Libor Market Models

Xiaorong Zou (xiaorzou@gmail.com)

Aug. 18, 2018

Abstract. We have implemented Libor Market Model in Python and the python source code including a unit test with relevant inputs can be found in [2]. This document provides some background and implementation specification.

Keywords: Libor Market Model, Forward Libor Rate, Interest Rate Swap, Cap/floor, Caplet/floorlet, Swaption, Instantaneous Volatility, Instantaneous Correlation, Term Volatility, Term Correlation, Term Structure.

1. Introduction

Libor market model, also termed Lognormal Forward-LIBOR Model (LFM), is one of the most popular families of interest rate model due to a few advantages. First of all, it aligns with the cap/floor markets in the sense that the LFM prices caps with Black's cap formula, which is standard formula employed in the cap market. Although LFM is not theocratically consistent with swaption market in the sense that forward swap rates no longer follow lognormal distribution under LFM framework, numerical analysis shows that the model still can be calibrated seasonally well for the swaption market. Secondly, the model can be calibrated directly to market observable instruments and the analytic formula makes model calibration effectively. Thirdly, the model is transparent such that the model parameters, mainly instantaneous volatilities and instantaneous correlations, are explicitly linked to terminal volatilities and terminal correlations which directly reflect the market conditions. Finally, a particular parametrization of model can be used to establish one-to-one correspondence between LFM parameters and swaption (ATM) volatilities, such that the calibration can be conducted immediately by solving a cascade of quadratic equations so no optimization is necessary. The algorithm, termed Cascade Calibration Algorithm (CCA), not only make the calibration instantaneous, but also can be used to diagnose possible misalignment of input target market data.

In this documentation, we provide some background on LFM and model specifications for a python implementation. The current implementation focus on the materials in chapter 6 and 7 of the the reference book [1]. As such, the model does not cover the volatility smile/skew. We plan to include volatility smile/skew for the next phase using Shifted-Lognormal model with Uncertain Parameters (SLMUP) covered in chapter 12.

The implementation is not complete in the sense that we skip most of parameterizations methods on instantaneous volatilise and instantaneous correlations presented in chapter 6 and 7 of [1] to focus on implement CCA. It should be relatively easy to add those methods on top of the current implementation.

2. Setup

We adapt the same notations as in [1]:

1. $\{T_{-1}, T_0, \ldots, T_M\}$ a set of time spot, $T_{-1} = 0$ is considered as the current time in general.

2. $L(S, T)$: the simply compound spot prevailing at time S for the maturity T. It often represents LIBOR rate.

3. $F(t; T, S)$: the forward rate for $L(S, T)$ prevailing at time t. $F_k(t) := F(t; T_{k-1}, T_k)$ and $F_k = L(T_{k-1}, T_k)$. $k = 0, 1, \ldots, M$

4. $\tau_i = T_i - T_{i-1}, 0 \leq i \leq M$: the year fraction for the period (T_{i-1}, T_i) based on certain rule.

5. $P(t, T_k)$: zero-coupon bond value at time t, maturity at time T_k.

6. Q^k: forward measure with numeraire $P(t, T_k)$.

7. $Z^k(t) = (Z_1^k, \ldots, Z_M^k)^T$: M-dim Brian motion vector under Q^k. Notice that we need $t \leq T_k$ since $P(t, T_k)$ expires at T_k.

8. $dZ_i^k \cdot dZ_j^k = \rho_{ij} dt$. notice that ρ_{ij} does not depend on k. i.e. it does not depend on the selection of the measure.

9. if $T_{m-2} < t \leq T_{m-1}$, define $\beta(t) = m$. $1 \leq m \leq M$

10. $S_{a,b}(t)$: The forward swap rate prevailing at time t, the floating-leg rate is reset at $T_a, \ldots, T_{b-1}$ and paid at time $T_{a+1}, \ldots T_b$. Assuming $0 < a < b$. Notice that time points T_i used in swap rate might not be same as used in $F(t; T_k - 1, T_k)$.

11. $C_{a,b} = \sum_{i=a+1}^{b} \tau_i P(t, T_i)$: forward annuity prevailing at t covering period $[T_a, T_b]$

2.1. LFM setup

Notice that for a given $1 \le i \le M$, $t \le T_i$

$$F_i(t) = \frac{P(t,T_{i-1}) - P(t,T_i)}{P(t,T_i)\tau_i},$$

Using $P(t,T_i)$ as numeraire and the forward measure Q^i, $F_i(t)$ is a Martingale. Under Lognormal forward-LIBOR model(LFM), we assume that

$$dF_i(t) = \sigma_i(t)F_i(t)dZ_i^i$$

Given $Q^i, 1 \le i \le M$, other F_k are not necessarily martingale. One can derive the SDE followed by F_k under the fixed Q^i based on

$$\begin{aligned}
dF_k(t) &= \sigma_k(t)F_k(t)dZ_k^k, \quad 1 \le k \le M \\
\ln \frac{P(t,T_k)}{P(t,T_i)} &= -\sum_{i+1<j\le k} \ln(1+F_j(t)), \quad 1 \le i < k \\
\ln \frac{P(t,T_k)}{P(t,T_i)} &= \sum_{k+1<j\le i} \ln(1+F_j(t)), \quad 1 \le k < i
\end{aligned}$$

and obtain [1]

$$\begin{aligned}
dF_i(t) &= \sigma_i(t)F_i(t)dZ_i \\
dF_k(t) &= -F_k(t)\sigma_k(t)\sum_{k<j\le i} \frac{\rho_{jk}F_j(t)\sigma_j(t)\tau_j}{1+\tau_jF_j(t)}dt + \sigma_k(t)F_k(t)dZ_k, \quad 1 \le k < i \\
dF_k(t) &= F_k(t)\sigma_k(t)\sum_{i<j\le k} \frac{\rho_{jk}F_j(t)\sigma_j(t)\tau_j}{1+\tau_jF_j(t)}dt + \sigma_k(t)F_k(t)dZ_k, \quad 1 \le i < k
\end{aligned}$$

REMARK 1. *If we move from Q^k to Q^{k+1}, then*

$$dZ^{k+1}(t) = dZ^k(t) + \frac{\tau_{k+1}F_{k+1}(t)}{1+\tau_{k+1}F_{k+1}(t)}\sigma_{k+1}(t)(\rho_{1,k+1},\ldots,\rho_{M,k+1})^T dt$$

2.2. LFM under Spot-LIBOR-measure

Define $B_d(t)$ as:

1. $B(0) = 1$

[1] to simplify the notation, we write $Z_i := Z_k^i$ when the forward measure Q^i is clear.

2. at each time T_j, invest the all balance to the bond that is matured at T_{j+1}.

It can prove under Q^d, for $T_{m-2} < t \leq T_{m-1}$

$$dF_k(t) = F_k(t)\sigma_k(t) \sum_{m \leq j \leq k} \frac{\rho_{jk} F_j(t) \sigma_j(t) \tau_j}{1 + \tau_j F_j(t)} dt + \sigma_k(t) F_k(t) dZ_k^d \tag{1}$$

3. Calibration of LFM to Caps and Floors

Assume that $T_{-1} = 0$ is the valuation date and T_a is the first reset date and T_b is the last payment date. T_{i-1}-caplet is for the period T_{i-1}, T_i.

REMARK 2. *usually,* $T_i = (i+1) \times 3M$ *or* $T_i = (i+1) \times 6M$*, for* $i = -1, 0, \ldots$.

Notice that Caplet can be valuated by Black formula with

$$v^2_{T_{i-1}-caplet} := \frac{1}{T_{i-1}} \int_0^{T_{i-1}} \sigma_i(t)^2 dt, \quad v_i^2 := T_{i-1} v^2_{T_{i-t}-caplet}$$

Cap Quotes in the market. let $\tilde{T}_j = [T_0, \ldots, T_j]$

$$Cap^{MKT}(0, \tilde{T}_j, K) := \sum_{1 \leq i \leq j} \tau_i P(0, T_i) Bl(K, F_i(0), \sqrt{T_{i-1}} v_{T_j-cap})$$

v_{T_j-cap} is same for all terms in the summation and is called forward volatility.

REMARK 3.

1. *It is clear to define ATM caplet. We use the swap rate to define ATM cap.*

2. *The Term structure of Volatility. At time* $t = T_j$*, the volatility term structure is given by*

$$T_k \to V(T_j, T_{k-1}), \quad V(T_j, T_{k-1})^2 := \frac{1}{T_{k-1} - T_j} \int_{T_j}^{T_{k-1}} \sigma_k^2(t) dt$$

3. *the volatility term structure typically has a humped shape. The actually shape of the market term structure does not change too much over time.*

3.1. Piecewise-constant instantaneous Volatility Structures

We assume that $\sigma_i(t)$ is constant on each of time period $T_{m-2}, T_{m-1}, 1 \leq m \leq i$ and denote the constant by $\sigma_{i,m}$, or use β notation in Section 2

$$\sigma_i(t) = \sigma_{i,\beta(t)}, \quad t \leq T_{i-1}, 1 \leq i \leq M$$

At time 0, we can calibrate the constant to caplet volatilities that can be derived from market observable cap volatilities. In [1], the following parametrization are proposed:

1. $$\sigma_i(t) = \sigma_{i,m}, \qquad T_{m-2} < t \leq T_{m-1}, m = 1, 2, \ldots, M \tag{2}$$

Table I. Evolution of instantaneous volatilities

Fwd Rate	$t \in [0, T_0]$	$t \in (T_0, T_1]$	$t \in (T_1, T_2]$	$\ldots$	$t \in (T_{M-2}, T_{M-1}]$
$F_1(t)$	$\sigma_{1,1}$	dead	dead	$\ldots$	dead
$F_2(t)$	$\sigma_{2,1}$	$\sigma_{2,2}$	dead	$\ldots$	dead
$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$F_M(t)$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	$\ldots$	$\sigma_{M,M}$

2. $\sigma_i(t) = \eta_{i-\beta(t)+1}$: volatility term structure remains the same. a desired feature. the model implies $1 \leq \frac{V(0,T_{i-1})}{V(0,T_i)} \leq \sqrt{\frac{i}{i-1}} \to 1$, so the term structure gets almost flat at the end. The model should be avoided if observed otherwise.

3. $\sigma_i(t) = s_i$: constant regardless of t, less desirable since change of volatility term structure

4. $\sigma_i(t) = \Phi_i \Psi_{\beta(t)}$: change of pattern of term structure, less desirable than previous

5. $\sigma_i(t) = \Phi_i \psi_{i-\beta(t)+1}$: flexible to capture the humped shape and keep term structure shape

6. $\sigma_i(t) = [a(T_{i-1}-t)+d]e^{-b(T_{i-1}-t)}+c$: not necessarily humped, but it shall keep it if initial term structure is humped.

7. $\sigma_i(t) = \Phi_i\{[a(T_{i-1}-t)+d]e^{-b(T_{i-1}-t)}+c\}$: flexible. suggested by Rebonato (1999)

REMARK 4. *Notice that the terminal correlation depends on instantaneous correlation and instantaneous volatilities.*

3.2. THE STRUCTURE OF INSTANTANEOUS CORRELATIONS

instantaneous correlation is:

$$\rho_{ij}dt = dZ_i \cdot dZ_j$$

some desired features on ρ_{ij}. For a fixed i,

$$\begin{aligned} \rho_{ij} &\geq 0 \\ \rho_{i,j} &\geq \rho_{i,j+1}, \quad \forall j \geq i \\ \rho_{i+j+1,j+1} &\geq \rho_{i+j,j}, \quad \forall j \geq 0 \end{aligned}$$

1. $\rho_{i,h} := \frac{c_i}{c_j}$ where $1 \leq c_i < \ldots < c_M$ and $\frac{c_1}{c_{i+1}}$ is increasing. such correlation is viable.

2.

$$\begin{aligned} \rho_{i,j} &= \quad \rho_\infty + (1-\rho_\infty)e^{-\beta|i-j|} \quad \beta \geq 0 \\ \rho_{i,j} &= \quad \rho_\infty + (1-\rho_\infty)e^{-|i-j|[\beta-\alpha(i\vee j-1)]} \quad \beta, \alpha \geq 0 \end{aligned} \tag{3}$$

3.

$$\rho_{i,j} = \cos(\theta_i, \theta_j), \quad 1 \leq i, j \leq M \tag{4}$$

REMARK 5.

1. *More selections about correlations can be found in [1].*

2. *Eq 3 is Rebonato's three parameters full rank parameterizations. it has desirable property of being increasing along sub-diagonals. Not necessary positive definite, but hardly violate it in practical situations. parameters can be experessed as functions of specific elements of the target correlation matrix. the specific correlations are called pivot points. One can show*

$$\alpha = \frac{\ln \frac{\rho_{1,2}-\rho_\infty}{\rho_{M-1,M}-\rho_\infty}}{2-M}, \quad \beta = \alpha - \ln \frac{\rho_{1,2}-\rho_\infty}{1-\rho_\infty}$$

Notice that $\rho_{1,M}$ and $\rho_{M-1,M}$ are both extreme cases. The expression makes the Rebonato's approach attractive and recommend in [1]. This allows operators to incorporate personal views, recent market shifts, or to carry out scenario analysis for risk management and hedging purposes. The task can be performed by modifying the relevant pivot points. The matrix will vary as desired, while maintaining all regularity properties of the selected parametric form. this is valuable feature, since it is not easy to maintain a correlation matrix when some entries are modified. another reason why we should use parametric correlation matrix.

4. Lognormal Forward-Swap model (LSM)

Applying no arbitragy argument, one can show

$$P(t,T_a) - P(t,T_b) = S_{a,b}(t)C_{a,b}(t) = \sum_{a+1\le i\le b} \tau_i P(t,T_i)F_i(t)$$

Approximation by freeing coefficients

$$\begin{aligned} S_{a,b}(t) &= \sum_{a+1\le i\le b} \frac{\tau_i P(t,T_i)}{C_{a,b}(t)} F_i(t) \\ &= \sum_{a+1\le i\le b} w_i(t)F_i(t) \approx \sum_{a+1\le i\le b} w_i(0)F_i(t) \end{aligned} \tag{5}$$

Payoff at T_a for a swapion with strike K in US market:

$$(S_{a,b}(T_a) - K)^+ C_{a,b}(T_a)$$

In the Euro market, the payoff at T_a is (called "cash settled swaption")

$$(S_{a,b}(T_a) - K)^+ \sum_{a+1\le i\le b} \tau_i \frac{1}{(1+S_{a,b}(T_a))^{\tau_i}}$$

DEFINITION 4.1. *Under the measure $Q^{a,b}$, called forward swap measure or swap measure, with the numeraire $C_{a,b}(t)$ $(0 \le t \le T_a)$, LSM assumes*

$$dS_{a,b}(t) = \sigma_{a,b}(t)S_{a,b}dW^{a,b}(t) \tag{6}$$

We have Black formula for swaption

$$PS^{LSM}(0,T_a,[T_a,\dots,T_b],K) = C_{a,b}(0)Bl(K,S_{a,b}(0),v_{a,b}(0,T_a)) \tag{7}$$

where

$$v_{a,b}(t,T)^2 = \int_0^T \sigma^2_{a,b}(x)dx$$

$v_{a,b}(t,T)$ is called average percentage variance of $S_{a,b}(t)$ over $[t,T]$.

4.1. Swaptions Hedging

Let

$$V(t) = C_{a,b}(t)Bl(K, S_{a,b}(t), v_{a,b}(t,T_a))$$

and $V(t)$ can be written as

$$V(t) = d_+(t)[P(t,T_a) - P(t,T_b)] - Kd_-(t)C_{a,b}(t)$$

Jamshidian shows that under the assumption of deterministic $v_{a,b}(t,T_a)$

$$dV(t) = d_+(t)[dP(t,T_a) - dP(t,T_b)] - K \sum_{a+1 \le i \le b} \tau_i dP(t,T_i),$$

implies that $V(t)$ is self-finance portfolio. As such, $V(0)$ can be risk free with a perfect hedging.

REMARK 6. *One should distinguish interest rate fluctuation and interest rate risk (or volatility risk). Interest rate is not a deterministic quantity, its uncertainty can be hedged if the pattern of uncertainty is certain, i.e. if the rate follows a known distribution. In risk management, interest rate risk refers to the risk caused by uncertain of the distribution, rather than the uncertainty of the value of the rate. Above perfect hedging is a good example to say that we do not face interest rate risk if there is only uncertainty but not risk.*

"In practice, a trader will always try and hedge her position to volatility risk, which can even be the largest portion of the risk involved in a real. However, there is no universal recipe for this and hedging strategies are usually constructed so as to be insensitive to local variations of the risk parameters. With this respect, the trader's experience and sensibility is invaluable and cannot be replaced with the sheer output of any quantitative model". (page 242-243) [1]

4.2. Incompatibility between LFM and LSM

Under $Q^{a,b}$, by LSM, $S_{a,b}(t)$ follows lognormal distribution. One can derive SDE for $S_{a,b}(t)$ under Q^a based on the standard toolkit of changing numeraire

$$m_t^S dt - m_t^U dt = d \ln \frac{S_t}{U_t} \cdot d \ln X_t.$$

One can calculate the drift term of $S_{a,b}(t)$ under Q^{T_a}, which depends on $P(t,T_i), F_k(t)$. As such, $S_{a,b}$ no longer follow lognormal distribution. As mentioned in [1], even under Q^{T_a}, the difference is theoretical and the swap is not far from being lognormal in practice under Q^{T_a}.

REMARK 7. *Here is a brief review about changing numeraire. Let X be a n-vector diffusion process and whose dynamics are given under two difference measures Q^S and Q^U with the numeraire S and U respectively.*

$$\begin{aligned}
dX &= diag(X)m_t^U dt + diag(X)diag(\sigma^X)CdW^U \\
&= diag(X)m_t^S dt + diag(X)diag(\sigma^S)CdW^U \\
dS &= (\cdot)dt + S\cdot(\sigma^S)^T CdW^U \\
dU &= (\cdot)dt + U\cdot(\sigma^U)^T CdW^U
\end{aligned}$$

where $C \cdot C^T = \rho$ and W^U and W^S are n-dim standard Brownian motion under Q^U and W^S.

4.3. LFM formula for swaption volatilities

Applying 5 to get the SDE of $S_{a,b}(t)$ based on SDE of $F_i(t)$, we have

$$dS_{a,b}(t) \approx (\cdot)dt + \sum_{a+1\le i\le b} w_i(0)F_i(t)\sigma_i(t)dZ_i(t)$$

and

$$dS_{a,b}(t)dS_{a,b}(t) \approx \sum_{a+1\le i,j\le b} w_i(0)F_i(t)\sigma_i(t)w_j(0)F_j(t)\sigma_j(t)\rho_{ij}dt$$

Notice that

$$d\ln S_{a,b}(t)d\ln S_{a,b}(t) = \sigma_{a,b}^2 S_{a,b}^2 dt$$

and

$$v_{a,b,LSM}^2(T) = \int_0^T \sigma_{a,b}(t)^2 dt$$

One can estimate

$$v_{a,b,LSM}^2 \approx v_{a,b,LFM}^2 := \sum_{i,j=a+1}^{b} \frac{w_i(0)w_j(0)F_i(0)F_j(0)\rho_{i,j}}{S_{a,b}^2(0)} \int_0^{T_a} \sigma_i(t)\sigma_j(t)dt \tag{8}$$

REMARK 8.

1. *Eq 8 is called Rebonato's formula.*

2. *$v_{a,b,LFM}$ is a reasonable proxy for the Black volatility $v_{a,b,LSM}$ of the swap rate $S_{a,b}(t)$.*

3. *the formula does not depend on the selection of numeraire!*

4.4. Terminal correlation

Using freeing trick, under a forward measure Q^i, one can have

$$d \ln F_k(T) = (\cdot)dt + \int_0^T \sigma_k(t)^2 dZ_k$$

by solving the equation and with some algebra,

$$F_i(T_a)F_j(T_a) = C(t)e^{\int_0^{T_a} \sigma_i(t)dZ_i + \int_0^{T_a} \sigma_j(t)dZ_j}$$

where $C(t)$ is deterministic. One can obtain

$$Corr^{REB}(F_i(T_a), F_j(T_a)) = \rho_{ij} \frac{\int_0^{T_a} \sigma_i(t)\sigma_j(t)dt}{\sqrt{\int_0^{T_a} \sigma_i(t)^2 dt}\sqrt{\int_0^{T_a} \sigma_j(t)^2 dt}} \tag{9}$$

REMARK 9.

1. *Approximation 9 is called Rebonato's terminal correlation formula.*

2. *It is clear $|Corr^{REB}(F_i(T_a), F_j(T_a))| < \rho_{i,j}$.*

3. *It does not depend on the choice of numeraire.*

4. *swaption volatility $v_{a,b,LFM}$ (Eq. 8) in LIBOR market model are more directly linked with terminal correlations, rather than with instantaneous ones.*

5. Calibration to Swaption prices

5.1. some comments

1. Computing market (plain vanilla) swaption prices is not the purpose of an IR model. At most, LFM can be used to determine the price of illiquid swaption or of standard swaption for which thh Black volatility is not quoted or is judged to be not completely reliable.

2. A model should incorporate as many standard swaption prices as possible.

3. $x \times y$ swaption is the swaption expired at x with tenor y. So the underlying swap is maturied at $x + y$.

4. One should not completely rely on the swaption matrix provided by a single broker, and, in any case, should not take it for granted. The most liquid swaptions are updated regularly, whereas some entries of the matrix refer to older market situations.

5. Historical estimations are usually characterized by problems such as outliers, non-synchronous data and, in particular with reference to IR, discontinuities in correlation surfaces due to the use of discount factors extracted from different financial instruments. As such, it was proposed to use possibly parameterized correlation matrices designed to remain smooth, regular and desired properties as outlined before.

6. European swaption prices are relatively insensitive to correlation details. Simple correlation matrix can provide good approximation as complex correlation matrix, on the other hand, a more regular correlation structure can be helpful in calibration to obtain improved diagnostics : more regular volatilities, a more stable evolution of the volatiity term structure, and better terminal correlations.

7. In the special case of the swaption cascade calibration, swaption prices are recovered exactly and are not altered by correlations, while the regularity of the volatilities may depend on the smoothness and regularity of the instantaneous correlation ρ. For CCA, one can use historically estimated exogenous correlation matrices, and smooth it using parametric forms.

8. Developing a closed-form-formula calibration to swaption volatilise and establish a one-to-one correspondence between swaption volatilities and LFM covariance parameters. (limiting to ATM makes this possible)

9. Dealing with missing input swaption volatilities that are inconsistent with the assumed LIBOR market model dynamics by avoiding interpolation

10. jointly calibrate caps and swaption when taking the semi-annual tenor of caps into account.

5.2. Connecting caplet and $S \times 1$ swaption volatilities

Notice that the cap market forward rates are mostly semi-annual rates whereas those entering the forward-swap-rate expressions are typically annual rates. We need reconcile volatilities of semi-annual forward rates and volatilities of annual forward rates. Denote for $t < S < T < U$

$$F_1(t) := F(t; S, T), \quad F_2(t) := F(t; T, U), \quad F(t) := F(t; S, U),$$

where $F_1(t)$ and $F_2(t)$ are semi-annual forward rates and $F(t)$ is annual forward rate. It is clear

$$F(t) = \frac{P(t,S)}{P(t,U)} - 1 = \frac{P(t,S)}{P(t,T)} \frac{P(t,T)}{P(t,U)} - 1 = \frac{F_1(t) + F_2(t)}{2} + \frac{F_1(t)F_2(t)}{4}$$

So we have

$$\begin{aligned} dF(t) &= \sigma_1(t)(\frac{F_1(t)}{2} + \frac{F_1(t)F_2(t)}{4})dZ_1(t) \\ &+ \sigma_2(t)(\frac{F_2(t)}{2} + \frac{F_1(t)F_2(t)}{4})dZ_2(t) + (\cdot)dt \end{aligned}$$

One can drive the percentage volatiity $\sigma(t)$ of $F(t)$

$$\sigma^2(t) = u_1^2(t)\sigma_1^2(t) + u_2^2(t)\sigma_2^2(t) + 2\rho u_1(t)u_2(t)\sigma_1(t)\sigma_(t)$$

where $u_1(t)$ and $u_2(t)$ meet

$$u_1(t)F(t) = \frac{F_1(t)}{2} + \frac{F_1(t)F_2(t)}{4}, \quad u_2(t)F(t) = \frac{F_2(t)}{2} + \frac{F_1(t)F_2(t)}{4},$$

Using approximation by freezing coefficients

$$\sigma_{app}^2(t) = u_1^2(0)\sigma_1^2(t) + u_2^2(0)\sigma_2^2(t) + 2\rho u_1(0)u_2(0)\sigma_1(t)\sigma_(t)$$

So the squared Black's volatility

$$v_{Black}^2 \approx \frac{1}{S}\int_0^S \sigma_{app}^2(t)dt \tag{10}$$

Under the assumption that $\sigma_1(t)$ and $\sigma_2(t)$ are constant,

$$v_{Black}^2 \approx u_1^2(0)v_{S-caplet}^2 + u_2^2(0)v_{T-caplet}^2 + 2\rho u_1(0)u_2(0)v_{S-caplet}v_{T-caplet}$$

REMARK 10. *The approximation above can give volatilities that are slightly larger than the actual ones.*

5.3. Calibration: Using formulation (3) for instantaneous volatilities and (2) for instantaneous correlation

In this section, we shall follow the following notations.

1. $F_i(t)$: the forward rate observed at time t, with reset time T_i and tensor $T_{i+1} - T_i$, $i = 0, 1, \cdots M-1$. Again, we assume that $T_{-1} = 0$. Notice that $F_i(t)$ represents $F(t; T_{i-1}, T_i)$. The new notation will make it easier for us to formulate the implementation algorithm in python since python array index start with 0. Since the tenor of forward rate is fixed, we shall refer to a forward rate only be the reset time and current time. i.e $F(t; reset, reset + tenor)$ will be denoted by $F(t; reset)$ or $F_{reset}(t)$.

2. $T_0, T_1, \cdots, T_{M-1}$ represents the reset times of M forward rates such that T_i is the reset time for the i-th forward rate and is the maturity of the $i-1$th forward rate.

3. $S_{resetTime,tenor}$ represent the swap rate such that it resets at T with the tensor $tenor$. The swaption tensors is on the set $t_0, t_1, \cdots t_{N-1}$ and reset time is a subset of $T_0, T_1, \cdots T_{M-1}$, and shall be denoted by $H_0, H_1, \cdots, H_{K-1}$. We shall use $S_{n,k}$ to denote S_{H_k, t_n}, $0 \leq k \leq K-1, 0 \leq n \leq N-1$.

4. $\sigma_{reset,t}$ or $\sigma_{reset}(t)$ will denote the instantaneous volatility of the rate $F_{reset}(t)$ and $\sigma_i(t) := \sigma_{T_i}(t)$. For each fixed reset time $T_i, 0 \leq i \leq M-1$, we assume that $\sigma_i(t)$ is piecewise constant over each time period $(T_{k-1}, T_k]$ for $k = 0, \cdots, i$

$$\sigma_i(t) := \sigma_{i,k}, \quad t \in (T_{k-1}, T_k]$$

   We define $\tau_i := T_i - T_{i-1}$.

The above notations provide a convenient notation for implementation. For a i-th annual forward contact: $F(t, T_i, 1Y)$ and the associated instantaneous volatility is

$$\sigma_i(t) = \sigma_{i,k} \quad t \in (T_k - \tau_k, T_k], \quad 0 \leq k \leq i.$$

$v_{k,n}$ denote the Black volatility associated to the swap $S_{k,n}$, then by Rabonato's formula

$$T_k S_{k,n}^2 v_{k,n}^2 = \sum_{k \leq i,j \leq k+n} w_i(0) w_j(0) F_i(0) F_j(0) \rho_{i,j} \int_0^{T_k} \sigma_i(t)\sigma_j(t) dt \quad (11)$$

Notice that w_i should write as $w_i^{(k,n)}$ since it depends on individual swap. we follow the convention as in the book to simplify the notation. By the assumption, we have

$$\int_0^{T_k} \sigma_i(t)\sigma_j(t) dt = \sum_{m=0}^{k} (T_m - T_{m-1}) \sigma_{i,m}\sigma_{j,m} = \sum_{m=0}^{k} \tau_m \sigma_{i,m}\sigma_{j,m}$$

5.4. Exact Swaption "Cascade" calibration: Cascade Calibration Algorithm (CCA)

We shall assume that the swaption are a square matrix such that $H_0 = T_0, H_1 = T_1, \cdots, H_{s-1} = T_{s-1}$ and for each reset time T_i, the tensors covered included also $T_0, T_1, ..., T_{s-1}$ in years. Following the idea in [1], we calibrate σ_{ij} in the lower triangle part according to the order demonstrated as follows.

$$\begin{pmatrix} (v_{0,0}, 0) & (v_{0,1}, 1) & (v_{0,2}, 3) & (v_{0,3}, 6) \\ (v_{1,0}, 2) & (v_{1,1}, 4) & (v_{1,2}, 7) & (v_{1,3}, NE) \\ (v_{2,0}, 5) & (v_{2,1}, 8) & (v_{2,2}, NE) & (v_{2,3}, NE) \\ (v_{3,0}, 9) & (v_{3,1}, NE) & (v_{3,2}, NE) & (v_{3,3}, NE) \end{pmatrix}$$

$$\begin{pmatrix} (\sigma_{0,0}, 0) & (\sigma_{0,0}, NA) & (\sigma_{0,0}, NA) & (\sigma_{0,0}, NA) \\ (\sigma_{1,0}, 1) & (\sigma_{1,1}, 2) & (\sigma_{1,2}, NA) & (\sigma_{1,3}, NA) \\ (\sigma_{2,0}, 3) & (\sigma_{2,1}, 4) & (\sigma_{2,2}, 5) & (\sigma_{2,3}, NA) \\ (\sigma_{3,0}, 6) & (\sigma_{3,1}, 7) & (\sigma_{3,2}, 8) & (\sigma_{3,3}, 9) \end{pmatrix}$$

where NE and NA refer to "No Need" and "Not applicable" respectively. More specifically, assuming that the matrix has the shape $s \times s$. We use $\{v_{ij}\}$ to conduct one-to-one calibration to obtain $\{\sigma_{k,l}\}$ in an anti-diagonal order.

$$\begin{aligned} v_{0,0} &\rightarrow \sigma_{0,0} \\ v_{0,1}, v_{1,0} &\rightarrow \sigma_{1,0}, \sigma_{1,1} \\ \cdots &\rightarrow \cdots \\ v_{0,m+n}, \cdots, v_{m+n,0} &\rightarrow \sigma_{m+n,0}, \cdots, \sigma_{m+n,m+n} \\ \cdots &\rightarrow \cdots \\ v_{0,s-1}, \cdots, v_{s-1,0} &\rightarrow \sigma_{s-1,0}, \cdots, \sigma_{s-1,s-1} \end{aligned}$$

Suppose that we need deal with $v_{m,n}$, then

ALGORITHM 5.1.

1. *if* $(m, n) = (m, 0)$*, i.e. we reach to the end of the* m*-th anti-diagonal, we start to the beginning position* $(0, m+1)$ *of the next anti-diagonal.*

2. *otherwise* $n > 0$*, we move from* $(m-1, n+1)$ *to the next position* (m, n) *on the same anti-diagonal. we solve* $\sigma_{m+n,m}$ *in following way:*

$$\begin{aligned} A &= F_{m+n}^2(0) w_{m+n}^2 \tau_{m+n} \\ B &= 2F_{m+n}(0) w_{m+n} \sum_{m \le j \le m+n-1} w_j F_j(0) \sigma_{j,m} \rho_{m+n,j} \tau_m \\ C &= K_{m,n} - T_m v_{m,n}^2 S_{m,n}^2 \end{aligned}$$

where

$$
\begin{aligned}
K_{m,n} &= \sum_{m \le i,j \le m+n} w_i^{(m,n)}(0) w_j^{(m,n)}(0) F_i(0) F_j(0) \rho_{i,j} \\
&\times \int_0^{T_m} \sigma_i^{(m-1,n+1)}(t) \sigma_j^{(m-1,n+1)}(t) dt \\
&= \sum_{m \le i,j \le m+n} w_i^{(m,n)}(0) w_j^{(m,n)}(0) F_i(0) F_j(0) \rho_{i,j} \\
&\times \sum_{0 \le p \le m} \sigma_{ip}^{m-1,n+1} \sigma_{jp}^{m-1,n+1} \\
&= ((FW^{(m,n)})^T) \cdot ((\sigma^{(m-1,n+1)} \cdot (\sigma^{(m-1,n+1)})^T) \otimes \rho^{(m,n)}) \cdot (FW^{(m,n)})
\end{aligned}
$$

where

a) $\sigma^{(m-1,n+1)}(t)$ *are based on the updated* σ *in the step* $(m-1, n+1)$*, but take the block that is associated to involving reset times and tenors in the volatility matrix. Initially, it is assigned to be zero matrix, and the upper index* (m,n) *in* w *emphasizes that the weight vector is based on the swap* $S_{m,n}$*.*

b) $\rho^{(m,n)}$ *is the block in the correlation matrix associated to the relevant forward rates.*

c) $FW^{(m,n)}$ *is the column vector by multiplying* F *and* $W^{(m,n)}$*, but taking the relevant block with related* F*.*

d) $A \otimes B$ *denotes the piecewise multiplication of two matrices with same shape.*

With A, B *and* C*,* $\sigma_{m+n,m}$ *can be solved by*

$$Ax^2 + Bx + C = 0$$

or

$$\sigma_{m+n,m} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \tag{12}$$

REMARK 11.

1. *Cascade calibration uses fixed correlation parameters* ρ*.*

2. *Eq (12) leads to a positive solution if and only if* $C < 0$*. (with positive correlation). Practical experience is that* $C < 0$ *is always met for non-pathological swaptions data. It can be negative on some artificial inputs.*

3. *For the items* $\sigma_{m,n}$ *where* $m \geq s$*, The equation for* $v_{m,n}$ *will contain more than one unknown* $\sigma_{m+n,k}$*. One can assume that all involved unknowns are equal to estimate the solution for* $\sigma_{m,n}$*. The details can be found in Algorithm 7.4.2 on page 330 in [1].*

4. *CCA maps swaption volatilities to model parameters (instantaneous volatility) and the testing results shows that the model can detect misaligned market data by negative or non-real model instantaneous volatilities.*

5. *Real market calibration (calibrated to observable) also shows negative signs in instantaneous volatilities although there are only a few. The negative outputs might be due to the misalignments of the quotes (liquid or reported not in the same date). To avoid this inconvenience, a smoothing method is proposed in [1] by assuming the following parameterizations of the market swaption matrix:*

$$v_{swaption}(S,T) = \gamma(S) + (\frac{e^{f \ln(T)}}{eS} + D(S))e^{-\beta e^{p \ln(T)}} \tag{13}$$

where

$$\begin{aligned} \gamma(S) &= c + (e^{h \ln(S)} + d)e^{-be^{m \ln(S)}} \\ D(S) &= \delta + (qe^{g \ln(S)} + r)e^{-se^{t \ln(S)}} \end{aligned}$$

The smoothed swaption volatilities are fairly close to the target ones and absolute difference are all less than 0.005.[2]*. Running CAA based on smoothed swaptions provides all positive instantaneous volatilities, suggesting that irregularity and illiquid in the input swaption matrix can cause negative or even imaginary values in the calibration instantaneous volatilise. By smoothing the input data before calibration, this undesired features can be avoided. Smoothing inputs also improves the terminal correlations to make the correlation monotonic. So the model is detecting a misalignment in the market, instead of concluding that the model is not suitable for the calibration.*

6. *Notice that smoothing method does not improve much to reduce the worrying features that the instantaneous volatility can reach to* 30% *along the diagonal. Also the desired humped shape are not preserved in the etiolation.*

7. *One-to-one correspondence helps n computing sensitivities with respect to swaptions volatilities since one knows on which* σ*'s one needs to act in order to influence a single swaption volatility.*

[2] See Table 7.7 in [1]

5.5. A Numerical testing result

The cap market and swaption market information are from Table 7.1 and Table 7.4 respectively. Using the rank 2 correlation parametrization $\rho_{ij} = \cos(\theta_i - \theta_j)$ with the following selection (page 331 in [1]).

$$\begin{aligned}\theta \ = \ & [0.0147, 0.0643, 0.1032, 0.1502, 0.1969, 0.2239,\\ & 0.2771, 0.2950, 0.3630, 0.3810, 0.4217, 0.4836,\\ & 0.5204, 0.5418, 0.5791, 0.6496, 0.6679, 0.7126, 0.7659]\end{aligned}$$

Table II. Output of CCA

T	1	2	3	4	5	6	7	8	9	10
1	0.1800									
2	0.1548	0.2039								
3	0.1285	0.1559	0.2329							
4	0.1178	0.1042	0.1656	0.2437						
5	0.1091	0.0988	0.0973	0.1606	0.2483					
6	0.1131	0.0734	0.0781	0.1009	0.1618	0.2627				
7	0.1040	0.0984	0.0502	0.0737	0.1128	0.1670	**0.2610**			
8	0.0940	0.1052	0.0938	0.0319	0.0864	0.0991	0.1671	0.2743		
9	0.1065	0.0790	0.0857	0.0822	0.0684	0.0563	0.0905	0.1724	0.2859	
10	0.1013	0.0916	0.0579	0.1030	0.1514	-0.0413	0.0448	0.0823	0.1676	0.2737

REMARK 12.

1. *we only provide the instantaneous volatilise that can be uniquely determined. One can generalize the CCA based on equal multiple unknowns assumption to leverage all market volatility information to cover the matrix shown in Table 7.5 on page* 333 *[1].*
2. *The testing results are same to Table* 7.5 *on page* 333 *for all items with maturity less than* 7. *Small difference are observed when longer maturities. those errors might be due to large amount of numerical calculating and possible different algorithm for the solution of quadratic equations.*

5.6. Further discussion about CCA

1. Two common shape pattern for $[\sigma_{T_i,t_k}, k \leq i \leq M-1]$ for a fixed $k (0 \leq k \leq M-1)$: humped (normal) and decreasing.

2. LFM owns its popularity to the compatibility with Black's market formula for caps. and, under the deterministic assumption, there is a unique solution reproducing the ap quotes. On the other hand, due to incompatible with Black formula used in Swaption market, it is not possible solve the LFM calibration problem to swaption by the simple inversion of a BS formula. This fact implies persistence of calibration errors. More importantly, considering the high number of parameters typical of this model, one could suspect the problem to be largely undetermined and the solution found to be of limited meaning. Although we use Reb formula for efficient calibration methods, those methods are still based on minimizing loss function expressing distance between market and model prices. the methods are far from the "unique solation", "immediate" and "exact" calibration type of the Black formula, which market model would like to incorporate.

3. we would like to obtain an immediate calibration of the LFM to swaption, thus having a unique central market model respecting the philosophy for which market models have been designed. (if non-negligible calibration error is tolerated, short rate model are at times preferred because can easily incorporate explicitly features that are considered to be economically important such as for example means reversion.). CCA is a step to achieve "exact immediate calibration" feature for both cap and swaption. Through closed form formulas, it gives an analytical, partially instantaneously and unique solution to the calibration problems. The other side of the coin is that CCA is not flexible and no constraints can be set on the outputs. (so we have negative instantaneous volatility if not smoothing data). It is important to realize that the smoothing is not a solution, since it alters market quotations and introduces not negligible calibration errors. it only points out that inconsistence or misalignments in input data could be responsible for the encountered troubles.

4. trade off between TSV (term structure of volatility) and TC (Terminal correlation). more sophisticated structured correlation (with high rank) leads to more regular TC, but with a less regular and less stable evolution of TSV.

5. CCA produces negative imaginary $\sigma_{i,j}$ in correspondence of linearly interpolated swaptions volatiities. As such, log-linear (or power) is proposed:

$$v_{T,t} = \beta_t T^{-\alpha_t}, or \ln(v_{T,t}) = \ln \beta_t - \alpha_t \ln T$$

By using linear log interpolation, CCA no longer produces negative outputs (using Rebonato three-parameters pivots correlation algorithm). An evidence that linear interpolation introduce inconsistencies and CCA detects such inconsistencies by producing negative/imginary σs.

6. No exogenous interpolation, such as log-linear, has proven enough to be trouble-free to avoid numerical problems in a complete calibration with a broad range of model dimensions. Generally satisfactory results have been ensured only through modifications of market data, making clear the importance of input consistency but spoiling exact calibration, and leaving room for model specification.

7. Modify CCA so it can work with missing holes. Filling the holes by interpolation based on internal qualities of the model obtained from the calibration to the directly available data only, thus allowing artificial values to be coherent with the underlying model by construction. The idea behind is "equal multiple unknowns".

References

1. Damiano Brigo and Fabio Mercurio (2006).*Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit*2nd Edition, Springer
2. Xiaorong Zou (2018) *Libor Model Implemenation in Python*, https://github.com/xiaorzou/LiborMarketModel