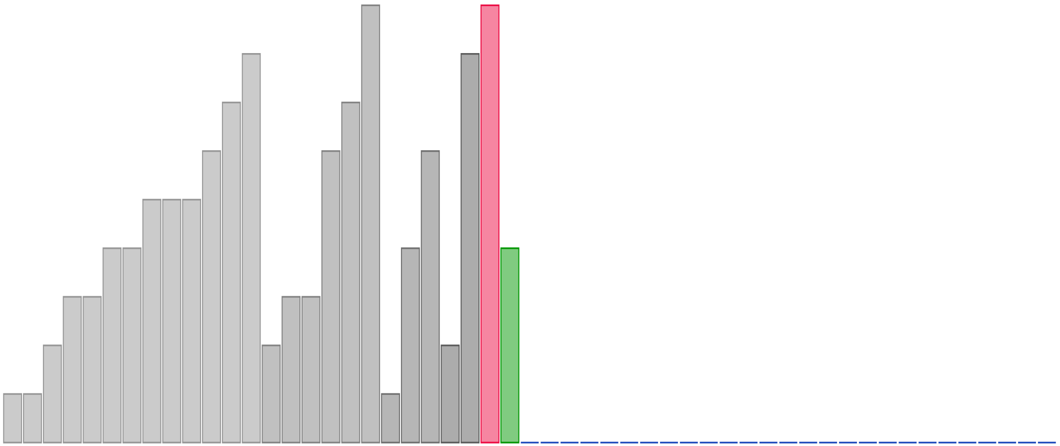


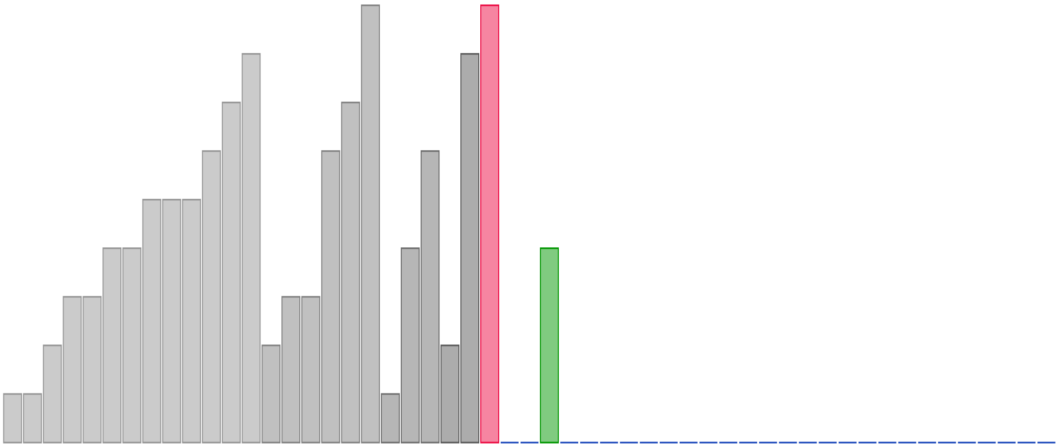


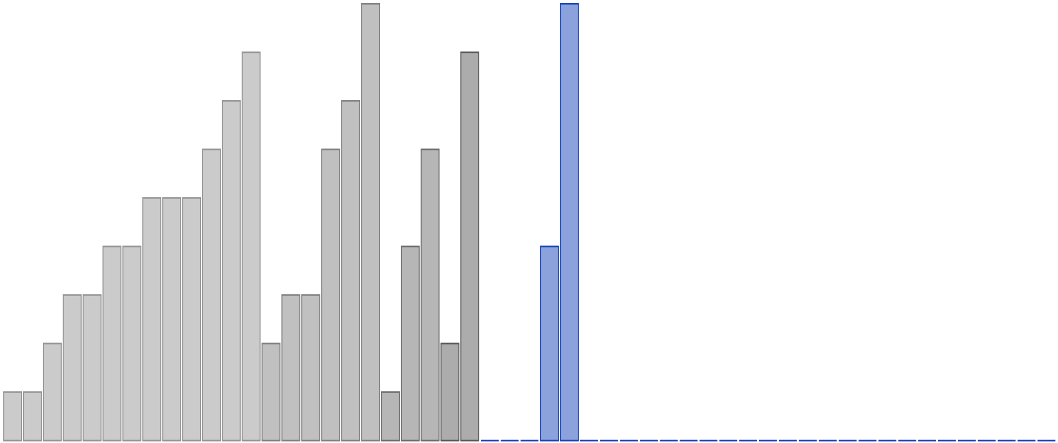


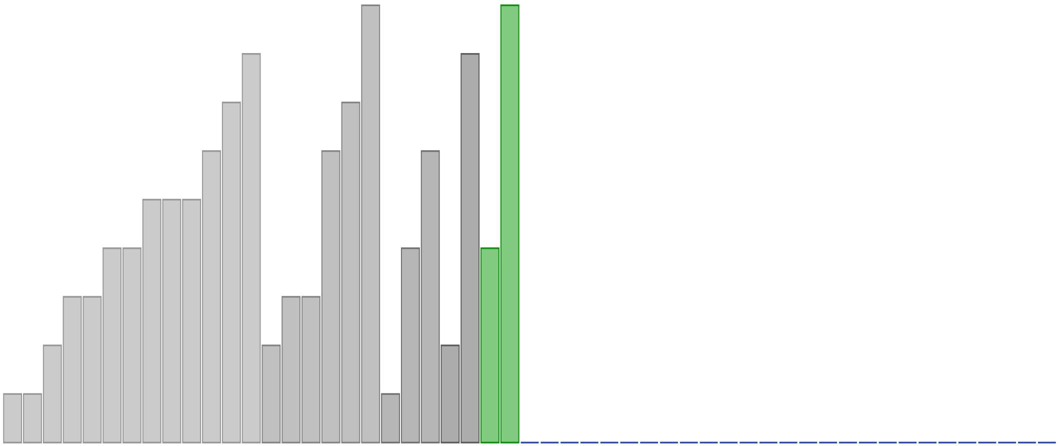


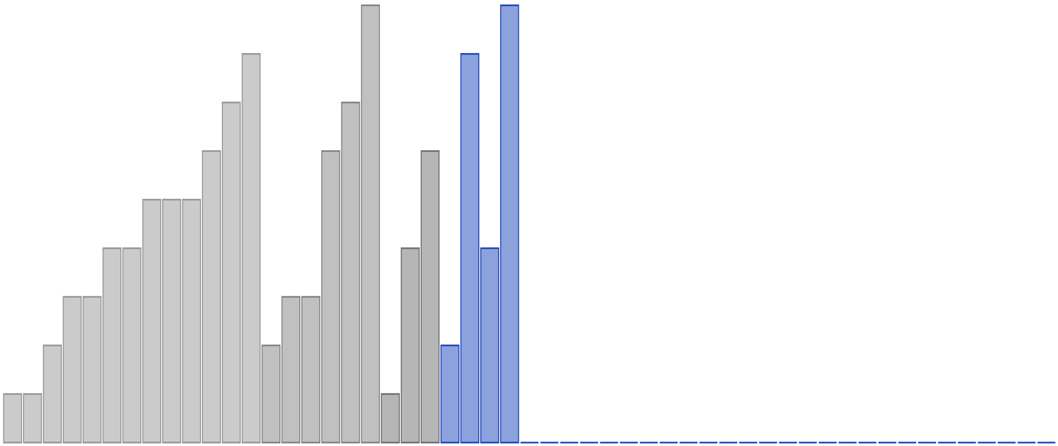


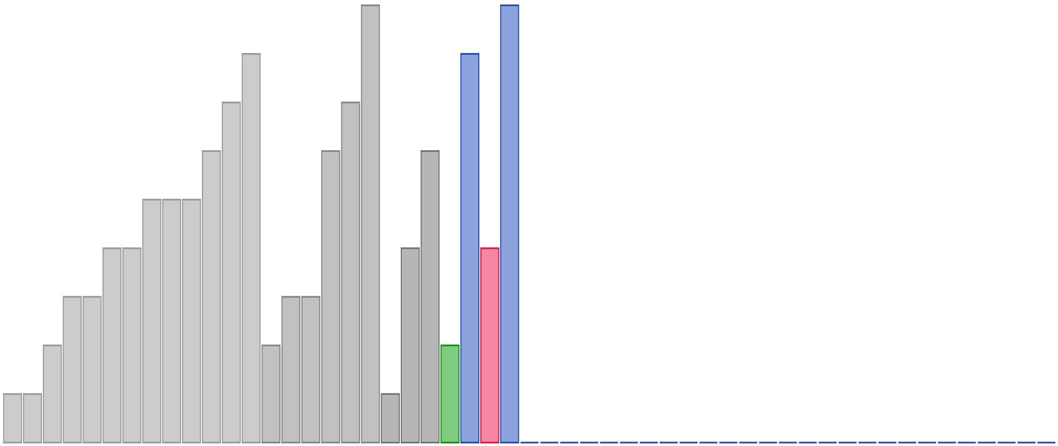


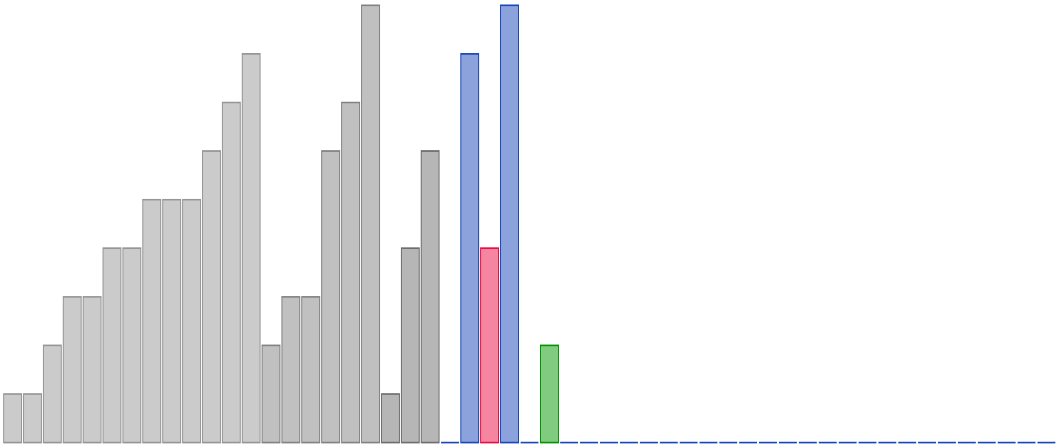


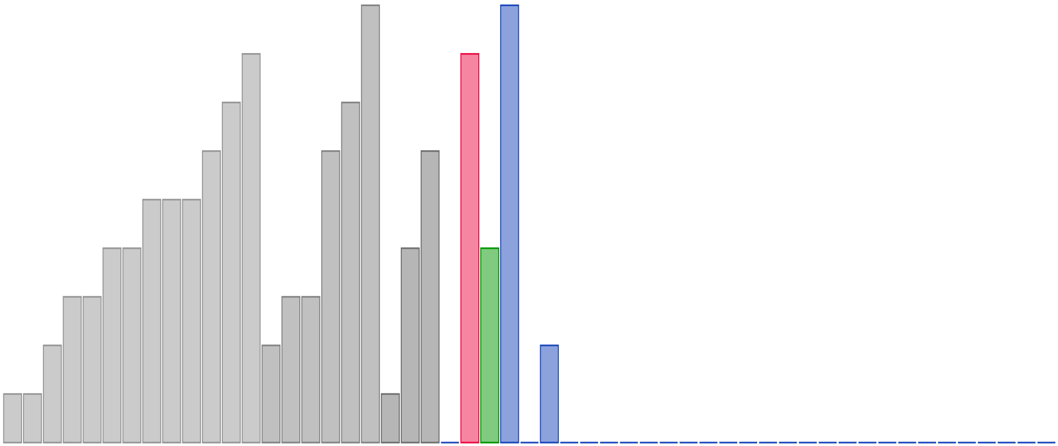


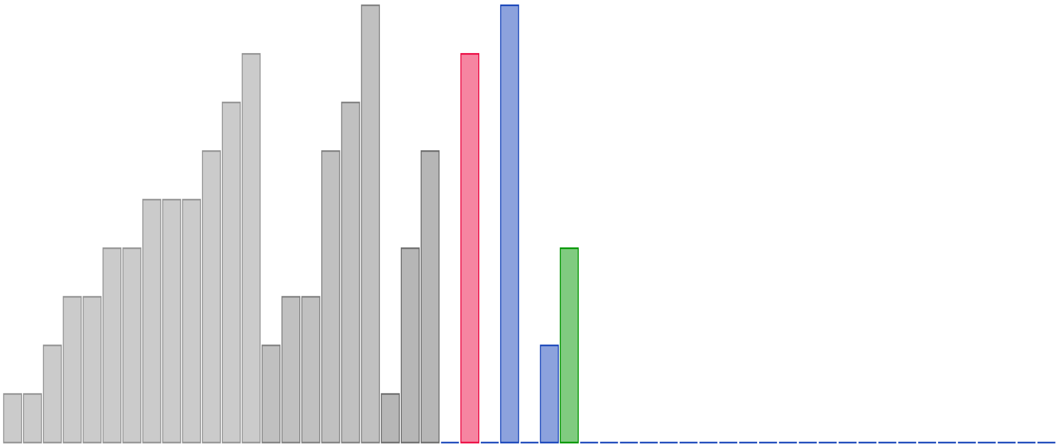


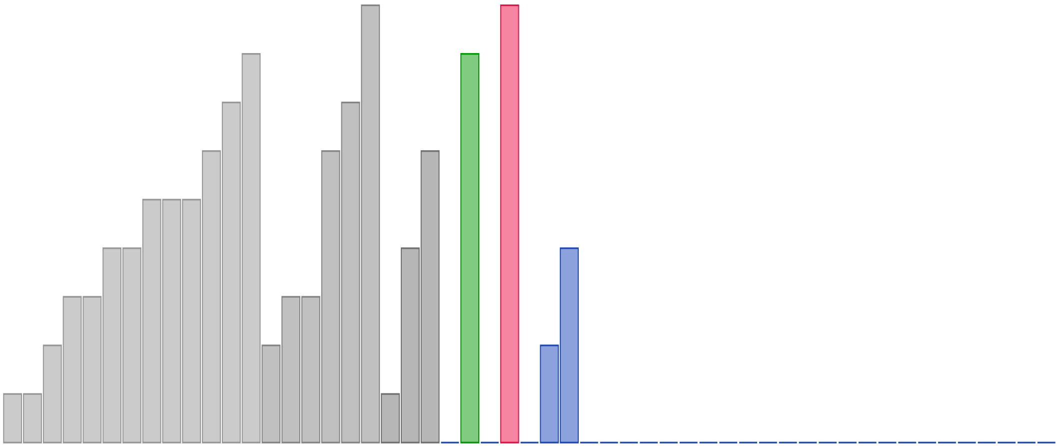


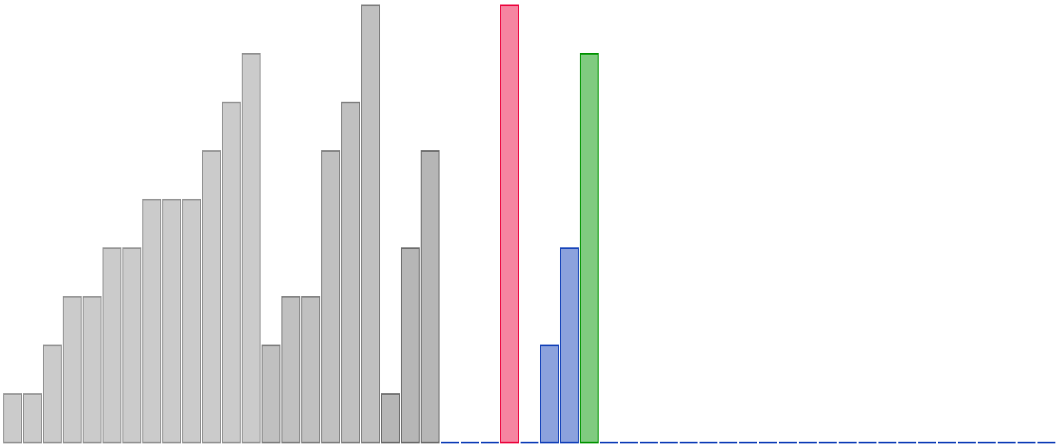


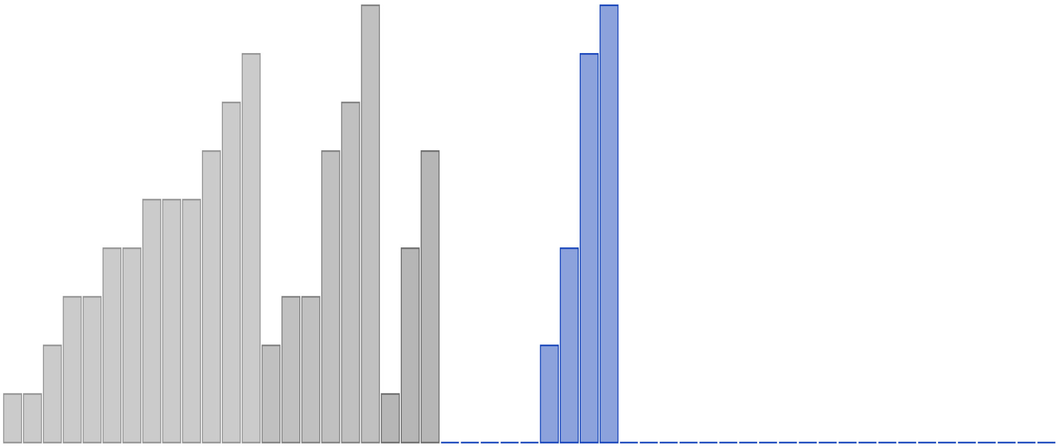


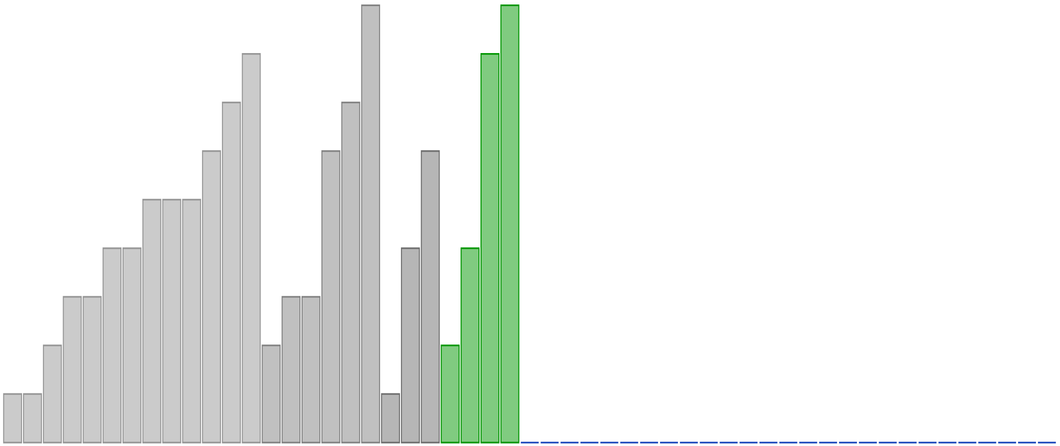


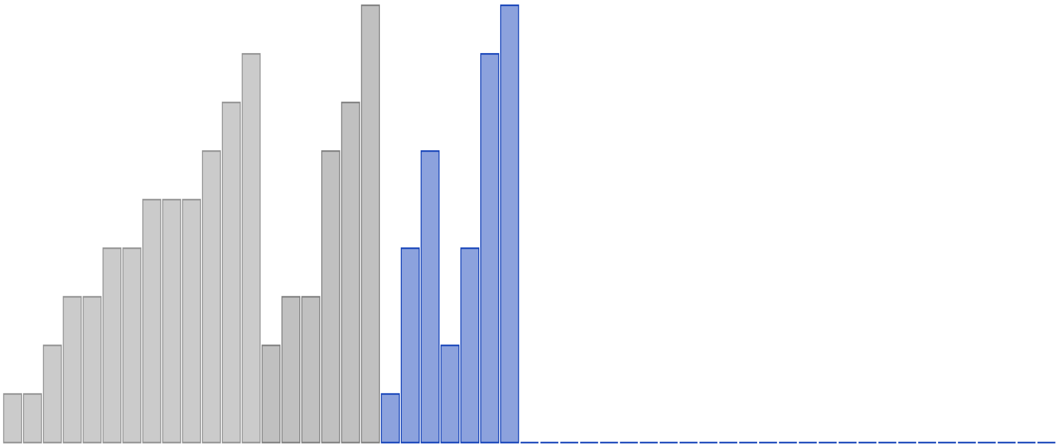


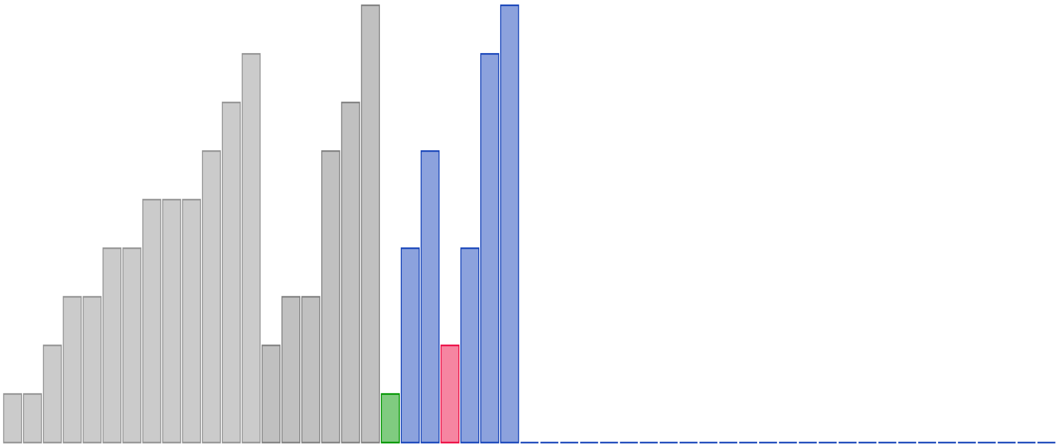


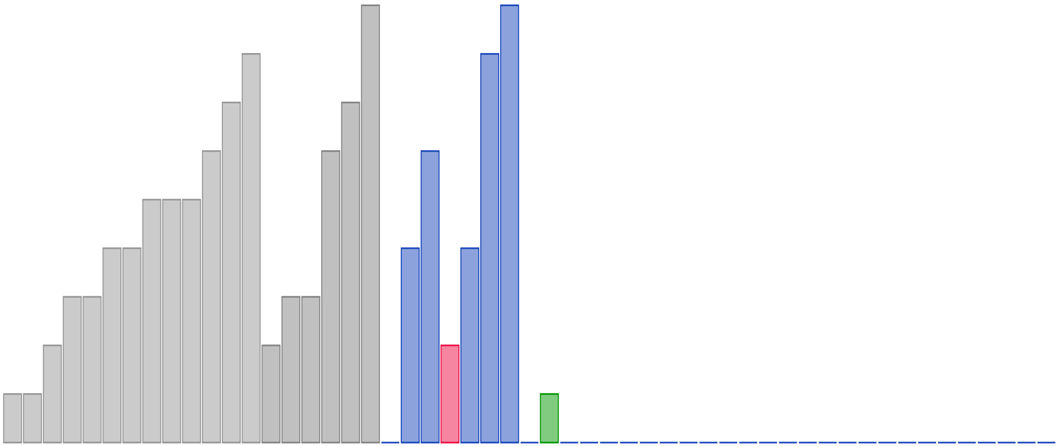


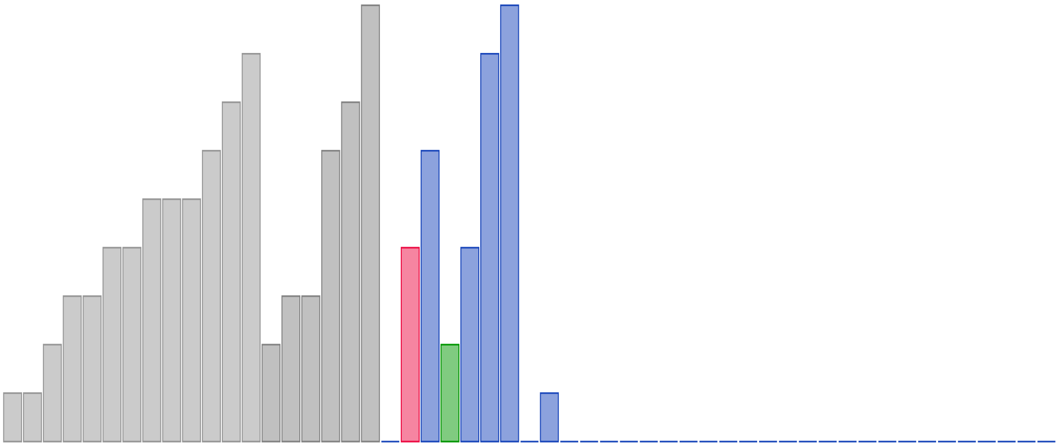


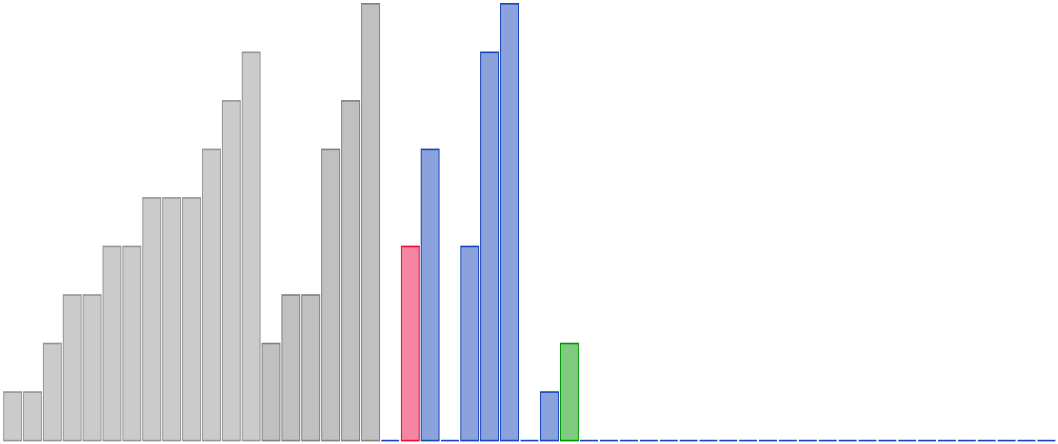


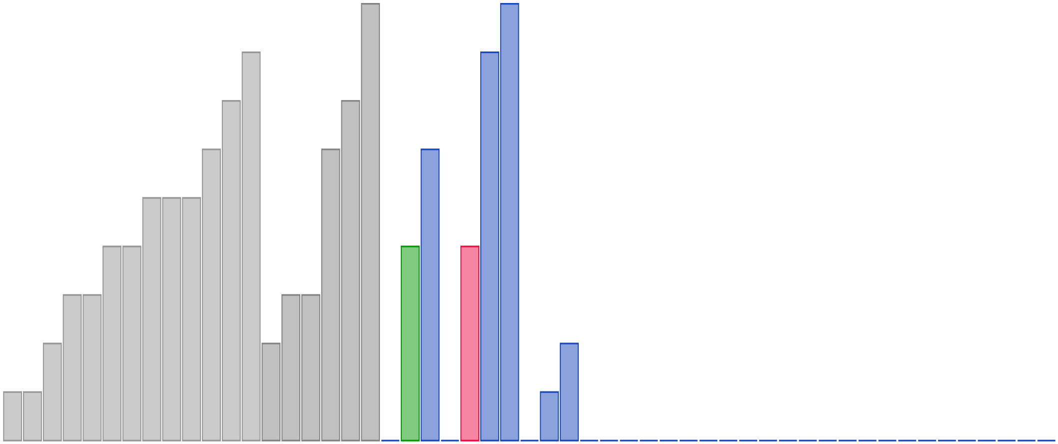


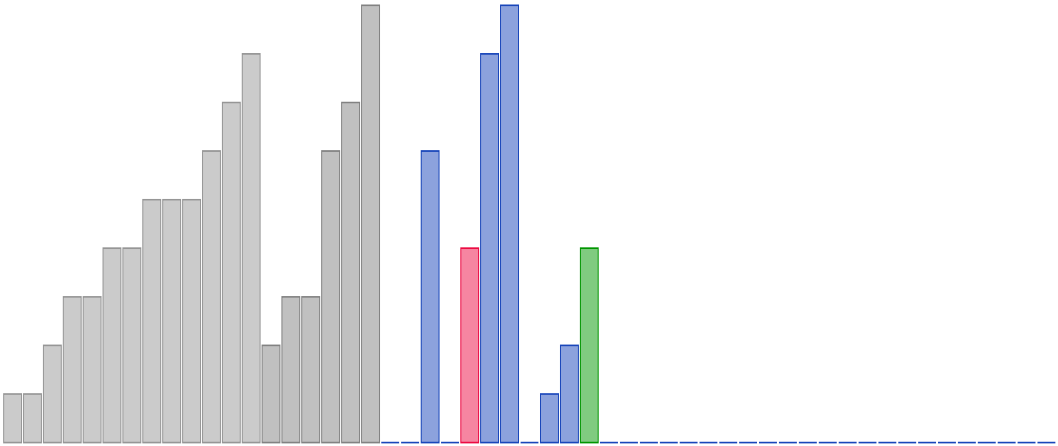


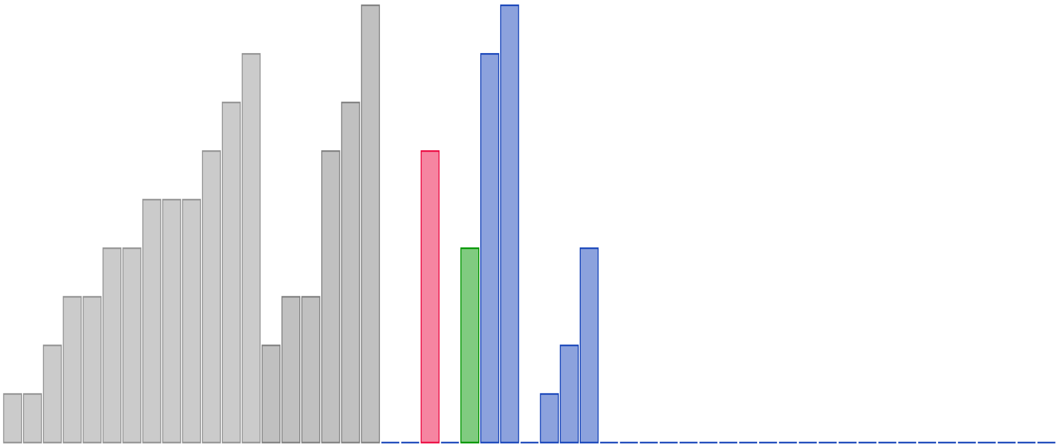


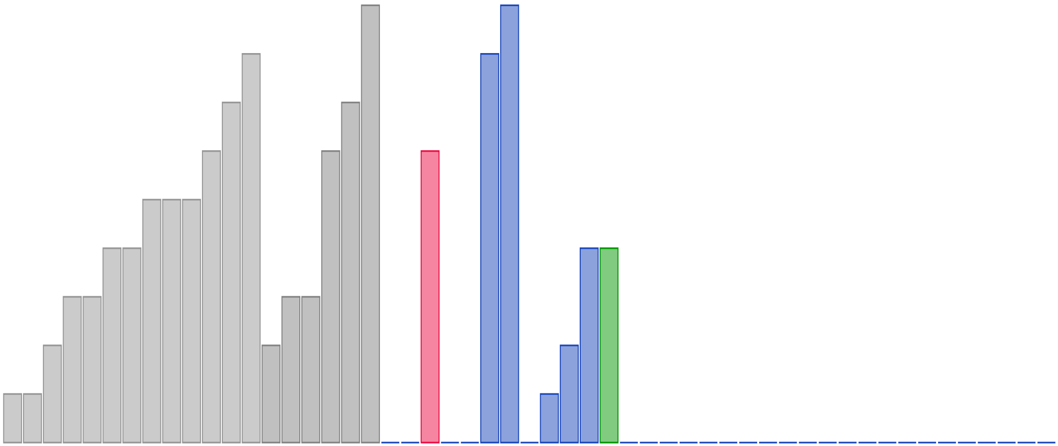


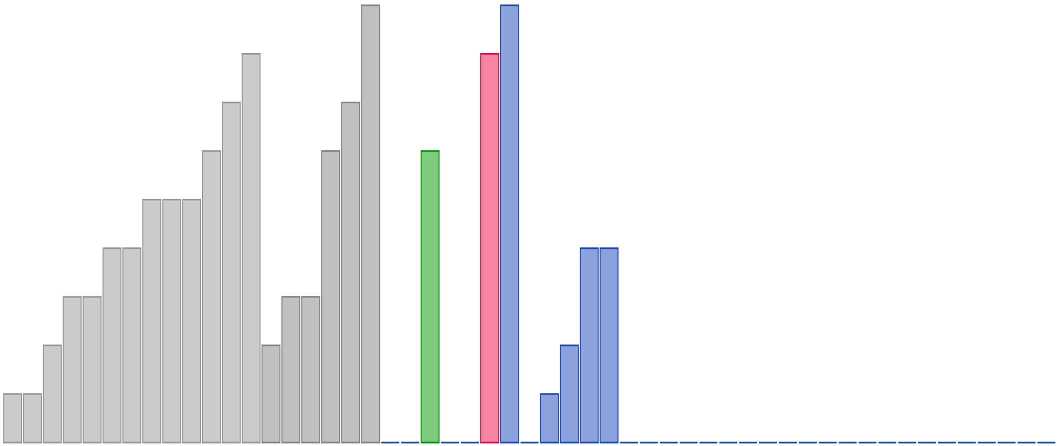


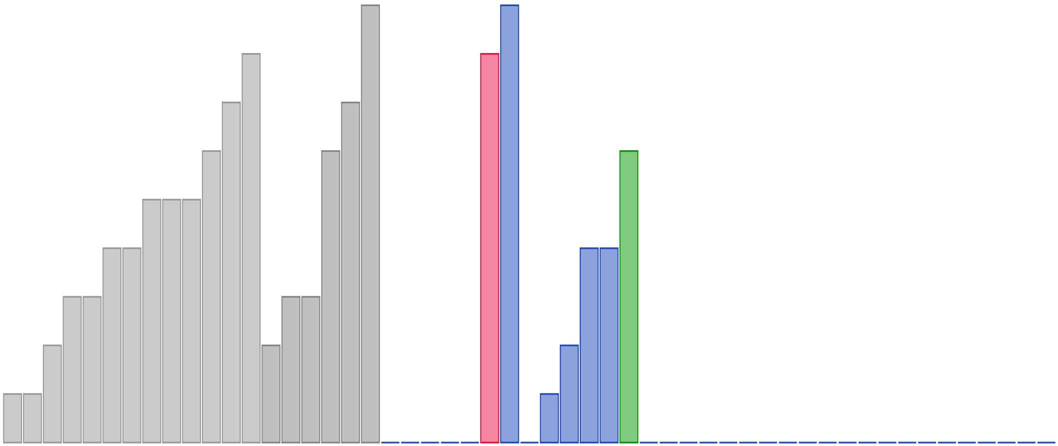


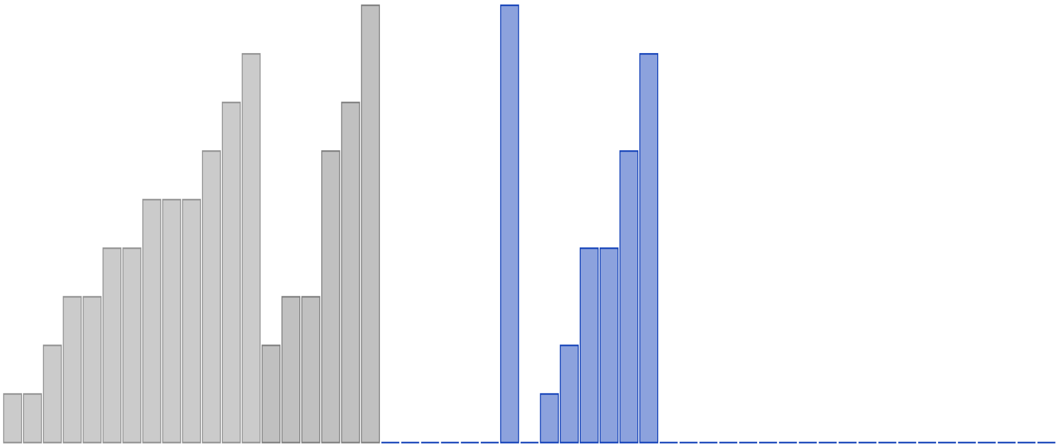


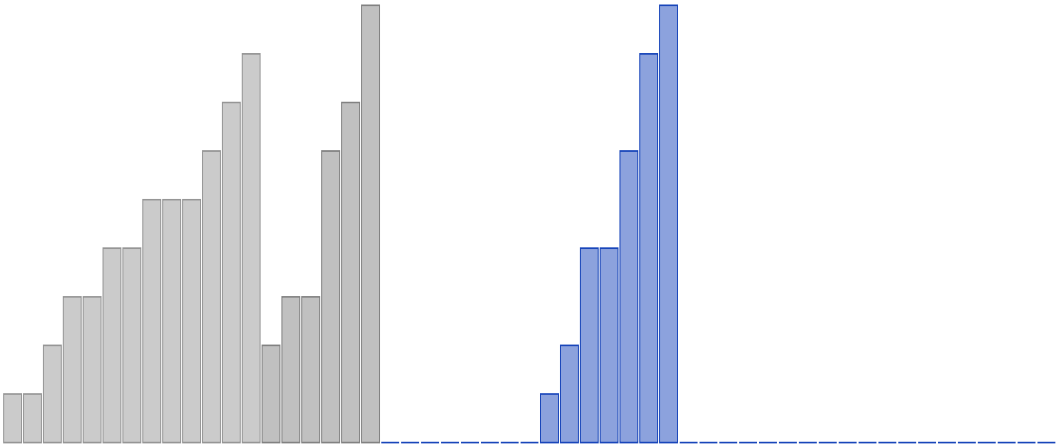


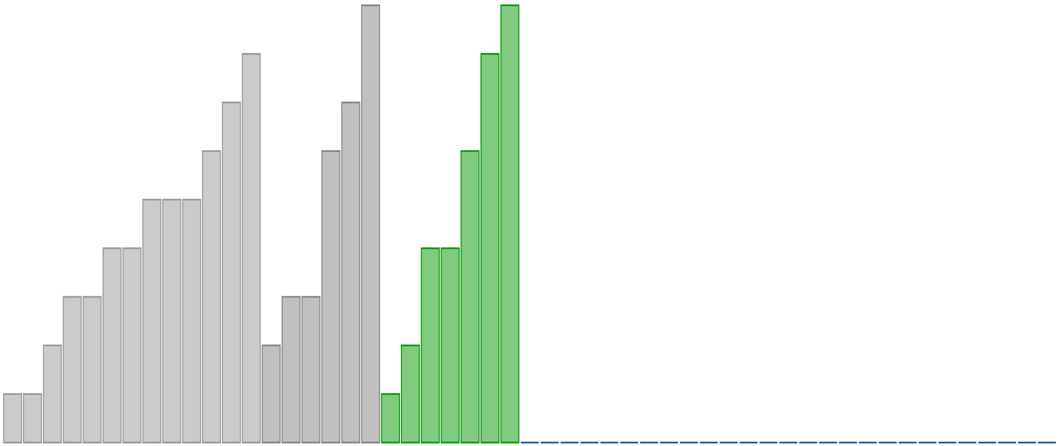


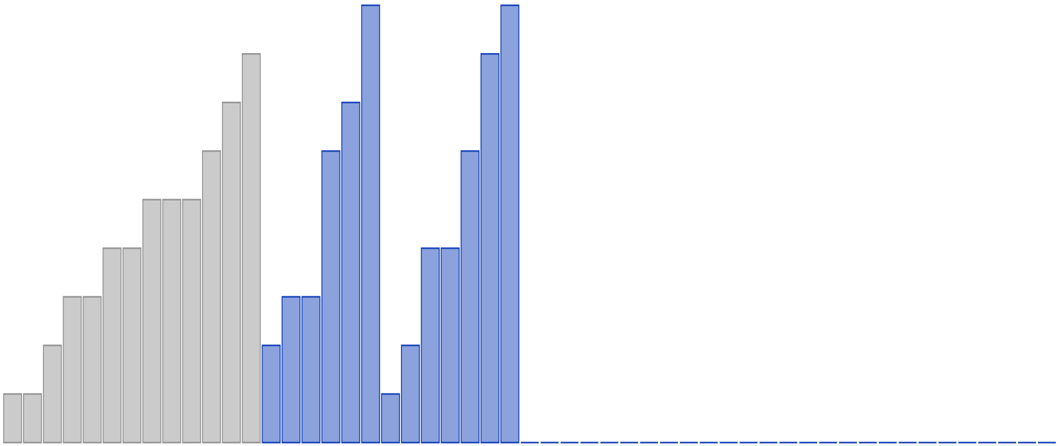


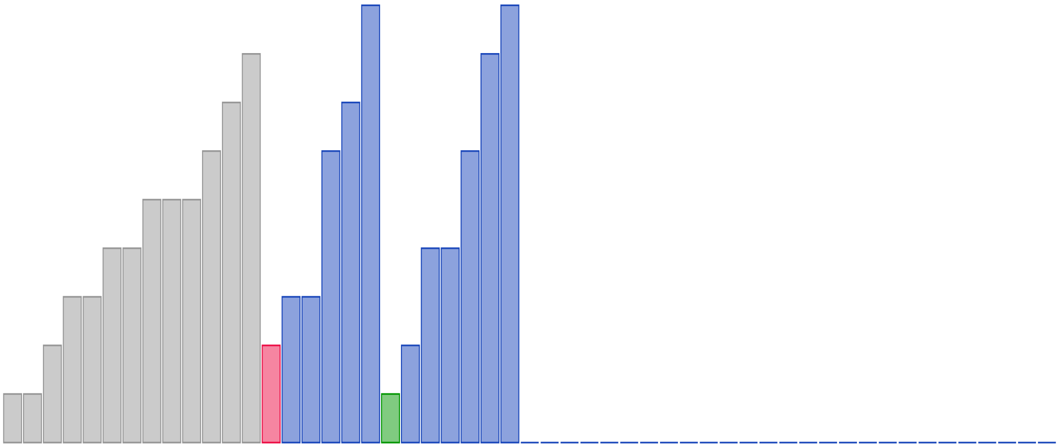


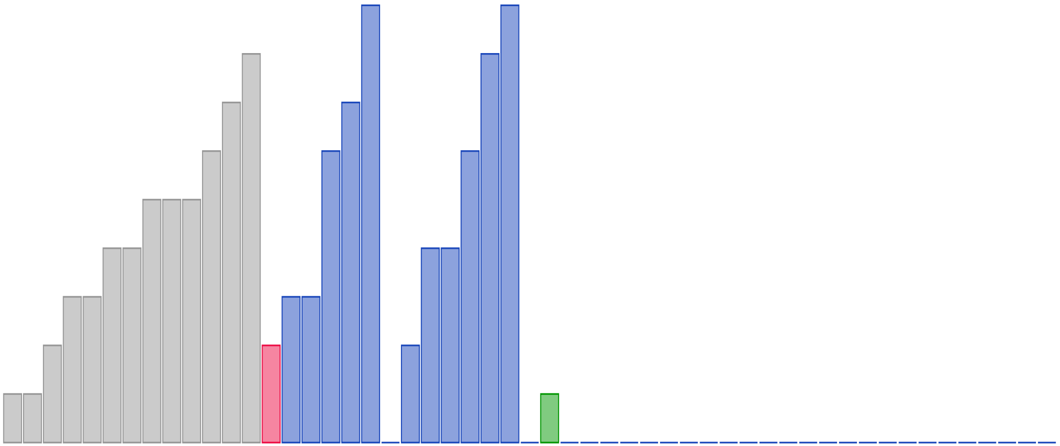


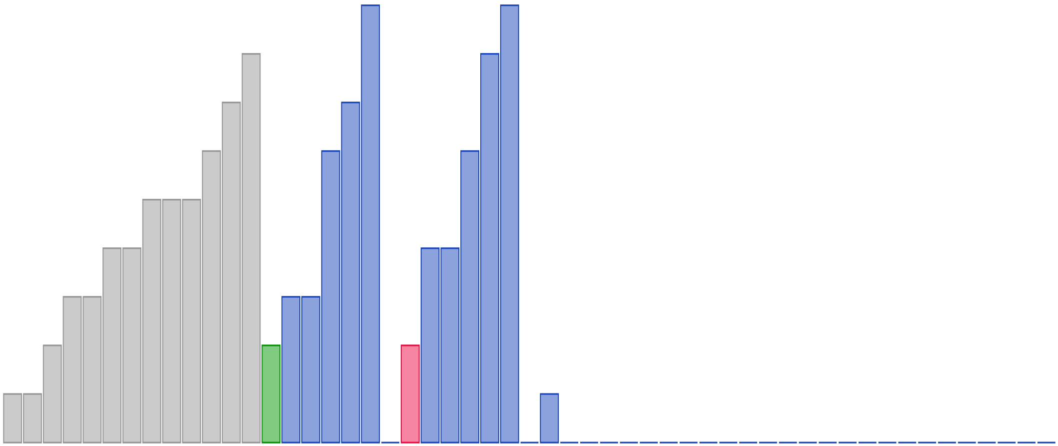


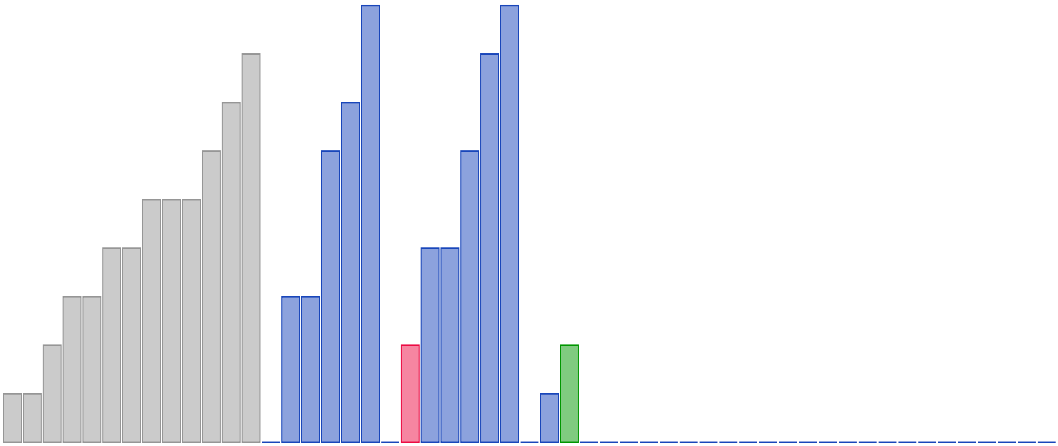


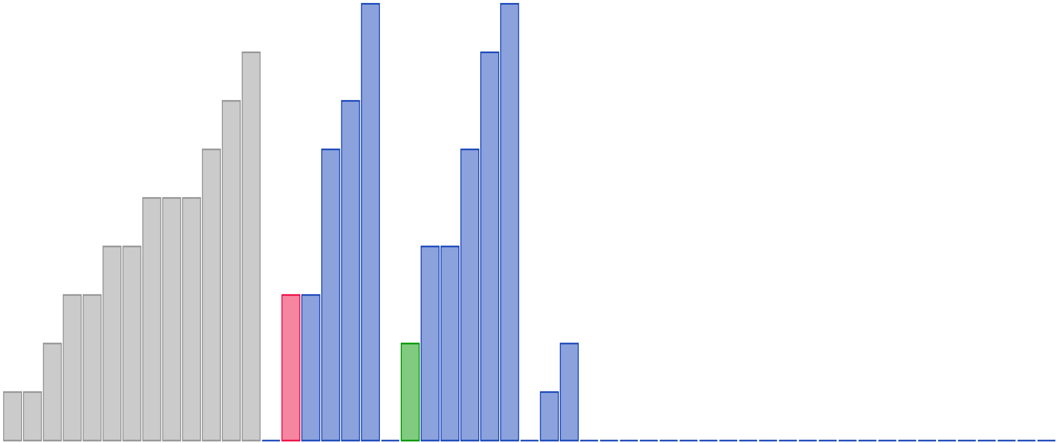


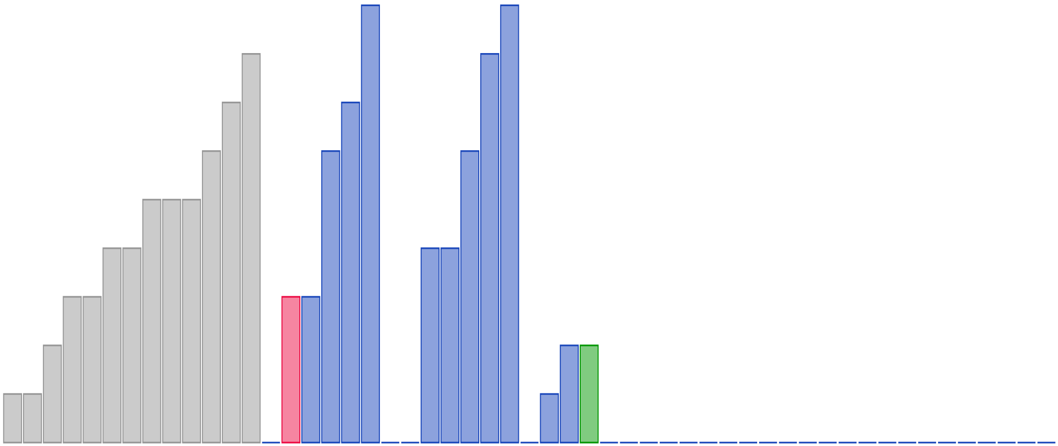


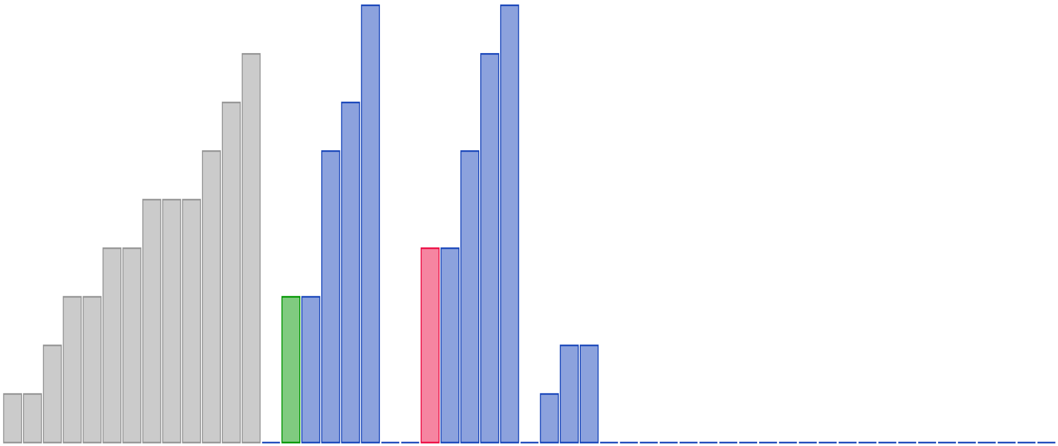


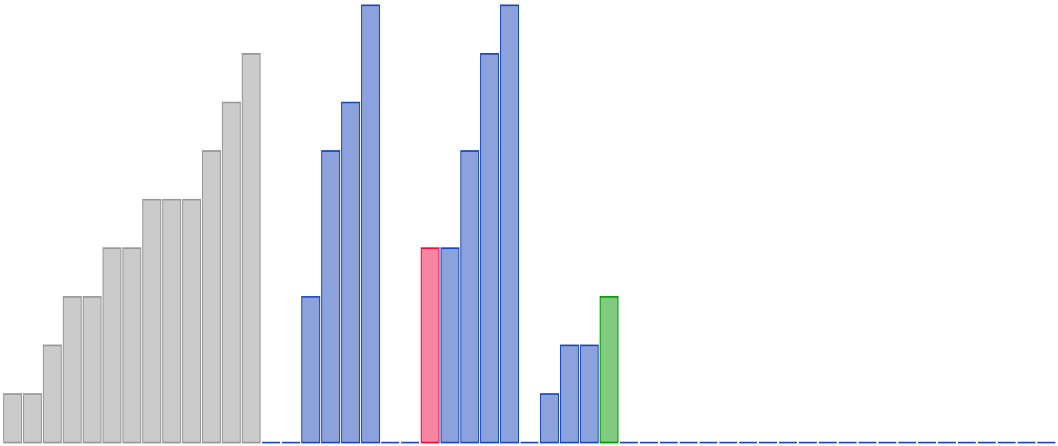


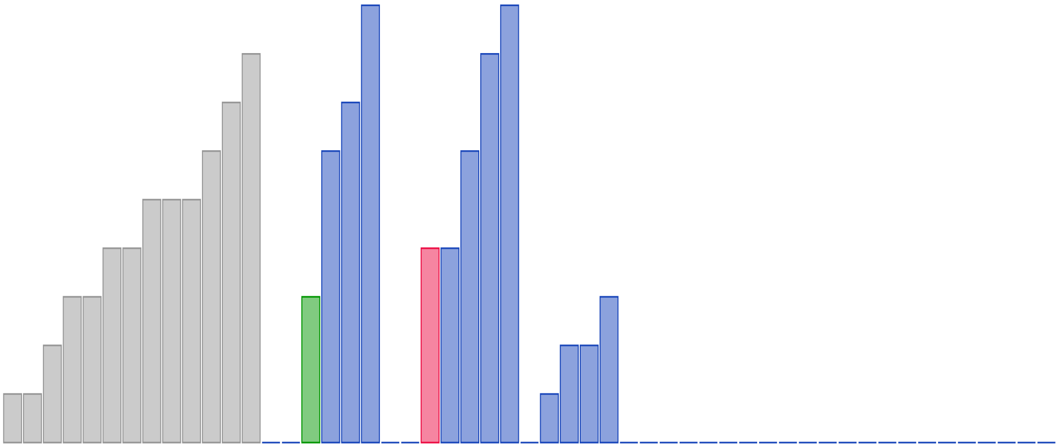


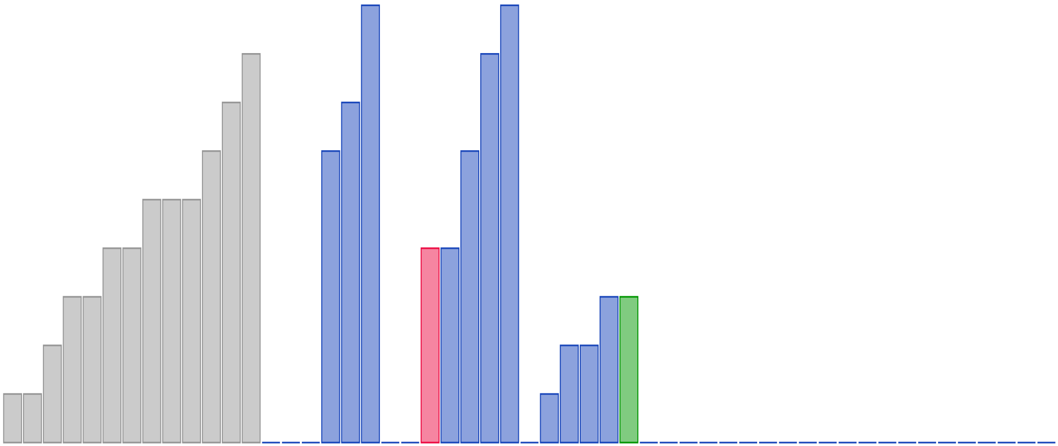


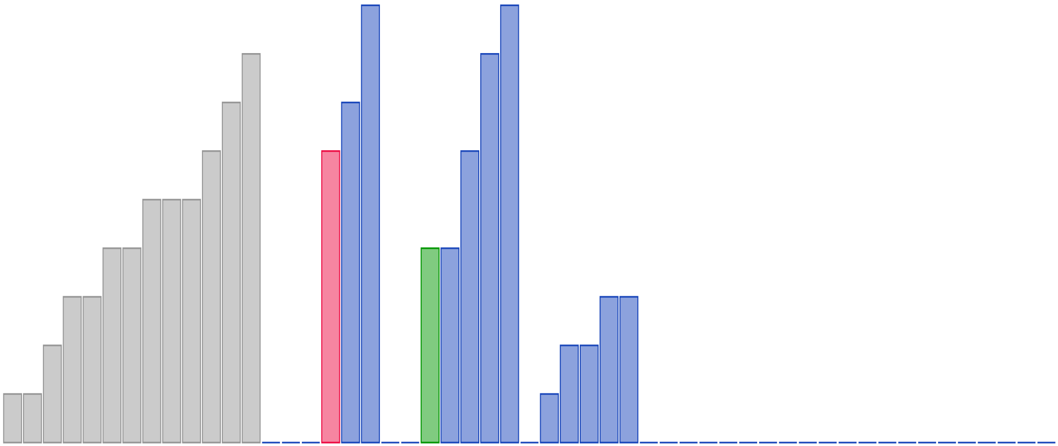


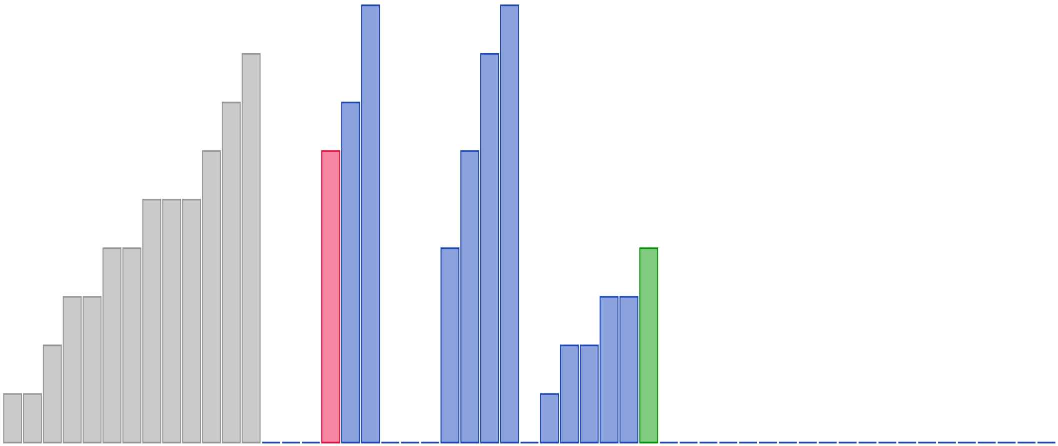


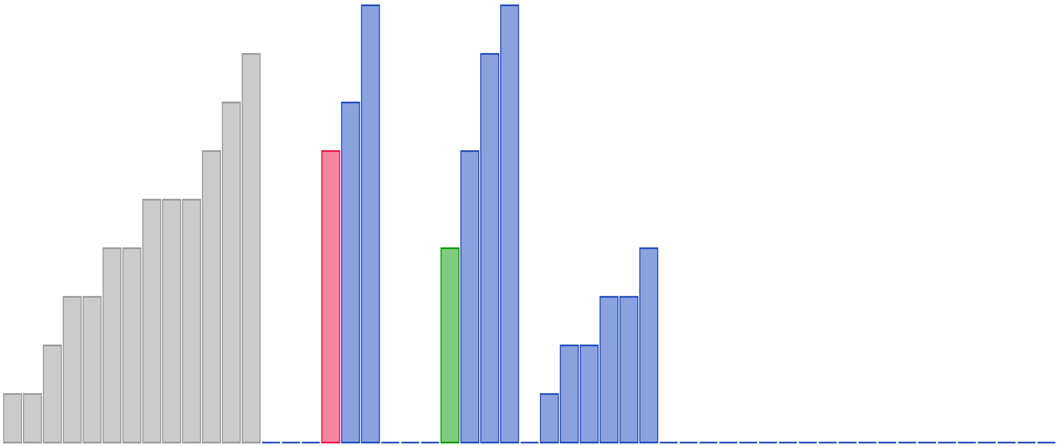


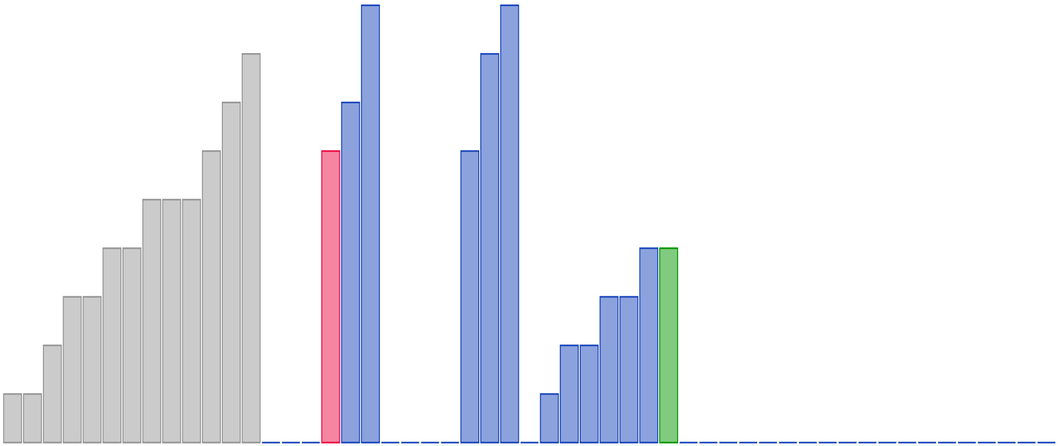


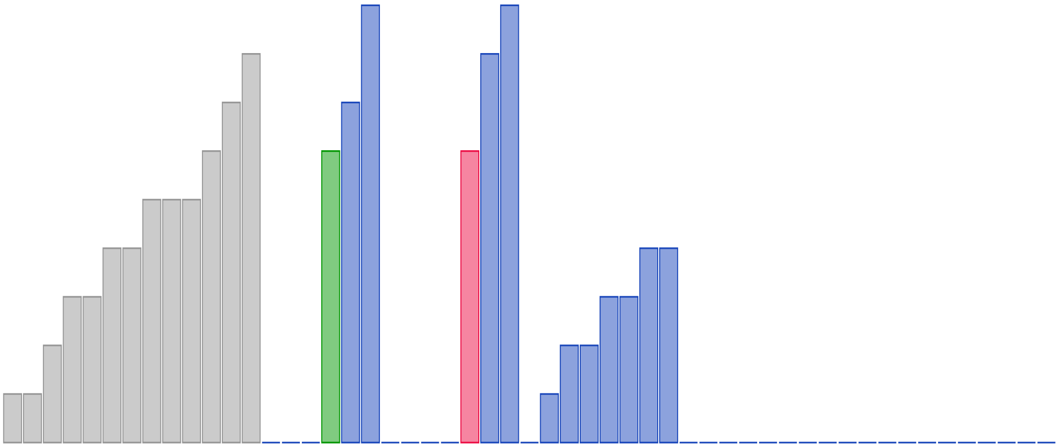


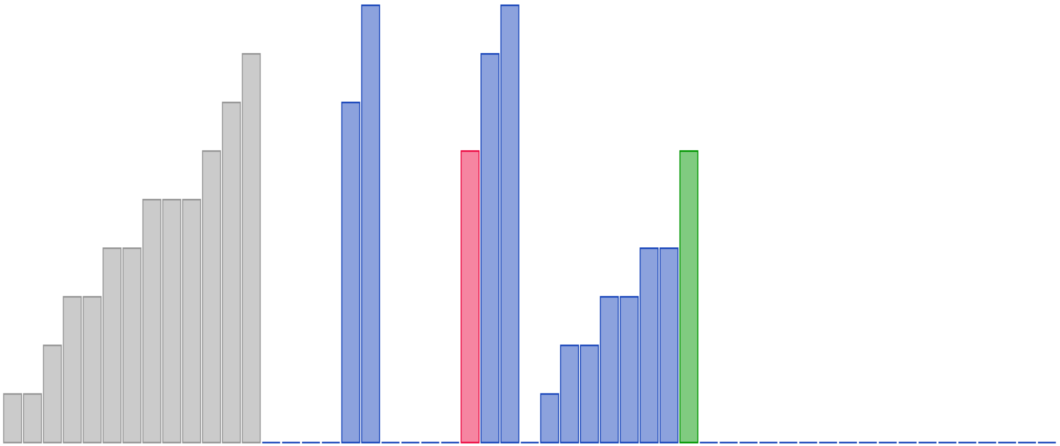


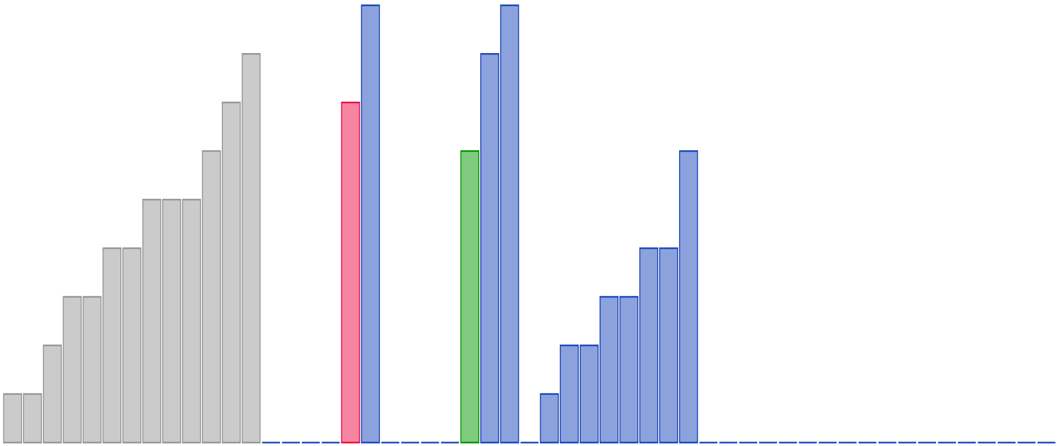


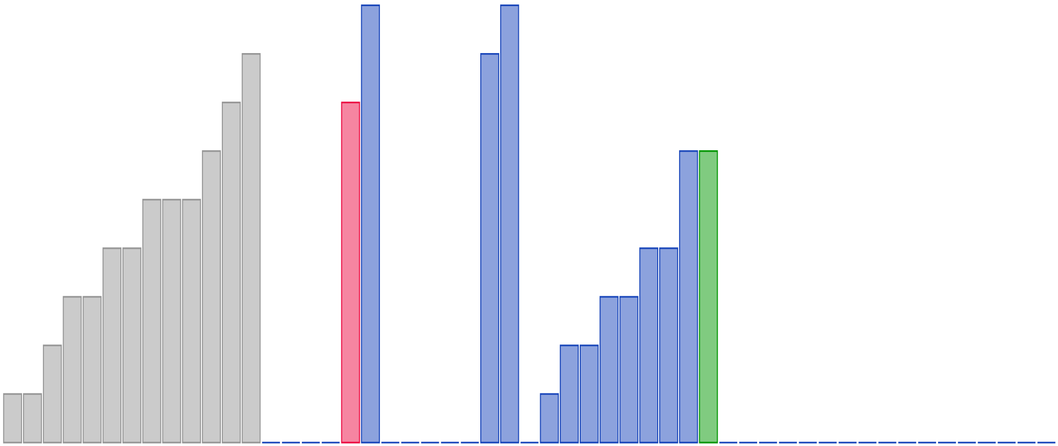


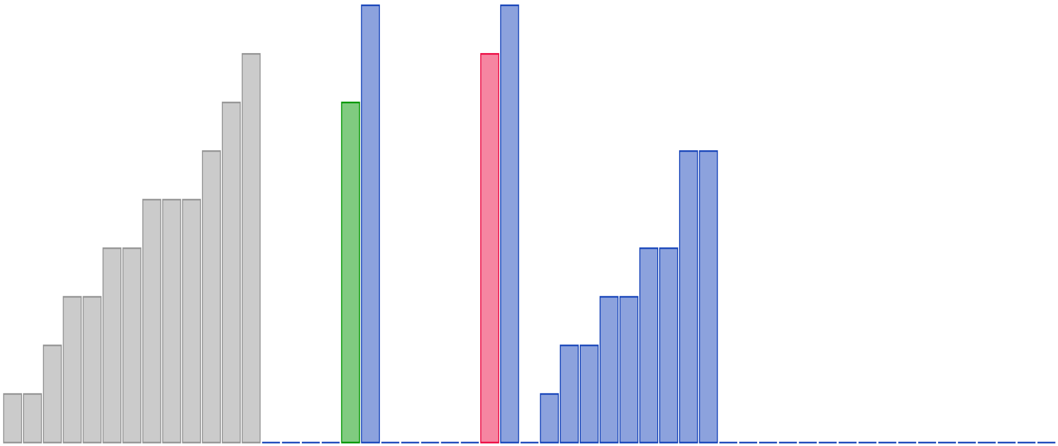


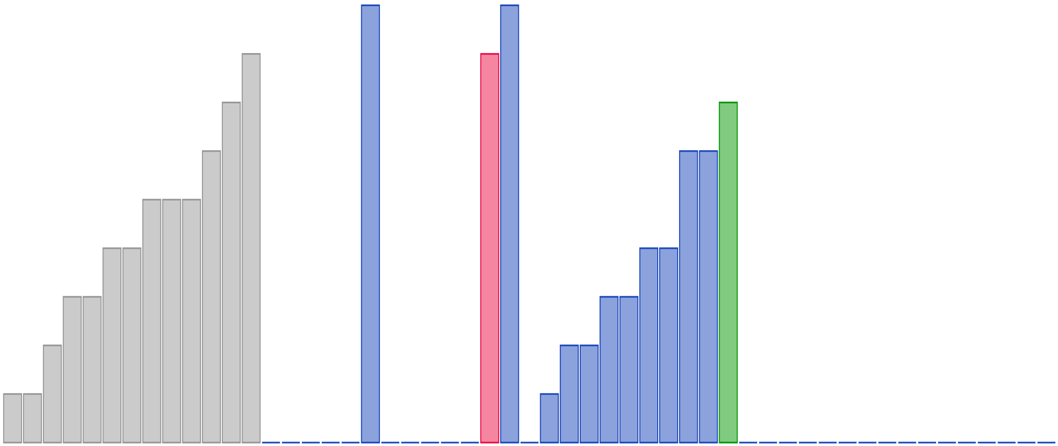


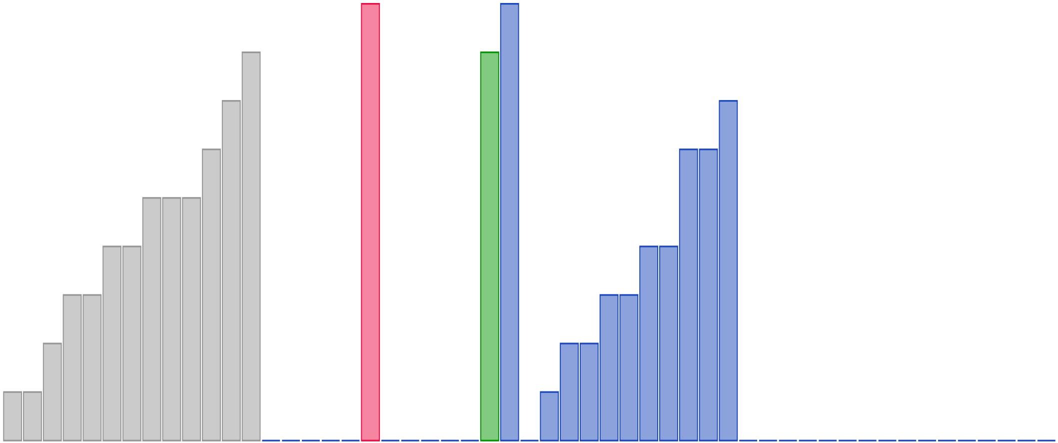


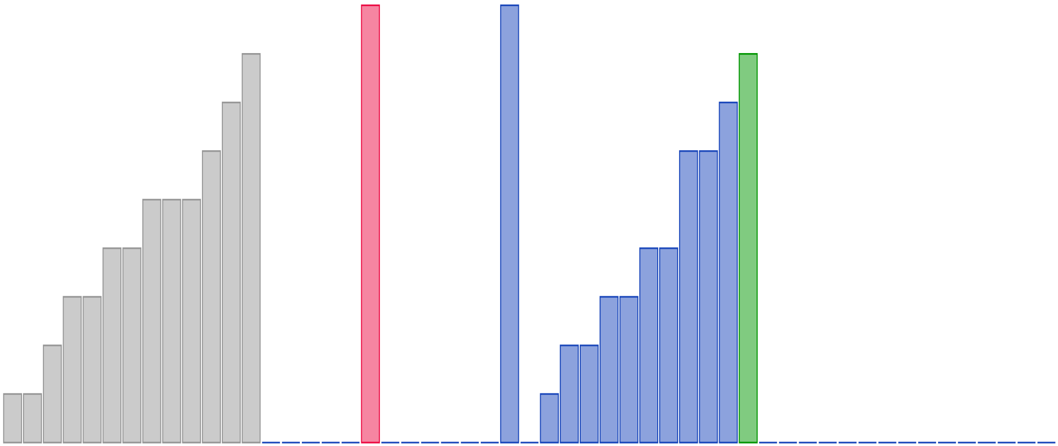


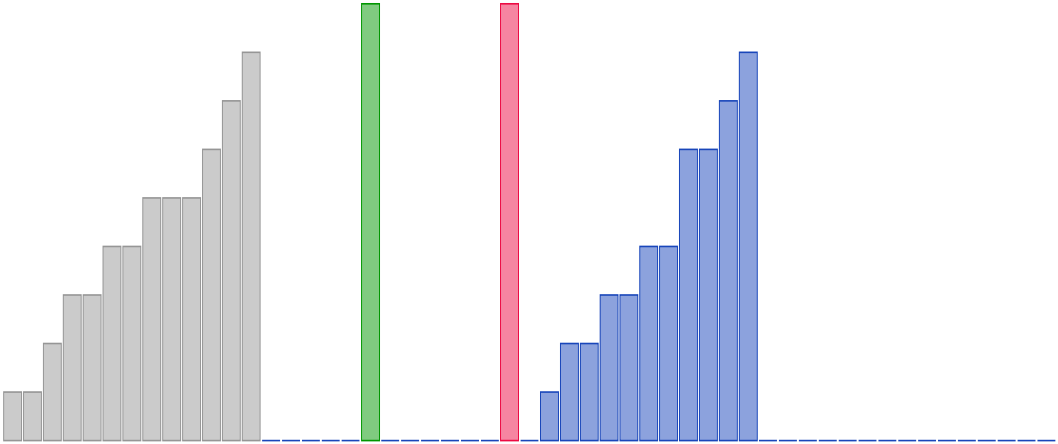


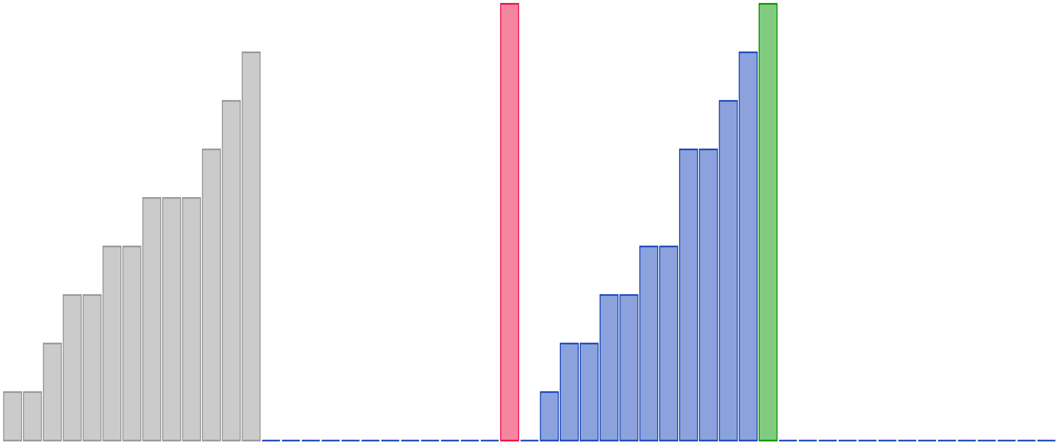


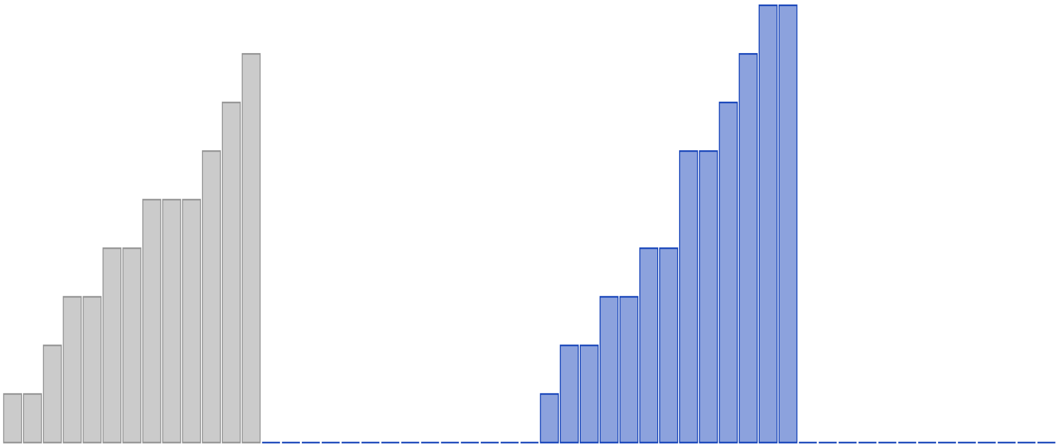


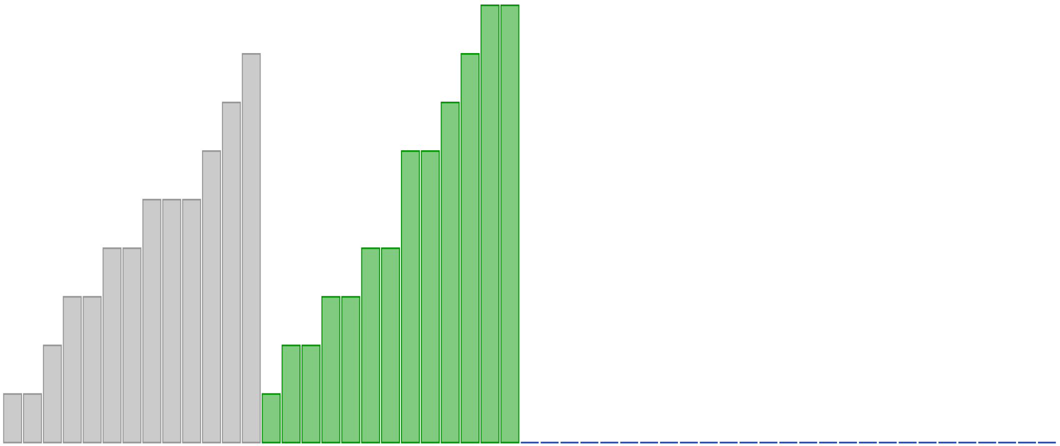


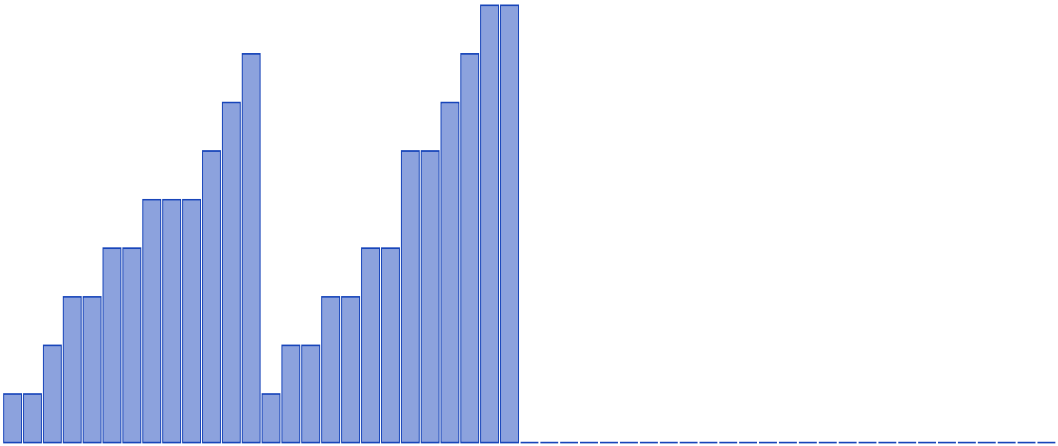


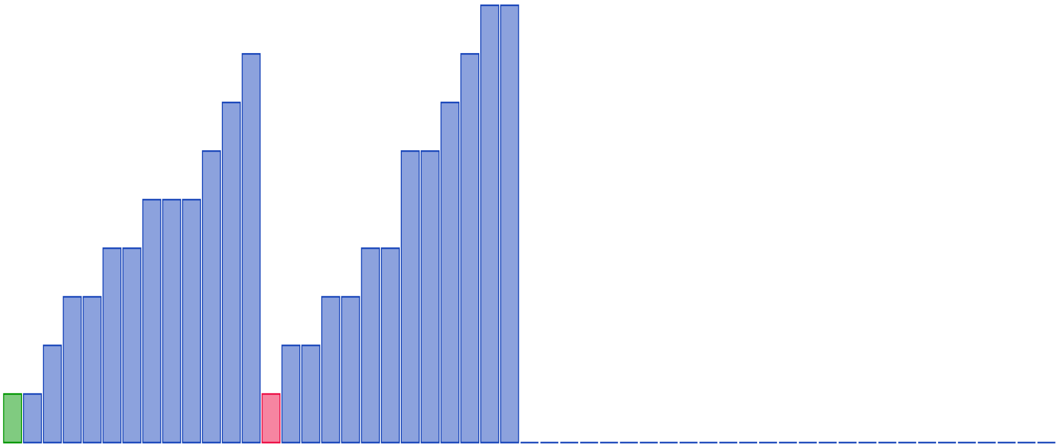


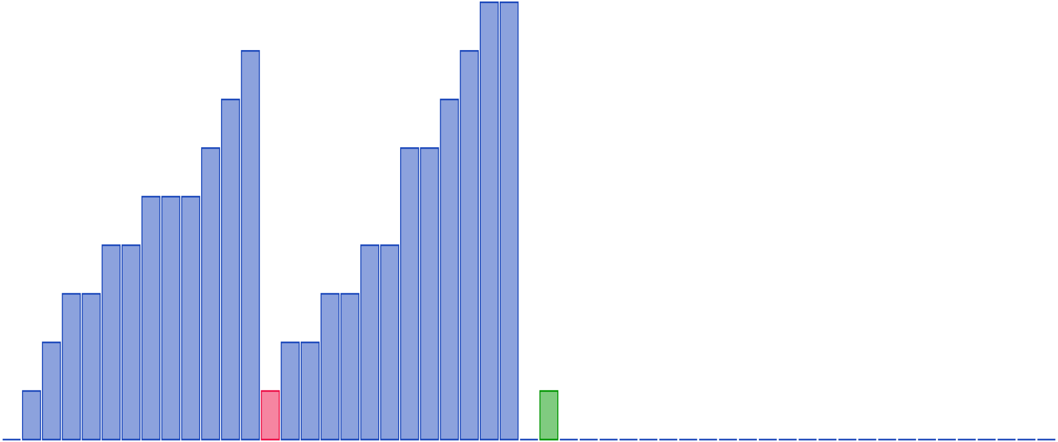


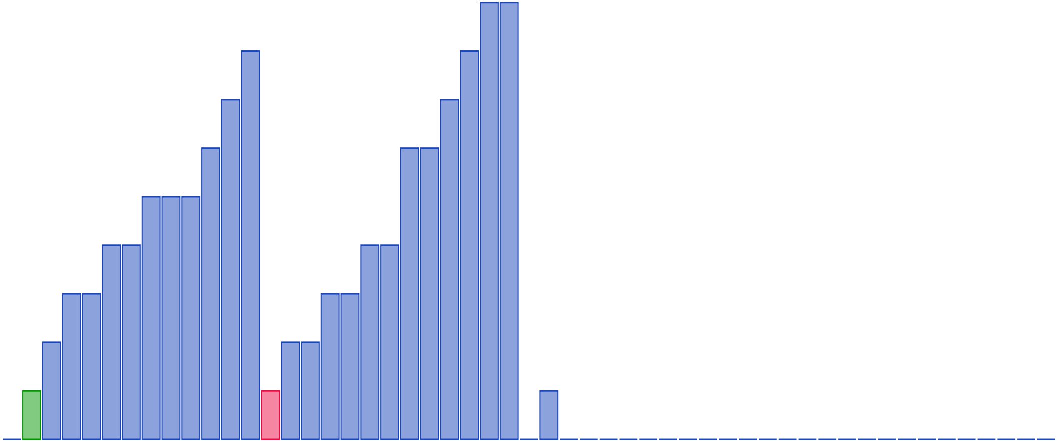


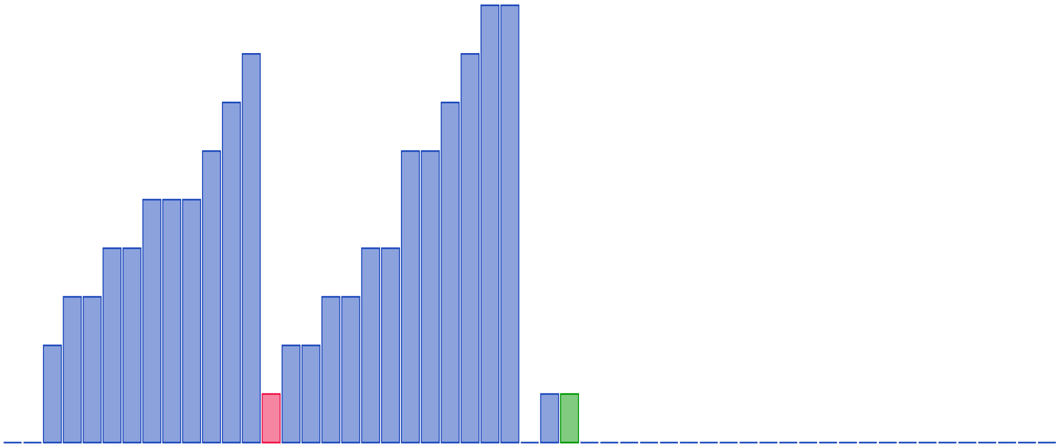


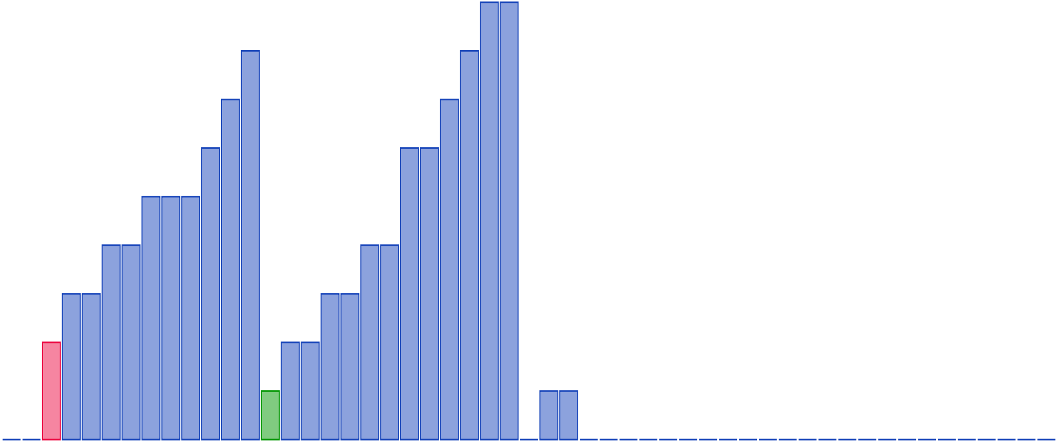


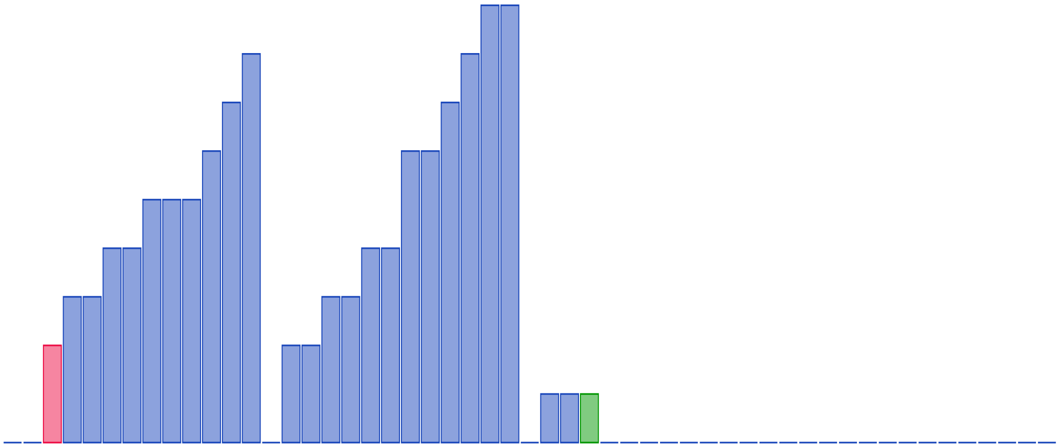


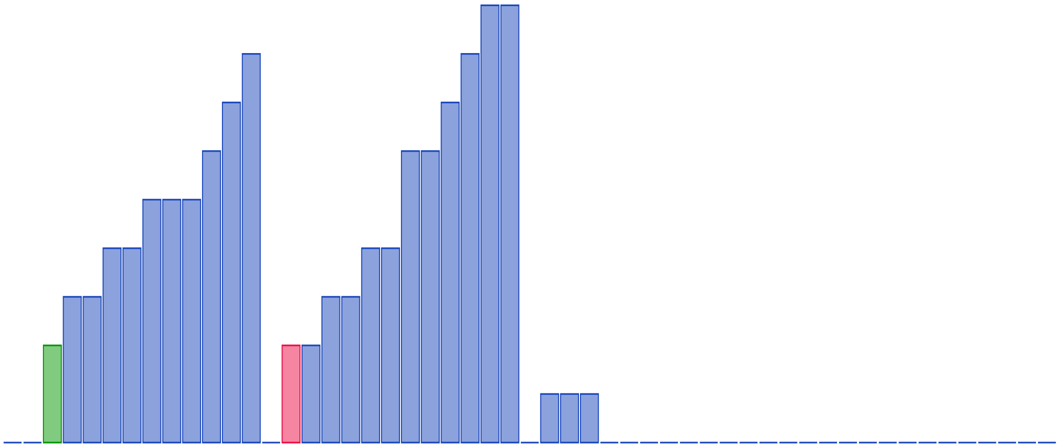


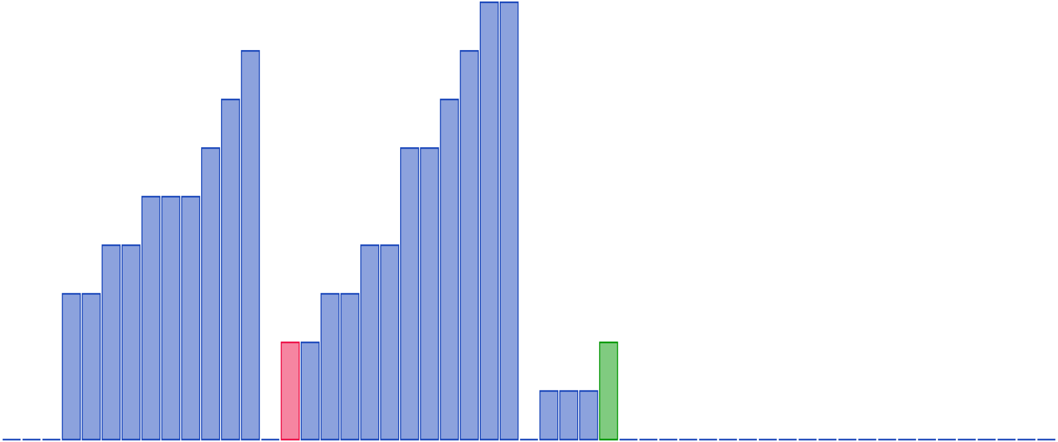


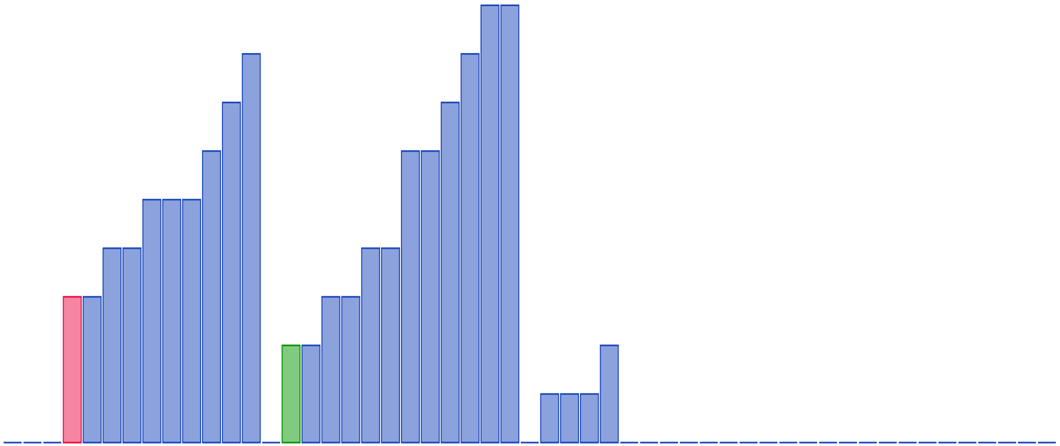


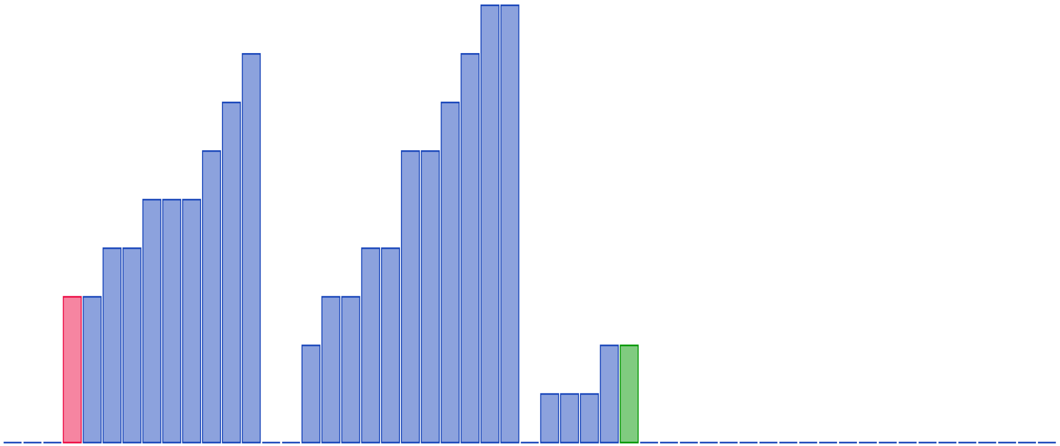


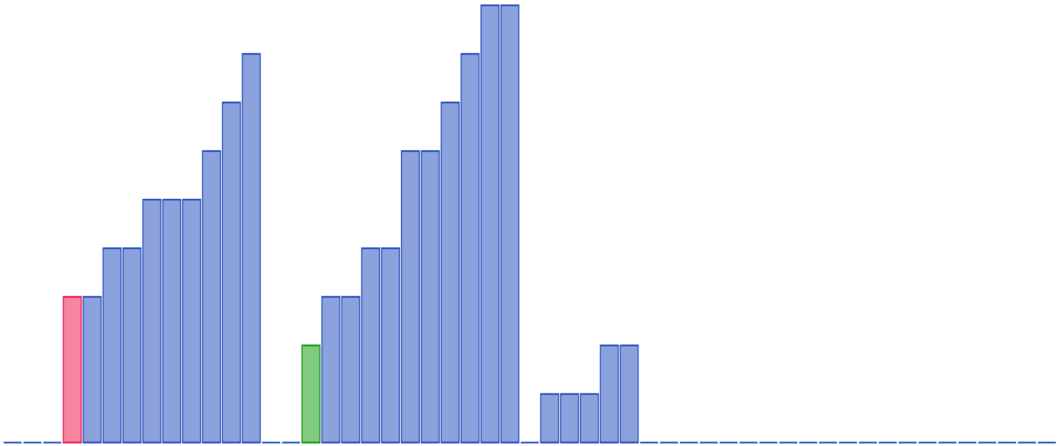


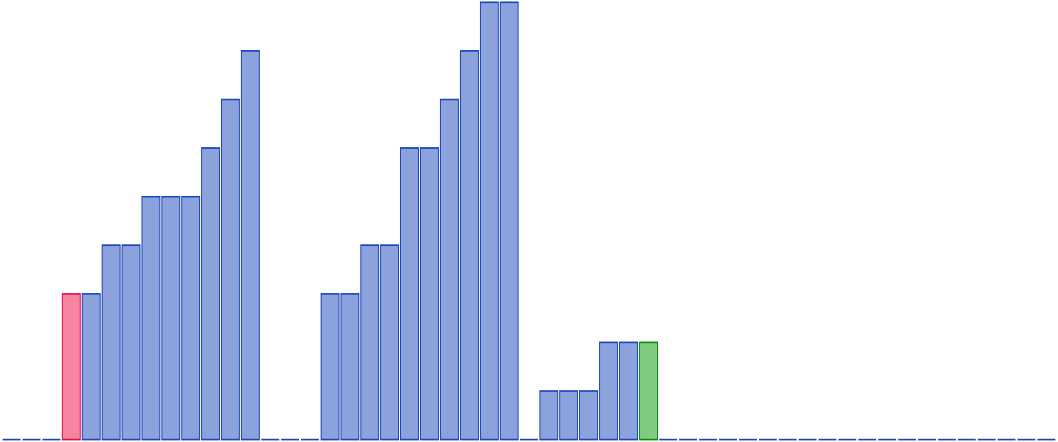


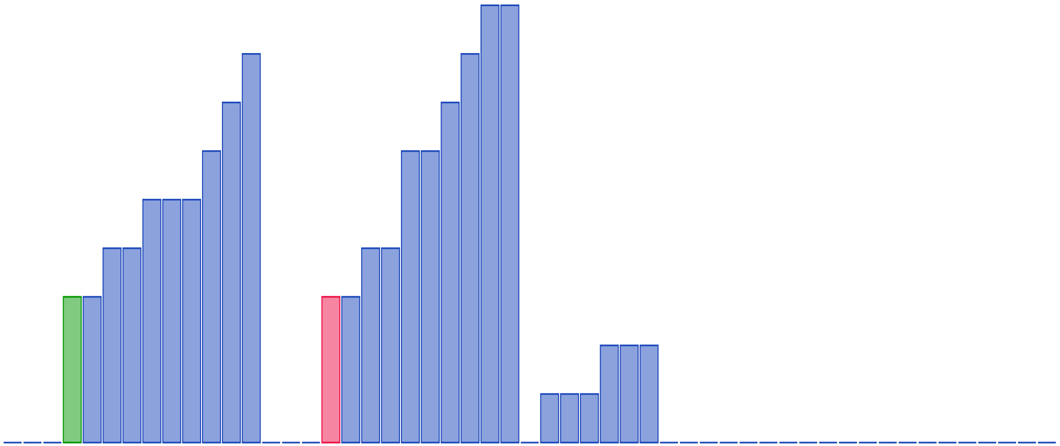


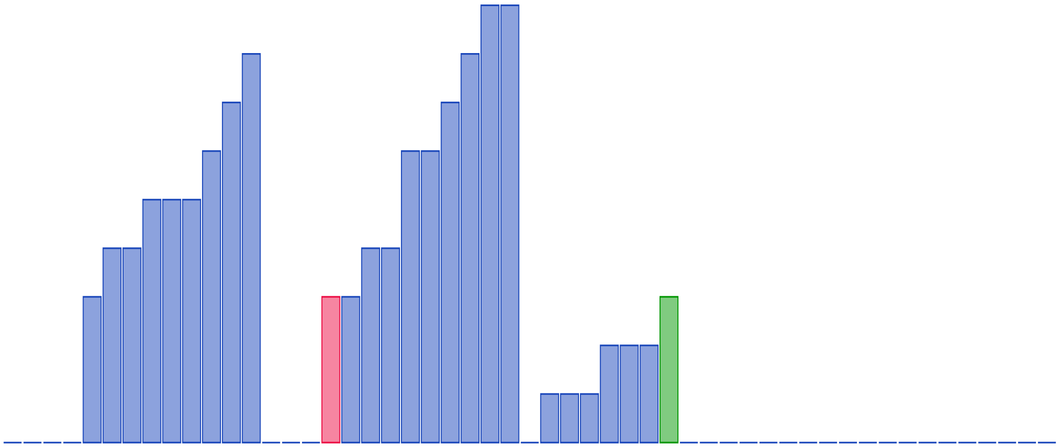


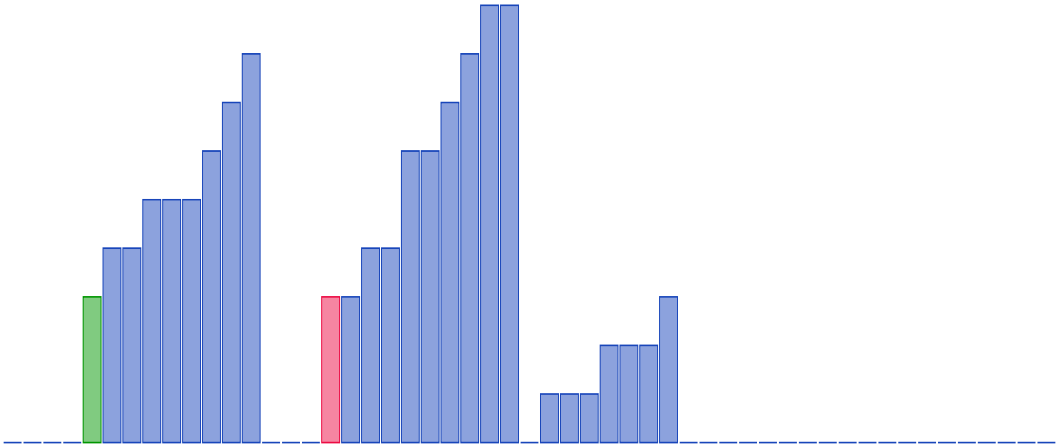


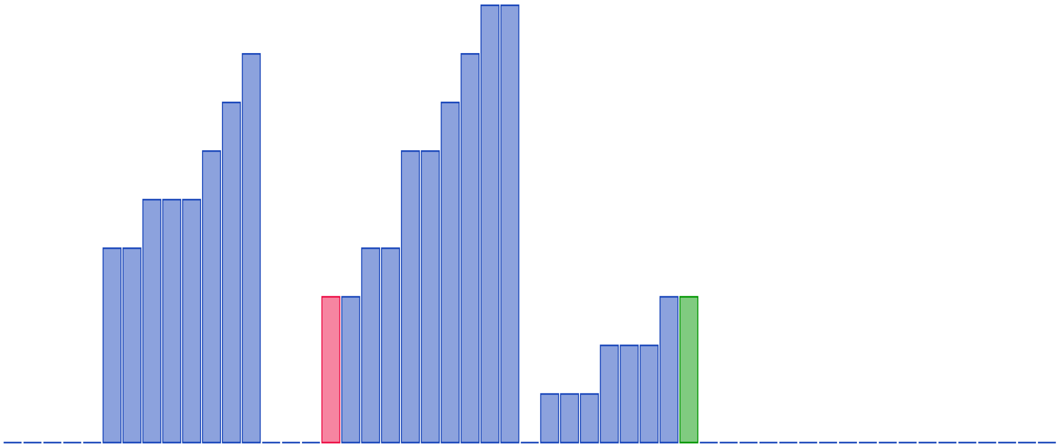


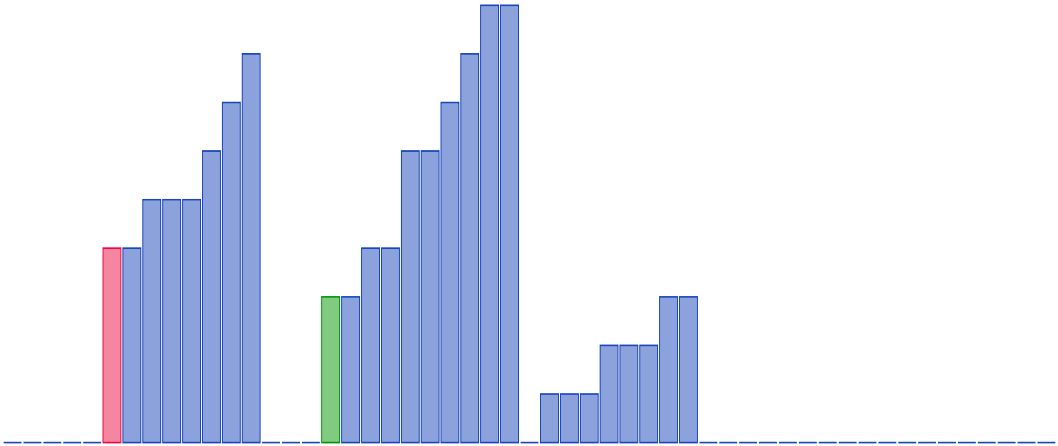


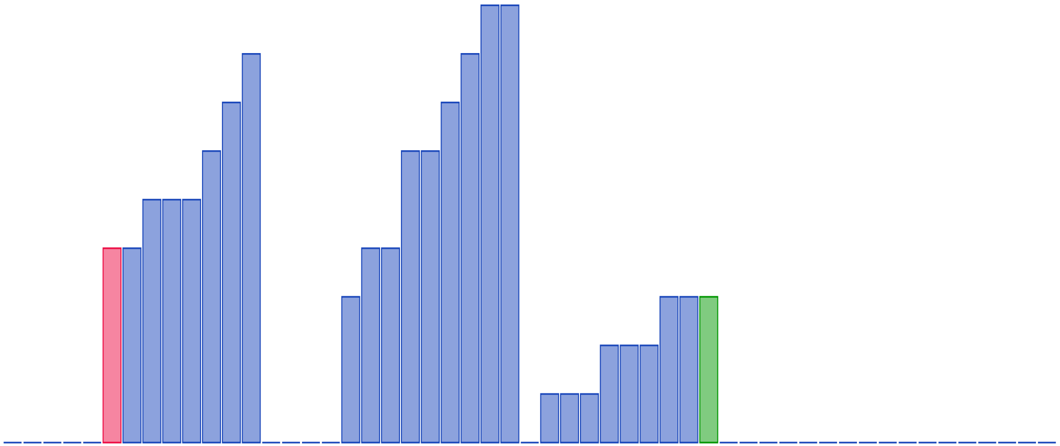


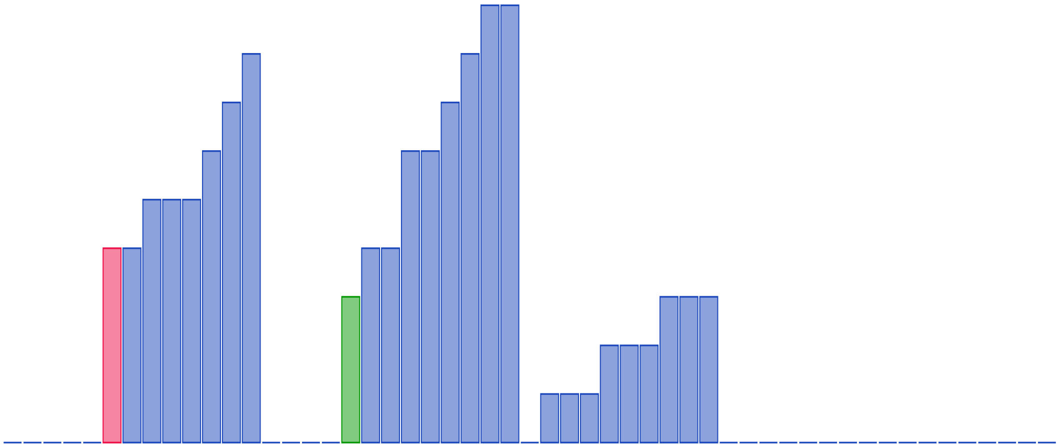


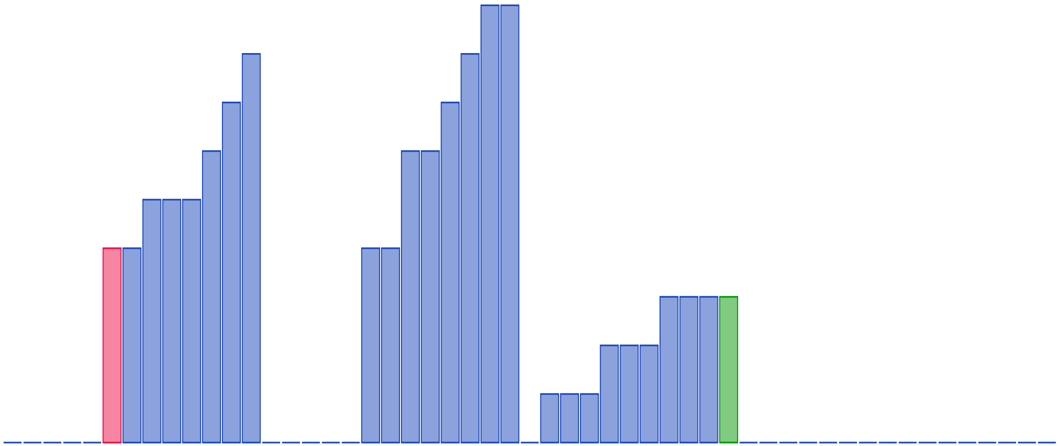


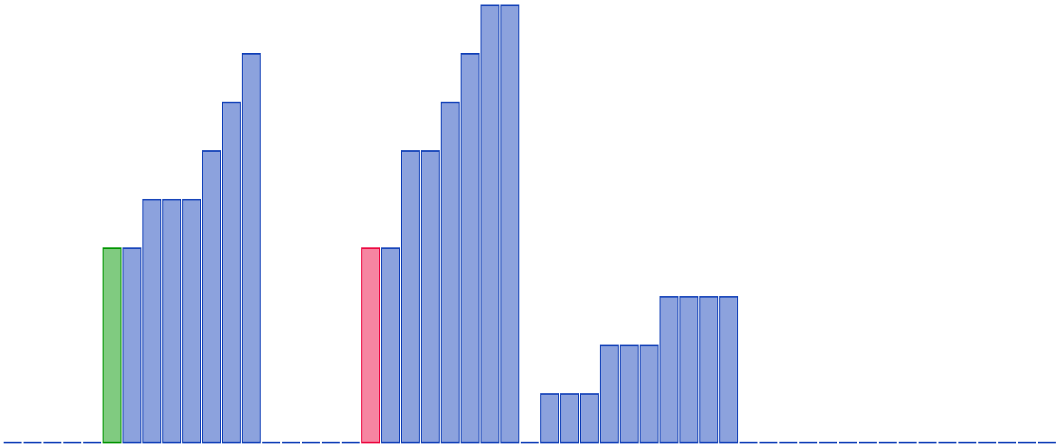


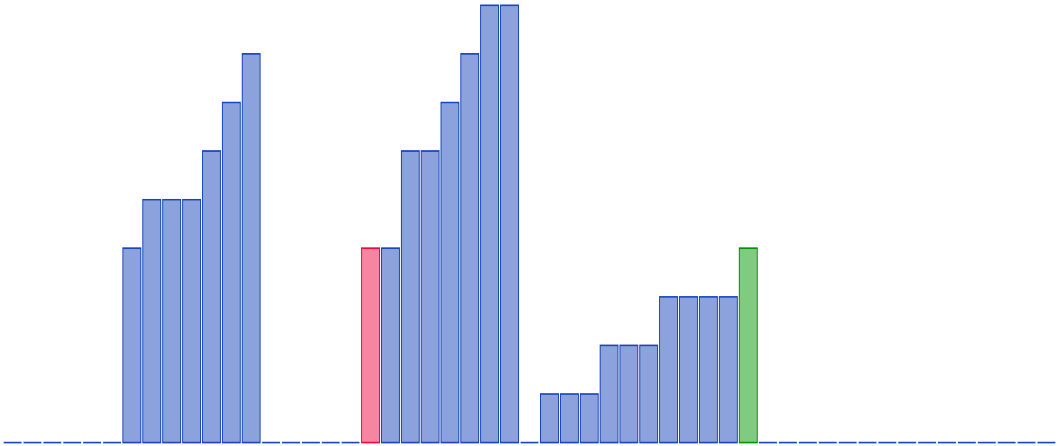


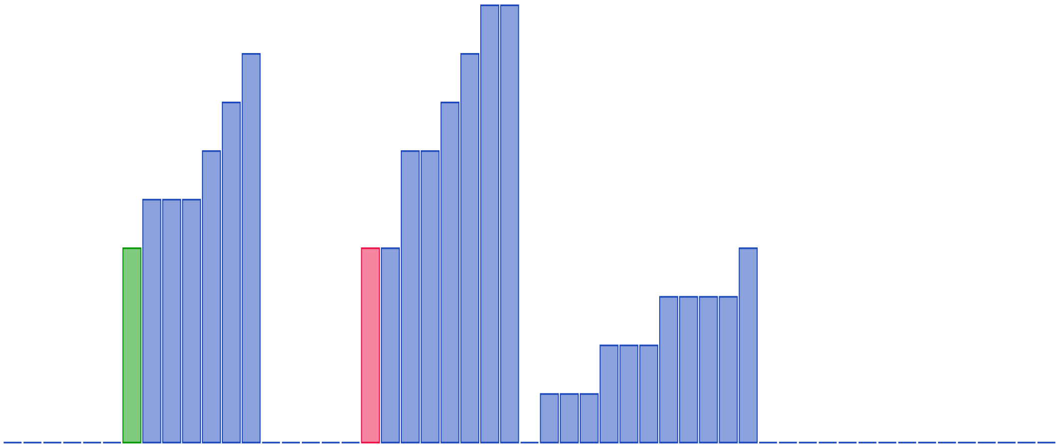


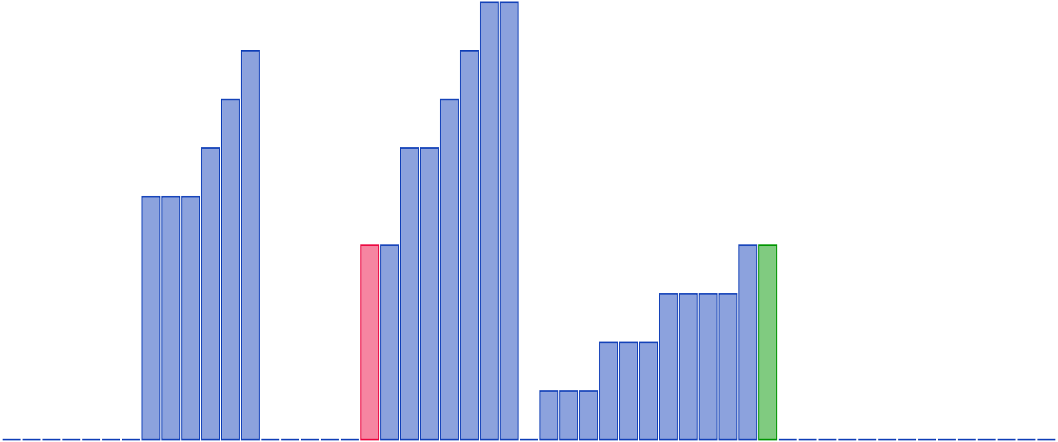


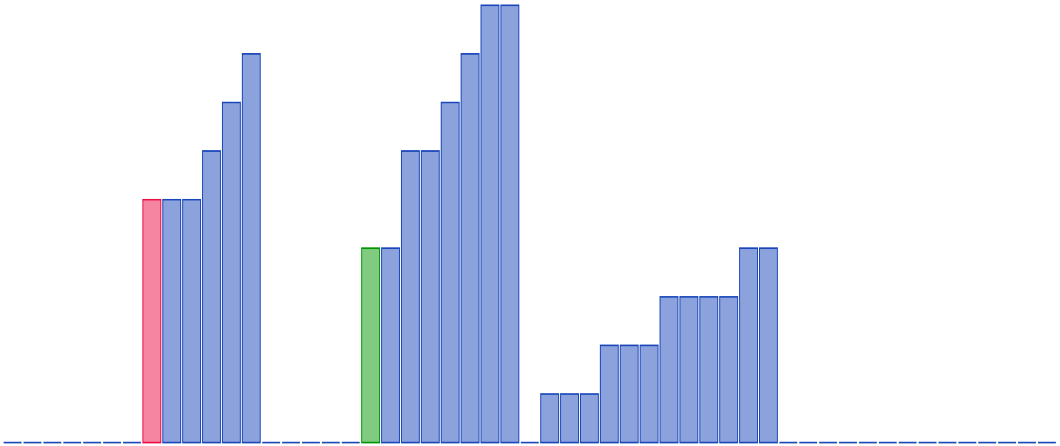


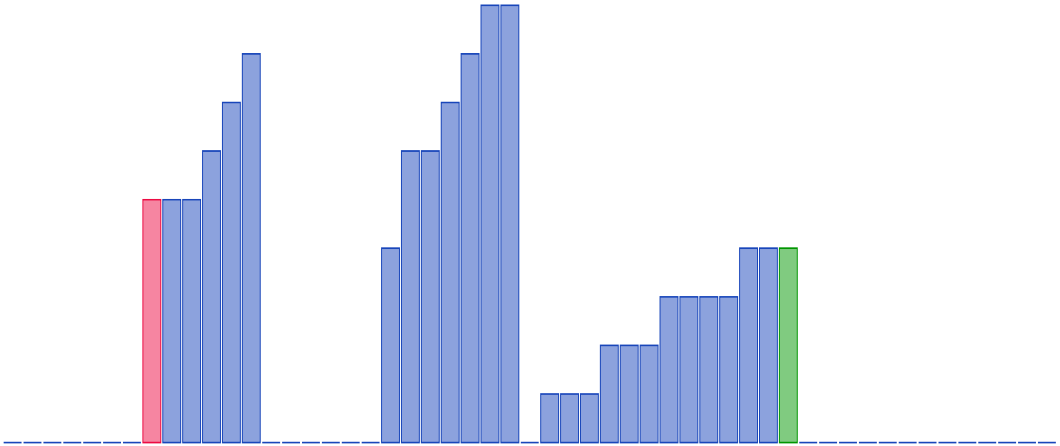


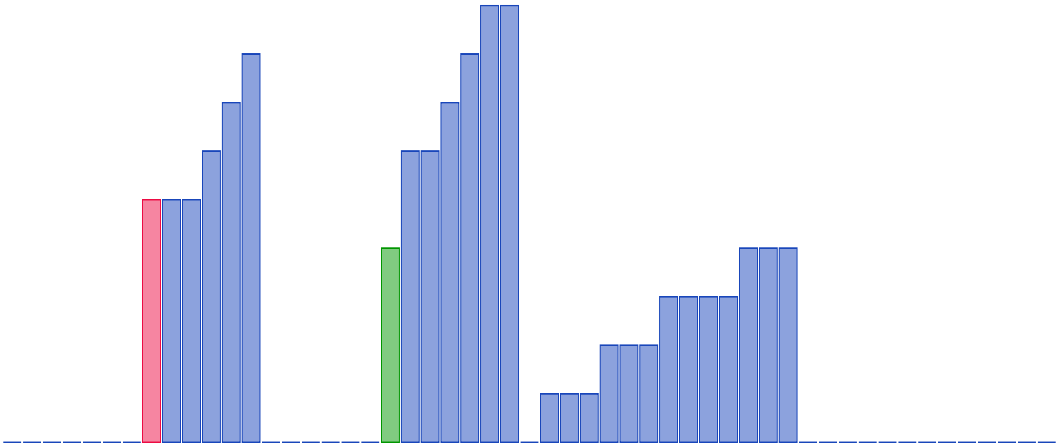


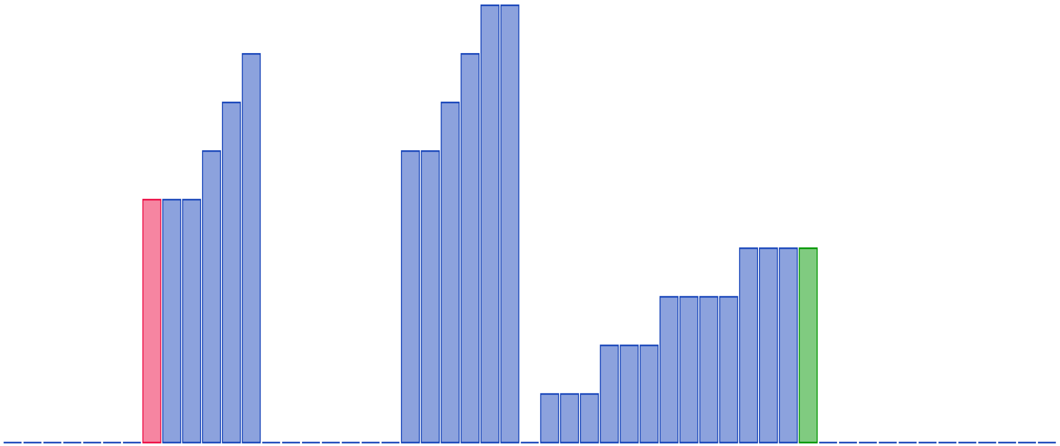


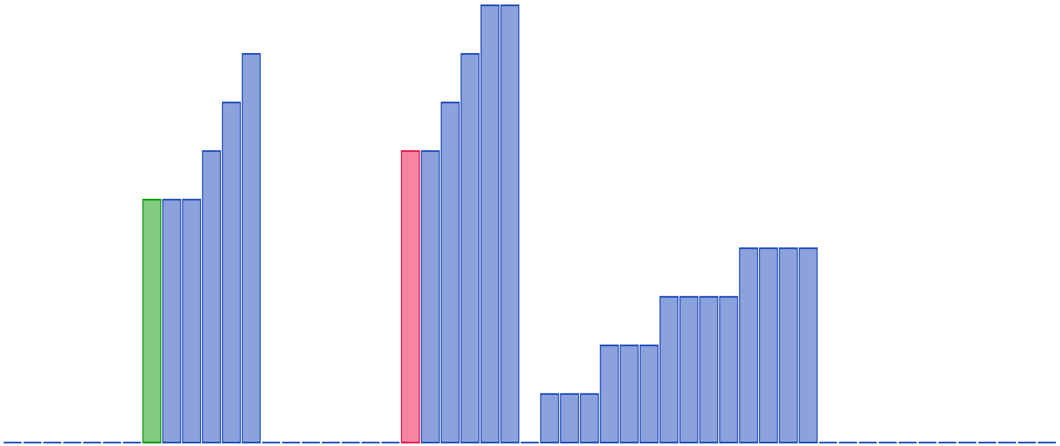


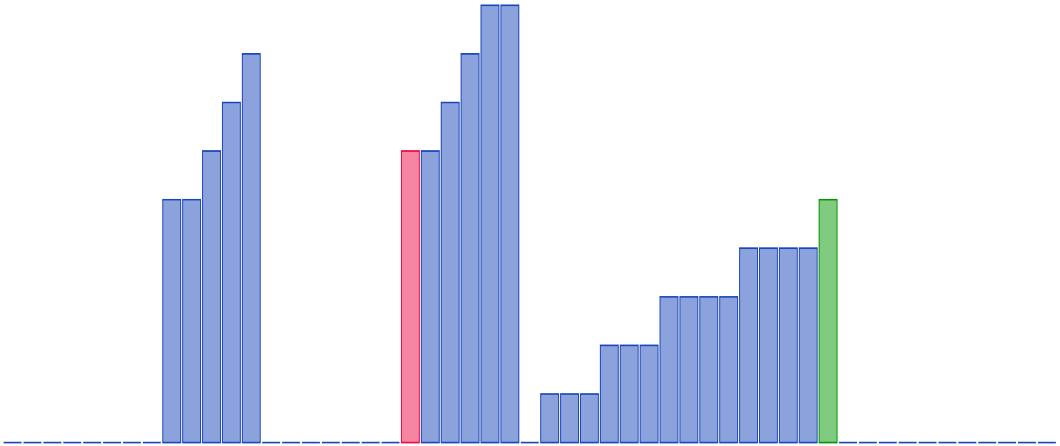


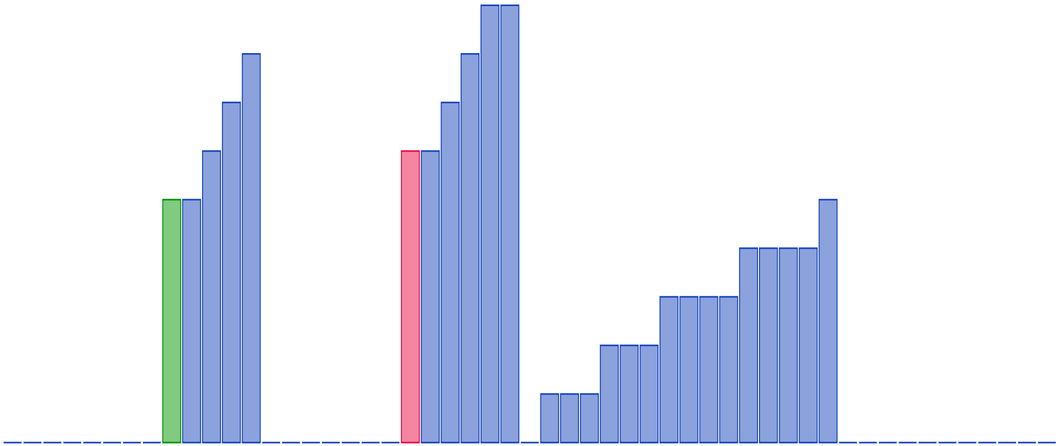


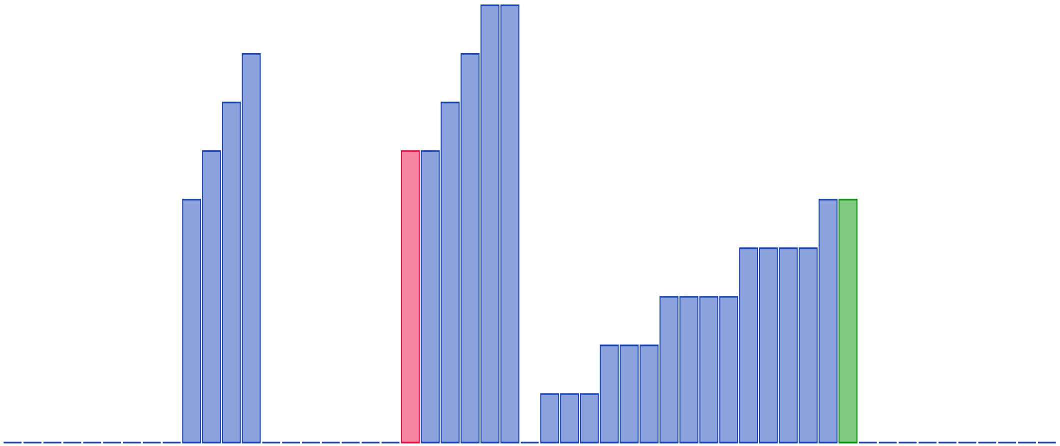


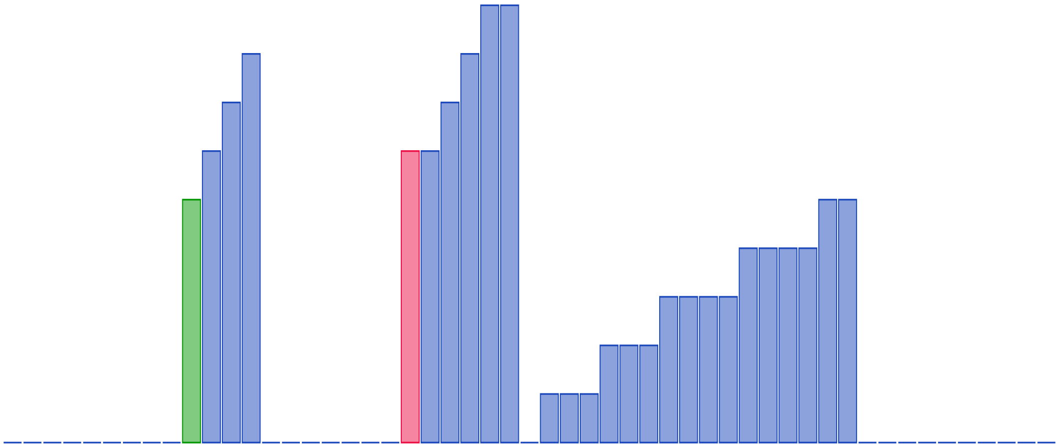


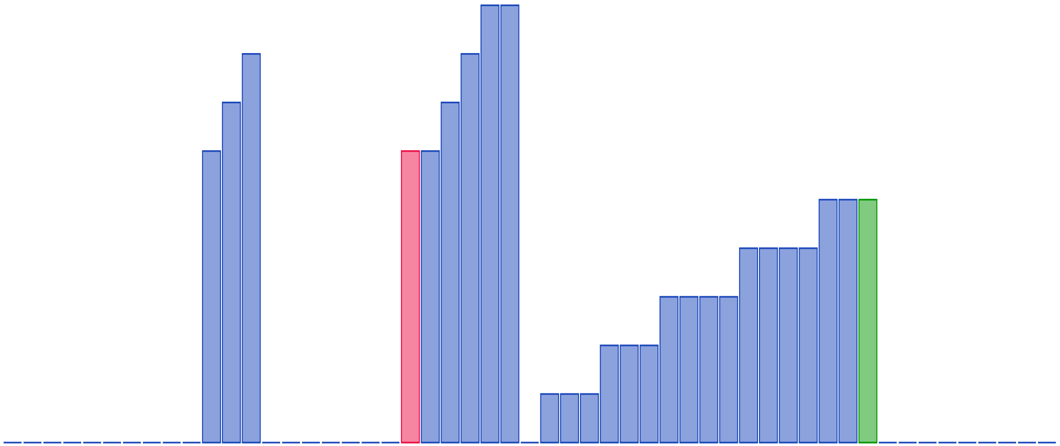


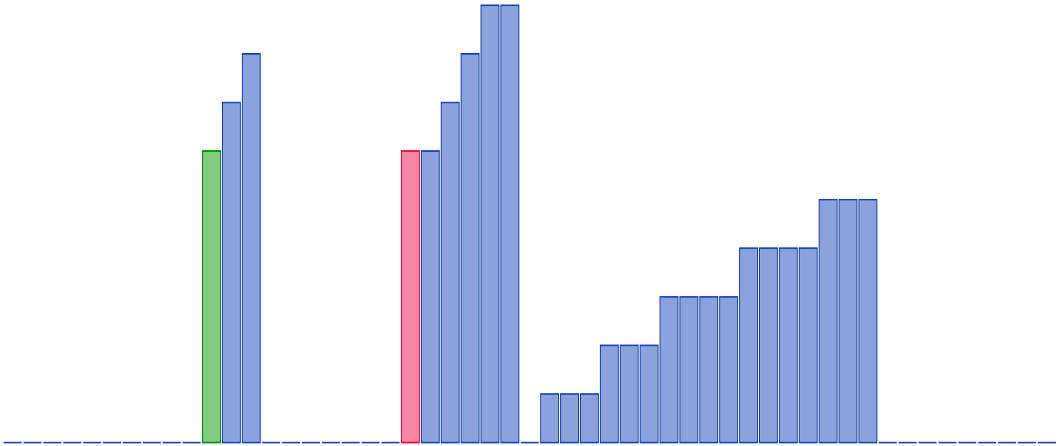


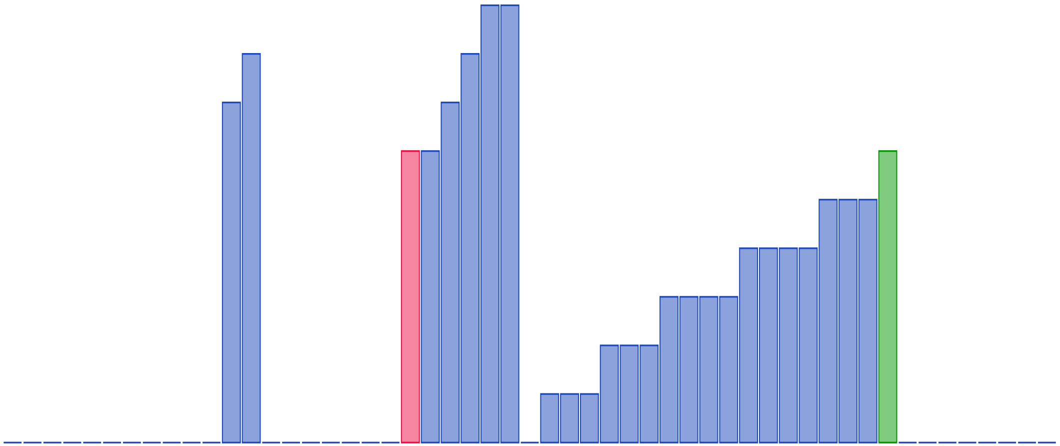


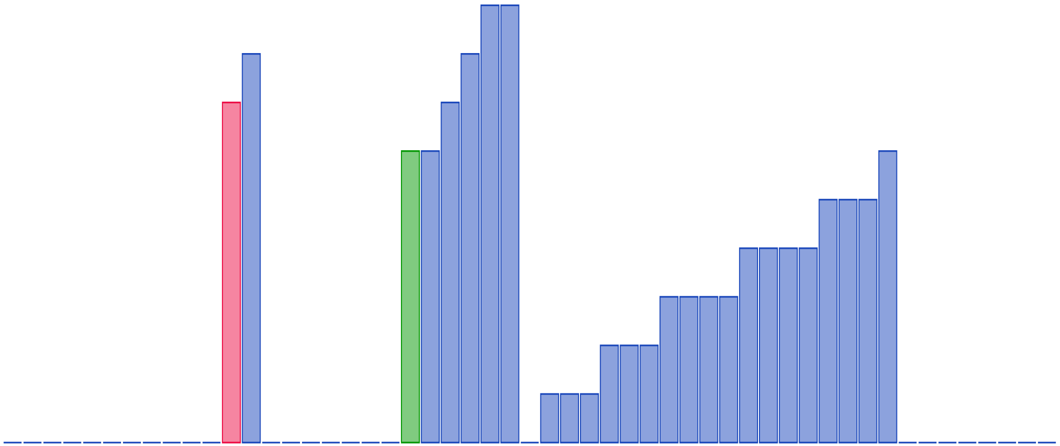


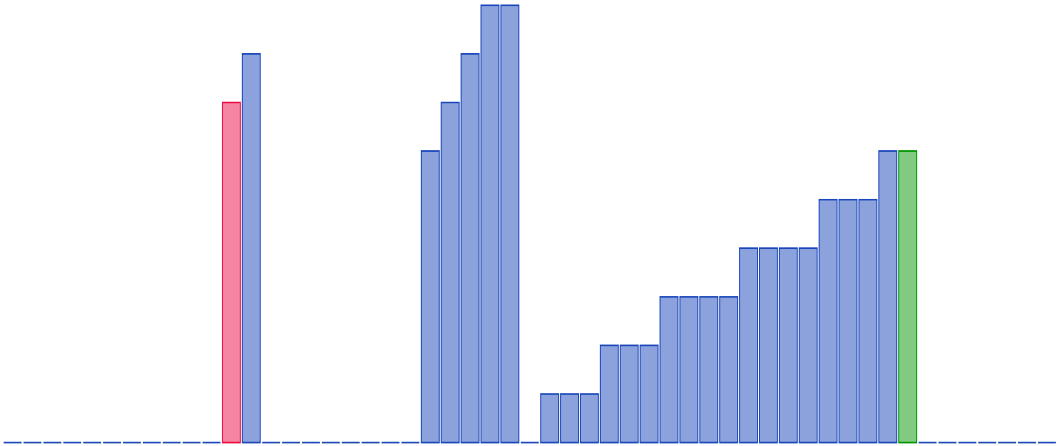


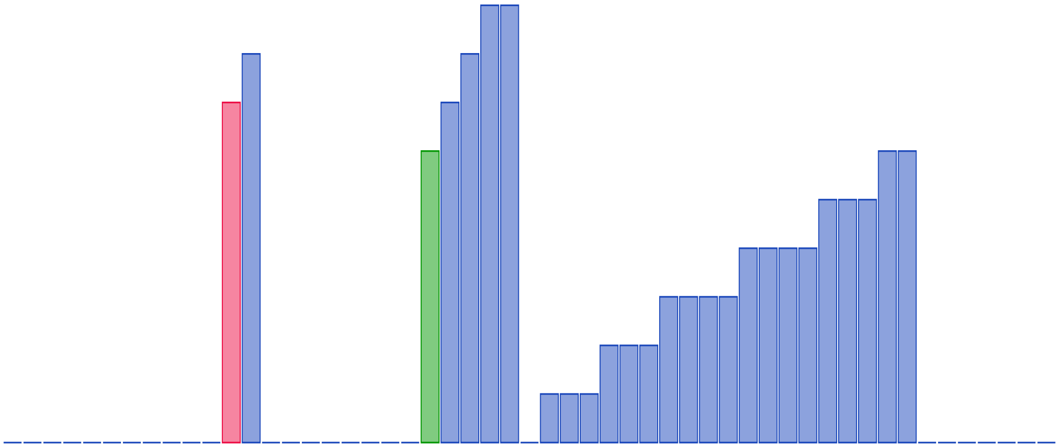


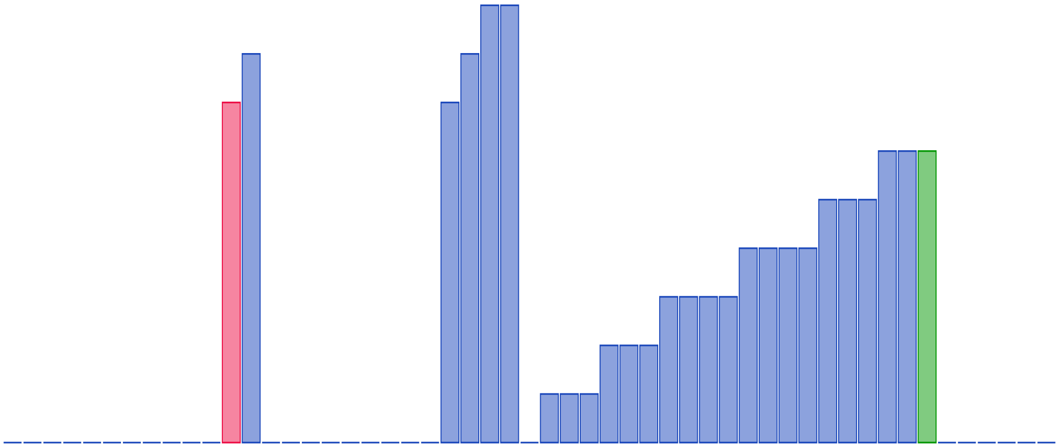


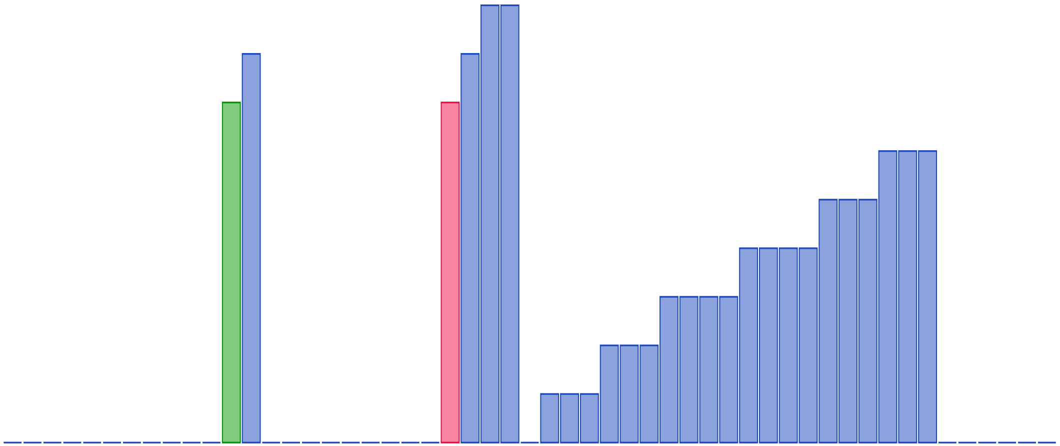


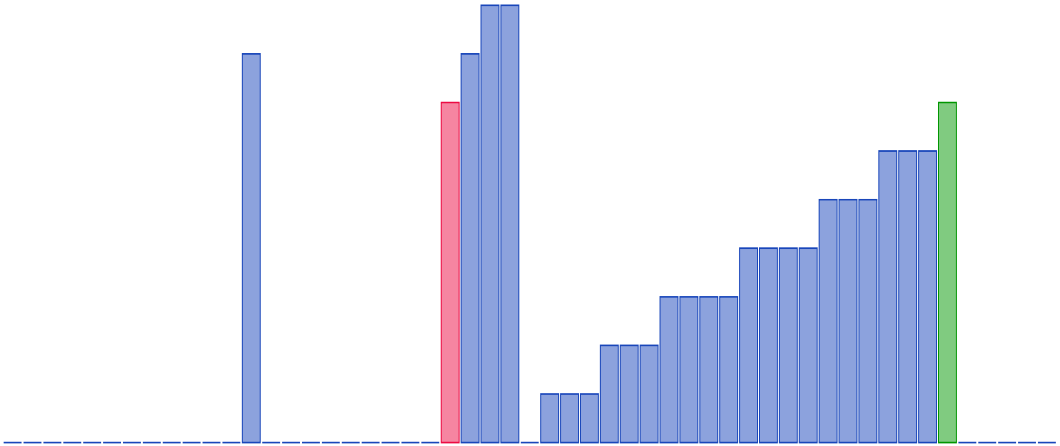


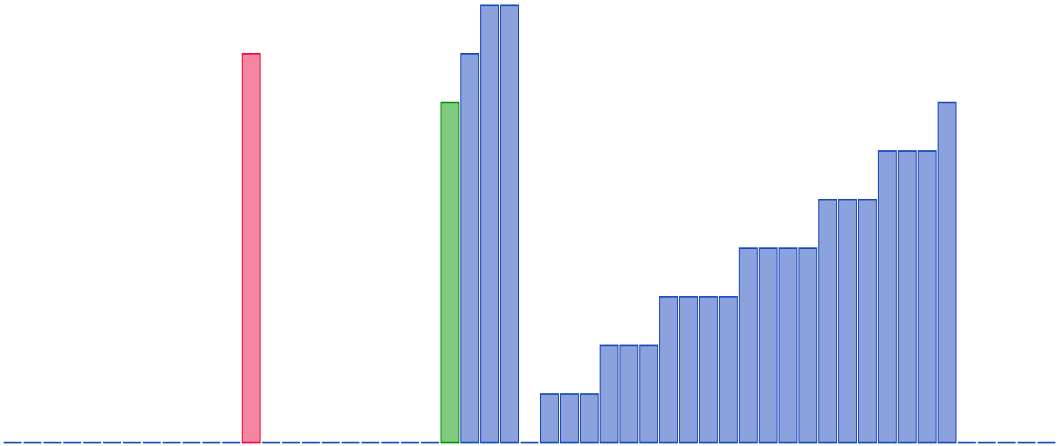


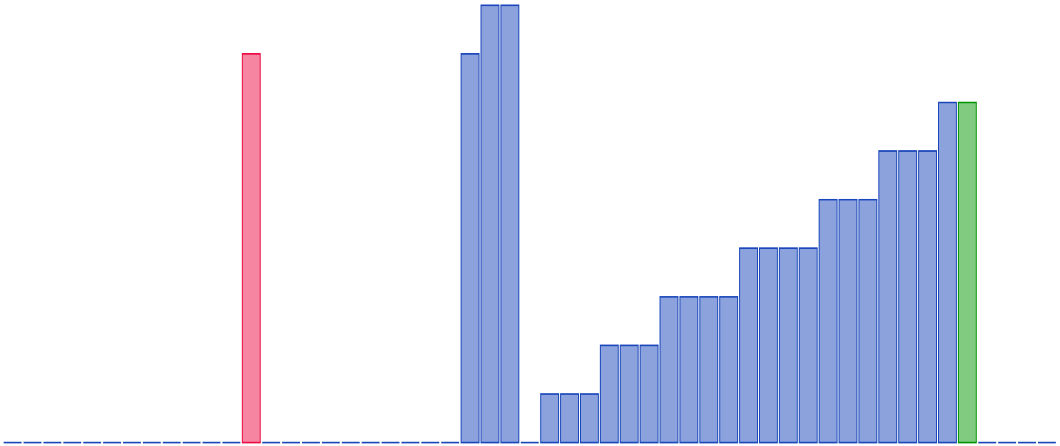


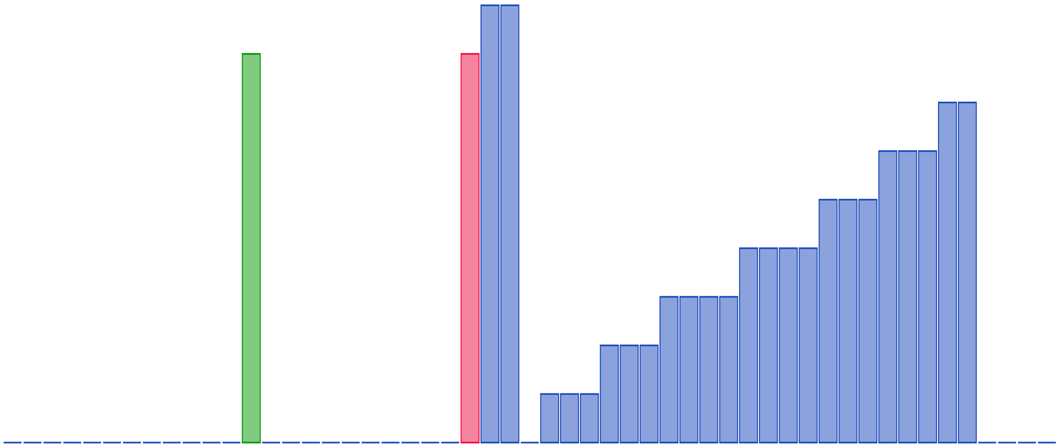


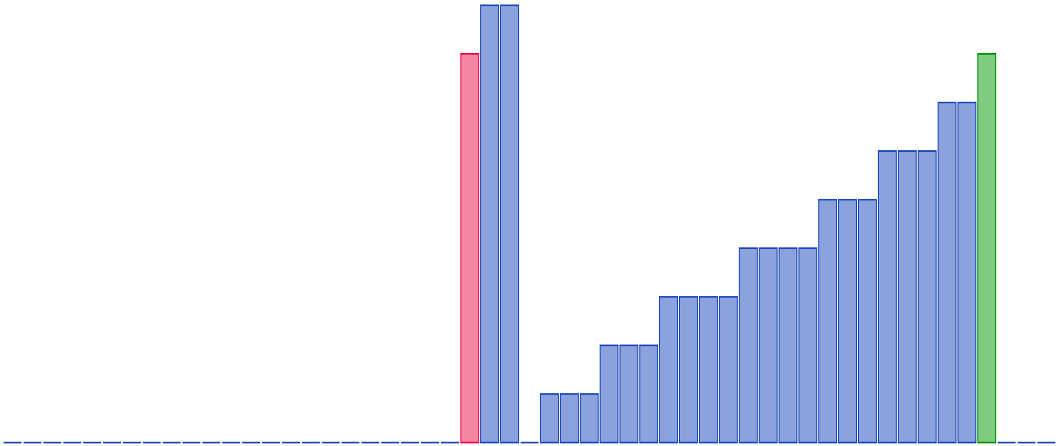


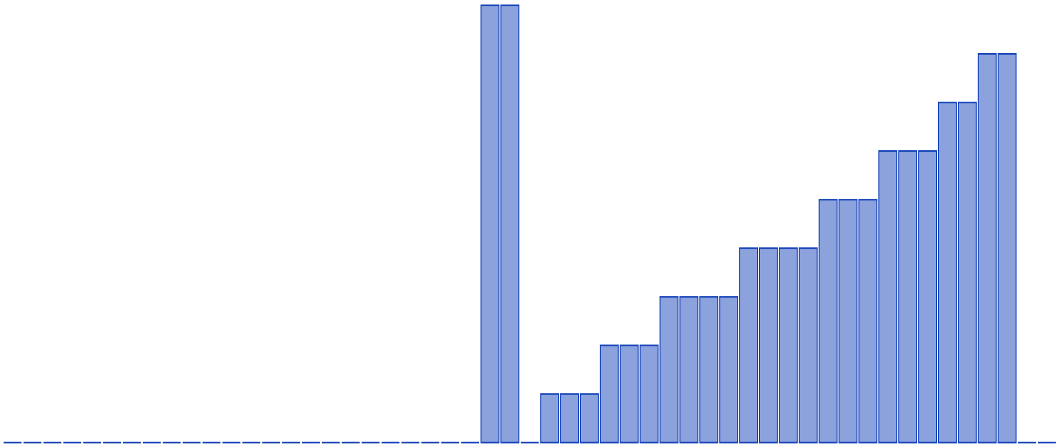


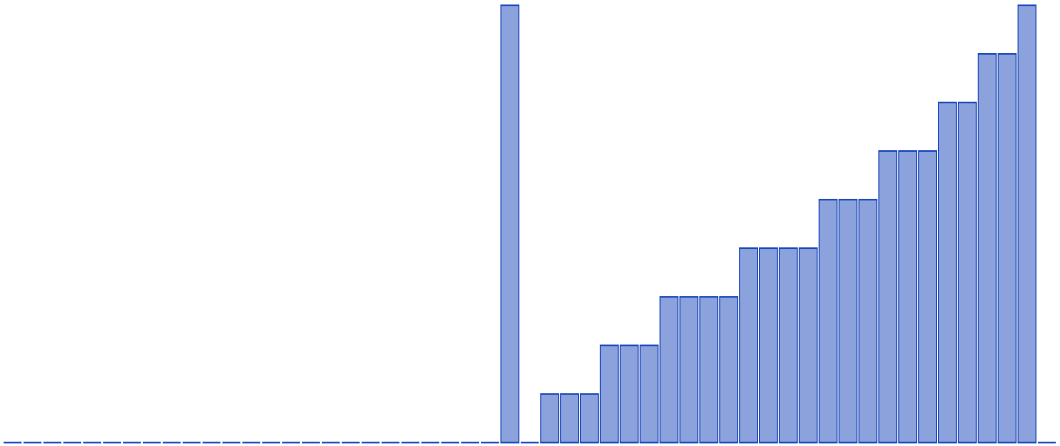


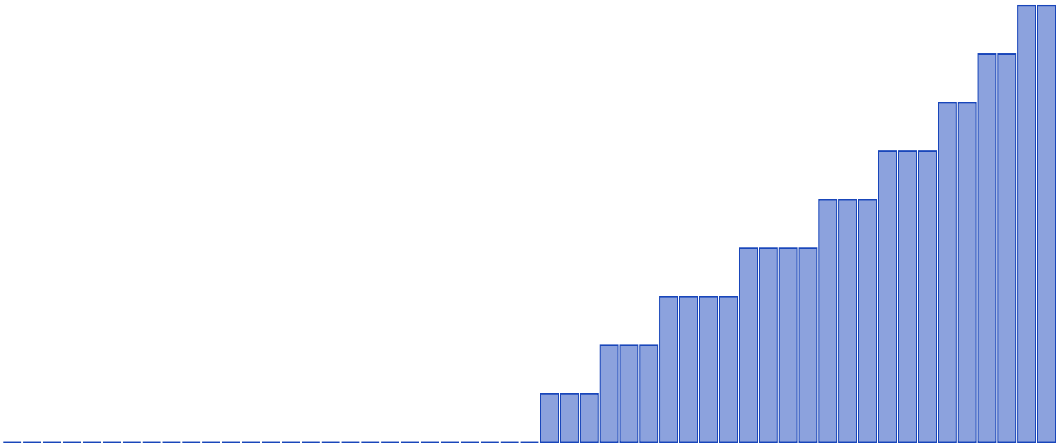


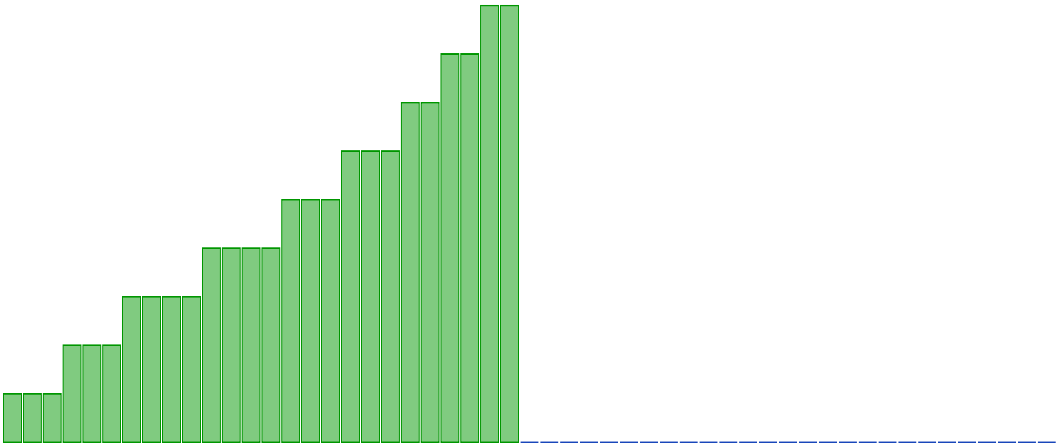












- What should the merge procedure do?
  - Given two sorted arrays, construct a sorted array from them

$i \leftarrow s$, $j \leftarrow m$, $k \leftarrow s$
while $i < m$ **or** $j < t$ **do**
    if $j = t$ **or** $(i < m$ **and** $A[i] \preceq A[j])$ **then**
        $W[t] \leftarrow A[i]$, $i \leftarrow i + 1$, $t \leftarrow t + 1$
    else
        $W[t] \leftarrow A[j]$, $j \leftarrow j + 1$, $t \leftarrow t + 1$
    end if
end while

- What should the merge procedure do?
  - Given two sorted arrays, construct a sorted array from them
- Why does it do this?
  - When $A[i]$ with $s \le i < m$ is moved?
    - After all not greater elements $s \le t < i$
    - After all elements from $[m; e)$ which are smaller than $A[i]$

$i \leftarrow s$, $j \leftarrow m$, $k \leftarrow s$
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    - After all not greater elements $m \le t < j$
    - After all elements from $[s; m)$ which are not greater than $A[j]$

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  - So, all elements are moved precisely in the sorted order
- The overall correctness of mergesort simply follows

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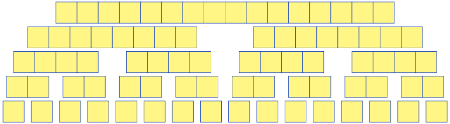

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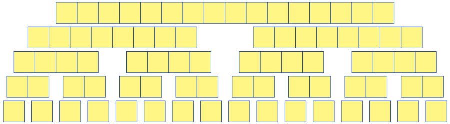

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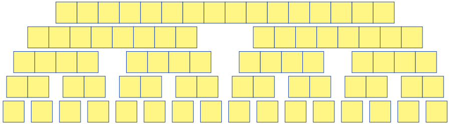

Running time is always $\Theta(N \log N)$.
Proof:

- Look at the call tree to the right
- Maximum depth: $\Theta(\log N)$, as every subarray size is exactly half of its parent's size
- All merges at given depth run in $\Theta(N)$
- Overall running time: $\Theta(N \log N)$