Solution to Homework 07

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1.

(a) The linears function in the hidden layer are

$$\mathbf{z}^{\mathrm{H}} = \mathbf{W}^{\mathrm{H}}\mathbf{x} + \mathbf{b}^{\mathrm{H}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_3 \\ x_1 + x_2 - 1 \\ x_1 + x_2 + x_3 + 1 \end{bmatrix}$$

So the activation functions are

$$\mathbf{u}^{\mathrm{H}} = g_{\mathrm{act}}(\mathbf{z}^{\mathrm{H}}) = \begin{bmatrix} g_{\mathrm{act}}(x_1 + x_3) \\ g_{\mathrm{act}}(x_2 + x_3) \\ g_{\mathrm{act}}(x_1 + x_2 - 1) \\ g_{\mathrm{act}}(x_1 + x_2 + x_3 + 1) \end{bmatrix} = \begin{bmatrix} \mathbb{1}_{(x_1+x_3 \geq 0)} \\ \mathbb{1}_{(x_2+x_3 \geq 0)} \\ \mathbb{1}_{(x_1+x_2 \geq 1)} \\ \mathbb{1}_{(x_1+x_2+x_3 \geq -1)} \end{bmatrix}$$

(b)

$$\begin{aligned} z^{\circ} &= W^{\circ}\mathbf{u}^{\mathrm{H}} + b^{\circ} = \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \mathbb{1}_{(x_1+x_3 \geq 0)} \\ \mathbb{1}_{(x_2+x_3 \geq 0)} \\ \mathbb{1}_{(x_1+x_2 \geq 1)} \\ \mathbb{1}_{(x_1+x_2+x_3 \geq -1)} \end{bmatrix} - 1.5 \\ &= \mathbb{1}_{(x_1+x_3 \geq 0)} + \mathbb{1}_{(x_2+x_3 \geq 0)} - \mathbb{1}_{(x_1+x_2 \geq 1)} - \mathbb{1}_{(x_1+x_2+x_3 \geq -1)} - 1.5 \end{aligned}$$

In the region that

$$x_1 + x_3 \geq 0, x_2 + x_3 \geq 0, x_1 + x_2 - 1 < 0, x_1 + x_2 + x_3 + 1 < 0$$

we have

$$z^{\circ} = 1 + 1 - 0 - 0 - 1.5 = 0.5$$

Outside the region, we have $z^{\circ} < 0$. So

$$\hat{y} = \begin{cases} 1 & x_1 + x_3 \geq 0, x_2 + x_3 \geq 0, x_1 + x_2 - 1 < 0, x_1 + x_2 + x_3 + 1 < 0 \\ 0 & \text{otherwise} \end{cases}$$

2.

(a) Since N_h has 3 outputs, $N_h = 3$.

$$\mathbf{z}^{\mathrm{H}} = \mathbf{W}^{\mathrm{H}}\mathbf{x} + \mathbf{b}^{\mathrm{H}} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -x - 1 \\ x + 1 \\ x - 2 \end{bmatrix}$$

The activation outputs are

$$\mathbf{u}^{\mathrm{H}} = \begin{bmatrix} \max\{0, -x - 1\} \\ \max\{0, x + 1\} \\ \max\{0, x - 2\} \end{bmatrix}$$

(b)

$$\hat{y} = g_{\text{out}}(z^\circ) = z^\circ$$

So the loss function could be

$$L = ||y - \hat{y}||^2 = ||y - z^\circ||^2$$

where

$$\begin{aligned} z^\circ &= \sum_{k=1}^{3} W_k^\circ u_k^{\mathrm{H}} + b^\circ \\ &= W_1^\circ \max\{0, -x-1\} + W_2^\circ \max\{0, x+1\} + W_3^\circ \max\{0, x-2\} + b^\circ \end{aligned}$$

(c) Assume that

$$\mathbf{A} = \begin{bmatrix} b^\circ & W^\circ \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} \mathbf{1} \\ \mathbf{u}^{\mathrm{H}} \end{bmatrix}$$

Then we have

$$y^\circ = z^\circ = \mathbf{AX}$$

So

$$L = ||y - z^\circ||^2 = ||y - \mathbf{AX}||^2$$

Let

$$\frac{\partial L}{\partial A} = -2(\mathbf{y} - \mathbf{AX})\mathbf{X}^{\mathrm{T}} = 0$$

Thus

$$\begin{bmatrix} b^\circ & W^\circ \end{bmatrix} = \mathbf{A} = \mathbf{y}\mathbf{X}^{\mathrm{T}}(\mathbf{X}\mathbf{X}^{\mathrm{T}})^{-1}$$

Functions in Python3:

```
def get_table():
    return np.matrix([-2, -1, 0, 3, 3.5]), np.matrix([0, 0, 1, 3, 3])

def get_uH():
    x, y = get_table()
    WH = np.matrix([-1, 1, 1]).T
    bH = np.matrix([-1, 1, -2]).T
    zH = WH * x + bH
    uH = zH
    uH[uH<0] = 0
    return uH

def get_Wb():
    _, y = get_table()
    uH = get_uH()
    X = np.vstack((uH, np.ones((1,5))))
    X = np.matrix(X)
    A = y * X.T * (X * X.T).I
    return np.ravel(A[0,:-1]), A[0,-1]

def p2_c():
    Wo, bo = get_Wb()
    print('The bias is', bo)
    print('The weights are', Wo)
```

Output:

```
$ python3 HW7.py p2_c
The bias is 2.886579864025407e-15
The weights are [-2.88657986e-15  1.00000000e+00 -1.00000000e+00]
```

(d) Functions in Python3:

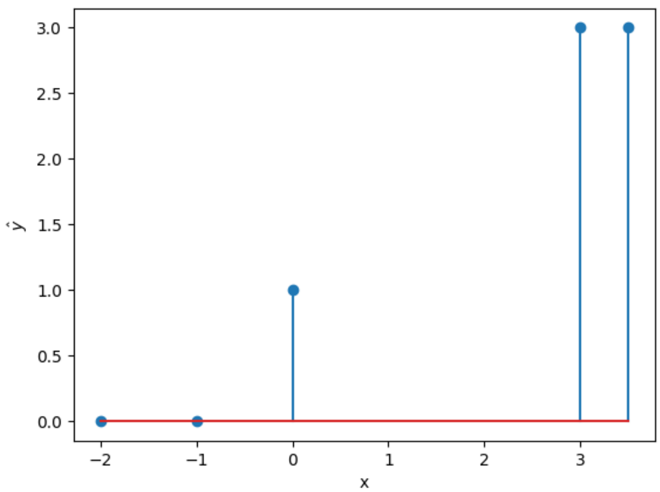
```python
def get_table():
    return np.matrix([-2, -1, 0, 3, 3.5]), np.matrix([0, 0, 1, 3, 3])

def get_uH():
    x, y = get_table()
    WH = np.matrix([-1, 1, 1]).T
    bH = np.matrix([-1, 1, -2]).T
    zH = WH * x + bH
    uH = zH
    uH[uH<0] = 0
    return uH

def get_Wb():
    _, y = get_table()
    uH = get_uH()
    X = np.vstack((uH, np.ones((1,5))))
    X = np.matrix(X)
    A = y * X.T * (X * X.T).I
    return np.ravel(A[0,:-1]), A[0,-1]

def p2_d():
    x, _ = get_table()
    Wo, bo = get_Wb()
    uH = get_uH()
    yhat = Wo * uH + bo
    x = np.ravel(x)
    yhat = np.ravel(yhat)
    plt.stem(x, yhat)
    plt.xlabel('x')
    plt.ylabel(r'$\hat{y}$')
    plt.savefig('image/2d.png')
```

Output:



(e) Function in Python3:

```python
def predict(x, y, WH, bH, Wo, bo):
    zH = WH * x + bH
    uH = zH
    uH[uH<0] = 0
    yhat = Wo * uH + bo
    return yhat
```

Note that all of the arguments are in the type of `numpy.matrix`, where `x` and `y` have only one row, `WH` and `bH` have only one column, `Wo` and `bo` have only one row.

3.

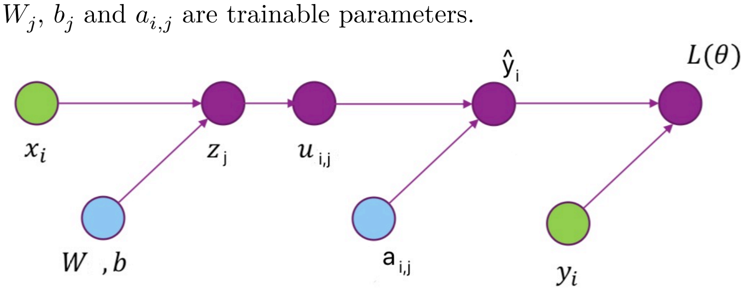
(a)

$$z_{i,j} = \sum_{k=1}^{N_i} W_{jk} x_{i,k} + b_j$$

$$u_{i,j} = \frac{1}{1 + e^{-z_{i,j}}}$$

$$\hat{y_i} = \frac{\sum_{j=1}^{M} a_{i,j} u_{i,j}}{\sum_{j=1}^{M} u_{i,j}}$$

(b) W_j, b_j and $a_{i,j}$ are trainable parameters.



(c)

$$\frac{\partial L}{\partial \hat{y}_i} = -2(y_i - \hat{y}_i)$$

(d) It can be computed by chain rule, i.e.

$$\frac{\partial L}{\partial \mathbf{u}} = \frac{\partial L}{\partial \hat{\mathrm{y}}} \frac{\partial \hat{\mathrm{y}}}{\partial u} = \frac{\partial L}{\partial \hat{\mathrm{y}}} \frac{a_{i,j} \sum_{j=1}^{M} u_{i,j} - \sum_{j=1}^{M} a_{i,j} u_{i,j}}{(\sum_{j=1}^{M} u_{i,j})^2}$$

(e) Using chain rule.

$$\frac{\partial L}{\partial \mathbf{z}} = \frac{\partial L}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial z} = \frac{\partial L}{\partial \mathbf{u}} \frac{e^{-z_{i,j}}}{(1 + e^{-z_{i,j}})^2}$$

(f)

$$\frac{\partial L}{\partial W_{jk}} = \frac{\partial L}{\partial \mathbf{z}} \frac{\partial \mathbf{z}}{\partial W_{jk}} = x_{ik} \frac{\partial L}{\partial \mathbf{z}}$$

$$\frac{\partial L}{\partial b_j} = \frac{\partial L}{\partial \mathbf{z}} \frac{\partial \mathbf{z}}{\partial b_j} = \frac{\partial L}{\partial \mathbf{z}}$$

(g) In conclusion,

$$\frac{\partial L}{\partial W_{jk}} = -2(y_i - \hat{y}_i) \frac{a_{i,j} \sum_{j=1}^{M} u_{i,j} - \sum_{j=1}^{M} a_{i,j} u_{i,j}}{(\sum_{j=1}^{M} u_{i,j})^2} \frac{e^{-z_{i,j}}}{(1 + e^{-z_{i,j}})^2} x_{ik}$$

$$\frac{\partial L}{\partial b_j} = -2(y_i - \hat{y}_i) \frac{a_{i,j} \sum_{j=1}^{M} u_{i,j} - \sum_{j=1}^{M} a_{i,j} u_{i,j}}{(\sum_{j=1}^{M} u_{i,j})^2} \frac{e^{-z_{i,j}}}{(1 + e^{-z_{i,j}})^2}$$

(h) Function in Python3:

```python
import numpy as np
def compute_grad(u, a, pL_py):
    py_pu = (a * np.sum(u, axis=0) - np.sum(a*u, axis=0)) / (np.sum(u, axis=0))**2
    return pL_py * py_pu
```