

Introduction to Machine Learning

Homework 4: Logistic Regression

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1. Suggest possible response variables and predictors for the following classification problems. For each problem, indicate how many classes there are. There is no single correct answer.
 - (a) Given an audio sample, to detect the gender of the voice.
 - (b) A electronic writing pad records motion of a stylus and it is desired to determine which letter or number was written. Assume a segmentation algorithm is already run which indicates very reliably the beginning and end time of the writing of each character.

2. Suppose that a logistic regression model for a binary class label $y = 0, 1$ is given by

$$P(y = 1|\mathbf{x}) = \frac{1}{1 + e^{-z}}, \quad z = \beta_0 + \beta_1 x_1 + \beta_2 x_2,$$

where $\beta = [1, 2, 3]^\top$. Describe the following sets:

- (a) The set of \mathbf{x} such that $P(y = 1|\mathbf{x}) > P(y = 0|\mathbf{x})$.
 - (b) The set of \mathbf{x} such that $P(y = 1|\mathbf{x}) > 0.8$.
 - (c) The set of x_1 such that $P(y = 1|\mathbf{x}) > 0.8$ and $x_2 = 0.5$.
3. A data scientist is hired by a political candidate to predict who will donate money. The data scientist decides to use two predictors for each possible donor:
 - x_1 = the income of the person(in thousands of dollars), and
 - x_2 = the number of websites with similar political views as the candidate the person follow on Facebook.

To train the model, the scientist tries to solicit donations from a randomly selected subset of people and records who donates or not. She obtains the following data:

Income (thousands \$), x_{i1}	30	50	70	80	100
Num websites, x_{i2}	0	1	1	2	1
Donate (1=yes or 0=no), y_i	0	1	0	1	1

- (a) Draw a scatter plot of the data labeling the two classes with different markers.

- (b) Find a linear classifier that makes at most one error on the training data. The classifier should be of the form,

$$\hat{y}_i = \begin{cases} 1 & \text{if } z_i > 0 \\ 0 & \text{if } z_i < 0, \end{cases} \quad z_i = \mathbf{w}^\top \mathbf{x}_i + b.$$

What is the weight vector \mathbf{w} and bias b in your classifier?

- (c) Now consider a logistic model of the form,

$$P(y_i = 1 | \mathbf{x}_i) = \frac{1}{1 + e^{-z_i}}, \quad z_i = \mathbf{w}^\top \mathbf{x}_i + b.$$

Using \mathbf{w} and b from the previous part, which sample i is the *least* likely (i.e. $P(y_i | \mathbf{x}_i)$ is the smallest). If you do the calculations correctly, you should not need a calculator.

- (d) Now consider a new set of parameters

$$\mathbf{w}' = \alpha \mathbf{w}, \quad b' = \alpha b,$$

where $\alpha > 0$ is a positive scalar. Would using the new parameters change the values \hat{y} in part (b)? Would they change the likelihoods $P(y_i | \mathbf{x}_i)$ in part (c)? If they do not change, state why. If they do change, qualitatively describe the change as a function of α .

4. Suppose we collect data for a group of students in a machine learning class with variables X_1 = hours studied, X_2 = undergrad GPA, and Y = receive an A. We fit a logistic regression and produce estimated coefficient, $\beta_0 = -6$, $\beta_1 = 0.05$, $\beta_2 = 1$.

- (a) Estimate the probability that a student who studies for 40 h and has an undergrad GPA of 3.5 gets an A in the class.
- (b) How many hours would the student in part (a) need to study to have a 50 % chance of getting an A in the class?

5. The loss function for logistic regression is the binary cross entropy defined as

$$J(\boldsymbol{\beta}) = \sum_{i=1}^N \ln(1 + e^{z_i}) - y_i z_i$$

where $z_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i}$ for two features \mathbf{x}_1 and \mathbf{x}_2 .

- (a) What are the partial derivatives of z_i with respect to β_0 , β_1 , and β_2 .
- (b) Compute the partial derivatives of $J(\boldsymbol{\beta})$ with respect to β_0 , β_1 , and β_2 . You should use the chain rule of differentiation.
- (c) Can you find the close form expressions for the optimal parameters $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\beta}_2$ by putting the derivatives of $J(\boldsymbol{\beta})$ to 0? What methods can be used to optimize the loss function $J(\boldsymbol{\beta})$?