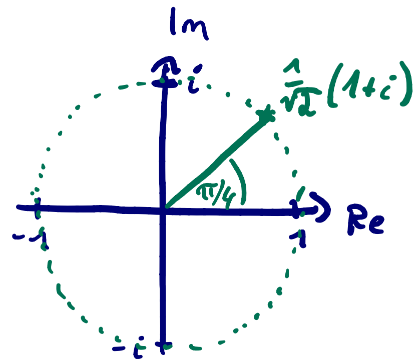
Hint: consider $H^{\otimes 4} U H^{\otimes 4} =: V$



$$\rightarrow V = \mathrm{diag}\Big(1,\ \tfrac{1}{\sqrt{2}}(1+i),\ \tfrac{1}{\sqrt{2}}(-1-i),\ i,\ i,\ \tfrac{1}{\sqrt{2}}(1-i),\ \tfrac{1}{\sqrt{2}}(-1+i),\ -1,\ \tfrac{1}{\sqrt{2}}(-1+i),\ -1,\ -1,\ \tfrac{1}{\sqrt{2}}(1+i),\ \tfrac{1}{\sqrt{2}}(-1-i),\ i,\ i,\ \tfrac{1}{\sqrt{2}}(-1+i)\Big)$$

define $\varepsilon_k := e^{\pi i k/4}$ $\Rightarrow$ $V = \mathrm{diag}(\varepsilon_0, \varepsilon_1, \varepsilon_5, \varepsilon_2, \varepsilon_2, \varepsilon_7, \varepsilon_3, \varepsilon_4, \varepsilon_3, \varepsilon_4, \varepsilon_4, \varepsilon_1, \varepsilon_5, \varepsilon_2, \varepsilon_2, \varepsilon_3)$

$\Rightarrow$ **goal:** create V

first step: consider single-qubit diagonal unitaries, i.e.

$$R_z(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

$$V_1 := R_z\left(\frac{\pi k_3}{4}\right) \otimes R_z\left(\frac{\pi k_2}{4}\right) \otimes R_z\left(\frac{\pi k_1}{4}\right) \otimes R_z\left(\frac{\pi k_0}{4}\right)$$

$$= \mathrm{diag}\big(\varepsilon_0, \varepsilon_{k_0}, \varepsilon_{k_1}, \varepsilon_{k_0+k_1}, \varepsilon_{k_2}, \varepsilon_{k_0+k_2}, \varepsilon_{k_1+k_2}, \varepsilon_{k_0+k_1+k_2}, \varepsilon_{k_3}, \varepsilon_{k_0+k_3}, \varepsilon_{k_1+k_3}, \varepsilon_{k_0+k_1+k_3}, \varepsilon_{k_2+k_3}, \varepsilon_{k_0+k_2+k_3}, \varepsilon_{k_1+k_2+k_3}, \varepsilon_{k_0+k_1+k_2+k_3}\big)$$

$\Rightarrow$ choose $k_0 = 1$, $k_1 = 5$, $k_2 = 2$, $k_3 = 3$

$$\rightarrow V_1 = \mathrm{diag}(\varepsilon_0, \varepsilon_1, \varepsilon_5, \varepsilon_6, \varepsilon_2, \varepsilon_3, \varepsilon_7, \varepsilon_0, \varepsilon_3, \varepsilon_4, \varepsilon_0, \varepsilon_1, \varepsilon_5, \varepsilon_6, \varepsilon_2, \varepsilon_3)$$

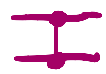
define $\tilde{V}$, s.t. $V := V_1 \cdot \tilde{V}$, i.e. what is "left to do" is

$$\tilde{V} = \mathrm{diag}(\begin{array}{cccc} \varepsilon_0, & \varepsilon_0, & \varepsilon_0, & \varepsilon_4, \\ \varepsilon_0, & \varepsilon_4, & \varepsilon_4, & \varepsilon_4, \\ \varepsilon_0, & \varepsilon_0, & \varepsilon_4, & \varepsilon_0, \\ \varepsilon_0, & \varepsilon_4, & \varepsilon_0, & \varepsilon_0 \end{array}) = \mathrm{diag}(\begin{array}{cccc} 1, & 1, & 1, & -1, \\ 1, & -1, & -1, & -1, \\ 1, & 1, & -1, & 1, \\ 1, & -1, & 1, & 1 \end{array})$$

<u>second step:</u> consider controlled - R_z rotations on two qubits, i.e.

$$C\text{-}R_z(\theta) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{i\theta} \end{pmatrix}$$

→ note that we only have ε_4 "left", so it is enough to consider

<u>CZ-gates ($\theta = \pi$)</u> 

$$V_2 = CR_z(\pi \cdot x_{01})_{0,1} \cdot CR_z(\pi \cdot x_{02})_{0,2} \cdot CR_z(\pi \cdot x_{03})_{0,3} \cdot \underbrace{CR_z(\pi \cdot x_{12})_{1,2}}_{\text{acting on qubits 1,2}}$$
$$\cdot\, CR_z(\pi \cdot x_{13})_{1,3} \cdot CR_z(\pi \cdot x_{23})_{2,3}$$

$$V_2 = \mathrm{diag}(\begin{array}{cccc} 1, & 1, & 1, & (-1)^{x_{01}}, \\ 1, & (-1)^{x_{02}}, & (-1)^{x_{12}}, & (-1)^{x_{01}+x_{02}+x_{12}}, \\ 1, & (-1)^{x_{03}}, & (-1)^{x_{13}}, & (-1)^{x_{01}+x_{03}+x_{13}}, \\ (-1)^{x_{23}}, & (-1)^{x_{02}+x_{03}+x_{23}}, & (-1)^{x_{12}+x_{13}+x_{23}}, & (-1)^{x_{01}+x_{02}+x_{03}+x_{12}+x_{13}+x_{23}} \end{array})$$

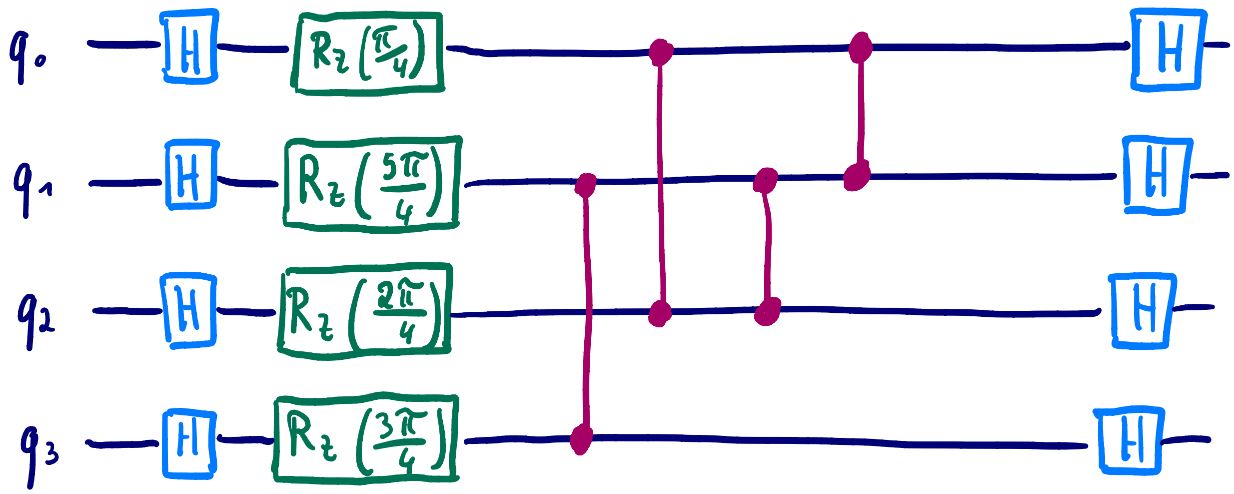
$\Rightarrow$ choose $x_{01} = 1$, $x_{02} = 1$, $\underline{x_{03} = 0}$, $x_{12} = 1$, $x_{13} = 1$, $\underline{x_{23} = 0}$

not needed!

$$\Rightarrow V_2 = \mathrm{diag}(\begin{array}{cccc} 1, & 1, & 1, & -1, \\ 1, & -1, & -1, & -1, \\ 1, & 1, & -1, & 1, \\ 1, & -1, & 1, & 1 \end{array}) = \tilde{V}$$

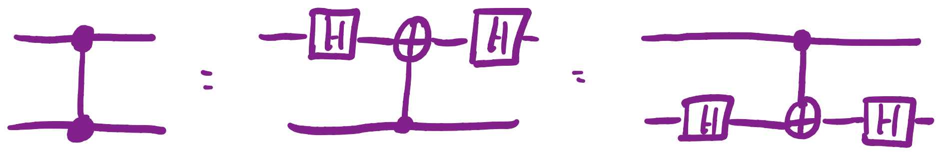
$\Rightarrow$ done!

⇒ circuit:

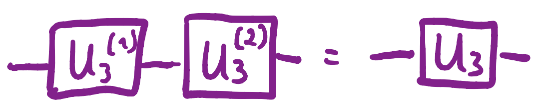


BUT: need to decompose into single-qubit unitaries & CNOT gates

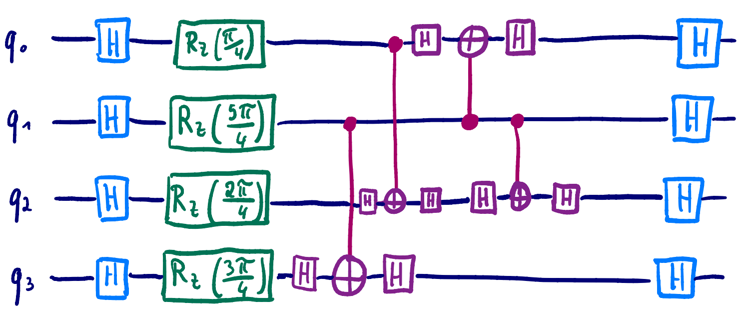
⇒ use gate identities: $CZ = H \cdot CNOT \cdot H$

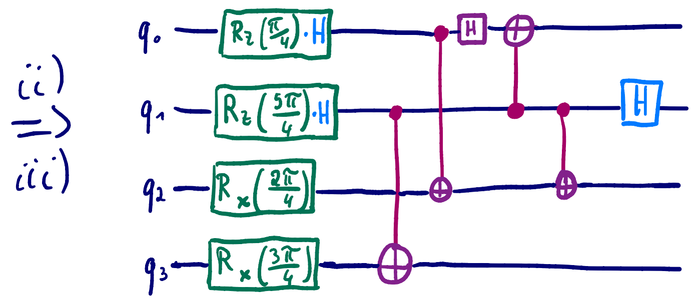
i) 

ii) 

iii) 

⇓ i)

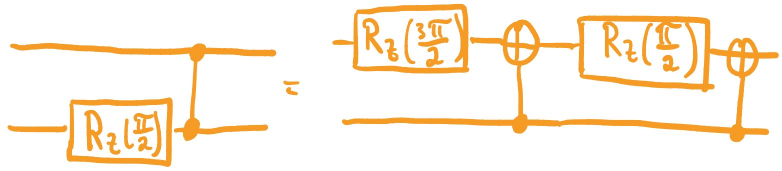




⇒ cost: $4 \times$ CNOT $+ 6 \times U_3 = 46$

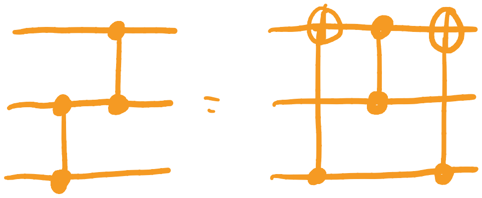
=> to reach 45, we need to make use of other identities first:

iv)

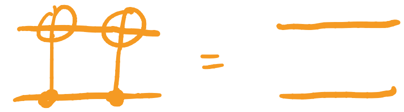


$$= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & i & \\ & & & -i \end{pmatrix}$$

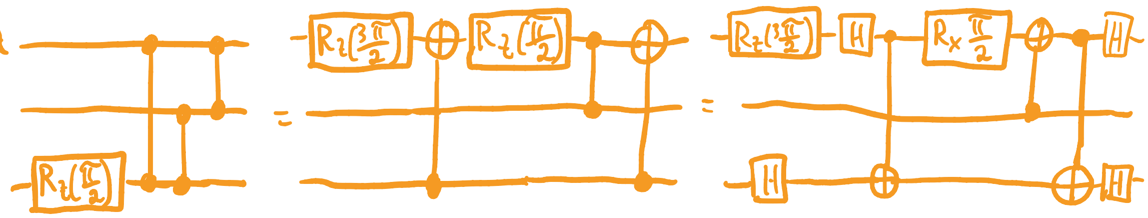
v)

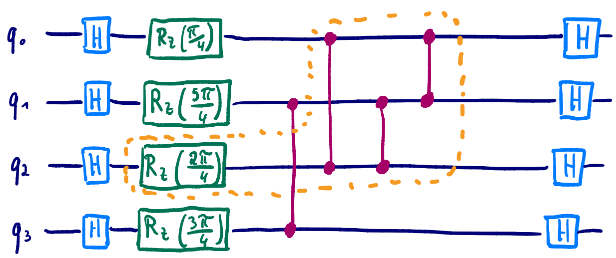


vi)

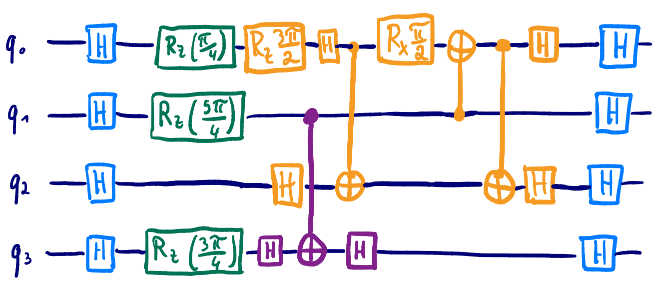


combined =>

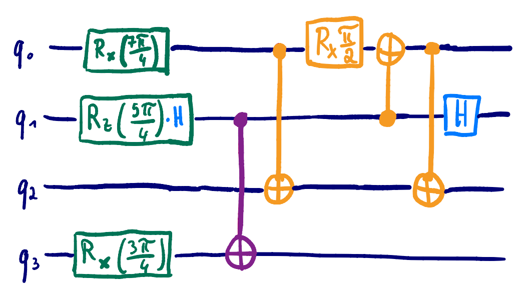




⇓



=>



=> cost: 4 × CNOT

+ 5 × U_3 = $\underline{\underline{45}}$