Feynman Simplified

4A: MATH FOR PHYSICISTS

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Everyone's Guide to the Feynman Lectures

Robert L. Piccioni, Ph.D.

Real Science Publishing

Feynman Simplified

4A: Math for Physicists

Everyone's Guide to the *Feynman Lectures on Physics*

by
Robert L. Piccioni, Ph.D.

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Everyone's Guide to the Feynman Lectures on Physics

Feynman Simplified gives mere mortals access to the fabled ***Feynman Lectures on Physics***.

This Book

Feynman Simplified: 4A explores all the math needed to understand ***The Feynman Lectures on Physics*** and much more.
The topics we explore include:

- Trigonometric Functions & Identities
- Rectilinear, Polar, Cylindrical & Spherical Coordinates
- Real & Complex Numbers; Scientific Notation
- Quadratic & Polynomial Equations & Solutions
- Dimensional Analysis & Approximation Methods
- Finite & Infinite Series
- Zeno's Paradox & Mortgage Payments
- Exponentials, Logarithms & Hyperbolic Functions
- Permutations, Combinations & Binomial Coefficients
- Discrete & Continuous Probabilities
- Poisson, Gaussian, and Chi-Squared Distributions
- Rotation & Velocity Transformations
- Vector Algebra, Identities & Theorems
- Differential, Integral & Variational Calculus
- Differential Equations
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Chapter 1

Review of Basic Math

I presume you have previously studied geometry, trigonometry, and basic algebra. The first few chapters of this eBook provide a quick review of that material and a definitions of key terms.

Trigonometry quantifies the geometric relationships among angles and distances, and is most often employed in analyzing triangles.

Let's begin by reviewing some basic shapes of Euclidean geometry and their key properties. Figure 1-1 shows five two-dimensional shapes.

In the upper left is a *triangle*, a figure bounded by three straight line segments; its three internal angles sum to π radians (180 degrees), and its area equals hw/2, where h is its height, and w is its width; h and w are mutually perpendicular.

In the upper right is a *square*, a figure bounded by four line segments of equal length, with four internal angles that are each 90 degrees (π/2 *radians*); its area equals h^2 (h=w).

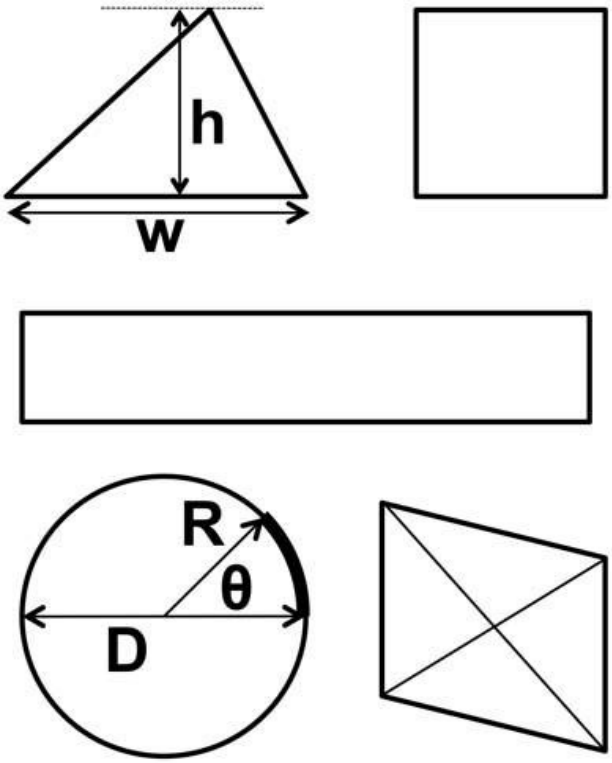


Figure 1-1 Two-Dimensional Shapes

In the middle is a rectangle, a figure bounded by four line segments with opposite sides of equal length, and four internal angles that are each 90 degrees; its area equals hw. (All squares are rectangles, but not all rectangles are squares.)

In the lower left is a *circle* of *radius* R and *diameter* D=2R, which is the *locus* (collection) of all points that are a distance R from the circle's center. The circle's circumference (length of its perimeter) equals πD, and its area equals πR^2. The length of the bolded arc that *subtends* angle θ equals θR, when θ is measured in radians. This makes sense: for $\theta=2\pi$, the arc becomes the circle's circumference whose length is $2\pi R$. (This is why God invented radians.)

The area enclosed by a circle is called a *disk*.

Lastly, in the lower right is a parallelogram, a figure bounded by four line segments with opposite sides of equal length, and opposite angles equal; its area equals one-half the product of the lengths of its two diagonals. (All rectangles are parallelograms, but not all parallelograms are rectangles.)

Moving to three dimensions, Figure 1-2 shows three common shapes.

In the upper right is a *cuboid* bounded by six rectangles, with all internal angles being equal. Its height h, width w, and length L may be different. Its enclosed volume equals hwL, and its surface area equals 2(hw+hL+wL). In the upper left is a *cube*, a cuboid in which h=w=L; its enclosed volume equals h^3 and its surface area equals $6h^2$, where h is the length of any side. (All cubes are cuboids, but not all cuboids are cubes.)

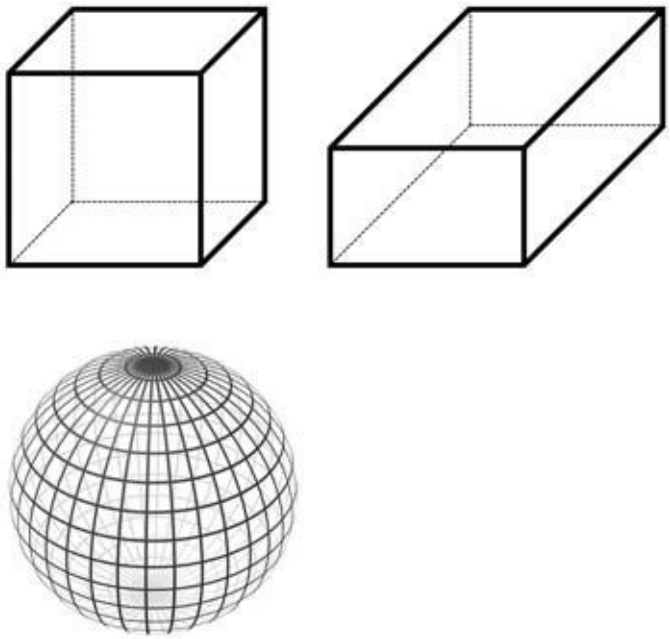


Figure 1-2 Three-dimensional Shapes

Lastly, in the lower image, is a *sphere*, the locus of all points that are a distance R from the sphere's center. The sphere's area equals $4\pi R^2$, and its enclosed volume equals $4\pi R^3/3$. Proper mathematical terminology defines a sphere as the 2-D surface that encloses a 3-D volume called a *ball*.

We wish to examine some quantitative relationships established by trigonometry. But first, we must discuss functions.

Functions

In mathematics, functions define relationships between variables. Since physics is all about relationships, functions are the bread and butter of mathematical physics.

Variables are quantities whose values change; they can change with location,

change over time, or change for some other reason. Temperature is a variable that changes with both location and time. We can describe how temperature T varies with location x and time t by using the function f:

$T = f(x,t)$

Here, x and t are called *independent variables*, and T is called a *dependent variable*. Functions can have one or more independent variables, but they must have exactly one dependent variable. In this case, T is a function of both x and t.

As the terms suggest, we are free to choose the values of x and t, and those values uniquely determine the value of T. Some prefer to think of functions as being "black boxes": when x and t are input into f, f outputs T. A more elegant mathematical description is: f *maps* (x,t) to T.

The essential characteristic of functions is that for each combination of independent variables there is one and only one value of the dependent variable.

In general, there may be more than one combination of independent variables that produce the same value of the dependent variable. For example, the temperature in Fairbanks, Alaska in mid-August might be the same as the temperature in Miami, Florida in mid-February. We can describe this mathematically by saying: there is a one-to-one mapping from (x,t) to T, but there is not a one-to-one mapping from T to (x,t).

Graphing Functions

Graphs are visual representations that can be extremely helpful in understanding the key properties of functions. Graphs typically plot a function's dependent variable vertically, and the function's independent variable horizontally.

We will discuss *sine* functions shortly and *exponential* functions in Chapter 7, but for now suffice it to say that both are very important functions in physics. Here, we will discuss graphs of these two functions.

The upper graph in Figure 1-3 plots the value of Y that corresponds to each value of X, as defined by the exponential function:

$Y = A + B\, e^{x}$

Here, A and B are constants.

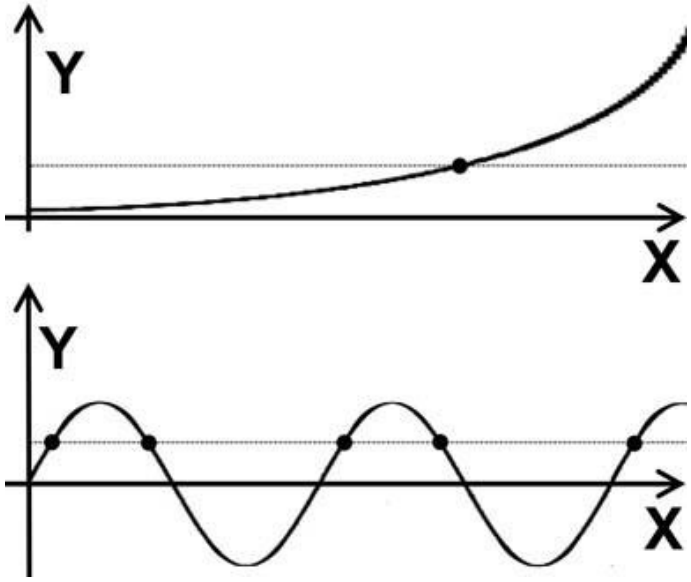


Figure 1-3 Exponential & Sine Functions

The lower graph plots the value of Y that corresponds to each value of X, as defined by the sine function:

$Y = A \sin(X)$

In the lower graph, the 5 black dots along the dotted horizontal line indicate 5 values of X for which sin(X) has the same value of Y. Like the prior example of the temperature in Fairbanks and Miami, Y=Asin(X) provides a one-to-one mapping from X to Y, but not a one-to-one mapping from Y to X.

Conversely, in the upper graph, there is only one black dot along the dotted line. In fact, for any Y value there is only one value of X for which $Y=A+Be^x$. This means exponentials provide one-to-one mappings from X to Y *and* from Y to X. Any function f with this special property has an *inverse function* g, such that:

if $y = f(x)$
then $g(y) = x$
and $g(f(x)) = x$
Again, the key property of such functions is that the mapping and the inverse mapping are both one-to-one.

Trig Functions

Now let's see how functions are used in trigonometry.

Figure 1-4 shows a triangle whose longest side has length r, whose vertical side has length y, and whose horizontal side has length x. Because the vertical and horizontal sides are *orthogonal* (perpendicular to one another), this is a *right* triangle and the longest side is the *hypotenuse*.

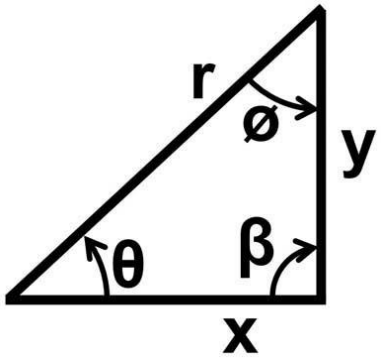


Figure 1-4 Angles & Sides of a Right Triangle

The angle β is a *right angle*, equal to 90 degrees (π/2 radians). Angles θ and ø can have any values that sum to 90 degrees.

The three primary trig functions are listed below with their English names, mathematical notations, and defining equations.

sine: $\sin(\theta) = y / r$
cosine: $\cos(\theta) = x / r$
tangent: $\tan(\theta) = y / x$

As we learned above, the sine function does not have a well-defined inverse function throughout the entire range of all possible angles. Indeed, this applies to all trig functions, because all are periodic, all repeat exactly at regular intervals. More precisely, for any integer n:

$\sin(2n\pi+\theta) = \sin(2n\pi+\pi-\theta) = \sin(\theta)$
$\cos(2n\pi+\theta) = \cos(2n\pi-\theta) = \cos(\theta)$
$\tan(n\pi+\theta) = \tan(\theta)$

Well-defined inverse functions do exist if we restrict the range of θ. The conventional allowed ranges, English names, mathematical notations, and defining equations of the inverse trig functions are:

$-\pi/2<\theta\leq+\pi/2$: arc sine: $\arcsin(y/r) = \theta$
$+0 \leq \theta < +\pi$: arc cosine: $\arccos(x/r) = \theta$
$-\pi/2<\theta\leq+\pi/2$: arc tangent: $\arctan(y/x) = \theta$

These inverse functions are sometimes written:
arcsine: $\sin^{-1}(y/r) = \theta$
arccosine: $\cos^{-1}(x/r) = \theta$
arctangent: $\tan^{-1}(y/x) = \theta$
However, this notation can be confusing: is $\sin^{-1}$ the arcsin or the reciprocal 1/sin? Context often resolves this ambiguity: the argument of arcsin is a ratio of lengths, while the argument of sin is an angle. But since both arguments are dimensionless numbers, it may be better to avoid this ambiguity entirely. I will use $\sin^{-1}$ only to reduce clutter in very messy equations, and then only (I hope) to represent 1/sin.

The following reciprocal functions are less commonly used:
cotangent: $\cot(\theta) = 1/\tan(\theta) = x / y$
secant: $\sec(\theta) = 1/\cos(\theta) = r / x$
cosecant: $\csc(\theta) = 1/\sin(\theta) = r / y$
In all equations, angles must be in units of *radians*, with 2π radians equal to 360 degrees.

For the triangle in Figure 1-3, the **Pythagorean theorem** states:
$r^2 = x^2 + y^2$
With the above definitions, we replace x with $r\cos(\theta)$ and y with $r\sin(\theta)$, yielding the very important equation:
$1 = \cos^2\theta + \sin^2\theta$

Laws of Sines & Cosines

Figure 1-5 shows a triangle whose sides have lengths A, B, and C, and whose opposite angles are a, b, and c, respectively.

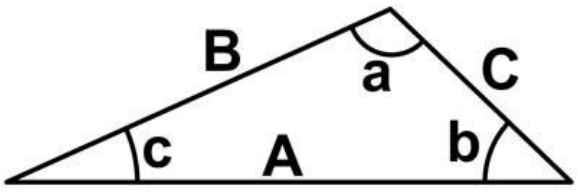


Figure 1-5 Triangle with Sides A, B, C

For any triangle:
$a + b + c = \pi$ radians = 180 degrees
Law of Sines: $A/\sin(a) = B/\sin(b) = C/\sin(c)$
Law of Cosines: $A^2 = B^2 + C^2 - 2BC\cos(a)$

Chapter 2

Coordinate Systems

Coordinates systems are human conventions that facilitate quantifying angles and distances. Nature has no inherent coordinate system. Hence we are free to choose whatever coordinates seem most convenient. Often the best choice in any particular situation is one that matches a natural symmetry.

The simplest coordinate system has only one dimension. For example, an object falling straight down in Earth's gravity can be described with only one dimension: let's call it height h. Figure 2-1 shows the *h-axis* pointing straight upward, with *tic-marks* indicating various values of h. A black ball is shown at h=3.

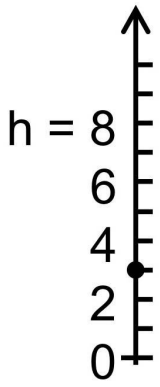


Figure 2-1 Ball at h=3 on h-axis

With this coordinate system, we can follow the ball's motion as h changes over time.

Now imagine a basketball player throwing a ball, hoping it goes through the hoop. Here, the motion is two-dimensional. As shown in Figure 2-2, Y is the vertical axis and X is the horizontal axis. This is called a *rectilinear* coordinate system because the axes are orthogonal to one another.

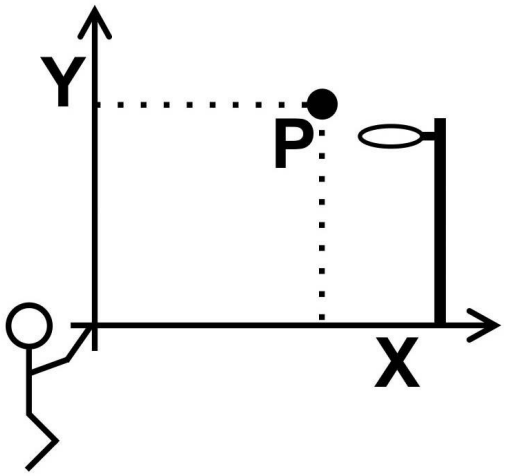


Figure 2-2 (X,Y) Coordinates in 2-D

Any point P in this 2-D space is uniquely specified by how far up P is in the Y-direction, and how far to the right P is in the X-direction. We can choose the *origin* of our coordinate system, the point with coordinates (0,0), to be anywhere we wish. Here, we choose the origin to be the point at which the ball was released. At any particular instant in time, the ball has height Y, horizontal distance X, and coordinates (X,Y).

A quite different situation is the motion of a lone planet around a star. This occurs in three dimensions of course, but due to spherical symmetry, the planet orbits entirely within a single plane. This allows us to analyze its motion in two dimensions. The most convenient approach employs *polar coordinates*, as illustrated in Figure 2-3.

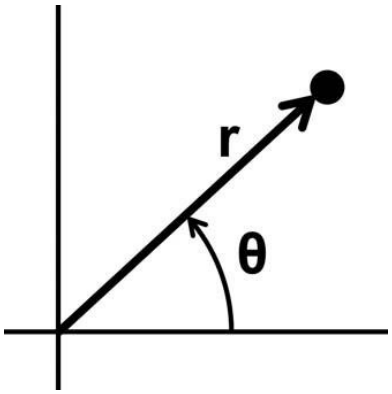


Figure 2-3 Polar Coordinates (r,θ)

Here, r is the ball's distance from the origin, the length of the *radial line* from the origin to the ball, and θ is the angle between the radial line and a chosen reference direction. Typically, θ ranges from 0 to 2π radians, although in verbal descriptions we often say 0 to 360 degrees. The most common reference

direction is the horizontal axis, as shown in Figure 2-3, and the most common convention is to measure the angle in the counterclockwise direction. Those choices are arbitrary, as is the choice of the origin.

If we know a ball's location (r,θ) in 2-D polar coordinates, we can calculate its location (x,y) in 2-D rectilinear coordinates, or we can do the reverse. The conversion equations are:

$x = r \cos\theta$

$y = r \sin\theta$

$r = \sqrt{(x^2 + y^2)}$

$\tan\theta = y / x$

Becoming Three-Dimensional

The screens of computers and eBook readers are two-dimensional, with pixels arranged in horizontal rows and vertical columns. A three-dimensional coordinate system adds depth. It has three independent directions. In physics, we most commonly employ a 3-D rectilinear system with three mutually orthogonal coordinate directions labeled x, y, and z, as illustrated in Figure 2-4. Here x is the horizontal right-left direction; y the vertical up-down direction, and z is the in-out direction perpendicular to the screen. This is often called a *Cartesian* coordinate system.

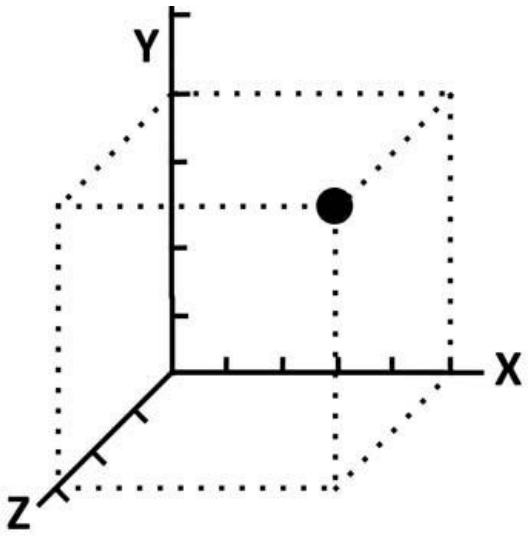


Figure 2-4 Rectilinear 3-D Coordinates

The coordinates of the black ball are its distance from three plane surfaces: x is the distance to the right of the x=0 plane; y is the distance above the y=0 plane; and z is the distance out from the z=0 plane (the screen). In this case, counting

tic-marks along each axis, we have: x=5, y=4, and z=3.

Figure 2-5 shows a 3-D *cylindrical* coordinate system, which might be useful in analyzing the tip of a corkscrew as it cuts into a cork. The three coordinates are: height z; radial distance from centerline r; and *azimuthal angle* ø.

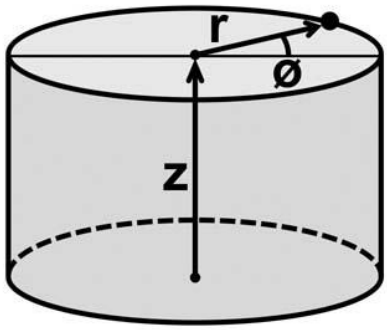


Figure 2-5 3-D Cylindrical Coordinates

The tip of the corkscrew has a constant r, an increasing ø, and a decreasing z. The ø and z change at rates whose ratio is determined by the pitch of the corkscrew. Again, ø ranges from 0 to 2π radians.

The coordinate transformations between 3-D rectilinear and 3-D cylindrical are:

$x = r \cos\theta$

$y = r \sin\theta$

$z = z$

$r = \sqrt{(x^2 + y^2)}$

$\tan\theta = y / x$

Another commonly used 3-D coordinate system is *spherical coordinates*, illustrated in Figure 2-6. The three coordinates are: the radial distance from the origin r; the *polar angle* θ; and the *azimuthal angle* ø.

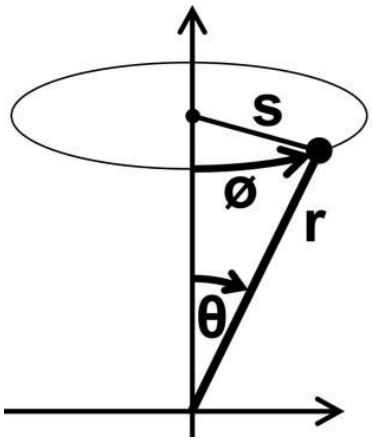


Figure 2-6 Spherical Coordinates

Here, ø ranges from 0 to 2π radians, as before, while θ ranges from 0 to π radians (0 to 180 degrees). Also shown is s, the radius of the indicated circle, whose length is rsinθ.

The coordinate transformations between 3-D rectilinear and 3-D spherical are:
$x = r \sin\theta \cos\phi$
$y = r \sin\theta \sin\phi$
$z = r \cos\theta$
$r = \sqrt{(x^2 + y^2 + z^2)}$
$\cos\theta = z / r$
$\tan\phi = y / x$
A modified version of spherical coordinates is routinely used to specify the position of objects near Earth's surface. In this case: θ is the angle between the equator and the object of interest, and corresponds to latitude; ø is measured relative to the Prime Meridian and corresponds to longitude; and (r minus Earth's radius) corresponds to elevation above sea level.

4-D Spacetime

Einstein proved the true geometry of our world is four dimensional: the customary three spatial dimensions plus the dimension of time.

Many authors consider time to be the zeroth coordinate and write the location of a *spacetime event* as (ct, x, y, z). Most adopt units in which c, the speed of light, equals 1, and write (t, x, y, z). It was once fashionable to define the time axis as ict, where $i=\sqrt{(-1)}$. The latter was motivated by the fact that the "distance" d between two events in 4-D spacetime is given by:

$$d^2 = -c^2(t_2-t_1)^2 + (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2$$

The minus sign on the time differences is required to ensure the speed of light is the same in all reference frames. Multiplying time values by ic automatically provides the $-c^2$ factor in the distance equation. The modern approach assigns such factors to a metric, as we explore in Chapter 16.

Chapter 3

Numbers

Science strives to understand the fundamental principles that govern nature. Essential to that effort are quantitative observations of natural phenomena, for which we must use and understand numbers.

The simplest numbers, *natural numbers*, are the positive integers: 1, 2, 3, … . Mankind employed natural numbers long before the beginning of recorded history; indeed it would be hard to record history without numbers.

In perhaps the fifth century B.C., some societies fully developed the concept of *zero*. That sufficed for most human needs until the advent of credit card debt, which requires *negative numbers*. Debt makes even zero look good.

Commerce spurred an understanding of *fractions*, ultimately opening our eyes to *rational numbers*, those expressible as the ratio of two integers, such as 4/5ths. This was followed by the introduction of *irrational numbers*, such as e and π, which are not equal to the ratio of any pair of integers and whose decimal expansions go on to infinity. Together the rational and irrational numbers comprise the set of *real numbers*.

Real numbers seemed sufficient: every arithmetic operation — addition, subtraction, multiplication, and division — performed on any pair of real numbers resulted in another real number. That is until someone tried to calculate $\sqrt{(-1)}$, the square root of –1. Imagine that.

Complex Numbers

Imaginary numbers mirror real numbers: there is a one-to-one correspondence between each real number x and its imaginary counterpart $x\sqrt{(-1)}$, which we write ix.

Combining imaginary numbers with real numbers yields the set of *complex*

numbers, each of which can be expressed as: x+iy, where x and y are real numbers.

What's next? Nothing, it seems. All known arithmetic operations performed on any combination of complex numbers yield another complex number. Arithmetically, the complex numbers form a *closed set*.

The most convenient way to represent and manipulate complex numbers employs 2-D polar coordinates and exponentials. Figure 3-1 shows the polar and rectilinear representations of the complex number z.

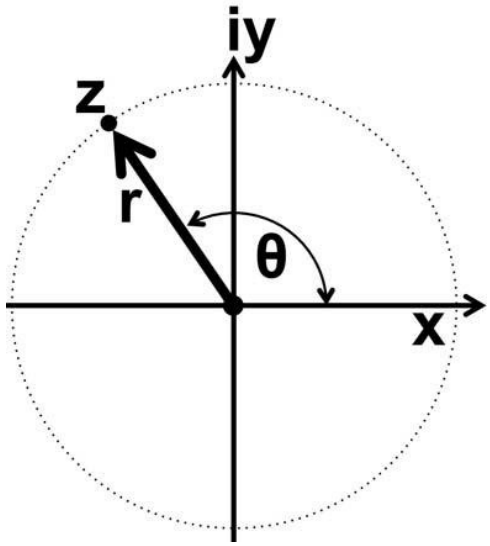


Figure 3-1 Complex Number Z

For z=x+iy, z's rectilinear coordinates are (x,y), while z's polar coordinates are (r,θ), where:

$r = \sqrt{(x^2+y^2)}$

$\theta = \arctan(y/x)$

$x = r\cos\theta$

$y = r\sin\theta$

The *complex conjugate* of z, written z*, equals x–iy. To get the complex conjugate of any complex expression, simply replace every "i" with "–i".

Figure 3-2 shows the evolution of the complex number z=(r,θ) as θ goes from 0 to 2π radians. The upper line of images is the polar plot of z for θ=0, π/2, π, and 3π/2 radians.

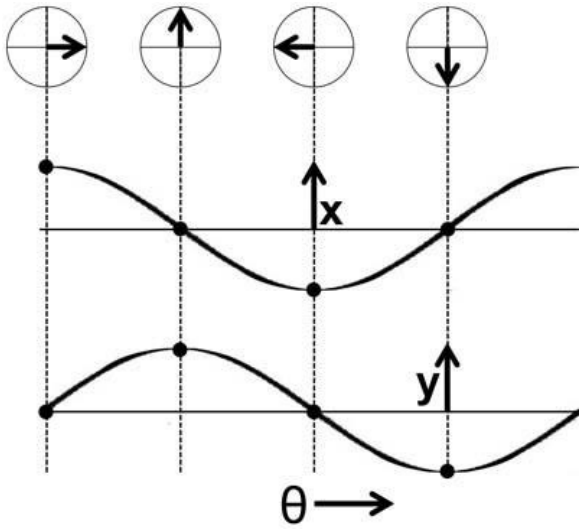


Figure 3-2 Z, X and Y vs. θ

The middle portion of Figure 3-2 graphs the x-coordinate of z, while the lower portion graphs its y-coordinate.

As we will discuss is Chapter 7, complex numbers can be represented by exponentials with imaginary exponents.

$z = r\, e^{i\theta} = r\cos\theta + i\, r \sin\theta$

One great advantage of $re^{i\theta}$ is that it is very easy to differentiate and integrate, as we will learn in the chapters on calculus. Simplifying the math of physics is always a good thing.

Scientific Notation

The following are three different ways of writing the number one hundred and twenty three:

$123 = 1.23\times10^{2} = 1.23E{+}2$

The first is the common English format. The second is called **scientific notation**, and the third is a standard computer notation. In the second and third forms, any number of digits may follow the decimal point, but only one digit should precede it. The exponent of 10, 2 in this case, may have any number of digits, and may be positive or negative. An example of a negative exponent is:

$0.000{,}0123 = 1.23\times10^{-5} = 1.23E{-}5$

Scientific and computer notations are convenient for expressing very large and very small quantities.

Ideally, any number quoted for scientific purposes should be accompanied by an explicit uncertainty, such as:

123 ± 4
Unless otherwise specified, this means the true value is:
68.3% likely to be between 119 and 127
95.5% likely to be between 115 and 131
99.7% likely to be between 111 and 135…
This assumes a Gaussian distribution with mean 123 and standard deviation 4, as we will explore comprehensively in Chapter 9.

Often, numbers are quoted without explicit uncertainties. In such cases, we presume a standard scientific convention that numbers are rounded off to their least significant digits. Thus a quantity quoted as being 1200 is presumed to lie between 1150 and 1249; this presumes the two zeros serve only to specify the quantity's magnitude. Conversely, a quantity quoted as being 1203 is presumed to lie between 1202.50 and 1203.49.

Thus for integers, we presume the uncertainty equals one-half of a 1 in the least significant nonzero digit. Hence:
1200 means 1200 ± 50
1203 means 1203 ± 0.5
If you wish to specify a quantity that you are confident lies between 1199.50 and 1200.49, you should write: 1200±0.5. Otherwise people will assume the uncertainty is 100 times larger.

When a decimal fraction is specified, we presume the uncertainty equals one-half of a 1 in the least significant digit, whether it is zero or nonzero. Thus:
1.20 means 1.20 ± 0.005
1.2 means 1.2 ± 0.05

Chapter 4

Advanced Algebra

This is a good time to introduce these symbols:
A<B, means A is less than B
A>B, means A is greater than B
A<<B, means A is much less than B
A>>B, means A is much greater than B.
A<=B, means A is less than or equal to B.
A>=B, means A is greater than or equal to B.

Absolute Value / Magnitude

Often the sign of a quantity is less important than its magnitude. In such cases, we may be interested in its *absolute value*, its unsigned magnitude, denoted by the symbols | |..
The absolute values of any real (not complex or imaginary) quantity x is:
$|x| = +x$, if $x >= 0$
$|x| = -x$, if $x < 0$
The absolute values of any complex quantity z, equal to x+iy, is:
$|z| = +\sqrt{(x^2+y^2)}$
For any Q, |Q| is never negative.

We also use the symbols | | to denote the magnitude of a vector, as we will discuss in Chapter 11.

Factorials

The symbol n!, read “n-factorial”, equals the product of the first n integers.
$n! = 1 \times 2 \times 3 \times 4 \times \ldots \times n$
For convenience, we define 0! = 1.
For large n, we can approximate n! by:
$n! \sim n^n \exp\{-n\} \sqrt{(2\pi n)}$
This approximation is off by only 0.7% for n=12, and becomes more accurate as n increases.

Polynomials

Polynomials are sums of *terms*, with each term comprised of a constant multiplied by an independent variable raised to an integral power. The general form is:

$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$

Polynomials are ubiquitous in physics equations, and are easily differentiated and integrated, as we will discuss when we get to calculus. Polynomial expressions and equations may not contain square roots, logarithms, trig functions, or other complex mathematical entities.

A polynomial is said to be *nth-order* if n is the highest power of the independent variable in the sum. In the prior equation, the constants a_0 through a_{n-1} can have any values, but a_n cannot be zero if this is an nth-order polynomial.

Every nth-order polynomial has n *roots*, n values of the independent variable at which the polynomial equals zero. The roots need not all be different. This means we can *factor* any nth-order polynomial, restating it as the product of n terms, each of which has the form: (x–root). Consider an example:

$0 = x^4 + x^3 - 6x^2$

The roots are: x = 0, 0, 2, and –3. The factored equation is:

$0 = (x)(x)(x-2)(x+3)$

Unfortunately, there is no general procedure for finding the roots or factors of an arbitrary polynomial. One must rely on experience, informed guesswork, and brute force arithmetic.

Quadratic Equations

A *quadratic equation* is a second-order polynomial equation. The general form is:

$0 = ax^2 + bx + c$

$0 = x^2 + (b/a)x + c/a$

Here, b and c can have any values, but a cannot be zero. Being a second order polynomial, every quadratic equation has two roots, which we define as:

Roots: $x = \lambda+\beta$, and $x = \lambda-\beta$

With this definition, the quadratic equation becomes:

$0 = [x - (\lambda+\beta)][x - (\lambda-\beta)]$

$0 = x^2 - 2\lambda x + \lambda^2 - \beta^2$

Matching the last equation with the general form yields:

$b/a = -2\lambda$

$\lambda = -b/2a$
$c/a = \lambda^2 - \beta^2$
$c/a = b^2/4a^2 - \beta^2$
$\beta^2 = b^2/4a^2 - c/a$
Hence the two solutions to the quadratic equation are:
$x = \lambda \pm \beta$
$x = \{-b \pm \sqrt{(b^2 - 4ac)}\} / 2a$

Circular Orbit Condition

We wish to find the acceleration required to maintain a small satellite in a circular orbit around a much more massive body. Our result will apply equally to any body orbiting a central isotropic force. Figure 4-1 shows an *arc*, a portion of a circle of radius R that subtends angle θ at the circle's center. The three lines R, R–d, and b form a right triangle, as do the three lines b, d, and s. Figure 4-2 contains an enlarged image of the upper portion of Figure 4-1.

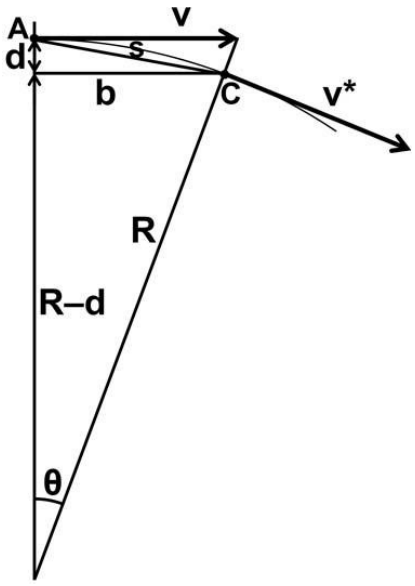


Figure 4-1 Portion of Circular Orbit

In a time interval Δt, the satellite moves from point A to point C, and its velocity changes from ***v*** to ***v****. In a circular orbit, the satellite's velocity has constant magnitude, but its direction of motion is continuously changing.

Applying the Pythagorean theorem to right triangle R-b-(R–d) yields:
$R^2 = b^2 + (R-d)^2$
$b^2 = R^2 - (R-d)^2$
$b^2 = 2Rd - d^2$

$b^2 = d\,(2R - d)$

This is a useful relationship. You might want to memorize it. I find it is easier, however, to remember how such important equations are derived. By re-deriving them each time, I reduce the chance of forgetting a factor of 2, *etc.*

In the lower image of Figure 4-2, we have moved ***v*** and ***v**** to a common vertex without changing the orientation of either vector. The angle between ***v*** and ***v**** is θ; it must be the same as the angle through which the satellite turns in Figure 4-1. Here, ***Δv*** equals the change in ***v*** during time Δt.

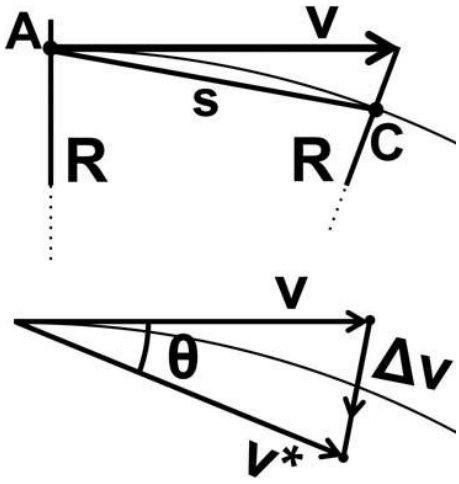


Figure 4-2 Enlarged Orbit Details

In the upper portion of this figure, s is the line that joins points A and C. Comparing Figures 4-1 and 4-2, we can identify two similar isosceles triangles: R-s-R and v-Δv-v*.

In both isosceles triangles, the angle between the two equal sides is θ. This means:

$\Delta v\, v = s\, R$

In the upper image of Figure 4-2, the satellite follows the circular arc, which is slightly longer than the straight line s. In fact, the arc length is Rθ, and the length of s equals $2R\sin(\theta/2)$. But for extremely small Δt, these lengths are nearly identical. In that limit:

$s = v\,\Delta t$

$\Delta v\, v = v\,\Delta t\, R$

$a = \Delta v / \Delta t = v^2 / R$

Here, a is the acceleration required to keep the satellite in a circular orbit of radius R.

Chapter 5

Dimensional Analysis

Units of Measure

This is a good time to talk about equations and units of measure. Quantities of interest in physics commonly are expressed as some number of units appropriate to that type of quantity. For example, distance can be stated as being X microns, Y meters, or Z light-years; each of those units is a standard distance, although their sizes are very different. Clearly, each is appropriate in quite different circumstances.

Time is typically measured in seconds or years. Electric charge is typically measured in coulombs or multiples of the charge of a single proton. Mass is typically measured in grams or kilograms; pound is a poor choice, because that word is used for both mass and force.

Velocity, which is distance moved divided by time to move, could be stated as X microns per nanosecond, Y meters per second, or Z light-years per year. In the latter case, we normally expect Z to be less than or equal to 1.

No choice of units is wrong per se. You could even use miles per month, or fathoms per fortnight. But if you follow standard conventions, you will probably minimize mistakes and facilitate communication with others in your field.

The set of units that scientists most commonly use has long been called the mks system, which stands for meter-kilogram-second. This system of units has evolved into the *Système International d'unités* metric system, which is abbreviated SI. Most fields of science define additional units appropriate to their unique needs.

SI Units

Seven *Base Units* are considered the most fundamental. These are:

- m: meter, unit of length
- kg: kilogram, unit of mass
- s: second, unit of time

- A: ampere, unit of electric current
- K: Kelvin, unit of temperature on absolute scale
- mole: number of atoms in 12 grams of carbon-12
- cd: candela, unit of luminosity

Additionally, the following are *Derived Units*:

- Hz: Hertz, unit of frequency, 1 cycle per second
- N: newton, unit of force, 1 N = 1 kg m / s^2
- Pa: pascal, unit of pressure, 1 Pa = 1 N / m^2
- J: joule, unit of energy, 1 J = 1 N m
- W: watt, unit of power, 1 W = 1 J / s
- C: coulomb, unit of charge, 1 C = 1 A s
- V: volt, unit of electric field strength, 1 V = 1 W / A
- F: farad, unit of capacitance, 1 F = 1 C / V
- Ω: ohm, unit of electrical resistance, 1 Ω = 1 V / A
- Wb: weber, unit of magnetic flux, 1 Wb = 1 V s
- T: tesla, unit of magnetic field strength, 1 T = 1 Wb / m^2
- H: henry, unit of electrical inductance, 1 H = 1 Wb / A

Factors of Ten

The key advantage of the metric system is that all units scale by factors of ten — none of this 12 inches to 1 foot, 3 feet to 1 yard, 1760 yards to 1 mile, and all that nonsense. While one can always write 10^3 m or 10^{-9} m, there are standard

prefixes for various powers of ten. The same prefixes apply to the names of all units, and the same prefix symbols apply to the symbols for all units. The more commonly used prefixes are:

P: 10^{+15}: peta, as in Pm, petameter
T: 10^{+12}: tera, as in Ts, terasecond
G: 10^{+9}: giga, as in Gcd, gigacandela
M: 10^{+6}: mega, as in MK, megakelvin
k: 10^{+3}: kilo, as in kg, kilogram

c: 10^{-2}: centi, as in cm, centimeter
m: 10^{-3}: milli, as in mm, millimeter
µ: 10^{-6}: micro, as in µm, micrometer or micron
n: 10^{-9}: nano, as in ns, nanosecond
p: 10^{-12}: pico, as in pA, picoamperes
f: 10^{-15}: femto, as in fm, fentometer
a: 10^{-18}: atto, as in as, attosecond

Natural Units

Generally, we believe the choice of units and coordinate systems are entirely arbitrary: we are free to choose whatever we please, because nature has no intrinsic units or coordinates.
However, there may be some exceptions, some instances in which nature specifies the “right” choice.
One example is the speed of light. Einstein’s theory of special relativity, the most precisely and extensively validated concept of all human thought, insists that the speed of light is invariant, that it has the same value c in every circumstance, in every location, and at every moment in time. The speed c is a fundamental property of our universe that establishes a natural unit of velocity.
Max Planck also found combinations of fundamental constants that have dimensions of length, time, and mass. These are:
Planck length: $\sqrt{(G\hbar/c^3)} = 1.6162\times10^{-35}$ m

Planck time: $\sqrt{(G\hbar/c^5)} = 5.3911\times10^{-44}$ s

Planck mass: $\sqrt{(\hbar c/G)} = 2.1765\times10^{-8}$ kg
Here, G sets the scale of gravitational phenomena, ħ sets the scale of quantum

phenomena, and c sets the scale of relativistic phenomena.
These so-called *natural units* are completely independent of people. They have nothing to do with the length of our feet, or our planet's size, day, or year.
While philosophically intriguing, no one has yet found a significant physical role for the Planck units. Some cosmologists propose that space and time are quantized on the Planck scale, but that research is still in its infancy.

Matching Units

Checking an equation's units is a great way to gain insight and minimize mistakes. This is generally called *dimensional analysis*. In every valid equation, both sides of the equal sign **must** have **exactly** the same units.
Consider a simple example:
$s = g\,t^2 / 2$
The left side of this equation is a snap; we know that s is a distance, so measuring it in meters makes sense. On the right side, the units are $(m/sec^2) \times (time)^2$. If we measure time in seconds, the units on the right side will be meters, just like the left side. If the right side had t instead of t^2, the units on the right would be $(m/sec^2) \times (sec) = m/sec$, which does not match the left side and is therefore wrong.
There is no partial credit for units that are only slightly wrong; only a perfect match is acceptable. Note that we could choose to measure time in minutes. But, that would give the wrong answer: $1g = 9.8\ m/sec^2 = 35{,}280\ m/min^2$, which is a very different number. The units must match *exactly* to get the right answer.
Units also tell us something about the physics: m/sec^2 means meters are changing with the square of time — that is fast.

Particle Physics Units

For brevity, Feynman adopts a convention normally employed by particle physicists: distance and time are measured in units that make the speed of light c equal to 1. One can do this by measuring time in seconds and distance in light-seconds (the distance light travels in one second). Astronomers prefer to measure time in years and distances in light-years. Some experimental high-energy physicists measure time in nanoseconds and distance in feet. Either way, c=1.
Feynman sometimes adopts c=1, but not consistently. You can restore missing c's in his electromagnetic equations by:
replacing each "t" with "ct"
replacing each "v" with "v/c"
replacing each "E" with "E/c"

replacing each “ø” with “ø/c”
replacing each “ρ” with “cρ”
After that, check the units on both sides of each equation and add c’s as necessary to dimensionally balance the equation.
Let me demonstrate this with an example. Consider an equation relating the magnetic field to changes in the electric field (assuming no electric currents). For now, don’t worry about what this equation means; just think about the units.
$\Delta B_z/\Delta y - \Delta B_y/\Delta z = \Delta E_x/\Delta t$ (with c=1)
Here the notation $\Delta Q/\Delta P$ means the small change in Q due to a small change in P.
Using the replacement list above, the equation becomes:
$\Delta B_z/\Delta y - \Delta B_y/\Delta z = \Delta(E_x/c)/c\Delta t$
$c^2 (\Delta B_z/\Delta y - \Delta B_y/\Delta z) = \Delta E_x/\Delta t$
The replacements correctly restored the missing c’s.
Alternatively, if you forget the replacement list, you can go directly to dimensional analysis. From the Lorentz force:
$F_x = q(E_x + v_y B_z - v_z B_y)$
We know that the units of E must be the same as the units of vB; let’s write that [E]=[vB].
In our original equation, the units are:
$\Delta B_z/\Delta y - \Delta B_y/\Delta z = \Delta E_x/\Delta t$
[B] *[x] = [E]* [t]
The units on the left are magnetic field B divided by distance. The units on the right are electric field E divided by time t. Substituting [E]=[vB] yields:
$[B] = [x] [vB] / [t] = [v]^2 [B]$
The units of distance [x] divided by time [t] are the units of velocity [v]. To balance this equation, we must multiply the left side by a velocity squared. Since we left out the c’s, this means the restored equation is:
$c^2 (\Delta B_z/\Delta y - \Delta B_y/\Delta z) = \Delta E_x/\Delta t$
Try this on a few equations that you already know. With a little practice, you will find that dimensional analysis is easy and highly effective.
If you want to be doubly sure, keep all the c’s, **and** do dimensional analysis on your results.

Chapter 6

Infinite Series

Zeno's Paradox

The most famous and perhaps oldest example of an infinite series is *Zeno's paradox*. In the fifth century B.C., the Greek philosopher Zeno proposed a seemingly unresolvable contradiction about a hypothetical race between the great Achilles and a tortoise that I will call Yertle.

Zeno said: if Yertle had a head start when the race began, Achilles would *never* catch up. His logic was:

1. By the time Achilles reaches Yertle's starting point, Yertle has crept ahead by some distance.

2. By the time Achilles reaches Yertle's new position, Yertle has crept slightly further yet.

3. This sequence of Achilles reaching where Yertle was, while Yertle advances further, repeats forever.

4. No matter how far Achilles runs, Yertle will always be slightly ahead; hence, Achilles never catches up.

Zeno implicitly assumed that an infinite number of repetitions of step (2) would require an infinite amount of time, so Achilles could never catch up in a finite amount of time.

But we all know that if Achilles runs faster than Yertle, he will eventually catch Yertle. Therefore, Zeno's logic must be wrong, as was obvious to everyone, including Zeno.

The ancient Greeks, despite their great wisdom in many fields, could not solve this paradox.

But we can.

First, let's show that an infinite series can have a finite sum. Consider the series:
$1 + 1/10 + 1/100 + 1/1000 = 1.111$
No matter how many terms are added on the left side, and how many 1's are added to the decimal expansion on the right side, both sides will always be less than 1.12. In fact, as the number of terms increases without limit, the sum comes closer and closer to 10/9.

Back to Achilles. Define Yertle's initial head start to be a distance of 1 unit. Let Achilles' speed be v_{ach}, Yertle's speed be v_{yer}, and $x=v_{yer}/v_{ach}$. We assume $x<1$. Whenever Achilles runs a distance D, Yertle creeps a distance xD.

In "stage 1", Achilles runs the distance 1, while Yertle creeps a distance x. In "stage 2", Achilles runs that distance x, while Yertle creeps distance x^2. This cycle continues indefinitely. The total distance Achilles runs during four stages is:
$D = 1 + x + x^2 + x^3$
In four stages, Yertle creeps xD. Including its head start, Yertle's distance from Achilles' starting point is:
$xD + 1 = 1 + x + x^2 + x^3 + x^4$
Hence after four stages, Yertle is ahead by a distance x^4. During each subsequent stage, an additional term with one higher power of x is added to both sums, and Yertle's lead becomes smaller. After an infinite number of stages, Achilles and Yertle are neck and neck.

Define S to be the sum after an infinite number of stages.
$S = 1 + x + x^2 + x^3 + \ldots$
The "…" means "continue to infinity". Clearly, if x equals 1 or more, the series will have an infinite sum. But $x>1$ would mean Yertle is faster than Achilles, which is wrong. For $0<x<1$, we calculate S as follows:
$S = 1 + x + x^2 + x^3 + \ldots$
$xS = x + x^2 + x^3 + x^4 + \ldots$
$S–1= x + x^2 + x^3 + x^4 + \ldots$
$S –1 = xS$
$S – xS = 1$
$S = 1 / (1 – x) = v_{ach} / (v_{ach} – v_{yer})$

Just like that, we have added an infinite number of terms, and proven:
$1 + x + x^2 + x^3 + \ldots = 1/(1 - x)$
Now let's plug in some numbers for $x=v_{yer}/v_{ach}$.
for x = 0.9, S = 1 / 0.1 = 10
for x = 0.5, S = 1 / 0.5 = 2
for x = 0.2, S = 1 / 0.8 = 1.25
for x = 0.1, S = 1 / 0.9 = 1.11…
For any x<1, S is finite. The total time t required for Achilles to reach S is simply S divided by Achilles' speed: $t=S/v_{ach}$. Since S is finite, so is t. At time t, Achilles has run a total distance S, Yertle has crept a distance xS=S–1. So, with Yertle's head start, Achilles catches up at time t.

We solved the problem with an infinite sum in the manner that Zeno presented his paradox.

Now, let's examine a simpler solution, one using an approach that we often employ in relativity problems. The initial separation is 1, and the *closing velocity*, the rate at which that separation decreases, equals the velocity difference: $v_{ach}–v_{yer}$. Hence the separation becomes zero at time $t=1/(v_{ach}–v_{yer})$. In time t, Achilles runs a distance:
$S = t\, v_{ach} = v_{ach} / (v_{ach}–v_{yer})$
just as we found above.

Finite Series

We can also calculate the sum of a partial series.
For $|x|<1$, let $Q = 1 + x + x^2 + x^3 + \ldots + x^{n-1}$
We can think of Q being the difference between two infinite sums of powers of x. In the first sum, the lowest power of x is zero, while in the second sum, the lowest power of x is n. We write this:
$Q = 1 + x + x^2 + x^3 + \ldots$
$\quad – (x^n + x^{n+1} + x^{n+2} + \ldots)$
$Q = 1 + x + x^2 + x^3 + \ldots$
$\quad –x^n(1 + x + x^2 + x^3 + \ldots)$
$Q = 1/(1–x) – x^n/(1–x)$
$Q = (1 – x^n) / (1 – x)$
We can do this more elegantly, using $\Sigma_{k=a}^{b} \{Y\}$ to denote the sum of {Y} from k=a to k=b.

$Q = \Sigma_{k=0}^{n-1} \{x^k\}$
$Q = \Sigma_{k=0}^{\infty} \{x^k\} - \Sigma_{k=n}^{\infty} \{x^k\}$
$Q = \Sigma_{k=0}^{\infty} \{x^k\} - x^n \Sigma_{k=0}^{\infty} \{x^k\}$
$Q = (1 - x^n) / (1 - x)$

Monthly Loan Payments

Installment payment loans, such as car loans and home mortgages, are features of modern life. You borrow principal amount P at annual interest rate i, and make n monthly payments of M. To be clear, if the annual nominal (uncompounded) interest rate is 6%, i=0.06 and the monthly interest rate is i/12=0.005.

At the end of month #1, the amount you owe equals principal P, plus interest iP/12, minus your first payment M, which equals P(1+i/12)–M.

Define x=1+i/12 to reduce clutter, making the month #1 ending balance Px–M.

At the end of month #7, the amount you owe equals (amount owed at the end of month #6) multiplied by x and then reduced by M. Each month, we multiply the prior balance by x and then subtract M.

The month ending balances are:
Month 1: $Px - M$
Month 2: $Px^2 - Mx - M$
Month 3: $Px^3 - Mx^2 - Mx - M$
Month 4: $Px^4 - Mx^3 - Mx^2 - Mx - M$
Month k: $Px^k - M(x^{k-1} + x^{k-2} + \ldots + x + 1)$
The balance at the end of month #n, when the loan is paid off, is zero. So:
$0 = Px^n - M(x^{n-1} + x^{n-2} + \ldots + x + 1)$
We do not know how to calculate this partial sum directly, because x>1. So, we use a little trick. Define y=1/x, and multiply everything by y^n, which is the same as dividing everything by x^n.
$P = M (y + y^2 + y^3 + \ldots + y^n)$
$P = M\, y (1 + y + y^2 + \ldots + y^{n-1})$
Now, since y<1, we can use the sum calculated above for this partial series:
$P = M\, y (1 - y^n) / (1 - y)$
Now multiply the right side numerator and denominator by x^{n+1}, and then solve

for M:
$P = M (x^n - 1) / x^n (x - 1)$
$M = P (i/12) x^n / (x^n - 1)$

Sums of Integers Squared

Let's try some more interesting sums. Consider the sum of the squares of all integers up to n:
$S_n = 1 + 2^2 + 3^2 + \ldots + n^2 = \Sigma_{j=1}^{n} \{j^2\}$
For a few values of n, S_n is:
$S_1 = 1$
$S_2 = 1 + 4 = 5$
$S_3 = 1 + 4 + 9 = 14$
$S_4 = 1 + 4 + 9 + 16 = 30$
We seek an equation for S_n that is valid for all n. A clue comes from looking for prime factors. We find primes in S_2 (5) and S_3 (7), both of which equal (2n+1). If we factor (2n+1) out of each S_n, we can rewrite all these sums in a consistent form.
$S_1 = 1 = (2n+1) [1] /3$
$S_2 = 5 = (2n+1) [3] /3$
$S_3 = 14 = (2n+1) [6] /3$
$S_4 = 30 = (2n+1) [10] /3$

For each n, the numbers in [ ]'s are the sums of the integers from 1 to n; each equals n(n+1)/2. We therefore have:
$S_n = (2n+1) n (n+1) /6$
That works for n up to 4. To prove it for all n, we use the principle of *mathematical induction*.
To use mathematical induction to prove that a statement Ω is true for all integers, we must prove two things:

1. Ω is true for some integer n; and
2. Ω being true for any n proves Ω is true for n+1.

Let's see how this works. We showed above that Sn is the true sum for n=1 to n=4. Let's evaluate S for n+1:
$S_{n+1} = (n+1)(n+2)(2n+3)/6$
$S_{n+1} = (n+1)\{ 2n^2 + 7n + 6 \}/6$
$S_{n+1} = (n+1)\{ (2n^2+n) + 6(n+1) \}/6$

$S_{n+1} = (n+1)\{ n(2n+1) \}/6 + (n+1)(n+1)$
$S_{n+1} = S_n + (n+1)^2$
Hence, since S_n is correct for n=4, it is correct for n=5. Since S_n is correct for n=5, it is correct for n=6…and so on ad infinitum.
QED (Quod Erat Demonstrandum — Latin for "High-5")

Next consider the sum of the squares of even integers:
$E_n = 2^2 + 4^2 + 6^2 + \ldots + n^2$
$E_n = 4\{ 1^2 + 2^2 + 3^2 + \ldots + (n/2)^2 \}$
$E_n = 4\, S_{n/2}$
$E_n = 4\,(n/2)(1+n/2)(n+1)/6$
$E_n = n(n+1)(n+2)/6$
Next consider the sum of the squares of odd integers:
$O_n = 1^2 + 3^2 + 5^2 + \ldots + n^2$
This is the sum of all squares minus the sum of even squares.
$= \{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots + n^2\}$
$- \{2^2 + 4^2 + 6^2 + \ldots + (n-1)^2 \}$
$O_n = S_n - E_{n-1}$
$6O_n = n(n+1)(2n+1) - (n-1)(n)(n+1)$
$6O_n = n(n+1)\{ (2n+1) - (n-1) \}$
$O_n = n(n+1)(n+2)/6$
Perhaps surprisingly, $O_n = E_n$ — the sum of odd squares and the sum of even squares have the same equation, but of course with different values of n.

Bessel Functions

Bessel functions often arise in problems involving cylindrical symmetry. *Bessel functions of the first kind* are denoted $J_n(x)$. These are also called *cylindrical harmonics*. For integer, non-negative n, J_n can be written as a power series:
$J_n(2x) = \Sigma_{k=0}^{\infty} \{ (-1)^k x^{2k+n} / k!\,(k+n)! \}$
$J_{-n}(y) = (-1)^n J_n(y)$
For n = +1/2, $J_n(y) = \sin(y)\,\sqrt{(2/\pi y)}$
For n = –1/2, $J_n(y) = \cos(y)\,\sqrt{(2/\pi y)}$
For n = 0, $J_0(2x) = 1 - x^2/(1!)^2 + x^4/(2!)^2 - x^6/(3!)^2 + \ldots$

Chapter 7

Exponentials

Most physics equations define how things *change*.

For example, the laws of nature do not specify an object's velocity, but rather how rapidly that velocity changes. Each second, the velocity of a ball falling from the *Leaning Tower of Pisa* increases by a fixed amount: 9.8 meters per second.

Other changes are *proportional*. Each second, the population of bacteria in a Petri dish might increase by a fixed *multiple* of the population, such as 1%. As the population grows, it changes by an ever-increasing amount — this is an *exponential* phenomenon.

Proportionality is the essential characteristic of an exponential: each exponential function f(t) always changes by the same percentage for a given change in t.

Proportional change describes many diverse natural phenomena. A bacterial population might double in one hour, exhibiting exponential growth. Conversely, the number of surviving radioactive particles might decrease by 50% in one nanosecond, exhibiting exponential decay.

Exponentials are normally written:
b^u
Here, the *base* b is raised to the u *power*, with the exponent u being any constant or any function. You are probably familiar with expressions like:
$10^6 = 1{,}000{,}000$
Exponentials have several wonderful advantages. Firstly, they are easy to integrate and differentiate, as we discuss in the chapters on calculus. Secondly, exponentials with the same base are easy to multiply and divide, as shown by:
$b^u \times b^w = b^{u+w}$
$b^u / b^w = b^{u-w}$

Thirdly, exponentials with complex exponents greatly simplify the analysis of wave and harmonic phenomena, as we will discuss in the chapter on differential equations.

In physics, the most common base is e, the base of the natural logarithm that we will get to shortly. This is normally written:
e^u
But in eBooks, I prefer to write:
exp{u}
The latter format is easier to read in an eBook. Also, since eBooks are limited to one level each of superscripts and subscripts, the normal format cannot express many functions in physics that contain their own superscripts and subscripts.

Definition of e

The definition of e is:
$e = \text{limit as } n \to \infty \{ (1+1/n)^n \}$
e is an irrational number whose numerical value is 2.718281828459045… When I was learning how to use computers, I calculated the first 40,000 digits of e, so let me know if you need more digits.

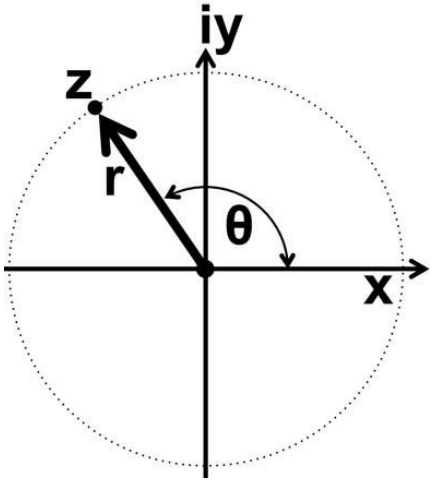
Exponential & Trig Series

Let's now consider e raised to the power x.
$e^x = \exp(x) = \text{limit } n \to \infty (1+1/n)^{xn}$
Let's evaluate $(1+1/n)^{xn}$ using the binomial expansion. The terms with the smallest powers of the infinitesimal quantity 1/n are:
$\exp(x) = 1 + (xn)(1/n) + (xn)(xn-1)/2!n^2$
$+ (xn)(xn-1)(xn-2)/3!n^3 + \ldots$
As $xn \to \infty$, this becomes:
$\exp(x) = 1 + x + x^2/2! + x^3/3! + \ldots$
$e^x = \Sigma_n x^n/n!$
This is the polynomial series expansion for e^x; it is a very useful equation.

Let's substitute x with iθ, recalling that $i=\sqrt{(-1)}$.
$\exp\{i\theta\} = \Sigma_n (i\theta)^n/n!$
$\exp\{i\theta\} = 1 + i\theta - \theta^2/2! - i\theta^3/3! + \theta^4/4! - \ldots$
Now separate the series into real terms and imaginary terms.
$\exp\{i\theta\} = \{1 - \theta^2/2! + \theta^4/4! - \ldots \}$

$+ i\{\theta - \theta^3/3! + \theta^5/5! - \ldots\}$

I show again the plot from Figure 3-1 of a complex number z in the x-iy plane.



Recall that for z=x+iy, z's rectilinear coordinates (x,y) are:

$r = \sqrt{(x^2+y^2)}$

$\theta = \arctan(y/x)$

$x = r\cos\theta$

$y = r\sin\theta$

For r=1, compare z with $\exp\{i\theta\}$:

$z = \cos\theta + i\sin\theta$

$\exp\{i\theta\}$

$= \{1 - \theta^2/2! + \theta^4/4! - \ldots\}$

$+ i\{\theta - \theta^3/3! + \theta^5/5! - \ldots\}$

We see that we can equate z with $\exp\{i\theta\}$ if:

$\sin\theta = \theta - \theta^3/3! + \theta^5/5! - \ldots$

$\cos\theta = 1 - \theta^2/2! + \theta^4/4! - \ldots$

From these, it can be shown that:

$\tan\theta = \theta + \theta^3/3! + 2\theta^5/15 - \ldots$

These are the polynomial series expansions for the sine, cosine, and tangent, three other very useful equations.

The *reciprocal* of an exponential is obtained by simply inverting the sign of the exponent.

$\exp(x)\exp\{-x\} = \exp\{x-x\} = 1$

$\exp\{-x\} = 1 / \exp\{x\}$

Natural Logarithms

Any good function has an inverse, a way to get back to your starting point. (You

wouldn't want to fly to Chicago in January without a return ticket, would you?)
The inverse function of e^x is the **natural logarithm** ln(x).
exp{ ln(x) } = x = ln(exp{x})
The natural logarithm and the ordinary base-10 logarithm are related by:
$\ln\{x\} = \log_{10}\{x\} * 2.3026\ldots$
$\log_{10}\{x\} = \ln\{x\} * 0.43429\ldots$
For –1<x<=+1, the polynomial series for the natural logarithm is:
$\ln\{1+x\} = x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \ldots$
Some useful logarithmic identities are:
ln{x*y} = ln{x} + ln{y}
ln{x/y} = ln{x} – ln{y}
$\ln\{x^a\} = a \ln\{x\}$

Exponential Decay

Consider the example of radioactive decay. Assume we start at time t=0 with 512 atoms that have a *half-life* of 2 seconds. While radioactive decay is probabilistic and therefore subject to statistical fluctuations, let's simplify our example by assuming that *exactly* 50% of all existing atoms decay during each half-life.

This means, the number of atoms at various times will be:
t = 0 sec, N = 512
t = 1 sec, N = 362
t = 2, sec, N = 256
t = 4, sec, N = 128
t = 10, sec, N = 16
t = 18, sec, N = 1
This behavior is described by the equation:
N(t) = N(0) exp{–t/T}
Here, T is *mean lifetime*, which is related to the half-life by:
T = half-life × ln{2}
In many contexts, including particle physics, physicists generally quote mean lifetimes. In other contexts, including radioactive nuclear decay, we generally quote the half-life.

During any small time interval Δt, some small number of atoms will decay. For small enough time intervals, the number of decays is proportional to Δt. We can

show this by letting ΔN be the change in the number of atoms during Δt. The equations are:

$\Delta N = N(0) \exp\{-(t+\Delta t)/T\} - N(0) [\exp\{-t/T\}$

$\Delta N = N(0) \exp\{-t/T\} [\exp\{-\Delta t/T\} - 1]$

$\Delta N = N(t) [(1 - \Delta t + \ldots) - 1]$

$\Delta N = - N (\Delta t / T)$

Above, "…" represents terms proportional to higher powers of Δt, which can be neglected for small enough Δt. This equation shows the proportional behavior; the rate of change of N is at all times a constant factor times N.

Hyperbolic Trig Functions

Related to the exponential are: sinh, the hyperbolic sine; cosh, the hyperbolic cosine; and tanh, the hyperbolic tangent. These are defined as:

$\sinh(x) = (\exp\{x\} - \exp\{-x\})/2$

$\cosh(x) = (\exp\{x\} + \exp\{-x\})/2$

$\tanh(x) = \sinh(x) / \cosh(x)$

Some useful hyperbolic identities are:

$\cosh(ix) = \cos x$

$\sinh(ix) = i \sin x$

$\tanh(ix) = i \tan x$

$\cosh^2 x - \sinh^2 x = 1$

$\sinh x = x + x^3/3! + x^5/5! - \ldots$

$\cosh x = 1 + x^2/2! + x^4/4! - \ldots$

$\tanh x = x - x^3/3 + 2x^5/15 - \ldots$

$\sinh(x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

$2 \sinh x \sinh y = \cosh(x+y) - \cosh(x-y)$

$2 \cosh x \cosh y = \cosh(x+y) + \cosh(x-y)$

$2 \sinh x \cosh y = \sinh(x+y) + \sinh(x-y)$

$\sinh x + \sinh y = 2 \sinh[(x+y)/2] \cosh[(x-y)/2]$

$\sinh x - \sinh y = 2 \sinh[(x-y)/2] \cosh[(x+y)/2]$

$\cosh x + \cosh y = 2 \cosh[(x+y)/2] \cosh[(x-y)/2]$

$\cosh x - \cosh y = 2 \sinh[(x+y)/2] \sinh[(x-y)/2]$

Chapter 8

Approximation Techniques

Virtually every mathematical analysis of natural phenomena employs numerous approximations. Successful physicists must know when and how to make the right approximations in the right way.

Even something as simple as measuring the force of Earth's gravity by dropping balls from the *Leaning Tower of Pisa* cannot be accomplished without approximations. General relativity says time proceeds more quickly at the Tower's top than at its base. The Sun's gravity retards the balls' fall at midday, but accelerates them at midnight. Both effects are extremely small — too small to impact any practical measurement — but they are not zero.

Physics courses are intended to teach students essential principles of science, and how these principles can be employed to understand a myriad of natural phenomena. If we attempt to include all effects that impact every situation, an ocean of minutia would obscure our understanding and make any mathematical analysis impossible. No one can solve the equations for everything all at once.

Approximations allow us to focus on just a few important issues at a time.

The key to making good approximations is knowing which terms and which effects are small enough to be neglected in any particular situation. In this chapter, we will explore some common approximation techniques.

Taylor Series

Taylor series are a common and easy approach to successful approximations. They provide a convenient means of simplifying complicated, potentially unsolvable functions.

Any physically realistic function f(x) can be represented by a Taylor series, a polynomial of the form:
$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$
Here, the a's are some set, possibly an infinite set, of constants.

Polynomials have a great advantage over some other functions: all polynomials are solvable, and all are easily differentiated and integrated, as we will discover in the chapters on calculus.

While there may be an infinite number of terms in the Taylor series, often it is possible to approximate f(x), perhaps within some limited range of x, by just a few terms. In such cases, we might write:
$f(x) = a_0 + a_1 x + a_2 x^2 + O(x^3)$
Here, $O(x^3)$ denotes any collection of terms, each of which is proportional to x to the third or higher power. If $|x|>1$, $O(x^3)$ could be enormous, perhaps even infinite. But if $|x|<<1$, terms proportional to higher powers of x will be much less than terms proportional to lower powers. If |x| is small enough, we can make the approximation that those higher power terms are zero. We might even be able to neglect the x^2 term as well, leaving us with a simple linear function:
$f(x) = a_0 + a_1 x$
In such cases, f(x) becomes linear, greatly simplifying the math.

Let's consider two examples. First, we examine e^x and its Taylor series.
$f(x) = \exp\{x\} = 1 + x + x^2/2! + x^3/3! + \ldots$
For $|x|<<1$, we can often approximate this as:
$f(x) = 1 + x$
Now consider:
$g(x) = \exp\{x\} - x = 1 + x^2/2! + x^3/3! + \ldots$
If, for $|x|<<1$, we drop all terms proportional to x^2 or higher powers, we would be left with g(x)=1. An approximation that crude might eliminate all the interesting physics this problem may entail. In general, we should avoid approximations that completely eliminate key variables. Let's instead make the approximation that $O(x^3)$ is zero, but $O(x^2)$ is not. We then have:
$g(x) = \exp\{x\} - x = 1 + x^2/2$
This shows that g(x) has a quadratic dependence on x near x=0 that could illuminate some interesting effects that a constant g(x) would not.

Let's next examine some common cases that lend themselves to Taylor series approximations. Many of these series are discussed in Chapter 6. In each series, we assume $|x|<<1$, and I provide here one more term than you will generally need.

$1/(1–x) = 1 + x + x^2/2 + O(x^3)$
$1/(1+x) = 1 – x + x^2/2 – O(x^3)$
$\sqrt{(1+x)} = 1 + x/2 – x^2/8 + O(x^3)$
$1/\sqrt{(1+x)} = 1 – x/2 + 3x^2/8 – O(x^3)$
$\exp\{x\} = 1 + x + x^2/2 + O(x^3)$
$\ln\{1+x\} = x – x^2/2 + x^3/3 – O(x^4)$

The following approximations are often used in relativistic calculations:

For modest velocities, where $v/c = \beta <<1$,
$\gamma = 1/\sqrt{(1–\beta^2)}$
$\gamma = 1 + \beta^2/2 + 3\beta^4/8 + O(\beta^6)$
For v very close to c, and $(1–\beta) <<1$,
$\gamma^2 = 1 / [(1+\beta)(1–\beta)]$
$\gamma^2 = 1 / [2(1–\beta)]$
$2\gamma^2 = 1 / (1–\beta)$
$1 / 2\gamma^2 = 1–\beta$
$\beta = 1 – 1 / 2\gamma^2$

Interpolation & Extrapolation

Imagine that we want to measure the orbit of Mercury.

Since Mercury is so close to our Sun, we can only briefly observe it shortly after sunset or shortly before sunrise. Of all the Sun's planets, Mercury has the fastest orbital velocity (averaging 170,000 km/hr or 107,000 mph), and the shortest orbital period (88 Earth days). Thus, Mercury's position changes more between measurement opportunities than any other major body in our Solar System.

Figure 8-1 illustrates a set of three consecutive hypothetical measurements of Mercury's position (black dots), taken near *aphelion*, its farthest distance from the Sun. Define the three measurement times to be $t = –1$, 0, and +1.

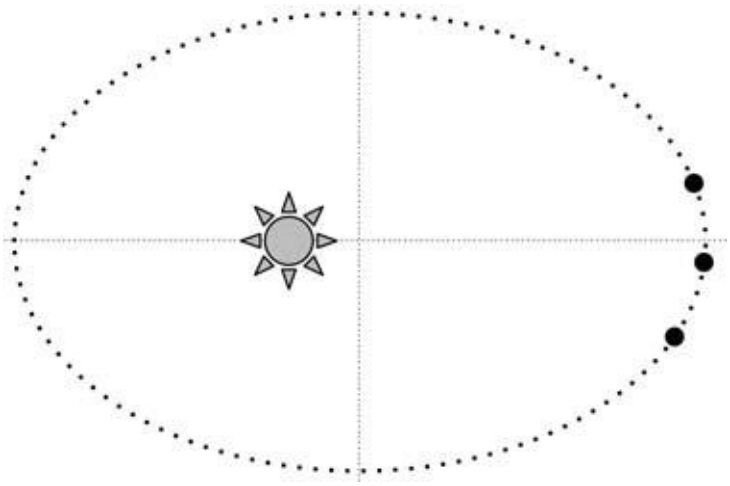


Figure 8-1 Mercury at 3 Times

To determine Mercury's maximum distance R form the Sun, we employ *interpolation*, estimating the value of a function *between* measured values.

The first step in interpolation is fitting a function to the measured data. For the sake of discussion, let's forget that Kepler taught us that planets orbit in ellipses. With three points, we instead choose a function of the form:

$r(t) = A + Bt + Ct^2$

We calculate A, B, and C, from the measured points as follows:

$r(0) = A$

$r(-1) = A - B + C$

$r(+1) = A + B + C$

$A = r(0)$

$B = [r(+1) - r(-1)]/2$

$C = [r(+1) + r(-1)]/2 - r(0)$

The next step is finding t_{aph}, the value of t at which r(t) reaches its maximum value. When r(t) is at a maximum value, r does not increase when t changes by a small amount, either a positive or negative amount. If r(t) did increase for any small change in t, r(t) would not be at its maximum. Consider r(t) and $r(t+\Delta t)$:

$r(t) = A + Bt + Ct^2$

$r(t+\Delta t) = A + B(t+\Delta t) + C(t+\Delta t)^2$

We now require that r(t) at t_{aph} is unchanged for small Δt:

$0 = r(t+\Delta t) - r(t)$

$0 = B\,\Delta t + 2C\,t_{aph}\,\Delta t + C\,\Delta t^2$

For very small Δt, we can approximate Δt^2 as being zero. This yields:

$0 = B + 2\,C\,t_{aph}$

$t_{aph} = -B / 2C$

Interpolation provides the best possible *estimate* based on the available data.

But, remember that we have not directly measured Mercury's actual aphelion; we relied on the assumption that its orbit is perfectly described by r(t). The actual orbit R(t) may deviate from r(t) by some unknown function Q. We can then write:

$R(t) = r(t) + Q$

$R(t) = A + Bt + Ct^2 + Q$

What can we say abut Q? Assuming the validity of our measurements, Q must be zero at t=–1, 0, and +1. We can therefore replace Q as follows:

$R(t) = A+Bt+Ct^2 + \{(t+1)(t)(t-1)\}\, g(t)$

The expression in { }s, which equals $\{t^3-t\}$, ensures that Q is zero at the measured times. Also, g(t) is another unknown function that we expand below in a Taylor series.

$R(t) = A+Bt+Ct^2 + \{t^3-t\}\, (a_0 + a_1 t + a_2 t^2 + \ldots)$

If we interpolate in the region near the center of the measured points, so that $|t| <1/2$, the maximum deviation from our fitted function r(t) is:

$|Q| < \{0.188\}\, (a_0 + a_1/2 + a_2/4 + \ldots)$

While some functions do oscillate wildly, most physical functions are relatively smooth. This means a_0, a_1, a_2, … are not likely to be enormous numbers. Furthermore, the higher order terms of g(t) are depressed by $|t|<1/2$, and the entire deviation is depressed by the {0.188} fitting factor. We therefore have a reasonable expectation that our interpolated estimate is not far off the true value.

By contrast, if we need the value of r(t) at t=2, we must *extrapolate*, estimate the value of r(t) *outside the range* of its measured values. We then have:

$Q = \{6\}\, (a_0 + 2a_1 + 4a_2 + \ldots)$

Here, the potential deviation is much greater. The { }-factor is 32 times larger, and the higher order terms of g(t) have much larger coefficients. Depending on the coefficients, extrapolating to t=2 could result in an error 100 times larger than interpolating to t=0.5. The farther we extrapolate beyond measured data, the greater the probability of an erroneous estimate. The figure of merit (or demerit) is t divided by (the distance between the farthest data points).

Consider a real example: calculating the position of Pluto when the New Horizons spacecraft flew by in July 2015. This $700 million mission was literally a shot in the dark that was saved by old-school astronomy.

When New Horizons launched in 2006, Pluto's orbit was known with a precision

of *only* 0.002%. That seems fine, but is actually not nearly good enough. Since Clyde Tombaugh discovered it in 1930, Pluto has completed only 1/3 of a single 248-year orbit. It has not been where it is now since 1767.

Modern high-precision observations of Pluto began fairly recently. Extrapolating this data to the flyby date left Pluto's location uncertain by 62,000 miles, 44 times Pluto's diameter.

To save weight, New Horizons was not equipped with any image analysis or search capabilities. It relied on mission controllers on Earth to tell it where to aim and when to collect images. Flying at 32,000 mph (51,000 km/hr), New Horizons moves one Pluto diameter in under 3 minutes. In a 9.5-year mission, this is a time window of 0.6 parts per million.

Six years after launch, a desperate Dr. Marc Buie visited the Lowell Observatory where Tombaugh discovered Pluto. Tombaugh's images were too blurry, but Buie happened to find archived photographs taken by another astronomer named Carl Lampland. For reasons that had nothing to do with Pluto, Lampland took 1000 pictures of the same piece of sky over a 20-year period ending in the 1950's. By sheer luck, Pluto just happened to be in Lampland's photographs. Using the best current technology, Buie meticulously digitized 1000 very old photographs, and — just in time — calculated Pluto's orbit with unprecedented precision.

On July 14, 2015, New Horizons took this image of Pluto that awed the world.

Buie still had to extrapolate, but thanks to Lampland, he had good data over a much longer orbital segment.

Functions near Extrema

Next, consider the behavior of an arbitrary function f(x) near an extremum, at a value of x at which f has a local minimum or maximum.

Let's shift the x-axis so that this extremum occurs at x=0, and expand f in a Taylor series.

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

I will now prove that $a_1=0$, because f has an extremum at x=0.

Examine what happens at infinitesimal values of x, either positive or negative. For $|x|<<1$, we can make the approximation that all terms proportional to x^2 and higher powers are negligible. In that limit:

for $|x|<<1$, $f(x) = a_0 + a_1 x$

If a_1 were >0, f(x) would be > f(0) for positive x; hence f(0) could not be a maximum. If a_1 were <0, f(x) would be > f(0) for negative x, and f(0) could not be a maximum. You can satisfy yourself that the same logic prevents f(0) from being a minimum for any nonzero a_1. Only if $a_1=0$, can f have an extremum at x=0.

This reduces our original Taylor series to:

$f(x) = a_0 + a_2 x^2 + a_3 x^3 + \ldots$

Let's now consider larger values of x, but still require that |x| is small. For small enough |x|, each term proportional to x^2 and higher powers is very small.

Hence, f changes very slowly near its extrema.

The mathematically elegant conclusion is:

Near any function's extrema,
its changes are only *second order*.

"Second order" means the change in f(x) is proportional to x^2.

This principle has great importance in several areas of physics.

In ***Feynman Simplified 2D***, Chapter 36, we explore the *Principle of Least Action*, which governs the motion of objects subject to forces; the distribution of electromagnetic fields; and the curvature of spacetime in general relativity. In each case, the solution to complex problems can be reduced to finding where the action reaches an extremum and varies most slowly.

The same mathematics arises in wave interference. The paths along which total phase change reaches an extremum are the paths with constructive wave interference. Those are the paths traversed by light and by elementary particles.

Chapter 9

Probability & Statistics

Successful scientists need to master probability, statistics, permutations, and combinations, because these are everywhere in nature and in our analyses of observations. All quantum theory is built on probabilities. All experimental data contain statistical fluctuations that must be understood and properly interpreted.

Permutations

Permutations are re-orderings of a set of objects. Consider a *set* S comprised of a finite number n of *elements*. For example, let S be the first five letters of the alphabet:

S = {A, B, C, D, E}

Each of the following is a *permutation* of S.

P1 = B A C D E

P2 = B C A D E

P5 = B C E D A

P1 is obtained from S with one exchange of adjacent pairs (A B goes to B A). P2 is obtained from S with two exchanges (A B C goes to B A C goes to B C A). P5 is obtained from S with five exchanges. P1 and P5 are *odd* permutations and P2 is an *even* permutation, because their numbers of exchanges of adjacent pairs are odd and even respectively.

If all elements of a set are different (no duplicates), the number of possible permutations P(n) of n elements is:

for n distinct elements, P(n) = n!

Recall that n! is n-factorial, the product of the integers from 1 to n.

This is easily proven. The first element of a permutation can be chosen in n different ways, because there are n distinct elements to choose from. The second element can be chosen in n–1 ways, because that is the number of remaining unchosen elements. This logic continues until the nth element of the permutation, which can only be chosen in 1 way. Since each choice is independent, the number of possible sequences of choices is:

$n \times (n-1) \times \ldots \times 2 \times 1 = n!$

Combinations

Combinatorics describes the number of ways in which k elements can be chosen from a set S of n elements. Note that we are not concerned here about the order of the elements: ABC and ACB are the same combination, although they are different permutations.

For example, consider again set S = {A, B, C, D, E}. In how many ways can we choose 2 of the 5 elements? The ten distinct *combinations* are:
AB, BC, AC, AD, AE
BD, BE, CD, CE, DE

We define C(n,k) to be the number of ways of choosing k elements from a set of n elements. Let's calculate C(n,k) using the example of five elements taken 2 at a time.

First, we show the 5! permutations of S in Figure 9-1.

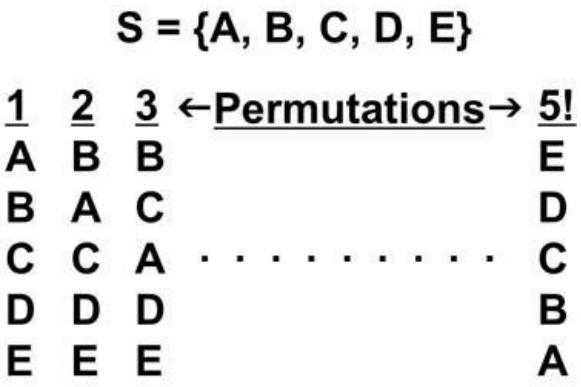


Figure 9-1 Permutations of 5 Elements

In general a set of n distinct elements has n! permutations.

Next, we divide each permutation into two parts: (1) the first two elements, and (2) the remaining three elements. We then sort all n! permutations into M groups, with each group containing all permutations that have the same first two elements in any order. As yet, we do not know the value of M.

The first **key point** is that each permutation belongs to *one and only one* group. The sorted groups are illustrated in Figure 9-2.

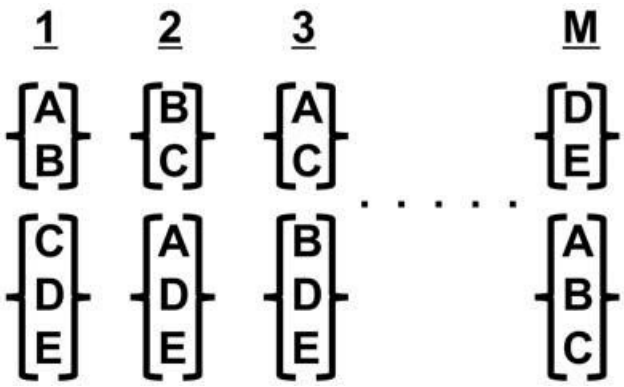


Figure 9-2 Groups Sorted By First Two Elements

The second **key point** is that *each group contains the same number of permutations*. Since the elements are all equivalent, we can relabel them in any way we please without changing M and without changing the number of permutations in any group. Therefore the groups must be the same except for the identity of the their elements.

Let's next determine the number of permutations per group. Figure 9-3 shows all the permutations in group #3. There are two permutations of the first two elements (shown horizontally), and 6 permutations of the remaining three elements (shown vertically).

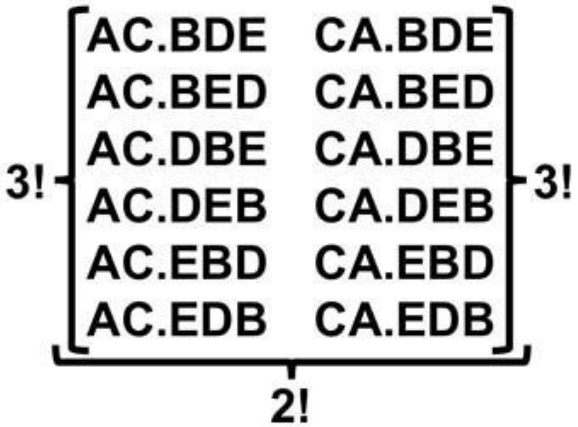


Figure 9-3 All Permutations in Group #3

The total number of permutations in group #3 is:
$2! \times 3! = 2 \times 6 = 12$
In general, in selecting k elements from n elements, there are k! permutations of the selected elements, and (n–k)! permutations of the remaining elements. This

means there are k!(n–k)! permutations in each of the M groups.

Since each of the n! permutations is a member of one and only one group, we have:
$n! = M\ k!(n–k)!$
$M = C(n,k) = n! / [\ k!\ (n–k)!\ ]$
Here, we note that M = C(n,k) = the number of ways of taking k elements from a set of n elements.
For 2 elements chosen from a set of 5, k=2 and n=5, hence:
$C(5,2) = 5! / [2!\ 3!] = 120 / 2x6 = 10$
Note that C(n,k) = C(n,n-k). The equation for C(n,k) demonstrates this, but logic also demands it. There is no mathematical distinction between: (a) choosing 2 elements from a set of 5, and leaving 3 unchosen; or (b) choosing 3 elements from a set of 5, and leaving 2 unchosen. Both (a) and (b) divide the original set of 5 elements into subsets of 2 and 3 elements. The distinction between elements that are "chosen" and those that are "unchosen" is only in our minds, not in the math.

Binomial Coefficients

An important result of combinatorics is the set of *binomial coefficients*. Consider the following *binomial equation* with with n being an integer.
$Z = (p + q)^n$
Let's expand this expression for several values of n.
For n = 0: $Z = (p+q)^n = 1$
For n = 1: $Z = (p+q)^1 = p+q$
For n = 2: $Z = (p+q)^2 = (p+q)\ (p+q)$
$= p^2+pq+qp+q^2$
The last Z is the sum of 4 *terms*. Recall that a term is any group of quantities that are multiplied or divided, but not added or subtracted. Two of these terms are equal, and the sum of the exponents in each term is 2.
For n = 3: $Z = (p+q)^3 = (p+q)\ (p+q)^2$
$= p^3+p^2q+qp^2+pq^2+qp^2+pq^2+q^2p+q^3$
Here, Z is the sum of 8 terms, several of which are equal, and the sum of the exponents in each term is 3.

In general, the expansion of $(p+q)^n$ is the sum of 2^n terms, many of which are equal. Also, the sum of the exponents in each term must be n; this is because there are n ( )'s, and each ( ) contributes one and only one factor, either p or q, to

each term. Because there are 2 ways to pick one factor from each ( )'s, there are 2^n possible ways to pick the n factors that comprise each term.

The 2^n terms contain powers of p that range from 0 to n, and the same is true for q. But in all terms, the exponent of q equals n minus the exponent of p. There are therefore only n+1 distinctly different terms; the rest are equal to other terms.

Therefore Z must have this form:
$Z = \Sigma_{k=0}^{n} A_k p^k q^{n-k}$
The set of A_k are the *binomial coefficients*.

To calculate A_k, we employ combinatorics.
In the original equation for Z:

$Z = (p+q)\ (p+q)\ (p+q)\ \ldots(p+q)$
There are n factors of p. The number of distinct ways to pick k p's from a set of n p's is C(n,k). Hence:
$A_k = C(n,k) = n! / [\ k!\ (n-k)!\ ]$
$Z = \Sigma_{k=0}^{n} p^k q^{n-k}\ n! / [\ k!\ (n-k)!\ ]$
Below is a table of the first few binomial coefficients, with each row having the same n and each column having the same k. Note the pattern that each entry is the sum of two entries in the preceding line: the entry directly above and the entry to its left (assuming 0's for positions other than those shown).

n:	0	1	2	3	4	<=k
0:	1					
1:	1	1				
2:	1	2	1			
3:	1	3	3	1		
4:	1	4	6	4	1	

Discrete Probabilities

The simplest probabilistic situation has only two possible outcomes. Let's consider flipping a coin that has probability p of coming up heads and probability q=1–p of coming up tails. A fair coin has p=q=0.5. However, the real world isn't always fair.

Using the binomial expansion, P(n,k) the probability of k heads occurring in n flips is:
$P(n,k) = p^k q^{n-k} n! / [k! (n-k)!]$
Note that we have the correct normalization: the probability of any number of heads is:
$\Sigma_{k=0}^{n} P(n,k) = \Sigma_{k=0}^{n} p^k q^{n-k} n! / [k! (n-k)!]$
$\Sigma_{k=0}^{n} P(n,k) = (p+q)^n = 1^n = 1$
For 4 flips of a fair coin, the P(4,k) are:
k=0: 1/16 = 6.25%
k=1: 4/16 = 25%
k=2: 6/16 = 37.5%
k=3: 4/16 = 25%
k=4: 1/16 = 6.25%
The sum of the probabilities of the five possible outcomes is 100%. As expected, the most probable result is 2 heads out of 4, and the probability of k heads is the same as the probability of k tails, for any k.

By contrast, if the probability of a head is p=0.25, the P(4,k) are:
k=0: $p^0 q^4 \times 1 = 31.64\%$
k=1: $p^1 q^3 \times 4 = 42.19\%$
k=2: $p^2 q^2 \times 6 = 21.09\%$
k=3: $p^3 q^1 \times 4 = 4.69\%$
k=4: $p^4 q^0 \times 1 = 3.91\%$
Again, the sum of probabilities is 100%. Here, the most probable result is 1 head. It is no coincidence that:
$1 = p \times n = 0.25 \times 4$
This brings us to the *expectation value* EV, also called the *mean* μ. The expectation value is the average result expected for a very large number of repetitions. For example, let a single trial consist of flipping a coin n times. After an infinite number of trials, the expectation value EV is:
$EV = \mu = \Sigma\, k\, P(n,k)$
Here, the sum is over all possible outcomes: k=0 to k=n.

For a binomial distribution this is:
$\mu = \Sigma_{k=0}^{n} k\, p^k q^{n-k} n! / [k! (n-k)!]$
Note that the k=0 term does not contribute to the sum. The sum now ranges over n values of k. Let's make these substitutions:
$m = n - 1$

$j = k - 1$
Note that:
$n-k = m-j$
$n! = n\ m!$
$k! = k\ j!$
We then have:
$\mu = \Sigma_{j=0}^{m}\ k\ p^{j+1}\ q^{m-j}\ n\ m!\ /\ [\ k\ j!\ (m-j)!\ ]$
$\mu = np\ \Sigma_{j=0}^{m}\ p^{j}\ q^{m-j}\ m!\ /\ [\ j!\ (m-j)!\ ]$
$\mu = np\ \Sigma_{j=0}^{m}\ P(m,j)$
$\mu = np\ [1]$
For a binomial distribution, the mean number (expectation value) of A outcomes in n trials equals np, where p is the probability of outcome A in one trial.

The other major characteristic of a probability distribution is its *variance* σ^2 defined by:
$\sigma^2 = \Sigma_{k=0}^{n}\ (k-\mu)^2\ P(n,k)$
For a population of independent *trials* or measurements, the *standard deviation* σ is defined to be the square root of the variance σ^2.

For a binomial distribution, σ^2 is:
$\sigma^2 = \Sigma_{k=0}^{n}\ (k-np)^2\ P(n,k)$
$\sigma^2 = \Sigma_{k=0}^{n}\ (k^2 - 2knp + n^2p^2)\ P(n,k)$
The n^2p^2 factor is a constant that we can move outside the summation; it is multiplied by the sum of P(n,k) that equals 1. Also, we previously derived $np=\Sigma_k\{kP(n,k)\}$, so the equation becomes:
$\sigma^2 = -2(np)np + n^2p^2 + \Sigma_{k=0}^{n}\ k^2\ P(n,k)$
Now we make a strange substitution: replace k^2 by $k(k-1)+k$.
$\sigma^2 = -n^2p^2 + \Sigma_{k=0}^{n}\ \{k(k-1) + k\}\ P(n,k)$
$\sigma^2 = -n^2p^2 + np + \Sigma_{k=0}^{n}\ k(k-1)\ P(n,k)$
Now, let's deal with the last sum. Note that the k=0 and k=1 terms are both zero. As above, we replace n with m+2, k with j+2, yielding:
$\Sigma_{k=2}^{n}\ k(k-1)\ P(n,k) = \Sigma_{j=0}^{m}\ Q\ p^{j+2}\ q^{m-j}$
$Q = k(k-1)\ n!\ /\ [k!\ (n-k)!]$
$Q = k(k-1)\ n(n-1)m!\ /\ [k(k-1)j!\ (m-j)!]$
$Q = n(n-1)\ m!/\ [\ j!\ (m-j)!]$
$\sigma^2 = -n^2p^2 + np + p^2\ n(n-1)\ \Sigma_{j=0}^{m}\ P(m,j)$
$\sigma^2 = -n^2p^2 + np + (n^2p^2 - np^2)\ (1)$

$\sigma^2 = +np - np^2$
$\sigma^2 = np(1-p)$
For a binomial distribution, the variance of the number of A outcomes in n trials equals npq, where p is the probability of outcome A in one trial, and q=1–p.

Next, let's consider a rare event with p<<1. We have previously set q=1–p. Here, it will be more mathematically convenient to set q=1. The binomial equation is still valid, but the sum has changed.
for q=1: $Z = \Sigma_{k=0}^{n} p^k \, n! / [\, k! \, (n-k)! \,] = (1+p)^n$
Since the mean $\mu = np$, we have in the limit of very large n:
$Z = \{1+\mu/n\}^n \rightarrow \exp\{\mu\}$ as $n \rightarrow \infty$
Also, for k<<n, in the limit that $n \rightarrow \infty$:
$n!/(n-k)! = n \times (n-1) \times \ldots \times (n-k+1) \rightarrow n^k$
This means the probability of k occurrences of a rare outcome A becomes:
$P(k) = p^k \, n^k \exp\{-\mu\} / k!$
$P(k) = \mu^k \exp\{-\mu\} / k!$
This is the *Poisson* probability distribution, illustrated in Figure 9-4. It is what the universally correct binomial distribution becomes in the limit of p<<1 and very large n.

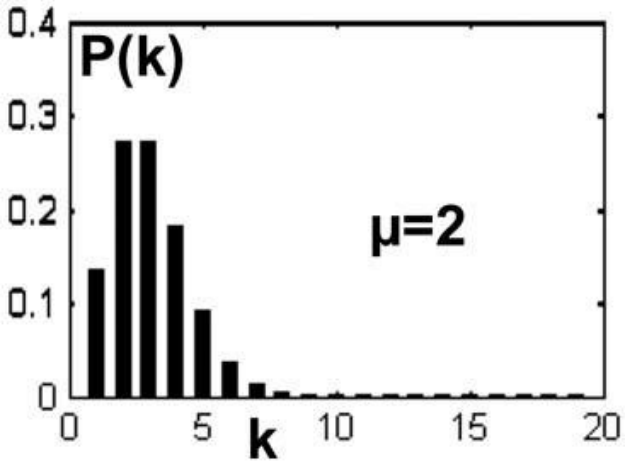


Figure 9-4 Poisson Distribution for μ=2

Note that the Poisson distribution is not symmetric; it has a long tail extending to large k values.

Some early applications of Poisson's probability distribution were: estimating the number of wrongful criminal convictions (1837); and investigating the number of soldiers accidentally killed by horse kicks (1898). The mean and

variance of the Poisson distribution are:
$\mu = np$
$\sigma^2 = np$
This is a **very important** result for physicists studying rare events. Whether counting radioactive decays, photons from remote quasars, or groups of particles with a combined effective mass near 127 GeV (Higgs boson), each event is extremely rare. Hence, $p<<1$, and the variance equals the number of detected events.

In most stochastic processes, such as those mentioned above, one normally quotes σ as the *one standard deviation statistical uncertainty* (more on this later). Hence, if the number of detected events is N, one normally quotes the result with its statistical uncertainty as:
$N \pm \sqrt{N}$
This provides a standard measure of the level of confidence in the result. Compare two cases: one with N=100 and the other with N=10,000.
100 ± 10 has 10% precision
$10{,}000 \pm 100$ has 1% precision
To make the precision 10 times better, one must collect 100 times more data. No one said this was easy.

We have so far discussed stochastic processes with two possible outcomes. Now let's consider a situation with multiple possible outcomes.

We turn to dice, and roll a pair of dice, each a cube with faces numbered 1 through 6. The table below shows the sum of the numbers on the top faces of die X (vertically) and die Y (horizontally) for each of the 36 possible outcomes of one roll. To keep the columns straight, I use "a" for 10, "b" for 11, and "c" for 12; if you know hexadecimal, this is familiar.

X|	1	2	3	4	5	6	<=Y
1|	2	3	4	5	6	7	
2|	3	4	5	6	7	8	
3|	4	5	6	7	8	9	
4|	5	6	7	8	9	a	
5|	6	7	8	9	a	b	
6|	7	8	9	a	b	c	

As you see, there are 11 possible sums: the integers from 2 to 12. The

probability of a specific sum equals the number of combinations that yield that sum divided by 36. The probabilities are:
Sum = 2: 1/36 = 2.78%
Sum = 3: 2/36 = 5.55%
Sum = 4: 3/36 = 8.33%
Sum = 5: 4/36 = 11.11%
Sum = 6: 5/36 = 13.89%
Sum = 7: 6/36 = 16.67%, most probable
Sum = 8: 5/36 = 13.89%
Sum = 9: 4/36 = 11.11%
Sum=10: 3/36 = 8.33%
Sum=11: 2/36 = 5.55%
Sum=12: 1/36 = 2.78%

Continuous Probabilities

Coins and dice produce a small number of discrete outcomes. By contrast, most scientific measurements — the energy of cosmic rays, the luminosity of quasars, the mass of the Higgs boson, or the mass of a black hole — result in continuous outcomes. These outcomes can be distributed in many different ways that are determined by the natural phenomena themselves and by how we measure them.

Let's take a "simple" example: measuring the mass of a black hole. While the measurement may be extremely challenging in practice, it is simple conceptually because we seek one quantity that has a specific value that changes imperceptibly over the time scale of our measurement.

Our measurement strategy is to identify objects that orbit the black hole, measure their orbital radii and periods, and use Kepler's "123" law:
$M^1 T^2 = R^3$
Here, M is the black hole's mass divided by the mass of our Sun, T is the orbital period in years, and R is the orbital semi-major axis in AU (1 Astronomical Unit = Earth's mean orbital radius).

The measurement challenges include precisely imaging very remote objects in obscured locations, and repeating those observations on a consistent basis over the many-year span of celestial orbits.

In nearly all scientific measurements, our instruments have limited precision. In this case, our measurements of T and R are inevitably uncertain to some degree. Experience, and the *central limit theorem* of probability theory, show that if the uncertainties are *stochastic* — random, independent fluctuations — a large number of measurements have a *Gaussian distribution*, also called a *normal distribution*, as illustrated in Figure 9-5.

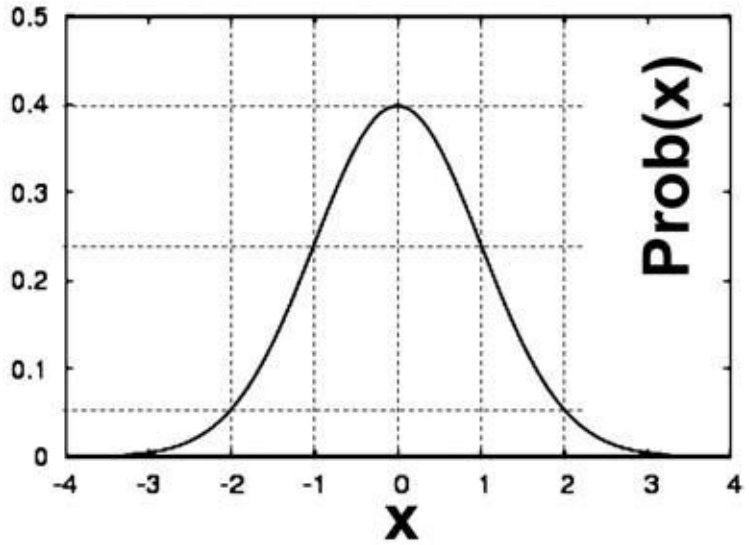


Figure 9-5 Gaussian Distribution

Here, Prob(x) is the probability that a quantity whose true value is zero will be measured to have value x if the measurement variance is 1. The equation for the curve in Figure 9-5 is:

$\text{Prob}(x) = \exp\{-x^2/2\} / \sqrt{(2\pi)}$

The constants in this equation are chosen so that: the total probability of all x values equals 1; and the variance of x also equals 1. You may enjoy proofs of these statements that are presented in the Integration by Parts section of Chapter 13.

Note that the distribution is symmetric about its most probable value, x=0. Also, the probability of x decreases monotonically and substantially as |x| increases. At x=0, the *probability density* is nearly 40%. For an infinitesimal dx, this means:

$\text{Prob}(-dx/2 < x < +dx/2) = \text{Prob}(x=0)\ dx$

$\text{Prob}(-dx/2 < x < +dx/2) = 0.3989\ dx$

The good news is: the most probable measured value is the true value. The bad news is: actual measurements are spread over a significant range.

As the deviation from the true value increases, the probability density decreases.

Some examples are:
Prob(x=±1) = 24.20%
Prob(x=±2) = 5.40%
Prob(x=±3) = 0.44%
Prob(x=±4) = 0.013%
Note that at large deviations, the probability drops precipitously with increasing x. From x=0 to x=1, the probability density drops by a factor of 1.6; from x=1 to x=2, it drops by a factor of 4.5; from x=2 to x=3 it drops by a factor of 12; and from x=3 to x=4, it drops by a factor of 34.

What is generally most important to physicists is the total probability that a measurement deviates, in either direction, from the true value (zero in this case) by more than Z. That probability is:
Prob(|x|>Z) = Prob(x>+Z) + Prob(x<–Z)
= 1 – Prob(–Z < x < +Z)
This is often expressed in terms of erf, the *error function*, as:
Prob(|x|>Z) = 1 – erf(Z/√2)
This equation says erf(Y) equals the probability that any measured x value deviates from the mean of a Gaussian distribution by less than Y/√2 standard deviations.

We show here the definition of the error function, which involves an integral. We will learn all about integrals in Chapter 13.
$\mathrm{erf}(Y) = (2/\sqrt{\pi}) \int_0^Y \exp\{-u^2\}\, du$
The equation for erf(Y) cannot be reduced to anything simple. One must obtain its values from tables.

In almost all cases, we are interested in probabilities rather than error functions. I provide below two short tables that are also given in Appendix 3. More comprehensive tables are readily available online.

Some values for Z, Prob(|x|>Z), and the odds of |x|>Z are:

Z	P(|x|>Z)	Odds of |x|>Z
0.5	0.61708	1 in 1.6204
0.7	0.48392	1 in 2.0665
1	0.31731	1 in 3.1515
2	0.04550	1 in 21.978
3	0.00270	1 in 370.40
4	6.33E–5	1 in 15787.
5	5.73E–7	1 in 1.74E+6
6	1.97E–9	1 in 5.07E+8

In the first line, Z=0.5 and P(|x|>Z) is the probability of a measurement x deviating, either positively or negatively, from the mean of a Gaussian distribution by *more* than 0.5 standard deviations; P(|x|>0.5)=61.708%, and the *odds* of x deviating by *more* than 0.5 are 1 in 1.6204 = 1/0.61708.
The last line says |x| exceeds 6 in 1.97 of one billion data sets, and the odds that |x| exceeds 6 are 1 in 507 million.
Often physicists assume that for all practical purposes |x| never exceeds 6: you would have to make half a billion measurements before expecting to have one |x|>6. For example, the discovery of the Higgs boson was almost universally accepted when the probability of it *not existing* corresponded to x>6. This condition is sometimes described as a “six sigma” confidence level. I say “almost”, because in a democracy, there is always at least one dissenting opinion.

The following table lists Z values corresponding to various values of Prob(|x|<Z). For example, the top entry on the left side states that 80% of all measured x values are in the interval:
–1.28156 < x < +1.28156

P(|x|<Z)	Z	P(|x|<Z)	Z
0.80	1.28156	3 9’s	3.29053
0.90	1.64485	4 9’s	3.89059
0.95	1.95996	5 9’s	4.41717
0.98	2.32635	6 9’s	4.89164
0.99	2.57583	7 9’s	5.32672
0.995	2.80703	8 9’s	5.73073
0.998	3.09023	9 9’s	6.10941

On the right side, “5 9’s” means 0.99999 — five nines following the decimal point. The bottom entry on the right side means that 99.9999999% of all measured x values are in the interval:
$-6.10941 < x < +6.10941$
We have so far discussed a Gaussian distribution with zero mean and unit variance. In general, a Gaussian distribution has mean μ and variance σ^2, with this equation:
$\text{Prob}(y) = \exp\{ -(y-\mu)^2 / 2\sigma^2 \} / \sqrt{(2\pi\sigma^2)}$

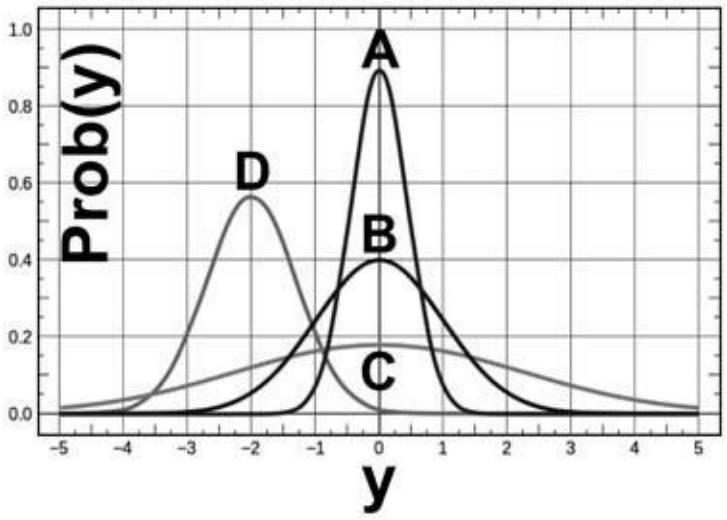


Figure 9-6 Four Gaussian Distributions

The four curves in Figure 9-6 have the following means and variances:
A: $\mu = 0$, $\sigma^2=0.2$
B: $\mu = 0$, $\sigma^2=1.0$
C: $\mu = 0$, $\sigma^2=5.0$
D: $\mu=-2$, $\sigma^2=0.5$
The effect of μ is simply to shift the curve left for $\mu>0$, or right for $\mu<0$. The effect of σ is to stretch the curve horizontally for $\sigma>1$, or compress it for $\sigma<1$. To maintain proper normalization, to keep the probabilities of all y values summing to 100%, the peak rises for $\sigma<1$ and drops for $\sigma>1$.

Everything that we said above about Prob(x) is equally valid for Prob($[y-\mu]/\sigma$).

Let’s review the **key points** of Gaussian distributions that are most important to physicists. Firstly, averages of independent measurements approach a Gaussian distribution as the number of measurements n becomes large. Secondly, as n increases, the average measured value μ approaches the true value of the

quantity being measured. Thirdly, the uncertainty in μ equals $\sigma/\sqrt{n}$, where the standard deviation σ is the square root of the variance σ^2. The standard format for quoting the result of a group of measurements is:
Result = $\mu \pm \varepsilon$, with $\varepsilon = \sigma/\sqrt{n}$
For a measurement whose uncertainties are entirely stochastic, the true value is within 1ε of μ 68% of the time, within 2ε 95% of the time, within 3ε 99.7% of the time, and within 4ε 99.994% of the time.

Mathematics cannot address *systematic* measurement errors or uncertainties. Math cannot fix faulty equipment or wrong interpretations.

Combining Uncertainties

Virtually every scientific measurement process entails some degree of uncertainty. Quantum mechanics insists that uncertainty is a fundamental characteristic of nature. All instruments have limited precision, and the many measurements involved in counting rare events are all subject to statistical fluctuations.

Often these uncertainties are improperly called “errors” — scientists speak of the error function, error bars, and standard errors — but none of these are due to human mistakes. Rather, all of these are truly random statistical variations that are intrinsically unavoidable.

We have discussed the uncertainties that arise from statistical fluctuations in the measurement of specific quantities. Now let’s examine the impact of such uncertainties in more complex analyses.

What happens when we combine two quantities, each with their own uncertainties?

Let’s assume two quantities m and n are the results of two measurement processes. Assume each process has a finite number of discrete outcomes, with probabilities p(m) and q(n).

We repeatedly add measured m and n values, and we wish to know the statistical properties of that result. The arithmetic operation is:

$Q = m + n$

We seek the mean and variance of Q. The variance of the left side of this equation must equal the variance of the right side. Recall the general equations for mean and variance:

mean(m): $\langle m \rangle = \Sigma_m \, m \, p(m)$

$\text{var}(m) = \Sigma_m \, (m - \langle m \rangle)^2 \, p(m)$

Similarly for n and q(n). Let p and q be normalized so that:

$\Sigma_m \, p(m) = \Sigma_n \, q(n) = 1$

We compute the mean of Q by summing m+n over all values of m and n, with each combination weighted by its probability.

$\text{mean}(Q) = \Sigma_{mn} \, (m+n) \, p(m) \, q(n)$

$= \Sigma_{mn} \, m \, p(m) \, q(n) + \Sigma_{mn} \, n \, p(m) \, q(n)$

$= \Sigma_m \, m \, p(m) + \Sigma_n \, n \, q(n)$

$= \langle m \rangle + \langle n \rangle$

We next compute the variance of Q:

$\text{var}(Q) = \Sigma_{mn} \, [(m+n) - (\langle m \rangle + \langle n \rangle)]^2 \, p(m) \, q(n)$

$= \Sigma_{mn} \, [(m - \langle m \rangle) + (n - \langle n \rangle)]^2 \, p(m) \, q(n)$

$= \Sigma_{mn} \, (m - \langle m \rangle)^2 \, p(m) \, q(n)$

$\quad + \Sigma_{mn} \, (n - \langle n \rangle)^2 \, p(m) \, q(n)$

$\quad + 2 \, \Sigma_{mn} \, (m - \langle m \rangle) \, (n - \langle n \rangle) \, p(m) \, q(n)$

$= \text{var}(m) + \text{var}(n)$

$\quad + 2 \, \{\Sigma_m (m - \langle m \rangle) \, p(m)\} \, \{\Sigma_n (n - \langle n \rangle) \, q(n)\}$

$\text{var}(Q) = \text{var}(m) + \text{var}(n)$

$\quad + 2 \, (\langle m \rangle - \langle m \rangle) \, (\langle n \rangle - \langle n \rangle)$

Thus we have the key results: (1) the mean of the sum equals the sum of the means; and (2) the variance of the sum equals the sum of the variances.

$\text{mean}(Q) = \text{mean}(m) + \text{mean}(n)$

$\text{var}(Q) = \text{var}(m) + \text{var}(n)$

$\sigma_Q^2 = \sigma_m^2 + \sigma_n^2$

If m and n are the counts of rare events, their variances are m and n, and we can write the summation equation as:

$(m+n) \pm \sigma = (m \pm \sqrt{m}) + (n \pm \sqrt{n})$

with $\sigma = \sqrt{(m^2 + n^2)}$

Here, σ is the one standard deviation uncertainty in Q, assuming all uncertainties are independent and purely random fluctuations.

Subtraction is equivalent to adding m and (–n). Inverting the polarity of n inverts

the polarity of its mean but leaves its variance unchanged. The result for subtraction is: (1) the mean of the difference equals the difference of the means; and (2) the variance of the difference equals the *sum* of the variances.

We have covered addition and subtraction; let's turn next to multiplication and division of quantities with uncertainties.

Consider the quantities A and B that have standard deviations a and b:
$Q = (A \pm a)\,(B \pm b)$
$Q = AB\,(1 \pm a/A)\,(1 \pm b/B)$
$Q = AB\,(1 \pm a/A \pm b/B \pm ab/AB)$
If we assume $a \ll A$ and $b \ll B$, we can drop the last term, reducing this to:
$Q = AB\,(1 \pm a/A \pm b/B)$
From the rule for adding quantities with uncertainties, this becomes:
$Q = AB\,\{1 \pm \sqrt{(a^2/A^2 + b^2/B^2)}\}$
The rule for division is:
$Q = (A \pm a) / (B \pm b)$

$Q = (A/B)\,\{1 \pm \sqrt{(a^2/A^2 + b^2/B^2)}\}$

Next, let's examine powers.
$Q = (A \pm a)^n$
This time the uncertainties in the product terms are not independent; they are in fact all identical. We can evaluate Q by expanding the right side using binomial coefficients.
$Q = A^n \pm nA^{n-1}a + n(n-1)A^{n-2}a^2/2 + \ldots$
Dropping terms of order $(a/A)^2$ and higher yields:
$Q = A^n\,(1 \pm na/A)$
Now let's try some more complex functions: sine, cosine, and tangent. For these examples, we assume $|A| \ll 1$ and $|a| \ll 1$, and we will keep only the most significant terms in the small quantities.
$\sin(A \pm a) = A \pm a$
$\cos(A \pm a) = 1 - (A \pm a)^2/2$
$\cos(A \pm a) = 1 - A^2/2 \pm Aa - a^2/2$
$\cos(A \pm a) = \cos(A) \pm Aa - a^2/2$
$\tan(A \pm a + \pi/2) = ?$
The last equation shows that, for small ε, $\tan(\pm\varepsilon + \pi/2)$ can be anything from $-\infty$

to $+\infty$. Even a tiny uncertainty in an angle near $\pi/2$ leads to a completely uncertain value of the tangent. I am sure you see that uncertainties can be subtle.

In very complex situations, the propagation of uncertainties from the raw measurements to the final results are often evaluated using *Monte Carlo* methods, as we explore in Chapter 20.

Recall that when we subtracted two quantities with uncertainties, their variances added. This has important consequences, particularly when subtracting quantities of similar magnitude. For example, consider the difference in intensity of two gamma ray bursts that have 100 photon counts and 95 photon counts.

$Q = (100 \pm 10.0) - (95 \pm 9.7)$

$Q = +5 \pm 14$

The initial two numbers, known with 10% precision, have a difference that has an uncertainty of 280%. We do not even know which gamma ray burst was brighter.

This highlights a critical issue in experimental science. Often it is important to compare two quantities A and B that are very similar, but that have a crucial distinction. We want to know if they are exactly equal, or if one is slightly larger. As we have just shown, it is very difficult to measure A, measure B, subtract the measurements, and obtain an extremely precise result. It is far better, if possible, to directly measure A–B.

Let's consider two real world examples. The famous 1887 Michelson-Morley experiment sent two light beams along two identical but orthogonal paths, and determined the difference in their travel times. If Michelson had measured each travel time and subtracted those measurements, measurement uncertainties would have dominated his results, rendering them meaningless. Michelson's genius was devising a method of directly measuring the *difference* in travel time by using wave interference. His high-precision measurements prove that Earth's velocity through the luminiferous ether, if it exists, is no more than 2.5% of Earth's orbital velocity. Michelson's measurement thus definitively refutes the model of ether as the medium of light, although it took most scientists decades to accept that conclusion.

Conversely, my thesis experiment confirmed the violation of matter-antimatter symmetry by showing that neutral kaons are 0.28% more likely to decay to $\pi^-\mu^+\nu$ than to $\pi^+\mu^-\bar{\nu}$. Unable to measure the 0.28% difference directly, we were forced to collect 10 million decays of each type in order to measure each decay rate with 0.035% precision. That determined their difference to a precision of only 18%. Forty years ago, that was the best measurement of this quantity, and it is still the second best measurement. But after collecting all that data, it was very painful to see 0.035% precision degrade to 18% when two nearly equal numbers were subtracted.

Chi-Square Analysis

Chi-square (χ^2) analysis is the standard method of comparing data sets to one another, or comparing a data set to a theoretical hypothesis.

Imagine taking N measurements of a quantity X, with each measured value x in a fixed range given by:
for all x: xmin<x<xmax
Let's now sort our measured x values into K *bins*, with the bin number j of each measurement given by:
j = 1 + K (x – xmin) / (xmax – xmin)
j = 1 through K
We define n(j) to be the number of measurements in bin #j. This means p(j), the probability of a measured value falling into bin #j equals n(j)/N, and q(j) = 1–p(j) = [N–n(j)]/N. As discussed above, for large N, the one standard deviation uncertainty in n(j) equals:
σ of n(j) = √{N p(j) q(j)}

σ of n(j) = √{ n(j) [N–n(j)] / N}

if n(j) << N, σ of n(j) = √{ n(j) }

For simplicity, we will assume all n(j)<<N.

Let's now compare the set of numbers n(j), with a theoretical hypothesis $\hat{H}$. Typically, $\hat{H}$ predicts a probability distribution from which we can calculate p(j), the probability that a single measured x falls into bin #j. For N total measurements, $\hat{H}$ predicts that bin #j will contain Np(j) measurements.

We then compute:
$\chi^2 = \Sigma_j \{ Np(j) - n(j) \}^2 / Np(j)$
Here, we sum from j=1 to j=K, excluding any j for which p(j)=0. Define k to be the number of bins for which p(j)>0.
We now define #df to be the *number of degrees of freedom*: #df equals the number of independent comparisons in the χ^2 summation minus the number of adjustable parameters.

In χ^2 summation, there are k terms — k comparisons between data and theory. But, each theoretical prediction p(j) is multiplied by N, and N is determined by the data: $[N=\Sigma_j n(j)]$. Hence there is one adjustable parameter, making #df = k–1.

To clarify this point, consider the special case of k=1, in which the summation reduces to:
for k=1: $\chi^2 = \{ N - N \}^2 / N = 0$
Here, there are zero degrees of freedom, and no meaningful comparison between data and theory.

The figure of merit in a chi-squared analysis is:
χ^2 / #df
If $\hat{H}$ were a perfect description of nature, if the data were free of systematic errors, and if k>>1, the summation terms would on average equal 1, and χ^2 would on average equal #df.

If χ^2/#df is less than or equal to 1, theory $\hat{H}$ is considered to be consistent with the measured data. Consistency is not sufficient to claim that $\hat{H}$ is proven, but it is a significant step in that direction. Consistency with multiple sets of data by multiple observers testing multiple predictions of $\hat{H}$ leads to wider and wider acceptance of $\hat{H}$ as an effective model.

Conversely, if χ^2/#df is much more than 1, the theory and the data are inconsistent, which typically means one or both are wrong. A theory is considered *falsified* if it is strongly inconsistent with several independent sets of data from different observers, or is strongly inconsistent with one indisputable set of data.

Clearly, there is a substantial gray area between acceptance and falsification. Physicists seek to resolve such quandaries with more data, taken by more observers, testing more predictions. “We need more data” is a common refrain.

Figure 9-7, created by Mikael Haggstrom, plots χ^2 horizontally vs. *p-value* vertically for five indicated values of #df. The horizontal dotted line is at a p-value of 0.5.

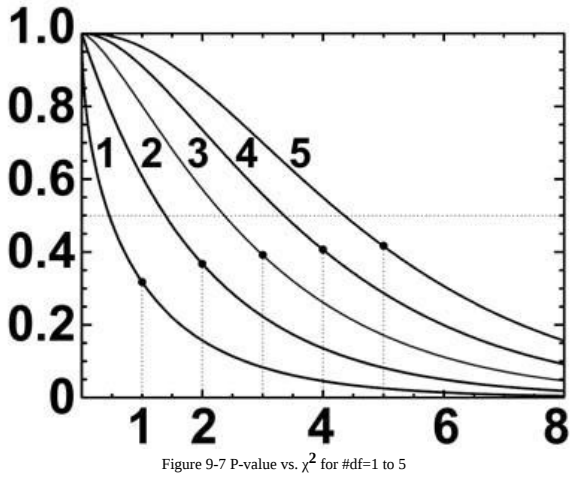


Figure 9-7 P-value vs. χ^2 for #df=1 to 5

Appendix 4 contains a more comprehensive table of χ^2/#df.

P-value(χ^2) is the fraction of other independent data sets that have a larger χ^2. The fraction of data sets with χ^2>#df is:
#df=1, 31% have χ^2>1
#df=2, 36% have χ^2>2
#df=3, 39% have χ^2>3
#df=4, 40% have χ^2>4
#df=5, 41% have χ^2>5
For #df>>1, 50% of data sets have χ^2>#df.

We can also use χ^2 to compare two data sets and test the hypothesis that they consistently measure the same natural phenomenon.

In comparing data with theory, we assumed above that the theoretical predictions had zero statistical uncertainties. This is often, but not always, true. If we measure muon lifetimes versus velocity, and compare our data to the predictions

of special relativity, the predicted values have zero uncertainty. But if we measure Alnico V's magnetization versus an external field, the theoretical predictions will necessarily incorporate numerous measured material properties whose uncertainties make the theoretical predictions uncertain, sometimes in very complex ways.

Whether the uncertainties are instrumental or theoretical, we can compare data set n(j) with data set m(j) by computing:

$$\chi^2 = \Sigma_j \{ m(j) - n(j) \}^2 / \sqrt{[\sigma_m(j)^2 + \sigma_n(j)^2]}$$

If both m(j) and n(j) are numbers of rare events with purely statistical uncertainties, this equation reduces to:

$$\chi^2 = \Sigma_j \{ m(j) - n(j) \}^2 / \sqrt{[m(j)^2 + n(j)^2]}$$

Again, the figure of merit is given by χ^2/#df. If χ^2>>#df, the data sets may be measuring different phenomena, or they may have different systematic errors.

Chapter 10

Rotation & Velocity Transformations

This chapter explores transforming coordinate systems. Two types of coordinate transformations are common in physics: (1) rotating the coordinate axes; and (2) changing the velocity of a coordinate system.

Each such transformation has an inverse: a second transformation that goes back to the original coordinate system.

Imagine that we wish to analyze a rocket's trajectory during a one-hour flight from one point on Earth's surface to another.

We could analyze this motion using Earth-bound coordinates: latitude, longitude, and elevation. That coordinate system continuously rotates as Earth spins on its axis. Earth's meridians, lines of constant longitude, turn 15 degrees per hour.

Alternatively, we could analyze the rocket's motion in a coordinate system whose axes remain stationary as Earth turns.

One transformation converts the rocket's coordinates in the Earth-bound system into coordinates in the stationary system. An inverse transformation converts stationary coordinates into Earth-bound coordinates.

Since the laws of nature are independent of our choice of coordinate systems, we can employ any coordinate system that seems convenient. Proper transformations ensure that measurements in any coordinate system are consistent with those laws.

We can view transformations in two equivalent ways. We can rotate the

coordinate axes by angle +θ while keeping the rocket stationary, or we can rotate the rocket by angle –θ while keeping the axes stationary. In the case of boosts, we can boost our reference frame by velocity +***v*** while keeping the rocket's velocity fixed, or we can boost the rocket by velocity –***v*** while keeping our reference frame fixed.

Throughout this chapter, and almost all of the ***Feynman Lectures***, transformations will change the coordinate axes and reference frames while leaving fixed the physical entities that we are describing.

Simple Rotations

The simplest coordinate rotation is shown in Figure 10-1. Here, point P remains stationary while the xy-axes are rotated by angle θ to become the x*y*-axes.

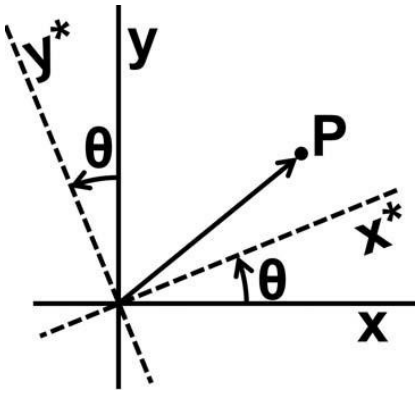


Figure 10-1 Rotation in xy-Plane

We can also describe this in 3-D, with the z-axis perpendicular to the screen, pointing outward in accordance with the right hand rule (see Chapter 11). We say the coordinate system is rotated about the z-axis by the positive angle θ. Again employing the right hand rule, we point our right thumb in the +z-direction, and note that our right fingers curl counterclockwise, which defines the direction of positive rotation angle.

Note that both x and y axes rotate in the same direction.

For a rotation of the xy-axes by **angle θ about the z-axis**, the transformation equations that relate the coordinates of point P in the xyz system to P's coordinates in the x*y*z* system are:

$x^* = + x \cos\theta + y \sin\theta$

$y^* = – x \sin\theta + y \cos\theta$

$z^* = z$

Note that the transformation equations contain one minus sign.

As we should expect, the **inverse transformation** is the same, but with θ replaced by –θ:

$x = + x^* \cos\theta - y^* \sin\theta$

$y = + x^* \sin\theta + y^* \cos\theta$

$z = z^*$

For a rotation of the yz-axes by **angle θ about the x-axis**, the transformation equations are:

$x^* = x$

$y^* = + y \cos\theta + z \sin\theta$

$z^* = - y \sin\theta + z \cos\theta$

For a rotation of the xz-axes by **angle θ about the y-axis**, the transformation equations are:

$x^* = + x \cos\theta - z \sin\theta$

$y^* = y$

$z^* = + x \sin\theta + z \cos\theta$

Rotation by Euler Angles

Any arbitrary rotation — any angles about any axes — can be achieved with three simple rotations of the form described above. Such rotations are defined using the *Euler angles* shown in Figure 10-2.

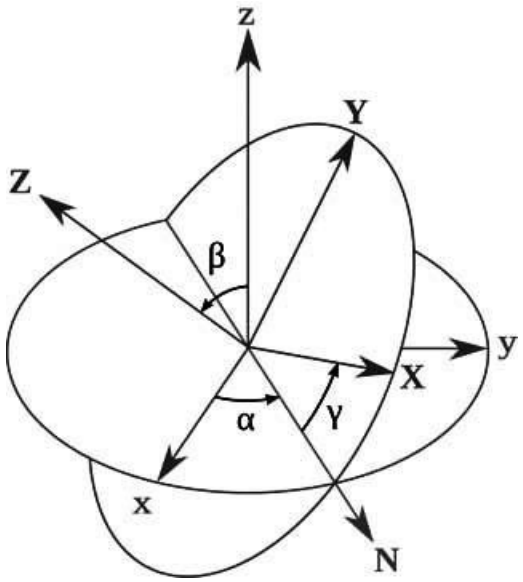


Figure 10-2 Euler Angles β, α, γ

An Euler rotation consists of three steps in this specific sequence:

1. rotate about the z-axis by angle β
2. rotate about N the new x-axis by angle α
3. rotate about Z the new z-axis by angle γ

The transformation equations that relate the coordinates of point P in the xyz system to P's coordinates in the x*y*z* system are:

$x^* = x(\cos\gamma\cos\beta - \cos\alpha\sin\gamma\sin\beta) + y(\cos\gamma\sin\beta + \cos\alpha\cos\beta\sin\gamma) + z(\sin\alpha\sin\gamma)$

$y^* = x(-\cos\beta\sin\gamma - \cos\alpha\cos\gamma\sin\beta) + y(\cos\gamma\cos\alpha\cos\beta - \sin\gamma\sin\beta) + z(\cos\gamma\sin\alpha)$

$z^* = x(\sin\beta\sin\alpha) + y(-\cos\beta\sin\alpha) + z(\cos\alpha)$

Boosts

A *boost* is a change in velocity: reference frame S is transformed to frame S* that is moving at velocity $+\boldsymbol{v}$ relative to S. Figure 10-3 shows a boost in which $\boldsymbol{v}$ is in the +x-direction. Here, point P remains stationary, while the x-ct axes are boosted to become the x*-ct* axes.

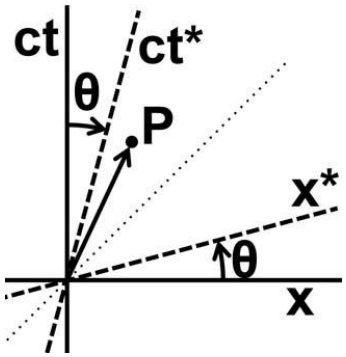


Figure 10-3 Boost in tx-plane

Note that the x* and ct* axes tilt in opposite directions: both tilt toward the 45-degree line by the same angle θ.

The transformation equations that relate the coordinates of point P in the x-ct frame to P's coordinates in the x*-ct* frame are:

$x^* = +x\gamma - ct\gamma\beta$

$ct^* = -x\gamma\beta + ct\gamma$
Here, $\beta=v/c$ and $\gamma=1/\sqrt{(1-\beta^2)}$ are the usual relativistic factors.

As we should expect, the inverse transformation is the same, but with β replaced by –β:
$x = +x^*\gamma + ct^*\gamma\beta$
$ct = +x^*\gamma\beta + ct^*\gamma$
Note that the transformation equations contain an even number of minus signs.

The angle θ is given by:
$\tanh(\theta) = \beta$
In most advanced physics books and papers, authors adopt units in which c=1, so the corresponding graphs and equations do not include c.

Rotations of Quantum Spin States

We tabulate here the equations for several rotational transformations between a set S of basis states and a rotated set T of basis states. Basis states in quantum mechanics have a similar role to normal coordinate axes.

»»For **spin 1/2** particles, each basis set has two linearly independent states, denoted + and –.

• To rotate about the **z-axis by angle ø**:
$|+T\rangle = \exp\{+i\text{ø}/2\}\ |+S\rangle$
$|-T\rangle = \exp\{-i\text{ø}/2\}\ |-S\rangle$

• To rotate about the **y-axis by angle θ**:
$|+T\rangle = +\cos(\theta/2)\ |+S\rangle + \sin(\theta/2)\ |-S\rangle$
$|-T\rangle = -\sin(\theta/2)\ |+S\rangle + \cos(\theta/2)\ |-S\rangle$

• To rotate about the **x-axis by angle θ**:
$|+T\rangle = \cos(\theta/2)\ |+S\rangle + i\sin(\theta/2)\ |-S\rangle$
$|-T\rangle = i\sin(\theta/2)\ |+S\rangle + \cos(\theta/2)\ |-S\rangle$

• For **Euler angles**: rotate first about the **z-axis by angle β**,
then rotate about the **new x-axis by angle α**,
then rotate about the **new z-axis by angle γ**:
$\langle+T|+S\rangle = \cos(\alpha/2)\exp\{+i(\beta+\gamma)/2\}$
$\langle+T|-S\rangle = i\sin(\alpha/2)\exp\{-i(\beta-\gamma)/2\}$

$\langle -T|+S\rangle = i\sin(\alpha/2)\exp\{+i(\beta-\gamma)/2\}$
$\langle -T|-S\rangle = \cos(\alpha/2)\exp\{-i(\beta+\gamma)/2\}$

»»For **spin 1 particles**, each basis set has three linearly independent states, denoted +, 0, –.

• To rotate about the **y-axis by angle θ**:

$\langle +T|+S\rangle = (1+\cos\theta)/2$
$\langle 0T|+S\rangle = -(\sin\theta)/\sqrt{2}$
$\langle -T|+S\rangle = (1-\cos\theta)/2$
$\langle +T|0S\rangle = +(\sin\theta)/\sqrt{2}$
$\langle 0T|0S\rangle = +\cos\theta$
$\langle -T|0S\rangle = -(\sin\theta)/\sqrt{2}$
$\langle +T|-S\rangle = (1-\cos\theta)/2$
$\langle 0T|-S\rangle = +(\sin\theta)/\sqrt{2}$
$\langle -T|-S\rangle = (1+\cos\theta)/2$

• To rotate about the **z-axis by angle θ**:

$\langle +T|+S\rangle = \exp\{+i\theta\}$
$\langle 0T|0S\rangle = 1$
$\langle -T|-S\rangle = \exp\{-i\theta\}$

The other six amplitudes are zero.

• To rotate about the **x-axis by angle θ**:

$\langle +T|+S\rangle = (\cos\theta+1)/2$
$\langle 0T|+S\rangle = +i(\sin\theta)/\sqrt{2}$
$\langle -T|+S\rangle = (\cos\theta-1)/2$
$\langle +T|0S\rangle = +i(\sin\theta)/\sqrt{2}$
$\langle 0T|0S\rangle = +\cos\theta$
$\langle -T|0S\rangle = +i(\sin\theta)/\sqrt{2}$
$\langle +T|-S\rangle = (\cos\theta-1)/2$
$\langle 0T|-S\rangle = +i(\sin\theta)/\sqrt{2}$
$\langle -T|-S\rangle = (\cos\theta+1)/2$

Chapter 11

Vector Algebra

In three dimensions, we can analyze generalized motion with three independent sets of equations, one for each coordinate:

Coord.	Accel.	Speed	Coordinate Change
X	a_x	$v_x = \int a_x\, dt$	$dx = v_x\, dt$
Y	a_y	$v_y = \int a_y\, dt$	$dy = v_y\, dt$
Z	a_z	$v_z = \int a_z\, dt$	$dz = v_z\, dt$

The total speed v and total distance traveled, ds, in time dt are:

$ds = \sqrt{[\, dx^2 + dy^2 + dz^2]}$

$ds = dt\, \sqrt{[(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2]}$

$ds = dt\, \sqrt{[(v_x)^2 + (v_y)^2 + (v_z)^2]}$

$v = \sqrt{[v_x^{\,2}+v_y^{\,2}+v_z^{\,2}]}$

Hence $ds = v\, dt$.

While we could write any physics equation explicitly, listing all the components in each dimension, mathematicians have devised a more compact notation: *vectors*. Compare the force equations below that are first written in component notation and then in vector notation. The fourth equation contains the same information as do the first three equations combined.

$F_x = m\, a_x = m\, dv_x/dt = m\, d^2x/dt^2$

$F_y = m\, a_y = m\, dv_y/dt = m\, d^2y/dt^2$

$F_z = m\, a_z = m\, dv_z/dt = m\, d^2z/dt^2$

$\boldsymbol{F} = m\, \boldsymbol{a} = m\, d\boldsymbol{v}/dt = m\, d^2\boldsymbol{r}/dt^2$

In this eBook, we denote vectors with bold italics.

In three dimensions, a vector is an ordered triplet of quantities, one component for each dimension of space, that *transform properly under rotations*. To understand what “transform properly”means, consider the position vector ***r***:

Position vector: $\boldsymbol{r} = (x, y, z)$

For a point P, whose rectilinear coordinates are x, y, and z, ***r*** is the vector from the origin of the coordinate system to point P, as shown in Figure 11-1.

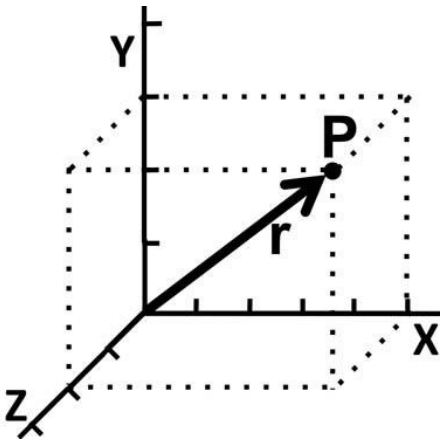


Figure 11-1 Position Vector ***r*** to Point P

The distance between the origin and P is the length of ***r***, which is given by:

$r = |\boldsymbol{r}| = \sqrt{(x^2+y^2+z^2)}$

Now imagine what happens when the coordinate axes rotate, as they would if the axes are defined relative to Earth's surface. As Earth turns, the coordinates of point P, which are the components of ***r***, change in a very well defined manner, as we discussed in Chapter 10. However, rotations never change the length of any vector, nor do rotations change the relative orientations of one vector to another. If the angle between two vectors is θ before rotation, it will be θ after any rotation. This is what we mean by transforming properly.

Not all quantities can be combined to form vectors. Your height H, weight W, and age A do not transform properly under rotation (they each remain constant), hence (H,W,A) is not a vector.

Other important vectors are:

<u>Velocity vector</u>

$\boldsymbol{v} = (dx/dt, dy/dt, dz/dt)$

$\boldsymbol{v} = (v_x, v_y, v_z)$

$\boldsymbol{v} = d\boldsymbol{s}/dt$

<u>Acceleration vector</u>

$\boldsymbol{a} = d\boldsymbol{v}/dt$

Force vector

$\boldsymbol{F} = m\,\boldsymbol{a}$

The force equation states that each of the three components of $\boldsymbol{F}$ equals m multiplied by the corresponding component of $\boldsymbol{a}$. This demonstrates two advantages of using vectors: (1) less writing; and (2) less clutter to distract our focus from the important physics underlying the equations.

Two important vector operations are the *dot product* and *cross product*:

$\boldsymbol{A} \bullet \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z$

$\boldsymbol{A} \bullet \boldsymbol{B} = \boldsymbol{B} \bullet \boldsymbol{A}$

$\boldsymbol{A} \times \boldsymbol{B} = (A_y B_z - A_z B_y,\ A_z B_x - A_x B_z,\ A_x B_y - A_y B_x)$

$\boldsymbol{A} \times \boldsymbol{B} = -\boldsymbol{B} \times \boldsymbol{A}$

The cross product produces a vector that has graphic significance, as shown on the left side of Figure 11-2. If one moves vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ to a common origin, without changing their directions, the two vectors define a parallelogram. The vector $\boldsymbol{A}\times\boldsymbol{B}$ is perpendicular to the plane of the parallelogram, and its length equals the parallelogram's area: $A\,B\sin\theta$.

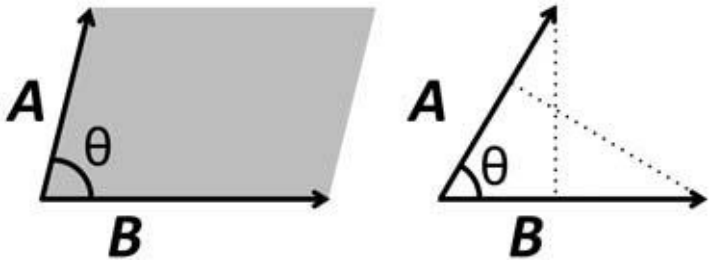


Figure 11-2 Left: $\boldsymbol{A}\times\boldsymbol{B}$. Right: $\boldsymbol{A}\bullet\boldsymbol{B}$

The dot product $\boldsymbol{A}\bullet\boldsymbol{B}$ equals $A\,B\cos\theta$, which is a scalar, an ordinary quantity without dimensionality. The geometric significance of the dot product is shown on the right side of Figure 11-2. The dot product is the normal projection of A onto B (indicated by the vertical dotted line), or equivalently, the normal projection of B onto A (indicated by the diagonal dotted line).

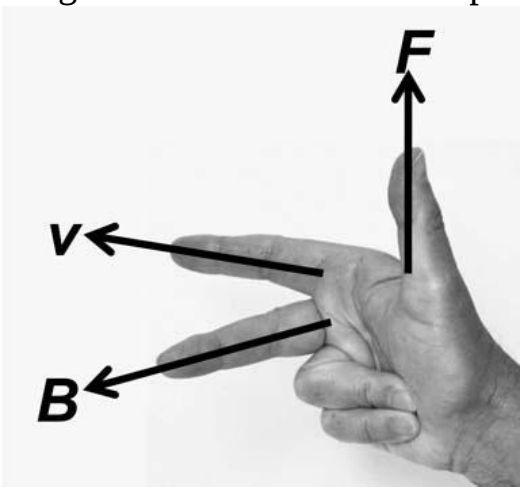
To determine whether or not $\boldsymbol{s}$ is a proper vector: calculate $\boldsymbol{s}\bullet\boldsymbol{r}$, for any known vector $\boldsymbol{r}$. The dot product $\boldsymbol{s}\bullet\boldsymbol{r}$ is an invariant scalar that is unchanged by any rotation, if and only if $\boldsymbol{s}$ is a proper vector. We prove this statement in Chapter 16.

Right Hand Rule

In many mathematical analyses — including those of vector algebra, trigonometry, and rotations — an ambiguity arises between two equally valid choices: one "left-handed" and the other "right-handed". In Figure 11-2 for example, the vector $\boldsymbol{A}\times\boldsymbol{B}$ is perpendicular to both $\boldsymbol{A}$ and $\boldsymbol{B}$, which means it is perpendicular to your screen. But, does it point toward you or away from you?

While both alternatives are equally valid mathematically, we must choose such alternatives consistently to avoid errors. Mathematicians and scientists have universally adopted the *right hand rule*. You lefties out there may feel umbrage, but you should note one advantage that lefties enjoy from this convention: you can use your right hand to determine answers while simultaneously writing equations with your left hand. Righties have to switch back and forth. More than once have I seen right-handed students groan in anguish midway through an exam during which they had repeatedly twisted their left hands in various directions.

Here is how I employ the right hand rule for the cross product: $\boldsymbol{F} = q\boldsymbol{v}\times\boldsymbol{B}$.



I point my index finger along the left vector of $\boldsymbol{v}\times\boldsymbol{B}$, and point my middle finger along the right vector of that product; my thumb is then pointed in the direction of the resultant vector $\boldsymbol{F}$. Works every time, but it can sometimes be a bit awkward. You should practice this until it becomes automatic. If you are unable to use your right hand, use your left and then take the direction opposite to your thumb.

The right hand rule also defines the polarity of angles, angular velocities and

momenta, torques, rotation transformations, and much more. Nothing in physics employs a left hand rule.

Polar vs. Axial Vectors

The vectors we have so far discussed are all *polar vectors*. There is another type: *axial vectors*. The two types behave identically under rotations, but behave oppositely under coordinate reflections or inversions. Axial vectors, often called *pseudovectors*, generally represent mathematical concepts rather than physical observables, and are typically defined by vector cross products. Torque and angular momentum are examples of axial vectors.

We will first examine a reflection of the x-axis. Let a mirror lie in the yz-plane, perpendicular to the x-axis, and assume real actions occur to the left of the mirror, as illustrated in Figure 11-3. From the left side, we see the action's reflected image, with all the same y-and z-coordinates, but with all x-coordinates having the opposite polarity. In the upper real image, a black dot moves with velocity v in the +y, +x direction; in the mirror image, the black dot moves in the +y, –x direction. Velocity, like all other polar vectors, transforms under x-axis reflection according to:

Polar Vector

Real Action: $\boldsymbol{v} = (v_x, v_y, v_z)$

Mirror Image: $\boldsymbol{v} = (-v_x, v_y, v_z)$

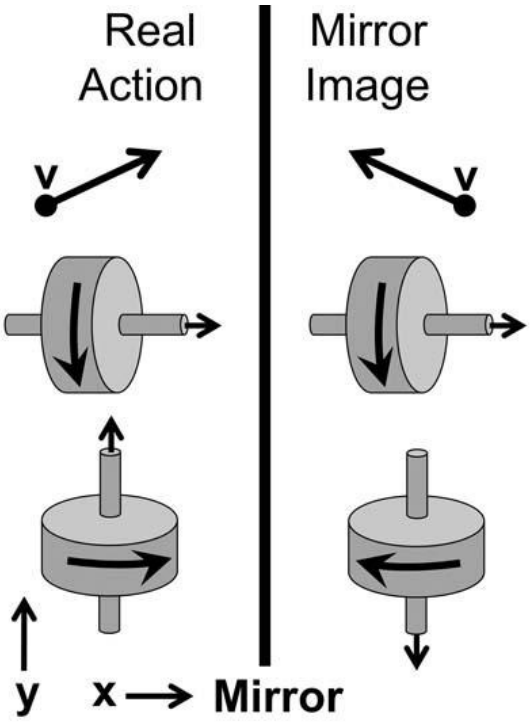


Figure 11-3 Reflection of X-axis in Mirror

Now let's examine the angular momenta of two flywheels. Recall that angular momentum ***L*** equals (position ***r***) × (linear momentum ***p***). The flywheel's axis is horizontal in the middle image and vertical in the lower image.

In the middle real image, the flywheel's near side moves downward (long arrow); so ***L*** (short arrow) points toward +x, by the right hand rule. In the mirror image, the flywheel's near side still moves downward, because the y-component of its velocity does not change sign; so ***L*** still points toward +x.

In the lower real image, the flywheel's near side moves toward +x; so ***L*** points toward +y. In the mirror image, the flywheel's near side moves toward –x, because the x-component of its velocity does change sign; so ***L*** points toward –y.

Under x-axis reflection, the x-components of polar vectors flip sign, while the y- and z-components of axial vectors flip sign.

Let's see how the math works for a different axial vector: the angular velocity defined according to the right-hand rule as:

$\boldsymbol{\omega} = \boldsymbol{v} \times \boldsymbol{r}$

In component notation this is:

$\omega_x = +v_y r_z - v_z r_y$
$\omega_y = +v_z r_x - v_x r_z$
$\omega_z = +v_x r_y - v_y r_x$
Exchanging x with –x changes v_x to $-v_x$, and r_x to $-r_x$. The mirror image ω, call it ω*, is:
$\omega^*_x = +v_y r_z - v_z r_y$
$\omega^*_y = +v_z (-r_x) - (-v_x) r_z$
$\omega^*_z = +(-v_x) r_y - v_y (-r_x)$
Axial Vector
Real Action: $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$
Mirror Image: $\boldsymbol{\omega} = (\omega_x, -\omega_y, -\omega_z)$
All that was for mirror reflections that reverse only one axis. A *coordinate inversion* reverses all three axes. For polar and axial vectors, the results of an inversion are:
Polar Vector
Original: $\boldsymbol{v} = (v_x, v_y, v_z)$
Inversion: $\boldsymbol{v} = (-v_x, -v_y, -v_z)$
Axial Vector
Original: $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$
Inversion: $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$

Chapter 12

Differential Calculus

Welcome to Calculus!

In the next three chapters, we explore the most basic procedures of calculus: derivatives and integrals. This branch of mathematics is both beautiful and essential to scientific literacy.

For physicists, knowing how to differentiate and integrate is a survival skill. Calculus will not only help you pass exams and succeed in your careers, but it will also change your worldview. After learning calculus, you will appreciate the majesty of the universe more profoundly than ever before.

For those discovering calculus for the first time, this may be exciting but perhaps a bit frightening. Do not expect to fully appreciate calculus after just one reading. Changing your worldview will take some time. For me, and for others I know, calculus was a dense fog that one-day seemed to clear in a single instant. I still vividly recall that one moment.

Enjoy this. Learning calculus is one of the great experiences of a scientific education.

Thank you for sharing this experience with me.

The Need For Speed

Derivatives describe rate of change.

Since everything in the universe changes, derivatives are the foundation upon which science strives to describe all natural phenomena.

Most physical laws describe how quantities change over time. Consider, for

example, a ball dropped from the *Leaning Tower of Pisa*. Newton's law of gravity does not tell us the ball's height or speed, but it does tell us how rapidly its speed is changing. To make use of these laws, we must be able to quantify rate of change, and that requires differential calculus.

Let's talk about speed: what exactly do physicists mean by "speed"?

Speed is distance traveled divided by travel time. The speedometer in your car tells you the car's speed. If it reads 100 kilometers per hour (km/hr), it means you would travel 100 km in the next hour if you maintained a constant speed. How does the speedometer know how far you would travel in one hour if you just started driving? If you happen to be only 2 km from your destination, how can you be going 100 km/hr? Clearly, "100 km/hr" cannot be the whole story.

What the speedometer measures is how far the car's tires turn in some small interval of time, such as one second. The turning rate, combined with the known tire circumference, determines how many meters your car travels in one second. If the speedometer reads 100 km/hr, what it really means is that your car is moving at this instant at the rate of 27.778 meters per second (m/sec).

Averaging over one second might be good enough for a car, but what about a ball falling from a great height? It turns out, that 9 seconds after being dropped, the ball's speed is 88 m/sec. This increases to 98 m/sec one second later. Since that is a substantial change, we should probably compute its speed over a time interval that is even less than one second.

The ultimate answer is to define speed in terms of the *infinitesimal distance* traveled during an *infinitesimal time* interval. This concept, developed independently by Isaac Newton (1643-1727) and Gottfried Leibniz (1646-1716), is the basis of *differential calculus*, the first "new math."

Calculus is a rich and beautiful branch of mathematics that was essential to the development of physics. It allows us to deal sensibly with infinitesimal quantities and also to properly accumulate them to describe macroscopic results.

Taking The Limit

By convention, we use the symbol *ds* for an infinitesimal change in distance and *dt* for an infinitesimal change in time. Calculus provides a precise definition of speed that we denote with the letter v:

$$v = \lim_{dt \to 0} \frac{ds}{dt}$$

This equation states: v equals the *limit* of the ratio ds/dt as *dt goes to zero*.

Well-behaved ratios come closer and closer to the final value of v as we compute the ratio for smaller and smaller time intervals dt. The value v is the *asymptotic limit* of the ratio ds/dt.

Not all ratios are well behaved. The ratio 1/x is not well behaved as x approaches 0, because it grows ever-larger and is infinite at x=0. But the ratio (sinx)/x is well behaved as x approaches 0, because the numerator and denominator both approach 0 *at the same rate*. For every infinitesimal but nonzero value of x, (sinx)/x equals 1. As we get closer and closer to x=0, but not at x=0, the ratio remains 1. This means the *limit* of this ratio equals 1.

Let's consider another example of this concept of limits: what is the speed of a falling ball 8 seconds after its release? From Newton's laws, we know that the equation relating s, the distance dropped, and t, the time lapsed since release, is:
$s(t) = g\ t^2 /2$
Here, g is the acceleration of gravity near Earth's surface, which equals 9.8 m/sec^2. We write s(t) to denote the distance s at time t, emphasizing that s is a *function* of t.

To compute speed, we calculate the distance dropped at time t, and also at an infinitesimally later time t+dt.
$s(t) = g\ t^2 /2$
$s(t+dt) = g\ (t+dt)^2 /2$
The infinitesimal distance traveled, ds, is the change in distance during dt.
$ds = s(t+dt) - s(t)$
$ds = g\ [\ (t+dt)^2 - t^2\ ]/2$
$ds = g\ [\ t^2 + 2t\ dt + dt^2 - t^2\ ]/2$
$ds = g\ [\ 2t\ dt + dt^2\ ]/2$

$ds/dt = g\,[2t + dt]/2$

Now, we take advantage of dt being extremely small, $dt \ll 1$. We can discard the dt term in the [ ]'s in the last equation because $dt \ll 2t$. Now we can compute the speed as a function of time, v(t), and evaluate it at t = 8 sec:

$v(t) = ds/dt = g\,[2t]\,/2$

$v(t) = g\,t$

$v(8\ \text{sec}) = [9.8\ \text{m/sec}^2]\,[8\ \text{sec}] = 78\ \text{m/sec}$

Differentiation

The procedure performed above on s(t) is called *differentiation* or more specifically *taking the derivative of* s *with respect to* t. Differentiation is important enough to merit an entire symbology: dq is not d times q, but is rather a single symbol denoting a *differential*, a tiny increment in the variable q or a tiny range of values of q.

The d-symbology applies to any variable, but coordinate differentials — such as dt, dx, dy, dz — are the most common.

The expression ds/dt is the ratio of two differentials, and is called the *derivative* of s with respect to t. The two d's in ds/dt **do not** cancel one another to leave s/t; ds/dt and s/t are entirely different expressions.

Any normal function or equation in physics can be differentiated. To differentiate X with respect to z, compute:

$dX/dz = [\ X(z+dz) - X(z)\ ] / dz$

and take the limit as dz goes to zero.

Derivatives have graphic significance. Figure 12-1 shows a plot of y(x) = sinx. The derivative of y(x) at each value of x is the slope of sinx at that x.

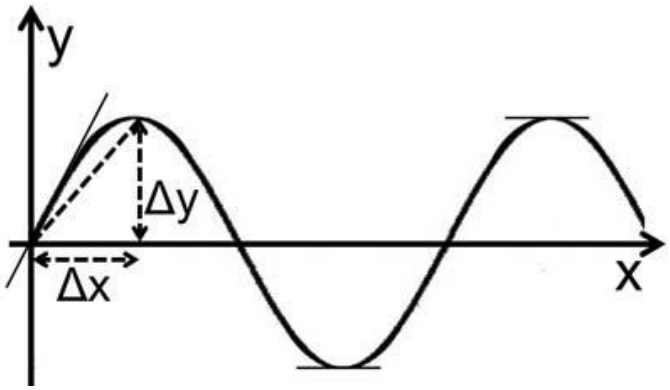


Figure 12-1 y(x) = sin(x)

At two values of x, the slope of y(x) is zero: at the minimum value of y(x) about halfway along the x-axis; and also at the maximum value of y(x) near the right end of the x-axis. At both points, the tangent lines are horizontal, and the derivative, dy(x)/dx, equals 0.

Figure 12-1 also shows two increments from x=y=0, Δy and Δx that are indicated by dashed lines. We see that for a substantial value of Δx, the ratio Δy/Δx, the slanted dashed line, differs from the tangent line at x=0. We show below that Δy/Δx comes closer and closer to the tangent line as Δx gets closer and closer to zero.

In Chapter 7, we found that the sine function can be expressed as an infinite series of which the first three terms are:
$\sin(x) = x - x^3/6 + x^5/120 - \ldots$
The derivative at x=0 is the limit as Δx goes to zero of:
$d\sin(0)/dx = \{ \sin(\Delta x) - \sin(0) \} / \Delta x$
$d\sin(0)/dx = \sin(\Delta x) / \Delta x$
$d\sin(0)/dx = \{\Delta x - \Delta x^3/6 + \Delta x^5/120 - \ldots.\} / \Delta x$
$d\sin(0)/dx = 1 - \Delta x^2/6 + \Delta x^4/120 - \ldots.$
We see that for moderate values of Δx, the right side of the last equation is less than 1. But as Δx goes to zero, Δx^2 and higher powers of Δx become negligible, and the expression approaches 1, the true derivative.

The maxima and minima of any function are always at points at which the derivative of the function is zero.

Now, let's step this up a notch, and consider acceleration a, the derivative of speed with respect to time. Continuing from above:

$v(t) = g\,t$
$v(t+dt) = g\,[t+dt]$
$dv = g\,[t+dt] - g\,t = g\,dt$
$a = dv/dt = g = 9.8 m/sec^2$
Since acceleration is the derivative of speed, and speed is the derivative of distance, we say acceleration is the *second derivative* of distance. Here, "second" means we differentiate twice. The symbology denoting the second derivative of s with respect to t is:
$a = dv/dt = d^2s/dt^2$
First derivatives, like ds/dt, are ubiquitous in physics, and we often omit the word "first." Second derivatives, like d^2s/dt^2, are less common; third and higher order derivatives are rare. The third derivative of distance, d^3s/dt^3, is called *jerk*. If your car speeds up with constant acceleration, you will be pressed back in your seat, but will not be unduly uncomfortable. But if the driver alternately floors and releases the gas pedal, the resulting *changes* in acceleration will toss you back and forth, which is very uncomfortable. That discomfort is due to jerk. Smooth rides are all about minimizing jerk.

Partial Derivatives

Physicists often deal with functions that vary with two or more variables. Temperature is a simple example: it is a function of the three dimensions of space and also time.

Earth's highest recorded air temperature was 134ºF (56.7ºC) in mid-afternoon, on July 10, 1913 at Furnace Creek, in Death Valley, California. But, we might be more interested in knowing where was the highest-ever noon temperature, or we might wish to know when was the highest-ever temperature at the North Pole. The highest-ever air temperature yields neither. To get what we want, we need to find the maximum temperature at a fixed position or at a fixed time. This is what partial derivatives do.

Next, consider an example from general relativity shown in Figure 12-2: the curved space near a massive body. The spatial curvature, shown as a downward deflection, is clearly a function of both x and y.

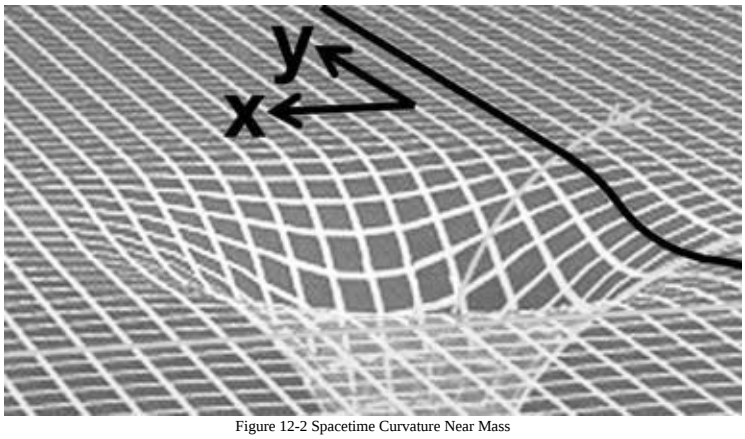


Figure 12-2 Spacetime Curvature Near Mass

The maximum curvature is deep down within the chasm. But we might be interested in the maximum curvature along the indicated black line parallel to the y-axis. We can obtain that by taking the partial derivative of the curvature function with respect to y, *while holding x fixed*. Holding x fixed ensures finding the greatest curvature for that value of x only, thus avoiding the even greater curvature deep within the chasm.

Mathematically, taking a partial derivative is identical to taking the derivative of a function of only one variable. Said another way: for functions of only one variable, there is no difference between the normal derivative and the partial derivative.

Here is the definition of the partial derivative for functions of multiple variables. Consider a function f of n+1 variables:
$f(x, y_1, y_2, \ldots, y_n)$
Here, x is one variable, and $y_1, y_2, \ldots, y_n$ are n other independent variables. The partial derivative of f with respect to x is:
$\partial f/\partial x = \partial f(x, y_1, y_2, \ldots, y_n)/\partial x$
$\partial f/\partial x$ = limit of {Q/dx} as dx goes to 0, with
$Q = f(x+dx, y_1, y_2, \ldots, y_n) - f(x, y_1, y_2, \ldots, y_n)$
Note the notation: $\partial/\partial x$ rather than d/dx. Also note that none of the y's change.

General Rules of Differentiation

Differentiation is a *linear* operation that obeys these rules:

For any constant a and variable q:
$da/dq = 0$

For any functions F & G, constants a & b, and variable q:
$d(aF+bG)/dq = a\ dF/dq + b\ dG/dq$
$d(FG)/dq = G\ dF/dq + F\ dG/dq$
$d(FG)/dq = d(GF)/dq$
$d(F/G)/dq = (1/G)dF/dq - (F/G^2)dG/dq$
$d(F^n)/dq = n\ F^{n-1}\ dF/dq$

Derivatives of Common Functions

Here, we list the derivatives of the most common functions of physics: polynomials, trig functions, and exponentials. In the following sections, we show the proofs of these equations. Learning requires practice, so if this material is new to you, study the proofs carefully, and then try some similar functions on your own.

$d\ \mathbf{x^n} / dt = n\ x^{n-1}\ dx/dt$
$d\ \mathbf{sin(x)}\ dt = +\ cos(x)dx dt$
$d\ \mathbf{cos(x)}\ dt = -\ sin(x)\ dx dt$
$d\ \mathbf{tan(x)} / dt = \cos^{-2}(x)\ dx/dt$
$d\ \mathbf{e^x} / dt = e^x\ dx/dt$
$d\ \mathbf{ln\{x\}}\ dt = (1x)\ dx/dt$
$d\ \mathbf{sinh(x)}\ dt = cosh(x)\ dx dt$
$d\ \mathbf{cosh(x)}\ dt = sinh(x)\ dx dt$
$d\ \mathbf{tanh(x)} / dt = \cosh^{-2}(x)\ dx/dt$

Vector Differential Operators

We define the vector operator $\boldsymbol{\check{D}}$, in rectilinear coordinates, as:
$\boldsymbol{\check{D}} = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$
This operator, called *del*, is most often denoted by an inverted Δ, the Greek capital letter delta. However, that symbol is not supported by all eBooks, so I use $\boldsymbol{\check{D}}$.

We next consider the operation of $\boldsymbol{\check{D}}$ in 3-D rectilinear coordinates.

Applying $\boldsymbol{\check{D}}$ to any scalar function f yields the *gradient* of f:
$\boldsymbol{\check{D}}f = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$
The gradient is a vector that points in the direction in which f is increasing most rapidly with distance; the magnitude of that vector equals the derivative of f in that direction.

Taking the dot product of $\check{\boldsymbol{D}}$ with any vector field $\boldsymbol{A}$ yields a scalar, the *divergence* of $\boldsymbol{A}$:

$\check{\boldsymbol{D}}\bullet\boldsymbol{A} = \partial A_x/\partial x + \partial A_y/\partial y + \partial A_z/\partial z$

Taking the cross product of $\check{\boldsymbol{D}}$ with any vector field $\boldsymbol{A}$ yields a vector, the *curl* of $\boldsymbol{A}$:

$\check{\boldsymbol{D}}\times\boldsymbol{A} = (\partial A_z/\partial y - \partial A_y/\partial z,$
$\partial A_x/\partial z - \partial A_z/\partial x,$
$\partial A_y/\partial x - \partial A_x/\partial y)$

Lastly, we apply the *Laplacian* $\check{\boldsymbol{D}}\bullet\check{\boldsymbol{D}} = \check{D}^2$ to any scalar field f, yielding:

$\check{D}^2 f = \partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2$

These operators have different forms in other coordinate systems. Below, I show these operators in the two most common non-rectilinear systems. Due to their greater complexity, it is best to use these operators in rectilinear coordinates whenever possible.

<u>3-D Cylindrical Coordinates</u>

$\check{\boldsymbol{D}}$f is the vector (R, Φ, Z), with:

$R = \partial f/\partial r$

$\Phi = (1/r)\ \partial f/\partial \phi$

$Z = \partial f/\partial z$

$\check{\boldsymbol{D}}\bullet\boldsymbol{A} = (1/r)\ \partial(r\ A_r)/\partial r + (1/r)\ \partial A_\phi/\partial \phi + \partial A_z/\partial z$

$\check{\boldsymbol{D}}\times\boldsymbol{A}$ is the vector (R, Φ, Z), with:

$R = (1/r)\ \partial A_z/\partial \phi - \partial A_\phi/\partial z$

$\Phi = \partial A_r/\partial z - \partial A_z/\partial r$

$Z = (1/r)\ \partial(\ r\ A_\phi\ )/\partial r - (1/r)\ \partial A_r/\partial \phi$

$\check{D}^2 f = (1/r)\ \partial(\ r\ \partial f/\partial r\ )/\partial r + (1/r^2)\ \partial^2 f/\partial \phi^2 + \partial^2 f/\partial z^2$

<u>3-D Spherical Coordinates</u>

$\check{\boldsymbol{D}}$f is the vector (R, Θ, Φ), with:

$R = \partial f/\partial r$

$\Theta = (1/r)\ \partial f/\partial \theta$

$\Phi = (1/r\ \sin\theta)\ \partial f/\partial \phi$

$\check{\boldsymbol{D}}\bullet\boldsymbol{A} = (1/r^2)\ \partial(\ r^2\ A_r\ )/\partial r$
$+ (1/r\ \sin\theta)\ \partial(\ \sin\theta\ A_\theta\ )/\partial \theta$
$+ (1/r\ \sin\theta)\ \partial A_\phi/\partial \phi$

$\check{\boldsymbol{D}}\times\boldsymbol{A}$ = is the vector (R, Θ, Φ), with:

$R = (1/r\ \sin\theta)\ \{\ \partial\ (\sin\theta\ A_\phi\ )/\partial \theta - \partial A_\theta/\partial \phi\ \}$

$\Theta = (1/r \sin\theta)\, \partial A_r/\partial \phi - (1/r)\, \partial(r A_\phi)/\partial r$
$\Phi = (1/r)\, \partial(r A_\theta)/\partial r - (1/r)\, \partial A_r/\partial \theta$
$\check{D}^2 f = (1/r^2)\, \partial(r^2\, \partial f/\partial r)/\partial r$
$+ (1/r^2 \sin\theta)\, \partial(\sin\theta\, \partial f/\partial \theta)/\theta$
$+ (1/r^2 \sin^2\theta)\, \partial^2 f/\partial \phi^2$

Directional Derivative

We now know how to take derivatives along any coordinate axis. But sometimes, we also need to take derivatives along other directions, such as the direction specified by a vector ***v***. The derivative of function f(***r***) along ***v*** is defined using the gradient ***Ď*** by:

$\boldsymbol{v \ddot{Y} \check{D}}\, f(\boldsymbol{r}) = \lim dx \text{–>} 0\ \{ [f(\boldsymbol{r}+\boldsymbol{v}dx) - f(\boldsymbol{r})] / dx \}$

This equation is valid even if ***v*** varies with location. Often ***v*** is chosen to be a unit vector, such as the unit normal to a surface.

Proof of General Rules

Here, we prove the general rules of differentiation.

For brevity, I will write "P–>Q" to denote "in the limit that P goes to Q".

The definition of the derivative of F with respect to q is:
as dq–>0: $dF/dq = \{ F(q+dq) - F(q) \} / dq$
Hence,
$F(q+dq) = F(q) + dq\, dF/dq$
Also for brevity, F and G will mean F(q) and G(q).

»» **d(aF+bG)/dq**
$= \{ aF(q+dq) - aF + bG(q+dq) - bG \}/dq$
$= \{ aF + dq\, a\, dF/dq - aF$
$+ bG + dq\, b\, dG/dq - bG \}/dq$
$= a\, dF/dq + b\, dG/dq$

»» **d(FG)/dq** $= \{ F(q+dq)G(q+dq) - FG \}/dq$
$= \{ [F + dq\, dF/dq] [G + dq\, dG/dq] - FG \}/dq$
$= \{ FG + dq\, G\, dF/dq + dq\, dG/dq$
$+ dq^2\, dF/dq\, dG/dq - FG \}/dq$
As dq–>0, we can drop terms of order dq^2, yielding:
$= \{ dq\, G\, dF/dq + dq\, F\, dG/dq \}/dq$

= G dF/dq + F dG/dq

»» **d(GF)/dq** = { G(q+dq)F(q+dq) – GF}/dq
d(GF)/dq = { F(q+dq)G(q+dq) – FG}/dq
d(GF)/dq = d(FG)/dq

»» **d(F/G)/dq** = { F(q+dq) *G(q+dq) – F*G}/dq
= { [F + dq dF/dq] *[G + dq dG*dq] – F/G}/dq
= { [F + dq dF/dq]G – F[G + dq dG/dq] }
*{ [G + dq dG*dq] G dq}
As dq–>0, the denominator goes to G^2dq.
= { FG + dq G dF/dq – FG – dq F dG/dq }
/ { G^2 dq}
= (1/G) dF/dq – (F/G^2) dG/dq

»» **d(F^n)/dq** = {F^n(q+dq) – F^n}/dq
= { $(F + dq\ dF/dq)^n$ – F^n}/dq
Keeping only the lowest order terms in dq yields:
= { (F^n + n dq F^{n-1} dF/dq) – F^n}/dq
= n F^{n-1} dF/dq
The remainder of this chapter derives the derivatives of the most common functions used by physicists. A listing of common derivatives is in Appendix 6.

Derivative of x^n

dx^n/dx = limit dx–>0 { $[(x+dx)^n – x^n]$ / dx}

As we found in Chapter 9, we can rewrite $(x+dx)^n$ as:
$(x+dx)^n = \Sigma_k \{x^{n-k}\ dx^k\ n!\ /\ k!\ (n–k)!\ \}$
Here, Σ_k represents the sum over all values of k from k=0 to k=n. Dropping all terms of order dx^2 and higher reduces the sum to:
$(x+dx)^n = x^n + nx^{n-1}dx$
We now put that into the derivative equation.
dx^n/dx = limit dx –> 0 { $[(x+dx)^n – x^n]$ / dx}
= { $[x^n + nx^{n-1}dx – x^n]$ /dx}
$dx^n/dx = nx^{n-1}$
dx^n/dx (dx/dt) = nx^{n-1} (dx/dt)

$dx^n/dt = nx^{n-1}\ dx/dt$
This result is valid even if n is not an integer, as we prove later.

Derivatives of Trig Functions

Recall the trig relations:
sin(A+B) = sin(A) cos(B) + sin(B) cos(A)
cos(A+B) = cos(A) cos(B) – sin(B) sin(A)

Sine

d sin(x) *dx = {sin(x+dx) – sin(x) }* dx
sin(x+dx) = sin(x) cos(dx) + sin(dx) cos(x)
As dx–>0, cos(dx)–>1 and sin(dx)–>dx. Hence, as dx–>0:
sin(x+dx)–sin(x) –> sin(x)+dx cos(x) – sin(x)
sin(x+dx)–sin(x) –> dx cos(x)
d sin(x) / dx = cos(x)

Cosine

d cos(x) / dx = {cos(x+dx) – cos(x) } / dx
cos(x+dx) = cos(x) cos(dx) – sin(dx) sin(x)
As dx–>0, cos(dx)–>1 and sin(dx)–>dx. Hence:
cos(x+dx) –> cos(x) – dx sin(x)
cos(x+dx) – cos(x) –> – dx sin(x)
d cos(x) / dx = – sin(x)

Tangent

d tan(x) *dx = d [sin(x)*cos(x)] / dx
= [1/cos(x)] dsin(x)/dx
  +[sin(x)] $d\cos^{-1}(x)/dx$
= [1/cos(x)] cos(x)
  + (–1)[sin(x)] $\cos^{-2}(x)$ [–sin(x)]
= $\cos^{-2}(x)$ {$\cos^2(x)$ + $\sin^2(x)$}
d tan(x) / dx = $\cos^{-2}(x) = 1/\cos^2(x)$

Derivative of e^x

The definition of e is:
e = limit n–>∞ { $(1+1/n)^n$ }
e^x = exp(x) = lim n–>∞ $(1+1/n)^{xn}$
Since exponents in physics can sometimes be quite elaborate, I typically use exp{x} instead to e^x to improve eBook readability.

We now evaluate $(1+1/n)^{xn}$ using the binomial expansion. The terms with the smallest powers of the infinitesimal quantity 1/n are:

$\exp(x) = 1 + (xn)(1/n) + (xn)(xn–1)/2n^2$

$+ (xn)(xn–1)(xn–2)/3!n^3 + \ldots$

As xn–>∞, this becomes:

$\exp(x) = 1 + x + x^2/2 + x^3/3! + x^4/4! + \ldots$

d exp(x) *dx = 0 + 1 + 2x*2 + $3x^2/3! + 4x^3/4! + \ldots$

d exp(x) / dx = $1 + x + x^2/2 + x^3/3! + \ldots$

d exp(x) / dx = exp(x)

Derivative of Natural Logarithm

By definition of the natural logarithm ln:

x = exp(ln[x])

Taking the derivative of both sides with respect to x yields:

dx/dx = exp(ln[x]) d(ln[x])/dx

1 = x d(ln[x])/dx

d(ln[x])/dx = 1 / x

Derivatives of Hyperbolics

Recall the definitions of sinh, cosh, and tanh:

sinh(x) = (exp{x} – exp{–x})/2

cosh(x) = (exp{x} + exp{–x})/2

tanh(x) = sinh(x) / cosh(x)

Sinh

d sinh(x) / dx = (d exp{x} *dx – d exp{–x}* dx)/2

d sinh(x) *dx = (exp{x} + exp{–x})* 2

d sinh(x) / dx = cosh(x)

Cosh

d cosh(x) / dx = (d exp{x} *dx + d exp{–x}* dx)/2

d cosh(x) *dx = (exp{x} – exp{–x})* 2

d cosh(x) / dx = sinh(x)

Tanh

d tanh(x) / dx = d [sinh(x)/cosh(x)] / dx

= [1/cosh(x)] dsinh(x)/dx +[sinh(x)] $d\cosh^{-1}(x)/dx$

= [1/cosh(x)] cosh(x)

+ (–1)[sinh(x)] $\cosh^{-2}(x)$ [sinh(x)]

= $\cosh^{-2}(x)$ {$\cosh^2(x) – \sinh^2(x)$}

d tan(x) / dx = $\cosh^{-2}(x) = 1/\cosh^{2}(x)$

Derivative of x^a

For any x and constant a:
d (x^a) *dx = d (exp{ a ln[x] })* dx
= (exp{ a ln[x] }) d(a ln[x])/dx
= (x^a) a/x = a x^{a-1}

Chapter 13

Integral Calculus

It All Adds Up

Just as addition is the inverse of subtraction, integration is the inverse of differentiation. If the derivative of X equals Y, then the integral of Y equals X — well almost. Since the derivative of any constant is zero, a more precise statement is:

If the derivative of (X + any constant) = Y,
then the integral of Y = (X + any constant)

Generally, the so-called *arbitrary constant of integration* is determined by initial conditions, as we shall soon see.

Integrals solve problems that are the reverse of the problems that derivatives solve.

For example, consider a falling ball. In the prior chapter, we calculated the ball's velocity at time t from the equation for the distance the ball has dropped by time t. Now, let's ask: how far does the ball drop between time t=4 seconds and t=8 seconds?

Let's first use an equation quoted in the prior chapter for the distance s that a ball drops in time t.

$s(t) = g\ t^2 /2$

The distance fallen between t=4 seconds to t=8 seconds is:

$s(8\ \text{sec}) - s(4\ \text{sec}) = (g/2)\ [64\ \text{sec}^2 - 16\ \text{sec}^2]$

$s(8\ \text{sec}) - s(4\ \text{sec}) = 4.9\ \text{m/sec}^2\ [48\ \text{sec}^2]$

$s(8\ \text{sec}) - s(4\ \text{sec}) = 235.2\ \text{m}$

That is the correct answer, but now let's pretend that we know only the ball's velocity equation and not its distance equation. We know that distance equals velocity multiplied by time, but how can we calculate distance if the velocity is continuously changing?

The velocity equation is:

$v(t) = g\,t$, with $g = 9.8\ m/sec^2$

Let's imagine that the velocity is constant for a short time, and then changes to some other velocity for another short time, and so on. In that case, we could sum (velocity multiplied by time) for each "short time" and get the total distance traveled. Let's see how this works out, assuming the "short time" is 2 seconds.

Figure 13-1 shows a plot of the speed of a falling ball versus time, with the dotted line representing the true equation: $v(t) = g\,t$.

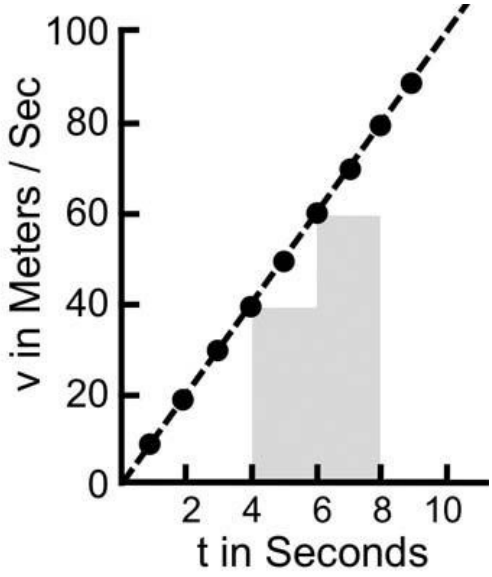


Figure 13-1 Plot of Velocity versus Time

The two shaded rectangles have heights of v(4sec) and v(6sec), and both have widths of 2 seconds. The area of each rectangle has units of meters/sec × sec = meters. Since we seek the distance traveled, we should sum the areas of these rectangles. The total area of both rectangles is:

Area = v(4sec)×2sec + v(6sec)×2sec

Area = (g×4sec)×2sec + (g×6sec)×2sec

Area = 9.8m×(8+12) = 196m

This total area of 196m is a rough approximation to the distance the ball actually traveled between 4 and 8 seconds, which we calculated above to be 235.2m. The area is only approximate because the two rectangles do not cover the entire region under the line in Figure 13-1. The ball's velocity changes substantially during 2 seconds, leaving gaps above the rectangles. We can do better using shorter time intervals. With 4 rectangles each 1 second wide, the total area covered is:

Area = [v(4)+v(5)+v(6)+v(7)]×1sec

Area = g×[4+5+6+7]×1
Area = 9.8 × 22 = 216m
With 8 rectangles, each 1/2 second wide, the area is 9.8 × 23 = 225m. With 16 rectangles, each 1/4 second wide, the area is 9.8 ×23.5 = 230m.

We are getting closer. Clearly the thing to do is to use an infinitesimal time interval dt, as shown in Figure 13-2.

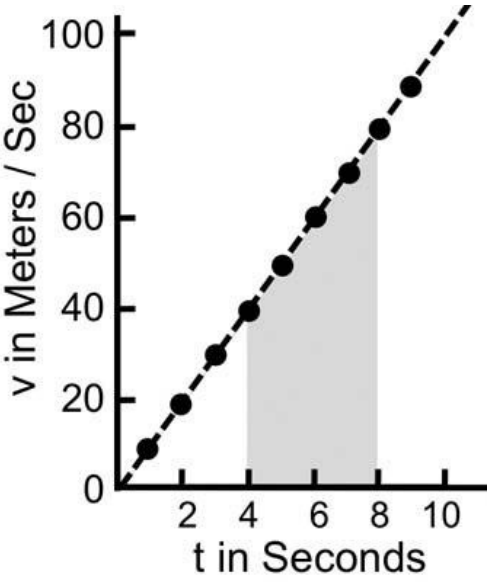


Figure 13-2 Shaded Area = Definite Integral

The shaded area under the line is the sum of an extremely large number of extremely thin rectangles, each of width Δt, which we label n = 1, 2, 3, …

We can now write:
Area = sum of $v(t_n) \times \Delta t$
Here, the sum is over all values of n

If we let n go to infinity, which means letting Δt go to zero, the sum of the areas of all the rectangles becomes an *integral* yielding the correct distance.

The operation of *integration* is denoted with the symbol ∫, an enlarged S derived from the Latin word *summa*. The general form of an integral is:
$Q = \int f(u)\, du$
Here, f(u) is the *integrand*, the quantity being summed, u is the independent variable, du is the infinitesimal change in u, and Q is the *integral*, the sum of the

areas of an infinite number of infinitesimal rectangles, each of height f(u) and width du.

The equation for finding the distance s traveled by an object with speed v(t), which need not be constant, is:
$s(t) = \int ds(t) = \int v(t)\, dt$
For the case of a falling ball:
$s(t) = \int g\, t\, dt = g\, t^2/2 + C$
The arbitrary integration constant C represents our arbitrary choice in defining the location of s=0. The equation says the ball will accelerate with the same time dependence from any initial height. In each specific situation, we set that initial height with C. Here, we will choose s=0 at time t=0, which makes C=0.

We know the value of the above integral because we found in Chapter 12 that the derivative of t^2 equals 2t, thus the integral of t equals $t^2/2$.

Integrals correspond to areas "under the curve" of a function, whereas derivatives correspond to the slope of that curve.

Definite & Indefinite Integrals

There are two types of integrals that are closely related. The integral that we just discussed:
$s(t) = \int v(t)\, dt$
is called an *indefinite integral*. We integrate the function v(t) with respect to t, and get another function s(t), which like any normal function has a value at each value of t.

The other type of integral is called a *definite integral*. Here we select two values of t, A and B, and the definite integral yields the area under the curve between t=A and t=B. The definite integral is written:
$\int_A^B v(t)\, dt = s(t=B) - s(t=A)$
In the example in Figure 13-2, the definite integral from t=4 sec to t=8 sec is the shaded area under the curve, which we compute as follows:
$\int_4^8 v(t)\, dt = s(8) - s(4)$
$= (g/2)\ \{\ (8\ \text{sec})^2 - (4\ \text{sec})^2\ \}$
$= (4.9\ \text{m/sec}^2)\ \{\ 48\ \text{sec}^2\ \}$
$= 235.2\text{m}$

The result of every indefinite integral includes an arbitrary constant. The result of every definite integral has no arbitrary constant; the constant is the same at both limits A and B, and therefore cancels.

How to Integrate

Summing rectangles, as we did above, is an integration procedure that works, but it must be done numerically, generally using a computer. We discuss the best approaches to numerical integration in Chapter 17. Numerical integration yields the area under the curve, but it cannot yield an analytical function. In our example, numerical integration of the ball's velocity yields a drop distance of 235.2 meters, but does not yield the equation for distance versus time: $s(t) = gt^2/2$.

In the prior chapter, we described an analytical procedure to differentiate any expression. Unfortunately, there is no corresponding general analytical procedure for integration. We learn how to do integrals with a haphazard reverse process: if we know that B is the derivative of A, then we know that A is the integral of B.

Mathematicians have differentiated a vast menagerie of functions and tabulated their results. You can search these tables hoping to find the answer to a challenging integral. If it is not listed, you must employ trickery or resort to numerical integration.

Most physicists memorize many common integrals and keep extensive tables handy for others.
Here are two simple but very useful integrals:

$$\int x^n \, dx = x^{n+1} / (n+1)$$

$$\int \exp\{u\} \, du = \exp\{u\}$$

Appendix 7 contains a short table of common integrals. In additional, I will present some useful tricks.

Integration by Parts

Our first trick is *integration by parts*. This allows us to transfer a derivative from

one function in an integrand to another. For any two functions u and v:
$\int v\, du = uv - \int u\, dv$
Proof:
$d(uv)/dx = u\, dv/dx + v\, du/dx$
$\int [d(uv)/dx]\, dx = \int [u\, dv/dx]\, dx + \int [v\, du/dx]\, dx$
$\int d(uv) = \int u\, dv + \int v\, du$
$uv - \int u\, dv = \int v\, du$
An example of integration by parts is the calculation of the variance σ^2 of the Gaussian distribution, which we discuss in Chapter 9. Recall the probability distribution:
$Prob(x) = \exp\{-x^2/2\} / \sqrt{(2\pi)}$
We first show that the total probability of all x values equals 1.
Let $Z = \int_{-\infty}^{+\infty} \exp\{-x^2/2\}\, dx / \sqrt{(2\pi)}$

As it turns out, Z^2 is easier to calculate than Z.
$Z^2 = \{\int \exp\{-x^2/2\}\, dx\}\, \{\int \exp\{-y^2/2\}\, dy\}/ 2\pi$
$2\pi\, Z^2 = \iint_{-\infty}^{+\infty} \exp\{-(x^2+y^2)/2\}\, dx\, dy$
Now switch to polar coordinates with:
$x = r \cos\theta$
$y = r \sin\theta$
$r^2 = x^2+y^2$
r goes from 0 to $+\infty$, and θ goes from 0 to 2π.

As explained in the section on Volume Elements in Chapter 14:

dx dy gets replaced by $r\, d\theta\, dr$

The integral in polar coordinates becomes:
$2\pi\, Z^2 = \int_0^{+\infty} \int_0^{2\pi} r \exp\{-r^2/2\}\, d\theta\, dr$
$2\pi\, Z^2 = 2\pi \int_0^{+\infty} r \exp\{-r^2/2\}\, dr$

Now let $u = -r^2/2$, with $du = -r\, dr$.

$2\pi\, Z^2 = -2\pi \int_0^{+\infty} \exp\{u\}\, du$
$2\pi\, Z^2 = -2\pi \exp\{u\} \big|_0^{+\infty} = 2\pi$
$Z = 1$
This confirms the the normalization factor.

Now let's calculate the variance.
$\sigma^2 = \int x^2 \, \text{Prob}(x) \, dx$

Let's begin by defining W:

$W = \sigma^2 \sqrt{(2\pi)} = \int_{-\infty}^{+\infty} x^2 \exp\{-x^2/2\} \, dx$
Choose u and v such that:
$v = x$
$dv = dx$

$u = -\exp\{-x^2/2\}$
$du = +x \exp\{-x^2/2\}$
We now integrate W by parts.
$W = \int v \, [du] = v \, [u] - \int u \, dv$
$W = \int_{-\infty}^{+\infty} x \, [x \exp\{-x^2/2\}] \, dx$
$W = -x \exp\{-x^2/2\} \, |_{-\infty}^{+\infty} + \int \exp\{-x^2/2\} \, dx$
$W = 0 + \sqrt{(2\pi)}$
$\sigma^2 = W / \sqrt{(2\pi)} = 1$
QED

Completing the Square

Another trick applies to integrals like:
$Q = \int \exp\{x^2+bx\} \, dx$
Here, b is either a constant or is not a function of x.

Without the bx term, the exponent would be a perfect square and we would have the answer from the prior section. The strategy of *completing the square* is to convert the exponent into a perfect square. We do this as follows:
$x^2 + bx = x^2 + bx + b^2/4 - b^2/4$
$x^2 + bx = (x + b/2)^2 - b^2/4$
The first term on the right is a perfect square and the second is a constant. We can therefore write:
$Q = \exp\{-b^2/4\} \int \exp\{ (x+b/2)^2 \} \, dx$
Now define u such that:
$(x+b/2) = u/\sqrt{2}$
$dx = du/\sqrt{2}$
With this substitution the integral becomes:
$Q = \exp\{-b^2/4\} \int \exp\{ u^2/2 \} \, du / \sqrt{2}$

$$Q = \exp\{-b^2/4\} \sqrt{(2\pi)} / \sqrt{2}$$
$$Q = \exp\{-b^2/4\} \sqrt{\pi}$$

Chapter 14

More Calculus

Path & Loop Integrals

Another type of integral is the *path integral*, also called a *line integral*. The path or line need not be a straight line. Figure 14-1 shows a path Γ of length L that starts at point A and ends at point B. The path may be arbitrarily complicated; path Γ may be a *closed* loop, in which case A=B, or it may be *open*, as shown in the figure.

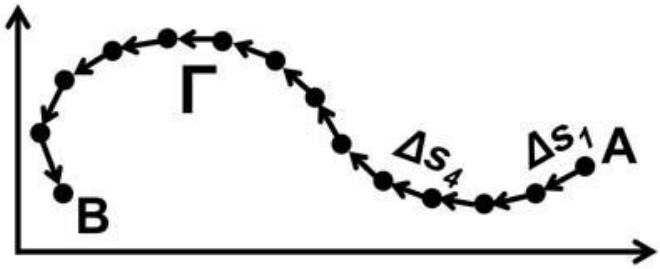


Figure 14-1 A to B Path Integral

Like other integrals, a path integral is the sum of N steps, each of length L/N, in the limit that N goes to infinity and the step size L/N goes to zero.

Each step can be represented by vector $\boldsymbol{\Delta s}$, which we number $\boldsymbol{\Delta s}_1$ through $\boldsymbol{\Delta s}_N$; $\boldsymbol{\Delta s}_1$ and $\boldsymbol{\Delta s}_4$ are labeled in Figure 14-1. Consider the integral Q:

$Q = \int_\Gamma ds$ = limit N–>∞ $\{\Sigma_k \Delta s_k\}$

Here, the subscript Γ on the integral sign denotes the path, and Σ_k is the sum from k=1 to k=N. In the limit that N goes to ∞, Σ_k becomes an infinite sum of infinitesimal step lengths, which is the total length L of path Γ.

Another example is the path integral of the tangential component of a vector field $\boldsymbol{C}$:

$Q = \int_\Gamma \boldsymbol{C} \bullet \boldsymbol{ds}$ = limit N–>∞ $\{\Sigma_k \boldsymbol{C} \bullet \boldsymbol{\Delta s}_k\}$

Here, $\boldsymbol{C} \bullet \boldsymbol{\Delta s}_k$ is the tangential component of $\boldsymbol{C}$ at each step, the component of $\boldsymbol{C}$ parallel to each infinitesimal step $\boldsymbol{\Delta s}_k$. The result Q is the sum of those

infinitesimal tangential components along the total path Γ.

A *loop integral* is a path integral in which the path forms a closed loop, as shown in Figure 14-2.

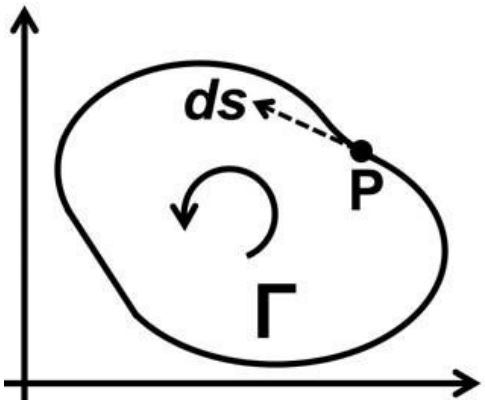


Figure 14-2 Integral Around Loop Γ

The normal sign convention for a loop integral is to go around the loop counterclockwise. This is another instance of the right hand rule: with your right thumb pointing toward you (out of the screen), the fingers of your right hand curl in the counterclockwise direction. If one performs the integral in the clockwise direction, one must multiply the result by –1.

Area & Volume Integrals

For simplicity of discussion, I will use "volume" here to describe any defined region with any number of dimensions, whether it is a 2-D area, a 3-D volume, or an n-D hyper-volume.

In rectilinear coordinate systems, integrating over an n-dimensional volume simply requires doing n one-dimensional integrals. For example, consider an integral over x, over y, and over z with these limits:
x: 0 to X
y: 0 to Y
z: 0 to Z
This integral is over the entire volume of a cuboid whose opposite corners are at (0,0,0) and (X,Y,Z). To make this as simple as possible, let the integrand be f, a constant.

$Q = \iiint f \, dx \, dy \, dz$

$Q = \iint (f |_0^X) \, dy \, dz = X \iint f \, dy \, dz$

$Q = X \int (f |_0^Y) \, dz = X \, Y \int f \, dz$

$Q = f\ X\ Y\ Z = f\ V$
Here, $V = X\ Y\ Z$ is the cuboid's volume.

If we now let f be a function of the coordinates, Q becomes:
$Q = \langle f \rangle\ V$
Here, $\langle f \rangle$ represents the average value of f throughout the cuboid.

This procedure works for any rectilinear coordinate system with any number of dimensions.

Let's next examine the integral of f across a disk of radius R that is centered at the origin. Assume for now that f is constant.

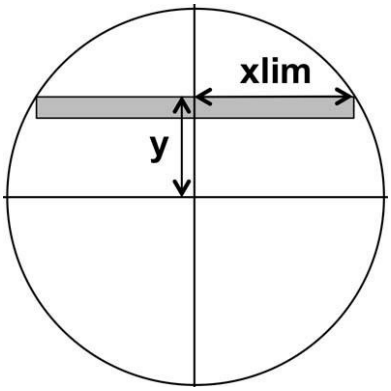


Figure 14-3 Integral in Rectilinear Coordinates

Figure 14-3 shows how we will do this: for each y value, from y=–R to y=+R, we sum a rectangle of height dy and width 2xlim, where $xlim = \sqrt{(R^2 - y^2)}$.
The integral is:
$Q = \iint f\ dx\ d$
$Q = f \int_{-R}^{+R} x\Big|_{-xlim}^{+xlim} dy$
$Q = f \int 2\ \sqrt{(R^2 - y^2)}\ dy$
From tables, this integral is:
$Q = f\ \{\ y\sqrt{(R^2 - y^2)} + R^2 \arcsin(y/R)\ \}\Big|_{-R}^{+R}$
$Q = f\ \{\ 0 + R^2\ (\pi/2 + \pi/2)\ \}$
$Q = f\ \pi\ R^2$
The result Q is exactly what we should expect: the integral equals f multiplied by the area within the circle of radius R.

Volume Elements

Let’s try the previous integral again, but now using polar coordinates. With r going from 0 to R, and θ going from 0 to 2π, the integral seems to be:
$Q = \int_0^{2\pi} \int_0^R f \, dr \, d\theta$
$Q = f \int_0^{2\pi} R \, d\theta = f \, 2\pi \, R$, WRONG
This calculation is wrong because dr×dθ is not the area enclosed when the radius sweeps from r to r+dr and the angle sweeps from θ to θ+dθ. In fact dr×dθ does not even have the units of area; its units are distance × radians. The problem is illustrated in 14-4.

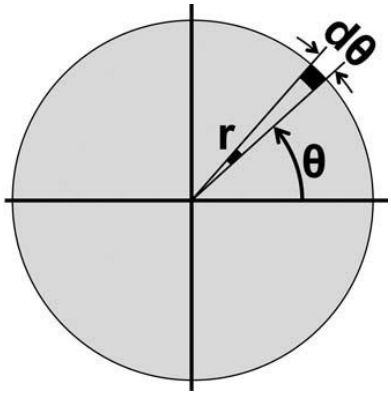


Figure 14-4 Integral in Polar Coordinates

Here, two black enclosed regions have the same width dθ and the same length dr, but clearly have very different areas. The proper width is not dθ, but rather rdθ, and the incremental area is:
for 2-D polar coordinates: r dr dθ
This expression is one example of a *volume element*, the measure of the space enclosed by infinitesimal changes in each coordinate axis. This name is commonly used regardless of the dimensionality of that space. Every coordinate system has a volume element; for rectilinear coordinates, the volume element is the product of the coordinate differentials, with no additional factors.

Let’s redo the prior integral with the proper volume element:
$Q = \int_0^{2\pi} \int_0^R \mathbf{r \, dr \, d\theta} = (R^2/2)\,(2\pi) = \pi R^2$
This is the correct result.

Here are the volume elements for all coordinate systems that physicists commonly use.
2-D polar: r dr dθ
2-D rectilinear: dx dy
3-D rectilinear: dx dy dz

3-D cylindrical: r dr dø dz
3-D spherical: r^2 sinθ dr dθ dø
4-D spacetime: c Δt Δx Δy Δz
We now know how to integrate in two other non-rectilinear coordinate systems: 3-D cylindrical and 3-D spherical, denoted C and S respectively.
$C = \int_0^Z \int_0^{2\pi} \int_0^R r\, dr\, d\text{ø}\, dz = (R^2/2)\,(2\pi)\, Z$
$C = \pi R^2 Z$
$S = \int_0^{2\pi} \int_0^{\pi} \int_0^R r^2 \sin\theta\, dr\, d\theta\, d\text{ø}$
$S = (R^3/3)\,(2)\,(2\pi) = 4\pi R^3/3$
Q, C, and S are respectively: the area of a disk of radius R; the volume of a cylinder of radius R and length Z; and the volume of a ball of radius R. If we multiply each integrand by f, the integrals would become <f>×volume for the disk, cylinder, or ball. This is exactly what integrals are supposed to do: sum an expression over a defined space, or equivalently, find an expression's average value multiplied by the volume of that space.

For any other coordinate system W, the volume element equals the determinant of the *Jacobian* matrix: $J_{kn} = \partial W_k / \partial X_n$, where W_k are the coordinates of W, and X_n are the rectilinear coordinates. We discuss matrices and determinants in Chapter 16.

Variational Calculus

In many situations, we seek the value of x at which a function f(x) reaches a maximum or minimum (an extremum). We find the answer by setting $\partial f/\partial x = 0$, and solving this equation for x.

Variational calculus solves more complex problems, those in which the entity that varies is more than simply a single quantity x. A typical example is: what path from point A to point B minimizes some function Q? We cannot simply take the derivative of Q in one direction; we must instead be able to compare alternative paths in multiple dimensions.

To do this, we need a more powerful tool: the *calculus of variations*.

Here is how variational calculus works.

Imagine two paths that start at the same x and t, and end at the same x and t, as shown in Figure 14-5. The true path (what we seek) is represented by the solid curve, and one alternate path is represented by the dashed curve.

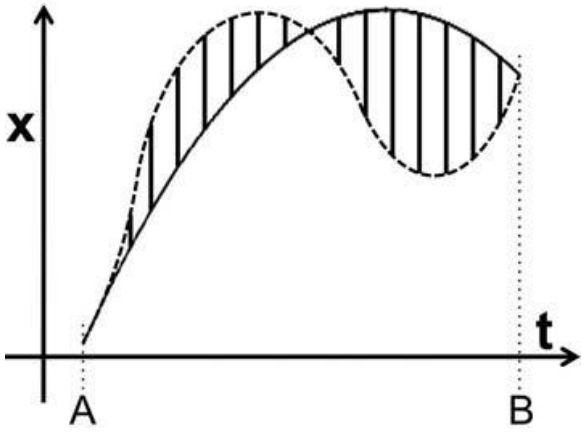


Figure 14-5 Path Difference: True vs. Alternate

We define x(t) and w(t) such that:
true path: x(t)
alternate: x(t) + w(t)
Thus w(t) is the difference between the true path and the alternate path. The length of each vertical line in Figure 14-5 represents the value of w(t) at selected times. In our analysis, we will consider alternate paths that deviate only slightly from the true path, which means:

$|w(t)| << |x(t)|$ for all t.

Let's consider the example of minimizing the *action* S, which equals an object's kinetic energy minus its potential energy U, as given by:
$S = mv^2/2 - U(x)$
Along the true path, the integral of S is:
$S_{true} = \int_A^B \{ (dx/dt)^2 \, m/2 - U(x) \} \, dt$
Along the alternate path, the integral of S is:
$S_{alt} = \int_A^B \{ (d[x+w]/dt)^2 \, m/2 - U(x+w) \} \, dt$
Since the true path has the least action, S_{alt} must be greater than or equal to S_{true}.
We define the *variation in S*, δS, to be:
$\delta S = S_{alt} - S_{true} >= 0$
$\delta S = \int_A^B \{ [(d[x+w]/dt)^2 - (dx/dt)^2] \, (m/2)$
$- U(x+w) + U(x) \} \, dt$
Let's simplify this piece by piece, beginning with the U terms. Since w is small,

we will drop terms proportional to w^2 and higher powers. From the definition of a derivative, we have:

$U(x+w) = U(x) + (dU/dx)\, w$

$-U(x+w) + U(x) = -(dU/dx)\, w$

Now, let's simplify the difference of the second derivatives in the δS equation.

$[(d[x+w]/dt)^2 - (dx/dt)^2]$

$= (dx/dt + dw/dt)^2 - (dx/dt)^2$

$= 2\,(dx/dt)\,(dw/dt)$

We then have:

$\delta S = \int_A^B \{\, [m(dx/dt)(dw/dt)] - (dU/dx)w\}dt$

Let's ignore the clutter for a moment and look at the "big picture". The integrand has this form:

$P\, dw/dt - Q\, w$

If it were $Pw - Qw$, we would immediately have a solution. So, the way forward is to turn Pdw/dt into Rw, for some R. But how?

Here is the **first key step:** *integration by parts*. Recall that for any two functions u and v:

$\int_A^B u\, dv = uv\,|_A^B - \int_A^B v\, du$

In the present case, set:

$v = w$

$u = dx/dt.$

$dv/dt = dw$

$du/dt = d^2x/dt^2$

Integration by parts yields:

$\int_A^B (dx/dt)\,(dw/dt)\, dt$

$= (dx/dt)\, w\,|_A^B - \int_A^B w\,(d^2x/dt^2)\, dt$

Here is **the second key step**: $w(A) = w(B) = 0$. The alternate path and the true path both start at the same x and t and end at the same x and t. We vary the path *between* the endpoints, but not *at* the endpoints. The δS equation reduces to:

$\delta S = -\int_A^B \{\, m\,(d^2x/dt^2) + (dU/dx)\,\}\, w\, dt$

As we know, functions change very slowly near their extrema. When the alternate path is the same as the true path, when $w=0$ everywhere, $\delta S=0$. When the alternate path is close to the true path, δS will be very close to zero, with δS deviating from zero only in the second order.

Our problem boils down to finding the x(t) for which $\delta S = 0$ for any small path deviation w(t).
The third key step: along the true path the term in { }'s *is zero everywhere*. Why?

Consider a function w(t) that is nonzero only between t* and t*+Δt. If Δt is small enough, we can approximate w(t) as constant, and get:
$\delta S = -\{ m [d^2x(t^*)/dt^2] + [dU(t^*)/dx] \} w(t^*) \Delta t$
If the term in { }'s is not zero at t*, we can make S smaller by choosing some nonzero w(t*). But this contradicts the definition that x(t) is the true path, which requires $\partial S >= 0$ for any nonzero w(t*). Since this applies to every value of t*, the true path is defined by:
for all t: $0 = \{ m [d^2x(t)/dt^2] + [dU(t)/dx] \}$
We therefore have our solution:
for all t: $m\, d^2x/dt^2 = - dU/dx$
In one dimension, $F = - dU/dx$, resulting in Newton's familiar equation:
$m a = F$
With variational calculus we proved that, for any conservative force F, every body moves according to Newton's second law: F = ma.

Feynman provides sage advice on this variational method:

> "It turns out that the whole trick of the calculus of variations consists of writing down the variation of S and then integrating by parts so that the derivatives of [w] disappear. It is always the same in every problem in which derivatives appear. … [Next] comes something which always happens—the integrated part disappears."

Project, Divide & Conquer

Another approach to solving complicated equations is to divide them into multiple simpler equations.

One such technique employs projecting a vector equation onto any selected vector using the dot product. In ***Feynman Simplified 2D***, Chapter 44, Feynman derives the following frightening equation for the steady flow of a non-viscous, incompressible fluid:

$0 = (\check{\boldsymbol{D}}\times\boldsymbol{v})\times\boldsymbol{v} + \check{\boldsymbol{D}}(\boldsymbol{v}\bullet\boldsymbol{v})/2 + \check{\boldsymbol{D}}P/\mu + \check{\boldsymbol{D}}ø$

Here, $\boldsymbol{v}$ is the fluid velocity, μ is its mass density, ø is its potential energy per unit mass, and P its pressure. We can learn something about fluid flow from the dot product of this equation with $\boldsymbol{v}$.

$0 = \boldsymbol{v}\bullet\{(\check{\boldsymbol{D}}\times\boldsymbol{v})\times\boldsymbol{v} + \check{\boldsymbol{D}}(\boldsymbol{v}\bullet\boldsymbol{v})/2 + \check{\boldsymbol{D}}P/\mu + \check{\boldsymbol{D}}ø\}$

The first term is zero since (anything)×$\boldsymbol{v}$ is always orthogonal to $\boldsymbol{v}$. This eliminates the most challenging term in the equation, and provides Bernoulli's theorem:

$0 = \boldsymbol{v}\bullet\check{\boldsymbol{D}}\ \{\ \boldsymbol{v}\bullet\boldsymbol{v}/2 + P/\mu + ø\ \}$

This says the quantity in { }'s, which is the fluid's energy per unit mass, is constant along every fluid streamline.

We can also project a vector equation onto the surface perpendicular to any vector $\boldsymbol{n}$. For simplicity, assume $\boldsymbol{n}$ is a unit vector. We take the dot product of the equation with $\boldsymbol{n}$, and then subtract the result from the initial equation. We can write all this as the vector operator:

$(1 - \boldsymbol{n}\bullet\ )$

Here is a trivial example, with $\boldsymbol{n}$ = (1, 0, 0) operating on vector (A, B, C).

$(1 - \boldsymbol{n}\bullet\ )$ (A, B, C) = (A, B, C) – (A, 0, 0) = (0, B, C)

The final result is a vector within the yz-plane that is perpendicular to x.

Both projection procedures can be generalized to tensor equations.

A related technique divides a vector field $\boldsymbol{u}$ into two parts: vector field $\boldsymbol{u}_{zd}$ that has zero divergence; and vector field $\boldsymbol{u}_{zc}$ that has zero curl.

The equation for a disturbed solid, perhaps our planet after an earthquake, is:

$\rho\ \partial^2\boldsymbol{u}/\partial t^2 - \mu\ \check{D}^2\boldsymbol{u} = (\lambda+\mu)\ \check{\boldsymbol{D}}(\check{\boldsymbol{D}}\bullet\boldsymbol{u})$

Here, ρ is the mass density, λ and μ are elasticity constants, and $\boldsymbol{u}$ is the displacement vector field. This looks like a harmonic equation, except for the term on the right.

Using $\boldsymbol{u} = \boldsymbol{u}_{zd} + \boldsymbol{u}_{zc}$, we obtain an equation that I will label ZDZC::

(ZDZC): $(\lambda+\mu)\ \check{\boldsymbol{D}}(\check{\boldsymbol{D}}\bullet\boldsymbol{u}_{zc})$

$= \rho\ \partial^2(\boldsymbol{u}_{zd}+\boldsymbol{u}_{zc})/\partial t^2 - \mu\check{D}^2(\boldsymbol{u}_{zd}+\boldsymbol{u}_{zc})$

Taking the divergence of (ZDZC) eliminates $\boldsymbol{u}_{zd}$.

$0 = \rho\ \check{\boldsymbol{D}}\bullet\partial^2(\boldsymbol{u}_{zc})/\partial t^2 - (\lambda+2\mu)\ \check{D}^2(\check{\boldsymbol{D}}\bullet\boldsymbol{u}_{zc})$

$0 = \check{\boldsymbol{D}}\bullet\{\ \rho\ \partial^2\boldsymbol{u}_{zc}/\partial t^2 - (\lambda+2\mu)\ \check{D}^2\boldsymbol{u}_{zc}\ \}$

Since $\boldsymbol{u}_{zc}$ has zero curl, the expression in { }'s has both zero divergence and zero curl everywhere. A solution that satisfies these requirements is the harmonic equation:

$0 = \rho\ \partial^2\boldsymbol{u}_{zc}/\partial t^2 - (\lambda+2\mu)\ \check{D}^2\boldsymbol{u}_{zc}$

Now taking the curl of (ZDZC) eliminates $\boldsymbol{u}_{zc}$.

$0 = \rho\ \check{\boldsymbol{D}}\times\partial^2(\boldsymbol{u}_{zd})/\partial t^2 - \mu\check{\boldsymbol{D}}\times\check{D}^2(\boldsymbol{u}_{zd})$

Again, we have a field with zero divergence and zero curl everywhere. A solution is the harmonic equation:

$0 = \rho\ \partial^2\boldsymbol{u}_{zd}/\partial t^2 - \mu\check{D}^2\boldsymbol{u}_{zd}$

Thus, $\boldsymbol{u}_{zd}$ corresponds to *transverse* or *shear* waves with velocity $C_{zd} = \sqrt{(\mu/\rho)}$, and $\boldsymbol{u}_{zc}$ corresponds to *compression* or *longitudinal* waves with the greater velocity $C_{zc} = \sqrt{(\lambda+2\mu)/\rho}$.

One More Trick

I want to share another fine example of Feynman's mathematical trickery. As shown in ***Feynman Simplified 1B***, Chapter 20, the intensity of Planck black body radiation is proportional to:

$I(z) = z^3 / (\ \exp\{z\} - 1)$

with $z = \hbar\omega / kT$.

Feynman shows us how to integrate I(z) from z=0 to z=∞.

We can rewrite Planck's equation as:

$[z^3\ exp\{z)]\ [1(1 - \exp\{-z\}]$

Since $\exp\{-z\}<1$, for $z>0$, and $I(z=0)=0$, we can employ the infinite series:

$1 / (1-x) = 1 + x + x^2 + x^3 + \ldots$

We then have:

$I(z) = [z^3 \exp\{-z\}]\ [1 + \exp\{-z\} + \exp\{-2z\} + \ldots\ ]$

$I(z) = z^3\ [\ \exp\{-z\} + \exp\{-2z\} + \exp\{-3z\} + \ldots\ ]$

$I(z) = z^3\ \Sigma_n\ \exp\{-nz\}$, sum from n=1 to ∞

Now, for each n, we need to calculate:

$I_n = \int z^3\ \exp\{-nz\}\ dz$

Let's start with a simpler integral:

$\int_0^\infty \exp\{-nz\}\ dz = (-1/n)\ \exp\{-nz\}\ |_0^\infty = 1/n$

Next comes a great trick. Take the *third derivative* of this equation *with respect to n*, yielding:

$\int_0^\infty (-z)^3\ \exp\{-nz\}\ dz = d^3\ (n^{-1})\ /dn^3$

$\int_0^\infty (-z)^3 \exp\{-nz\}\, dz = (-1)(-2)(-3)\, n^{-4}$

$\int_0^\infty z^3 \exp\{-nz\}\, dz = 6\, n^{-4}$

Hence:

$\int I(z)dz = \int z^3 \Sigma_n \exp\{-nz\}\, dz$

$\int I(z)dz = 6\, \Sigma_n\, n^{-4} = 6\,(1 + 1/2^4 + 1/3^4 + \ldots)$

In Appendix 2, the sum in ( )'s is given as $\pi^4/90$, which makes our integral:

$\int I(z)dz = \pi^4 / 15$

Chapter 15

Differential Equations

Almost all physics equations are differential equations, equations that define relationships among derivatives of quantities of interest. There is no one simple procedure for solving differential equations, because there is no one simple procedure for solving integrals.

We learn how to solve differential equations on a case-by-case basis, and by experience. For theoretical physicists, solving different equations is a survival art.

Linear Differential Equations

Let's begin with linear differential equations because: (1) we can actually solve these equations; and (2) many fundamental laws of physics are linear, or approximately so.

Perhaps the most common and most important differential equation describes **harmonic motion**:

$d^2x/dt^2 + \omega^2 x = 0$

The solution to this equation can be written in two different forms:

$x(t) = A \cos(\omega t+\phi)$

$x(t) = \mathrm{Re}\,[\, A \exp\{i\omega t\}\,]$

In both solutions, the system oscillates with amplitude A and frequency ω. In the first solution, ø is a constant phase shift, and A is a real constant. In the second solution, A may be complex (thus providing a phase shift), and x is the real part of the complex function in [ ]'s.

The general class of linear differential equations with constant coefficients has the form:

$a_n d^nx/dt^n + \ldots + a_1 dx/dt + a_0 x = f(t)$

If j is the largest index for which a_j is nonzero, we say the equation is jth *order*. In physics, we generally deal with second order differential equations, because forces are linked to accelerations, which are the second order derivatives of

position. For any order, we can define an *operator* that encompasses in one symbol all the derivative operations that are the guts of our equation. We define Λ so that:

$\Lambda(x) = a_n d^n x/dt^n + \ldots + a_1 dx/dt + a_0 x$

All the physics is encoded into Λ, in the number of coefficients and their values. All that remains is to plug in whatever function x we wish.

The most important feature of linear systems is *linear superposition*: the ability to combine solutions by simple addition. Any linear system specified by Λ, with two solutions x and y, and any two constants a and b, has another solution ax+by, as shown here:

$\Lambda(ax+by) = a\Lambda(x) + b\Lambda(y)$

Furthermore, let u be a solution of this system when it is unforced, when f(t)=0. This means Λ(u)=0. Now, if x is a solution when an external driving force is applied, then:

$\Lambda(x+u) = \Lambda(x) + \Lambda(u)$

$\Lambda(x+u) = \Lambda(x) + 0$

$\Lambda(x+u) = \Lambda(x)$

Hence, we can simply add any *transient solution* u to any forced solution x. Transient solutions are necessary to describe situations in which a driving force ceases. Transient solutions are also required to describe what happens when a driving force begins. An initially stationary system does not instantaneously transition from zero motion to periodic motion. At the start, the motion is a combination of periodic and transient motions, with the latter diminishing exponentially.

Thus we can linearly sum all types of solutions to solve a wide range of problems. This is true even when the driving force F is not a simple harmonic function.

Linear System Example

Let's consider an example: a mass m on a spring. For small displacements x(t), we can approximate the spring force F with Hooke's law, F=–kx, making this a linear system. In isolation, this system is a simple harmonic oscillator, whose differential equation and solution were presented at the start of this chapter:

$m\, d^2x/dt^2 + k\, x = 0$

$x(t) = A \cos(\omega t+ø)$

with $\omega = \sqrt{(k/m)}$

If that mass is now subject to an external driving force fcos(ßt), the system's differential equation becomes:

$m\, d^2x/dt^2 = -k\, x + f \cos(\beta t)$

A trial solution $x(t) = D \cos(\beta t)$ yields:

$-ß^2\, mD \cos(\beta t) = -k\, D \cos(\beta t) + f \cos(\beta t)$

$(k/m - ß^2)\, mD = f$

$D = (f/m) / (\omega^2 - ß^2)$

Let's now add a damping force F, such as friction, that is proportional to the velocity of the mass, according to:

damping force: $F = -\mu m (dx/dt)$

The system's differential equation becomes:

$m\, d^2x/dt^2 = -k\, x + f \cos(\beta t) - \mu m\, dx/dt$

We will now employ complex numbers and variables to more conveniently describe this motion.

A trial solution $x(t) = \mathrm{Re}\,(D \exp\{i\beta t\})$ yields:

$-ß^2\, m\, x = -kx + f \cos(\beta t) - i\beta\mu m\, x$

$(k/m - ß^2 + i\beta\mu)\, mD = f$

$D = (f/m) / (\omega^2 - ß^2 + i\mu\beta)$

In this case, we find D is complex, with amplitude r and lag angle θ, according to:

$r = (f/m)/\sqrt{[(\omega^2 - ß^2)^2 + (\mu ß)^2]}$

$\sin\theta = mr\mu ß / f$

at $ß = \omega$, $r = f/\mu m ß$ and $\theta = \pi/2$

Peak-to-width ratio $Q = \omega/\mu$

If an oscillator's driving force stops at t= 0, one of three behaviors results depending on the values of ω and μ. Figure 15-1 shows the three possible outcomes: curve O is the overdamped response, curve U is the underdamped response, and curve C is the critically damped response.

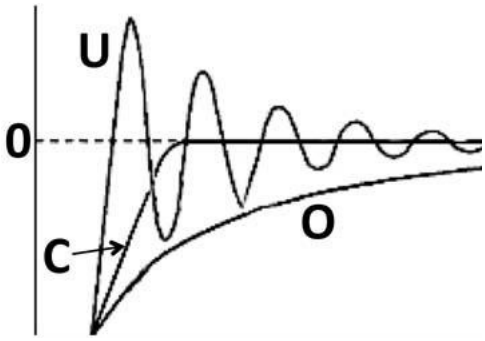


Figure 15-1 Three
Damped Transient Responses

Let: $\Omega^2 = \omega^2 - \mu^2/4$

If $\Omega^2 > 0$, system is **underdamped** (curve U)
$x = A \exp\{-t\mu/2\} \cos(a+\Omega t)$
motion oscillates while decreasing slowly
If $\Omega^2 = 0$, system is **critically damped** (curve C)
$x = A \exp\{-t\mu/2\}$
motion decreases rapidly
If $\Omega^2 < 0$, system is **overdamped** (curve O)
define $\beta^2 = -4\Omega^2 = \mu^2 - 4\omega^2 > 0$
$x = A \exp\{-t(\mu+\beta)/2\} + B \exp\{-t(\mu-\beta)/2\}$
motion decreases slowly

Quasi-Linear System Example

So far, we have analyzed systems that were assumed to be linear. This is an idealization that may be only approximately correct for many real systems. While the most general nonlinear systems cannot be analyzed simply, we can gain insight into the interesting behaviors of *quasi-linear systems*, those that are only slightly nonlinear.

Consider an electronic device with an input x(t) that produces an output y(t). In a nonlinear device, the relationship may have the form:
$y(t) = K [x(t) + \varepsilon x^2(t)]$
If ε is small enough, the nonlinear term $\varepsilon x^2(t)$ is small compared with x(t).

Consider a sinusoidal input x, and the output y, described by:
$x(t) = \cos(\omega t)$
$y(t) = K[\cos(\omega t) + \varepsilon \cos^2(\omega t)]$
If ε were zero, the system would be governed by a linear differential equation, resulting in the thin, sinusoidal curve in Figure 15-2. If ε is slightly greater than zero, the response is slightly nonlinear, resulting in the bold curve that is more complex than the sinusoidal curve

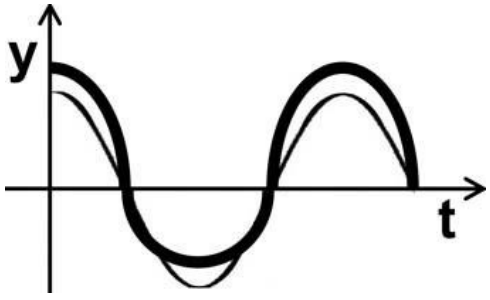


Figure 15-2 Linear (light) vs.
Nonlinear (dark) Responses

Recall that:
$\cos(2A) = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$
With that, we can rewrite the y(t) equation as:
$y(t) = K\cos(\omega t) + K\varepsilon/2 + (K\varepsilon/2)\cos(2\omega t)$
The first of the three terms on the right side of this equation is the normal linear response at the same frequency ω as the input x.

The second term adds a constant offset to y(t), shifting its average value. This shifting of the entire response is called *rectification*.

The third term adds a higher frequency harmonic to y(t). A nonlinearity proportional to x^2 results in a *second harmonic* at frequency 2ω. Nonlinearities proportional to x^3 or x^4 would add third or fourth harmonics at frequencies 3ω or 4ω, respectively. The most general nonlinearity would introduce an entire spectrum of harmonics. Adding harmonics is called *modulation*.

Now consider the response of a quasi-linear device to an input with two components of different frequency and amplitude.
$x(t) = A\cos(\omega t) + B\cos(\Omega t)$
$y(t) = K[x(t) + \varepsilon x^2(t)]$
$y(t) = K\,x(t) + K\varepsilon[A\cos(\omega t) + B\cos(\Omega t)]^2$
$y(t) = K\,x(t) + K\varepsilon[A^2\cos^2(\omega t) + B^2\cos^2(\Omega t)]$
$\quad + 2K\varepsilon AB\cos(\omega t)\cos(\Omega t)$
The term in [ ]'s is the same as the prior example; it produces second harmonics. The last term, containing the product of cosines of different frequencies, produces sidebands with cosines of the sum and difference of ω and Ω. We can rewrite this equation as:
$y(t) = KA\cos(\omega t) + KB\cos(\Omega t) + (K\varepsilon/2)[A^2+B^2]$
$\quad - (K\varepsilon/2)[A^2\cos(2\omega t) + B^2\cos(2\Omega t)]$
$\quad + K\varepsilon AB\{\cos([\omega+\Omega]t) + \cos([\omega-\Omega]t)\}$
We have here a combination of interesting effects:
linear response: $KA\cos(\omega t) + KB\cos(\Omega t)$
rectification: $(K\varepsilon/2)[A^2+B^2]$
harmonics: $A^2\cos(2\omega t) + B^2\cos(2\Omega t)$
and sidebands: $\cos([\omega+\Omega]t) + \cos([\omega-\Omega]t)$
If ω is nearly equal to Ω, the sidebands include a term with a frequency of about

2ω and another at ω–Ω. If ω>>Ω, the two sidebands are at nearly the same frequency. An entirely equivalent way of looking at this term comes from considering its prior form: cos(ωt) cos(Ωt). If ω is nearly equal to Ω, this term produces *beats*. If ω>>Ω, y(t) oscillates at ω, while being slowly modulated at frequency Ω. Both descriptions are perfectly correct.

Note that all the nonlinear effects are proportional to the second power of amplitudes: A^2, B^2, or AB. This means nonlinear effects are more important for larger inputs.

These nonlinear effects — rectification, harmonics, modulation, sum and difference frequencies — have many practical implications. It is believed that the human ear is somewhat nonlinear. Very loud sounds give us the sensation of harmonics, and sum and difference frequencies, even when the input is monotonic.

Separating Coupled Differential Equations

We sometimes encounter problems described by two coupled differential equations, such as the following equations for the voltage V and current J in a transmission line whose inductance and capacitance per unit length are L and C.

$$\partial V/\partial x = - L\, \partial J/\partial t$$
$$- \partial J/\partial x = C\, \partial V/\partial t$$

We separate V and J by differentiating the first equation with respect to x and the second with respect to t, so that both contain the term $\partial^2 J/\partial t \partial x$.

$$\partial^2 V/\partial x^2 / L = - \partial^2 J/\partial t \partial x$$
$$- \partial^2 J/\partial t \partial x = C\, \partial^2 V/\partial t^2$$
$$\partial^2 V/\partial x^2 - L\, C\, \partial^2 V/\partial t^2 = 0$$

The same trick, done the other way, yields two equations containing $\partial^2 V/\partial t \partial x$.

$$\partial^2 V/\partial x \partial t = - L\, \partial^2 J/\partial t^2$$
$$- \partial^2 J/\partial x^2 / C = \partial^2 V/\partial x \partial t$$
$$\partial^2 J/\partial x^2 - C\, L\, \partial^2 J/\partial t^2 = 0$$

We see that both V and J satisfy the 1-D wave equation with $v^2 = 1/LC$:

$$\partial^2 \psi/\partial x^2 - \partial^2 \psi/\partial t^2 / v^2 = 0$$

Separation of Variables – By Axes

Another approach to divide and conquer is called *separation of variables*. An

example is separating a complex function of multiple coordinates into simpler functions, each describing the dependence along a different coordinate axis.
Consider the harmonic equation:
$\check{D}^2\psi = c^{-2}\,\partial^2\psi/\partial t^2$
Since this is a linear differential equation, finding a complete basis set of solutions is sufficient. All other solutions will be linear combinations of those basis set solutions. To search for solutions in polar coordinates, we begin with a trial solution that is the product of two new functions X(***r***) and T(t), as follows:
$\psi(\boldsymbol{r},t) = X(\boldsymbol{r})\,T(t)$
We insert this into the prior differential equation, then divide by ψ, yielding:
$[1/X(\boldsymbol{r})]\,\check{D}^2X(\boldsymbol{r}) = [1/c^2T(t)]\,\partial^2T/\partial t^2$
The left side is a function of ***r*** only, while the right side is a function of t only. The two sides can be equal only if both are equal to some constant, which we define to be $-k^2$. This trial solution has thus divided a complex differential equation into two simpler equations:
$\check{D}^2X(\boldsymbol{r}) = -k^2\,X(\boldsymbol{r})$
$\partial^2T(t)/\partial t^2 = -k^2\,c^2\,T(t)$
The T equation has simple sinusoidal solutions.

We next expand the X equation in spherical coordinates as:
$0 = k^2\,X + (1/r^2)\,\partial(\,r\,\partial X/\partial r\,)/\partial r$
$\quad + (1/r^2\sin\theta)\,\partial(\,\sin\theta\,\partial X/\partial\theta\,)/\partial\theta$
$\quad + (1/r^2\sin^2\theta)\,\partial^2X/\partial ø^2$
We repeat the separation of variables strategy by defining a trial solution that is the product of three new functions R(r), Θ(θ), and Φ(ø), as follows:
$X(\boldsymbol{r}) = R(r)\,\Theta(\theta)\,\Phi(ø)$
We insert this into the prior equation, then multiply by r^2/X, yielding:
$0 = k^2\,r^2 + (1/R)\,\partial(\,r\,\partial R/\partial r\,)/\partial r$
$\quad + (1/\Theta\sin\theta\,)\,\partial(\,\sin\theta\,\partial\Theta/\partial\theta\,)/\partial\theta$
$\quad + (1/\Phi\sin^2\theta)\,\partial^2\Phi/\partial ø^2$
If we were to multiply everything by $\sin^2\theta$, the last term would be a function of ø only, while the other terms would be independent of ø. This is possible only if:
$\partial^2\Phi/\partial ø^2\;/\;\Phi = -m^2$
Here, m is some constant. Hence, Φ also has simple sinusoidal solutions.

We insert this into our equation, yielding:
$0 = k^2\,r^2 + (1/R)\,\partial(\,r\,\partial R/\partial r\,)/\partial r$
$\quad + (1/\Theta\sin\theta\,)\,\partial(\,\sin\theta\,\partial\Theta/\partial\theta\,)/\partial\theta$

$- m^2 / \sin^2\theta$

This separates into two equations, for some constant A.

$+A = k^2 r^2 + (1/R)\, \partial(r\, \partial R/\partial r)/\partial r$

$-A = (1/\Theta \sin\theta)\, \partial(\sin\theta\, \partial\Theta/\partial\theta)/\partial\theta - m^2 / \sin^2\theta$

The first is a function of r only, while the second is a function of θ only.

This process converts one very complex equation into two very simple equations and two solvable equations.

Separation of Variables – By Scale

Another separation of variables approach addresses complex functions that have both small scale and large scale structure. Here, we separate the complex function into simpler functions, each describing the dependence on a different scale.

Consider a difficult differential equation: Schrödinger's equation for the orbit of an electron in an atom.

$\partial^2(\rho\psi)/\partial\rho^2 = -(\varepsilon + 2/\rho)\, \rho\psi$

Here, ε is the electron energy, ρ is the radial coordinate, and ψ is the electron wave amplitude.

At small distances from the nucleus, ψ may oscillate rapidly, but at large distances, ψ must decrease exponentially with r. This is because, at large distances, a bound electron's potential energy is greater than its total energy. With a negative kinetic energy, wave number k becomes imaginary and the normal oscillatory exp{ikr} term becomes exp{–Kr}, where K=k/i.

We separate the large-scale exponential from the small-scale oscillations by making this substitution:

$\rho\psi = g(\rho) \exp\{-\beta\rho\}$

Here, β is an arbitrary constant, and g(ρ) is the unknown small-scale function of distance. After a lot of math, our differential equation becomes:

$\partial^2 g/\partial\rho^2 - 2\beta\, \partial g/\partial\rho + (\beta^2 + \varepsilon + 2/\rho)\, g = 0$

Believe it or not, this is progress. Let's choose $\beta^2 = -\varepsilon$, reducing our equation to:

$\partial^2 g/\partial\rho^2 - 2\beta\, \partial g/\partial\rho + 2g/\rho = 0$

If not for the ρ in the denominator of the third term, this would be a simple equation. But we can solve this with a Taylor series.

Let: $g(\rho) = \Sigma_k\, a_k\, \rho^k$

This technique will work if the coefficients a_k approach zero for large k. Putting the Taylor series into our equation yields:

$0 = \Sigma_n \{(n+1)na_{n+1} - 2\beta na_n + 2a_n\} \rho^{n-1}$

Here the sum is from n=1 to n=+∞. The above equation is valid for all values of ρ. This can only be true if the coefficient of each power of ρ is zero. This is an important rule for polynomials that is well worth remembering. After rearranging, we obtain:

for all n>0: $a_{n+1} = a_n \{2(\beta n - 1)/n(n+1)\}$

With any choice of a_1, we can recursively calculate a_2, a_3, … in terms of β, which is related to the electron's energy. For the electron to be bound to the nucleus, Feynman shows that β must equal 1/n for some integer n.

This is one of the most important discoveries of science: **electrons in atoms have quantized energies**.

This is the basis of the Periodic Table, chemistry, biology, solid state physics, digital electronics, and everything else we know about atoms. And without this, we would not know that the universe is expanding.

Solving Laplace's 2-D Equation

Using functions of complex variables, we can solve many interesting physical phenomena that are governed by the two-dimensional Laplace equation, which is:

$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = 0$

All the usual mathematical functions of real variables can be extended to become functions of complex variables. We have done this before, for example, in analyzing harmonic phenomena with exponentials with complex exponents.

Any function F(β) of complex variable β=x+iy can be expressed as the sum of its real and imaginary parts. Recall this example:

$\exp\{\beta\} = \exp\{x+iy\} = \exp\{x\} (\cos y + i \sin y)$

Similarly, for two real functions U and V, let:

$F(\beta) = U(x,y) + i\, V(x,y)$

For example:

$F(\beta) = \beta^2 = x^2 + 2ixy - y^2$

$U(x,y) = x^2 - y^2$

$V(x,y) = 2xy$

Feynman says: "… a miraculous mathematical theorem [proves that] U and V *automatically* satisfy the relations:"

$\partial U/\partial x = + \partial V/\partial y$

$\partial V/\partial x = - \partial U/\partial y$

We can confirm that our example satisfies these relations.

$\partial U/\partial x = + \partial V/\partial y = +2x$

$\partial V/\partial x = - \partial U/\partial y = +2y$

Taking the second order partial derivatives of the prior pair of equations yields:

$\partial^2 U/\partial x^2 = \partial^2 V/\partial x \partial y = - \partial^2 U/\partial y^2$

$\partial^2 U/\partial x^2 + \partial^2 U/\partial y^2 = 0$

$\partial^2 V/\partial x^2 = - \partial^2 U/\partial x \partial y = - \partial^2 V/\partial y^2$

$\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0$

Thus, we can pick any function $F(\beta)$ and immediately have two solutions, U and V, to the Laplace equation. Feynman says: "We can write down as many solutions as we wish—by just making up functions—then we just have to find the *problem* that goes with each solution. It may sound backwards, but it's a possible approach."

Let's take an example.

$F(\beta) = \beta^2 = (x^2 + iy)^2$

$U(x,y) = x^2 - y^2 =$ some constant A

$V(x,y) = 2xy =$ some constant B

Both the U and V equations are solved by hyperbolae, with the U solutions everywhere orthogonal to the V solutions. They describe the equipotentials of a quadrupole lens, in two different configurations.

Cylindrical Harmonics

Bessel functions often arise in problems involving cylindrical symmetry. *Bessel functions of the first kind* are denoted $J_n(x)$, and solve the differential equation:

$x^2 \, d^2y/dx^2 + x \, dy/dx + (x^2 - n^2) \, y = 0$

These are also called *cylindrical harmonics*.

Chapter 16

Tensors & Matrices

The mathematics of matrices and tensors is essential in analyzing complex systems, particularly for quantum mechanics, general relativity, and multi-dimensional situations.

What is a Matrix?

Matrices are rectangular arrays of components laid out in rows and columns. Each component may be an ordinary number, a complex number, a function, or a quantum mechanical amplitude. An n×m ("n-by-m") matrix has n rows, m columns, and n times m components.

In physics, the most common matrices have the same number of rows and columns; these are called **square matrices**.
Let's consider the example of a 3×3 matrix M. The components of M are denoted M_{ij}, where the first index i denotes the component's row number and the second index j denotes the column number. We often use M to denote the entire matrix, but sometimes we write M_{ij} to emphasize that it is a matrix. The layout of matrix M is shown below.

$$M_{ij} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$$

It is essential to remember that in matrix mathematics, M_{ij} is **not** equal to M_{ji} in general.

A **diagonal matrix** is a square matrix in which the only nonzero components are those with equal row and column numbers ($M_{ij} = 0$ if i does not equal j).

Multiplication by a scalar is the simplest matrix operation. A *scalar* is a single

entity that might be a number, a function, or an amplitude. You can think of a scalar as being a 1×1 matrix. Multiplying matrix M by scalar s just means multiplying each component of M by s.

$$s\,M = \begin{pmatrix} sM_{11} & sM_{12} & sM_{13} \\ sM_{21} & sM_{22} & sM_{23} \\ sM_{31} & sM_{32} & sM_{33} \end{pmatrix}$$

Matrix addition is defined only for matrices with the same number of rows and the same number of columns. If matrix A = matrix M plus matrix N, then $A_{ij}=M_{ij}+N_{ij}$, for all i and j.

$$M+N = \begin{pmatrix} M_{11}+N_{11} & M_{12}+N_{12} & M_{13}+N_{13} \\ M_{21}+N_{21} & M_{22}+N_{22} & M_{23}+N_{23} \\ M_{31}+N_{31} & M_{32}+N_{32} & M_{33}+N_{33} \end{pmatrix}$$

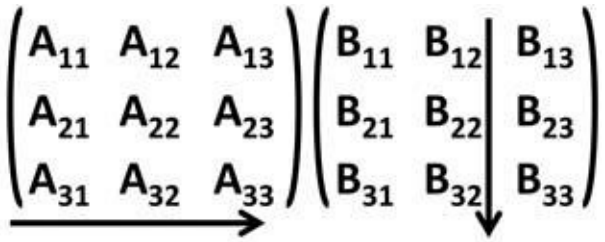
Matrix subtraction is simply multiplying one matrix by the scalar –1, followed by matrix addition.

Matrix multiplication is a bit trickier. If matrix C equals the product of matrix A times matrix B:

$C_{ij} = \Sigma_k A_{ik} B_{kj}$

Here we sum over all values of k for each combination of i and j.

This product is defined only when the number of columns in A equals the number of rows in B, which ensures that each term in the above sum over k is well defined. The product matrix C has the same number of rows as A and the same number of columns as B. The calculation of C_{32} is schematically illustrated below: the summation runs across row 3 of A and down column 2 of B.



$$= \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & \boxed{C_{32}} & C_{33} \end{pmatrix}$$

$$\Sigma_k A_{3k} B_{k2} = C_{32}$$

Let's try an example: multiply a 2×4 matrix by a 4×3 matrix to produce a 2×3 matrix, as shown below.

$$\begin{pmatrix} 1 & 4 & 5 & 3 \\ 3 & 5 & 6 & 9 \end{pmatrix} \begin{pmatrix} 2 & 7 & 8 \\ 4 & 1 & 2 \\ 3 & 2 & 4 \\ 5 & 9 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 48 & 48 & 39 \\ 89 & 119 & 67 \end{pmatrix}$$

You might check this yourself to see if I got the right answer. The row 2, column 2 component is:

3×7 + 5×1 + 6×2 + 9×9

= 21 + 5 + 12 + 81

= 119

Multiplication order is critical: A times B is **not** equal to B times A in general. When several matrices are multiplied together, we always multiply from right to left. To evaluate the matrix product ABC, multiply B times C, then multiply that result by A.

Vectors with n components can be considered either n×1 matrices or 1×n matrices. Vectors and matrices can be multiplied together. For example, consider a 3×3 matrix R and two 3-component vectors ø and Ψ:

$\Psi_i = \Sigma_j R_{ij} \phi_j$

Here Ψ and ø could be spin state vectors of a spin 1 particle, and R could be the rotation operator that transforms states from one coordinate basis to another. The particle's state in one basis, ø, is transformed into its state in another basis, Ψ, by multiplying ø by R.

The ***identity matrix***, also called the ***unit matrix***, is denoted by the *Kronecker delta* δ_{ij}.

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This may seem trivial, but the unit matrix is just as useful as the number 1.

Matrix equations can be valid only if every term is a matrix with the same number of rows and the same number of columns. The equation $M=0*\delta_{ij}$ means that every component of matrix M is zero. In such equations, δ_{ij} is sometimes not explicitly shown, but is implicitly assumed.

Matrix components can be complex, particularly in quantum mechanics. They can be complex both in the sense of being complicated, and in the sense of having real and imaginary parts (x+iy). Matrices are **Hermitian** if $M_{ij} = M_{ji}*$ for each ij, where $M_{ji}*$ is the complex conjugate of M_{ji}. In general the sum of two Hermitian matrices *is* Hermitian, but their product *is not*.

Matrix division is not defined. However, most matrices of interest in physics have inverses. If A^{-1} is the ***inverse*** of matrix A, then:

$A\,A^{-1} = A^{-1}\,A = \delta_{ij}$

Matrix A has an inverse if and only if it is a *nonsingular*, n×n matrix. In this case, A^{-1} is also a nonsingular, n×n matrix. A matrix is singular if any of its rows equals a linear combination of its other rows, or if any of its columns equals a linear combination of its other columns. Conversely, if the rows of a matrix are all linearly independent, and if its columns are also all linearly independent, the matrix is nonsingular and has an inverse.

Simple examples of singular matrices are those whose components are all zero

across an entire row or an entire column.

Each n×n matrix has a *determinant*. The determinant of a nonsingular matrix is nonzero, while the determinant of a singular matrix is zero.

Matrix Determinants

Determinants are defined only for N×N matrices.

Determinants have geometric significance. Consider each column (or each row) to be an independent vector. The determinant of a 2×2 matrix equals ± the area of the parallelogram whose two sides are the two column vectors, with the sign determined by the order of the column vectors. The determinant of a 3×3 matrix equals ± the volume of the parallelepiped whose three sides are the three column vectors. The determinant of a N×N matrix equals ± the measure of the N-dimensional space enclosed by the N column vectors.

The simplest way to calculate the determinant of a matrix is by iteration.

The determinant of a 1×1 matrix simply equals its sole component.
for M = (M_{11}), Det |M| = M_{11}
The determinant of a 2×2 matrix M equals the product of the upper-left and lower-right components minus the product of the other two components, as shown below:

$$\text{Det } |M_{ij}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Det $|M_{ij}|$ = ad – cb
The determinant of a 3×3 matrix M has three contributions. Begin by picking any row or any column; all choices yield the same determinant. In the example below, we chose row 1. We proceed to step across the row, sequentially selecting each component and evaluating its contribution. The first contribution equals the first component of row 1, M_{11}=a, multiplied by the determinant of the *minor* of M_{11}. The minor of M_{11} is the 2×2 matrix formed by deleting M_{11}'s row and column. In the example below, the minor for each contribution is shown in bold type.

$$\text{Det}\,|M_{ij}| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix}$$

$$= a\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix} - b\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix} + c\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & j \end{vmatrix}$$

Det $|M_{ij}|$ = +a(ej–hf) –b(dj–gf) +c(dh–ge)

In this example, the minor of M_{11}=a is the matrix formed by components e, f, h, and j. The second contribution equals b (row 1, column 2) multiplied by the determinant of its minor, which is the matrix formed by d, f, g, and j. The third contribution equals c (row 1, column 3) multiplied by the determinant of its minor, the matrix d, e, g, h.

Each of the three contributions must be summed with the proper sign: $(-1)^{r+c}$, where r is the row number and c is the column number of the selected component. For example, for the first contribution we selected row 1 column 1, which has sign $(-1)^{1+1}=+1$. For the second contribution we selected row 1 column 2, which has sign $(-1)^{1+2}=-1$. The signs alternate as one proceeds across the chosen row or column.

For larger matrices, repeat this procedure iteratively. Pick any row (or column). Multiply each component in that row (or column) by the determinant of its minor and by its proper sign. Sum the contributions across the entire row (or column).

An equivalent equation for the determinant of the N×N matrix M is:

Det $|M_{ij}| = \Sigma\ \text{Sign}(abc\ldots)\bullet M_{1a}\bullet M_{2b}\bullet M_{3c}\bullet\ldots$

Here, abc… is a permutation of the integers 1 through N. Each term in the sum is the product of N components, with one selected from each row and one selected from each column. The sum extends over all permutations of the integers 1 through N, and Sign(abc..) equals +1 for even permutations and –1 for odd permutations. A permutation is even (or odd) if it is obtained from the sequence 1, 2, 3, … N by an even (or odd) number of swaps of adjacent integers. For

example, 1243 is obtained from 1234 with one swap, so Sign(1243)=–1, while 1423 requires two swaps, so Sign(1423)=+1.

Matrix Inverses

Calculating the inverse of a matrix M is only a bit more work than calculating its determinant, Det(M). The equation is:

$M^{-1}_{ij} = (-1)^{i+j}$ Det(minor of M_{ij}) / Det(M)

For the prior 3×3 matrix M, the inverse M^{-1} is:

$$M_{ij} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$$

$$M^{-1}_{ij} = \begin{pmatrix} ej-hf & gf-dj & dh-ge \\ hc-bj & aj-gc & gb-ah \\ bf-ec & dc-af & ae-db \end{pmatrix} / \text{Det}(M)$$

Eigenvalues & Eigenvectors

For an n×n matrix M and an n-component vector ***V***, it sometimes occurs that:

M ***V*** = E ***V***

for some constant E. In component notation this is:

$\Sigma_k M_{ik} V_k = E V_i$

This means multiplication by M may change the length and polarity of ***V*** but not its orientation.

In this case, ***V*** is called an *eigenvector* and E is the corresponding *eigenvalue* of M. The terms are derived from the German word “eigen” that denotes an intrinsic characteristic. Since these are linear equations, if ***V*** is a solution to the above equation so must be 17***V***. Therefore, we generally normalize each eigenvector so that it has unit length (***V***•***V***=1).

An n×n matrix M may have up to n linearly independent eigenvectors; it will have n eigenvalues, some of which may be the same.

If matrix M has n linearly independent eigenvectors V_j, j=1 to n, each with a

corresponding eigenvalue E_j, we can construct an n×n matrix Q whose jth column is M's jth eigenvector. This means:
$Q_{ij} = (V_i)_j$ for each value of i.
Here, $(V_i)_j$ means the ith component of the jth eigenvector. We can rewrite the prior eigenvector equation to encompass all n eigenvectors.
$\Sigma_k M_{ik} Q_{km} = E_m V_{im}$
Note this equation is valid for each value of i and m; we are not summing over m. Since the eigenvectors are linearly independent, Q has an inverse Q^{-1}. Multiplying the prior equation on the left by Q^{-1} yields:
$\Sigma_i Q^{-1}_{ni} \Sigma_k M_{ik} Q_{km} = E_m \Sigma_i Q^{-1}_{ni} V_{im}$
$\Sigma_{ik} Q^{-1}_{ni} M_{ik} Q_{km} = \Sigma_i E_m Q^{-1}_{ni} Q_{im}$
$(Q^{-1} M Q)_{nm} = E_m \delta_{nm}$
$(Q^{-1} M Q)_{nm} = E_n$ if n=m, else = 0
This means the matrix $(Q^{-1}MQ)$ is a diagonal matrix whose diagonal components are the eigenvalues of M. In physics, we interpret this as: the phenomenon described by M has its most natural representation in a coordinate system whose axes are M's eigenvectors. In that system, the action along each axis is independent of the actions along the other axes.

For example, in a crystalline solid, an external electric field displaces electrons in the crystal's atoms from their equilibrium positions. For a given electric field strength, the amount of displacement may be different in different directions because of the types of atoms and their locations within the crystal structure. In an arbitrary xyz coordinate system, an electric field in the x-direction may cause electron displacements along all three axes, complicating the mathematics. We can represent that by a 3×3 polarization matrix Π, which may have 9 nonzero components.

However, there is always a coordinate system whose axes align with the crystal's *principal axes*, the directions in which an electric field produces displacements that are entirely parallel to that field. In this coordinate system, Π is a diagonal matrix with only 3 nonzero components.

Whenever possible, complicated matrix problems are greatly simplified by finding the eigenvalues and eigenvectors of a matrix, and then diagonalizing that matrix.

Characteristic Polynomial

Here is one approach for finding the eigenvalues and eigenvectors of a nonsingular, square matrix. For an N×N matrix H, an N-component vector $\boldsymbol{V}$, and a constant E, consider the equation:

$0 = (H_{jk} - \delta_{jk}E)\,\boldsymbol{V}$

The left side is a matrix with all zero components, and hence its determinant equals zero. For the equation to be valid, the right side must also be a matrix with all zero components and zero determinant.

We can write the equation for the determinant of $(H_{jk} - \delta_{jk}E)$ and set it equal to zero. This results in the **characteristic polynomial** of $(H_{jk} - \delta_{jk}E)$, with E as the independent variable. In general, this polynomial contains all powers of E from the zeroth power to the Nth power. As we know from the algebra of complex numbers, such equations have N roots, some of which might be equal. Each root is an eigenvalue of the matrix H.

With the N eigenvalues denoted E_n, n=1…N, we then have N equations of the form:

$H_{jk}\boldsymbol{V}_n = \delta_{jk}\,E_n\boldsymbol{V}_n$

Here, $\boldsymbol{V}_n$ is the eigenvector corresponding to eigenvalue E_n. These equations can be solved to yield the N eigenvectors; however, the required effort is proportional to N^2. With the eigenvectors, we can diagonalize the matrix H.

Solving a Sample Problem

Let's see how these ideas are employed to solve a simple matrix problem: the stationary states of a two-state system. This is a common problem in quantum mechanics. It also arises in macroscopic problems, such as two pendulums joined with a spring.

In the latter case, if one pendulum is set in motion while the other is stationary, the first exerts a force through the spring on the second. The spring transfers energy from the first to the second. Eventually, the second swings and the first stops. The process then repeats in reverse. Over time, the first pendulum swings, then stops, then swings again, while the second does the opposite.

What we seek are the *stationary states* of this system: the states of motion that

repeat at a single frequency with a definite energy. We know the solutions for the motion of a single pendulum: $\omega=\sqrt{(g/L)}$, with ω being the oscillation frequency in radians per second, g being Earth's gravitational acceleration, and L being the pendulum length. For two identical pendulums, we define two *basis states*:

$|1\rangle$ = pendulum #1 swinging at frequency ω

$|2\rangle$ = pendulum #2 swinging at frequency ω

Like two coordinate axes, any motion of two pendulums is a linear combination of $|1\rangle$ and $|2\rangle$, which we can write in vector form as:

$(p, q) = p|1\rangle + q|2\rangle$

This is logically equivalent to:

$(x, y) = x|e_x\rangle + y|e_y\rangle$

with $|e_x\rangle$ and $|e_y\rangle$ being unit vectors in the x-and y-directions.

The equation of motion can then be written in matrix form as:

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = E \begin{pmatrix} p \\ q \end{pmatrix}$$

Here, E is the energy of motion, and the matrix H is called the *Hamiltonian*.

We calculate the eigenvalues by setting the determinant of $(H_{jk}–\delta_{jk}E)$ equal to zero.

$$M_{jk} = H_{jk} - \delta_{jk}E = \begin{pmatrix} H_{11}–E & H_{12} \\ H_{21} & H_{22}–E \end{pmatrix}$$

$0 = \text{Det } |M| = (H_{11}–E)\bullet(H_{22}–E) –H_{21}\bullet H_{12}$

$0 = H_{11}\bullet H_{22} –E\bullet(H_{11}+H_{22}) + E^2 –H_{21}\bullet H_{12}$

$E = E_0 \pm E^*$

Here the two values of E are the eigenvalues, and E_0 and E* are given by:

$E_0 = (H_{11}+H_{22})/2$

$E^* = \sqrt{\{E_0^2 –H_{11}H_{22}+H_{21}H_{12}\}}$

Now let's calculate the eigenvectors. For energy E, the equation of motion produces two equations, one from each row. We show both, but we actually only need one.

$H_{11}\, p + H_{12}\, q = E\, p$

$H_{21}\,p + H_{22}\,q = E\,q$
$H_{12}\,q = (E - H_{11})\,p$
$p/q = H_{12} / (E - H_{11})$
For each of the two eigenvalues, the ratios of eigenvector components are:
for $E = E_0+E^*$: $p/q = H_{12} / (E_0+E^* - H_{11})$
for $E = E_0-E^*$: $p/q = H_{12} / (E_0-E^* - H_{11})$
In each case, we normalize p and q such that $|p|^2+|q|^2=1$.

Let's examine the simplest case, where $H_{11}=H_{22}=E_0$ and $H_{21}=H_{12}=A$.
$E^* = \sqrt{\{E_0^{\,2} - E_0^{\,2} + A^2\}} = A$
for $E=E_0+A$: $p/q = -A / (E_0+A - E_0) = -1$
for $E=E_0-A$: $p/q = -A / (E_0-A - E_0) = +1$
for E_0+A, $p=-q$; pendulums swing oppositely
for E_0-A, $p=+q$; pendulums swing together
For each state of motion, the pendulums swing with the same amplitude and frequency. In the higher energy state ($E=E_0+A$), the pendulums always swing in opposite directions. In the lower energy state ($E=E_0-A$), the pendulums always swing in the same direction.

Rotations as Matrices

Recall from Chapter 10 the equations for the rotations of coordinate axes. For a rotation of the xy-axes by angle θ about the z-axis, the transformation equations that relate the coordinates of a point P in the xyz system to P's coordinates in the x*y*z* system are:
$x^* = +\,x\cos\theta + y\sin\theta$
$y^* = -\,x\sin\theta + y\cos\theta$
$z^* = z$
We can write this rotation transformation as a matrix R(θ), whose components are:

$$\begin{array}{c} \\ j=x^* \\ y^* \\ z^* \end{array} \begin{array}{c} \begin{array}{ccc} k=\ x & y & z \end{array} \\ \begin{pmatrix} +\cos\theta & +\sin\theta & 0 \\ -\sin\theta & +\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

Note that $R^{-1}(\theta)$, the inverse of R(θ), is simply R(–θ), which is the *transpose* of

R:
$R_{jk}^{-1} = R_{kj}$
The vector Q transforms into vector Q* when the coordinate axes rotate by angle θ about the z-axis. In matrix notation, this is written:
$Q^*_j = \Sigma_k Z_{jk} Q_k$
In Chapter 10, we said that we can determine if some vector ***s*** is a proper vector by taking its dot product with a known vector ***r***. If, and only if, ***s*** is a proper vector will ***s•r*** be an invariant scalar that is unchanged by any rotation. Here is the proof.

Take any rotation matrix R that transforms ***r*** into ***r**** and ***s*** into ***s****, and let $R^{-1}_{jk} = R_{kj}$ be R's inverse matrix, the matrix that transforms vectors back to the original coordinate system. Now let's compare ***s•r*** in the rotated and original systems.
$\boldsymbol{s^*\bullet r^*} = \Sigma_k s^*_k r^*_k$
Now replace ***r**** by R***r.***
$\boldsymbol{s^*\bullet r^*} = \Sigma_k s^*_k (\Sigma_j R_{kj} r_j)$
Next, rearrange the sums.
$\boldsymbol{s^*\bullet r^*} = \Sigma_{kj} R_{kj} s^*_k r_j$
And, finally, we replace R_{kj} with R^{-1}_{jk}.
$\boldsymbol{s^*\bullet r^*} = \Sigma_j r_j (\Sigma_k R^{-1}_{jk} s^*_k)$
The right hand side will equal:
$\Sigma_j s_j r_j = \boldsymbol{s\bullet r}$
if and only if:
$\Sigma_k (R^{-1}_{jk} s^*_k) = s_j$
The last equation is valid if, and only if, ***s*** is a proper vector.

What is a Tensor?

Tensors are a generalization of vectors and matrices. Tensor calculus is a beautiful branch of mathematics that empowers us to elegantly and effectively describe many complex, multi-dimensional phenomena.

Tensors are essential in general relativity, where 4-D spacetime curves, twists, and stretches differently at every point, at every instant, and in every direction.

Tensors are also employed in 3-D analyses of mechanics and wave propagation in anisotropic materials, those whose properties are different in different

directions.

The most important thing to know about tensors is that any tensor equation that is valid in one coordinate system is automatically valid without any modifications in all coordinate systems, regardless of their rotation or motion relative to the original coordinate system.

That generality is one reason that the mathematics of general relativity is so challenging, but it is also one of the most powerful tools in solving problems. If we can identify a coordinate system in which we can solve a complex problem with a tensor equation, we have immediately solved the problem in all coordinate systems. General relativity is the only major branch of physics in which tensor equations are universally employed.

Tensors are arrays of components that transform properly between coordinate systems. In 3-D, they transform according to Euclidian coordinate rotations. In 4-D spacetime, they also transform according to the Lorentz transformation.

Tensors can have one component or millions of components, each being a different function of all coordinates.

Let’s consider some quantities that are not tensors.

Temperature is a simple quantity that changes with time and location, making it a function of the four coordinates of spacetime. Its values are different in different coordinate systems, but these values do not change according to the rotation matrices or the Lorentz transformation. Hence, temperature is not a tensor.

Similarly, energy by itself is not a tensor. But, the proper combination of energy and momentum — $(E/c, p_x, p_y, p_z)$ — is a tensor because its components do transform properly.

Tensors are characterized by their *rank* and by the dimensionality of the space in which they are defined. In physics, the most common spaces are Euclidian 3-D,

and 4-D spacetime. The most common tensors have rank 0, 1, 2, or 4. The largest meaningful tensor I know is a rank 10 tensor with 1,048,576 components; don't worry — I will not share this tensor with you in this eBook.

The simplest tensors are scalars; these are *rank 0* tensors. These include π, 7, 0, and your age — all numbers that have the same values in all coordinate systems.

We are also very familiar with *rank 1* tensors: vectors. Every proper 3-vector is a 3-D rank 1 tensor, and every proper 4-vector is a 4-D rank 1 tensor.

An example of a 4-D *rank 2* tensor is the Faraday tensor shown below.

$$F_{\mu\sigma}$$

	$\sigma = t$	x	y	z
$\mu = t$	0	$-E_x/c$	$-E_y/c$	$-E_z/c$
x	E_x/c	0	$-B_z$	$+B_y$
y	E_y/c	$+B_z$	0	$-B_x$
z	E_z/c	$-B_y$	$+B_x$	0

When a tensor is shown as an array of components, its components are generally enclosed in square brackets [ ], as above.

The Faraday tensor has the special property of being *antisymmetric*. This means $F_{\mu\sigma} = -F_{\sigma\mu}$ for all combinations of indices μ and σ. As a result, all the diagonal components are zero, and components on opposite sides of the diagonal are the same, but with the opposite sign. With 4 zero components and 6 redundant components, the Faraday tensor has only 16–4–6 = 6 independent components. These are the 3 components of ***E*** and the 3 components of ***B***.

Tensors of rank 2 and greater can have interesting symmetry properties. Some tensors are symmetric, meaning that $G_{\mu\sigma} = +G_{\sigma\mu}$ for all combinations of μ and σ. A rank 2, symmetric tensor has 6+4 = 10 independent components, and 6 others that are redundant.

Tensor Indices

Since tensors can have so many components, we use indices to avoid writing them all out individually. Above, we used two indices to identify the components of a rank 2 tensor. Let's now discuss tensor indices in general.

A rank N tensor has N indices that each range over the same set of allowed values. A rank 2 tensor, for example, must have the same number of rows and columns, unlike a matrix that may have a different number of rows and columns.

A rank N tensor has 4^N components in 4-D, and 3^N components in 3-D.

The sums and differences of tensors with the same rank and indices are also tensors. The product of two tensors is a tensor, but the quotient of two tensors is not generally a tensor.

In 3-D and flat (non-curved) 4-D spacetime, tensor indices are written as subscripts. For example, consider two alternative notations for the components of the rank 1, position tensor:
$r_1 = r_x = x$
$r_2 = r_y = y$
$r_3 = r_z = z$
$r_0 = r_t = ct$
$r_\sigma = (ct, x, y, z)$
Here, $\sigma = 0, 1, 2, 3$, or if you prefer, $\sigma = t, x, y, z$.

These equations demonstrate a critical difference between the *free index* σ, which can have any value in the allowed range, and the *fixed indices* 0, 1, 2, 3, x, y, z, and t. The latter refer to specific components, whereas the former refers to any component corresponding to any possible value of σ.

Free indices are just labels. The specific letters we choose have no significance mathematically or physically: x_σ and x_μ mean exactly the same thing, as do $A_{\mu\sigma}$ and $A_{\beta\alpha}$. The significance of free indices lies in the relationships they establish. For example: $A_{\mu\sigma} = B_{\sigma\mu}$ means that every component of B equals the component of A on the opposite side of the diagonal. Just like a vector equation, this tensor equation would mean exactly the same thing for any letters we might substitute

for μ and σ.

Tensor Algebra

A common tensor equation is:

$L_\mu = 0$

This establishes the same equation for all values of the free index μ. In this case, it means each component of L equals zero. Every tensor equation is valid for all values of every *unmatched* free index. By "unmatched", we mean a free index that appears no more than once in each product term. Hence:

$A_{\sigma\mu} = 1$

means every component of tensor A equals 1.

We said above that the product of two tensors is itself a tensor. For example, if A_μ and B_σ are two position 4-vectors (rank 1 tensors), their product $A_\mu B_\sigma$ is a rank 2 tensor $(AB)_{\mu\sigma}$. The product of a rank N tensor with a rank M tensor is always a tensor of rank N+M.

When a free index appears twice in one term, the *Einstein summation convention* directs us to sum over all values of the repeated index. This is similar to a vector dot product, and is called a *tensor contraction* (contraction in the sense that the number of indices and the tensor's rank decrease). For example:

$A_\mu B_\mu = A_0 B_0 + A_1 B_1 + A_2 B_2 + A_3 B_3$

Here μ appears twice in the product term AB, which directs us to sum over all values of μ. All quantities on the right side are components, hence their sum is a scalar with no free indices. The tensor contraction of a rank N tensor with a rank M tensor has rank N+M–2.

In a proper tensor equation, each nonzero term must have the same set of unmatched free indices. For example:

valid for all σ: $A_{\sigma\mu} x_\mu + C_\sigma = B_\sigma$

invalid: $A_{\sigma\mu} x_\mu + C_\beta = B_\mu$

σ, μ, and β are not in every term

The *unit tensor* [1] is as important in tensor calculus as 1 is in arithmetic. The components of the unit tensor are 1 when all indices are the same, and 0 whenever any two indices are different. The rank 2, 4-D unit tensor is defined by:

$\delta_{\mu\sigma} = 1$ if $\mu=\sigma$ and zero otherwise.
In 4-D, the tensor contraction $\delta_{\mu\mu}$ equals:
$\delta_{\mu\mu} = \delta_{tt} + \delta_{xx} + \delta_{yy} + \delta_{zz} = 4$
You will recognize $\delta_{\mu\sigma}$, the *Kronecker delta*. In tensor calculus, we can extend the *Kronecker delta* to any number of indices, such as the rank 4 tensor $\delta_{\mu\sigma\beta\gamma}$.

Like matrices, all nonsingular tensors have inverses: $A^{-1}_{\mu\sigma}$ is the inverse of $A_{\mu\sigma}$. The contraction of a tensor with its inverse always equals the unit tensor.
$A^{-1}_{\sigma\mu} A_{\mu\beta} = A_{\sigma\mu} A^{-1}_{\mu\beta} = [1] = \delta_{\sigma\beta}$
This is as close as tensor calculus gets to dividing by a tensor. For tensors A and C, and a nonsingular tensor B:

If $A\ B = C$, then $A = C\ B^{-1}$
We define tensors as arrays of components that transform properly. In 3-D, that means they transform according to the rules of Euclidean rotations. As discussed above, each 3-D rotation R has a corresponding 3×3 matrix R_{jk}.

Rotating coordinate axes transforms vector Q into vector Q* according to:
$Q^*_j = R_{jk} Q_k$
The requirement that a tensor transform properly under any rotation R can be written:
rank 1: $A^*_j = R_{jk} A_k$
rank 2: $A^*_{jk} = R_{jn} R_{km} A_{nm}$
rank 3: $A^*_{jk\mu} = R_{jn} R_{km} R_{\mu\sigma} A_{nm\sigma}$
In each line, A* is the transformed tensor of the original tensor A. In 3-D for rank 1, we sum over all three values of k. For rank 2, we sum over all 3×3=9 values of n and m. For rank 3, we sum over all 3×3×3=27 possible values of n, m, and σ.

For rank 2, the above transform can also be written:
$A^*_{jk} = R_{jn} R_{km} A_{nm} = R_{jn} R^{-1}_{mk} A_{nm}$
$A^* = R\ A\ R^{-1}$
Note that since tensor operations are fully defined by paired free indices, the order of product terms is irrelevant, unlike matrix multiplication.

We can form tensors by combining any proper polar vectors (but not axial

vectors). For example, if A_j and B_k are proper polar 3-vectors, then:
$C_{jk} = A_j B_k$ is a proper rank 2, 3-D tensor.
Let's show that C transforms properly for any rotation R by transforming each vector.
$(A^*_n)\,(B^*_m) = (R_{nj}\,A_j)\,(R_{mk}\,B_k)$
$(A^*_n)\,(B^*_m) = R_{nj}\,R_{mk}\,A_j\,B_k$
$(A^*_n)\,(B^*_m) = R_{nj}\,R_{mk}\,C_{jk} = C^*_{nm}$
Hence, C transforms properly and is therefore a tensor. The same logic applies to tensors of any rank. For example:
$F_{\mu\sigma}\,\Lambda_{\alpha\beta}\,R_{\delta\varepsilon}$
is a proper rank 6 tensor, although it has no physical meaning as far as I know.

To understand the ***Feynman Lectures***, that is as much as you need to know about tensors.

Those who wish a glimpse of general relativistic tensor calculus can enjoy the next section (there's no exam), while others can skip to the following section.

Tensor Calculus in Curved Spacetime

In 4-D curved spacetime, the dot product cannot be simply the sum of the products of corresponding components. This is because coordinate axes may change directions and rulers may change lengths, and all that can happen differently at every location and instant in time.

We therefore need a *metric* $g_{\mu\sigma}$ to reveal the geometry at each *event* (ct,x,y,z) in spacetime. That metric specifies the *invariant interval*, the "true distance", between any two nearby events. It turns out that knowing the interval between all nearby events completely determines the geometry everywhere.

In curved space, the dot product of two 4-vectors A_μ and B_μ is:
$A_\mu B^\mu = A^\mu B_\mu = g_{\mu\sigma} A^\mu B^\sigma = g_{\mu\sigma} A^\sigma B^\mu =$
$+ g_{tt} A^t B^t + g_{tx} A^t B^x + g_{ty} A^t B^y + g_{tz} A^t B^z$
$+g_{xt} A^x B^t +g_{xx} A^x B^x +g_{xy} A^x B^y +g_{xz} A^x B^z$
$+g_{yt} A^y B^t +g_{yx} A^y B^x +g_{yy} A^y B^y +g_{yz} A^y B^z$
$+g_{zt} A^z B^t + g_{zx} A^z B^x +g_{zy} A^z B^y +g_{zz} A^z B^z$
If ds_μ is the separation 4-vector between two nearby events, the invariant interval

ds^2 between those points is:
$ds_\mu = (cdt, dx, dy, dz)$
$ds^2 = ds_\mu\, ds^\mu = ds^\mu\, ds_\mu = g_{\mu\sigma}\, ds^\mu\, ds^\sigma$
Here, ds^2 is an invariant scalar — it measures the separation between nearby events, and has the same value in any coordinate system. In Feynman's sign convention, ds^2 equals $c^2d\tau^2$ where τ is *proper time*, the time measured by an ideal clock moving between these nearby events.

Note that some indices are subscripted while others are superscripted. The former are called *covariant indices*, while the latter are called *contravariant indices*. Since superscripts look exactly like exponents, we try to avoid using exponents in tensor calculus whenever confusion might arise: x^2 always means the second component of x^μ, while $(x)^2$ means x-squared. Because the square of coordinate differentials occur so frequently, an exception to this rule is dx^2, which always means $(dx)^2$.

In 4-D curved spacetime, we only sum repeated free indices if one is covariant and the other is contravariant. The difference between the two is demonstrated by the covariant and contravariant position 4-vectors (in Feynman's sign convention).
$x^\alpha = (ct, -x, -y, -z)$ is contravariant
$x_\alpha = (ct, x, y, z)$ is covariant
The metric in flat spacetime, in Feynman's sign convention, is:

$g_{\mu\sigma}$

	$\sigma = t$	x	y	z
$\mu = t$	1	0	0	0
x	0	–1	0	0
y	0	0	–1	0
z	0	0	0	–1

In polar coordinates, the metric near a black hole in Feynman's sign convention is:

$g_{\mu\sigma}$

	$\sigma = t$	r	θ	ϕ
$\mu = t$	Ω	0	0	0
r	0	$-1/\Omega$	0	0
θ	0	0	$-r^2$	0
ϕ	0	0	0	$-r^2\sin^2\theta$

Here, the $g_{\theta\theta}$ and $g_{\phi\phi}$ components are the normal polar coordinate factors, which are unaffected by gravity. Gravity dilates time and stretches space through the factor $\Omega=1–2GM/c^2r$, where G is Newton's gravitational constant, M is the black hole's mass, and r is the distance from its center.

Note that odd things happens when $\Omega=0$ at $r=2GM/c^2$, the location of the black hole's event horizon. One interesting effect is that the event horizon is timeless — the passage of time has no effect whatsoever on the event horizon, because g_{tt} is zero at that radius.

In the most common modern notation, the metric $g_{\mu\sigma}$ has a minus sign on the time component and plus signs on the three spatial components, the opposite of Feynman's convention.
The Lorentz transformation tensor is:

$\Lambda^{\mu}{}_{\sigma}$

	$\sigma = t$	x	y	z
$\mu = t$	γ	$-\beta\gamma$	0	0
x	$-\beta\gamma$	γ	0	0
y	0	0	1	0
z	0	0	0	1

Here, $\beta = v/c$ and $\gamma = 1/\sqrt{(1–\beta^2)}$.
Some typical index operations are:

1. Lowering an Index: $x_{\mu} = g_{\mu\sigma}\, x^{\sigma}$
2. Raising an Index: $x^{\mu} = g^{\mu\sigma}\, x_{\sigma}$
3. Lorentz Transform: $X_{\sigma} = \Lambda^{\beta}{}_{\sigma}\, x_{\beta}$

Like other square matrices, the metric tensor for most geometries can be *diagonalized*, meaning all non-diagonal components can be made zero with suitable transformations. The invariant interval is then reduced to 4 terms, and the inverse metric is simply $g^{\alpha\alpha} = 1/g_{\alpha\alpha}$.

Diagonalizing the metric tensor can mix the coordinates in surprising ways. For example, the time coordinate t and radial distance coordinate r might be replaced by the coordinates u=ct+r and w=ct–r, leaving nothing that represents pure time. But since the tensor calculus of general relativity works in any coordinate system, such mixing is mathematically valid; it can simplify our calculations even when it defies our intuition.

When one becomes comfortable with tensor notation, it is possible to drop the indices altogether, as we do in vector algebra. We can then write equation (3) as:
$X = \Lambda x$

Einstein's Field Equations

The ultimate equation of general relativity, and Einstein's greatest contribution to mankind, are his *field equations*, which are written:
$G = 8\pi T$
Here, G represents the geometry of spacetime, and T represents all forms of energy, including mass and momentum. G is now called the *Einstein tensor*, and T is called the *mass-energy-stress tensor*. Both are symmetric, rank 2 tensors. We say equation**s**, the plural, because $G=8\pi T$ represents 16 component equations: 4 describe the conservation of energy and momentum; 6 relate energy density to spacetime curvature; and the remaining 6 are redundant.

John Archibald Wheeler said the meaning of $G = 8\pi T$ is:

> "The geometry of spacetime tells mass and energy how to move, while mass and energy tell space how to curve."

Brian Greene describes Einstein's field equations as the choreography of the cosmic ballet of the universe. It is a duet in which both parties lead one another.

I hope you found this brief taste of general relativity intriguing. For a thorough yet accessible explanation of the most profound theory of science see *General*

Cross Product as a Tensor

The cross product of two vectors should properly be considered a tensor. For example, the equation for torque is:

$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$

We can write this as:

$\tau_{jk} = r_j F_k - r_k F_j$

Here we see that τ_{jk} is formed by two pairs of proper polar 3-vectors. As we discussed in the **Tensor Algebra** section, each product term on the right side is a proper rank 2 tensor, and so therefore is their difference.

Clearly, τ_{jk} is antisymmetric: $\tau_{jk} = -\tau_{kj}$. In 3-D, this means τ_{jk} has only three independent nonzero components. Feynman says it is "almost by accident" that these three components form a proper axial 3-vector. He says "accident" because this is true only in 3-D. In 4-D, for example, antisymmetric tensors have 6 nonzero components, which clearly cannot make a 4-vector.

The same logic applies to any vector cross product; each can be written as a rank 2, antisymmetric tensor.

Recall that vectors are divided into two classes, polar and axial, depending on how they change in reflection (inverting the polarity of one coordinate axis). Recall also that almost all axial vectors are defined by cross products. Similarly, tensors are divided into two classes, normal tensors and pseudo-tensors, depending on how they change in reflection. The cross product is a pseudo-tensor.

Chapter 17

Numerical Integration

Numerical integration is necessary when we can compute y(x) for any x, but cannot obtain an analytical solution to ∫y(x)dx.

The standard procedure is to divide the total range of x into N subintervals, each of width Δx. For each subinterval, define x_j to be its x-value and $y_j=y(x_j)$. The integral is then estimated to be the sum of the areas of N rectangles, with the jth rectangle having area $y_j \times \Delta x$.

In the limit that Δx goes to zero and N goes to ∞, we obtain an exact integral.

While N=∞ is impossible, integration precision improves as N increases, encouraging us to choose the largest affordable N. Since most simple functions are analytically integrable, numerical integration is typically employed for quite complicated functions, where the primary cost is in calculating the y_j values.

As an example of standard numerical integration, consider integrating $y(x)=x^2$ from x=0 to x=8. I deliberately chose a function that is easy to integrate analytically:
$S = \int_0^8 x^2\,dx = x^3/3\,|_0^8 = 512/3 = 170.666\ldots$
Let's see how well the standard numerical integration procedure works.

We being with Δx=2. Figure 17-1 shows three rectangles of width Δx=2 and heights y(2)=4, y(4)=16, and y(6)=36. The complete integral also includes a fourth rectangle of height y(0)=0, which is invisible.

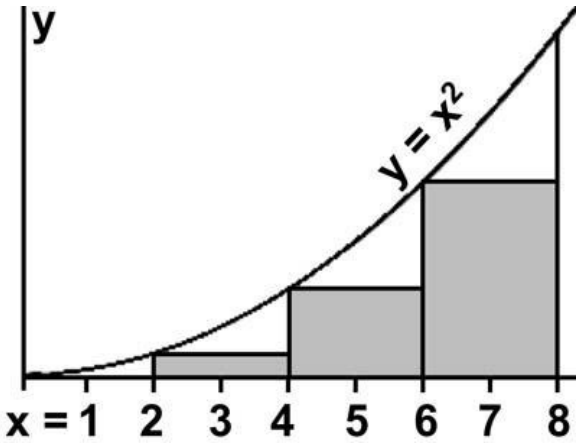


Figure 17-1 Integrating with Rectangles

The areas of these four rectangles sum to:

$S_4 = 2 \times (0 + 4 + 16 + 36) = 112$

This is 30% less than the correct value. The problem is obvious from the figure: the rectangles do not cover the entire area under the curve. For better precision, we must add more rectangles.

A good strategy is to double the number of rectangles. This way, we can reuse the y-values that we already computed. To go from 4 to 8 rectangles, we only need to compute 4 more y-values.

If we do not know the true value of the integral, how will we know when to stop doubling the number of rectangles? Generally, one stops when the change in the final result from the last doubling is acceptably "small enough". Clearly, "small enough" depends on the cost versus value of greater precision for the specific circumstances.

For this integral, the results with N rectangles, for various values of N, are:

N = 4: $S_4 = 112$

N = 8: $S_8 = 140$

N=16: $S_{16} = 155$

N=32: $S_{32} = 162.75$

Each time N doubles, the error decreases by about a factor of 2. But even at N=32, the standard procedure is still nearly 5% below the true value of 170.666….

Trapezoidal Integration

A simple way to improve on the standard procedure is to employ trapezoids rather than rectangles, as shown in Figure 17-2.

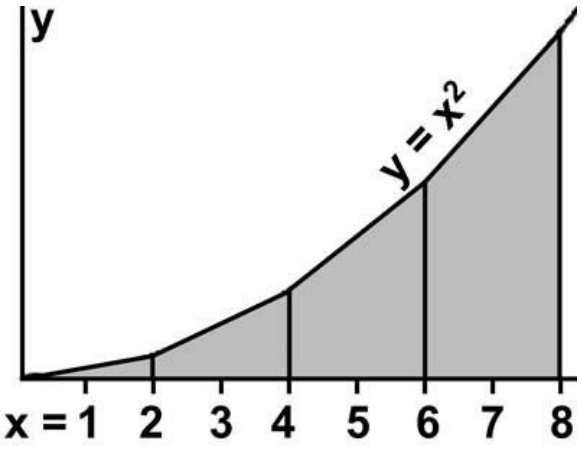


Figure 17-2 Trapezoids Replacing Rectangles

The area of a trapezoid equals its base multiplied by its average height, so the area of the largest trapezoid in the figure is:

$\Delta x\,[y(6) + y(8)]\,/2$

The equation for the total area of the 4 trapezoids in Figure 17-2 is:

$$S_4 = \Delta x\,[y(0) + y(2)]\,/2 + \Delta x\,[y(2) + y(4)]\,/2 + \Delta x\,[y(4) + y(6)]\,/2 + \Delta x\,[y(6) + y(8)]\,/2$$

$$S_4 = \Delta x\,[y(0) + 2y(2) + 2y(4) + 2y(6) + y(8)]\,/2$$

The endpoints, y(0) and y(8), each have half the weighting of the interior points. The difference between summing rectangles and summing trapezoids is that the rectangle sum includes only one endpoint, at full weight, while the trapezoid sum includes both endpoints at half-weight. We can rewrite this above sum as:

$$S_4 = \Delta x\,[y(0) + y(2) + y(4) + y(6)] + \Delta x\,[y(8) - y(0)]\,/2$$

The sum in the upper line is the area of 4 rectangles, exactly matching our prior procedure. Switching this numerical integral from rectangles to trapezoids simply adds the expression in the lower line.

In general, for any value of N, switching from rectangles to trapezoids requires one more y-value and increases the integral by:

$\Delta x\,[y(\text{x-max}) - y(\text{x-min})]\,/2$

Here, x-min and x-max are the integration endpoints.

Recalculating the integral of $y=x^2$ for N trapezoids, for the same values of N, yields:

N = 4: $S_4 = 176$

N = 8: $S_8 = 172$

N=16: $S_{16} = 171$

N=32: $S_{32} = 170.75$

For N=32, trapezoids reduce the error from 5% to 0.05%. The improvement is not always so dramatic. Trapezoids provide exact integrals for linear functions, which are trivial to integrate in any case. The example we choose here is a quadratic function of x, whose deviation from linearity decreases with the square of Δx, playing to trapezoids' strength. For more complicated functions, the improvement may be less significant. But the cost is minor: only 1 more y-value than the standard rectangle procedure.

Romberg Integration

Romberg integration improves the precision and minimizes the cost of numerical integration.

Let's consider integrating the function y(x)=sin(x) from x=0 to π/2. This function is also easy to integrate analytically:

$S = \int_0^{\pi/2} \sin(x)\,dx = -\cos(x)\,|_0^{\pi/2} = 0 - (-1) = 1$

Let's see how well various methods can match the correct value of 1.

Figure 17-3 shows an example of the standard procedure: numerically summing four rectangles, each of width π/8, with heights y(0), y(π/8), y(2π/8), and y(3π/8). The first rectangle is invisible since it has zero height.

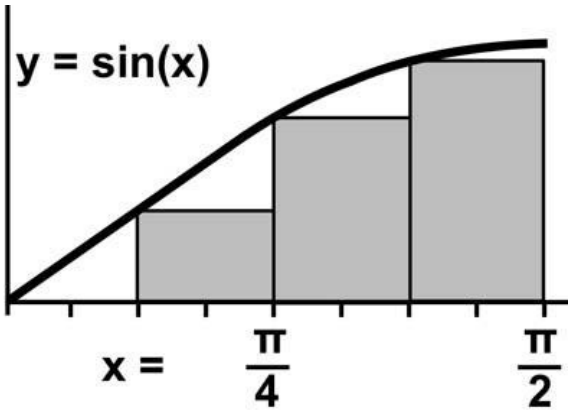


Figure 17-1 Rectangular Integration of sin(x)

The sum of the areas of these 4 rectangles is:
$S_4 = (\pi/8)\times(0+0.38268+0.70711+0.92388)$
$S_4 = 0.79077$
Continuing with the standard approach, we try more and more, narrower and narrower rectangles. Here are the results obtained with N rectangles, for various N:
N = 2: $\Delta x = \pi/4$: $S_2 = 0.55536$
N = 4: $\Delta x = \pi/8$: $S_4 = 0.79077$
N = 8: $\Delta x=\pi/16$: $S_8 = 0.89861$
N=16: $\Delta x=\pi/32$: $S_{16}= 0.95011$
N=32: $\Delta x=\pi/64$: $S_{32}= 0.97526$
Even with 32 values of sin(x), the numerical integral is off by 2.5%.

Using trapezoids instead of rectangles substantially improves these results, yielding:
N = 2: $\Delta x = \pi/4$: $S_2 = 0.94806$
N = 4: $\Delta x = \pi/8$: $S_4 = 0.98712$
N = 8: $\Delta x=\pi/16$: $S_8 = 0.99679$
N=16: $\Delta x=\pi/32$: $S_{16}= 0.99920$
N=32: $\Delta x=\pi/64$: $S_{32}= 0.99980$
For N=32, trapezoids are 100 better times than rectangles.

But Romberg integration is even better.
Figure 17-4 plots the S-values for five values of Δx.

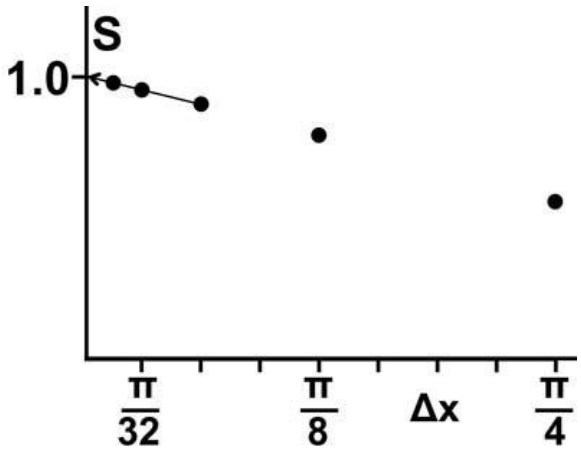


Figure 17-4 Extrapolating S to Δx=0

The Romberg scheme is to fit a curve to S(Δx), and extrapolate that curve to

$\Delta x=0$. In the figure, the fitted curve is shown passing through 3 S-values, with an arrow pointing to its intercept at $\Delta x=0$.

In this case, a quadratic curve is an excellent choice. Curve fitting is an art onto itself, as we will discuss in subsequent chapters. Fitting a quadratic to three data points is straightforward. As I prove below, a quadratic that passes through points f(z=1), f(z=2), and f(z=4) has a value at z=0 given by:
$f(0) = \{ 8f(1) – 6f(2) + f(4) \} / 3$
Fitting a quadratic and extrapolating to $\Delta x=0$ yields results that are extremely close to the correct answer, as these data show:
Set (2, 4, 8) : Romberg = 1 – 118 ppm
Set (4, 8,16): Romberg = 1 – 7.3 ppm
Set (8,16,32): Romberg = 1 – 0.5 ppm
Here, Set(J,K,L) means fitting a quadratic to the numerical integrals with J, K, and L rectangles. Also, “ppm” means parts per million, so 1 – 118 ppm = 0.999,882.

The power of the Romberg method is evident: it provides greater precision with less calculation.

With 8 values of the integrand, the standard procedure has an error of 10%. But with the Romberg method, the error is nearly 900 times smaller. Romberg with 8 rectangles is 200 times better than the standard rectangle method with 32 rectangles, and 2 times better than 32 trapezoids.

Note that in the Romberg scheme, there is no advantage to substituting trapezoids for rectangles. Since trapezoids increase each S-value by an amount that is linearly proportional to Δx, using them would change the slope of the Romberg curve, but not its intercept at $\Delta x=0$.

Next, I will show you how to fit a quadratic to three points spaced in the ratio 1:2:4. If you wish, you can skip the derivation, and just remember the result.

You can use these quadratic fit coefficients on any function, if you have its values at the right spacings. You can also use this procedure to calculate other coefficients for different spacings.

Define function $f(z) = A + Bz + Cz^2$. Our objective is to calculate A, the value of f at z=0.

$f(1) = A + B + C$

$f(2) = A + 2B + 4C$

$f(4) = A + 4B + 16C$

Subtract the upper and lower pairs.

$f(2)–f(1) = B + 3C$

$f(4)–f(2) = 2B + 12C$

Solve for C, then B, and finally A.

$\{f(4)–f(2)\} – 2\{f(2)–f(1)\} = 6C$

$C = \{f(4) –3f(2) +2f(1)\}/6$

$B = \{f(2)–f(1)\} – 3C$

$B = \{f(2)–f(1)\} – \{f(4)–3f(2)+2f(1)\}/2$

$B = \{5f(2) –f(4) –4f(1)\}/2$

$A = f(1) – B – C$

$6A = 6f(1)$
$– 15f(2) + 3f(4) +12f(1)$
$– f(4) + 3f(2) – 2f(1)$

$A = \{ 16f(1) – 12f(2) +2f(4) \}/6$

$A = \{ 8f(1) – 6f(2) +f(4) \} / 3$

QED

Chapter 18

Data Fitting

Imagine that we wish to measure the luminosity of a remote star as a function of time. We might be interested in the star's intrinsic properties — its variability, the frequency of flares, or the intensity of star-spots. We might also be searching for periodic luminosity dips that reveal orbiting planets.

Ideally, measurements would yield L(t), the luminosity at all times. But real world measurements actually yield L_j, luminosities with instrumental uncertainties, taken at a set of discrete times t_j.

To properly interpret real data, physicists often attempt to represent such data by continuous analytic functions. Called *curve fitting*, this craft is a blend of science and art. We sometimes fit data to theoretical functions, and other times to empirical functions, such as Taylor series.

Let's consider some examples.

If N competent physicists measure the mass of individual oxygen atoms, we might sum their results and divide by N to obtain an average value. In doing so, we have chosen to represent this data by a single constant, the simplest possible function. This seems entirely reasonable since oxygen atoms should have the same mass in everyone's laboratory…unless those atoms are moving at relativistic velocities. We would then be more successful using a more complex function that included each atom's velocity.
Next, consider measurements of the distance x_j that an airplane has traveled by time t_j, as it flies from Los Angeles to London. We might fit that data with a linear function: $x(t) = A+Bt$. Or we might try a more complicated function that addresses a slow climb followed by increasing speed as burning fuel decreases weight.

How do we know which function to use, and whether or not we have a "good"

fit?

Fit quality is generally evaluated with the ratio: $\chi^2/\#df$, chi-square per degree of freedom (also discussed in Chapter 9). For N data points, x_j with j=1 to N, and corresponding values of a fitting function f_j, χ^2 is given by:

$$\chi^2 = \Sigma_j \{ x_j - f_j \}^2 w_j / u_j$$

Here, we sum over j=1 to N. The w_j are optional weighting factors that are entirely subjective; you can employ them to stress data that you deem more important or more valuable than others. Most commonly, we treat all data points equally and set all weighting factors to 1.

The u_j's are the one standard deviation uncertainties in $\{x_j - f_j\}$. If the f-values have zero uncertainty, and the x_j's are the numbers of rare events, $u_j = \sqrt{x_j}$, if x_j is not zero. If some x_j's are zero, you can remove those j's from the summation, or set u_j to 0 or 1, depending on the circumstances.

When the collected data has gaps, those zeros should not be included in computing χ^2. Conversely, if you are absolutely sure that f_j must be precisely zero for some j, a nonzero x_j is definitive proof of an erroneous fit, and a zero or tiny value of u_j is appropriate. This decision can be as much art as science.

Next, let's discuss the *number of degrees of freedom*, which equals the number of terms in the summation of χ^2 minus the number of *adjustable parameters* in the fit.

If we are fitting our data to a straight line, there are two adjustable parameters: the line's slope and intercept. If we are fitting our data to a fifth-order polynomial, there are six adjustable parameters: A through F in:

$$f(t) = A + Bt + Ct^2 + Dt^3 + Et^4 + Ft^5$$

In almost all fits, there is at least one adjustable parameter. We must scale the fitting function to match the total number of events in the data, which is determined by budgets and manpower rather than by the laws of nature. We therefore use one adjustable parameter to ensure:

$$\Sigma_j f_j = \Sigma_j x_j$$

Let's first examine how we use $\chi^2/\#df$ to fit functions to data, and later discuss

what χ^2/#df says about the acceptability of that fit.

Imagine that we wish to experimentally determine a function x(t). We first measure data points x_j at times t_j, each with uncertainty u_j, for j=1 to N. We next try to fit that data with a quadratic polynomial:

$f(t) = A + Bt + Ct^2$

This fit has N–3 degrees of freedom. The equation for χ^2 is:

$\chi^2 = \Sigma_j \{ x_j - A - Bt_j - Ct_j^2 \}^2 / u_j$

If χ^2 were zero, the function f would represent the data perfectly. Evidently, the best fit corresponds to the values of parameters A, B, and C that minimize χ^2. We therefore compute the partial derivatives of χ^2 with respect to each parameter, set those derivatives equal to zero, and solve for A, B, and C.

$0 = \partial\chi^2/\partial A = -2 \Sigma_j \{ x_j - A - Bt_j - Ct_j^2 \} / u_j$

$0 = \partial\chi^2/\partial B = -2 \Sigma_j \{ x_j - A - Bt_j - Ct_j^2 \} t_j / u_j$

$0 = \partial\chi^2/\partial C = -2 \Sigma_j \{ x_j - A - Bt_j - Ct_j^2 \} t_j^2 / u_j$

We can rearrange these equations to read:

$\Sigma_j A /u_j + \Sigma_j B t_j/u_j + \Sigma_j C t_j^2/u_j = \Sigma_j x_j /u_j$

$\Sigma_j A t_j/u_j + \Sigma_j B t_j^2/u_j + \Sigma_j C t_j^3/u_j = \Sigma_j x_j t_j/u_j$

$\Sigma_j A t_j^2/u_j + \Sigma_j B t_j^3/u_j + \Sigma_j C t_j^4/u_j = \Sigma_j x_j t_j^2/u_j$

To reduce clutter, define:

$T_n = \{ \Sigma_j t_j^n/u_j \} / \{ \Sigma_j 1/u_j \}$

$X_n = \{ \Sigma_j x_j t_j^n/u_j \} / \{ \Sigma_j 1/u_j \}$

With those definitions, the prior equations become:

$A T_0 + B T_1 + C T_2 = X_0$

$A T_1 + B T_2 + C T_3 = X_1$

$A T_2 + B T_3 + C T_4 = X_2$

These equations can be written in matrix form as follows:

$$(A \; B \; C) \begin{pmatrix} T_0 & T_1 & T_2 \\ T_1 & T_2 & T_3 \\ T_2 & T_3 & T_4 \end{pmatrix} = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}$$

If the matrix, call it M, is singular (if its determinant is zero), the equations are under-constrained; they do not provide enough information to solve for A, B, and C.

But if M is non-singular, it has an inverse matrix M^{-1}. We can then multiply both

sides of this equation on their right ends by M^{-1}. Since MM^{-1} equals the unit matrix, the left side reduces to the vector (A, B, C), while the right side becomes a vector whose components are the best fit values of A, B, and C.

This procedure is valid for any number N of data points and any nth order polynomial, provided n+1 does not exceed N. The procedure can also be straightforwardly generalized for fitting functions other than polynomials.

Let's try an example: fit five data points with the quadratic, $f(t)=A+Bt+Ct^2$.

Let N = 5, all $u_j = 1$, and $t_j = (-2, -1, 0, +1, +2)$. These messy equations then reduce to:

$T_0 = (+1+1+1+1+1)/5 = 1$
$T_1 = (-2-1+0+1+2)/5 = 0$
$T_2 = (+4+1+0+1+4)/5 = 2$
$T_3 = (-8-1+0+1+8)/5 = 0$
$T_4 = (+16+1+0+1+16)/5 = 6.8$
$X_0 = (+x_1+x_2+x_3+x_4+x_5)/5$
$X_1 = (-2x_1-x_2+x_4+2x_5)/5$
$X_2 = (+4x_1-x_2+x_4+4x_5)/5$

The matrix M and its inverse M^{-1} are:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 6.8 \end{pmatrix} \qquad \begin{pmatrix} 13.6 & 0 & -4 \\ 0 & 2.8 & 0 \\ -4 & 0 & 2 \end{pmatrix}$$

$$\mathbf{M} \qquad\qquad \mathbf{5.6\times M^{-1}}$$

Note the determinant of M:

Det M = $1(2\times6.8-0) -0 +2(0-2\times2)$
Det M = 13.6 – 8 = 5.6

Multiplying the vector (X_0, X_1, X_2) by M^{-1} yields:

$A = \{ 13.6(+x_1+x_2+x_3+x_4+x_5) + 0$
$\quad -4(+4x_1-x_2+x_4+4x_5) \}/(5*5.6)$
$B = \{ 0 +2.8(-2x_1-x_2+x_4+2x_5) +0 \}/28$
$C = \{ -4(+x_1+x_2+x_3+x_4+x_5) + 0$

$+2(+4x_1 – x_2 + x_4 + 4x_5)\ \}/28$

These reduce to:

$A = (–6x_1 + 24x_2 + 34x_3 + 24x_4 – 6x_5)/70$

$B = \{–2x_1 – x_2 + x_4 + 2x_5\ \}/10$

$C = (+2x_1 – x_2 – 2x_3 – x_4 + 2x_5)/14$

Before we continue, note that the endpoints, x_1 and x_5, dominate the equations for B and C. This is generally true for all polynomial fits and also for some other functions. Statistical fluctuations or systematic errors in endpoint data can dramatically alter the fit coefficients.

After all that math, it is worthwhile checking some simple cases.

If the x_j are all equal ($x_j = Q$), the fit coefficients are:

$A = 28Q / 28 = Q$

$B = (–3Q + 3Q)/10 = 0$

$C = (+8Q – 8Q)/28 = 0$

If the x_j vary linearly ($x_j = Qt_j$), the fit coefficients are:

$A = (+4.8Q – 9.6Q + 9.6Q – 4.8Q)/28 = 0$

$B = (+4Q + Q + Q + 4Q)/10 = Q$

$C = (–8Q + 2Q – 2Q + 8Q)/28 = 0$

If the x_j vary quadratically ($x_j = Qt_j^2$), the fit coefficients are:

$A = (–9.6Q + 9.6Q + 9.6Q – 9.6Q)/28 = 0$

$B = (–8Q – Q + Q + 8Q)/10 = 0$

$C = (+16Q – 2Q – 2Q + 16Q)/28 = Q$

All three simple test cases yield the proper results. This means we probably have not made a big mistake.

Here is a practice problem if you wish: fit a quadratic to three data points: x_1, x_2, and x_3 measured at times –1, 0, and +1, respectively. Try doing this without looking at the preceding example. The answers are at the end of this chapter.

Having derived the fitting function, we can then calculate the figure of merit $\chi^2/\#df$, unless #df=0. In the latter case, we have no means of judging the quality of fit. Any three data points can be fit by a quadratic with $\chi^2=0$; but because #df=0, we have no basis for believing that the true function is well-represented by that quadratic.

In general, a “good” fit has χ^2/#df less than 1, or not “too much” more than 1. If χ^2/#df is much greater than 1, the fitted function is a bad match to the data. As #df increases, the expected χ^2 becomes closer and closer to 1, and what constitutes “too much” decreases rapidly. Appendix 4 contains a table quantifying the statistical expectations for χ^2 for various numbers of degrees of freedom.

Curve Fitting Cautions

The results of curve fitting should be considered suggestive rather than definitive; the results must be judiciously interpreted.

In searching for a “good” fit, one generally starts with simple functions, and then adds more complex terms as needed. If χ^2/#df = 6 for a quadratic polynomial, one might try a third-order polynomial. One might keep adding higher-order terms until χ^2/#df becomes “good”. But do not add more terms than necessary.

The coefficients in polynomial fitting are not stable. This means, the coefficient of t^3 in a fourth-order fit can be very different from the coefficient of t^3 in a fifth-order fit: the fit coefficients *do not* converge asymptotically to “true” values. Lower-order polynomial fits are generally more stable.

As discussed above, beware of endpoint issues. Any systematic problems that impact the endpoints will be magnified in a polynomial fit. Even unavoidable statistical fluctuations in endpoint values can substantially alter the fit. It is often wise to try alternative fits with the endpoints removed; if the fit function changes dramatically, you are on thin ice and may wish to reexamine the endpoint data.
Fits are much less sensitive to inner data points, but these too deserve attention. In addition to evaluating the sum of χ^2 over all points, look at the χ^2 contribution from each data point individually. χ^2 is the square of (the actual deviation divided by one standard deviation). For a single data point:
$\chi^2 > 1$, in 32% of data points
$\chi^2 > 4$, in 5% of data points
$\chi^2 > 9$, in 0.3% of data points
Data points with χ^2 well beyond these statistical expectations merit judicious scrutiny. The objective is to avoid biased data; biases can be statistical, systematic, or human (people have a tendency to get rid of “ugly” data).

Generally, one should throw out data only if it is almost certainly wrong. In an attempt to minimize sensitivity to statistical fluctuations, some researchers always throw out the maximum and minimum measured values in each data bin. If you do adopt that or any similar policy, you must be consistent to avoid unintentional biases.

Before publishing results, you might test your fits for stability. If you are organizing data into bins, check to see if you get a similar fit with twice as many bins, or with half as many bins. The entire fitting process may well be erroneous if the fit changes substantially depending on arbitrary analysis choices, such as bin size.

An Australian physicist suggests calculating medians instead of averages, because a median (the value that half the data are greater than and half are less than) is less impacted by statistical fluctuations than is an average or mean (the sum of all data values divided by their number). To my knowledge, that interesting idea never caught on.

Answer to Quadratic Fit

The challenge was to fit $x(t)=A+Bt+Ct^2$ to three data points: x_1, x_2, and x_3 measured at times –1, 0, and +1, respectively.

$T_0 = (+1+1+1)/3 = 1$

$T_1 = (–1+0+1)/3 = 0$

$T_2 = (+1+0+1)/3 = 2/3$

$T_3 = (–1+0+1)/3 = 0$

$T_4 = (+1+0+1)/3 = 2/3$

$X_0 = (+x_1+x_2+x_3)/3$

$X_1 = (–x_1+x_3)/3$

$X_2 = (+x_1+x_3)/3$

With these quantities, the matrix M and its inverse M^{-1} are:

$$3M = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \qquad M^{-1} = \begin{pmatrix} +3 & 0 & -3 \\ 0 & 3/2 & 0 \\ -3 & 0 & 9/2 \end{pmatrix}$$

The determinant of M equals 4/27. Multiplying the vector (X_0, X_1, X_2) by M^{-1} yields:

$A = 3(+x_1+x_2+x_3)/3 +0 –3(+x_1+x_3)/3$

$B = 0 +(3/2)(–x_1+x_3)/3 +0$

$C = –3(+x_1+x_2+x_3)/3 +0 +(9/2)(+x_1+x_3)/3$

These reduce to:

$A = x_2$

$B = (–x_1 +x_3)/2$

$C = (x_1 –2x_2 +x_3)/2$

QED

Chapter 19

Transforms & Fourier Series

In the prior chapter, we fit polynomials to represent data. In this chapter, we will discover how to represent functions with sinusoids, transforms, and spherical harmonics (sinusoids in two dimensions).

Fourier Series

Any physically realistic waveform, however complex, can be represented by a linear sum of sinusoidal functions. Let's now discover the mathematical procedure by which this is accomplished: the Fourier series.

Consider a violin string, vibrating between two fixed points, the neck stop and the violinist's finger. Define those fixed points to be x=0 and x=L. The violin bow excites a transverse oscillation that must have zero amplitude at each fixed point. The equation for string displacement f(x,t) is:

$f(x,t) = A \sin(\omega t+ø) \sin(n\pi x/L)$

$\lambda_n = 2L / n$

$\omega_n = 2\pi v / \lambda_n$

Here, ω_n is the oscillation frequency, A is the oscillation amplitude, ø is the initial phase, and λ_n is the wave length. The wave velocity v is determined by the string's mass density and tension, both of which has a constant value (after tuning). Frequency ω_n is determined by v and λ_n according to a standard wave equation.

The right-most sine function in f(x,t) ensures that f(x=0,t)=0. But we must also require that the oscillation *mode number* n be an integer so that f(x=L,t)=0. Since the sine function is zero twice in each wavelength, there must be an integral number of half-wavelengths in length L:

$L = n \lambda_n /2$

Each value of n yields a valid solution to the string's wave equation. Since this is

a linear system, any linear sum of such solutions is also a solution. Hence, the most general solution is:

$f(x,t) = \Sigma_n A_n \sin(\omega_n t+ø_n) \sin(n\pi x/L)$

$\omega_n = nv\pi/L$

for any integer $n>0$

Let's now define $\omega=v\pi/L$, the frequency at which all modes repeat, which means:

$\omega_n = n\omega = nv\pi/L$

We can also eliminate the phase angles $ø_n$ by using:

$\sin(n\omega t+ø_n) = \sin(n\omega t) \cos(ø_n) + \sin(ø_n) \cos(n\omega t)$

Let's next consider the wave form at a fixed value of x, and define:

$a_n = A_n \cos(ø_n) \sin(n\pi x/L)$

$b_n = A_n \sin(ø_n) \sin(n\pi x/L)$

With these definitions, we have another expression for the most general waveform at x:

$f(t) = \Sigma_n \{ a_n \cos(n\omega t) + b_n \sin(n\omega t) \}$

The above expression is called the *Fourier series* for f(t), a function that repeats with frequency ω.

In musical terms, the n=1 term is the *first harmonic*, n=2 is the *second harmonic*, *etc.*

Waves are almost always expressed with an average value of zero, such as sin(x), but the above sums can accommodate waves with offsets, such as 1+sin(x), by including an n=0 term, which is simply the constant a_0.

Figure 19-1 shows how increasing the number of terms in a Fourier series better approximates a true square wave shown in the upper third of this figure.

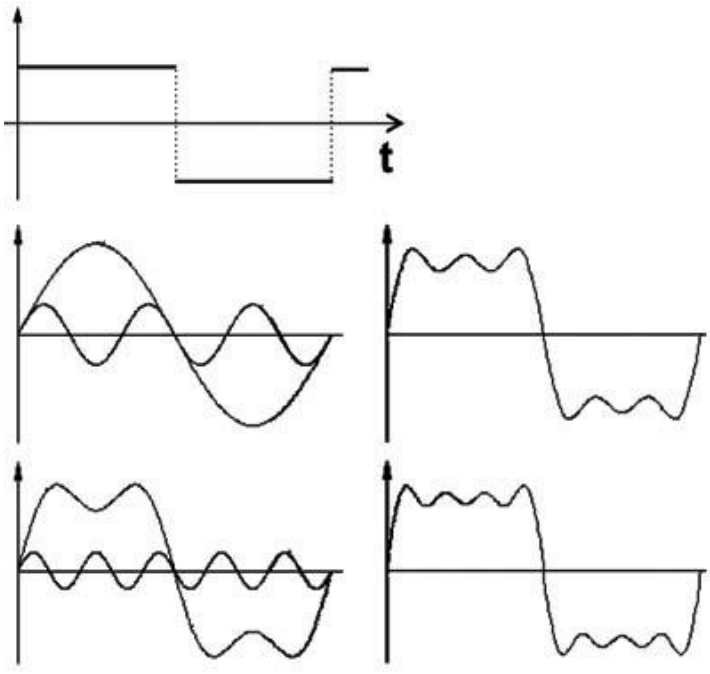


Figure 19-1 Fourier Fit To Square Wave

On the left side of Figure 19-1, the middle image plots the n=1 and n=3 modes [sin(ωt) and sin(3ωt)], while the lower image plots the best Fourier fit with modes n=0 to 3. On the right side, are the best fits with modes n=0 to 5 in the middle image, and n=0 to 7 in the lower image.

Musical Quality & Consonance

Fourier series help us describe the musical concepts of *quality* and *consonance*.

Even when the tones from a violin and an oboe have the same pitch (the same ω), the differing characteristics of these instruments produce sounds with different *tone qualities*, due to differing coefficients of higher frequencies in their Fourier series representations.

Sounds that contain only one harmonic are called *pure tones*. Sounds composed of many strong harmonics are called *rich tones*.

Calculating Fourier Coefficients

Fourier analysis is a powerful tool for linear systems because any periodic function f is a sum of sine and cosine functions, with some set of Fourier coefficients. We can often solve difficult equations for the special case of simple sinusoidal functions. The solution for the complex function f is then simply the sum of the sinusoidal solutions, with the appropriate Fourier coefficients.

All we need is a procedure to compute the Fourier coefficients. Not surprisingly, the person who developed this procedure was Jean-Baptiste Joseph Fourier (1768-1830). Consider again the Fourier series:

$f(t) = \Sigma_n \{ a_n \cos(n\omega t) + b_n \sin(n\omega t) \}$

Define T to be one full repetition period: $T = 2\pi/\omega$.

The integral of any sinusoid over a full cycle is zero. Hence only the cosine part of the n=0 term remains on the right hand side after that integration.

$\int_T f(t)\, dt = \int_T a_0 \cos(0)\, dt = T\, a_0$

$a_0 = (1/T) \int_T f(t)\, dt$

Fourier discovered that calculating the other coefficients is not much harder. Multiply the Fourier series by cos(jωt) for some integer j, with j>0, and then integrate over T:

$\int_T f(t) \cos(j\omega t)\, dt$

$= \Sigma_n \{ \int_T a_n \cos(n\omega t) \cos(j\omega t)\, dt \}$

$+ \Sigma_n \{ \int_T b_n \sin(n\omega t) \cos(j\omega t)\, dt \}$

Note that:

$2 \cos(n\omega t) \cos(j\omega t) = \cos([n+j]\omega t) + \cos([n-j]\omega t)$

$2 \sin(n\omega t) \cos(j\omega t) = \sin([n+j]\omega t) + \sin([n-j]\omega t)$

$2 \sin(n\omega t) \sin(j\omega t) = \cos([n-j]\omega t) - \cos([n+j]\omega t)$

Each term on the right side of each equation is a sinusoidal function of frequency mω, for m either n+j or n–j. If m is nonzero, the integral over period T is zero. If m is zero, sin(mωt) is zero in the middle equation. Therefore, we have shown that after integrating from t=0 to $t=T=2\pi/\omega$:

$\int \cos(n\omega t) \cos(j\omega t)\, dt = T/2$ if n=j, else = 0

$\int \sin(n\omega t) \cos(j\omega t)\, dt = 0$ for any n and j

$\int \sin(n\omega t) \sin(j\omega t)\, dt = T/2$ if n=j, else = 0

The prior integral becomes:

$\int_T f(t) \cos(j\omega t)\, dt = \int_T a_j \cos(j\omega t) \cos(j\omega t)$

$\int_T f(t) \cos(j\omega t)\, dt = a_j\, T/2$

$a_j = (2/T) \int_T f(t) \cos(j\omega t)\, dt$

We can repeat this logic multiplying f(t) by sin(jωt), and integrating over T. The result is:

$b_j = (2/T) \int_T f(t) \sin(j\omega t)\, dt$

We have been very successful analyzing many repetitive motion problems using exponentials with complex exponents. We can employ that technique here as

well, and rewrite the Fourier series equations as:
$f(t)$ = Real part of $\Sigma_n \{ z_n \exp(in\omega t) \}$, with
$z_n = a_n + ib_n$
$z_n = (2/T) \int_T f(t) \exp(-in\omega t)\, dt$, for $n>0$
$a_0 = (1/T) \int_T f(t)\, dt$, and $b_0 = 0$

Evaluating a Fit

We showed above how to calculate the Fourier coefficients a_n and b_n. But how do we know when to stop calculating — how many terms in the Fourier series do we need?

Generally, one calculates Fourier coefficients for larger and larger values of n until their values become "small" — until a_N and b_N are much smaller in absolute value than the a_n and b_n for all $n<N$. Then evaluate the fit by calculating the *rms, root mean square* deviation:
$(rms)^2 = (1/T) \int_T \{ f(t) - F(t) \}^2\, dt$
with $F(t) = \Sigma_n \{ a_n \cos(n\omega t) + b_n \sin(n\omega t) \}$
Here, the sum is from $n=0$ to $n=N$. We cannot use the χ^2/#df tables to evaluate rms, because f and F have no statistical fluctuations, But rms does provide a quantitative measure of the fit precision. Ideally, rms is zero, indicating a perfect fit. With a nonzero rms, you can decide whether the benefit of potentially greater precision is worth the cost and effort of calculating more coefficients.

Fourier Series of Square Wave

With these equations we can calculate the Fourier series for a square wave. Let the square wave be a periodic function, repeating with period T and defined by:
$f(t) = +1$ for $0 \le t < T/2$
$f(t) = -1$ for $T/2 \le t < T$
$f(t+T) = f(t)$ for any t
The square wave function is shown in Figure 19-2.

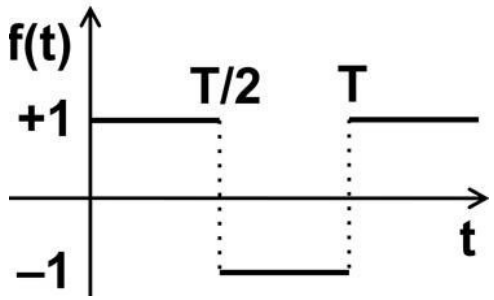


Figure 19-2. Square Wave Function

Clearly a_0, the average value of f(t), is zero. To compute the other coefficients, we must separate each integral into two parts: (1) the integral from t=0 to t=T/2, where f(t)=1; and (2) the integral from t=T/2 to t=T, where f(t)=–1. Recall ωT=2π.

$a_j = (2/T) \{ \int_{(1)} \cos(j\omega t)\, dt - \int_{(2)} \cos(j\omega t)\, dt \}$

$a_j = (2/Tj\omega)$

$\{ \sin(j\omega T/2) - 0 - \sin(j\omega T) + \sin(j\omega T/2) \}$

$a_j = (1/j\pi) \{ 2\sin(j\pi) - \sin(j2\pi) \}$

$a_j = (1/j\pi) \{ 0 - 0 \}$

$b_j = (2/T) \{ \int_{(1)} \sin(j\omega t)\, dt - \int_{(2)} \sin(j\omega t)\, dt \}$

$b_j = (2/Tj\omega)$

$\{-\cos(j\omega T/2)+1 + \cos(j\omega T)-\cos(j\omega T/2) \}$

$b_j = (1/j\pi) \{ 1 -2\cos(j\pi) + \cos(j2\pi) \}$

$b_j = (1/j\pi) \{ 2 -2\cos(j\pi) \}$

$a_j = 0$ for all j

$b_j = 0$ for j even

$b_j = (4/j\pi)$ for j odd

$f(t) = (4/\pi) \{\sin(\omega t) + \sin(3\omega t)/3 + \sin(5\omega t)/5 + \dots \}$

Since all sinusoids are continuous functions, the Fourier series of a square wave cannot exactly match the square wave at its discontinuity (t=T/2 in this case). Here we find:

$f(T/2) = (4/\pi) \{\sin(\pi) + \sin(3\pi)/3 + \sin(5\pi)/5 + \dots \} = 0$

The Fourier series yields the value half way between the square wave's value at t<T/2 and at t>T/2. This seems reasonable. Natural phenomena are almost never discontinuous. Any physically realistic function that goes from +1 to –1 must pass through zero.

Fourier Transform

Fourier series are appropriate for waves confined to a finite space. Let's now extend the Fourier methodology to an infinite space.

We showed above that the equations for the Fourier series representation of f(t) can be written:
f(t) = Real part of Σ_n { $z_n \exp(in\omega t)$ }, with
$z_n = a_n + ib_n$
$z_n = (2/T) \int_T f(t) \exp(-in\omega t)\, dt$, for $n>0$
$a_0 = (1/T) \int_T f(t)\, dt$, and $b_0 = 0$
Here the Fourier series is the sum of a set, perhaps an infinite set, of terms with discrete frequencies. For a space of length L, this discrete set includes all frequencies that are multiples of $\omega=\pi v/L$.

We can generalize this by extending the Fourier sum to a continuous sum, an integral of terms with a continuous range of frequencies.

For any physically realistic function s(t), the **Fourier transform** S(f) and its inverse transform are:
$S(f) = \int s(t) \exp\{-i2\pi f t\} dt$
$s(t) = \int S(f) \exp\{+i2\pi f t\} df$
The last equation shows that the Fourier transform has an inverse operation that brings us back to s(t). We need to specify the range of the above integrals. The range of both of the above integrals is from $-\infty$ to $+\infty$, if:
$\int_{-\infty}^{+\infty} |s(t)|\, dt$ is finite
Note that s(t) is a function of time, whereas S(f) is a function of frequency. Roughly speaking, S(f) is the amount of frequency f in the sum of complex exponentials that equals s(t).

For example, consider a function s(t) and its Fourier transform S(f):
$s(t) = [\sin(t) + \cos(3t)]\, D$

with $D = \exp\{-t^2/10000\}$
We included D to make the integral of s(t) finite, while very slowly reducing the amplitude of s(t). In this way, D does not change significantly within one oscillation of either sin(t) or cos(3t).
$S(f) = \int s(t) \exp\{-i2\pi f t\} dt$
$S(f) = \int s(t) [\cos(2\pi f t) - i \sin(2\pi f t)] dt$
$S(f) = \int D \sin(t) \cos(2\pi f t) dt$
$+ \int D \cos(3t) \cos(2\pi f t) dt$

$-i \int D \sin(t) \sin(2\pi ft)dt$
$-i \int D \cos(3t) \sin(2\pi ft)dt$

We can make these approximations:

$\int D \sin(t) \cos(2\pi ft)dt = 0$

$\int D \cos(3t) \sin(2\pi ft)dt = 0$

$\int D \cos(3t) \cos(2\pi ft)dt = 0$ unless $2\pi f=3$

$\int D \sin(t) \sin(2\pi ft)dt = 0$ unless $2\pi f=1$

For $2\pi f = 1$: $\int D \sin(t) \sin(2\pi ft)dt$

$= \int D \sin^2(t)\, dt$

$= \int (1/2) \exp\{-t^2/10000\}\, dt = \sqrt{(\pi/2)}$

For $2\pi f = 3$: $\int D \cos(3t) \cos(2\pi ft)dt$

$= \int D \cos^2(3t)\, dt$

$= \int (1/2) \exp\{-t^2/10000\}\, dt = \sqrt{(\pi/2)}$

To this approximation, S(f) is nonzero at only two values of f, because only two sinusoidal functions need be summed to equal s(t).

Since we know how to solve linear differential equations for sinusoidal forces, we know how to solve them for any series, even an infinite one, of sinusoids. Thus, with Fourier series and transforms, we can solve linear differential equations for any realistic function.

Fourier Transform of a Gaussian

Now let's find the Fourier representation of a Gaussian distribution. Gaussians are very important because many natural phenomena follow such distributions. The equation for a Gaussian distribution, G(x), with mean zero and standard deviation σ is:

$G(x) = \exp\{-x^2/2\sigma^2\} / \sqrt{(2\pi\sigma^2)}$

The Fourier transform of function G(x) is:

$S(k) = \int_{-\infty}^{+\infty} \exp\{-x^2/2\sigma^2 - ikx\}\, dx / (2\pi\sigma)$

This is not the prettiest exponent, but it is integrable with a neat trick called *completing the square*. We can make the exponent of the integrand a perfect square by adding the right constant A to the exponent of G(x).

$-(x/\sigma\sqrt{2} + A)^2 = -x^2/2\sigma^2 - 2xA/\sigma\sqrt{2} - A^2$

To complete the square, choose A such that:

$2xA/\sigma\sqrt{2} = ikx$

$A = ik\sigma/\sqrt{2}$

We can then rewrite the exponent of the integrand as:

$-x^2/2\sigma^2 - ikx = -(x/\sigma+ik\sigma)^2/2 - k^2\sigma^2/2$

The exponential then becomes:
$\exp\{-(x/\sigma+ik\sigma)^2/2\} \exp\{-k^2\sigma^2/2\}$
To reduce clutter, define $u = x/\sigma+ik\sigma$, which means $du = dx/\sigma$.

The integral then becomes:
$S(k) = \exp\{-k^2\sigma^2/2\} \int \exp\{-u^2/2\} \sigma du / (2\pi\sigma)$
$S(k) = \exp\{-k^2\sigma^2/2\} \sqrt{(2\pi)} / (2\pi)$
$S(k) = \exp\{-k^2\sigma^2/2\} / \sqrt{(2\pi)}$
We see that S(k), the Fourier transform of the Gaussian G(x), is also a Gaussian. Also note that the standard deviation of G(x) is σ, while the standard deviation of S(k) is $1/\sigma$.

For any Gaussian distribution, about 50% of the population is contained within $1/\sqrt{2}$ standard deviations of the mean. In the context of a wave packet, we can view $1/\sqrt{2}$ standard deviations as being the uncertainties Δx and Δk in the values of x and k, respectively. The product of these two uncertainties is:
$\Delta x \Delta k = (\sigma)/\sqrt{2} (1/\sigma)/\sqrt{2} = 1/2$
This analysis proves that reducing Δx increases Δk, and vice versa, demonstrating the unavoidable tradeoff between the uncertainty of a wave packet's location and the uncertainty of its wave number.

Since quantum mechanics equates momentum p with $\hbar k$, we have proven:
$\Delta x \Delta p = \hbar/2$, for Gaussian distributions
which is the Heisenberg Uncertainty principle of quantum mechanics.

Green's Function

Let's now examine the Fourier transform of a strange but particularly useful function. The *Dirac delta function* δ(x) equals zero for all x except x=0, but is, loosely speaking, infinite at x=0. To be more precise, for any function f(x):
$\int_{-\infty}^{+\infty} \delta(x) f(x) dx = f(0)$
The Dirac delta function effectively picks out the single value of f at x = 0. The Dirac delta function can represent an instantaneous force: a sudden impact that transfers energy and momentum but whose duration is infinitesimal. This is an idealization that is never fully realized in practice, but is very useful mathematically and conceptually.
The Fourier transform of the Dirac delta function is simple:
$\boldsymbol{S}(f) = \int_{-\infty}^{+\infty} \delta(t) \exp\{-i2\pi ft\}dt$

$\boldsymbol{S}(f) = e^{-0}$

$\boldsymbol{S}(f) = 1$

This means the Fourier transform of the Dirac delta function includes all frequencies equally. Since we can solve linear differential equations for sinusoids, we can solve them for an instantaneous impulsive force, as represented by the Dirac delta function.

We can now take that another step forward. As sketched in Figure 19-3, any function F(t) can be broken down into a series of thin rectangles, each having width dt. The rectangle at time t has height F(t).

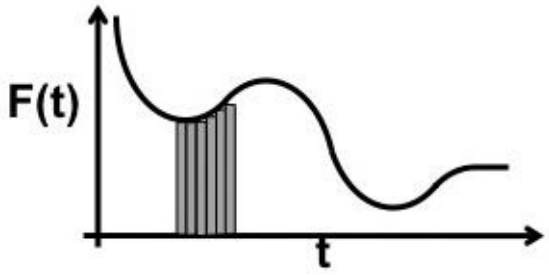


Figure 19-3 Function F(t) Decomposed Into Rectangles

In the limit that dt goes to zero, the rectangles become a series of instantaneous impulses, which we now know we can solve. This approach is called the *Green's function method*, named after George Green (1793 – 1841).

Spherical Harmonics

Spherical harmonics occur frequently in the analysis of both atomic scale and cosmic scale phenomena when using 3-D spherical coordinates. Spherical harmonics are typically written:

$Y_{j,m}(\theta,\phi)$

Here, θ is the polar angle, and ø is the azimuthal angle (see Figure 2-6). In addition, j is the total angular momentum of this distribution, and m is the component of j in the θ=0 direction.

Another common representation of spherical harmonics uses the *associated Legendre functions* $P_j^m(\cos\theta)$:

$P_j^m(\cos\theta)\exp\{im\phi\} = Y_{j,m}(\theta,\phi)$

It is helpful to know both representations, since both are often referenced and tabulated in many books and online.

The spherical harmonics $Y_{j,m}(\theta,\phi)$ are mutually orthogonal and form a complete set. This means any distribution on the surface of a sphere can be represented by a linear sum of spherical harmonics.

For example, let Q be a function of angles θ and ø, and express Q as a linear superposition of spherical harmonics. This is:

$$Q(\theta,\phi) = \Sigma_{jm}\, A_{jm}\, Y_{j,m}(\theta,\phi)$$

Now multiply this equation by $Y^*_{J,M}(\theta,\phi)$, the complex conjugate of any selected harmonic $Y_{J,M}(\theta,\phi)$, and integrate over all values of the angles (θ=0 to π; ø=0 to 2π).

$$\int Q(\theta,\phi)\, Y^*_{J,M}(\theta,\phi)\, d\theta\, d\phi$$
$$= \int \Sigma_{jm}\, A_{jm}\, Y_{j,m}(\theta,\phi)\, Y^*_{J,M}(\theta,\phi)\, d\theta\, d\phi$$

Since the Y's are mutually orthogonal, each jm term in the summation vanishes when integrated, except the one term with j=J and m=M. Therefore, we have:

$$A_{JM} = \int Q(\theta,\phi)\, Y^*_{J,M}(\theta,\phi)\, d\theta\, d\phi \;/\; \int Y_{J,M}(\theta,\phi)\, Y^*_{J,M}(\theta,\phi)\, d\theta\, d\phi$$

Repeating this process for all required values of J and M yields the complete set of A's needed to represent Q. One can evaluate the quality of fit, and determine how many spherical harmonics are required, in the same manner that we described above for evaluating a Fourier series fit.

The $Y_{j,m}$ functions for j=0, 1, and 2 are tabulated here.

P	j	m	$Y_{j,m}(\theta,\phi)$
s	+ 0	0	1
p	− 1	+1	$-\sin\theta\, e^{+i\phi}/\sqrt{2}$
		0	$\cos\theta$
		−1	$+\sin\theta\, e^{-i\phi}/\sqrt{2}$

P	j	m	$Y_{j,m}(\theta,\phi)$
d	+ 2	+2	$+\sin^2\theta\, e^{+2i\phi}\sqrt{(3/8)}$
		+1	$+\sin\theta\cos\theta\, e^{+i\phi}\sqrt{(3/2)}$
		0	$(3\cos^2\theta - 1)/2$
		−1	$-\sin\theta\cos\theta\, e^{-i\phi}\sqrt{(3/2)}$
		−2	$-\sin^2\theta\, e^{+2i\phi}\sqrt{(3/8)}$

The unlabeled left-most column above shows the historical symbol assigned to

each value of j. These symbols originated well before anyone understood the underlying physics. While somewhat archaic, they remain in frequent use.

The column labeled P is the *parity* of Y, which equals $(-1)^j$. Upon mirror reflection, an even or positive parity harmonic remains unchanged, while an odd or negative parity harmonic flips polarity.

The above $Y_{j,m}(\theta,\phi)$ are not normalized. This is adequate for most purposes, because spherical harmonics are often combined with functions of radius and time, with normalization occurring thereafter.

Chapter 20

Advanced Data Analysis

A notable physicist once said: “With 8 parameters, I can fit an elephant.” While exaggerated for dramatic effect, the message is: with enough data manipulation, almost any conclusion is possible. Data manipulation is a case of less is better than more.

Clever data analysis drives science forward, but it is a hazardous road. It is best to exercise great care and remain skeptical.

An interesting example is a 2013 paper on exoplanets. The paper admits that, among 42,000 Sun-like stars, they actually detected ZERO Earth-like planets in Earth-like orbits. Yet, after extensive analysis, they claim to prove that 12% of Sun-like stars have an Earth-like planet in the habitable zone. Is it really possible to prove that more than 5000 Earths exist where none are found? To evaluate this claim, you as a scientist must understand the powers and pitfalls of many different data analysis techniques.

When doing your own data analysis, be the devil’s advocate. Ask yourself: How could my analysis be wrong? Do not stop asking until you have solid answers to refute every imaginable objection. Be equally zealous in questioning results that you “know” are true as you would results that you find disturbing. It is best to be the first to ask penetrating questions — you certainly will not be the last interrogator, and not all of those will be friendly.

In this chapter, we explore some cutting-edge data analysis tools, and how to avoid leaving your blood on those sharp edges.

Monte Carlo Methods

Many very complicated problems in physics are solved using *Monte Carlo* methods. The most common application is estimating the detection efficiency of

experimental apparatus.

Monte Carlo calculations have similarities to numerical integration, but are much more versatile.

Let's start with a simple example: finding the area A of the gray region in Figure 20-1.

This is possible for any A on two conditions: (1) for any x and y, we must be able to determine whether or not the point (x,y) is within A; and (2) A must be contained within known limits.

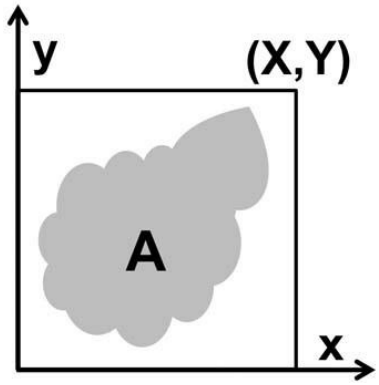


Figure 20-1 Finding Area A

In this case, the known limits of A are [0,X] horizontally and [0,Y] vertically.

The Monte Carlo procedure seems incredibly simple: randomly select N points within the known limits, and determine the number n that fall within A. The area of A is then:

$A = (n/N)\ XY$

Here, XY is the total area within the known limits, and (n/N) is the fraction that are within A.

This is equivalent to randomly throwing darts at the total area XY and counting the fraction that hit A.

What could be simpler?

A Real Monte Carlo Example

Now consider a more challenging real-world situation: radiation cancer therapy. A beam of electrons with a known energy distribution and cross-section enters a patient whose life is threatened by a malignant tumor. The beam enters the patient at a known location and angle. These electrons lose energy and scatter as they pass through tissues of various densities and shapes. While each individual scattering event is random, physicists have carefully measured the probability distributions of scattering angles and energy loss.

We need to calculate how much energy is deposited in the tumor, and how much in the patient's vital organs. While the challenge varies with the type of tumor and type of healthy tissue, the general rule is: if the radiation dose is 2% too low there is a 50% chance of not killing the tumor, which thereby kills the patient indirectly; if the dose is 2% too high there is a 50% chance of the treatment killing the patient directly. Our calculations must be very precise.

With extreme care, such precision is possible, but only with Monte Carlo methods and extensive particle physics measurements.

Electrons lose energy and also scatter as they penetrate matter. Most of their energy is deposited where they come to a stop. Both energy loss and scattering depend on the amount and type of matter traversed. For most human tissue, energy loss and scattering are proportional to distance traveled multiplied by the density of matter; this product has units of grams/cm^2. Energy loss and scattering rates increase in matter with a high atomic number, such as bones rich in calcium.

An electron's interaction probability within range R is given by:

$p(R) = \exp\{-R/\lambda\}$

Here, λ is the mean free path, which varies with electron energy and material type. Both R and λ are measured in grams/cm^2. If one randomly picks a probability p between 0 and 1, the range R with the correct distribution is given by:

$R = -\lambda \ln\{p\}$

Deriving an optimal treatment plan begins with a CT-scan that produces a high-resolution cross-sectional map of the mass density within the patient. This map is typically hundreds of pixels wide by hundreds of pixels tall. An example is

shown in Figure 20-2.

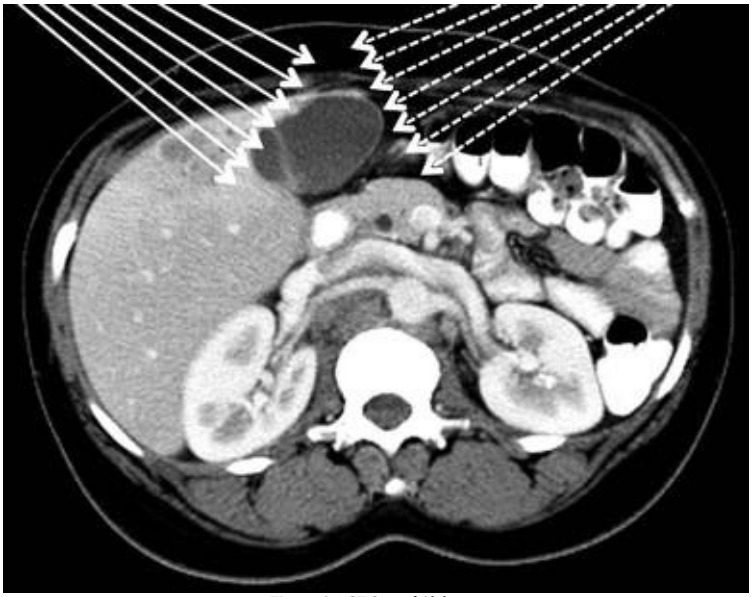


Figure 20-2 CT Scan of Abdomen

The white arrows indicate two possible beam orientations aimed at a tumor in the patient's liver. We wish to maximize the radiation delivered to the tumor while minimizing radiation delivered to vital organs, such as the spinal cord. Also seen in the image are both kidneys, the descending aorta, muscles of the back, intestines partially filled with fluid, six ribs in cross-section, and an ample layer of fat under the skin.

We seek to calculate a dose map of the energy deposited in each pixel from an exposure to N electrons with average energy E and entry angle θ.

The key steps in such calculations are:

1. Set number of simulated electrons n=0.
   Zero the dose map.

2. Add 1 to n. For electron #n, randomly select energy E & position within beam profile according to their distributions.

3. Project electron #n onto the patient's skin.

4. Randomly select interaction depth R.

5. Project electron #n forward 1 pixel.
   If electron has exited patient, go to #9.
   Set ΔR = mass density × pixel width.
   Calculate energy loss ΔE in ΔR.

6. Add ΔE to dose map; subtract ΔE from E.
   If E<=0 go to #9.

7. Subtract ΔR from R; if R>0 go to #5

8. Randomly scatter electron #n:
   Compute new electron direction and scattering energy loss ΔE.
   Add ΔE to dose map, and go to #4.

9. Electron #n completed.
   If n<N go to #2

10. Dose map is complete.

Having calculated the dose map for one beam energy and angle, we repeat this procedure to obtain dose maps for several other beam conditions. We then search for the optimum treatment plan: the combination of exposure intensity at each beam condition that maximizes tumor dose and minimizes vital organ dose. We will discuss search procedures shortly.

Clearly, this application of physical principles and techniques is quite different from typical particle physics research. But both employ the same analysis methods, which must overcome the same pitfalls.

This is a good time to ask: How do we estimate the precision of a Monte Carlo analysis? When is N, the number of simulated electrons, large enough?

Most likely, K, the number of electrons passing through any particular pixel, has a Gaussian distribution (most things do). For rare events, one standard deviation of K equals $\sqrt{K}$. But, since Monte Carlo calculations are often Big Black Boxes, it may be worth verifying that the results truly do have a Gaussian distribution.
One simple test is comparing separate trials. Imagine that we decide to calculate a dose map with N=100,000. We can pause after every 10,000 simulated

electrons, and save the dose map for that group. When we have ten maps, we can average them to create a master map, and then plot the deviations of each of the ten maps from the master. If those deviations have a Gaussian distribution, we can calculate the rms (root mean square) differences between the ten maps and the master map. The estimated standard deviation of the master dose map is that rms divided by √(10–1). It's 10–1 because we removed one degree of freedom in calculating the master. If the deviations are not Gaussian, there may be a systematic effect that should be identified.

You can then decide if the benefit of halving the standard deviation justifies the cost of quadrupling N.

In oncology, we want to drive the dose uncertainties well below the ±2% figure quoted earlier.

In a physics experiment, your final result might be n, the number of detected particles, divided by ε, the Monte Carlo-computed detection efficiency. The uncertainty in your final result will be the square root of the sum of the squares of the fractional uncertainties in n and ε. Since physics experiments are typically much more expensive than computer simulations, you will want the fractional uncertainty in ε to be well below the statistical fractional uncertainty in n.

You no doubt noticed that Monte Carlo methods rely heavily on randomly selected values of key variables. All major computer languages have subroutines that provide random numbers; typically, they respond to each request with one random number between 0 and 1.
One potential pitfall is that computer-generated random numbers are only *pseudo-random*.

True random numbers are *independent* samples from a *fixed* distribution. In this case, they have an equal probability of having any value between 0 and 1. Pseudo-random numbers have the proper distribution, but they are not entirely independent: they may be *correlated* with one another in various ways.
The essence of current computer technology is absolute determinism: computers invariably and precisely execute specified instructions. This makes it impossible for any existing computer to produce a truly random result. That capability may

come with quantum computers.

In some applications, each trial, such as simulating a coin flip, might require only one random number, so correlations might not be a problem. But most Monte Carlo applications use many random numbers to simulate each trial. In the oncology example, we might need more than 10 random numbers to track a single electron through a patient's body. In such cases, random number correlations can lead to serious biases.

Consider a particle physics example: a Monte Carlo simulation of the decay:
$K^+ \rightarrow \pi^+ + \pi^+ + \pi^-$
Here, a kaon decays into three charged pions. In the kaon's rest frame, the three pions share 75 MeV of kinetic energy. This means there are two degrees of freedom for how that energy is distributed: the kinetic energy of pion #1, and the kinetic energy of pion #2. Having selected those, the kinetic energy of pion #3 is fixed. There are three additional degrees of freedom associated with angular orientations. We can examine the distribution of pion momenta using a Dalitz plot.

The upper portion of Figure 20-3 is a Dalitz plot of real experimental measurements reported by Brian Lindquist. Each dot represents one decay. We see that the dots are distributed uniformly, as they should be.

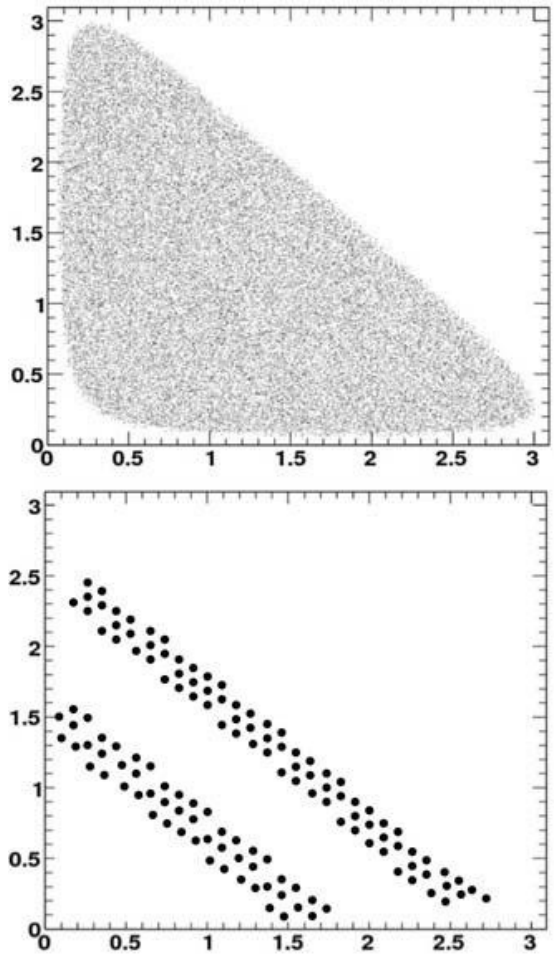


Figure 20-3 Dalitz Plots

The lower image is a possible Dalitz plot of a Monte Carlo simulation of this decay, where pseudo-random number correlations are obvious. Pairs of consecutive computer-generated random numbers have related values — they are not independent. The “random” pairs do not fill the space uniformly, causing subtle errors in the simulation.

Without explicitly looking for this effect, you might well never notice this systematic error. This actually happened in my thesis experiment, but fortunately a wise colleague found it early on. His solution was to get a first pseudo-random number and use it to determine how many of the following pseudo-random numbers to discard before actually using the next one. That brute force fix eliminated any detectable correlations.

Another pitfall of Monte Carlo methods is human: unjustified confidence. Because computers do arithmetic perfectly, it is easy to think that computers are always correct. Never assume that. Remember the computer adage: GIGO —

Garbage In, Garbage Out.
Monte Carlo analyses typically require an enormous amount of input data, which computers accept without question. If the input data has an error, the output will be wrong. If your programming instructions have an error, the output will be wrong. If you forget an important physical effect, the output will be wrong.

Monte Carlo analysis is a powerful tool that must be used with great skill and scrutiny.

Searching for Optima

The oncology example leads us to another analysis challenge: searching for optima. In cancer therapy, we seek the greatest tumor-to-vital-organ dose ratio. In calculating electron distributions in complex molecules, we seek the lowest energy levels. In designing physics experiments, we use Monte Carlo simulations to achieve the greatest detection efficiency for a fixed budget.

In these cases and many more, if there are N adjustable parameters (variables), we must search an N-dimensional space. This can become enormously expensive as N increases.

Let's consider a simple example: maximizing a function f within some range of two variables x_1 and x_2. (If you need to minimize a function g, just maximize f = –g.)

Figure 20-4 shows an example, with the largest values of f shown in the darkest shades of gray. Such plots are often called *landscapes*, and the optimization procedure is call *hill climbing*.

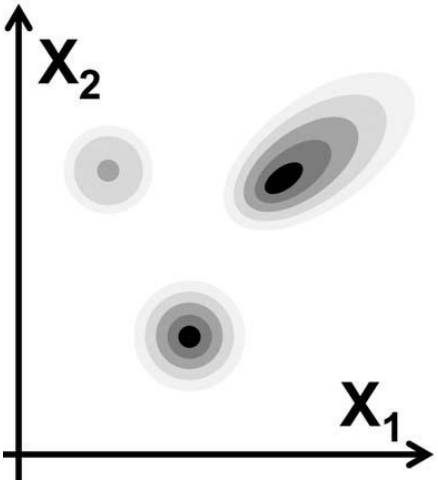


Figure 20-4 Search Landscape

Before discussing how to find the maximum, examine three challenges that are illustrated in the figure: (1) *local maxima*; (2) *plateaus*; and (3) *skewed ridges*.

The modest hill in the upper left is a local maximum but not a global maximum. Its peak is a better solution than nearby alternatives, but it is not the best solution in the landscape.

A second challenge is the plateau: most of the landscape is flat. At any randomly selected point, neighboring points have a very similar f. This means there is no clear guidance as to where to search next.

The third challenge is in the upper right: a ridge that is much wider in one direction than in others. The problem this ridge poses is that it is skewed, extended in a direction quite different from either axis. As hill climbing algorithms typically step along one axis at a time, it will take a great many small steps to zigzag back and forth along this ridge.

Let's discover how these algorithms typically work.

Hill climbing algorithms start at one set of values of all the variables. They cycle through the variables, and sequentially evaluate f at larger and smaller values of each variable. Whenever they find a larger value of f, they restart their search at that set of variable values. The most basic algorithm is:

#1 Choose an initial step size Δ_j for each variable X_j, j=1 to N. Each Δ_j is

typically some percentage of X_j's allowed range. Set each X_j to some starting value, which may be chosen randomly, or may be the center of X_j's range. Calculate $F=f(X_1,\ldots,X_N)$ and set MAX=F. Set END to whatever number of search loops you choose, and set LOOP=0.

1. Set $Y_j = X_j$ for all j, and set J=1

2. Add Δ_J to Y_J
   Calculate $F=f(Y_1,\ldots,Y_N)$.
   If F>MAX, go to #3, else
   subtract Δ_J from Y_J, and go to #4

3. Set MAX=F. Set $X_J = Y_J$.
   Go to #2

4. Subtract Δ_J from Y_J
   Calculate $F=f(Y_1,\ldots,Y_N)$
   If F>MAX, go to #5, else
   add Δ_J to Y_J, and go to #6

5. Set MAX=F. Set $X_J = Y_J$
   Go to #4

6. Add 1 to J, if J<=N go to #2

7. Divide all Δ_j's by 4. Add 1 to LOOP
   If LOOP<END go to #1

8. Search complete
   Maximum f equals MAX at coordinates X_j

By reducing the step sizes in #7, the search zooms in on the maximum.

There are countless refinements that can be made to the above algorithm. However, no search algorithm is guaranteed to find the global maximum in a finite number of loops.

To avoid the local maxima and plateau problems, some search procedures begin

by calculating f at an array of trial points that are equally spaced along each of the N axes. Hill climbing is then launched at the best trial value.

Another alternative is periodically evaluating f at a randomly chosen point in N-space. If that point is better, the search is restarted there.

Some algorithms attempt to minimize skewed ridge problems by calculating f at all neighboring points in N-space, and then moving to the best of those, simultaneously changing as many variables as required.

The performance of a computer search can be greatly improved by employing physical intuition. Often, some variables are linked by natural phenomena, or have special symmetry properties. These considerations may make (X_1+X_2) and $(X_1–X_2)$ better search variables than X_1 and X_2.

Also, number the variables with care. Variables with the greatest expected impact on function f should have the smallest indices so that they get searched first. This reduces wasted steps in searching for the values of less important variables.

Many sophisticated search algorithms are described online; you might start looking by googling *function minimization*.

As with any complex tool, you will be more successful using a Search Black Box if you have an understanding of how it works and how it can fail.

Edge Degradation

Every scientific instrument has imperfections, and most physics experiments involve a variety of instruments, with a variety of imperfections that can combine in intricate ways to bias the final measurement. Such biases, called *systematic errors*, are quite different from the unavoidable *statistical errors* resulting from measuring stochastic variables with a finite number of data points.

We discussed statistical errors in Chapter 9. Let’s now discuss systematic errors: how to identify them, and how to properly correct your data.

Let’s examine a simple systematic error that often arises due to instrumental imprecision. Assume we are interested in a natural phenomenon with a sharp edge, a rapid change in a small distance. Now imagine trying to quantify that behavior using an instrument that measures distance imprecisely.

To be specific: let the true values Y(x) of some physical entity be constant from x=1 to x=5; and let Y drop precipitously to zero for x>5, as shown by the gray rectangles in Figure 20-5.

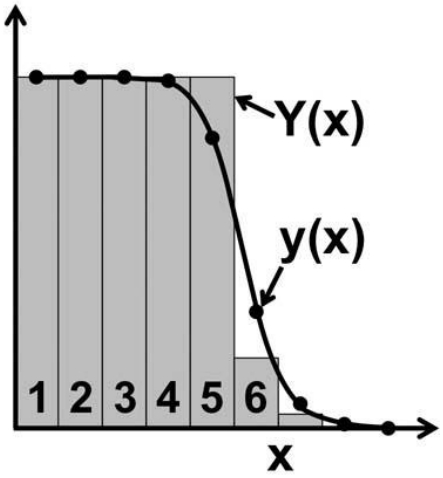


Figure 20-5 Degraded Edge

Because of instrumental uncertainty in our measurement of x, we actually measure y(x), the black dots that are connected by a black curve. Due to mismeasured x-values, the measured y values decline more gradually than do the true Y values.

In this example, we assume: 58% of measured x values fall into the correct bin; 40% are shifted by 1 bin, 20% to a lower bin and 20% to a higher bin; and 2% are shifted by 2 bins, 1% to a lower bin and 1% to a higher bin.

The true Y values and measured y values at each x value are:
x = 1, Y=100, y=100
x = 2, Y=100, y=100
x = 3, Y=100, y=100
x = 4, Y=100, y= 99
x = 5, Y=100, y= 83

x = 6, Y= 20, y = 33
x = 7, Y = 4, y = 8
x = 8, Y = 1, y = 2
x = 9, Y = 0, y = 0
From x=5 to x=6, the true Y drops by a factor of 5, while the measured y drops by a factor of only 2.5.

Let me emphasize that this systematic error is an inescapable reality, not a human mistake. Since no instrument is ever perfectly precise, all our measurements are compromised to some degree.

There are at least two ways to try to rectify edge degradation. Both require a precise understanding of our instrument's *point spread function*: the distribution of values that our instrument would measure if the true input values were all the same. This can be determined, for example, by measuring something known to have a perfectly sharp edge.

A logically simple approach to rectifying edge degradation employs search optimization, as described in the prior section. For some assumed true values Z(x), we can calculate z(x), what our instrument would measure due to its point spread function.

If z(x) exactly matched y(x), what we actually measured, then the assumed Z(x) would exactly equal the true Y(x). In practice, z(x) and y(x) will differ. We then calculate a χ^2 for how well they match:
$\chi^2 = \Sigma\ [z(x) - y(x)]^2 / \sqrt{[z(x) + y(x)]}$
We define f, a function of the assumed Z's, with $f = -\chi^2$, and then do a search for the Z's that maximize f, thereby minimizing χ^2. The Z(x) that minimizes χ^2 are those that are closest to the true Y(x).

A more intellectually sophisticated approach to rectifying edge degradation employs Fourier transforms. It can be proven that S(y), the Fourier transform of the measured data y(x), equals the product of S(Y), the Fourier transform of the true values Y(x), multiplied by S(psf), the Fourier transform of the point spread function. We write this:
S(y) = S(Y) S(psf)

The true values Y(x) are therefore equal to the inverse Fourier transform of S(y)/S(psf). We write this:

$Y = S^{-1} \{ S(y) / S(psf) \}$

In any real experiment, there will be many measurement imprecision issues, each with an effect like the point spread function discussed above. Monte Carlo methods are the most effective way to calculate and compensate for the impact of these effects on your measured data.

Appendix 1

Trigonometric Identities

For any triangle with sides A, B, and C, each with opposite angles a, b, and c:
$a + b + c = \pi$ radians = 180 degrees
Law of Sines: $A/\sin(a) = B/\sin(b) = C/\sin(c)$
Law of Cosines: $A^2 = B^2 + C^2 - 2BC\cos(a)$

Trig Identities

All the following identities are valid for any angles A and B:

$\cos^2 A + \sin^2 A = 1$
$\sin(A+B) = \sin A \cos B + \cos A \sin B$
$\cos(A+B) = \cos A \cos B - \sin A \sin B$
$\sin(2A) = 2 \sin A \cos A$
$\cos(2A) = \cos^2 A - \sin^2 A$
$= 1 - 2\sin^2 A = 2\cos^2 A - 1$

$\sin^2(A/2) = (1 - \cos A)/2$
$\cos^2(A/2) = (1 + \cos A)/2$

$\sin A + \sin B = 2 \sin[(A+B)/2] \cos[(A-B)/2]$
$\cos A + \cos B = 2 \cos[(A+B)/2] \cos[(A-B)/2]$
$\sin^2 A - \sin^2 B = \sin(A+B) \sin(A-B)$
$\cos^2 A - \cos^2 B = \sin(A+B) \sin(B-A)$
$\cos^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
$= \cos(A+B) \cos(B-A)$

$\cosh(iA) = \cos(A)$
$\sinh(iA) = i \sin(A)$
$\tanh(iA) = i \tan(A)$

$\cosh^2 A - \sinh^2 A = 1$

Appendix 2

Sums of Common Series

Infinite Series

Below are infinite series that have finite sums provided that $|x|<1$:

$1/(1–x) = 1 + x + x^2 + x^3 + \ldots$

$\sqrt{(1+x)} = 1 + x/2 – x^2/8 + x^3/16 – \ldots$

$1/\sqrt{(1+x)} = 1 – x/2 + 3x^2/8 – 5x^3/16 + \ldots$

$1/\sqrt{(1–x^2)} = 1 + x^2/2 + 3x^4/8 + 5x^6/16 + \ldots$

$\ln\{1+x\} = x – x^2/2 + x^3/3 – x^4/4 + \ldots$

Finite Series

We can also calculate the sum of a partial series, for $|x|<1$.

Let $Q = 1 + x + x^2 + x^3 + \ldots + x^{n-1}$

$Q = (1 – x^n) / (1 – x)$

Monthly Loan Payments

Assume you borrow a principal amount P at an uncompounded annual interest rate i, and make n monthly payments in the amount of M. To be clear, if the annual interest rate is 6%, i=0.06, and the monthly interest rate is i/12=0.005. The equation relating M and P is:

$M = P\,(i/12)\,x^n / (x^n – 1)$ with $x=1+i/12$

Sums of Integers Squared

Here are sums of integers raised to various powers:

$1 + 2 + 3 + 4 + \ldots + n = n(n+1)/2$

$1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$

$1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2(n+1)^2/4$

$1^4 + 2^4 + 3^4 + \ldots + n^4 =$
$n(n+1)(2n+1)(3n2+3n–1)/30$

Evens: $2^2 + 4^2 + 6^2 + \ldots + n^2 = n(n+1)(n+2)/6$

Odds: $1^2 + 3^2 + 5^2 + \ldots + n^2 = n(n+1)(n+2)/6$

Finally, a few infinite sums of reciprocals:

$1 + 2^{-2} + 3^{-2} + 4^{-2} + \ldots = \pi^2/6$
$1 + 2^{-4} + 3^{-4} + 4^{-4} + \ldots = \pi^4/90$
$1 + 3^{-2} + 5^{-2} + 7^{-2} + \ldots = \pi^2/8$
$1 + 3^{-4} + 5^{-4} + 7^{-4} + \ldots = \pi^4/96$
$1 - 2^{-2} + 3^{-2} - 4^{-2} + \ldots = \pi^2/12$
$1 - 1/3 + 1/5 - 1/7 + \ldots = \pi/4$

Series for Common Functions

For –1<x<=+1, the natural logarithm is:
$\ln\{1+x\} = x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \ldots$
For any x:
$\exp(x) = 1 + x + x^2/2! + x^3/3! + \ldots$
$e^x = \Sigma_n\, x^n/n!$
$\sin(x) = x - x^3/3! + x^5/5! - \ldots$
$\cos(x) = 1 - x^2/2! + x^4/4! - \ldots$
$\tan(x) = x + x^3/3! + 2x^5/15 - \ldots$
$\sinh(x) = x + x^3/3! + x^5/5! - \ldots$
$\cosh(x) = 1 + x^2/2! + x^4/4! - \ldots$
$\tanh(x) = x - x^3/3! + 2x^5/15 - \ldots$

This next sum is a bit tricky.
$\Sigma_J\{J\, x^J\} = 0 + x + 2x^2 + 3x^3 + 4x^4 + \ldots$
$= x\,[1 + x + x^2 + x^3 + \ldots\,]$
$+ x^2\,[1 + x + x^2 + x^3 + \ldots\,]$
$+ x^3\,[1 + x + x^2 + x^3 + \ldots\,]$
$+ x^4\,[1 + x + x^2 + x^3 + \ldots\,]$
…
On each row above, the infinite series in [ ]'s equals 1/(1–x), as above. That leaves the infinite sum of rows:
$\Sigma_J\{J\, x^J\} = x\,(1 + x + x^2 + x^3 + \ldots)/(1-x)$
$\Sigma_J\{J\, x^J\} = x\,/(1-x)^2$

Appendix 3

Tables of Gaussian Probability

Recall the normal Gaussian probability distribution from Chapter 9:
$\text{Prob}(x) = \exp\{ -(x-\mu)^2/2\sigma^2 \} / \sqrt{(2\pi\sigma^2)}$
Here, Prob(x) is the probability of value x, μ is the mean or average value of x, and σ^2 is the variance of x.
For μ=0 and σ=1, the following table lists values for Z, Prob(|x|>Z), and the odds of |x|>Z are:

Z	P(|x|>Z)	Odds of |x|>Z
0.5	0.61708	1 in 1.6204
0.7	0.48392	1 in 2.0665
1	0.31731	1 in 3.1515
2	0.04550	1 in 21.978
3	0.00270	1 in 370.40
4	6.33E–5	1 in 15787.
5	5.73E–7	1 in 1.74E+6
6	1.97E–9	1 in 5.07E+8

In the first line, P(|x|>Z) for x=0.5 means the probability of a measurement x deviating, either positively or negatively, from the mean of a Gaussian distribution by more than 0.5 standard deviations is 61.71%, and the odds of x deviating by more than 0.5 are 1 in 1.6204 = 1/0.6171.
The last line says |x| exceeds 6 in 1.97 of one billion data sets, and the odds that |x| exceeds 6 are 1 in 507 million.
For μ=0 and σ=1, the following table lists Z values corresponding to various values of Prob(|x|<Z). For example, the top entry in the left column states that 80% of all measured x values are in the interval:
$-1.28156 < x < +1.28156$

P(|x|<Z)	Z	P(|x|<Z)	Z
0.80	1.28156	3 9's	3.29053
0.90	1.64485	4 9's	3.89059
0.95	1.95996	5 9's	4.41717
0.98	2.32635	6 9's	4.89164
0.99	2.57583	7 9's	5.32672
0.995	2.80703	8 9's	5.73073
0.998	3.09023	9 9's	6.10941

Here, "5 9's" means 0.99999 — five nines following the decimal point. The bottom entry in the right column means that 99.9999999% of all measured x values are in the interval:

$-6.10941 < x < +6.10941$

Appendix 4

χ^2 & Degrees of Freedom

Recall the discussions and equations related to χ^2 in chapters 9 and 19. Assume two sets of N quantities:
$n(j) \pm \sigma_n(j)$, for j=1 to N
$m(j) \pm \sigma_m(j)$, for j=1 to N
Here, the σ's are the one standard deviation uncertainties in each quantity.

To evaluate whether or not n(j) and m(j) are samples from the same probabilistic distribution, we compute:
$\chi^2 = \Sigma_j \{ m(j) - n(j) \}^2 / \sqrt{[\sigma_m(j)^2 + \sigma_n(j)^2]}$
If both m(j) and n(j) are the numbers of rare events with purely statistical uncertainties, this equation reduces to:
$\chi^2 = \Sigma_j \{ m(j) - n(j) \}^2 / \sqrt{[m(j)^2 + n(j)^2]}$
If the m(j) are numbers of rare events with purely statistical uncertainties, and the n(j) are based on a theory with no statistical uncertainty, this equation reduces to:
$\chi^2 = \Sigma_j \{ m(j) - n(j) \}^2 / m(j)$
The *number of degrees of freedom*, #df, equals N the number of quantities being compared minus the number of adjustable parameters. If the m(j) are scaled to have the same sum as the n(j), there is one adjustable parameter. If the m(j) are measured data and the n(j) represent a 4th-order polynomial fit to the m(j), there are 5 adjustable parameters.
The following table provides guidance in interpreting χ^2 for various numbers of degrees of freedom (#df). The table lists X values versus #df vertically and P horizontally, where P is the probability that χ^2/#df exceeds X. The table assumes χ^2 and #df are calculated as described above. It further assumes a comparison between two sets of quantities that both truly represent the same physical phenomenon. Lastly, it assumes all data fluctuations are entirely random, independent, and free of systematic errors.

#df	10%	5.0%	2.5%	1.0%	0.5%	0.1%
1	2.71	3.84	5.02	6.63	7.88	10.8
2	2.30	3.00	3.69	4.61	5.20	6.91
3	2.08	2.60	3.12	3.77	4.28	5.42
4	1.94	2.37	2.78	3.32	3.72	4.62
5	1.85	2.21	2.57	3.02	3.35	4.10
6	1.77	2.10	2.41	2.80	3.09	3.74
7	1.72	2.01	2.29	2.64	2.90	3.47
8	1.67	1.94	2.19	2.51	2.74	3.27
9	1.63	1.69	2.11	2.41	2.62	3.10
10	1.60	1.83	2.05	2.32	2.52	2.96
12	1.55	1.75	1.94	2.18	2.36	2.74
15	1.49	1.67	1.83	2.04	2.19	2.51
20	1.42	1.57	1.71	1.88	2.00	2.27
30	1.34	1.46	1.57	1.70	1.79	1.99
63	1.23	1.31	1.38	1.46	1.52	1.64
127	1.16	1.21	1.26	1.31	1.35	1.43
511	1.08	1.11	1.13	1.15	1.17	1.20

The last line of this table says: in a series of repeated data sets measuring the same phenomenon, for a fit with 511 degrees of freedom, $\chi^2/511$ will exceed 1.08 in 10% of data sets, and will exceed 1.20 in 0.1% of data sets.

In medical studies, the decision criterion is often set at 5%. For example, consider testing a new drug to determine if it can cure some dreadful disease. Two groups of N patients each are selected, with the patients being as nearly identical as possible. The *control group* is given a placebo with no therapeutic value, and the *test group* is given the new drug. Let the number of survivors in the control group be n_{CTL} and in the test group be n_{TST}. To evaluate the drug's effectiveness, we calculate the χ^2 for the null hypothesis that the drug has no effect.

$\chi^2 = (n_{TST} - n_{CTL})^2 / \sqrt{(n^2_{TST} + n^2_{CTL})}$

This test has 1 degree of freedom. The drug's effectiveness is consider statistically significant if the probability of this χ^2 with 1 degree of freedom is less than 5%. Of course, we also insist that $n_{TST} > n_{CTL}$; we don't want a new drug that is statistically significant in harming people.

Since physicists measure objects that are simpler and less precious than human beings, we generally set more stringent decision criteria. For the acceptance of a major new discovery, such as the Higgs boson, the common criterion is that the probability of a *false detection* must be less than 1 part per million. Here, false detection means a statistical fluctuation that mimics a new discovery where none truly exists.

Appendix 5

Vector Identities & Theorems

Vector Identities

These identities are valid for any vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, and any scalar functions f and g.

$f(\boldsymbol{A} + \boldsymbol{B}) = f\boldsymbol{A} + f\boldsymbol{B}$

$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$

$\boldsymbol{A}\bullet\boldsymbol{B} = \boldsymbol{B}\bullet\boldsymbol{A}$

$(\boldsymbol{A}+\boldsymbol{B})\bullet\boldsymbol{C} = \boldsymbol{A}\bullet\boldsymbol{C} + \boldsymbol{B}\bullet\boldsymbol{C}$

$\boldsymbol{A}\bullet(\boldsymbol{A}\times\boldsymbol{B}) = 0$

$\boldsymbol{A}\bullet(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}\bullet(\boldsymbol{C}\times\boldsymbol{A}) = (\boldsymbol{A}\times\boldsymbol{B})\bullet\boldsymbol{C}$

$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\bullet\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\bullet\boldsymbol{B})\boldsymbol{C}$

$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C} = (\boldsymbol{A}\bullet\boldsymbol{C})\boldsymbol{B} - \boldsymbol{A}(\boldsymbol{B}\bullet\boldsymbol{C})$

$(\boldsymbol{A}\times\boldsymbol{B})\bullet(\boldsymbol{C}\times\boldsymbol{D}) = (\boldsymbol{A}\bullet\boldsymbol{C})(\boldsymbol{B}\bullet\boldsymbol{D}) - (\boldsymbol{B}\bullet\boldsymbol{C})(\boldsymbol{A}\bullet\boldsymbol{D})$

Vector Operator Identities

These operator identities are valid for any vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, and any scalar functions f and g.

The gradient operator $\check{\boldsymbol{D}}$ equals $(\partial/\partial x, \partial/\partial y, \partial/\partial x)$

$\check{\boldsymbol{D}}\bullet(\check{\boldsymbol{D}}\times\boldsymbol{A}) = 0$

$\check{\boldsymbol{D}}\times(\check{\boldsymbol{D}}f) = 0$

$\check{\boldsymbol{D}}\bullet(f\boldsymbol{A}) = f\check{\boldsymbol{D}}\bullet\boldsymbol{A} + (\check{\boldsymbol{D}}f)\bullet\boldsymbol{A}$

$\check{\boldsymbol{D}}(A\bullet B)$
$= (\boldsymbol{A}\bullet\check{\boldsymbol{D}})\boldsymbol{B} + (\boldsymbol{B}\bullet\check{\boldsymbol{D}})\boldsymbol{A} + \boldsymbol{A}\times(\check{\boldsymbol{D}}\times\boldsymbol{B}) + \boldsymbol{B}\times(\check{\boldsymbol{D}}\times\boldsymbol{A})$

$\check{\boldsymbol{D}}\bullet(\boldsymbol{A}\times\boldsymbol{B}) = (\check{\boldsymbol{D}}\times\boldsymbol{A})\bullet\boldsymbol{B} - \boldsymbol{A}\bullet(\check{\boldsymbol{D}}\times\boldsymbol{B})$

$\check{\boldsymbol{D}}\times(f\boldsymbol{A}) = f\check{\boldsymbol{D}}\times\boldsymbol{A} + \check{\boldsymbol{D}}f\times\boldsymbol{A}$

$\check{\boldsymbol{D}}\times(\boldsymbol{A}\times\boldsymbol{B})$
$= \boldsymbol{A}(\check{\boldsymbol{D}}\bullet\boldsymbol{B}) - \boldsymbol{B}\bullet(\check{\boldsymbol{D}}\bullet\boldsymbol{A}) + (\boldsymbol{B}\bullet\check{\boldsymbol{D}})\boldsymbol{A} - (\boldsymbol{A}\bullet\check{\boldsymbol{D}})\boldsymbol{B}$

$f\check{D}^2g - g\check{D}^2f = \check{\boldsymbol{D}}\bullet(f\check{\boldsymbol{D}}g - g\check{\boldsymbol{D}}f)$

$\check{D}^2(fg) = f\check{D}^2g + 2\,\check{\boldsymbol{D}}f\,\check{\boldsymbol{D}}g + g\check{D}^2f$

$\check{D}^2(f\boldsymbol{A}) = \boldsymbol{A}\check{D}^2f + 2(\check{\boldsymbol{D}}f\bullet\check{\boldsymbol{D}})A + f\check{D}^2\boldsymbol{A}$

$\check{D}^2(A \bullet B)$
$= \boldsymbol{A} \bullet \check{D}^2\boldsymbol{B} - \boldsymbol{B} \bullet \check{D}^2\boldsymbol{A} + 2\boldsymbol{\check{D}} \bullet ([\boldsymbol{B} \bullet \boldsymbol{\check{D}}]\boldsymbol{A} + \boldsymbol{B} \times \boldsymbol{\check{D}} \times \boldsymbol{A})$

Vector Operator Theorems

In what follows, V represents a volume within a closed surface S, dV is an infinitesimal element of V, A is any vector field, f is any scalar field, da is an infinitesimal area within S and ***n*** is the vector of unit length that is normal (perpendicular) to da. For a closed surface S, ***n*** points outward. We often write ***da*** to represent ***n*** da.

Gauss' law equates the (*flux* of ***A*** through a closed surface S) to the (divergence of ***A*** within the volume V that S encloses). By "flux of A" we mean the outward component of ***A*** summed across the defined surface. If ***A*** represents the velocity of gas molecules, the flux of ***A*** through S is the rate at which gas flows outward through that surface.

Gauss's law: $\int_S \boldsymbol{A} \bullet \boldsymbol{n}\, da = \int_v (\boldsymbol{\check{D}} \bullet \boldsymbol{A})\, dV$

Stokes' theorem equates the *circulation* of ***A*** around a closed loop Γ to the curl of ***A*** normal to the surface that Γ encloses. By "circulation of ***A***" we mean the sum of the tangential component of ***A*** around loop Γ in the counterclockwise direction.

Stokes theorem: $\int_\Gamma \boldsymbol{A} \bullet \boldsymbol{ds} = \int_s (\boldsymbol{\check{D}} \times \boldsymbol{A}) \bullet \boldsymbol{n}\, da$

Appendix 6

Table of Common Derivatives

General Rules of Differentiation

For any constant a and variable q:

$da/dq = 0$

For any functions F & G, constants a & b, and variable q:

$d(aF+bG)/dq = a\,dF/dq + b\,dG/dq$

$d(FG)/dq = G\,dF/dq + F\,dG/dq$

$d(FG)/dq = d(GF)/dq$

$d(F/G)/dq = (1/G)dF/dq - (F/G^2)dG/dq$

$d(F^n)/dq = n\,F^{n-1}\,dF/dq$

Derivatives of Common Functions

$d\,\mathbf{x}^n / dt = n\,x^{n-1}\,dx/dt$

Trig Functions

$d\sin(x)\,dt = +\cos(x)\,dx dt$

$d\cos(x)\,dt = -\sin(x)\,dx dt$

$d\tan(x) / dt = \cos^{-2}(x)\,dx/dt$

Exponentials

$d\,e^x / dt = e^x\,dx/dt$

$d\ln\{x\}\,dt = (1x)\,dx/dt$

Hyperbolic Functions

$d\sinh(x)\,dt = \cosh(x)\,dx dt$

$d\cosh(x)\,dt = \sinh(x)\,dx dt$

$d\tanh(x) / dt = \cosh^{-2}(x)\,dx/dt$

Square Roots

For $\mathbf{r} = \sqrt{(\mathbf{x}^2 + \mathbf{a}^2)}$, with constant a:

$d\,r\,dx = x r$

$d\,(r)^{-1}\,dx = -x r^{3}$
$d\,(r)^{-2}\,dx = -2x r^{4}$
$d\,(r)^{-3}\,dx = -3x r^{5}$
$d\,(x/r)\,/dx = a^{2}/r^{3}$

For $\boldsymbol{r}_{jk} = \boldsymbol{r}_{j} - \boldsymbol{r}_{j}$; $r_{jk} = |\boldsymbol{r}_{jk}|$; $\boldsymbol{v}_{j} = d\boldsymbol{r}_{j}/dt$:
$d\,r_{jk}^{-1}\,/dt = \boldsymbol{r}_{jk} \bullet (\boldsymbol{v}_{k} - \boldsymbol{v}_{j})\,/\boldsymbol{r}_{jk}^{3}$

Differential Reciprocals

It is sometimes easier to calculate dy/dx than dx/dy. But if you can calculate either, you automatically have the answer to the other, because as one might guess from the notation:

$(dx/dy)\,(dy/dx) = 1$

Here is an example: an equation relating wave number k and wave frequency ω, in a material with natural frequency Ω.

$k = (1/c)\,\sqrt{\{\omega^{2} - \Omega^{2}\}}$

First calculate dk/dω:

$dk/d\omega = (1/c)\,(1/2)\,[\omega^{2} - \Omega^{2}]^{-1/2}\,(2\omega)$

$dk/d\omega = [1 - (\Omega/\omega)^{2}\,]^{-1/2} / c$

Now calculate the group velocity dω/dk:

$k^{2}c^{2} + \Omega^{2} = \omega^{2}$

$2\,kc^{2} = 2\omega\,d\omega/dk$

$d\omega/dk = k\,[\,c^{2} / \omega\,]$

$d\omega/dk = (1/c)\,\sqrt{\{\omega^{2} - \Omega^{2}\}}\,[\,c^{2} / \omega\,]$

$d\omega/dk = c\,[1 - (\Omega/\omega)^{2}]^{+1/2}$

Hence: $(d\omega/dk)\,(dk/d\omega) = 1$.

Appendix 7

Table of Common Integrals

Here are some useful integrals that will solve most physics problems.

Powers

$\int x^n \, dx = x^{(n+1)} / (n+1)$

Here, n need not be an integer, but n cannot be –1.

$\int x^{-1} \, dx = \int (1/x) \, dx = \ln\{x\}$

Trig Functions (x in radians)

$\int \sin(x) \, dx = -\cos(x)$

$\int \cos(x) \, dx = +\sin(x)$

$\int \sin^2(x) \, dx = x/2 - \sin(2x)/4$

$\int \cos^2(x) \, dx = x/2 + \sin(2x)/4$

$\int \sin(x) \cos(x) \, dx = \sin^2(x)/2$

Exponentials

$\int \exp\{x\} \, dx = \exp\{x\}$

$\int \ln\{x\} \, dx = x \ln\{x\} - x$

Hyperbolic Functions

$\int \sinh(x) \, dx = \cosh(x)$

$\int \cosh(x) \, dx = \sinh(x)$

$\int \tanh(x) \, dx = \ln\{ \cosh(x) \}$

Integrals with $r = \sqrt{(x^2+a^2)}$, with constant a

$\int (1/r) \, dx = \ln\{ x + r \}$

$\int (1/r^3) \, dx = x / (a^2 r)$

$\int (x/r) \, dx = r$

$\int (x/r^3) \, dx = -1 / r$

Appendix 8

Principal Physical Constants

Primary Mathematical and Physical Constants

π	3.141 592 653 59…	
e	2.718 284 590 45…	
ø	1.618 033 988 7…	Golden Ratio
c	$2.997\ 924\ 58 \times 10^{8}$ m / s	Speed of Light
h	$6.626\ 070 \times 10^{-34}$ J s	Planck's constant
$\hbar$	$1.054\ 571 \times 10^{-34}$ J s	h / 2π
G	6.674×10^{-11} m^3 / kg s^2	Newton's constant

Atomic and Particle Constants

Å	$1.00\ldots \times 10^{-10}$ m	Angström
a_0	$0.529\ 177\ 210 \times 10^{-10}$ m	Bohr radius
e	$1.602\ 176\ 62 \times 10^{-19}$ C	Proton Charge
α	1/ 137.035 999 1	Fine Structure
R_y	13.605 693 0 eV	Rydberg
r_p	0.88×10^{-15} m	Proton radius
m_p	$1.672\ 621\ 9 \times 10^{-27}$ kg	Proton mass
m_e	$9.109\ 383\ 5 \times 10^{-31}$ kg	Electron mass
	1836.152 673 9	p/e mass ratio
eV	$1.602\ 176\ 57 \times 10^{-19}$ J	Electron-Volt

Statistical Mechanics and Thermodynamics Constants

N_A	$6.022\ 140\ 8 \times 10^{23}$	Avogardo's Nmbr.
k	$1.380\ 649 \times 10^{-23}$ J/K	Boltzmann's Const.
R	8.314 46 J / mole K	Molar Gas Const.
σ	$5.670\ 367 \times 10^{-8}$ W/m^2K^4	Stephan-Boltzmann

Electrical and Gravitational Constants

Electrical Constants

ε_0	$8.854\ 187\ 817 \times 10^{-12}$ F / m	
μ_0	$1.256\ 637\ 061\ 4 \times 10^{-7}$ N / A^2	

Gravitational Constants

g	9.806 65 m / s	Std. Acceleration
	$3.986\ 004\ 42 \times 10^{14}$ m^3 / s^2	$G \times M_{Earth}$
	$1.327\ 124\ 40 \times 10^{20}$ m^3 / s^2	$G \times M_{Sun}$

Meet The Author

Congratulations and thank you for reading my book. I know your time is valuable, and I sincerely hope you enjoyed this experience.

I'd like to tell you something about myself and share some stories.

First, the obligatory bio (as if 3 "tweets"-worth can define anyone): I have a B.S. in physics from Caltech, a Ph.D. in high-energy particle physics from Stanford University, and was on the faculty of Harvard University. Now "retired," I teach at the Osher Institutes at UCLA and CSUCI, where students honored me as "Teacher of the Year." In between, I ran eight high-tech companies and hold patents in medical,x semiconductor, and energy technologies.

My goal is to help more people appreciate and enjoy science. We all know one doesn't have to be a world-class musician to appreciate great music — all of us can do that. I believe the same is true for science — everyone can enjoy the exciting discoveries and intriguing mysteries of our universe.

I've given 400+ presentations to general audiences of all ages and backgrounds, and have written 3 printed books and 29 eBooks. My books have won national and international competitions, and are among the highest rated physics books on Amazon.com. I'm delighted that two of these recently became the 2nd and 3rd best sellers in their fields.

Richard Feynman was a friend and colleague of my father, Oreste Piccioni, so I knew him well before entering Caltech. On several occasions, Feynman drove from Pasadena to San Diego to sail on our small boat and have dinner at our home. Feynman, my father, my brother and I once went to the movies to see *"Dr. Strangelove or: How I Learned to Stop Worrying and Love the Bomb."* It was particularly poignant watching this movie next to one of the Manhattan Project's key physicists.

At Caltech I was privileged to learn physics directly from this greatest scientist of our age. I absorbed all I could. His style and enthusiasm were as important as

the facts and equations. Top professors typically teach only upper-level graduate classes. But Feynman realized traditional introductory physics didn't well prepare students for modern physics. He thought even beginners should be exposed to relativity, quantum mechanics, and particles physics. So he created a whole new curriculum and personally taught freshman and sophomore physics in the academic years 1961-62 and 1962-63.

The best students thrived on a cornucopia of exciting frontier science, but many others did not. Although Caltech may be the world's most selective science school, about half its elite and eager students drowned in Feynman's class. Even a classmate, who decades later received the Nobel Prize in Physics, struggled in this class. Feynman once told me that students sometimes gave him the "stink eye" — he added: "Me thinks he didn't understand angular momentum."

Some mundane factors made the class very tough: Feynman's book wasn't written yet; class notes came out many weeks late; and traditional helpers (teaching assistants and upper classmen) didn't understand physics the way Feynman taught it.

But the biggest problem was that so much challenging material flew by so quickly. Like most elite scientists, Feynman's teaching mission was to inspire the one or two students who might become leading physicists of the next generation. He said in his preface that he was surprised and delighted that 10% of the class did very well.

My goal is to reach the other 90%.

It's a great shame that so many had so much difficulty with the original course — there is so much great science to enjoy. I hope to help change that and bring Feynman's genius to a wider audience.

Please let me know how I can make ***Feynman Simplified*** even better — contact me through my WEBSITE.
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