## Solution to Homework 1

Name: Chen Shen

NetID: cs5236

**1.**

(a) For example, GPA.

(b) It is discrete-valued.

(c) For instance, the students' GPA in high school and the rank of their high schools.

(d) Yes, a linear model is reasonable. Besides, there should be a positive correlation.

**2.**

(a) The sample means are

$$\bar{x} = \frac{1}{5}\sum_{i=1}^{5} x_i = 2$$

$$\bar{y} = \frac{1}{5}\sum_{i=1}^{5} y_i = 6$$

(b) The sample variances and co-variances are

$$s_x^2 = \frac{1}{5}\sum_{i=1}^{5}(x_i - \bar{x})^2 = 2$$

$$s_y^2 = \frac{1}{5}\sum_{i=1}^{5}(y_i - \bar{y})^2 = 37.2$$

$$s_{xy} = \frac{1}{5}\sum_{i=1}^{5}(x_i - \bar{x})(y_i - \bar{y}) = 8$$

(c) The least squares parameters are

$$\beta_1 = \frac{s_{xy}}{s_x^2} = 4$$

$$\beta_0 = \bar{y} - \beta_1\bar{x} = -2$$

(d) The predicted value at $x = 2.5$ is

$$\hat{y} = \beta_0 + \beta_1 x|_{x=2.5} = -2 + 4 \cdot 2.5 = 8$$

**3.**

(a) In the following fomula, $z_0$ and $\alpha$ appear linearly.

$$\ln(z(t)) = \ln(z_0 e^{-\alpha t}) = \ln z_0 - \alpha t$$

(b) Let $y = \ln(z(t))$, $\beta_0 = \ln z_0$, $\beta_1 = -\alpha$ and $x = t$. Then we can rewrite the fomula as

$$y = \beta_0 + \beta_1 x$$

where

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i,\ \ \bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$$

$$s_x^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2,\ \ s_y^2 = \frac{1}{N}\sum_{i=1}^{N}(y_i - \bar{y})^2$$

$$s_{xy} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})$$

$$\beta_1 = \frac{s_{xy}}{s_x^2},\ \ \beta_0 = \bar{y} - \beta_1 \bar{x}$$

(c) The code could be

```
# Define x and y
x = t
y = np.log(z)

# Compute variance and covariance
sxx = np.var(x)
sxy = np.cov(np.append(x, x.mean()), np.append(y, y.mean()))[0][1]

# Compute the least squares parameters
b1 = sxy / sxx
b0 = y.mean() - b1 * x.mean()

# Compute alpha and z0 in the original fomula
alpha = -b1
z0 = exp(b0)
```

**4.**

(a) The cost function is

$$\mathrm{RSS}(\beta) = \sum_{i=1}^{N}(y_i - \beta x_i)^2$$

(b) The $\beta$ should be

$$
\begin{aligned}
&\frac{\partial \mathrm{RSS}(\beta)}{\partial \beta} = \sum_{i=1}^{N} 2(y_i - \beta x_i)(-x_i) = 0 \\
&\implies -\sum_{i=1}^{N} x_i y_i + \beta \sum_{i=1}^{N} x_i^2 = 0 \\
&\implies \beta = \frac{\sum\limits_{i=1}^{N} x_i y_i}{\sum\limits_{i=1}^{N} x_i^2}
\end{aligned}
$$