HW1

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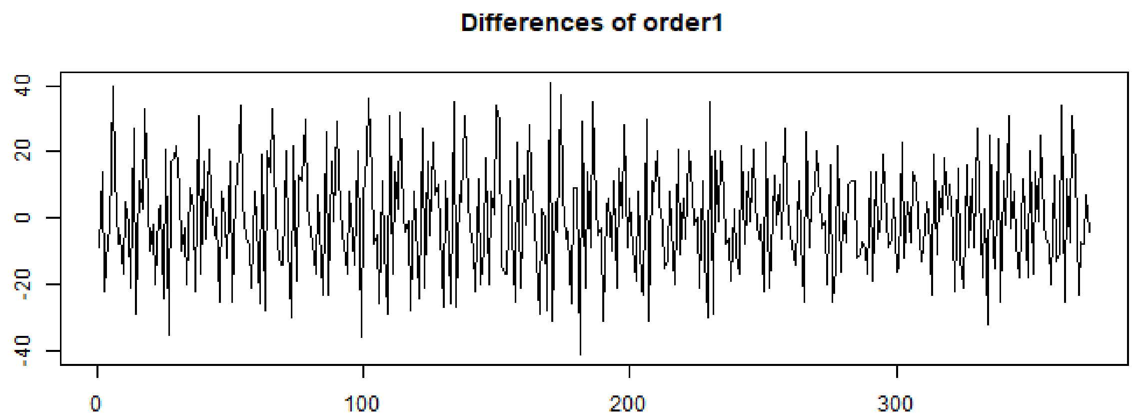
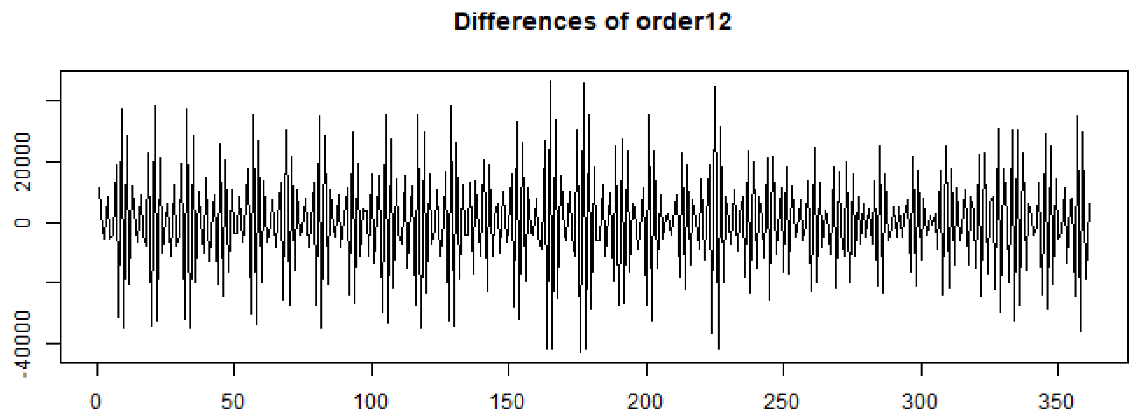
1. For the data set 1 on Moodle, write some R code to perform ∇ and ∇_{12} on the time series. In addition, plot the time series after the differencing. Report your R code.

```
> q1data <- read.table("HW1_dataset.txt", head = T)
> diff.func <- function(x, order = 1){
+   for(i in 1:order) x <- x[-1]-x[-length(x)]
+   return(x)
+ }
> diff.1 <- diff.func(q1data$birth_rates)
> diff.12 <- diff.func(q1data$birth_rates, 12)
> par(mfrow=c(2,1))
> plot(diff.1, type="l", xlab = "", ylab = "", main = "Differences of order1")
> plot(diff.12, type="l", xlab = "", ylab = "", main = "Differences of order12")
```





2. Consider the time series $X_t = \beta_0 + \beta_1 t + Z_t$, where β_0 and β_1 known constants and $Z_t \sim \mathrm{WN}(0, \sigma^2)$.
   (a) Determine whether X_t is stationary.

$$\mathrm{E}\,(X_t) = \mathrm{E}\,(\beta_0 + \beta_1 t + Z_t) = \beta_0 + \beta_1 t + \mathrm{E}\,(Z_t) = \beta_0 + \beta_1 t.$$

   Since $\mathrm{E}\,(X_t)$ depents on time, X_t is not stationary.

   (b) Show that the process $Y_t = X_t - X_{t-1}$ is stationary.

$$\mathrm{E}\,(Y_t) = \mathrm{E}\,(X_t - X_{t-1}) = \mathrm{E}\,(X_t) - \mathrm{E}\,(X_{t-1}) = \beta_0 + \beta_1 t - (\beta_0 + \beta_1 t - \beta_1) = \beta_1$$

   and let $s = t + h$

$$\begin{aligned}
\gamma(h) &= \mathrm{Cov}\,(Y_s, Y_t) = \mathrm{Cov}\,(Y_{t+h}, Y_t) = \mathrm{E}\,[(Z_{t+h} - Z_{t+h-1})(Z_t - Z_{t-1})] \\
&= \mathrm{E}\,(Z_{t+h}Z_t - Z_{t+h}Z_{t-1} - Z_{t+h-1}Z_t + Z_{t+h-1}Z_{t-1}) \\
&= \begin{cases} 2\sigma^2 & , h = 0 \\ -\sigma^2 & , h = 1 \\ 0 & , h > 1. \end{cases}
\end{aligned}$$

   Same result as $s = t - h$ would be obtained. Hence, Y_t is stationary.

   (c) Show that the mean of the moving average $V_t = \frac{1}{2q+1}\sum_{i=-q}^{q} X_{t-i}$ is $\beta_0 + \beta_1 t$.

$$\begin{aligned}
\mathrm{E}\,(V_t) &= \mathrm{E}\left(\tfrac{1}{2q+1}\textstyle\sum_{i=-q}^{q} X_{t-i}\right) \\
&= \tfrac{1}{2q+1}\textstyle\sum_{i=-q}^{q}(\beta_0 + \beta_1 t - \beta_1 i) \\
&= \tfrac{1}{2q+1}\textstyle\sum_{i=-q}^{q}(\beta_0 + \beta_1 t) - \beta_1 \sum_{i=-q}^{q} i \\
&= \tfrac{1}{2q+1}\left[(2q+1)(\beta_0 + \beta_1 t)\right] - 0 \\
&= \beta_0 + \beta_1 t.
\end{aligned}$$

3. Verify the calculations made in Example 3.4 as follows.
   (a) Let $X_t = \phi X_{t-1} + Z_t$, where $|\phi| > 1$ and $Z_t \sim i.i.d.\mathrm{N}(0, \sigma^2)$. Show that

$$\mathrm{E}\,(X_t) = 0 \text{ and } \gamma_X(h) = \frac{\sigma^2\phi^{-2}\phi^{-h}}{1-\phi^{-2}} \text{ for } h \geq 0.$$

   Let $X_{t+1} = \phi X_t + Z_{t+1}$, then

$$X_t = \phi^{-1}X_{t+1} - \phi^{-1}Z_{t+1} = \phi^{-1}(\phi^{-1}X_{t+2} - \phi^{-1}Z_{t+2}) - \phi^{-1}Z_{t+1} = \ldots = -\sum_{j=1}^{\infty}\phi^{-j}Z_{t+j}.$$

   Hence,

$$\mathrm{E}\,(X_t) = \mathrm{E}\left(-\sum_{j=1}^{\infty}\phi^{-j}Z_{t+j}\right) = -\sum_{j=1}^{\infty}\phi^{-j}\mathrm{E}\,(Z_{t+j}) = 0$$

   and

$$\begin{aligned}
\gamma_X(h) &= \mathrm{Cov}\,(X_{t+h}, X_t) = \mathrm{E}\left[\left(-\textstyle\sum_{j=1}^{\infty}\phi^{-j}Z_{t+h+j}\right)\left(-\sum_{k=1}^{\infty}\phi^{-k}Z_{t+k}\right)\right] \\
&= \mathrm{E}\left[(\phi^{-1}Z_{t+h+1} + \phi^{-2}Z_{t+h+2} + \ldots)(\phi^{-1}Z_{t+1} + \ldots + \phi^{-(h+1)}Z_{t+h+1} + \phi^{-(h+2)}Z_{t+h+2} + \ldots)\right] \\
&= \phi^{-(h+2)}\sigma^2 + \phi^{-(h+4)}\sigma^2 + \ldots \\
&= \sigma^2\phi^{-h}\textstyle\sum_{i=1}^{\infty}\phi^{-2i} \\
&= \tfrac{\sigma^2\phi^{-2}\phi^{-h}}{1-\phi^{-2}}.
\end{aligned}$$

(b) Let $Y_t = \phi^{-1}Y_{t-1} + W_t$, where $W_t \sim i.i.d.\mathrm{N}(0, \sigma^2\phi^{-2})$. Argue that Y_t is causal with the same mean function and autocovariance function as X_t.

$$Y_t = \phi^{-1}Y_{t-1} + W_t = \phi^{-1}(\phi^{-1}Y_{t-2} + W_{t-1}) + W_t = \ldots = \sum_{j=0}^{\infty} \phi^{-j} W_{t-j}.$$

$$\mathrm{E}\,(Y_t) = \mathrm{E}\left(\sum_{j=0}^{\infty} \phi^{-j} W_{t-j}\right) = \sum_{j=0}^{\infty} \phi^{-j} \mathrm{E}\,(W_{t-j}) = 0.$$

$$\begin{aligned}
\gamma_Y(h) &= \mathrm{Cov}\,(Y_{t+h}, Y_t) = \mathrm{E}\left[\left(\textstyle\sum_{j=0}^{\infty} \phi^{-j} W_{t+h-j}\right)\left(\textstyle\sum_{k=0}^{\infty} \phi^{-k} W_{t-k}\right)\right] \\
&= \mathrm{E}\left[(W_{t+h} + \ldots + \phi^{-h}W_t + \phi^{-(h+1)}W_{t-1} + \ldots)(W_t + \phi^{-1}W_{t-1} + \ldots)\right] \\
&= \sigma^2\phi^{-2}\phi^{-h}\textstyle\sum_{i=0}^{\infty} \phi^{-2i} \\
&= \frac{\sigma^2\phi^{-2}\phi^{-h}}{1-\phi^{-2}}.
\end{aligned}$$

4. (a) Derive the autocovariance function and ACF for the MA(1) process

$$X_t = Z_t + \theta Z_{t-1}, Z_t \sim \mathrm{WN}(0, \sigma^2)$$

$$\gamma(0) = \mathrm{Cov}\,(X_t, X_t) = \mathrm{E}\,[(Z_t + \theta Z_{t-1})(Z_t + \theta Z_{t-1})] = \mathrm{E}\left(Z_t^2 + \theta^2 Z_{t-1}^2\right) = (1+\theta^2)\sigma^2,$$

$$\gamma(1) = \mathrm{Cov}\,(X_{t+1}, X_t) = \mathrm{E}\,[(Z_{t+1} + \theta Z_t)(Z_t + \theta Z_{t-1})] = \mathrm{E}\left(\theta Z_t^2\right) = \theta\sigma^2,$$

$$\gamma(2) = \mathrm{Cov}\,(X_{t+2}, X_t) = \mathrm{E}\,[(Z_{t+2} + \theta Z_{t+1})(Z_t + \theta Z_{t-1})] = 0.$$

Hence, the autocovariance function is

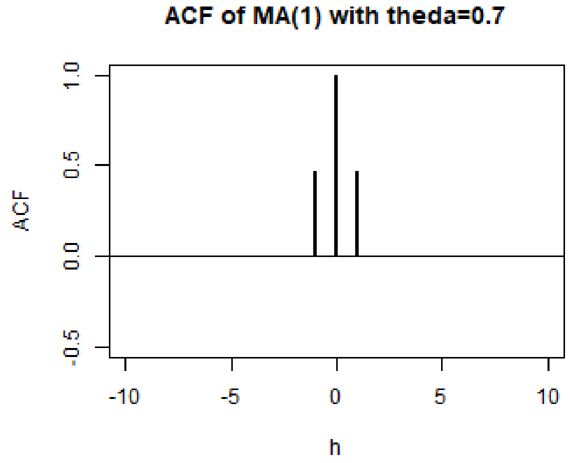
$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma^2 & , h = 0 \\ \theta\sigma^2 & , h = 1 \\ 0 & , h > 1. \end{cases}$$

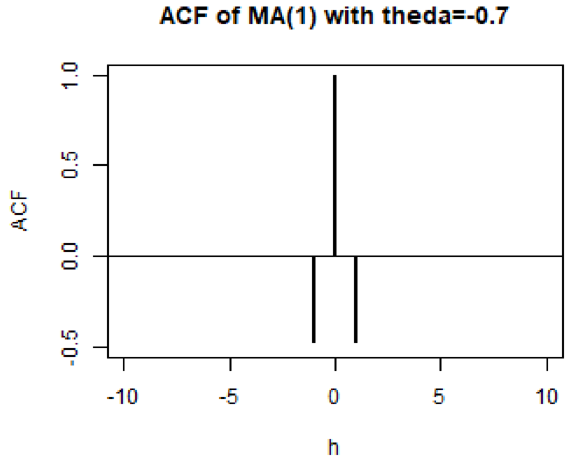
and the ACF is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \begin{cases} 1 & , h = 0 \\ \frac{\theta}{1+\theta^2} & , h = 1 \\ 0 & , h > 1. \end{cases}$$

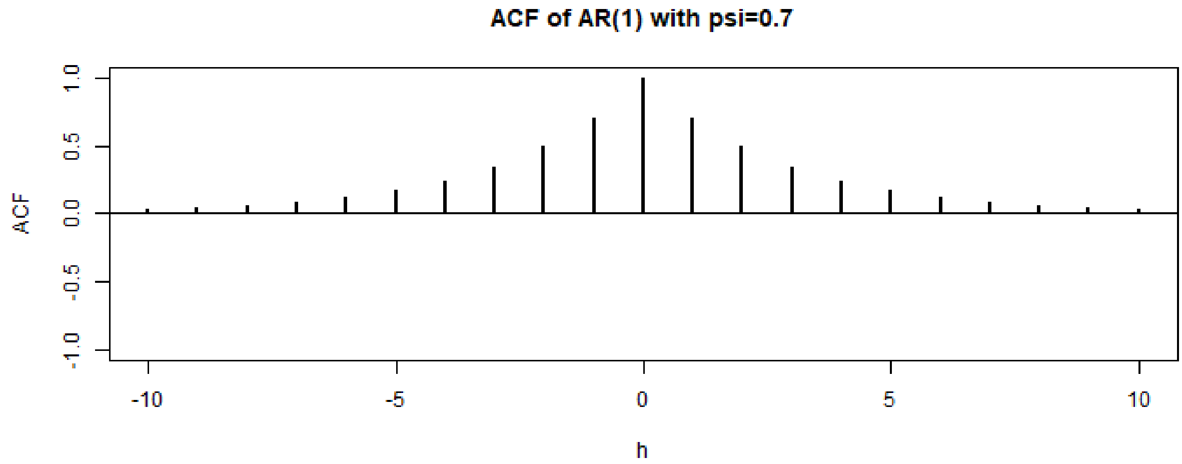
(b) Suppose further that $\theta = 0.7$, plot the ACF for $h \in Z$.

(c) Repeat (b) for $\theta = -0.7$





(d) The ACF for an AR(1) process has been derived in lecture notes. Based on the function, plot the ACF of $X_t = 0.7X_{t-1} + Z_t$ for $h \in Z$.



(e) What is the difference between the ACF for the MA(1) and AR(1) in (b) and (d)?

As $h(h > 0)$ increases, the ACF of (b) decreases faster than that of (d).

Appendix: all the code for the work

```
### Homework1: Time Series Analysis (110下). Due 2022.03.24 ####################
## Q1
q1data <- read.table("HW1_dataset.txt", head = T)
diff.func <- function(x, order = 1){
  for(i in 1:order) x <- x[-1]-x[-length(x)]
  return(x)
}
diff.1 <- diff.func(q1data$birth_rates)
diff.12 <- diff.func(q1data$birth_rates, 12)
png(file="Rplothw101.png", width=700, height=600)
par(mfrow=c(2,1))
plot(diff.1, type="l", xlab = "", ylab = "", main = "Differences of order1")
plot(diff.12, type="l", xlab = "", ylab = "", main = "Differences of order12")
dev.off()

## Q4
ACF.MA1.func <- function(theda, range){
  x <- range[1]:range[2]
  y <- rep(0, length(x))
  y[x == 0] <- 1
  y[abs(x) == 1] <- theda/(1+theda^2)
  return(cbind(x, y))
}
png(file="Rplothw102.png", width=700, height=300)
par(mfrow=c(1,2))
plot(ACF.MA1.func(0.7, c(-10, 10)), type = 'h', lwd = 2, ylim = c(-0.5, 1),
     ylab = "ACF", xlab = "h", main = "ACF of MA(1) with theda=0.7")
abline(h = 0)
plot(ACF.MA1.func(-0.7, c(-10, 10)), type="h", lwd = 2, ylim = c(-0.5, 1),
     ylab = "ACF", xlab = "h", main = "ACF of MA(1) with theda=-0.7")
abline(h = 0)
dev.off()

ACF.AR1.func <- function(theda, range){
  x <- range[1]:range[2]
  y <- theda^abs(x)
  return(cbind(x, y))
}
png(file="Rplothw103.png", width=700, height=300)
par(mfrow=c(1,1))
plot(ACF.AR1.func(0.7, c(-10, 10)), type="h", lwd = 2, ylim = c(-1, 1),
     xlab = "h", ylab = "ACF", main = "ACF of AR(1) with psi=0.7")
abline(h = 0)
dev.off()
```