CSC 665: Artificial Intelligence

Probability

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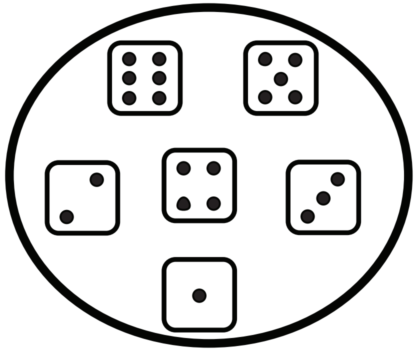
Today

- Probability
  - Random Variables
  - Joint and Marginal Distributions
  - Conditional Distribution
  - Product Rule, Chain Rule, Bayes' Rule
  - Independence

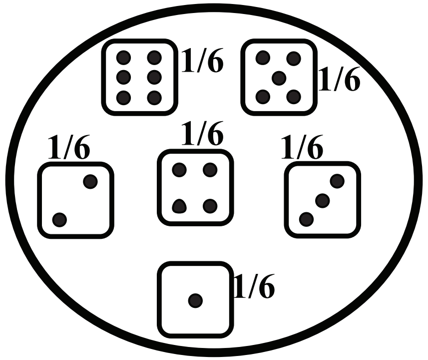
- You'll need all this stuff A LOT for the next couple of weeks, so make sure you go over it now!

Basic Laws of Probability

- Begin with a set Ω of possible worlds (outcomes)
  - E.g., 6 possible rolls of a die: {1, 2, 3, 4, 5, 6}



- A **Probability Model** assigns a number $P(\omega)$ to each world ω
  - E.g., $P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = 1/6$.



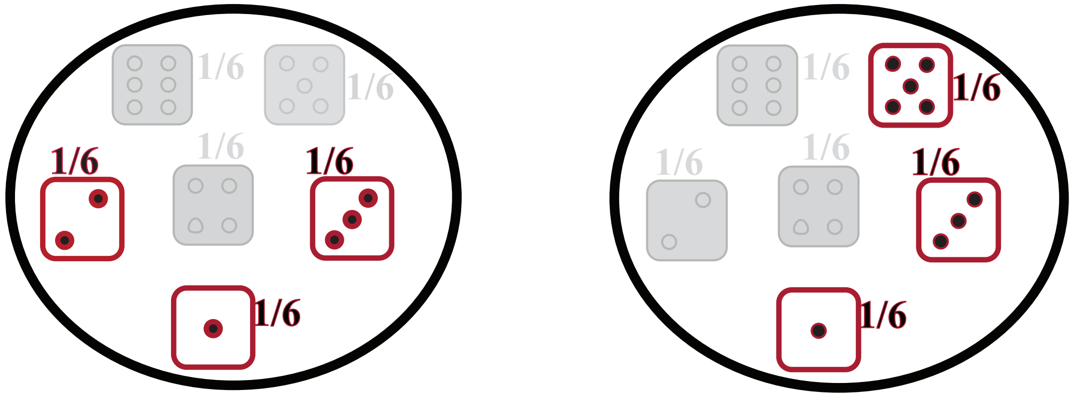
- These numbers must satisfy

$$0 \leq P(\omega) \leq 1$$

$$\sum_{\omega \in \Omega} P(\omega) = 1$$

Basic Laws of Probability

- An Event is any subset of Ω
  - E.g., “roll < 4” is the set {1,2,3}
  - E.g., “roll is odd” is the set {1,3,5}



- The probability of an event is the sum of probabilities over its worlds:

$$P(E) = \sum_{\omega \in E} P(\omega)$$

  - E.g., $P(roll < 4) = P(1) + P(2) + P(3) = ½$

Random Variables

- A random variable X is some aspect of the world about which we (may) have uncertainty
  - Odd = Is the die roll an odd number?
  - T = Is it hot or cold?
  - D = How long will it take to drive to work?
  - L = Where is the ghost?
- We denote random variables with capital letters
- Random variables have domains (or values)
  - Odd in {true, false}
    - Odd(1) = true, Odd(2)=false
  - T in {hot, cold}
  - D in $[0, \infty)$
  - L in possible locations, maybe {(0,0), (0,1), …}

Probability Distributions

- Associate a probability with each value: sums to 1 $\sum_x P(X=x) = 1$

Temperature:

$P(T)$

T	P
hot	0.5
cold	0.5

Weather:

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

- A probability is a single number
- A distribution is a TABLE of probabilities of values

Shorthand notation:

$$P(hot) = P(T = hot),$$
$$P(cold) = P(T = cold),$$
$$P(rain) = P(W = rain),$$
$$\ldots$$

Joint Distributions

- A joint distribution over a set of random variables: $X_1, X_2, \ldots X_n$ specifies a real number for each assignment (or world or *outcome*):

Long form: $P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n)$

Short form: $P(x_1, x_2, \ldots x_n)$

- Must obey:

$$P(x_1, x_2, \ldots x_n) \geq 0$$

$$\sum_{(x_1, x_2, \ldots x_n)} P(x_1, x_2, \ldots x_n) = 1$$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Probabilities of Events in JPD

- From a joint distribution, we can calculate the probability of any event
  - Probability that it's hot AND sunny? 0.4
  - Probability that it's hot? 0.5
  - Probability that it's hot OR sunny? 0.7
- Typically, the events we care about are partial assignments, like P(T=hot)

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Quiz: Probabilities of Events in JPD

- P(X=true, Y=true) ? 0.2

- P(X=true) ? 0.5

- P(Y=false OR X=true) ? 0.6

$$P(X, Y)$$

X	Y	P
true	true	0.2
true	false	0.3
false	true	0.4
false	false	0.1

Marginalization

- Marginalization (summing out): Given the joint distribution, we can compute distributions over smaller sets of variables.

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$P(T)$

T	P
hot	0.5
cold	0.5

$P(W)$

W	P
sun	0.6
rain	0.4

$$P(X_1 = x_1) = \sum_{x_2} P(X_1 = x_1, X_2 = x_2)$$

Quiz: Marginalization

$P(X, Y)$

X	Y	P
true	true	0.2
true	false	0.3
false	true	0.4
false	false	0.1

$$P(x) = \sum_y P(x, y)$$

$$P(y) = \sum_x P(x, y)$$

$P(X)$

X	P
true	
false	

$P(Y)$

Y	P
true	
false	

Quiz: Marginalization

$P(X, Y)$

X	Y	P
true	true	0.2
true	false	0.3
false	true	0.4
false	false	0.1

$$P(x) = \sum_y P(x, y)$$

$$P(y) = \sum_x P(x, y)$$

$P(X)$

X	P
true	0.5
false	0.5

$P(Y)$

Y	P
true	0.6
false	0.4

Conditional Probabilities

- A simple relation between joint and conditional probabilities

$$P(a|b) = \frac{P(a,b)}{P(b)}$$

$P(T, W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$P(W=s|T=c) = \frac{P(W=s, T=c)}{P(T=c)} = \frac{0.2}{0.5} = 0.4$$

$$= P(W=s, T=c) + P(W=r, T=c)$$
$$= 0.2 + 0.3 \quad = 0.5$$

Quiz: Conditional Probabilities

$P(X, Y)$

X	Y	P
true	true	0.2
true	false	0.3
false	true	0.4
false	false	0.1

- P(X=true | Y=true) ? 0.333
- P(X=false | Y=true) ? 0.667
- P(Y=false | X=true) ? 0.6

Conditional Distributions

- Conditional distributions are probability distributions over some variables given fixed values of others

Joint Distribution

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

Conditional Distributions

$P(W|T)$

$$P(W|T = hot)$$

W	P
sun	0.8
rain	0.2

$$P(W|T = cold)$$

W	P
sun	0.4
rain	0.6

Conditional Distributions: Normalization Trick

$$P(T, W)$$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

$$\begin{aligned} P(W = s|T = c) &= \frac{P(W = s, T = c)}{P(T = c)} \\ &= \frac{P(W = s, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\ &= \frac{0.2}{0.2 + 0.3} = 0.4 \end{aligned}$$

$$\begin{aligned} P(W = r|T = c) &= \frac{P(W = r, T = c)}{P(T = c)} \\ &= \frac{P(W = r, T = c)}{P(W = s, T = c) + P(W = r, T = c)} \\ &= \frac{0.3}{0.2 + 0.3} = 0.6 \end{aligned}$$

$$P(W|T = c)$$

W	P
sun	0.4
rain	0.6

Conditional Distributions: Normalization Trick

$P(W|T=c)$ = ?

$$\begin{aligned} P(W=s|T=c) &= \frac{P(W=s,T=c)}{P(T=c)} \\ &= \frac{P(W=s,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\ &= \frac{0.2}{0.2+0.3} = 0.4 \end{aligned}$$

$P(T,W)$

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

SELECT the joint probabilities matching the evidence

$P(c,W)$

T	W	P
cold	sun	0.2
cold	rain	0.3

NORMALIZE the selection (make it sum to one)

$P(W|T=c)$

W	P
sun	0.4
rain	0.6

$$\begin{aligned} P(W=r|T=c) &= \frac{P(W=r,T=c)}{P(T=c)} \\ &= \frac{P(W=r,T=c)}{P(W=s,T=c)+P(W=r,T=c)} \\ &= \frac{0.3}{0.2+0.3} = 0.6 \end{aligned}$$

Quiz: Normalization Trick

- P(X | Y=false) ?

$$P(X, Y)$$

X	Y	P
true	true	0.2
true	false	0.3
false	true	0.4
false	false	0.1

SELECT the joint probabilities matching the evidence

X	Y	P
true	false	0.3
false	false	0.1

NORMALIZE the selection (make it sum to one)

X	P
true	0.75
false	0.25

To Normalize

- (Dictionary) To bring or restore to a normal condition

All entries sum to ONE

- Procedure:
  - Step 1: Compute Z = sum over all entries
  - Step 2: Divide every entry by Z

- Example 1

W	P
sun	0.2
rain	0.3

Normalize

Z = 0.5

W	P
sun	0.4
rain	0.6

- Example 2

T	W	P
hot	sun	20
hot	rain	5
cold	sun	10
cold	rain	15

Normalize

Z = 50

T	W	P
hot	sun	0.4
hot	rain	0.1
cold	sun	0.2
cold	rain	0.3

The Product Rule

- Sometimes have conditional distributions but want the joint

$$P(x|y) = \frac{P(x,y)}{P(y)} \quad \Longleftrightarrow \quad P(y)P(x|y) = P(x,y)$$

The Product Rule

$$P(y)P(x|y) = P(x,y)$$

- Example:

$P(W)$

W	P
sun	0.8
rain	0.2

$P(D|W)$

D	W	P
wet	sun	0.1
dry	sun	0.9
wet	rain	0.7
dry	rain	0.3

$P(D,W)$

D	W	P
wet	sun	
dry	sun	
wet	rain	
dry	rain	

The Chain Rule

- More generally, can always write any joint distribution as an incremental product of conditional distributions

$$P(x_1, x_2, x_3) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2)$$

$$P(x_1, x_2, \ldots x_n) = \prod_i P(x_i|x_1 \ldots x_{i-1})$$

Bayes' Rule

Bayes' Rule

- Two ways to factor a joint distribution over two variables:

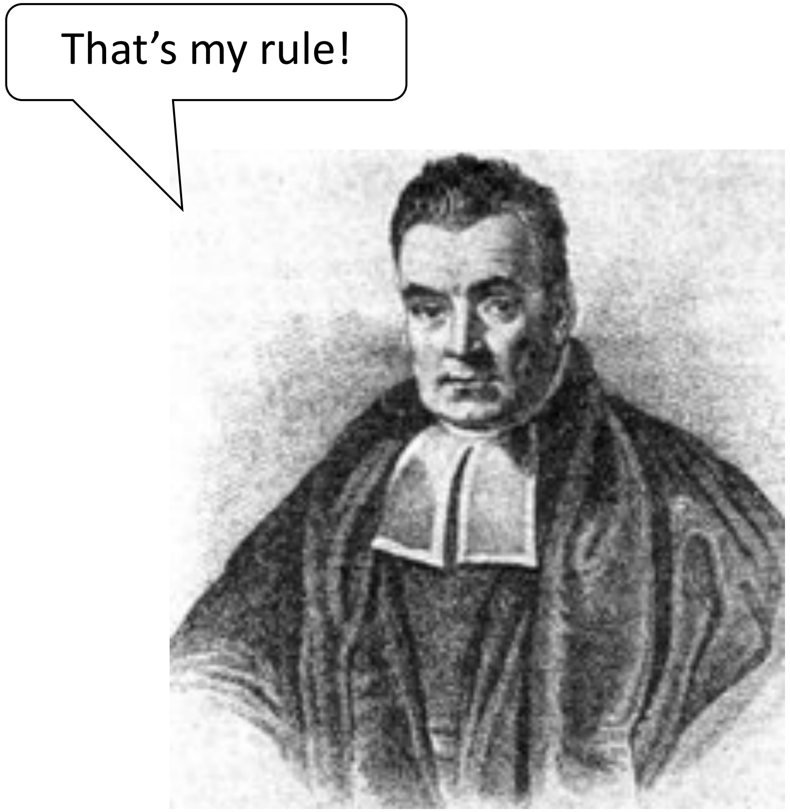
$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

- Dividing, we get:

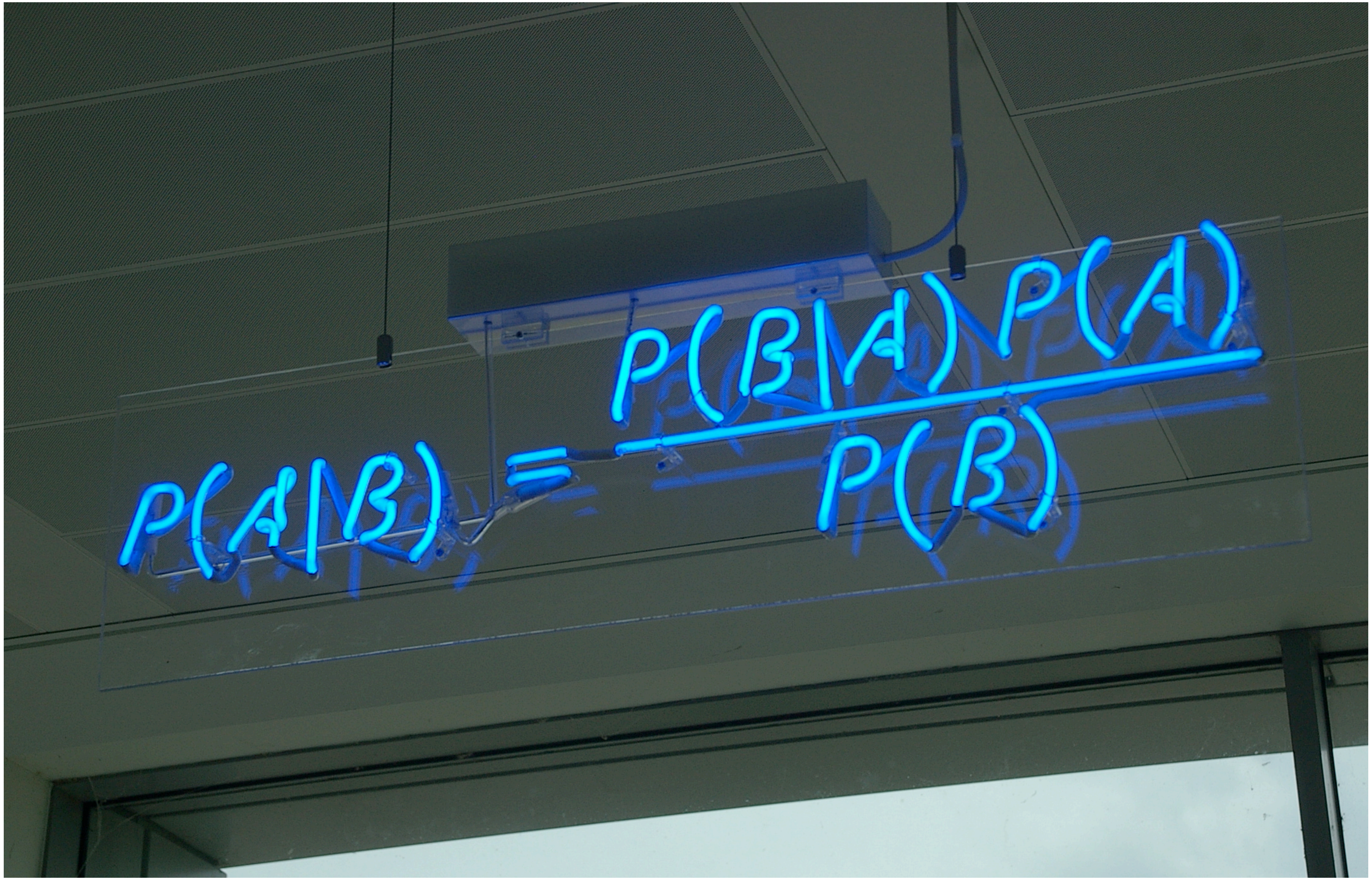
$$P(x|y) = \frac{P(y|x)}{P(y)}P(x)$$

- Why is this at all helpful?
  - Lets us build one conditional from its reverse
  - Often one conditional is tricky but the other one is simple

- In the running for most important AI equation!



Bayes' Rule



Example: Bayes' Rule

- On Average, the alarm rings once a year
- If there is a fire, the alarm will almost always ring
- On average, we have a fire every 10 years
- The fire alarm rings, what is the probability there is a fire?

Example: Bayes' Rule

- On Average, the alarm rings once a year
  - P(alarm) = 1/365
- If there is a fire, the alarm will almost always ring
  - P(alarm | fire) = 0.999
- On average, we have a fire every 10 years
  - P(fire) = 1/3650
- The fire alarm rings, what is the probability there is a fire?

0.999 0.9 0.0999 0.01

Example: Bayes' Rule

- On Average, the alarm rings once a year
  - P(alarm) = 1/365
- If there is a fire, the alarm will almost always ring
  - P(alarm | fire) = 0.999
- On average, we have a fire every 10 years
  - P(fire) = 1/3650
- The fire alarm rings, what is the probability there is a fire?

$$P(fire|\text{alarm}) = \frac{P(alarm|fire)\times P(fire)}{P(alarm)} = \frac{0.999 \times 1/3650}{1/365} = 0.0999$$

Even though the alarm rings the chance for a fire is only about 10%

Independence

Independence

- Two variables X and Y are *independent* if:

$$\forall x, y : P(x|y) = P(x)$$

  - Knowing the value of one does not tell you anything about the other

  - Another form:

$$\forall x, y : P(x, y) = P(x)P(y)$$

  - This says that their joint distribution *factors* into a product of two simpler distributions

  - We write:

$$X \perp\!\!\!\perp Y$$

Example: Independence

- Variables W (weather) and R (result of a die throw)
- Let’s compare P(W) vs. P(W | R = 6)
- What is P(W=cloudy) ?

0.066 0.1 0.4 0.6

Weather W	Result R	P(W,R)
sunny	1	0.066
sunny	2	0.066
sunny	3	0.066
sunny	4	0.066
sunny	5	0.066
sunny	6	0.066
cloudy	1	0.1
cloudy	2	0.1
cloudy	3	0.1
cloudy	4	0.1
cloudy	5	0.1
cloudy	6	0.1

Example: Independence

- Variables W (weather) and R (result of a die throw)

- Let's compare P(W) vs. P(W | R = 6)

- What is P(W=cloudy) ?
  - $P(W=cloudy) = \Sigma_{r \in R} P(W=cloudy, R = r)$
    $= 0.1+0.1+0.1+0.1+0.1+0.1 = 0.6$

- What is P(W=cloudy|R=6) ?

0.066/0.166	0.1/0.166
0.066+0.1	0.1/0.6

Weather W	Result R	P(W,R)
sunny	1	0.066
sunny	2	0.066
sunny	3	0.066
sunny	4	0.066
sunny	5	0.066
sunny	6	0.066
cloudy	1	0.1
cloudy	2	0.1
cloudy	3	0.1
cloudy	4	0.1
cloudy	5	0.1
cloudy	6	0.1

Example: Independence

- Variables W (weather) and R (result of a die throw)

- Let's compare P(W) vs. P(W | R = 6)

- What is P(W=cloudy) ?
  - $P(W=cloudy) = \Sigma_{r \in R} P(W=cloudy, R = r)$
    $= 0.1+0.1+0.1+0.1+0.1+0.1 = 0.6$

- What is P(W=cloudy|R=6) ?

$$P(W=cloudy|R=6) = \frac{P(W=cloudy \wedge R=6)}{P(R=6)}$$

$P(W=cloudy \wedge R=6) = 0.1$ (from table)
$P(R=6) = 0.166$ (marginal, 0.1+0.066)
Thus, $P(W=cloudy|R=6) = 0.1/0.166 = 0.6$

Weather W	Result R	P(W,R)
sunny	1	0.066
sunny	2	0.066
sunny	3	0.066
sunny	4	0.066
sunny	5	0.066
sunny	6	0.066
cloudy	1	0.1
cloudy	2	0.1
cloudy	3	0.1
cloudy	4	0.1
cloudy	5	0.1
cloudy	6	0.1

Example: Independence

- Variables W (weather) and R (result of a die throw)

- Let's compare P(W) vs. P(W | R = 6)

- What is P(W=cloudy) ?
  - $P(W=cloudy) = \Sigma_{r \in R} P(W=cloudy, R = r)$
    $= 0.1+0.1+0.1+0.1+0.1+0.1 = 0.6$

- What is P(W=cloudy|R=6) ?

$$P(W=cloudy|R=6) = \frac{P(W=cloudy \wedge R=6)}{P(R=6)}$$

$P(W=cloudy \wedge R=6) = 0.1$ (from table)

$P(R=6) = 0.166$ (marginal, 0.1+0.066)

Thus, $P(W=cloudy|R=6) = 0.1/0.166 = 0.6$

Weather W	**Result R**	**P(W,R)**
sunny	1	0.066
sunny	2	0.066
sunny	3	0.066
sunny	4	0.066
sunny	5	0.066
sunny	6	0.066
cloudy	1	0.1
cloudy	2	0.1
cloudy	3	0.1
cloudy	4	0.1
cloudy	5	0.1
cloudy	6	0.1

Example: Independence

- Variables W (weather) and R (result of a die throw)
- Let's compare P(W) vs. P(W | R = 6)
- The two distributions are identical: Knowing the result of the die does not change our belief in the weather

Weather W	P(W)
sunny	0.4
cloudy	0.6

Weather W	P(W|R=6)
sunny	0.066/0.166=0.4
cloudy	0.1/0.166=0.6

Weather W	Result R	P(W,R)
sunny	1	0.066
sunny	2	0.066
sunny	3	0.066
sunny	4	0.066
sunny	5	0.066
sunny	6	0.066
cloudy	1	0.1
cloudy	2	0.1
cloudy	3	0.1
cloudy	4	0.1
cloudy	5	0.1
cloudy	6	0.1

Exploiting Independence

- If $X_1, X_2, \ldots, X_n$ are independent, then we can represent their JPD as a product of marginal distributions

$$P(X_1, \ldots, Xn) = \prod_{i=1}^{n} P(Xi)$$

Conditional Independence

Conditional Independence

- X is conditionally independent of Y given Z $\quad X \perp\!\!\!\perp Y | Z$

  if and only if:

$$\forall x, y, z : P(x|z, y) = P(x|z)$$

- learning that Y=y does not change your belief in X when we already know Z=z
- and this is true for all values y that Y could take and all values z that Z could take

- or, equivalently, if and only if

$$\forall x, y, z : P(x, y|z) = P(x|z)P(y|z)$$

Conditional Independence

- What about this domain:
  - Traffic
  - Umbrella
  - Raining

Traffic ind Umbrella | Raining

Conditional Independence

- What about this domain:
  - Fire
  - Smoke
  - Alarm

Fire ind Alarm | Smoke

Conditional Independence and the Chain Rule

- Chain rule: $P(X_1, X_2, \ldots X_n) = P(X_1)P(X_2|X_1)P(X_3|X_1, X_2)\ldots$

- Trivial decomposition:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) =$$
$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain}, \text{Traffic})$$

- With assumption of conditional independence:

$$P(\text{Traffic}, \text{Rain}, \text{Umbrella}) =$$
$$P(\text{Rain})P(\text{Traffic}|\text{Rain})P(\text{Umbrella}|\text{Rain})$$

- Conditional Independence allows us to write them compactly

Probability Recap

- Conditional probability $$P(x|y) = \frac{P(x,y)}{P(y)}$$
- Product rule $$P(x,y) = P(x|y)P(y)$$
- Chain rule
$$\begin{aligned} P(X_1, X_2, \ldots X_n) &= P(X_1)P(X_2|X_1)P(X_3|X_1,X_2)\ldots \\ &= \prod_{i=1}^{n} P(X_i|X_1,\ldots,X_{i-1}) \end{aligned}$$
- X, Y independent if and only if:
$$\forall x,y : P(x|y) = P(x)$$
$$\forall x,y : P(x,y) = P(x)P(y)$$
- X and Y are conditionally independent given Z if and only if:
$$\forall x,y,z : P(x|z,y) = P(x|z)$$
$$\forall x,y,z : P(x,y|z) = P(x|z)P(y|z)$$

Reading

- Read Sections 12.1-5 in the AIMA textbook