Potential Theory in Solving Elliptic PDEs

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Abstract

The purpose of this paper is to give solutions in probabilistic form of discrete and continuous boundary problems, especially related to second-order elliptic partial differential equations (PDEs). Although finding analytical solutions of boundary problems is an important task in mathematics, they are not available in most cases. Since the classical potential theory provides us with insights about the origin of *Poisson problems* in physics, we construct the solutions of more general boundary problems in potential form, which can be numerically approximated by Monte Carlo methods. On the other hand, the continuity behavior of solutions in potential form is another important issue. In this paper, we discuss one special boundary which can ensure the continuity of our solutions.

Keywords: Markov chains, Potential theory, Elliptic PDEs, Electric networks, Monte-Carlo method

Contents

1 Introduction

The genesis of classical potential theory comes from physics in the 19th century. At that time, two fundamental forces of nature, gravity and electrostatic force, were found to be derived from potentials which satisfy *Poisson equation*. In modern time, potential theory has strong connection with probability and Markov chain. In this paper, we extend the application of potential theory to more general boundary problem, especially for the second-order elliptic PDEs.

In fact, the motivation is clear from the following observations. Some boundary problems, such as *Poisson problem*, have physics background. For example, the solution of *Laplace equation*, which is a typical elliptic PDEs, is the electric potential. In [5], [4] and [3], they have given solutions of *Poisson problem* in potential form and discuss the continuity behaviors of these solutions on boundary. Hence, it is reasonable to apply potential theory to elliptic PDEs. In [6] and [2], they have showed some applications of Markov chains and random walk in electric networks. In this paper, we give probabilistic interpretations of these electric characteristics from a potential theory insight, e.g. in §2.3, we give a probabilistic interpretation of *current* and *voltage*.

Although the solution in potential form in Theorem 3.2.1 may not give insight into derivatives and gradient, we can actually get an approximation of solutions by *Monte Carlo* methods. As examples in §4.2 show, the analytical solution and numerical solution provided by our method quite well match. Therefore, we may say that our method can solve these boundary problems numerically whose analytical solution is not available.

2 Potential Theory in Discrete Case

2.1 Discrete-time Markov chain

Markov chain is critical in computing the potential, we give a brief introduction here. Besides, we will see the definition of potential in the next section, see equation (5), hence our key goal is to compute the expectation of path integration with boundary value, which is driven by a stochastic process (Brownian motion or random walk, etc.). Hence, it is reasonable to discuss some theorems related to stopping time and Markov chain.

Definition 2.1.1 (Markov chain)**.** *A random process $(X_n)_{n\geq 0}$ is called a Markov chain with countable state space I, initial distribution $\lambda = (\lambda_i)_{i\in I}$ and transition matrix $P = \{p_{i,j}\}_{i,j\in I}$, if it satisfies*

1. *X_0 has distribution λ, that is, $\mathbb{P}(X_0 = i) = \lambda_i$.*
2. *If $X_n = i \in I$, X_{n+1} has distribution $(p_{i,j} : j \in I)$, which is independent of $X_0, ..., X_{n-1}$.*

One simple example of discrete-time Markov chain is random walk, which we will describe in the next section. Before that, we see some important properties related to Markov chain.

Proposition 2.1.1. *Let $(X_n)_{n\geq 0}$ be a Markov chain with transition matrix P and $X_m = i$. Then the random process $(X_{m+n})_{n\geq 0}$ is also a Markov chain with the same transition matrix, which is independent of $X_0, ..., X_{m-1}$.*

Remark 2.1.1. *If we change m by a stopping time T, $(X_{T+n})_{n\geq 0}$ is also a Markov chain with the same transition matrix, which is independent of $X_0, ..., X_{T-1}$. It is called the strong Markov property. In this paper, the most common stopping time is $T = \min_{n\geq 0}\{X_n \in \partial D\}$, where ∂D is called boundary as in §2.2.*

Now for any non-empty subset $A \subset I$, we define the hitting time of A as a random variable $H^A : I \to \mathbb{N}$ defined by

$$H^A(\omega) = \min\{n : X_n(\omega) \in A, \omega \in I\}.$$

More specifically, we are interested in the following two quantities.

$$\begin{aligned} h_i^A &= \mathbb{P}_i(H^A < \infty) := \mathbb{P}(H^A < \infty | X_0 = i). \\ k_i^A &= \mathbb{E}_i[H^A] := \mathbb{E}_i\left[H^A | X_0 = i\right]. \end{aligned}$$

Here, $h^A = (h_i^A : i \in i)$ and $k^A = (k_i^A : i \in I)$, which can be calculated by the following.

Theorem 2.1.1 ([6], Theorem 1.3.2, p.13). *h^A is the minimal non-negative solution of the following equation*

$$\begin{cases} h_i^A = 1. & i \in A \\ h_i^A = \sum_{j\in I} p_{i,j} h_j^A. & i \notin A \end{cases} \tag{1}$$

Proof. It is clear that $h_i^A = 1$ for $i \in A$. By Markov property, for $i \notin A$, we can get

$$\begin{aligned} h_i^A = P_i(H^A < \infty) &= \sum_{j\in I} P_i(H^A < \infty | X_1 = j). \\ &= \sum_{j\in I} p_{i,j} P_j(H^A < \infty). \end{aligned} \tag{2}$$

Hence, h^A satisfies equation (2). Suppose $x = (x_i : i \in I)$ is another non-negative solution, for $i \notin A$, we have

$$\begin{aligned} x_i &= \sum_{j\in I} p_{i,j} x_j = \sum_{j\in A} p_{i,j} + \sum_{j\in I\setminus A} p_{i,j} x_j \\ &= P_i(H^A = 1) + \sum_{j\in I\setminus A}\sum_{k\in A} p_{i,j} p_{j,k} + \sum_{j\in I\setminus A}\sum_{k\in I\setminus A} p_{i,j} p_{j,k} x_k \\ &= \sum_{k=1}^{m} P_i(H^A = k) + \sum_{j_1,...,j_m \in I\setminus A} \left(\prod_{l=1}^{m} p_{j_{l-1},j_l}\right) x_{j_m}. \end{aligned} \tag{3}$$

where $j_0 = i$. Finally, let $m \to \infty$, we conclude

$$x_i \geq \sum_{k=1}^{\infty} P_i(H^A = k) = P_i(H^A < \infty).$$

$\square$

Theorem 2.1.2 ([6], Theorem 1.3.5, p.17)**.** *k^A is the minimal non-negative solution of the following equation*

$$\begin{cases} k_i^A = 0. & i \in A \\ k_i^A = 1 + \sum_{j\in I} p_{i,j} k_j^A. & i \notin A \end{cases} \tag{4}$$

Proof being quite similar to that of Theorem 2.1.1, we omit it.

2.2 Potential theory in discrete case

In this subsection, we first give the definition of potential in discrete case.

Definition 2.2.1 (Stopping time)**.** *$(X_n)_{n\geq 0}$ be a discrete-time Markov chain with transition matrix P and state-space I. We divide I into two parts once and for all: the interior D and boundary $\partial D = I \backslash D$. Besides, $c(i) = c_i$ for $i \in D$ and $f(j) = f_j$ for $j \in \partial D$ are two non-negative functions, then the potential at state $i \in I$ is*

$$\phi_i = \mathbb{E}_i \left[\sum_{n=0}^{T} c(X_n) + f(X_T) \right]. \tag{5}$$

where $T = \min_{n\geq 0}\{X_n \in \partial D\}$ is the stopping (hitting) time.

For simpleness, we call $c(\cdot)$ and $f(\cdot)$ is interior and boundary function respectively in all following sections. According to Definition 2.2.1, all states in boundary are absorbing, i.e. for $i \in \partial D$, $p_{i,j} = \delta_i^j$ where δ_i^j is the Kronecker delta. Next theorem shows how to compute potential in a 'matrix' form equation.

Theorem 2.2.1 (Solution of discrete potential problems)**.** *With all conditions in Definition 2.2.1 being satisfied, we suppose $P_i(T < \infty) = 1$ for all $i \in I$. Then the potential $\phi = (\phi_i : i \in D, f_j : j \in \partial D)^T$ is the unique bounded solution of the following equation*

$$\phi = P\phi + c. \tag{6}$$

where $c = (c_i : i \in D, 0)^T$ and $P = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$ is the transition Matrix.

Proof. Since $p_{i,j} = \delta_i^j$, we can re-arrange the transition matrix P into the following form

$$P = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$$

For $i \in \partial D$, it is clear that $\phi_i = f_i$. Let $i \in D$, according to equation (5) and Markov property, we have

$$\begin{aligned} \phi_i &= c_i + \mathbb{E}\left[\sum_{n=1}^{T} C(X_n) + f(X_T) | X_1 \right] \\ &= c_i + \sum_{j\in I} p_{i,j} \mathbb{E}\left[\sum_{n=1}^{T} C(X_n) + f(X_T) \Big| X_1 = j \in I \right] \\ &= c_i + \sum_{j\in I} p_{i,j}\phi_j. \end{aligned} \tag{7}$$

We concludes that $\phi = P\phi + c$. Next, suppose φ is another bounded solution of (6). We define $\phi_i(m) = \mathbb{E}_i[\sum_{n=0}^{m} C(X_n)1_{n<T} + f(X_T)1_{T\leq m}]$ and $\phi(m) = (\phi_i(m) : i \in I)$, we can see $\phi_i(m)$ monotonously converge to ϕ_i as $m \to \infty$.

$$
\begin{aligned}
\varphi_i &= c_i + \sum_{j\in I} p_{i,j}\varphi_j \\
&= c_i + \sum_{j\in\partial D} p_{i,j}f_j + \sum_{j\in D} p_{i,j}\left(c_j + \sum_{k\in I} p_{j,k}\varphi_k\right) \\
&= c_i + \sum_{j\in\partial D} p_{i,j}f_j + \sum_{j\in D} p_{i,j}p_{j,k}f_k + \sum_{j\in D}\sum_{k\in D} p_{i,j}p_{j,k}\varphi_k \\
&= \mathbb{E}_i\left[C(X_0)1_{0<T} + C(X_1)1_{1<T} + f(X_1)1_{T=1}\right] + \sum_{j\in D}\sum_{k\in I} p_{i,j}p_{j,k}\varphi_k \\
&= \phi_i(1) + \mathbb{E}_i\left[\varphi_{X_1}1_{1<T}\right]. \qquad (8)
\end{aligned}
$$

According to the above process, we can obtain $\varphi_i = \phi_i(m) + \mathbb{E}_i[\varphi_{X_m}1_{m<T}]$ for all $n \in \mathbb{N}$ by induction. Since we assume φ is bounded and $P_i(T < \infty) = 1$, we conclude that

$$
|\mathbb{E}_i[\varphi_{X_m}1_{m<T}]| \leq MP_i(T > m)
$$

Moreover $\lim_{m\to\infty} P_i(T > m) = 0$ and we conclude that

$$
\varphi_i = \lim_{m\to\infty} \phi_i(m) = \phi_i
$$

In other words, we show that $\phi \equiv \varphi$. $\square$

Example 2.2.1. *Suppose $(X_n)_{n\geq 0}$ in Definition 2.2.1 has finite state space I and transition matrix P, then we solve equation (6). In detail, the equation (6) can be rewritten by*

$$
\phi = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix} \phi + \begin{pmatrix} c \\ 0 \end{pmatrix}. \qquad (9)
$$

where $\phi = (\phi_D, f)^T$, $f = (f_j)_{j\in\partial D}$ and $\phi_D = (\phi_i)_{i\in D}$. Our goal is to find ϕ_D. In fact, we can obtain

$$
\phi = P^n\phi + \sum_{i=0}^{n-1} P^i \begin{pmatrix} c \\ 0 \end{pmatrix}. \qquad (10)
$$

where

$$
P^n = \begin{pmatrix} A^n & \sum_{i=0}^{n-1} A^iB \\ 0 & I \end{pmatrix}
$$

Since $||A||_\infty < 1$, we have $\sum_{i=0}^{\infty} A^iB = (I - A)^{-1}B := Q$. Finally, letting $n \to \infty$, we conclude $\phi_D = Qf + (I - A)^{-1}c$.

If a Markov chain has countable infinite states, the following example also shows one way to find ϕ_D.

Example 2.2.2 ([6], Example 4.2.2, p. 137)**.** *Suppose $(X_n)_{n\geq 0}$ is a discrete random walk on $\mathbb{Z}$ with transition probabilities $p_{i,i-1} = q$ and $p_{i,i+1} = p$ for $i \in \mathbb{Z}$, where $p > q$ and $p + q = 1$. Since $X_\infty = +\infty$ with probability 1, the boundary is $\{+\infty\}$. Let $c_i = c^i$ and $f_\infty = 0$, where $c > 0$. And ϕ_i can be computed as*

$$\phi_i = \mathbb{E}_i \left[\sum_{i=0}^{\infty} c^{X_i} \right] = c^i + p\phi_{i+1} + q\phi_{i-1}. \tag{11}$$

On the other hand, denoting $Y_i = X_i - 1$, we also have

$$\phi_{i+1} = \mathbb{E}_{i+1} \left[\sum_{i=0}^{\infty} c^{X_i} \right] = c\mathbb{E}_i \left[\sum_{i=0}^{\infty} c^{Y_i} \right] = c\phi_i. \tag{12}$$

According to equations (11) and (12), we conclude that

$$\phi_i = c^i + (pc + q/c)\phi_i. \tag{13}$$

that is,

$$\phi_i = \frac{c^{i+1}}{c - pc^2 - q}. \tag{14}$$

Notice that the denominator of equation (14) has two roots 1 and q/p. Hence, the range of c is $(q/p, 1)$. Interestingly, ϕ_i will tend to infinity when c is much less than 1, since $X_i < 0$ will happen for infinitely times.

2.3 Electric network

In [6] and [2], they have discussed some applications of random walk and Markov chain in electric network. We will use a potential theoretic view to interpret this physical model. We make the following assumptions: A network with a countable *node* set I, some nodes are connected by *wires*. The *conductivity* and *resistance* between a and b defined by $c_{a,b} = c_{b,a}$ and $r_{a,b} = 1/c_{a,b}$. According to *Ohm's law*, we have

$$i_{a,b} = (\phi_a - \phi_b)c_{a,b}$$

where ϕ_a is the *voltage* of a and $i_{a,b}$ is the *text*current of charge flowing from a to b. In addition, we define the total current from node a to the whole network by

$$i_a = \sum_{b\in I} i_{a,b}$$

Here, we only consider equilibrium state of the charge, that is, i_a is a non-negative constant. In particular, $i_a = 0$ is the case that node a with no external connection (*Kirchhoff's Current Law*).

As above, we divide I into D and ∂D and we let f_j be the voltage of $j \in \partial D$. In addition, we define a Markov chain $(X_n)_{n\geq 0}$ with transition matrix $P = \{p_{x,y}\}_{x,y\in I}$, where $C_x = \sum_{y\in I} c_{x,y}$ and $p_{x,y} = c_{x,y}/C_x$.

Theorem 2.3.1 ([5], Theorem 4.3.2, p.153)**.** *Given a electric network with finite nodes and Markov chain $(X_n)_{n\geq 0}$ defined as above with $X_0 = a$*

(1) If we assign $\phi_a = 1$ and $\phi_b = 0$, the voltage of node k is

$$\phi_k = \mathbb{P}_k(T_a < T_b)$$

where T_a and T_b are the hitting times starting from node k.

(2) If we put $i_a = 1$ and $i_b = -1$, the current $i_{x,y}$ equals to

$$i_{x,y} = \mathbb{E}_a[\Gamma_{x,y} - \Gamma_{y,x}]$$

where $\Gamma_{x,y}$ is the number of times that $(X_n)_{n\geq 0}$ reach y from x before hitting b.

Proof. For the first item, let $\partial D = \{a, b\}$ here. By *Ohm's Law*, for $k \in D$, we have

$$0 = i_k = \sum_{l\in I} i_{k,l} = \sum_{l\in I} c_{k,l}(\phi_k - \phi_l).$$

We conclude that

$$\begin{cases} \phi_k = \sum_{l\in I} p_{k,l}\phi_l. & k \in D \\ \phi_k = 1 \text{ or } 0. & k \in \{a, b\} \end{cases} \tag{15}$$

On the other hand, it is easy to see that $\mathbb{P}_a(T_a < T_b) = 1$ and $\mathbb{P}_b(T_a < T_b) = 0$. In addition, we find that

$$\mathbb{P}_k(T_a < T_b) = \sum_{l\in I} p_{k,l}\mathbb{P}_l(T_a < T_b).$$

Hence, $\mathbb{P}_k(T_a < T_b)$ also satisfies equation (15). By Theorem 2.2.1, these two solutions coincide, i.e. $\phi_k = \mathbb{P}_k(T_a < T_b)$.

For the second item, let u_x be the expected number of $(X_n)_{n\geq 0}$ visiting node x before reaching node b. Then we see that $u_b = 0$ and $u_a = C_a$. For $x \in D$, since $p_{x,y}C_x = p_{y,x}C_y$, we have

$$u_x = \sum_{l\in I} u_l p_{l,x} = \sum_{l\in I} u_l \frac{p_{x,l}C_x}{C_l}.$$

or

$$\frac{u_x}{C_x} = \sum_{l\in I} p_{x,l}\frac{u_l}{C_l}.$$

By Theorem 2.2.1, we have $\phi_x = \frac{u_x}{C_x}$ and $\mathbb{E}_a[\Gamma_{x,y}] = u_x p_{x,y}$, , so that

$$i_{x,y} = (\phi_x - \phi_y)c_{x,y} = u_x p_{x,y} - u_y p_{y,x} = \mathbb{E}_a[\Gamma_{x,y} - \Gamma_{y,x}].$$

□

The following theorem shows a minimal property of electric potential, which is similar as Theorem 2.1.2. Here, we introduce the concept of energy related to potential ϕ, which is defined as the following

$$E(\phi) = \frac{1}{2} \sum_{x,y \in I} (\phi_x - \phi_y)^2 c_{x,y}. \tag{16}$$

Theorem 2.3.2 ([2], p. 51)**.** *If there is no current source in D, the potential ϕ with boundary value f has the minimal energy. In other words, for any φ with the same boundary value as ϕ, $E[\varphi] \geq E[\phi]$.*

Proof. Let $\delta = (\delta_i : i \in I)$ where $\delta_i = 0$ if $i \in \partial D$. We only need to show $E[\phi + \delta] \geq E[\phi]$. According to *Ohm's law* and *Kirchhoff's Current law*, we have

$$\sum_{x,y \in D} (\phi_x - \phi_y) i_{x,y} = 2 \sum_{x \in I} \phi_x i_x = 0.$$

$$\frac{1}{2} \sum_{x,y \in I} (\phi_x - \phi_y)^2 c_{x,y} = \frac{1}{2} \sum_{x,y \in I} (\phi_x - \phi_y) i_{x,y} = 0.$$

On the other hand, we compute the energy of $\phi + \delta$.

$$\begin{aligned} E[\phi + \delta] &= \sum_{x,y \in I} [(\phi_x - \phi_y) - (\delta_x - \delta_y)]^2 c_{x,y} \\ &= \frac{1}{2} \sum_{x,y \in I} (\delta_x - \delta_y)^2 c_{x,y} - \sum_{x,y \in I} (\delta_x - \delta_y) c_{x,y}. \end{aligned}$$

Similarly, we can show that $\sum_{x,y \in I} (\delta_x - \delta_y) c_{x,y} = 0$. Hence, $E[\phi + \delta] \geq 0 = E[\phi]$. $\square$

3 Potential Theory in Continuous Case

3.1 Continuous-time Markov chain

We have discussed discrete-time case of potential theory in §2. In this subsection, we will consider a continuous-time Markov chain $(X_t)_{t \geq 0}$ with state space I and transition matrix $P(t) = \exp(tQ) = \sum_{k=0}^{\infty} \frac{t^n Q^n}{n!}$, where $Q = [q_{i,j}]_{i,j \in I}$ is a matrix which has the following properties.

1. $0 \leq -q_{i,i} < \infty$ for all $i \in I$.
2. $q_{i,j} \geq 0$ for all $i \neq j$.
3. $\sum_{j \in I} q_{i,j} = 0$ for all $i \in I$.

As a result, the potential ϕ_i with boundary and interior functions will be defined by:

$$\phi_i = \mathbb{E}_i \left[\int_0^T c(X_s) ds + f(X_T) 1_{T < \infty} \right]. \tag{17}$$

where T is the hitting time of ∂D. Here, we introduce jump process of $(X_t)_{t \geq 0}$ in order to find the 'matrix' form equation (17) satisfied.

Proposition 3.1.1. *Suppose $(X_t)_{t\geq 0}$ changes its state from s_{k-1} to s_k at time $(t_k)_{k\geq 1}^N$. Then the jump process is a discrete Markov chain $(Y_k)_{k\geq 0}$ with transition matrix Π and state space I, where $Y_k = X_{t_k}$, $Y_N = X_T$ and $\Pi = \{\pi_{i,j}\}_{i,j\in I}$. Moreover, $\pi_{i,j} = \frac{q_{i,j}}{q_i}$.*

Proof. We only need to show $\pi_{i,j} = q_{i,j}/q_i$, according to infinitesimal definition, we have $P(X_{t+h} = j|X_t = i) = \delta_i^j + q_{i,j}h + o(h)$, where δ_i^j is the Kronecker delta. Besides, if $\widetilde{t}_1$ is the time of first changing state such that $X_{\widetilde{t}_1} = k \neq j$, then it follows that

$$\begin{aligned}\mathbb{P}_i(X_{t_1} = j) &= \int_0^\infty q_{i,j}e^{-q_{i,j}t}\mathbb{P}_i(\widetilde{t}_1 > t)dt \\ &= \int_0^\infty q_{i,j}e^{-q_{i,j}t}\prod_{k\neq j} e^{-q_{i,k}t}dt \\ &= \frac{q_{i,j}}{q_i}.\end{aligned}$$

On the other hand, $\pi_{i,j} = \mathbb{P}_i(X_{t_1} = j)$. □

Based on Proposition 3.1.1, the potential ϕ_i is rewritten as

$$\phi_i = \mathbb{E}_i\left[\sum_{k=1}^N (t_k - t_{k-1})c(Y_k) + f(Y_N)1_{N<\infty}\right].$$

Here, we write $\mathbb{E}[(t_k - t_{k-1})c(s_{k-1})|s_{k-1} = j] = \widetilde{c}_j$. Since the distribution of $(t_k - t_{k-1})_{s_{k-1}=j}$ is exponential with growth rate $q_j = \sum_{i\neq j} q_{i,j}$, we obtain

$$\widetilde{c}_j = c_j\mathbb{E}[t_k - t_{k-1}|s_{k-1} = j] = c_j\int_0^\infty tq_je^{-q_jt}dt = \frac{c_j}{q_j}.$$

According to Theorem 2.2.1, it follows that

Theorem 3.1.1. *The potential $\phi = (\phi_i : i \in D, f_j : j \in \partial D)^T$ satisfies the following equation*

$$\phi = \Pi\phi + \widetilde{c}. \tag{18}$$

where $\widetilde{c} = (\widetilde{c}_i : i \in D, 0)^T$. Moreover, if $\mathbb{P}_i(T < \infty) = 1$ for all $i \in D$, the above equation will only have at most one bounded solution.

Remark 3.1.1. *In fact, if potential ϕ of equation (18) is finite, it is also the minimal non-negative solution of the following*

$$-Q\phi = c. \tag{19}$$

Besides, if $\phi_i = \infty$ for some $i \in I$, the equation (18) has no non-negative finite solution.

3.2 Elliptic PDEs in potential theory

In [5], [4] and [3], they have showed how potential theory solve Schrödinger equation and Poisson problem. In addition, these books also discuss the continuity behavior on boundary of potential form solutions. Therefore, we will give probability interpretation of more general PDEs, that is, elliptic partial differential equations. First, let's see the definition of elliptic PDEs.

Definition 3.2.1 (Elliptic operator)**.** *Suppose Ω is a bounded and connected domain in $\mathbb{R}^d$ and $u(x) \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then the operator $\mathcal{L}$ is called an elliptical operator defined by*

$$\mathcal{L}u(x) = \sum_{i,j=1}^{d} a_{i,j}(x)\partial_{x_i,x_j}u(x) + \sum_{k=1}^{d} b_k(x)\partial_{x_k}u(x) + c(x)u(x). \tag{20}$$

where $a_{i,j}(\cdot)$, $b_k(\cdot)$ and $c(\cdot)$ are continuous in $\overline{\Omega}$. Generally, we suppose $c(\cdot)$ is non-positive.

Here, we focus on boundary problem in the following form.

$$\begin{cases} \mathcal{L}u(x) = f(x). & x \in \Omega \\ u(x) = \varphi(x). & x \in \partial\Omega \end{cases} \tag{21}$$

Theorem 3.2.1. *If the hitting time T is almost surely finite and the equation (21) has solution $u(x)$, then $u(x)$ is the unique solution in the following form*

$$\mathbb{E}_x\left[\varphi(X_T)e^{\int_0^T c(X_s)ds} - \int_0^T f(X_s)e^{\int_0^s c(X_t)dt}ds\right]. \tag{22}$$

Here $(X_t)_{t\geq 0}$ is a continuous stochastic process governed by the following differential equation

$$dX_t = \beta(X_t)dt + \alpha(X_t)dB_t. \tag{23}$$

where $\beta(x) = (b_1(x), ..., b_d(x))^T$ and $\frac{1}{2}\alpha(x)\alpha(x)^T = [a_{i,j}(x)]_{i,j=1}^d$.

Proof. Suppose $u(x)$ is a solution of (21). Let $D_n = \{x \in \mathring{D} : |x - y| > \frac{1}{n}\ for\ \forall y \in \partial D\}$ and T_n be the first exiting time of D_n. We write $\exp(\int_0^t c(X_s)ds) = C_t$ and $M_t^{(n)} = u(X_{t\wedge T_n})C_{t\wedge T_n} - \int_0^{t\wedge T_n} f(X_s)C_s ds$. By *Ito's lemma*, it concludes that

$$\begin{aligned} &u(X_{t\wedge T_n})C_{t\wedge T_n} - u(X_0) \\ &= \int_0^{t\wedge T_n} C_s\nabla u(X_s)\cdot dX_s + \int_0^{t\wedge T_n} c(X_s)u(X_s)C_s ds + \frac{1}{2}\int_0^{t\wedge T_n}\sum_{i,j=1}^{d} C_s\partial_{x_i,x_j}u(X_s)d\langle X^i, X^j\rangle_s \\ &= \int_0^{t\wedge T_n} C_s\mathcal{L}u(X_s)ds + \int_0^{t\wedge T_n} C_s\nabla u(X_s)\cdot\alpha(X_s)dB_s. \end{aligned}$$

Hence $M_t^{(n)}$ is an $\mathcal{F}_t$-martingale for $n \geq 1$. Besides, all $M_t^{(n)}$ are bounded by

$$\max_{x\in\Omega}|u(x)| + (t\wedge T_n)\max_{x\in\Omega}|f(x)|. \tag{24}$$

Let $n \to \infty$ and use the bounded convergence theorem. Then we obtain the martingale property of the process M_t. Moreover, since $T < \infty$ almost surely, letting $t \to \infty$, we conclude that

$$M_\infty = \varphi(X_T)C_T - \int_0^T f(X_s)C_s ds. \tag{25}$$

Now, by $\mathbb{E}_x[M_0] = \mathbb{E}_x[M_\infty]$, we obtain

$$u(x) = \mathbb{E}_x\left[\varphi(X_T)e^{\int_0^T c(X_s)ds} - \int_0^T f(X_s)e^{\int_0^s c(X_t)dt}ds\right].$$

which completes our proof. □

We can not claim that (22) must be the solution of (21), but the following theorem gives additional conditions which proves our claim.

Theorem 3.2.2. *If $\phi(x) = \mathbb{E}_x\left[\varphi(X_T)e^{\int_0^T c(X_s)ds} - \int_0^T f(X_s)e^{\int_0^s c(X_t)dt}ds\right]$ belongs to $C^2(\Omega)\cap C^1(\overline{\Omega})$, then $\phi(x)$ must satisfy equation (21).*

For the proof, we need the following lemma.

Lemma 3.2.1 ([4], Theorem 8.6, p.221)**.** *Let $(X_t)_{t\geq 0}$ satisfies differential equation (23) with the initial value $X_0 = x \in \Omega$. Then $(X_t)_{t\geq 0}$ and $\int_0^t g(X_s)ds$ has Markov property, where $g(\cdot)$ is a bounded measurable function in $\mathbb{R}^d$.*

Proof of Theorem 3.2.2. Since $a_{i,j}(\cdot)$, $b_k(\cdot)$ and $c(\cdot)$ are continuous in Ω, according to Lemma 3.2.1, it follows that on $\{T > s\}$

$$\phi(X_s) - \int_0^s f(X_t)C_t dt = \mathbb{E}_x\left[\varphi(X_T)C_T - \int_0^T f(X_t)C_t dt|\mathcal{F}_s\right].$$

The right side is a local martingale on $[0, T)$, so is the right side. Repeating the argument in the proof of Theorem 3.2.1, we deduce that

$$\phi(X_t) - \phi(X_0) - \int_0^t f(X_s)C_s ds = \int_0^t (\mathcal{L}\phi - f)(X_s)C_s ds + \text{local martingales}.$$

Since the left side is a local martingale, so is the right side. On the other hand, $\int_0^t(\mathcal{L}\phi - f)(X_s)C_s ds$ is continuous and locally of bounded variation, it must be $\equiv 0$. Since $\mathcal{L}\phi - f$ is continuous in Ω, it follows that $\mathcal{L}\phi \equiv f$. □

We are also concerned with the continuity behavior around boundary of our solutions. However, the solution may not continuously extend on boundary if the boundary $\partial\Omega$ is not 'regular' enough. Hence, we focus on a special kind of boundary here.

Definition 3.2.2 (Regular boundary)**.** *We say $y \in \partial\Omega$ is a regular point, if $P(T_y = 0) = 1$, where T_y is the hitting time of boundary $\partial\Omega$. Moreover, if every point on $\partial\Omega$ is regular, we call $\partial\Omega$ is regular.*

According to Definition 3.2.2, we can get Theorem 3.2.3. For the proof, we need the following lemmas.

Lemma 3.2.2. *Suppose $(X_t)_{t\geq 0}$ satisfies (23) and $T = \inf\{t \geq 0 : X_t \in \partial\Omega\}$. For $t > 0$ and $x_n \in \Omega \to y \in \partial\Omega$, it follows that*

$$\liminf_{n\to\infty} \mathbb{P}_{x_n}(T \leq t) \geq \mathbb{P}_y(T \leq t). \tag{26}$$

Moreover, if y is a regular point, then for $\delta > 0$ and the ball $\mathcal{B}(y, \delta)$

$$\lim_{n\to\infty} \mathbb{P}_{x_n}(T < \infty, X_T \in \mathcal{B}(y, \delta)) = 1. \tag{27}$$

Proof. According to Lemma 3.2.1, $(X_t)_{t\geq 0}$ has Markov property and so it follows that

$$\mathbb{P}_x(X_s \in \Omega^c \ for \ some \ s \in (\epsilon, t]) = \int_\Omega p_{x,y}(\epsilon)\mathbb{P}_y(T \leq T - \epsilon)dy.$$

where $p_{x,y}(\epsilon)$ is the transition probability. Since $y \to \mathbb{P}_y(T \leq T - \epsilon)$ is bounded and measurable. According to Proposition 3.1.1, $\lim_{h\to 0} \mathbb{P}_x(X_{x+h} = y) = \delta_x^y$. By dominated convergence theorem, we conclude that

$$x \to \mathbb{P}_x(X_s \in \Omega^c \text{ for some } s \in (\epsilon, t]).$$

is continuous for $\epsilon > 0$. Finally, by Fatou's lemma,

$$\begin{aligned}
\mathbb{P}_y(T \leq t) &= \lim_{\epsilon\to 0} \int_\Omega \lim_{x_n\to y} p_{x_n,z}(\epsilon)\mathbb{P}_z(T \leq T - \epsilon)dz \\
&\leq \liminf_{n\to\infty} \lim_{\epsilon\to 0} \int_\Omega p_{x_n,z}(\epsilon)\mathbb{P}_z(T \leq T - \epsilon)dz \\
&= \liminf_{n\to\infty} \mathbb{P}_{x_n}(T \leq t).
\end{aligned}$$

In addition, if $y \in \partial\Omega$ is regular, we have $\mathbb{P}_y(T = 0) = 1$. For any $\epsilon > 0$, we choose $t > 0$ such that $\mathbb{P}_0(\sup_{0\leq s\leq t} |X_s| > \delta/2) < \epsilon$. Since $\lim_{n\to\infty} \mathbb{P}_{x_n}(T \leq t) = 1$ by (26), we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}_{x_n}(T < \infty, X_T \in \mathcal{B}(y, \delta)) &\geq \lim_{n\to\infty} \mathbb{P}_{x_n}\left(T < t, \sup_{0\leq s\leq t} |X_s - x_n| \leq \frac{\delta}{2}\right) \\
&\geq \lim_{n\to\infty} \mathbb{P}_{x_n}(T \leq t) - \mathbb{P}_0\left(\sup_{0\leq s\leq t} |X_s| > \frac{\delta}{2}\right) \\
&> 1 - \epsilon.
\end{aligned}$$

Finally, letting $\epsilon \to 0$, follows (27). $\square$

Lemma 3.2.3. *Suppose Ω is bounded and $(X_t)_{t\geq 0}$ satisfies (23). Then we have*

$$\sup_{x\in\Omega} \mathbb{E}_x[T] < \infty. \tag{28}$$

where $T = \inf\{t \geq 0 : X_t \in \partial\Omega\}$.

Proof. Let $L = \sup\{|x-y| : x, y \in \Omega\}$. According to Lemma 3.2.1, $(X_t)_{t\geq 0}$ has Markov property, it follows that

$$\mathbb{P}_x(T < 1) \geq \mathbb{P}(|X_1 - x| > L) = \mathbb{P}_0(|X_1| > L) := \epsilon > 0.$$

Then we conclude $\mathbb{P}_x(T \geq k) \leq \prod_{i=1}^k \mathbb{P}_{X_{i-1}}(|X_i| \leq L) = (1-\epsilon)^k$ for any $k \in \mathbb{N}$ and $x \in \Omega$. Hence, for any $x \in \Omega$, we obtain

$$\mathbb{E}_x[T] = \int_0^\infty \mathbb{P}_x(T > t)dt \leq \sum_{i=0}^\infty \mathbb{P}_x(T \geq i) = \frac{1}{\epsilon} < \infty.$$

In other words, we conclude $\sup_{x\in\Omega} \mathbb{E}_x[T] = \epsilon^{-1} < \infty$. □

Theorem 3.2.3. *If the boundary $\partial\Omega$ is regular, then for any $y \in \partial\Omega$, $\phi(x)$ satisfies*

$$\lim_{x\to y \ in \ D} \phi(x) = \varphi(y) \tag{29}$$

Proof. Suppose $y \in \partial\Omega$ is regular. From Lemma 3.2.2, it follows that when $x_n \to y$, then $\lim_{n\to\infty} \mathbb{P}_{x_n}(T < \delta) = 1$ and $\lim_{n\to\infty} \mathbb{P}_{x_n}(X_T \in \mathcal{B}(y, \delta)) = 1$ for all $\delta > 0$. Since $a_{i,j}(\cdot)$, $b_k(\cdot)$ and $c(\cdot)$ are continuous in Ω, the bounded convergence theorem implies that

$$\lim_{n\to\infty} \mathbb{E}_{x_n}\left[\varphi(X_T)C_T - \int_0^T f(X_s)C_s ds : T < 1\right] = \varphi(y).$$

For the case $\{T \geq 1\}$, notice that the boundedness of all $a_{i,j}(\cdot)$, $b_k(\cdot)$ and $c(\cdot)$. Then using the Markov property and the boundedness of w established, we have

$$\begin{aligned} &\mathbb{E}_{x_n}\left[\varphi(X_T)C_T - \int_0^T f(X_s)C_s ds : T \geq 1\right] \\ &\leq ||\varphi||_\infty P_{x_n}(T > 1) + ||f||_\infty \mathbb{E}_{x_n}\left[\int_0^T |f(X_s)|ds : T \geq 1\right]. \end{aligned} \tag{30}$$

By Lemma 3.2.3, we have $\sup_{x\in\Omega} \mathbb{E}_x[T] = C < \infty$, whence that (30) is no more than $(||\varphi||_\infty + C||f||_\infty)\mathbb{P}_{x_n}(T > 1)$. In other words,

$$\lim_{n\to\infty} \mathbb{E}_{x_n}\left[\varphi(X_T)C_T - \int_0^T f(X_s)C_s ds : T \geq 1\right] = 0. \tag{31}$$

This completes our proof. □

4 Numerical Methods

4.1 Monte Carlo method in Continuous-time Case

In this subsection, we use Monte Carlo method to numerically solve elliptic PDEs. Consider a special case in Theorem 3.2.1, that is, $f \equiv c \equiv 0$. In order to simulate the stochastic process $(X_t)_{t\geq 0}$, where $X_0 = x$, we use the following approximation based on (23)

$$X_{t+\delta t} \approx X_t + \beta(X_t)\delta t + \alpha(X_t)\sqrt{\delta t}\mathcal{N}(0, 1). \tag{32}$$

where $\mathcal{N}(0,1)$ is the standard normal distribution. Suppose we assign N random walkers at point x and denoted by $\{w_k(t)\}_{k=1}^N$, where $w_k(\cdot)$ is simulated by (32) and $t_k = \inf\{n \in \mathbb{N} : w_k(n\delta t) \in \Omega^c\}$, which is the first time w_k leaving Ω. Besides, the actual exiting time denoted by $\widetilde{t_k}$ and $0 \le \widetilde{t_k} - t_k < \delta t$. Then we have the following estimation

$$\phi(x) = \mathbb{E}_x[\varphi(X_T)] \approx \frac{\sum_{k=1}^N \varphi(w_k(t_k))}{N}. \tag{33}$$

Since we simulate $(X_t)_{t\ge 0}$ by above approximation, we must consider the error. In general, there are three kinds of error in our method, which denoted by ϵ_1, ϵ_2 and ϵ_3, see [1]

$$\epsilon_1 = \mathbb{E}_x[\varphi(X_T)] - \frac{\sum_{k=1}^N \varphi(X_T^k)}{N} \tag{34}$$

$$\epsilon_2 = \frac{\sum_{k=1}^N \varphi(X_T^k)}{N} - \frac{\sum_{k=1}^N \varphi(w_k(\widetilde{t_k}))}{N} \tag{35}$$

$$\epsilon_3 = \frac{\sum_{k=1}^N \varphi(w_k(\widetilde{t_k}))}{N} - \frac{\sum_{k=1}^N \varphi(w_k(t_k))}{N}. \tag{36}$$

Here, X^k is the k-th ideal path that random walker w_k walks, which satisfies equation (23). In fact, ϵ_1 is the standard error of Monte Carlo method, by central limit theorem, we have

$$\lim_{N\to\infty} \mathbb{P}\left(-K < \frac{\sum_{m=1}^N \varphi(w_k(t_k)) - N\phi(x)}{\sqrt{N}\sigma_x} < K\right) = \int_{-K}^{K} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

where $\sigma_x = \mathbb{E}_x[\varphi(X_T)^2] - \phi(x)^2 < \infty$. Hence, $\epsilon_1 \sim \mathcal{O}(\frac{1}{\sqrt{N}})$ in probability. Next, we assume $\varphi(x)$ is Lipschitz continuous and obtain

$$\epsilon_2 \le \frac{L_\varphi}{N} \sum_{k=1}^N |X_T^k - w_k(\widetilde{t_k})|$$

where L_φ is the Lipschitz constant. According to large number theorem, we see that $\frac{1}{N}\sum_{k=1}^N |X_T^k - w_k(\widetilde{t_k})|$ converges to $\mathbb{E}[|X_T - w(\widetilde{t})|]$ in probability, where w is the simulation stochastic process and $\widetilde{t}$ is the hitting time. In fact, we have the following estimation.

Theorem 4.1.1 ([7], Theorem 10.2.2, p.342)**.** *Suppose $\alpha(\cdot)$ and $\beta(\cdot)$ in (23) are Lipschitz continuous with Lipschitz constant L_α and L_β. Besides, $|\alpha(x)| + |\beta(x)| < K(1+|x|)$ and $\mathbb{E}[|X_0|^p] < \infty$ for some $p > 0$, then we have*

$$\mathbb{E}\left[\sup_{s\le T} |X_s - w(s)|^p\right] < CN^{-\frac{p}{2}}. \tag{37}$$

where C is a constant only depended on p, T, $\mathbb{E}[|X_0|^p]$ and K.

According to above theorem, we set $p = 1$ and conclude $\epsilon_2 \sim \mathcal{O}(\frac{1}{\sqrt{N}})$. Finally, ϵ_3 is caused by the difference between actual exiting time and numerical estimating time. Hence, we get

$$\epsilon_3 \le \frac{L_\varphi}{N} \sum_{k=1}^N |w_k(\widetilde{t_k}) - w_k(t_k)| \le \frac{L_\varphi}{N} \sum_{k=1}^N L_k |\widetilde{t_k} - t_k|$$

Here, L_k is Lipschitz constant of w_k, then we define $L = \max_{1\leq k\leq N} L_k$. In addition, since $\beta(\cdot)$ and $\alpha(\cdot)$ are continuous in bounded domain Ω, the constant $L < \infty$ and we get

$$\epsilon_3 \sim \mathcal{O}(\delta t)$$

In conclusion, according to above argument, we do make sure that our approximation method can result in solving potential form solution of some special elliptic PDEs as the number of random walkers tends to infinite.

4.2 Numerical examples

In this part, we just show several examples about how Monte Carlo methods solve elliptic partial equations.

Example 4.2.1. *Let $\Omega = [0,1]\times[0,1]$ and $u(x,y)$ satisfies the following equation*

$$\begin{cases} u_{x,x} + u_{y,y} = 4(x^2+y^2)u. & x\in\Omega \\ u(x,y) = e^{x^2-y^2}. & x\in\partial\Omega \end{cases} \tag{38}$$

In fact, it is clear that the analytic solution of $u(x,y)$ is just $e^{x^2-y^2}$. In order to get the numerical solution, we make a grid $\{x_{i,j}\}_{i,j=1}^{M}$ such that $x_{i,j} = (i/M, j/M)$, then compute the potential of $x_{i,j}$ by equation (32) and (33). Next, we compare the numerical results and analytic solution of each grid point and find the maximal L^1 error. As in discussion of §4.1, we choose $\delta t = 10^{-3}$, $M = 10^2$ and $N = 10^4$. According to our program results, we get the maximal L^1 error is 0.085.

Example 4.2.2. *Let $\Omega = [0,1]\times[0,1]$ and $u(x,y)$ satisfies the following equation*

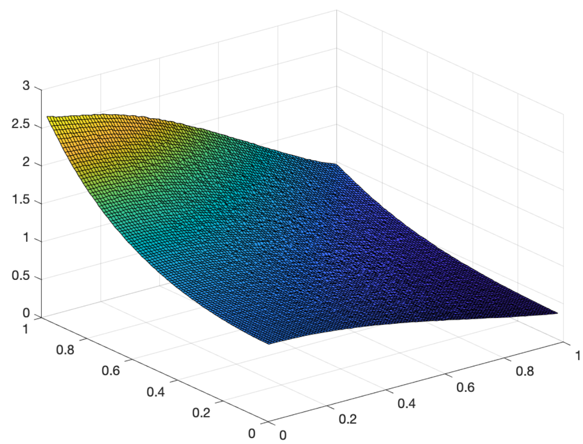
$$\begin{cases} u_{x,x} + 6x^2 u_{y,y} = 0. & x\in\Omega \\ u(x,y) = x^4 - y^2. & x\in\partial\Omega \end{cases} \tag{39}$$

It is still clear that the analytic solution is $u(x,y) = x^4 - y^2$. With other conditions as Example 4.2.1, we get the L^1 error is 0.084.

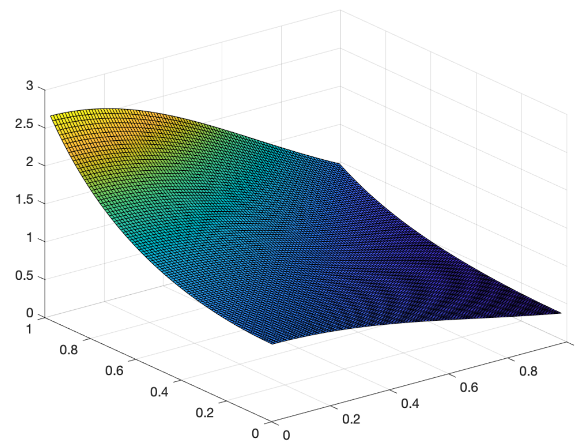
Example 4.2.3. *Let $\Omega = [0,1]\times[0,1]$ and $u(x,y)$ satisfies the following equation*

$$\begin{cases} \triangle u - \frac{1}{x+1}u_x - \frac{1}{y+1}u_y = 0. & x\in\Omega \\ u(x,y) = (x+1)^2 + (y+1)^2. & x\in\partial\Omega \end{cases} \tag{40}$$

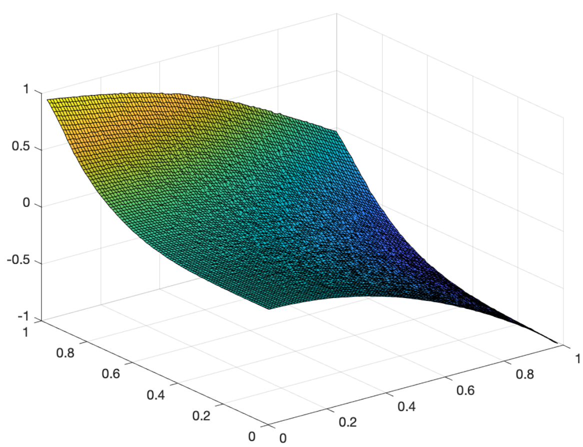
As before, the analytic solution is $(x+1)^2 + (y+1)^2$ and the L^1 error is 0.097. More details about these three examples are shown in Figure 1. In conclusion, we find the L^1 error of these three examples all constant times of $\frac{1}{\sqrt{N}}$, which fits our discussion in §4.1. There results show that numerical method actually has good performance. However, we should notice that Monte Carlo is quite time-consuming, in this case, it is a weakness of our method compared with standard Euler and other higher-order methods.



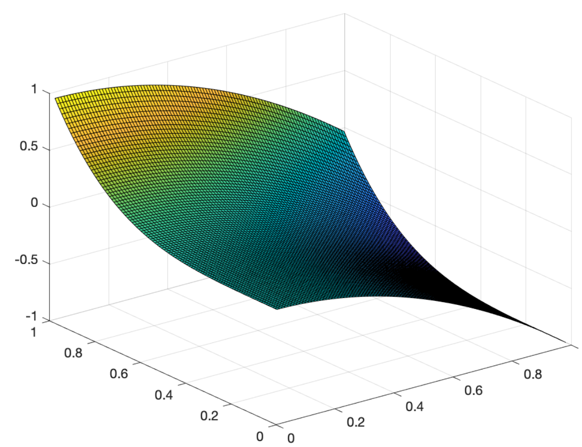
(a) Numerical solution of Example 4.2.1



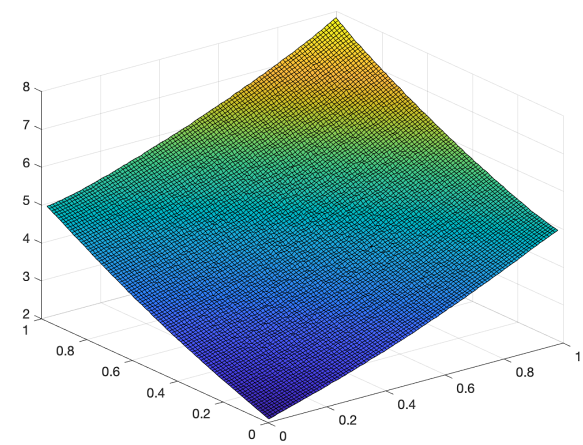
(b) Analytical solution of Example 4.2.1



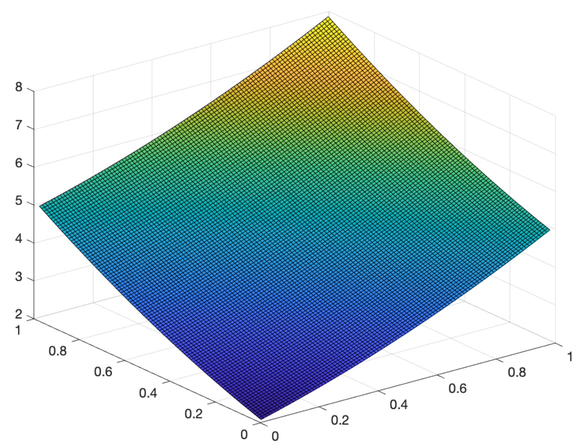
(c) Numerical solution of Example 4.2.2



(d) Analytical solution of Example 4.2.2



(e) Numerical solution of Example 4.2.3



(f) Analytical solution of Example 4.2.3

Figure 1: Figures in the left side show the solutions of numerical method based on Monte-Carlo simulation, comparing with the analytical solutions on the right side.

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