Solution to Homework 04

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1.

(a) A possible variable could be the frequency, i.e. a vector of STFT. There are two classes, male or female.

(b) A possible variable could be the position of the stylus, i.e. a vector of (x, y), which represents the coordinate. Since 0-9, a-z and A-Z are included, there are $10 + 26 + 26 = 62$ classes.

2.

(a) First, we have

$$P(y = 0|\mathbf{x}) = 1 - P(y = 1|\mathbf{x}) = 1 - \frac{1}{1 - e^{-z}}$$

Hence,

$$\begin{aligned} & P(y = 1|\mathbf{x}) > P(y = 0|\mathbf{x}) \\ \Longrightarrow \quad & \frac{1}{1 + e^{-z}} > \frac{e^{-z}}{1 + e^{-z}} \\ \Longrightarrow \quad & e^{-z} < 1 \\ \Longrightarrow \quad & z > 0 \end{aligned}$$

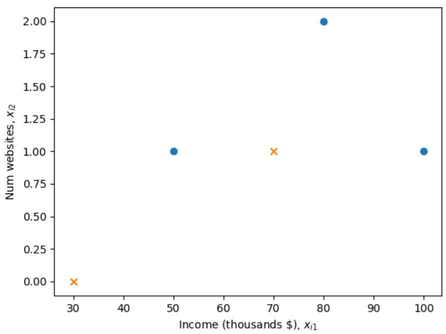
(b)

$$\begin{aligned} & P(y = 1|\mathbf{x}) > 0.8 \\ \Longrightarrow \quad & \frac{1}{1 + e^{-z}} > 0.8 \\ \Longrightarrow \quad & 1 + e^{-z} < 1.25 \\ \Longrightarrow \quad & e^{-z} < 0.25 \\ \Longrightarrow \quad & z > -\ln 0.25 = \ln 4 \end{aligned}$$

(c) Based on problem (b), there is $z > \ln 4$. Besides, $x_2 = 0.5$

$$\begin{aligned} & \beta_0 + \beta_1 x_1 + \beta_2 x_2 > \ln 4 \\ \Longrightarrow \quad & 1 + 2x_1 + 3 \cdot 0.5 > \ln 4 \\ \Longrightarrow \quad & x_1 > \ln 2 - 1.25 \end{aligned}$$

3.

(a) 

(b) For example, $x_{i2} = 0.5$. That is

$$z_i = \mathbf{w}^{\mathrm{T}}\mathbf{x}_i + b = x_{i2} - 0.5$$

So

$$\mathbf{w} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$b = -0.5$$

(c)

Income(thousands \$), x_{i1}	30	50	70	80	100
Num websites, x_{i2}	0	1	1	2	1
Donate (1=yes or 0=no), y_i	0	1	0	1	1
$z_i = x_{i2} - 0.5$	-0.5	0.5	0.5	1.5	0.5
$P(y_i \vert \mathbf{x}_i)$	$\frac{1}{1+e^{-0.5}}$	$\frac{1}{1+e^{-0.5}}$	$\frac{1}{1+e^{0.5}}$	$\frac{1}{1+e^{-1.5}}$	$\frac{1}{1+e^{-0.5}}$

From the table, we know that sample 3 is the least likely.

(d)

$$\begin{aligned} z_i' &= (\mathbf{w}')^{\mathrm{T}}\mathbf{x}_i + b' \\ &= \alpha \left[\mathbf{w}^{\mathrm{T}}\mathbf{x}_i + b\right] \\ &= \alpha z_i \end{aligned}$$

So the $\hat{y}$ will not change.
Since for $z_i > 0$, $z_i' > z_i$; for $z_i < 0$, $z_i' < z_i$, we can tell that for $P(y_i = 1|\mathbf{x}) > 0.5$, the probability will increase; for $P(y_i = 1|\mathbf{x}) < 0.5$, the probability will decrease.

4.

(a)

$$\begin{aligned} z_i &= \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} \\ &= -6 + 0.05 \cdot 40 + 1 \cdot 3.5 \quad = -0.5 \end{aligned}$$
$$P(Y) = \frac{1}{1 + e^{-z_i}} = \frac{1}{1 + e^{0.5}} = 0.378$$

(b) In order to make

$$P(Y) = \frac{1}{1 + e^{-z_i}} \geq 0.5$$

There must be

$$z_i \geq 0$$

Also we have

$$\begin{aligned} z_i &= \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} \\ &= -6 + 0.05 x_{1i} + 1 \cdot 3.5 \\ &= -2.5 + 0.05 x_{1i} \end{aligned}$$

Hence,

$$x_{1i} \geq 50$$

So the student needs to study 50 hours.

5.

(a)

$$\frac{\partial z_i}{\partial \beta_0} = 1$$

$$\frac{\partial z_i}{\partial \beta_1} = x_{1i}$$

$$\frac{\partial z_i}{\partial \beta_2} = x_{2i}$$

(b)

$$\frac{\partial J}{\partial \beta_0} = \frac{\partial J}{\partial z_i} \cdot \frac{\partial z_i}{\partial \beta_0} = \sum_{i=1}^{N} \left(\frac{1}{1+e^{z_i}} \cdot e^{z_i} \cdot 1 - y_i \cdot 1 \right) = \sum_{i=1}^{N} \left(\frac{e^{z_i}}{1+e^{z_i}} - y_i \right)$$

$$\frac{\partial J}{\partial \beta_1} = \frac{\partial J}{\partial z_i} \cdot \frac{\partial z_i}{\partial \beta_1} = \sum_{i=1}^{N} \left(\frac{1}{1+e^{z_i}} \cdot e^{z_i} \cdot x_{1i} - y_i \cdot x_{1i} \right) = \sum_{i=1}^{N} \left(\frac{e^{z_i} x_{1i}}{1+e^{z_i}} - y_i x_{1i} \right)$$

$$\frac{\partial J}{\partial \beta_2} = \frac{\partial J}{\partial z_i} \cdot \frac{\partial z_i}{\partial \beta_2} = \sum_{i=1}^{N} \left(\frac{1}{1+e^{z_i}} \cdot e^{z_i} \cdot x_{2i} - y_i \cdot x_{2i} \right) = \sum_{i=1}^{N} \left(\frac{e^{z_i} x_{2i}}{1+e^{z_i}} - y_i x_{2i} \right)$$

(c) Let $\frac{\partial J}{\partial \beta_0} = 0$, $\frac{\partial J}{\partial \beta_1} = 0$, $\frac{\partial J}{\partial \beta_2} = 0$. Then sum them all. We will have

$$\sum_{i=1}^{N} \left(\frac{e^{z_i} z_i}{1+e^{z_i}} - y_i z_i \right) = 0$$

This is a transcendental equation. So there is no analytical solutions. To optimize the loss function, we can use some numerical methods, such as gradient descent.