

Thesis

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Chapter 1

Froehlich Polaron

1.1 Electrons in crystals

The study of electrons inside a solid crystal is an important field due to their role in the determination of transport and optical properties of such materials.

Multiple experiments involving X-ray or electron scattering from suitable solid samples have demonstrated that a crystalline solid, whether a metal, an insulator or a semiconductor, is described a periodic arrangement of atoms consisting of a unit cell (usually the primitive one, defined as the smallest unit cell possible in a determined material) with a suitable atomic basis: some examples are Iron (Fe), described by a BCC (Body-Centered Cubic) unit cell with a single Fe atom basis, rock salt (NaCl), described by by a FCC (Face-Centered Cubic) unit cell with a basis composed of one Na atom and one Cl atom, and crystalline silicon (Si), which is characterized by the diamond structure, an FCC structure with a basis composed of 2 silicon atoms (geometrically different from that seen in rock salt).

Given the periodic nature of crystal structures, it is natural to consider the electrostatic potential generated by the crystal to be periodic too. Such is the basic assumption which underlies the treatment of electrons in crystals, that was first described by Felix Bloch in his famous 1928 paper *Über die Quantenmechanik der Elektronen in Kristallgittern* [1].

The model proposed by Bloch assumed independent electrons (no electron-electron interactions terms): each electron behaves as if only an average contribution to the potential from the other electrons (with the same periodicity of the lattice) exists, thus retrieving an **effective one-electron potential**.

The resulting Hamiltonian for the single independent electron is:

$$H\psi(\mathbf{r}) = \left(-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right) \psi(\mathbf{r}) = E\psi(\mathbf{r}), \quad (1.1.1)$$

where $V(r)$ is a periodic potential with the lattice periodicity, given a lattice translation vector

$$\mathbf{T} = n_1 \mathbf{a}_1 + n_2 \mathbf{a}_2 + n_3 \mathbf{a}_3, \quad n_1, n_2, n_3 \text{ integers}, \quad (1.1.2)$$

with \mathbf{a}_1 , \mathbf{a}_2 and \mathbf{a}_3 primitive translation vectors. Therefore, we have

$$V(\mathbf{r} + \mathbf{T}) = V(\mathbf{r}) \quad (1.1.3)$$

This means that the potential can be expanded in Fourier series

$$V(\mathbf{K}) = \sum_{\mathbf{K}} V_{\mathbf{K}} e^{i\mathbf{K} \cdot \mathbf{r}}, \quad (1.1.4)$$

and is consequently useful to expand the wavefunction in the same way. For this reason the Born-von Karman boundary conditions are applied [2]

$$\psi(\mathbf{r} + N_i \mathbf{a}_i) = \psi(\mathbf{r}), \quad i = 1, 2, 3. \quad (1.1.5)$$

where $N_i \mathbf{a}_i$ is a chosen vector such that $V(\mathbf{r} + N_i \mathbf{a}_i) = V(\mathbf{r})$ (it is a translation vector for the lattice). This is a normalizing condition for the wavefunction which corresponds to taking N_i unit cell in each direction and periodically repeating them. In this way it is possible to write the wavefunction in the following way:

$$\psi(\mathbf{r}) = \sum_{\mathbf{q}} c_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{r}}. \quad (1.1.6)$$

Going back to the periodic potential $U(\mathbf{r})$ we observe that the Fourier terms $U_{\mathbf{K}}$ are related to $U(\mathbf{r})$ by

$$U_{\mathbf{K}} = \frac{1}{V} \int_{cell} d\mathbf{r} e^{-i\mathbf{K} \cdot \mathbf{r}} U(\mathbf{r}) \quad (1.1.7)$$

with V unit cell volume and \mathbf{K} reciprocal lattice vector (vector in k-space such that $e^{i\mathbf{K} \cdot \mathbf{a}_i} = 1$). If it is assumed that the potential $V(\mathbf{r})$ is real and $V(-\mathbf{r}) = V(\mathbf{r})$ (valid for every Bravais lattice) it follows that the coefficients $U_{\mathbf{K}}$ are real.

Substituting in 1.1.1 we obtain:

$$(\mathbf{H} - E)\psi(\mathbf{r}) = \sum_{\mathbf{q}} \left(\frac{\hbar^2}{2m} q^2 - E \right) c_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{r}} + \sum_{\mathbf{K} \neq 0} U_{\mathbf{K}} c_{\mathbf{q}' - \mathbf{K}} e^{i\mathbf{q}' \cdot \mathbf{r}} = 0, \quad (1.1.8)$$

which yields the following result:

$$\left(\frac{\hbar^2}{2m} (\mathbf{k} - \mathbf{K})^2 - E \right) c_{\mathbf{k} - \mathbf{K}} + \sum_{\mathbf{K}' \neq 0} c_{\mathbf{K}' - \mathbf{K}} c_{\mathbf{k} - \mathbf{K}'} = 0. \quad (1.1.9)$$

From this equation it becomes clear that for a fixed k only the coefficients $c_{\mathbf{k}}, c_{\mathbf{k} - \mathbf{K}}, \dots$ whose wavevector differs from \mathbf{k} by a reciprocal lattice vector are coupled. In this way the original problem has been divided in N independent equations for each allowed value of \mathbf{k} (in the first Brillouin zone, the primitive unit cell in the reciprocal space).

Going back to the expansion 1.1.6 the wavefunction now becomes:

$$\psi_k(r) = \sum_{\mathbf{K}} c_{\mathbf{k} - \mathbf{K}} e^{i(\mathbf{k} - \mathbf{K}) \cdot \mathbf{r}}, \quad (1.1.10)$$

which can be recast as

$$\psi_k(r) = e^{i\mathbf{k} \cdot \mathbf{r}} \sum_{\mathbf{K}} c_{\mathbf{k} - \mathbf{K}} e^{-i\mathbf{K} \cdot \mathbf{r}}, \quad (1.1.11)$$

in this form it is easy to recognize that

$$u_{\mathbf{k}}(\mathbf{r}) = \sum_{\mathbf{K}} c_{\mathbf{k}-\mathbf{K}} e^{-i\mathbf{K}\cdot\mathbf{r}} \quad (1.1.12)$$

has the periodicity of the (reciprocal) Bravais lattice.

The shape of the Bloch wavefunction as stated in the Bloch theorem is thus retrieved

$$\psi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} u_{\mathbf{k}}(\mathbf{r}) \quad (1.1.13)$$

and its square modulus (from which important properties such as charge distribution follow) $|\psi_{\mathbf{k}}(\mathbf{r})|^2$ has the periodicity of the lattice.

The new Hamiltonian eigenvalues equation using Bloch wavefunction reads

$$H_{\mathbf{k}} u_{n\mathbf{k}} = \left(\frac{\hbar^2}{2m} \left(\frac{1}{i} \nabla + \mathbf{k} \right)^2 + U(\mathbf{r}) \right) u_{n\mathbf{k}}(\mathbf{r}) = E_{n\mathbf{k}} u_{n\mathbf{k}}(\mathbf{r}), \quad (1.1.14)$$

which yields a different energy eigenvalue for each \mathbf{k} in the first Brillouin zone. The index n arises from the fact that for each \mathbf{k} value infinite discrete solutions to the eigenvalue equation exist, in the same as in the particle in a box problem. Going to the thermodynamic limit ($N_i \rightarrow \infty$ for each component i) the entire first Brillouin zone is sampled and the typical band structure of the solids is recovered.

All the methods employed in solid state calculations rely on this "simple" equation to determine band structures, granted analytical form of the periodic potential $U(\mathbf{r})$ can be retrieved.

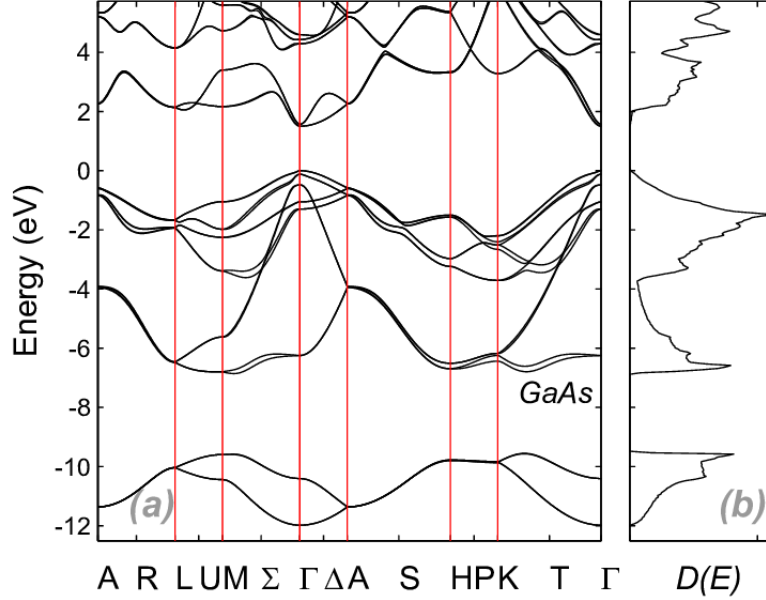
The most used and effective computation method is **Density Functional Theory** [3] or **DFT** for short, which does not directly compute the eigenvalue equation for each single k and n value, which would be impossible since the potential $U(\mathbf{r})$ is a many-body object that includes electron-electron interactions, but rather computes all the physical quantities of interest starting from the ground state charge density $\rho_0(\mathbf{r})$. This is made possible by the two Hohenberg-Kohn theorems [4], which respectively state that the external potential (the periodic $U(\mathbf{r})$ from the electron point of view) is uniquely determined up to a constant by the ground state charge density $\rho_0(\mathbf{r})$ and that this density can be computed variationally by minimizing a suitable functional $F[\rho]$.

Nevertheless, the functional $F[\rho]$ exact form is unknown: for this reason the **Kohn-Sham scheme** [5] is used. The idea at the basis of this practical setup is to substitute the real interacting system (many-body) with an auxiliary non-interacting one (single particle) which has the same ground state as the original. In this system the total potential $V(\mathbf{r}) = V_{ion}(\mathbf{r}) + V_{ee}(\mathbf{r})$ is replaced with an effective potential $V_{eff}(\mathbf{r})$ defined as

$$U_{eff}(\mathbf{r}) = U_{ion}(\mathbf{r}) + U_H(\mathbf{r}) + U_{xc}(\mathbf{r}), \quad (1.1.15)$$

where U_{ion} is the ionic potential, often modelled using pseudopotentials [6], $U_H(\mathbf{r})$ is the Hartree term, the electrostatic potential due to the mean field distribution of all the other electrons, and $U_{xc}(\mathbf{r})$ the exchange-correlation term, an approximate potential which includes Pauli exclusion principle (exchange), higher order electron-electron interactions (correlation) and many body corrections to kinetic energy. The exchange-correlation

term is usually computed using **LDA** (Local Density Approximation) or **GGA** (Generalized Gradient Approximation), both based on the the Uniform Electron Gas (UEG) [7]. Despite the inevitable approximations, DFT calculations are able to provide extremely accurate band structure calculations with few drawbacks, such as inaccurate gap estimations in insulators and semiconductors.



DFT calculated low-temperature GaAs Wurtzite structure with relative DOS [8].

As it can be seen in Figure 1.1 for GaAs, at the band extrema (at the edge between the allowed band states and the forbidden gap) the bands have a quadratic dispersion akin to that of a free electron: an electron in these regions only feel a very weak external potential $U(\mathbf{r})$ and can be considered as free provided that the free electron mass m is substituted with an effective mass m^* which takes into account the potential acting on it.

In a general case the effective mass is a tensorial quantity (bands are anisotropic in general with respect to the wavevector) and their value can be computed with

$$m_{ij}^{*-1} = \frac{1}{\hbar^2} \frac{\partial^2 E}{\partial k_i \partial k_j}, \quad (1.1.16)$$

provided that the quadratic approximation is valid. This approximation is really useful when calculating quantities near band extrema, providing an effective description of an otherwise complex interaction.

1.2 The polaron problem: the Froehlich model

The treatment described up until now for electrons in solids relies on a basic assumption: electro dynamics is much faster than ion dynamics. This is the so-called **Born-Oppenheimer approximation**, which assumes that the motion of the much heavier

ions is slow compared to the fast dynamics of the electrons (the characteristic time are of the order of ps for ions and fs for electrons). As a result, electrons are considered to respond instantaneously to changes in the positions of the ions, allowing the separation of electronic and ionic degrees of freedom in the analysis. Given M ions and n electrons the wavefunction $\Psi(\mathbf{R}_1, \dots, \mathbf{R}_M, \mathbf{r}_1, \dots, \mathbf{r}_n)$ with \mathbf{R}_α ionic positions and \mathbf{r}_i electron positions can be decomposed in the following way:

$$\Psi(\mathbf{R}_1, \dots, \mathbf{R}_M, \mathbf{r}_1, \dots, \mathbf{r}_n) = \chi(\mathbf{R}_1, \dots, \mathbf{R}_M) \psi(\mathbf{r}_1, \dots, \mathbf{r}_n), \quad (1.2.1)$$

and only the electronic wavefunction $\psi(\mathbf{r}_1, \dots, \mathbf{r}_n)$ is taken into consideration, decoupling ionic degrees of freedom and treating ionic potential as a constant energy surface $U_{ion}(\mathbf{r})$. This approximation, although extremely effective in computing electronic bands, fails by design to describe systems where there is a strong coupling between electrons and ions (for example electron-phonon interaction).

Such is the case of polar crystals: in these materials single electrons at the bottom of the conduction band (or holes at the top of the valence band) couple to the strongly polarized ions and distort the unit cell: an electron-phonon interaction is thus present in the lattice and the electron properties (such as energy and effective mass) are renormalized due to this coupling [9].

The basic assumption of the Froehlich model is that the polaron is "large", namely the characteristic size of the polaron (electron together with its phonon cloud) is much larger than the lattice constant a . In this way it is possible to ignore the atomic details of the actual material and treat it as a uniform dielectric medium.

It is assumed that the electron which takes part in the formation of the polaron lies at the bottom of the valence band, in this way it is expected to have a quadratic energy dispersion characterized (as already seen in 1.1.16) by an effective mass

$$\epsilon(k) = \frac{\hbar^2 k^2}{2m^*}, \quad (1.2.2)$$

where the effective mass m^* encapsulates all the interactions due to crystal structure (for a hole polaron the dispersion is similar considering a negative effective mass).

Given the fact that we are dealing with the large polaron model, it is possible to treat the material as an homogeneous continuum which is polarized by the excess charge (the conduction electron). We also assume that the material is isotropic. Let $\mathbf{P}(\mathbf{r})$ be the electric polarization at the point \mathbf{r} , the total electric field is then:

$$\begin{aligned} \mathbf{D}(\mathbf{r}) &= \epsilon_0 \mathbf{E}(\mathbf{r}) + 4\pi \mathbf{P}(\mathbf{r}), \\ \mathbf{D}(\mathbf{r}) &= \epsilon \mathbf{E}(\mathbf{r}). \end{aligned} \quad (1.2.3)$$

The only free charge present in the material is the conduction electron in excess, this means that $\mathbf{D}(\mathbf{r})$ is completely determined by it:

$$\begin{aligned} D(\mathbf{r}, \mathbf{r}_{el}) &= -\nabla \frac{e}{|\mathbf{r} - \mathbf{r}_{el}|}, \\ \nabla \cdot \mathbf{D}(\mathbf{r}, \mathbf{r}_{el}) &= 4\pi \delta(\mathbf{r} - \mathbf{r}_{el}), \end{aligned} \quad (1.2.4)$$

with \mathbf{r}_{el} the coordinate of the conduction electron. Developing the equations the following relation is obtained:

$$4\pi \mathbf{P}(\mathbf{r}) = \left(1 - \frac{1}{\epsilon}\right) \mathbf{D}(\mathbf{r}). \quad (1.2.5)$$

$\mathbf{P}(\mathbf{r})$ is the total polarization, which has the contribution both from electrons and from ionic displacements, since only the second contribution is relevant in our treatment, a method to tell apart this two contributions is needed.

To obtain this we imagine a system where a field is slowly applied and then rapidly switched off [10]: only electrons are able to keep up with the variation of the field. A formula connecting polarization and \mathbf{D} (dropping the \mathbf{r} dependence for simplicity since an isotropic and homogeneous medium is assumed) is then retrieved as

$$4\pi\delta\mathbf{P} = -\left(1 - \frac{1}{\epsilon_\infty}\right)\mathbf{D}, \quad (1.2.6)$$

it is then obtained the lattice contribution to the polarization:

$$4\pi\mathbf{P}_{lat} = 4\pi(\mathbf{P} - \delta\mathbf{P}) = \left(\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon_0}\right)\mathbf{D}. \quad (1.2.7)$$

We now consider a model for the polar crystal lattice: we assume it to be constituted of individual oscillating discrete dipoles located at fixed sites of the crystal with characteristic frequency ω . The energy of each one of these oscillators is

$$\frac{M}{2}(\dot{q}^2 + \omega^2 q^2). \quad (1.2.8)$$

The next step is to move from a discrete set of fixed oscillators to a continuum of dipoles: it is assumed that the dipoles are not coupled. Defining the effective charge as e^* , the following substitution is made:

$$e^*\mathbf{q}_n \rightarrow \mathbf{P}_{lat}(\mathbf{r}), \quad (1.2.9)$$

which relates the dipole moment to the polarization field. We also write the following relation:

$$\frac{M}{e^{*2}} = \gamma d^3 r, \quad (1.2.10)$$

where $M = nd^3 r$ is the mass density of the medium.

It is now possible to define the kinetic and potential energy of the freely oscillating polarization field:

$$\begin{aligned} T &= \int \frac{\gamma}{2} \dot{\mathbf{P}}_{lat}^2(\mathbf{r}) d^3 r, \\ V &= \int \frac{\gamma\omega^2}{2} \mathbf{P}_{lat}^2(\mathbf{r}) d^3 r. \end{aligned} \quad (1.2.11)$$

We identify ω with the optical phonon frequency ω_{LO} , but the constant γ has not yet been given a physical definition. For this reason we consider the interaction between electrons and dipole moment, which is given by

$$e \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{P}_{lat}, \quad (1.2.12)$$

which, using a continuous charge distribution and the polarization field, becomes

$$\rho(\mathbf{r}) d^3 r \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{P}_{lat}(\mathbf{r}') d^3 r'. \quad (1.2.13)$$

The interaction energy E_I is thus

$$E_I = \iint \rho(\mathbf{r}) \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{P}_{lat}(\mathbf{r}') d^3r d^3r'. \quad (1.2.14)$$

The Lagrangian is then built as follows:

$$L = T - U - E_I \quad (1.2.15)$$

and the equation of motion are found solving the Lagrange equations

$$\frac{d}{dt} \frac{\delta L}{\delta \dot{P}_{lat_i}} - \frac{\delta L}{\delta P_{lat_i}} = 0 \quad (1.2.16)$$

for $i = 1, 2, 3$. The explicit formula of the equation of motion is:

$$\gamma(\ddot{\mathbf{P}}_{lat}(\mathbf{r}') + \omega^2 \mathbf{P}_{lat}(\mathbf{r}')) = - \int \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \rho(\mathbf{r}) d^3r \quad (1.2.17)$$

keeping fixed $\rho(\mathbf{r})$. The term on the right side is simply the dielectric displacement $\mathbf{D}(\mathbf{r}')$ and, in the static limit, the equation simplifies to

$$\gamma\omega^2 \mathbf{P}_{lat}(\mathbf{r}') = \mathbf{D}(\mathbf{r}'). \quad (1.2.18)$$

Comparing this equation with 1.2.7, the value of γ is obtained:

$$\gamma = \frac{4\pi}{\omega^2} \left(\frac{1}{\epsilon_\infty - \frac{1}{\epsilon_0}} \right)^{-1}. \quad (1.2.19)$$

It is now time to define the full complete polaron Hamiltonian. We start by defining the free electron term (the interaction lies inside the effective mass m^*), considering that the electron charge density operator is defined as:

$$\rho(\mathbf{r}) = -e\psi^\dagger(\mathbf{r})\psi(\mathbf{r}) \quad (1.2.20)$$

using the second quantization formalism with field operators, then the electron kinetic term is

$$H_{el} = \int \psi^\dagger(\mathbf{r}) \left(-\frac{\hbar^2}{2m^*} \nabla^2 \right) \psi(\mathbf{r}) d^3r. \quad (1.2.21)$$

If we use a suitable basis for the field operator

$$\psi(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} c_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{r}} \quad (1.2.22)$$

with V (unit cell) volume, then 1.2.21 becomes

$$H_{el} = \sum_{\mathbf{k}} \frac{k^2}{2m^*} c_{\mathbf{k}}^\dagger c_{\mathbf{k}}, \quad (1.2.23)$$

with $c_{\mathbf{k}}^\dagger$ and $c_{\mathbf{k}}$ creation and annihilation operators for the free electron.

We now turn our attention to the polarization field, taking into account that it can be modelled as an harmonic oscillator a form similar to 1.2.21 can be recovered (P_{lat} is substituted by P in the following equations):

$$H_{ph} = \int \left(\frac{1}{2\gamma} \mathbf{\Pi}^2(\mathbf{r}) + \frac{\gamma}{2} \omega^2 \mathbf{P}^2(\mathbf{r}) \right) d^3r, \quad (1.2.24)$$

where $\mathbf{\Pi}(\mathbf{r})$ is the conjugate momentum to $\mathbf{P}(\mathbf{r})$.

Using the canonical substitution for the operator $\mathbf{P}(\mathbf{r})$

$$\mathbf{P}(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} \sqrt{\frac{\hbar}{2\gamma\omega}} \frac{\mathbf{q}}{q} e^{i\mathbf{q}\cdot\mathbf{r}} (a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger) \quad (1.2.25)$$

and the corresponding one for $\mathbf{\Pi}(\mathbf{r})$ we obtain the equation for a quantum harmonic oscillator:

$$H_{ph} = \hbar\omega \sum_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, \quad (1.2.26)$$

with $a_{\mathbf{q}}^\dagger$ and $a_{\mathbf{q}}$ bosonic phononic creation and annihilation operators.

Using the same formalism, the interaction term is instead written in the following way:

$$H_{EPC} = \iint \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) \frac{e}{|\mathbf{r} - \mathbf{r}'|} (-\nabla_{\mathbf{r}'} \cdot \mathbf{P})(\mathbf{r}') d^3r d^3r', \quad (1.2.27)$$

using the same substitution in 1.2.25 for $\mathbf{P}(\mathbf{r})$ the following form is obtained:

$$H_{EPC} = \int \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) \left[4\pi i \left(\frac{e^2 \hbar}{2\gamma\omega V} \right)^{1/2} \sum_{\mathbf{q}} \frac{1}{q} (a_{\mathbf{q}}^\dagger e^{-i\mathbf{q}\cdot\mathbf{r}} - a_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}}) d^3r \right]. \quad (1.2.28)$$

The term is simplified even more if we substitute the electron field operator using 1.2.25, then the final form of the coupling term is reached

$$H_{EPC} = \sum_{\mathbf{k}, \mathbf{q}} (V_{\mathbf{w}} c_{\mathbf{k}+\mathbf{q}}^\dagger c_{\mathbf{k}} a_{\mathbf{w}} + V_{\mathbf{w}}^* c_{\mathbf{k}-\mathbf{q}}^\dagger c_{\mathbf{k}} a_{\mathbf{q}}^\dagger), \quad (1.2.29)$$

where the coupling parameter is given as

$$V_{\mathbf{q}} = -4\pi i \left(\frac{e^2}{2\hbar\gamma\omega V} \right)^{1/2} \frac{1}{q}. \quad (1.2.30)$$

It is possible to rewrite the coupling parameter in terms of the dimensionless coupling strength α :

$$V_{\mathbf{q}} = i \left(\frac{4\pi\alpha}{V} \right)^{1/2} \frac{1}{q}, \quad (1.2.31)$$

$$\alpha = \frac{2\pi e^2}{\hbar\gamma\omega} \sqrt{\frac{2m^*\omega}{\hbar}} = \frac{1}{2} \left(\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon_0} \right) \frac{e^2}{\hbar\omega} \sqrt{\frac{2m^*\omega}{\hbar}}.$$

If we now identify ω with ω_{LO} angular frequency of the longitudinal optical phonon modes (the modes that effectively couple with the electron), we obtain the final form of the Froehlich Hamiltonian:

$$H^{Fr} = \sum_{\mathbf{k}} \frac{k^2}{2m^*} c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \hbar\omega_{LO} \sum_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + \sum_{\mathbf{k}, \mathbf{q}} (V_{\mathbf{q}} c_{\mathbf{k}+\mathbf{q}}^\dagger c_{\mathbf{k}} a_{\mathbf{q}} + V_{\mathbf{q}}^* c_{\mathbf{k}-\mathbf{q}} c_{\mathbf{k}} a_{\mathbf{q}}^\dagger) \quad (1.2.32)$$

where we have dropped the coupling parameter dependence on \mathbf{q} direction since it only depends on its modulus.

The main assumptions of this model are:

- Free electron with quadratic dispersion (up to a *scalar* effective mass m^*).
- Dispersionless optical mode with frequency ω_{LO} .
- "Large" polaron, with characteristic dimension d much greater than lattice parameter a (continuum approximation).

In this form the model correctly describes only a handful of real materials and it mainly serves as a toy model (we will briefly mention how it can be solved at weak coupling with perturbation theory and at strong coupling variationally), however we will focus on the **Diagrammatic Monte Carlo Method**, a computational method that is capable of solving the polaron problem using some tricks. We will also see how some of the main limitations of the model (namely the scalar effective mass m^* and the single phonon mode ω_{LO}) can be lifted without lifting the main assumption (the large polaron approximation). It will be illustrated in particular the case of an anisotropic band with effective mass $m^*(\mathbf{k})$ and multiple dispersionless optical modes ω_{LO_j} , already capable of modelling valence band minima in a wide range of materials.

1.3 Weak coupling limit: perturbation theory

In the weak coupling regime it is possible to apply perturbation theory to the Froehlich Hamiltonian, for this treatment a slightly modified Hamiltonian with respect to the one seen in 1.2.32 will be used [11]:

$$H^{Fr} = -\frac{\nabla^2}{2m^*} + \omega_{LO} \sum_{\mathbf{q}} (a_{\mathbf{q}}^\dagger a_{\mathbf{q}}) + \sum_{\mathbf{q}} (V_{\mathbf{q}} a_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} + V_{\mathbf{q}}^* a_{\mathbf{q}}^\dagger e^{-i\mathbf{q}\cdot\mathbf{r}}). \quad (1.3.1)$$

We will respectively define the non-interacting Hamiltonian H_0^{Fr} and the interaction term H_I^{Fr} as

$$\begin{aligned} H_0^{Fr} &= -\frac{\nabla^2}{2m^*} + \omega_{LO} \sum_{\mathbf{q}} (a_{\mathbf{q}}^\dagger a_{\mathbf{q}}), \\ H_I^{Fr} &= \sum_{\mathbf{q}} (V_{\mathbf{q}} a_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{r}} + V_{\mathbf{q}}^* a_{\mathbf{q}}^\dagger e^{-i\mathbf{q}\cdot\mathbf{r}}). \end{aligned} \quad (1.3.2)$$

The quantum state of the non-interacting electron is described by $|\mathbf{k}\rangle = \frac{1}{\sqrt{V}}e^{i\mathbf{k}\cdot\mathbf{r}}$, while for the non interacting phonons it is possible to define the average number of excitations $\langle a_{\mathbf{q}}^\dagger a_{\mathbf{q}} \rangle = \langle n_{\mathbf{q}} \rangle$, which is equal to 0 at the ground state. The total wavefunction of the non-interacting system then becomes:

$$|\mathbf{k}, 0\rangle = e^{i\mathbf{k}\cdot\mathbf{r}}|0\rangle. \quad (1.3.3)$$

The first excited state to which the ground non-interacting state can jump to is the one with electron wavevector $\mathbf{k} - \mathbf{q}$ and one phonon $n_{\mathbf{q}} = 1$, total energy is:

$$\frac{(\mathbf{k} + \mathbf{q})^2}{2m^*} + \omega_{LO} = \frac{k^2}{2m^*}, \quad (1.3.4)$$

the corresponding wavefunction is

$$|\mathbf{k} - \mathbf{q}, 1\rangle = e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{r}}|1\rangle. \quad (1.3.5)$$

At first order perturbation theory there is no correction to the energy, we then need to go to second order. We consider the matrix element

$$\langle \mathbf{k} - \mathbf{q}, 1 | H_I^{Fr} | \mathbf{k}, 0 \rangle = V_{\mathbf{q}}^*, \quad (1.3.6)$$

second order perturbation theory energy then becomes:

$$E_{\mathbf{k}P} = \frac{k^2}{2m^*} - \sum_{\mathbf{q}} \frac{|V_{\mathbf{q}}|^2}{(\mathbf{k} - \mathbf{q})^2/2m^* + \omega_{LO} - k^2/2m^*}, \quad (1.3.7)$$

evaluating the sum over the q wavevectors we arrive at

$$E_{\mathbf{k}P} = \frac{k^2}{2m^*} - \frac{\alpha\omega_{LO}\sqrt{2m^*\omega_{LO}}}{k} \arcsin \frac{k}{\sqrt{2m^*\omega_{LO}}}, \quad (1.3.8)$$

which for $k \ll \sqrt{2m^*\omega_{LO}}$ yields

$$E_{\mathbf{k}P} = \frac{k^2}{2m^*} - \alpha\omega_{LO}. \quad (1.3.9)$$

The polaron effective mass m_P^* is given by:

$$m_P^* = \frac{m^*}{\left(1 - \frac{\alpha}{6}\right)}. \quad (1.3.10)$$

From the equation it clearly results that the polaron effective mass diverges for $\alpha \rightarrow 6^-$ and is negative for $\alpha > 6$. This clearly signals that perturbation theory does not correctly describe the physical system at strong couplings.

1.4 Strong coupling limit: variational treatment

For the strong coupling case a different approach is taken: the main assumption of this model is a localized polaron wavefunction with a gaussian form [12], the Froehlich Hamiltonian is recast in order to write the phonon operators as displacement and conjugate moment operators:

$$Q_{\mathbf{q}} = \frac{1}{\sqrt{2}} (a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger), \quad P_{\mathbf{q}} = -\frac{i}{\sqrt{2}} (a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger), \quad (1.4.1)$$

so that the Hamiltonian in 1.2.32 can be rewritten as

$$H^{Fr} = \frac{p^2}{2m^*} + \frac{\omega_{LO}}{2} \sum_{\mathbf{q}} (P_{\mathbf{q}}^2 + Q_{\mathbf{q}}^2) + \sum_{\mathbf{q}} V_{\mathbf{q}} Q_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{r}}. \quad (1.4.2)$$

A wavefunction with dependence both on the electron position \mathbf{r} and the lattice displacement $Q_{\mathbf{q}}$ is needed, for this reason the following gaussian ansatz is used:

$$\begin{aligned} \Phi(\mathbf{r}, Q_{\mathbf{q}}) &= \phi(\mathbf{r}) \Psi_n(Q_{\mathbf{q}} + \delta Q_{\mathbf{q}}), \\ \phi(r) &= \left(\frac{\beta}{\sqrt{\pi}} \right)^{3/2} e^{-\frac{\beta^2}{2} r^2}, \end{aligned} \quad (1.4.3)$$

with β variational parameter, Ψ_n the harmonic oscillator wavefunction and $\delta Q_{\mathbf{q}}$ displacement to be calculated.

We now need to take the expectation value of the Hamiltonian operator:

$$H(Q_{\mathbf{q}}) = \langle \phi(r) | H | \phi^*(r) \rangle = \int \phi(r) H \phi(r) d^3r, \quad (1.4.4)$$

the terms with \mathbf{r} dependence are respectively evaluated as:

$$\begin{aligned} \int \phi^*(\mathbf{r}) \frac{p^2}{2m^*} \phi(\mathbf{r}) d^3r &= \frac{3\beta^2}{4m^*}, \\ \int \phi^*(\mathbf{r}) e^{i\mathbf{q} \cdot \mathbf{r}} \phi(\mathbf{r}) d^3r &= e^{-q^2/4\beta^2}. \end{aligned} \quad (1.4.5)$$

The following result is thus obtained:

$$H(Q_{\mathbf{q}}) = \frac{3\beta^2}{4m^*} + \frac{\omega_{LO}}{2} \sum_{\mathbf{q}} (P_{\mathbf{q}}^2 + Q_{\mathbf{q}}^2) + \sum_{\mathbf{q}} L_{\mathbf{q}} Q_{\mathbf{q}}, \quad L_{\mathbf{q}} = V_{\mathbf{q}} e^{-q^2/4\beta^2}. \quad (1.4.6)$$

To cancel out the linear term in $Q_{\mathbf{q}}$ it is important to choose the right equilibrium displacement $\delta Q_{\mathbf{q}}$ as

$$\delta Q_{\mathbf{q}} = \frac{L_{\mathbf{q}}}{\omega_{LO}}, \quad (1.4.7)$$

which yields the Hamiltonian

$$\begin{aligned} H(Q_{\mathbf{q}}) &= \frac{\omega_{LO}}{2} \sum_{\mathbf{q}} [P_{\mathbf{q}}^2 + (Q_{\mathbf{q}} + \delta Q_{\mathbf{q}})^2] + \frac{3\beta}{4m^*} - \frac{1}{2\omega_{LO}} \sum_{\mathbf{q}} L_{\mathbf{q}}^2 \\ &= \frac{\omega_{LO}}{2} \sum_{\mathbf{q}} [P_{\mathbf{q}}^2 + (Q_{\mathbf{q}} + \delta Q_{\mathbf{q}})^2] + E(\beta). \end{aligned} \quad (1.4.8)$$

The three different terms have precise physical meanings: the first term describes the harmonic oscillation of phonons around their new equilibrium position ($Q_{\mathbf{q}} + \delta Q_{\mathbf{q}}$), the second term represents the kinetic energy of the electron in the gaussian formalism, the third one the interaction energy between the electron and the phonons. To variationally retrieve the lowest energy the β parameter is minimized, we have:

$$\begin{aligned} \frac{1}{2\omega_{LO}} \sum_{\mathbf{q}} L_{\mathbf{q}}^2 &= \alpha \left(\frac{\beta^2 \omega_{LO}}{m^* \pi} \right)^{1/2}, \\ E(\beta) &= \frac{3\beta^2}{4m^*} - \alpha \left(\frac{\beta^2 \omega_{LO}}{m^* \pi} \right)^{1/2}. \end{aligned} \quad (1.4.9)$$

If we calculate the derivative with respect to β and we minimize it the following result is obtained:

$$\frac{dE}{d\beta} = \frac{3\beta}{2m^*} - \alpha \left(\frac{\omega_{LO}}{m^* \pi} \right)^{1/2} = 0, \quad (1.4.10)$$

which yields a value for β_0 :

$$\beta_0 = \frac{\alpha}{3} 2m^* \left(\frac{\omega_{LO}}{m^* \pi} \right)^{1/2}. \quad (1.4.11)$$

Using this result the minimum energy $E(\beta_0)$ is:

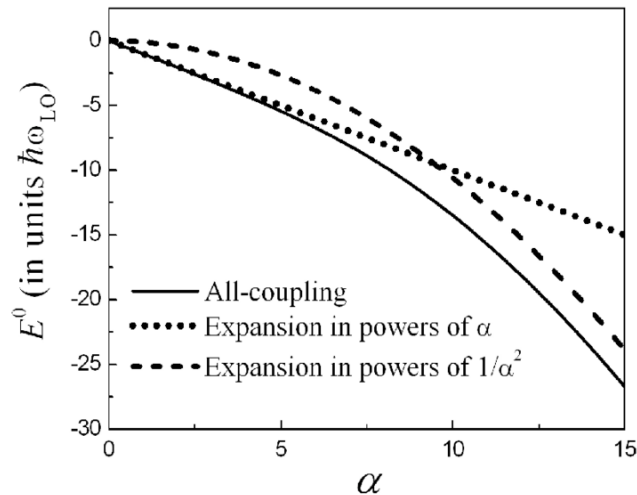
$$E(\beta_0) = -\frac{\alpha^2 \omega_{LO}}{3\pi} = -0.106 \alpha^2 \omega_{LO}, \quad (1.4.12)$$

with a quadratic dependence on the coupling strength α different from that found using perturbation theory (linear).

A more refined treatment of the strong coupling limit [13] yields the following result:

$$\lim_{\alpha \rightarrow \infty} E_0(\alpha) = -\omega_{LO} \left[-0.1085 \alpha^2 + 2.836 + O(1/\alpha^2) \right], \quad (1.4.13)$$

not so different from our obtained result.



Computed polaron energy using perturbation theory, strong coupling theory and Diagrammatic Monte Carlo [14].

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