Solution to Homework 3

Name: Chen Shen

NetID: cs5236

1.

(a) The model is linear.
There is no under-modeling.
The true parameters are $\beta 0 = 1$, $\beta 1 = 2$, $\beta 2 = 0$.

(b) The model in nonlinear.
The function can be rewritten as $f_0(x) = 1 + \frac{1}{2+3x} = \frac{3+3x}{2+3x}$.
So there is no under-modeling.
The true parameters are $a_0 = 3$, $a_1 = 3$, $b_0 = 2$, $b_1 = 3$.

(c) The model is linear. The function can be rewritten as $x_1^2 - 2x_1x_2 + x_2^2$.
There is under-modeling because the x_1x_2 term is missing.

2.

(a) The least-squares estimates should be

$$\hat{\beta}_1 = \frac{s_{xy}}{s_x^2},\ \hat{\beta}_0 = \hat{y} - \hat{\beta}_1\bar{x},$$

where

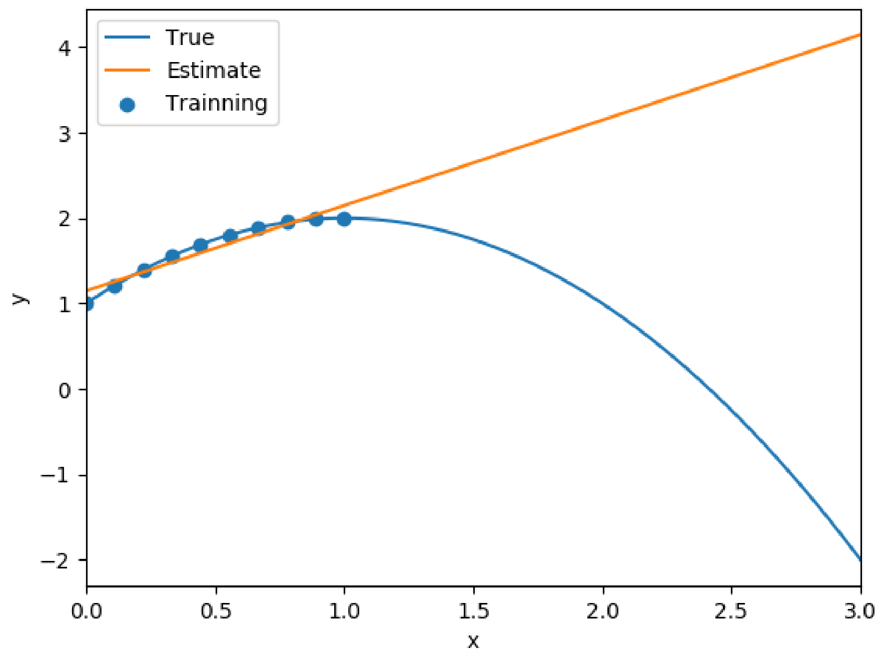
$$\bar{x} = \frac{1}{n}\sum_{i=1}^{N} x_i,\ \bar{y} = \frac{1}{n}\sum_{i=1}^{N} y_i$$

$$s_x^2 = \frac{1}{n}\sum_{i=1}^{N}(x_i - \bar{x})^2,\ s_y^2 = \frac{1}{n}\sum_{i=1}^{N}(y_i - \bar{y})^2$$

$$s_{xy} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

(b) The fomulas seem the same as the above expressions. But $y_i = \beta_{00} + \beta_{01}x_i + \beta_{02}x_i^2$.

(c) The figure is as following.



Code:

```
#!/usr/bin/python
# -*- coding: utf-8 -*-

import numpy as np
import matplotlib
matplotlib.use('Agg')
import matplotlib.pyplot as plt
import numpy.polynomial.polynomial as poly

def main():
    beta0 = np.array([1, 2, -1])
    x = np.linspace(0, 1, 10)
    y = poly.polyval(x, beta0)

    sxx = np.var(x)
    sxy = np.cov(np.append(x, x.mean()), np.append(y, y.mean()))[0][1]

    hat_beta_1 = sxy / sxx
    hat_beta_0 = y.mean() - hat_beta_1 * x.mean()

    xp = np.linspace(0, 3, 100)
    yp0 = poly.polyval(xp, beta0)
    hat_yp = hat_beta_0 + hat_beta_1 * xp

    plt.plot(xp, yp0, label = 'True')
    plt.plot(xp, hat_yp, label = 'Estimate')
    plt.scatter(x, y, label = 'Trainning')
    plt.legend()
```

```python
    plt.xlim([0, 3])
    plt.xlabel('x')
    plt.ylabel('y')
    plt.savefig('plot.png')

if __name__ == '__main__':
    main()
```

(d) It is easy to tell that the linear fit only tries to fit the data in $[0, 1]$. But the fit becomes worse and worse outside the region. So the bias error is largest at $x = 3$.

3.

(a) Define x_1 as the cancer volume, x_2 as the patient's age, x_3 as the cancer type with the following coding:

$$x_3 = \begin{cases} 0 & \text{Type I cancer} \\ 1 & \text{Type II cancer} \end{cases}$$

Thus, the models are

$$\begin{aligned} &\text{Model 1: } \hat{y} = \beta_0 + \beta_1 x_1 \\ &\text{Model 2: } \hat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 \\ &\text{Model 3: } \hat{y} = \beta_0 + \beta_1 x_1 x_3 + \beta_2 x_1 (1 - x_3) + \beta_3 x_2 \end{aligned}$$

(b) 2 parameters in Model 1; 3 parameters in Model 2; 4 parameters in Model 3. Model 3 is the most complex.

(c) For Model 1, the first three rows of matrix $\mathbf{A}$ should be

$$\mathbf{A} = \begin{bmatrix} 1 & x_{11} \\ 1 & x_{21} \\ 1 & x_{31} \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0.7 \\ 1 & 1.3 \\ 1 & 1.6 \\ \vdots & \vdots \end{bmatrix}.$$

For Model 2, the first three rows of matrix $\mathbf{A}$ should be

$$\mathbf{A} = \begin{bmatrix} 1 & x_{11} & x_{12} \\ 1 & x_{21} & x_{22} \\ 1 & x_{31} & x_{32} \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0.7 & 55 \\ 1 & 1.3 & 65 \\ 1 & 1.6 & 70 \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

For Model 3, the first three rows of matrix $\mathbf{A}$ should be

$$\mathbf{A} = \begin{bmatrix} 1 & x_{11}x_{13} & x_{11}(1 - x_{13}) & x_{12} \\ 1 & x_{21}x_{23} & x_{21}(1 - x_{23}) & x_{22} \\ 1 & x_{31}x_{33} & x_{31}(1 - x_{33}) & x_{32} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0.7 & 0 & 55 \\ 1 & 0 & 1.3 & 65 \\ 1 & 0 & 1.6 & 70 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

(d) The lowest mean test RSS is 0.70 in Model 3. So the standard error is

$$SE = \frac{0.05}{\sqrt{K - 1}} = \frac{0.05}{\sqrt{9}} = 0.0167.$$

As a result, the target RSS is $0.70 + 0.0167 = 0.7167$. Thus, Model 3 is the simplest model within one SE of minimum.