# Matrices and Vectors

Chris Johnson

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## 1 Introduction

Only a small amount of knowledge of matrcies and vectors (often known as *linear algerba*) is needed for this course and we will try to avoid abstract mathematical concepts. This short document deals with a few concepts and definitions which you will need to understand to get the most out of the course.

## 2 What are matrices and vectors?

On a computer a dense matrix (*ie* one with mostly non-zero elements) is typically stored as a 2D rectangular array of data and written on paper as follows:

$$\mathsf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix}$$

where we tend to use uppercase to denote the whole matrix (in this case $\mathsf{A}$) and lower case to denote the individual elements (in this case $a_{ij}$). Matrices typically consist of real of complex elements or a mixture of the two. Make sure you memorise which way round the indicies work (*ie* the difference between $a_{23}$ and $a_{32}$).

A matrix is considered to be a of size $n \times m$ if it has $n$ rows and $m$ columns. For example, the above is a $4 \times 3$ matrix. We will most often be dealing with square matrices where $n = m$.

There is a special case of a matrix where one of the dimensions is equal to 1. Such matrices are known as vectors and have many physical interpretations and are usually used to convey some notion of direction and magnitude. We tend to consider a vector as an $n \times 1$ column *eg*

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \cdots \\ v_n \end{pmatrix}$$

and the $(1 \times n)$ transpose of such a vector is written as $\mathbf{v}^T$ where

$$\mathbf{v}^T = (v_1 \, v_2 \, v_3 \, \cdots \, v_n)$$

but the distinction is arbitrary and really both are vectors and on a computer usually stored as 1D arrays with no sense of orientation.

# 3 Dot products

A vector can be "dotted" with another vector by multiplying the two vectors element-wise and then summing up the results, *eg*

$$\mathbf{v}.\mathbf{w} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \cdots \\ v_n \end{pmatrix} . \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \cdots \\ w_n \end{pmatrix} = v_1 w_1 + v_2 w_2 + v_3 w_3 + \ldots + v_n w_n \, .$$

Mathematically this is equivalent to

$$\sum_{i=1}^{n} v_i w_i \, .$$

Dot products are sometimes referred to as *scalar products* or *inner products*.

# 4 Square matrices

The following are special kinds of square matrices:

If a real matrix is symmetric about the diagonal (*ie* $a_{ij} = a_{ji}$) it is known as a **Symmetric matrix**. *Eg*

$$\mathsf{A} = \begin{pmatrix} 1.0 & -2.4 & 4.1 \\ -2.4 & -0.5 & 5.2 \\ 4.1 & 5.2 & 7.8 \end{pmatrix} .$$

If a matrix has the property that (*ie* $a_{ij} = a^*_{ji}$) it is known as a **Hermitian matrix**. Such matrices are symmetric in the real part and anti-symmetric in the imaginary part *Eg*

$$\mathsf{A} = \begin{pmatrix} 3.1 & 0.4 - 7.1i & 0.2 + 1.7i \\ 0.4 + 7.1i & -0.4 & -9.7 - 3.6i \\ 0.2 - 1.7i & -9.7 + 3.6i & 2.9 \end{pmatrix} .$$

If all the elements below the diagonal are zero the matrix is known as an **Upper triangular**. *Eg*

$$\mathsf{A} = \begin{pmatrix} 1.4 & -2.7 & -3.6 & 4.1 \\ 0.0 & -0.2 & 3.7 & 4.7 \\ 0.0 & 0.0 & -9.2 & -8.7 \\ 0.0 & 0.0 & 0.0 & 0.6 \end{pmatrix} .$$

Similarly if all the elements above the diagonal are zero the matrix is known as an **Lower triangular**. *Eg*

$$\mathsf{A} = \begin{pmatrix} 2.4 & 0.0 & 0.0 & 0.0 \\ -9.5 & -5.1 & 0.0 & 0.0 \\ 1.8 & 4.2 & -0.9 & 0.0 \\ 3.8 & -0.9 & -0.8 & 2.7 \end{pmatrix} .$$

If the only non-zero elements are on the diagonal, the matrix is known as a **Diagonal matrix**. *Eg*

$$\mathsf{A} = \begin{pmatrix} -0.8 & 0 & 0 & 0 \\ 0 & 1.8 & 0 & 0 \\ 0 & 0 & 4.2 & 0 \\ 0 & 0 & 0 & 0.8 \end{pmatrix} .$$

A matrix with where most of the elements are non-zero is known as a **dense matrix**. If most of the elements are zero the it is known as a **sparse matrix**. What we mean by "most" here will depend on the context.

# 5 Vectors

To measure the "length" or "magnitude" of a vector we consider its **norm**. There are several different kinds of norm but we will use the following definition (known as the $L_2$ norm)

$$||\mathbf{v}|| = \sqrt{(v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2)} .$$

Mathematically, this can be written as

$$\sum_{i=1}^{n} v_i^2 \, .$$

This is simply Phythagoras in $n$-dimensions. Another way of thinking of it is that it is the vector dotted with itself and the result square rooted.

To normalise a vector we simply divide it by it's norm:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{||\mathbf{v}||} \, .$$

The resulting vector then has norm equal to 1 and is known as a **unit vector** and we usually denote this with a "hat" (*eg* $\hat{\mathbf{v}}$).

# 6 Matrix-vector multiplication

When a matrix multiplies a vector the result is a new vector. Matrix-vector multiplication can look complicated but it actually not difficult once you get the hang of it. Mathematically, for a matrix $\mathsf{A}$ of size $n \times n$ and vector $\mathbf{v}$ of length $n$ the result would be the vector $\mathbf{w}$

$$w_{ik} = \sum_{j=1}^{n} a_{ij} v_{jk}$$

or expanding this out for $n = 3$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + a_{13}v_3 \\ a_{21}v_1 + a_{22}v_2 + a_{23}v_3 \\ a_{31}v_1 + a_{32}v_2 + a_{33}v_3 \end{pmatrix} \, .$$